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Abstract

The antenna problem consisting of exciting the infinite wire with an arbitrary distribution of field along the axis of the wire is discussed. The special case of excitation by a delta function voltage source is solved. In the solution to this problem both the discrete modes mentioned in Part I and the two sets of continuous modes must be included in order to have a complete set of modes. The far field and orthogonality properties of both the discrete and continuous modes are discussed. Finally, expressions are given for the current on the antenna. From this current the input conductance of the antenna can be obtained.
PART II

EXCITATION OF THE INFINITELY LONG ANTENNA

2.1. Introduction

The question of what discrete waveguide modes may exist on an open waveguide structure consisting of an infinite thin horizontal wire near a conducting earth has been answered in Part I. It is logical to ask to what extent each mode is excited when the infinitely long wire is driven by a specified source. The problem of driving an infinite wire by a delta function voltage source has been previously discussed by Chang[1]. In that work only the transmission line or slow wave mode was used since the earth was assumed to be very highly conducting and terms leading to the second discrete mode which represented the fast wave propagation were dropped. In addition to the discrete mode, a continuous spectrum of modes is also included. These modes are known to contribute to the radiation field of the excited wire.

In this part of the research, using the more accurate modal equation from Part I, the excitation of the two discrete modes by a delta function voltage source is found. In addition, the excitation of two sets of continuous modes, each associated with one of the branch cuts of the modal equation, is calculated. The input conductance of the wire and the far fields produced by each of the modes
are computed. Lastly, the set of modes consisting of the discrete and continuous modes is shown to be a complete orthogonal set.

2.2. Derivation of the Integral Equation for the Current Distribution on a Horizontal Antenna

Consider first, as depicted in Fig. 2.1, the field of a horizontal Hertzian dipole of moment \( \text{Idx} \) located in a dielectric medium with electrical constants \( \varepsilon_1, \mu_1 \) at a distance \( I \) (meters) above a homogeneous conducting half-space with electrical constants \( \varepsilon_2, \mu_2, \sigma_2 \). It is assumed that \( \varepsilon_1 = \varepsilon_0 \), the permittivity of free space and \( \mu_1 = \mu_2 = \mu_0 \), the permeability of free space. The formal solution for the \( x \) component of the electric field is known as \([2,3]\)

\[
E_x = \left( \frac{i\varepsilon_0}{4\pi} \right) \text{Idx} \left[ \frac{d}{dx} \left( G_{11} - G_{12} + \frac{2\pi}{N^2} \right) + k_1^2 \left( G_{11} - G_{12} + 2\pi \right) \right]
\]

(2.1a)

where

\[ G_{11} = \exp(ik_1R_{11})/(k_1R_{11}), \]

\[ R_{11} = [x^2 + y^2 + (z - d)^2]^{1/2} \]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-k_1u_1(z-d)\right\} \frac{\exp(-ik_1y\lambda)d\lambda}{u_1} \exp(ik_1x\alpha) \, d\alpha
\]

\[
= i\pi \int_{-\infty}^{\infty} H_0^{(1)}[k_1\zeta[(z-d)^2 + y^2]^{1/2}] \exp(k_1x\alpha) \, d\alpha
\]

(2.1b)
provided the branch cuts are chosen such that

\[ 0 < \arg \zeta < \pi. \]  \hfill (2.1c)

Also,

\[ G_{12} = \exp(ik_1 R_{12})/(k_1 R_{12}), \quad R_{12} = [(x^2 + y^2 + (z+d)^2)^{1/2} \]

\[ = i\pi \int_{-\infty}^{\infty} H_0^{(1)}[k_1\zeta[(z+d)^2 + y^2]^{1/2}] \exp(ik_1 x\alpha) \, d\alpha. \] \hfill (2.1d)

\[ Q = \int_{-\infty}^{\infty} \{ \int_{-\infty}^{\infty} (u_2 + n^2 u_1)^{-1} \exp[-k_1 u_1(z+d)] \]

\[ \exp(-ik_1 y\lambda) \, d\lambda \} \exp(ik_1 x\alpha) \, d\alpha \]

\[ = \frac{\pi n^2}{2\alpha^2} \int_{-\infty}^{\infty} Q(y,z;\alpha) \exp(ik_1 x\alpha) \, d\alpha. \] \hfill (2.1e)

\[ P = \int_{-\infty}^{\infty} \{ \int_{-\infty}^{\infty} (u_1 + u_2)^{-1} \exp[-k_1 u_1(z+d)] \]

\[ \exp(-ik_1 y\lambda) \, d\lambda \} \exp(ik_1 x\alpha) \, d\alpha \]

\[ = \frac{\pi}{2} \int_{-\infty}^{\infty} P(y,z;\alpha) \exp(ik_1 x\alpha) \, d\alpha. \] \hfill (2.1f)

\[ u_1 = (\lambda^2 + \alpha^2 - 1)^{1/2} = (\lambda^2 - \zeta^2)^{1/2} \]

\[ u_2 = (\lambda^2 + \alpha^2 - n^2)^{1/2}, \quad \text{Re} \, (u_1), \text{Re} \, (u_2) \geq 0 \] \hfill (2.1g)
\( n \) (the refractive index of the conducting medium)
\[
= [\varepsilon_r (1 + i\delta)]^{1/2}, \quad \varepsilon_r = \varepsilon_\infty / \varepsilon_0
\]

\( \delta \) (loss tangent) = \( \frac{\sigma}{\omega \varepsilon_r \varepsilon_0} \)

\( k_1 \) (the free space wave number) = \( \omega (\mu_0 \varepsilon_0)^{1/2} \)

\( \varepsilon_0 \) (the free space characteristic impedance) = 120\( \pi \) ohm

and

\( \exp(-i\omega t) \) is the suppressed time dependence.

Carrying out the differentiations in Eq. 2.1a, \( E_x \) can be expressed as

\[
E_x = \frac{-\varepsilon_0 k_1^2}{4} \text{Idx} \int_{-\infty}^{\infty} \{ \varepsilon^2 [H_0^{(1)}(k_1 \varepsilon ((z-d)^2 + y^2)^{1/2}] - H_0^{(1)}(k_1 \varepsilon ((z+d)^2 + y^2)^{1/2}] \}
\]

\[+ \text{P}(y,z;\alpha) - \text{Q}(y,z;\alpha) \exp(ik_1 x\alpha) \text{ d}\alpha, \quad (2.2)\]

Consider next the structure of a horizontal thin tubular antenna which has a radius \( a \) (meters), a half-length \( h \), and a height \( I \) above a conducting half-space (Fig. 2.2). A thin wire approximation \( (a << d, \lambda_0 \) (the free space wavelength)) is made so that the current has only an \( x \) component and is assumed to be angularly independent on the surface of the antenna. At the center of the antenna
Geometry of a Horizontal Hertzian Dipole Above the Earth

Geometry of a Horizontal Antenna Above the Earth
a delta function voltage generator of unity magnitude is applied such that the average tangential electric field on the antenna surface satisfies

$$E_x(x) = -\delta(x) \quad (2.3)$$

(Note that the solution to this problem for any source distribution can be obtained in the same manner as followed here. It is only necessary to know the axial component of the impressed source field distribution along the antenna surface.)

It is now possible to set up an integral equation for the current on the antenna using Eqs. 2.2 and 2.3 based upon the concept of superposition. Using these, the integral equation becomes

$$\int_{-h}^{h} I(x') M(x - x') \, dx' = \frac{4}{c_0 k_1} \delta(x) - h \leq x \leq h \quad (2.4a)$$

where

$$M(x - x') = \int_{-\infty}^{\infty} M(\alpha) \exp(ik_1(x - x')\alpha) \, d\alpha \quad (2.4b)$$

$$M(\alpha) = \xi^2 \left[ H_0^{(1)}(A\xi) J_0(A\xi) - H_0^{(1)}(2D\xi) \right] + P(\alpha) - Q(\alpha) \quad (2.4c)$$

where

$$P(\alpha) = P(o,d;\alpha) \text{ and } Q(\alpha) = Q(o,d;\alpha).$$

More specifically,
\[ P(\alpha) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-2Du_{1})}{u_{1} + u_{2}} \, d\lambda; \quad (2.4d) \]

\[ Q(\alpha) = \frac{2\alpha^{2}}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-2Du_{1})}{u_{2} + \beta^{2}u_{1}} \, d\lambda; \quad (2.4e) \]

where

\[ A = k_{1}a, \quad D = k_{1}d. \]

Equation 2.4c is just the exact modal equation of Chapter 1 given by Eq. 1.5.

2.3. Solution to the Integral Equation for the Infinite Antenna

In principal this integral equation can now serve as a basis for finding the total current distribution \( I(x) \) on an antenna of arbitrary length with a delta function source generator. In the case of an infinitely long antenna, an analytical solution to Eq. 2.4 is available from the convolution theorem\(^{[4]}\).

\[ I(x) = \frac{2}{\pi \zeta_{0}} \int_{-\infty}^{\infty} \frac{\exp(ik_{1}x\alpha)}{M(\alpha)} \, d\alpha \quad (2.5) \]

where the convolution integral

\[ \int_{-\infty}^{\infty} f(z') g(z-z')dz' = \int_{-\infty}^{\infty} F(\alpha) \, G(\alpha) \exp(ik_{1}x\alpha) \, d\alpha \quad (2.6) \]

was used. The Fourier transforms are defined as

\[ M(\alpha) = \frac{k_{1}}{2\pi} \int_{-\infty}^{\infty} M(x) \exp(-ik_{1}x\alpha) \, dx \quad (2.7) \]
The input conductance of the antenna can be obtained from Eq. 2.5 by evaluating the real part of the current at $x = 0$.

The approximate form of $M(\alpha)$ is used in the evaluation of Eq. 2.5. It is defined by Eqs. 2.4c, 1.9, 1.13, 1.14, 1.18, and B.3-B.6. Equation 2.5 is evaluated by deforming the integration contour from the real axis to the dashed line shown in Fig. 2.3. This line envelops the branch cuts of the two branch points of $M(\alpha)$; $\alpha = 1$ and $\alpha = \alpha_B = (1 - 1/n^2)^{1/2}$. Also enclosed by the new contour are the two zeros of $M(\alpha)$, $\alpha_{p1}$ the transmission line zero and $\alpha_{p2}$ the fast wave zero. Since each $\alpha$ represents a completely different distribution of electromagnetic field in the cross-sectional plane of the horizontal wire, the integral contributions can be interpreted physically as two discrete modes corresponding to the two pole residues and two continuous spectra of modes corresponding to the values of $\alpha$ along the two branch cuts. As will be clear in later sections, the individual modes of the continuous spectra do not obey the radiation condition. However, in any physical problem the sum of all the continuous modes must obey the radiation condition.

The current is represented as the sum of four terms; the residues of the poles at $\alpha_{p1}$ and $\alpha_{p2}$ and the two branch cut integrations. The two residue contributions are

$$I_{p1}(x) = \frac{4i}{\zeta_0} \frac{\exp(ik_1 x \alpha_{p1})}{M'(\alpha_{p1})}$$

and

$$I_{p2}(x) = \frac{4i}{\zeta_0} \frac{\exp(ik_1 x \alpha_{p2})}{M(\alpha_{p2})},$$

(2.8)
Fig. 2.3
Integration Contour in the Complex $\alpha$ Plane
where \( M'(\alpha) \) is the derivative of \( M(\alpha) \) with respect to \( \alpha \) and is given as

\[
M'(\alpha) = -2\alpha \left( H_0^{(1)}(A\zeta) - H_0^{(1)}(2D\zeta) \right) + \alpha\zeta \left( AH_1^{(1)}(A\zeta) \right)
- 2D H_1^{(1)}(2D\zeta) + P'(\alpha) - Q'(\alpha).
\] (2.9)

\[
P'(\alpha) = -\frac{2\alpha}{n^2 - 1} \left( H_0^{(1)}(2D\zeta)(i2D - 1) + 2D\zeta H_1^{(1)}(2D\zeta) \right).
\]

\[
Q'(\alpha) = \frac{4\alpha}{n^2} \left( H_0^{(1)}(2D\zeta) + \exp(-i2D/n)(W_0 + W_d)/n\pi \right)
+ \frac{2\alpha^3}{n^2\zeta} \left( 2D H_1^{(1)}(2D\zeta) - \exp(-i2D/n)(W_{op} + W_{dp})/n\pi \right).
\]

\[
W_0(\alpha) = \frac{2\pi i}{(\frac{1}{n^2} - \zeta^2)^{1/2}} - \frac{2}{(\frac{1}{n^2} - \zeta^2)^{1/2}} \{ \ln \left( \frac{1}{n} - i\left( \frac{1}{n^2} - \zeta^2 \right)^{1/2} \right)
- \ln \zeta + i\pi \}.
\]

\[
W_{op}(\alpha) = \frac{i\zeta}{(\frac{1}{n^2} - \zeta^2)^{1/2}} W_0(\alpha) - \frac{2}{(\frac{1}{n^2} - \zeta^2)^{1/2}} \}
\]

\[
\left( \frac{\frac{1}{n^2} - \zeta^2}{(\frac{1}{n^2} - \zeta^2)^{1/2}} \left( \frac{1}{n} - i\left( \frac{1}{n^2} - \zeta^2 \right)^{1/2} \right) - \frac{1}{\zeta} \right).
\]
\[ W_d = -i\pi \left[ -\left(2D\right)^2 H_1^{(1)}(2D\zeta)/\zeta + 2D H_0^{(1)}(2D\zeta)/\zeta^2 \right]/2n^2 \\
+ I_0(\alpha)(1 + 1/(2n^2\zeta^2) + iI_1(\alpha)/n) \]

\[ W_{dp} = -i\pi \left[ -\left(2D\right)^2 H_1^{(1)}(2D\zeta)/\zeta^2 + (2D)^3(H_0^{(1)}(2D\zeta)) \\
- H_1^{(1)}(2D\zeta)/2D\zeta)/\zeta - 4D H_0^{(1)}(2D\zeta)/\zeta^3 \\
- (2D)^2 H_1^{(1)}(2D\zeta)/\zeta^2)/2n^2 - I_0(\alpha)/n^2\zeta^3 \right. \\
+ \left. (1 + 1/2n^2\zeta^2) I_{op}(\alpha) + iI_1p(\alpha)/n \right]. \]

\[ I_0(\alpha) = 2D H_0^{(1)}(2D\zeta) + 2D \pi \left( H_0^{(1)}(2D\zeta) H_1^{(1)}(2D\zeta) \\
- H_1(2D\zeta) H_0^{(1)}(2D\zeta))/2 \right. \]

\[ I_1(\alpha) = (2D\zeta H_1^{(1)}(2D\zeta) + 2i/\pi)/\zeta^2 \]

\[ I_{op}(\alpha) = -\left(2D\right)^2 H_1^{(1)}(2D\zeta) + (2D)^2 \pi/2 \left(\frac{2}{\pi} - H_1(2D\zeta)) \right) H_1^{(1)}(2D\zeta) \\
+ H_0(2D\zeta)H_0^{(1)}(2D\zeta) - H_1^{(1)}(2D\zeta)/2D\zeta \]

\[ - (H_0(2D\zeta) - H_1(2D\zeta)/2D\zeta) H_0^{(1)}(2D\zeta) \]

\[ + H_1(2D\zeta) H_1^{(1)}(2D\zeta) \]
\[ I_{1p}(\alpha) = -2 (2D_\zeta H_1^{(1)}(2D_\zeta) + 2i/\pi)/\zeta^3 \]
\[ + (2D H_1^{(1)}(2D_\zeta) + (2D)^2 \zeta (H_0^{(1)}(2D_\zeta) \]
\[ - H_1^{(1)}(2D_\zeta/2D_\zeta))/\zeta^2. \]

\( H_0^{(1)}(2D_\zeta), H_1^{(1)}(2D_\zeta) \) are the zero and first order Hankel function of the first kind, respectively. \( H_0(2D_\zeta) \) and \( H_1(2D_\zeta) \) are the zero and first order Struve function, respectively. In Eq. 2.9 three terms of \( W_d(\alpha) \) as defined in Appendix B, Eq. 3.6 are used.

A better approximation can be obtained by using more terms of the series.

The integration along the branch cut terminating at \( \alpha = 1 \) can be evaluated by dividing the integration into two parts. It can be written as

\[ I_{B1}(x) = \frac{i}{\pi \zeta_0} \left( \int_0^1 \frac{M_\pi(\alpha) - M_\sigma(\alpha)}{M_\sigma(\alpha) M_\pi(\alpha)} \exp(ik_1x\alpha) \, d\alpha \right) \tag{2.10} \]
\[ - i \int_0^\infty \frac{M_\pi(i\alpha') - M_\sigma(i\alpha')}{M_\sigma(i\alpha') M_\pi(i\alpha')} \exp(-k_1x\alpha') \, d\alpha'. \]

The \( \pi \) and the \( \sigma \) subscripts on \( M \) in Eq. 2.10 refer to the argument of \( \zeta \).

The integration on the second branch cut can be transformed into an integration on a real path by the substitution
\[ \alpha = (\alpha_B^2 - S^2)^{1/2}, \, S \text{ real.} \] The integral becomes
\[ I_{B2}(x) = -\frac{i}{\pi \sigma} \int_0^\infty \frac{M_\pi(W) - M_0(W)}{W^{\frac{1}{2}} M_\pi(W) M_0(W)} \exp(ik_1xW) \, dW. \quad (2.11) \]

\[ W = (\alpha_B^2 - s^2)^{1/2}; \quad \text{Re}(W) > 0 \]

\[ s = (\alpha_B^2 - w^2)^{1/2}. \]

The subscripts 0 and \( \pi \) on \( M \) in Eq. 2.11 refer to the argument of \( S \).

2.4. Results

It can be shown that as \( n \to \infty \), the second integration in Eq. 2.10 is purely imaginary. It has been numerically verified that for large \( n \) this integration contributes very little to the real part of \( I_{B1} \). This fact is important since for \( x \) very small the second integration converges very slowly. Since we are only interested in the real part of \( I_{B1} \) near \( x = 0 \) (the input conductance), the second integration can be dropped with little error. Table 2.1 shows a typical error incurred by neglecting the second integration.

| TABLE 2.1 |
| EFFECT OF SECOND INTEGRATION ON INPUT CONDUCTANCE |

<table>
<thead>
<tr>
<th>Second integration</th>
<th>Input conductance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Not included</td>
<td>1.692</td>
</tr>
<tr>
<td>Included</td>
<td>1.667</td>
</tr>
<tr>
<td>( n = 7.43 + i6.73 ) \quad ( d = .45\lambda ) \quad ( a = .01\lambda )</td>
<td></td>
</tr>
</tbody>
</table>
Figures 2.4 and 2.5 display the four currents as a function of distance along the antenna from the source. In each case most of the phase variation of the current has been eliminated by multiplication by \( \exp(ik_1x) \). The first branch current decays rapidly away from the source. (Its imaginary part is unbounded at \( x = 0 \) due to the unphysical delta function source.) The second branch current decays more slowly. It is expected that far away from the source it will decay approximately as \( \text{Im}(1 - \frac{1}{n^2})^{1/2} \) since most of the contribution to the integration comes from near the branch point. This can be verified by multiplying the second branch current by \( \exp(ikx(1 - \frac{1}{n^2})^{1/2}) \) and noting that the result is asymptotically constant as \( x \) becomes large.

The current due to the fast wave zero decays more slowly than that due to the transmission zero. This is to be expected from the relative position of the zeros in the complex \( \alpha \) plane as shown in Fig. 1.9.

Figure 2.6 is a plot of the input conductance of the antenna as a function of its height above the ground for \( n = 7.43 + i 6.73 \). Also plotted are the values of the four component parts of the input conductance. Notice that as \( I \) becomes very small the transmission line contribution becomes dominant indicating that the antenna can be treated as a lossy transmission line. This coincides with the fact that for large \( n \) and small \( d \) radiation from the structure is minimized and that the \( Q(\alpha) \) term which is responsible for both the second branch current and the fast wave pole can be neglected. If \( Q(\alpha) \) is neglected then it can be shown that the transmission line pole reduces to the one found by Chang\(^1\). The total input conductance oscillates
CONTINUOUS SPECTRUM CURRENTS

\[ n = 7.43 + i \times 6.73 \]
\[ a = 0.01 \lambda \]
\[ d = 0.25 \lambda \]

Fig. 2.4 Currents Due to Branch Cut Integrations
Fig. 2.5 Currents Due to Residue Contributions
Fig. 2.6
Input Conductance of an Infinitely Long Wire Above Earth

\[ n = 7.43 + i \cdot 6.73 \]
\[ \alpha = 10^{-2} \lambda \]
around 3.11 m-mhos which agrees with the free space value of the
input conductance obtained by Shen, Wu, and King[5]. Eventually,
as d goes to infinity, the input conductance due to the first branch
integration should equal the free space value. This has been
numerically verified. Additional support to this conclusion is
found in the fact that for d greater than 1.3λ (Fig. 2.7), the input
conductance due to the fast wave term begins to decrease. The
seemingly unphysical negative input conductance contribution of the
second branch integration can be explained in light of the fact that
neither the first branch modes nor the second branch modes indi-
dually satisfy the radiation condition. It is believed that there is
no physical situation in which only the second branch modes can be
excited or a combination of the first and second branch modes such
that the input conductance is negative. In this regard, notice that
the sum of the first and second branch contributions is always posi-
tive in Fig. 2.6.

Figure 2.8 is a plot of the input conductance of the antenna
as a function of its height above the ground for n = 5.52 + i 4.53.
It can be seen that the total input conductance is not very sensitive
to ground conductivity. This conclusion is not true for the excita-
tion of the different parts of the current. For an earth with a
smaller refractive index, the excitation of the fast wave is increased
while the excitation of the transmission wave is decreased. This is
an important conclusion in light of the fact that for some applica-
tions it is desirable to couple energy as efficiently as possible
into the fast wave due to its small attenuation. The excitation of
EXCITATION OF THE FAST WAVE

\[ n = 7.43 + i6.73 \]
\[ a = 10 - 2\lambda \]

Fig. 2.7 Excitation of the Fast Wave by a Voltage Generator
Fig. 2.8
Input Conductance of an Infinitely Long Wire Above Earth
of the branch currents appears to be less sensitive to changes in the earth's refractive index.

2.5. Far Field Patterns of the Modes

The transverse magnetic fields of any one of the modes may be obtained via the Hertz Potentials from Eqs. 1.2, 1.3, and 1.4. The asymptotic expansions of the Hertz potentials are found in Appendix C.

An examination of Eqs. C.14 and C.20 reveals that any mode which has a propagation constant which is not located on one of the two branch cuts of the modal equation in the complex alpha plane will decay exponentially. Thus both of the discrete modes decay exponentially in the far field. They are said to be bound to the wire.

The behavior of modes of the continuous spectra must be considered more carefully. On the branch cut associated with the branch point at $\alpha = 1$, $\arg \xi = 0$ on one side of the cut and $\pi$ on the other side. A mode (single value of $\alpha$) consists of a superposition of the two solutions. Since $\arg \xi = 0, \pi$ only the first terms of Eq. C.14 and Eq. C.20 remain finite and the potentials can be written

$$\Pi_{1x} \sim \frac{G_0}{\sqrt{r}} \left\{ \exp(i k_1 r |\xi|) + R_{\alpha_1} \exp(-i k_1 r |\xi|) \right\}$$

(2.12)

and

$$\Pi_{1x}^* \sim \frac{G_0}{\sqrt{r}} \left\{ \exp(i k_1 r |\xi|) + R_{\alpha_1}^* \exp(-i k_1 r |\xi|) \right\}$$

(Branch associated with $\alpha = 1$ modes)
where $R_{\alpha 1}$ and $R_{\alpha 1}^*$ are two unknown functions of $z$. These modes exist in the far field for all values of $\theta$.

On the second branch cut only the second terms of Eqs. C.14 and C.20 remain finite. The potentials of a mode from this branch can be written

\[
\Pi_{1x} \sim C_e \exp(-i k_1 z/n) \{\exp(-i k_1 \lambda_p y) + R_{\alpha 2} \exp(i k_1 \lambda_p y)\}
\]

\[
(2.13)
\]

\[
\Pi_{1x}^* \sim C_M \exp(-i k_1 z/n) \{\exp(-i k_1 \lambda_p y) + R_{\alpha 2}^* \exp(i k_1 \lambda_p y)\}
\]

$y < 0, z > 0$

(Second branch modes)

These modes are surface bound modes since they only exist in a region very close to the surface of the earth.

2.6. Completeness and Orthogonality Properties of the Modes

It will now be shown that the set of modes described in this work forms a complete orthogonal set for this open waveguide structure. Thus, the total field for any source in the presence of the structure can be expanded as a weighted sum over the modes\^[6]. For example the total field of an electric dipole, modified by the presence of the structure, can be expanded as a weighted sum over the modes.

At least for any physically realizable source the problem of finding the fields and currents for this structure can be solved
using Fourier transform techniques identical to those just used for the delta function excitation case. The resulting expression is an inverse Fourier transform of the Fourier transform of the arbitrary source field distribution divided by M(α). (See note after Eq. 2.3.) By deforming the contour, the inverse Fourier transform can be written as a weighted sum over the discrete and continuous modes. Thus the set of modes is complete. The total fields on this structure for at least any physically realizable source can be expressed as a weighted sum over the modes. Building upon a great deal of previous work[7,8,9], an orthogonality property of the modes will now be presented so that a method is available for expanding an arbitrary source field as a weighted sum over the modes.

The waveguiding structure under consideration is an open waveguide inhomogeneously filled with lossy and perfectly conducting material. The guide is invariant in the x direction. Collin[6] has shown that for closed waveguides (with perfectly conducting walls) inhomogeneously filled with perfectly conducting, lossy or lossless material the following orthogonality property holds

$$\iiint_{S} E_{tN} \times H_{tM} \cdot \vec{a}_x \, dS = 0 \quad M \neq N \quad (2.14)$$

where the integration is over the cross section of the guide and the subscript t refers to the transverse fields. M and N are integers which designate the different modes.

The validity of Eq. 2.14 for the discrete and continuous modes of the open waveguiding structure will now be examined.
It can be shown[6] that the evaluation of the surface integral (Eq. 2.14) reduces to the evaluation of line integrals of products of the E and H fields of different modes over the perimeter of S. Since S is an infinite surface for the open structure it is necessary to know the asymptotic expressions for the E and H fields.

Manenkov[7] has shown that Eq. 2.14 holds for modes of any open waveguiding structure which decay exponentially for large values of \( r = (y^2 + z^2)^{1/2} \). As pointed out in Section 2.5, the discrete modes of the structure under study decay exponentially in the asymptotic range. Thus, they are orthogonal to each other in the sense of Eq. 2.14. It also follows from the exponential behavior of the discrete modes that any mode of the discrete spectrum is orthogonal to any mode of the continuous spectrum.

By considering S to be the area of a circle with a radius which approaches infinity in the limit, Manankov proved the orthogonality principle for any two modes of a spectrum of modes (designated by \( \chi \) and \( \chi' \), their respective transverse wavenumbers) which behave asymptotically as

\[
\frac{G(\theta)}{\sqrt{r}} \{\exp(i\chi r) + R_a(\chi) \exp(-i\chi r)\}
\]

(2.15)

\( \chi \) any real number

Thus, according to Eqs. 2.12 and 2.15, continuous modes located on the branch cut associated with \( \alpha = 1 \) in the complex plane are orthogonal to each other in the sense of Eq. 2.14.

Using a proof identical to Manenkov's with the exception that the large circular surface is replaced by a rectangular surface,
orthogonality can be shown to hold between any two modes (designated by $\lambda_p$ and $\lambda'_p$) which behave asymptotically as

$$\mathcal{C} \exp(bz) \{\exp(-ik_1\lambda_p y) + R_b(\lambda_p) \exp(ik_1\lambda_p y)\}$$

(2.16)

where $R_e(b) < 0$, $\lambda_p$ any real number

in the region $y < 0$, $z > 0$. The modes behave in a similar fashion in the other regions of space. Thus orthogonality is proved between modes of the continuous spectrum on the second branch cut in the complex $\alpha$ plane, according to Eqs. 2.13 and 2.16.

Lastly, it can be shown that orthogonality holds between a mode of the first continuous spectrum and one of the second continuous spectrum. The proof follows from the fact that continuous modes of the first branch exist over the whole far field and decay as $r^{-1/2}$ while modes of the second branch exist only near the surface of the earth. Line integrals over the perimeter of $S$ of products of these modes are thus zero.

Thus, orthogonality holds between any two modes of the set. The set of modes is thus complete and orthogonal.

2.7. Concluding Remarks

The excitation by a delta function voltage source of the different modes which can exist on an infinite horizontal wire near the earth has been investigated. It has been found that the fast wave mode which has a small attenuation can be a significant part of the current on the wire especially if the wire is not electrically
close to the earth and the ground is not too highly conducting. Since there are applications for which it is desired to couple energy into the fast wave mode efficiently, different methods of exciting the wire should be investigated. It appears from Figs. 1.7 and 1.8 of Chapter 1 that the fast wave mode may be more efficiently excited by a source which has its fields concentrated above the wire.

2.8. References


APPENDIX C

ASYMPTOTIC EXPANSION OF THE POTENTIALS

It is known that the complete electromagnetic field of any mode on the wire can be obtained from the x component of the electric and magnetic Hertz potentials ($\Pi_x$ and $\Pi_x^*$, respectively). The asymptotic expansions of these potentials will be obtained. The fields can be obtained from Eqs. 1.3. From Eqs. 1.2 it is known that for any mode

$$\Pi_{1x} = -\frac{Z_0 I(x)}{8k_1} \left[ \{H_0^{(1)}(k_1 \zeta R_1) - H_0^{(1)}(k_1 \zeta R_2)\} \right. + \left. \frac{1}{\zeta^2} \{P(y,z;\alpha) - Q(y,z;\alpha)\} \right] \tag{C.1}$$

where

$$R_1 = [(z - d)^2 + y^2]^{1/2}$$

and

$$R_2 = [(z + d)^2 + y^2]^{1/2}$$

Under the approximation $u_2 = -i$ in the dominant term of $P(y,z;\alpha)$ is

$$\frac{i 2\zeta(z + d)}{n R_2} H_1^{(1)}(k_1 \zeta R_2). \tag{C.2}$$

Using the same approximation $Q(y,z;\alpha)$ becomes
\[
\frac{2\alpha^2}{i\pi n^2} \int_{-\infty}^{\infty} \frac{\exp(-u_1 k_1 (z + d)}{u_1 - i/n} \exp(-ik_1 y \lambda) \, d\lambda
\]

\[= \frac{2\alpha^2}{i\pi n^2} \{i \pi H_0^{(1)}(k_1 \zeta R_2) + I\}
\]

where

\[I = \frac{i}{n} \int_{-\infty}^{\infty} \frac{\exp(-u_1 k_1 (z + d)}{(u_1 - i/n) u_1} \exp(-ik_1 y \lambda) \, d\lambda.
\]

I can be evaluated by deforming its contour around the branch cut in the upper half plane (valid if \(y < 0\)). I becomes the residue of the pole at \(\lambda = \lambda_p = (\zeta^2 - 1/n^2)^{1/2}\) plus the branch cut integration. Evaluating the residue and using the transformation \(\lambda = (\zeta^2 - t^2)^{1/2}\) in the branch integration, I becomes

\[I = \frac{i}{n} (I_R + I_B)
\]

where

\[I_R = \frac{2\pi i \exp(-ik_1(z + d)/n) \exp(-ik_1 \lambda_p y)}{\lambda_p}
\]

\[I_B = \int_{-\infty}^{\infty} \frac{\exp(-itk_1 d) \exp(-itk_1 z - ik_1 y(\zeta^2 - t^2)^{1/2}) \, dt}{(\zeta^2 - t^2)^{1/2} (t - 1/n)}
\]

\[0 < \arg(\zeta^2 - t^2)^{1/2} < \pi
\]

Expressing the integral in this form has the advantage that the pole of the integrand is fixed with respect to \(t\). The integral \(I_B\) will next be expanded by the saddle point method.
The argument of the second exponential can be written as

\[ f(t) = -ik_1r \left( t \cos \theta + (\zeta^2 - t^2)^{1/2} \sin \theta \right) \quad (C.7) \]

where

\[ z = r \cos \theta \quad \text{and} \quad y = r \sin \theta. \]

An examination of the derivative of this function reveals that there is a saddle point at \( t_0 = -\zeta \cos \theta \). If \( k_1r \) is large the integral can be evaluated asymptotically by deforming the contour of integration to the steepest descent path and using a form of the integrand valid near the saddle point. If in deforming to the steepest descent path the contour crosses the pole at 1/n, then the residue of this pole must be added. Near the saddle point the function \( f(t) \) can be expanded in a Taylor series as follows

\[ f(t) \approx +ik_1r\zeta - \frac{ik_1r}{\zeta \sin^2 \theta} (t - t_0)^2 \quad (C.8) \]

Deforming to the steepest descent path by using the substitution

\[ (t - t_0) = \frac{n \sin \theta \sqrt{\zeta}}{\exp(i\pi/4)} \quad (C.9) \]

and evaluating the remainder of the integrand at \( t_0 \) the asymptotic expansion of the integral becomes

\[ I_B = \left( \frac{-i\pi}{\zeta k_1r} \right)^{1/2} \exp(+i\zeta \cos \theta k_1d) \exp(+ik_1r\zeta) \left\{ \frac{\zeta \cos \theta + 1/n}{\zeta \cos \theta + 1/n} \right\} \quad (C.10) \]
It now must be determined whether in deforming to the steepest descent path the pole at \( \lambda = 1/n \) was crossed. It is only necessary to decide this question for the case \( \theta = \pi/2 \) since it can be shown that the residue, if added, can be finite in the asymptotic range only if \( z \) is small. At \( \theta = -\pi/2 \), the saddle point is at \( t = 0 \). For this angle the asymptotes of the steepest descent path are on the real axis in the \( t \) plane. The slope of the path at the saddle point is determined by \( \sqrt{\zeta} \exp(-i\pi/4) \). The only values of \( \alpha \) which lead to a finite asymptotic value for the pole if it is captured by the steepest descent path are

\[
\alpha = (\alpha_B^2 - \varepsilon)^{1/2} = (1 - 1/n^2 - \varepsilon)^{1/2} \quad \varepsilon > 0 \quad (C.11)
\]

for these values of \( \alpha \)

\[
\frac{\pi}{2} - \frac{1}{2} \text{arg } n < \text{arg } \sqrt{\zeta} < \frac{\pi}{2}
\]

Thus the possible slopes of the steepest descent paths at \( t = 0 \) are shown in Fig. C.1. The dotted lines show a typical continuation of the steepest descent to its asymptotes. Thus it can be seen that the pole is not captured. The integral I becomes

\[
I = \frac{i}{n} \left\{ \frac{2\pi i \exp(-ik_1(z + d)/n)\exp(-ik_1\lambda_p y)}{\lambda_p} \right. \\
+ \left. \left( -\frac{i\pi}{\zeta k_1 r} \right)^{1/2} \frac{\exp(+i\zeta \cos \theta k_1 d) \exp(+ik_1 r \zeta)}{\zeta \cos \theta + 1/n} \right\} \quad (C.12)
\]
Fig. C.1
Possible Steepest Descent Paths
The asymptotic expansion for $\Pi_{1x}$ is then

$$
\Pi_{1x} = \frac{-Z_0 I(x)}{8k_1} \left\{ \left( \frac{2}{\pi \xi k_1 r} \right)^{1/2} \exp(i k_1 r \xi - i \pi/4) \left[ \exp(-i k_1 d \xi \cos \theta) + \exp(i k_1 d \xi \cos \theta) \right] \right.

+ \exp(i k_1 d \xi \cos \theta) \left[ -1 + \frac{2 \cos \theta}{n \xi} - \frac{2 \alpha^2}{n^2} \left( 1 - \frac{1}{1 + \frac{1}{n \xi \cos \theta}} \right) \right] \\
+ \frac{4 \pi^2}{n^3 \xi^2 \lambda_p^2} \exp(-i k_1 (z + d)/n) \exp(-i k_1 \lambda_p y) \}

y < 0

\tag{C.13}

= G_\epsilon(\theta) \frac{\exp(i k_1 r \xi)}{\sqrt{r}} + C_\epsilon \exp(-i k_1 z/n) \exp(-i k_1 \lambda_p y) \tag{C.14}

The asymptotic value of $\Pi^{\ast}_{1x}$ may be obtained in the same manner as was that of $\Pi_{1x}$.

$$
\Pi^{\ast}_{1x} = \int_{-\infty}^{\infty} \exp(-u_1 k_1 (z + d)) \frac{M(\lambda)}{u_1} \exp(-i \lambda k_1 y) \, d\lambda \tag{C.15}
$$

where

$$
M(\lambda) = \frac{\lambda \alpha (n^2 - 1) u_1}{\zeta^2 2 \pi k_1 (u_2 + u_1)(u_2 + n^2 u_1)} \cdot I(x)
$$

Making the approximation $u_2 = -in$ in the integral can be written
\[ I(x) \propto \frac{(n^2 - 1)}{2\pi k_1 \zeta^2 n^2} \]

\[ \Pi_{1x}^* = \frac{I(x)\alpha(n^2 - 1)}{2\pi k_1 \zeta^2 n^2} W \]

where

\[ W = \int_{-\infty}^{\infty} \frac{\lambda \exp(-u_1 k_1(z + d)) \exp(-ik_1 y\lambda)}{(u_1 - in)(u_1 - i/n)} \, d\lambda \quad (C.16) \]

For \( y < 0 \) the contour can be deformed in the upper half plane as was \( \Pi_{1x}^* \)

\[ W = \frac{2\pi i}{n} \frac{\exp(-ik/n(z + d)) \exp(-ik_1 y\lambda_p)}{(1/n - n)} \]

\[ + \int_{-\infty}^{\infty} \frac{t \exp(-itk_1(z + d)) \exp(-ik_1 y(\zeta^2 - t^2)^{1/2})}{(t - n)(t - i/n)} \, dt \]

\[ = W_R + W_B \quad (C.17) \]

As before, the saddle point is at \( t_0 = -\zeta \cos\theta \). Following the previous derivation

\[ W_B = -\left(\frac{\pi}{k_1 n}\right)^{1/2} \frac{\zeta^2 \cos\theta \sin\theta}{(\zeta \cos\theta + n)(\zeta \cos\theta + i/n)} \]

\[ \exp(ik_1 r\zeta) \exp(ik_1 d\zeta\cos\theta) \quad (C.18) \]
Thus

\[
\Pi^{*}_{1x} = \frac{I(x)\alpha(n^2 - 1)}{2\pi n^2 k_1 \zeta^2} \left\{ -\frac{1}{k_1 r_\zeta} \frac{1/2}{\zeta^2 \cos \theta \sin \theta} \right\} \frac{2\pi i \exp(-ik_1 z/n) \exp(-ik_1 y \lambda_p)}{(1 - n^2)} \exp(ik_1 r_\zeta) \exp(ik_1 d \zeta \cos \theta)
\]

\[= G_M(\theta) \frac{\exp(ik_1 r_\zeta)}{\sqrt{\Gamma}} + C_M \exp(-ik_1 z/n) \exp(-ik_1 \lambda_p y) \quad (C.19)\]

The potentials valid beneath the interface can be evaluated in the following way

\[
\Pi_{2x} = \frac{(1 - \alpha^2)}{(\zeta^2 - \alpha^2)} \left[ \int_{-\infty}^{\infty} \exp(-u_1 d + u_2 z) \frac{\exp(-i\lambda y)(1 + R(\lambda))}{u_1} d\lambda \right] \quad (C.21)
\]

where \(1 + R(\lambda)\) is defined in Eq. 1.2. As before, let \(u_2 = -in\).

\[
\Pi_{2x} = \frac{(1 - \alpha^2)}{(n^2 - \alpha^2)} \exp(-inz) \Pi_{1x}(z = 0) \quad (C.22)
\]

Therefore \(\Pi_{2x}\) is known in terms of the value of \(\Pi_{1x}\) at the interface. Similarly,
\[ \Pi_2^* = \frac{1 - \alpha^2}{(n^2 - \alpha^2)} \exp(-inz) \Pi_1^*(z = 0) \] (C.23)

An examination of the results reveals that the two discrete modes decay exponentially in the far field. They are bound to the wire. The modes due to the branch cut associated with \( \alpha = 1 \) decay algebraically in the far field and are represented by the terms multiplied by \( G \) (the first term of Eqs. C.14 and C.20). They exist for all values of \( \theta \) in the far field. Similarly, the modes associated with the second branch cut are represented by the terms multiplied by \( C \) (the second term of Eqs. C.14 and C.20). They exist only near the surface of the earth and are thus surface bound modes.