THE INTERTWINEMENT OF ACTIVITY AND ARTIFACTS:
A CULTURAL PERSPECTIVE ON REALISTIC MATHEMATICS EDUCATION

by

FREDERICK A. PECK

B.A., Carnegie Mellon University, 2001
M.A., University of Colorado, 2006

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This thesis entitled: The intertwinement of activity and artifacts: A cultural perspective on Realistic Mathematics Education
written by Frederick A. Peck
has been approved for the School of Education

___________________________________
David Webb

___________________________________
Kris Gutiérrez

___________________________________
Victoria Hand

___________________________________
William Penuel

___________________________________
Luis Radford

Date__________________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

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The intertwinement of activity and artifacts: A cultural perspective on Realistic Mathematics Education
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Realistic mathematics education (RME) is guided by the notion that mathematics is the human activity of mathematizing the world. In much of the RME literature, mathematizing is theorized to be an individual activity, and the products of mathematizing—mathematical productions and mathematical knowledge—are theorized to be individual constructions and private knowledge. In this dissertation I extend these ideas and discuss how mathematizing is mediated by, and distributed across, cultural artifacts. I therefore argue for the importance of adopting a cultural perspective within RME. My arguments are theoretical and empirical. My empirical argument is drawn from design-based research in high-school classrooms. My theoretical argument is made on the basis of historical and contemporary texts in RME. Adopting a cultural perspective has implications for many of RME’s key principles. Exploring these implications is the next frontier in RME research.
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CHAPTER I

INTRODUCTION

Beginning in the late 1960s, the Dutch mathematician Hans Freudenthal (1968, 1983, 1987, 1991) began to sketch a vision for mathematics education based on the radical notion of mathematics as an activity, rather than a pre-existing structure or body of knowledge. This vision led Freudenthal to conclude that mathematics education should not be concerned with the transmission (or even the discovery) of mathematical structure, but rather with engaging students in the activity of structuring the world mathematically, which Freudenthal (1968, 1983) called “mathematizing.” As Freudenthal and his colleagues worked out the implications of this vision, they created a domain-specific instructional theory for mathematics known as Realistic Mathematics Education (RME; Gravemeijer, 1994a; Treffers, 1987, 1993; van den Heuvel-Panhuizen & Wijers, 2005). Although Freudenthal passed away in 1990, researchers have continued to make theoretical and practical contributions to RME.

With respect to practice, researchers have explored how RME can be applied in classrooms in elementary schools (e.g., Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997), middle schools (e.g., Webb, 2010), and universities (e.g., Rasmussen & King, 2000). With respect to theory, RME has benefitted from many theoretical advances. For example, Gravemeijer (1999) provided RME with a key theoretical tool when he showed how the notion of a chain of signification from semiotics could be incorporated into RME (see Section 2 for
more detail). Others have explored how psychological theories of learning and development can contribute to RME. In particular, researchers have argued that RME is compatible with constructivism (Cobb, Zhao, & Visnovska, 2008; Gravemeijer & Terwel, 2000) and social constructivism (Cobb & Yackel, 1996; Verschaffel & De Corte, 2011). In Section 2 I provide a more detailed summary of RME.

RME has always been a work in progress:

RME started out as a vision, or as a philosophy of mathematics education that still had to be worked out. This is being done in developmental research projects, in which each time the research question is: What would mathematics education, which fulfills the initial points of departure, look like for a given topic? (Gravemeijer, 1999, p. 159)

This quote captures the spirit of this dissertation. In this dissertation, I extend the work described above. With respect to practice, the work in this dissertation represents one of only very few studies of how RME can be applied to secondary mathematics classrooms in the US. Additionally, the mathematical topics that I report on, including how high school students learn about fractions and slope in Algebra I, have received very little scrutiny in the research literature. With respect to theory, I will argue that the focus on individual and social psychological processes in RME has been too narrow. I will show how RME is compatible with—and can benefit from—cultural perspectives that consider artifacts to be a constituent part of human activity and cognition. To do so I will draw on cultural theories of learning and development, including activity theory (Cole, 1996; Engeström, 1987; Kaptelinin & Nardi, 2006; LCHC, 2010; Roth & Lee, 2007), sociocultural psychology (Cole & Wertsch, 1996; Vygotsky, 1978; Wertsch,
In Section 2 I elaborate this framework in more detail. The three main chapters of the dissertation (Chapters 2, 3, and 4) are each written as stand-alone papers. In the rest of this introduction I will introduce each chapter, present a unifying framework that links the chapters together, and discuss how each chapter is related to the unifying framework.

Introduction to each paper/chapter

The dissertation is composed of the following three paper/chapters:

Chapter 2: Reinventing fractions and division as they are used in Algebra: The power of preformal productions

In chapter 2, my co-author Michael Matassa and I explore algebra students’ mathematical realities around fractions and division, and the ways in which students reinvented mathematical productions involving fractions and division. We find that algebra students’ initial realities do not include the fraction-as-quotient sub-construct. This can be problematic because in algebra, quotients are almost always represented as fractions and many algebraic concepts, such as slope, blur the line between fractions and division. Thus, understanding the relationship between fractions and division (codified in the fraction-as-quotient sub-construct) is vital for understanding algebraic notation and concepts. We therefore conducted a design-based research study, in which students progressively reinvented the fraction-as-quotient sub-construct. Analyzing this study, we find that a particular type of mathematical production, which we call preformal productions, played two key meditational roles for students: (1) they mediated mathematical activity, and (2) they mediated the reinvention of more formal mathematical
productions. We suggest that preformal productions may emerge even when they are not designed for, and we show how preformal productions embody historic classroom activity and social interaction. As such, we argue for a cultural perspective in which pre-formal productions are seen as cultural artifacts.

This chapter is a first authored paper with Michael Matassa, which is in press at *Educational Studies in Mathematics*.

**Chapter 3: Beyond rise over run: A local instructional theory for slope**

In this chapter I present a local instructional theory for slope that emerged during a design experiment in a high-school Algebra I classroom. In the design experiment, students explored situations related to making predictions. As students mathematized these situations, they reinvented and made meaningful multiple sub-constructs of slope. I show that this process involved the assembly and coordination of mathematical artifacts, and I introduce the theoretical construct of a *cascade of artifacts* to describe this process. As artifacts are inextricably bound with activity, I discuss the nature of the classroom activities that promoted the development of the cascade of artifacts.

This chapter is a single-authored paper. It will be submitted to *Cognition and Instruction*.

**Chapter 4: Mathematizing is a mediated activity: A cultural perspective on Realistic Mathematics Education**

Above I explained how RME is guided by the notion that mathematics is the human activity of mathematizing the world. As I explain below, in much of the RME literature, mathematizing is theorized to be an individual activity. In this chapter I extend these ideas and discuss how mathematizing is mediated by, and distributed across, cultural artifacts. I show that a cultural perspective for RME is a necessary consequence of RME’s first principles, and I
discuss implications to many of RME’s core principles. I argue that exploring these implications is the next frontier in RME research.

This chapter is a single-authored paper. It will be submitted to The Journal for Research in Mathematics Education.

**Unifying conceptual framework**

Above, I explained that I was interested in bringing cultural theories of learning and development to bear on RME. This is the theme that unifies the three papers in my dissertation. Below, I describe RME and cultural theories of learning in two separate sections. I then describe how I plan to address the relationship between these two constructs in the three proposed papers for my dissertation.

**Realistic mathematics education**

RME (Freudenthal, 1973, 1991) is based on Hans Freudenthal’s belief that mathematics is not a ready-made structure, but rather the human activity of structuring the world mathematically (which Freudenthal called “mathematizing”). This belief led Freudenthal to conclude that mathematics education should not be concerned with the transmission (or even the discovery) of formal mathematics, but rather with engaging students in the activity of mathematizing. He recognized that in order to be meaningful, such activities had to be rooted in the student’s reality. However, this did not mean that Freudenthal rejected formal and abstract mathematics. On the contrary, he observed that mathematicians use and discuss abstract (and in some sense, imaginary, Núñez, 2009) mathematical productions as if they were real objects. Indeed, for the mathematician, these imaginary productions are real objects, and formal mathematics is part of the mathematician’s reality.
In RME, mathematics is both an activity and a product (Gravemeijer & Terwel, 2000). First and foremost, mathematics is an activity: it is the human activity of mathematizing the world. As a product, mathematics refers to the set of productions (such as concepts, models, tools, symbols, and algorithms) that are built from—and used in—the activity of mathematizing. Mathematics education, then, should involve engaging students in the activity of mathematizing such that, in the course of the activity, students are guided to reinvent mathematical productions.

Broadly, mathematical productions can be categorized at three levels of formalization (Webb, Boswinkel, & Dekker, 2008). During educational activity, students begin by mathematizing contextual situations. As students engage with these problems, they create models of the situation (Gravemeijer, 1999). In RME, these are called informal models. Informal models are those that are particular to the situation. Through further problem solving, these informal models can themselves be mathematized, leading to more general models. These are models for learning (Gravemeijer, 1999), and they can be applied to problems beyond the problem at hand. These are called preformal models. Further problems encourage students to formalize preformal models into formal mathematics. Formal productions are stripped of all contextual referents, making them potentially very general, but also very abstract.

This process of progressive formalization (van Reeuwijk, 2001) has been theorized as a process of emergent modeling (Gravemeijer, 1999), in which models are reinvented through a chain of signification (Gravemeijer, 1999; Whitson, 1997). At first, informal models signify a particular context or situation. This context-model pair forms a “sign” that consists of a signifier (the informal model) and a signified (the problem context). In subsequent activity, this sign is itself signified by more general models. The process continues such that models at one level signify those signs that came before, and the new sign is signified by those that come after. For
example, in paper 1 we show how a chain of signification emerged as students engaged in activities that involved sharing multiple sandwiches to multiple people. As shown in Figure 1, initially the context of sharing sandwiches was signified by pictures of sandwiches. Together, this signified-signifier pair forms a sign.

\[
\begin{align*}
\text{picture of sandwiches} & \quad \text{sharing context} \\
\{ & \quad \text{sign}_1
\end{align*}
\]

Figure 1. A sign in which the sharing context is the signified, and the picture of the situation is the signifier.

In subsequent activity, this sign was itself signified by the fraction bar model, as shown in Figure 2.

\[
\begin{align*}
\text{fraction bar model} & \quad \text{informal models} \\
\{ & \quad \text{sharing sandwiches} \\
\{ & \quad \text{sign}_1
\end{align*}
\]

Figure 2 A sign in which \(\text{sign}_1\) is signified by the fraction bar model.

Problem contexts play a key role in emergent modeling: they set the stage for students’ initial informal productions, and they drive the creation of more formal productions. Problem contexts must meet three criteria. First, they must be able to be experienced as real by students (Freudenthal, 1987). Second, they must present problematic situations that, from the students’ perspective, are “begging to be organized” (Gravemeijer & Terwel, 2000, p. 787). Finally, they must be designed such that they guide students through a process of emergent modeling, where a model of a particular situation can be mathematized into a model for a new situation.
(Gravemeijer, 1999). For example as described above, the “fraction bar” can be a powerful preformal model for situations that involve rational numbers (Middleton, van den Heuvel-Panhuizen, & Shew, 1998). In Paper 1, we discuss how students were guided to reinvent this model by first engaging in an activity in which they had to determine whether various schemes of sharing a sub sandwich were equitable. Students’ informal pictures of the sharing activity were pictures of sub sandwiches. Notice that sub sandwiches are bar-shaped objects that can be cut. Thus, the informal models of the situation prefigure the more general bar model. As we describe in paper 1, further activities guided students to mathematize these informal models into the preformal bar model.

Interaction is also a key driver of emergent modeling. Often, students are guided towards more formal models by their peers. However, the actual process of reinvention is considered to be an individual process, with each student reinventing mathematical productions on their own, and only then sharing their productions with their peers (van den Heuvel-Panhuizen & Wijers, 2005). In paper 3 I draw on cultural theories of learning to critique this individualist perspective regarding RME, and show that reinvention is distributed across people and artifacts. Below I describe these cultural theories of learning.

**Cultural theories of learning: How artifacts play a central role in learning**

In this section I focus on four key concepts, *culture, mediation, history, and objectification*, to examine how artifacts play a central role in learning.

I take a process and product view towards culture. Cultural *processes* are those that “accumulate partial solutions to frequently encountered problems” (Hutchins, 1995a, pp. 354–355). The residua of these processes—the partial solutions themselves—exist in material and ideal form as cultural *artifacts* (Cole, 1996). For example, a function table is a common cultural
artifact in secondary mathematics. It is a partial solution to the frequently encountered problem of working with two quantities that are in a functional relationship. Function tables have an ideal form, but they are made material in use, often through inscription. The notion of a function table is ideal. An inscription—the written manifestation of a function table—is material.

Artifacts, such as function tables, serve to propagate historical achievements into the present, and the set of these artifacts constitutes culture-as-product: “the species-specific medium of human life” (Cole, 2010, p. 462). Notice in this quote that Cole refers to the “medium” constituted by culture. Human actions take place in a cultural milieu and as such they are mediated by culture. What this means is that human actions “involve not a direct action on the world but an indirect action, one that takes a bit of material matter used previously and incorporates it as an aspect of action” (Cole & Wertsch, 1996, p. 252). Here, the “bit of material matter” is a cultural artifact, which, recall are simultaneously material and ideal. The set of cultural artifacts present in a society creates the cultural milieu: “the species-specific medium of human life.”

The above discussion of the cultural milieu and its use by humans serves to introduce the difference between a cultural perspective and a purely social perspective. To see this difference further, consider the difference between human and non-human simian societies (Latour, 1996). Simian societies are highly social in that all primates have to interact with other primates in order to accomplish their goals. But non-human simians differ in that they have little-to-no way to preserve their social accomplishments, and thus these accomplishments must be reenacted anew in each interaction. This is manifestly not the case for humans. Why not? Because of culture. Cultural processes capture and preserve social accomplishments in the form of artifacts (cultural products) so that humans do not have to recreate these accomplishments in every interaction.
Humans don’t always have to go through other humans to accomplish a task, but they always have to go through culture. In doing so, humans delegate tasks that used to require human work to a non-human artifact. But the non-human artifact is still doing work: it’s doing the work that used to be human work. In order to do this work, the artifact requires certain behaviors from the human users, behaviors that simply would not have the same effect in the absence of the artifact (Johnson & Latour, 1998; Latour, 1992).

For example, consider the calculator. The calculator is a cultural artifact. It is a partial solution to the frequently encountered problem of doing arithmetic. As such, I can delegate to the calculator the (previously human) work of multiplying two numbers, but only if I acquiesce to the calculator’s demand that I tap in particular locations in a particular sequence. Notice that these behaviors do not have the same effect in the absence of a calculator. I can tap the table in front of me, hitting particular locations in a particular sequence and the result is a pleasing series of sounds. If I put a calculator on the table, the result of the same sequence of behaviors is a solution to a multiplication problem. I use the calculator and the calculator uses me. Together we form a system and the solution to the multiplication problem is an accomplishment of that system (A. Clark, 1998; Hutchins, 1995a, 1995b; Ingold, 1999).

The same is true for all human activity. When we use artifacts to mediate our activity we are participating in a system that marshals historical accomplishments to serve purposes in the present. Because artifacts are also participants in this system, they shape our behaviors such that we behave in ways that history would not recognize in order to create forms of activity that were historically not possible. In this way, mediating artifacts do more than simply facilitate or amplify an action that would otherwise exist. Rather, they enable new forms of human actions.
These actions are enabled by culture, and that is what separates humans from other social animals. The focus on the role of culture is what separates a cultural perspective from a purely social perspective. We can describe the activity of non-human primates by appealing to their social world. But the social world is simply not sufficient to account for human activity. We cannot describe human activity without accounting for the ways that the activity is shaped by the past in the form of mediating artifacts, and the ways in which the activity is contributing to future in the form new artifacts.

So far, I have used the word “history” rather loosely. History can be understood on many timescales (Wertsch, 1985). Mathematics has developed over human sociocultural history. Humans learn mathematics over the course of their lives—ontogenetic history. Classrooms too have a history, and often artifacts emerge in classrooms that embody the history of activity in that classroom (Schwarz & Hershkowitz, 2001). Regardless of the timescale, culture is present twice in any human activity. Cultural processes accumulate partial solutions to problems that are frequently encountered in activity, and cultural products—artifacts—are literally “in the middle” (Cole, 1996, p. 116), mediating the activity.

**Previewing the argument: Bringing cultural theories of learning to bear on RME**

From my perspective, there is a synergy in the frameworks presented above. If mathematics is the human activity of structuring the world, then—as with any human activity—culture is present twice: as a process and as a product. Cultural processes collect partial solutions to problems that are frequently encountered in structuring the world. The collection of these partial solutions is itself a structure: mathematics as product. Thus, mathematics-as-product is itself an artifact. Using Wartofsky’s (1979) hierarchy, mathematics-as-product is a tertiary artifact, one that has been “abstracted from [its] direct representational function” (p. 209), such
that it constitutes another world—related to the physical world but not bound to it. As people interact with this artifact it becomes part of their realities. This was Hans Freudenthal’s key insight when he claimed that mathematical activity should “start and stay within reality” (Freudenthal, 1987). As discussed above, this was not a call to limit mathematics to the “real world.” Rather, it was a call to expand students’ “real world” to include mathematics. This expanded reality, embodied in artifacts, creates new ways of seeing the physical world: “Once the visual picture can be ‘lived in’ perceptually, it can also come to color our perception of the ‘actual world,’ as envisioning possibilities in it not presently recognized” (Wartofsky, 1979, p. 209).

Learning mathematics, then, is a reality-expanding process, which does not happen exclusively “in the head” but rather in the cultural world. The result is not simply a mental structure but rather an expanded reality, composed of mathematical artifacts. These artifacts are then put to use, expanding perception, mediating future activity, and creating new possibilities for activity.

Chapters 2 and 3 explore these ideas empirically, via two design studies conducted in high-school algebra classrooms. In Chapter 2, my co-author Michael Matassa and I explore the meditational roles played by a particular type of mathematical artifact: those which the RME literature refers to as “preformal productions.” We show how that this type of artifact plays two key meditational roles as students reinvent fractions and division in Algebra: preformal productions mediate both current mathematics activity and the reinvention of more formal mathematical productions. We trace the development of preformal productions, showing how they emerge in the classroom as “partial solutions to frequently encountered problems,” and how they embody historical activity and social interaction in the classroom.
Chapter 3 examines both the process and product of learning from a cultural perspective. To do so, I introduce the theoretical construct of a *cascade of artifacts*. The cascade of artifacts is based on Latour’s (1986) notion of a cascade of inscriptions. Above I described how mathematical artifacts are often made material through inscription. For Latour, this material form is the source of the artifact’s power. An inscription is immutable and mobile, and can therefore be brought into coordination with other inscriptions. Together, inscriptions can be compared, contrasted, and co-manipulated. In other words, the inscriptions themselves can be operated on, often resulting in new inscriptions that contain within them the inscriptions that came before them. The process continues, creating a cascade of inscriptions, each containing within it more and more referents. Thus an inscription doesn’t just emerge out of thin air, it is built out of—and therefore derives its initial meaning from—other inscriptions. In Chapter 3 I describe how the same process applies to artifacts, specifically mathematical artifacts related to slope. For example, I describe how students reinvented the geometric ratio (“rise over run”) by assembling other artifacts such as number lines, coordinate graphs, and the algebraic ratio \( \frac{y_2 - y_1}{x_2 - x_1} \), and bringing these artifacts into coordination.

Chapters 2 and 3 provide empirical evidence that a cultural perspective is compatible with RME. In Chapter 4, I make a focused argument that a cultural perspective is a *necessary consequence* of the first principles of RME. As I discussed above, mathematization has been theorized as an individual or a social process (Cobb & Yackel, 1996; Keijzer & Terwel, 2001; Verschaffel & De Corte, 2011). In Chapter 4, I extend these ideas and show that mathematization is a *mediated* activity, and therefore it is located not just in people but also in artifacts. I argue that such a cultural perspective is a necessary consequence of the first principles of RME. As part of the discussion I draw on the connections that I alluded to above, including
the connection between activity and artifacts, and Wartofsky’s tertiary artifacts and Freudenthal’s mathematical reality.

**Organization of the dissertation**

Each of Chapters 2-4 is a stand-alone paper. The papers are presented in the order in which they were written. One way to read the dissertation, then, is to consider the chapters as stand-alone papers that are related because they all address a common theme, but which don’t build to anything greater.

On the other hand, it is possible to read the chapters as more than simply a collection of papers that happen to be related. In this second way of reading, the chapters, taken together, make a cohesive argument for the necessity and utility of adopting a cultural perspective in RME—an argument that is more than the sum of the parts. In the conclusion, I synthesize the main arguments from each paper into this larger, cohesive argument.

In what follows, I use the traditional dissertation language of “chapters” to refer to each paper. However, as I described above, each chapter should be read as representing a stand-alone paper.
The study reported in this paper was motivated by our observations—as educators and researchers in Algebra I classrooms—that there is often a mismatch between students’ prior experience with fractions and the way that teachers use fractions in introductory algebra. How are fractions used in algebra? Consider Equations 1 and 2 below, where, for each, the task is to solve for the unknown variable:

\[4x - 8 = 0\]  
\[9y - 7 = 0\]

Superficially, there seems to be little difference between the two equations. However, there is one key difference: Equation 1 has an integral solution, while Equation 2 does not. In our experience, this difference makes the two tasks very different for students in introductory algebra. The difficulty occurs at the “division step” in the traditional algorithm for solving equations (for example, in Equation 2, the division step would come after adding 7 to both sides of the equation, and it would involve dividing 7 by 9 in order to find \(y\)). Students’ solution processes up to the division step are similar for both equations. However, we see many more mistakes in the division step for equations without integral solutions, like Equation 2. These mistakes take many forms. Some students will state that the problem is not solvable, because, to
take Equation 2 as an example, “you can’t divide 7 by 9.” Other students do the division “backwards” (e.g., they solve Equation 2 by dividing 9 by 7).

Furthermore, in our experience nearly all introductory algebra students who attempt the division use the division symbol (÷) to represent the division operation, and use the long division algorithm to express their final solution in decimal notation. Few students use the fraction bar to represent the division operation or use fractions to represent their solutions. This last point may seem trivial: why should it matter if a correct answer is expressed in decimal notation or fraction notation? However, it is important because it suggests that students do not seem to recognize that a fraction can be used to represent, simultaneously, a division problem and the numerical result of the division problem (this is often referred to as the “quotient” subconstruct of rational number, Kieren, 1980; see the review of the literature on fractions as quotients below).

If this is true, it is especially concerning because the fraction bar will serve as a division symbol, and hence fractions will serve as quotients, for the rest of a student’s mathematical life (Rotman, 1991). Furthermore, fraction vocabulary will come to be synonymous with division vocabulary. For example, the slope of a linear function can be conceptualized as the change in the dependent variable per unit change in the independent variable. Such unit rates are calculated by dividing, but they are represented by fractions (e.g., \( \frac{\Delta y}{\Delta x} \)), and they are verbalized using language that only makes sense in terms of fractional representations (e.g., “delta y over delta x” or “rise over run,” where the word “over” designates a position in a fraction, but mathematically can be interpreted as a division operator).

It is therefore vital that students have experiences in which they come to understand that fractions can represent quotients, and that the fraction bar can be interpreted as a division
operator. Our observations (discussed above) suggest that many students who enter our introductory algebra course have not had such experiences. But is this really the case? If so, how can it be addressed in an algebra classroom? To explore these questions, we engaged in a design experiment (Cobb, Confrey, DiSessa, Lehrer, & Schauble, 2003; Steffe & Thompson, 2000), the results of which are summarized in this paper.

To begin, we will discuss our conceptual framework and then summarize the relevant literature on fractions-as-quotients and on design experiments on fractions. In doing so, we situate our study in the vast literature on how students learn fractions. We will then describe the design of our experiment and our specific research questions. We follow this with a description of the learning activities that emerged in the experiment. Finally, we discuss the pertinent results and highlight our contributions to theory and practice.

**Conceptual Framework and Literature Review**

**Realistic Mathematics Education**

Our design experiment was guided by the principles of Realistic Mathematics Education (RME; Freudenthal, 1973, 1991), which is based on Hans Freudenthal’s belief that mathematics is not a ready-made structure, but rather the human activity of structuring the world mathematically (which Freudenthal called “mathematizing”).

This belief led Freudenthal to conclude that mathematics education should not be concerned with the transmission (or even the discovery) of formal mathematics, but rather with engaging students in the activity of mathematizing. He recognized that in order to be meaningful, such activities had to be rooted in the student’s reality. However, this did not mean that Freudenthal rejected formal and abstract mathematics. On the contrary, he observed that mathematicians use and discuss abstract (and in some sense, imaginary; Núñez, 2009)
Mathematical productions as if they were real objects. Indeed, for the mathematician, these imaginary productions are real objects, and formal mathematics is reality.

Mathematics education, then, should involve engaging students in activity such that mathematical productions become real to students. Broadly, mathematical productions can be categorized at three levels of formalization (Webb, Boswinkel, & Dekker, 2008). Students begin by mathematizing contextual situations. As students engage with these problems, they draw pictures and create models. These are models of learning (Gravemeijer, 1999; Treffers, 1993), and are specific to the problem at hand. They are called informal models. Through further problem solving, these informal models can themselves be mathematized, leading to more general models. These are models for learning (Gravemeijer, 1999; Treffers, 1993), and they can be applied to problems beyond the problem at hand. These are called preformal models. Further problems encourage students to formalize preformal models into formal mathematics. Formal productions are stripped of all contextual clues, making them potentially very general, but also very abstract.

This process of progressive formalization (van Reeuwijk, 2001) has been theorized as a process of emergent modeling (Gravemeijer, 1999), in which models are reinvented through a chain of signification (Gravemeijer, 1999; Whitson, 1997). At first, informal models signify a particular context or situation. This forms a “sign” that consists of a signifier (the informal model) and a signified (the problem context). In subsequent activity, this sign is itself signified by more general models. The process continues such that models at one level signify those signs that came before, and the new sign is signified by those that come after. We draw on this notion to define learning as the reinvention of mathematical productions, and conceptualize the process of learning formal mathematics as one of emergent modeling via a chain of signification.
However, in this paper, we do not limit ourselves to models when we discuss the productions that form the chain of signification. Instead we demonstrate the value of a broader view, and we include all manner of productions, including models, tools, and strategies.

Chains of signification are created as students engage in mathematical activity. Problem situations play a key role: they set the stage for students’ initial informal productions, and they drive the creation of more formal productions. The focus on thoughtfully designed problem situations in RME is reminiscent of the Theory of Didactical Situations (TDS; G. Brousseau, 1997). In both RME and TDS, students are presented with problem situations that are “begging to be organized” (Gravemeijer & Terwel, 2000), and which the students can take ownership of. In TDS, these well-designed problem situations are called adidactical situations. The term “situation” has a broader meaning in TDS, encompassing the task, the students, the teacher, and the milieu in which teaching and learning take place. There is no question that the elements of this broader situation play important mediating roles in learning, and more recent conceptions of RME (Cobb et al., 2008) have attempted to incorporate this broader perspective. In our discussion section, we consider the mediating role of the teacher. However, for most of this paper we focus on the problem situations, the students, and the mathematical productions that emerged as students interacted with the problem situations.

**Literature review: Rational numbers as quotients**

Kieren (1980) identified five sub-constructs for rational numbers: (1) Part/whole, (2) Ratio, (3) Quotient, (4) Measurement, and (5) Operator. While these sub-constructs are related, the contexts in which they arise are very different. For example, consider how the fraction $\frac{3}{4}$ might be interpreted under the part/whole and quotient sub-constructs (Lamon, 2012):
**Part/whole:** I have one pizza, cut into four equal pieces. If I eat three of those pieces, I have eaten $\frac{3}{4}$ of the pizza.

**Quotient:** If three pizzas are shared equally amongst four people, each person receives $\frac{3}{4}$ of a pizza. Hence, $3 \div 4 = \frac{3}{4}$

Before discussing the literature on rational numbers as quotients, we first have to address an issue of terminology regarding the terms “fraction” and “rational number.” Following Lamon (2007), we define fraction as a form of notation that expresses a multiplicative relationship numerically as $\frac{a}{b}$. We define rational numbers as the set of numbers that can be expressed as the quotient of two integers. Rational numbers may be written as fractions, but they can be expressed in other ways as well (e.g., $\frac{1}{2}$ and 0.5). At the same time, two different fractions (e.g., $\frac{1}{2}$ and $\frac{2}{4}$) may represent the same rational number. Finally, it is possible to express non-rational numbers as fractions (e.g., $\frac{\pi}{2}$). In this paper, we are often interested in the fraction representation. When we mean to invoke the $\frac{a}{b}$ notation, we use the term “fraction.”

The part-whole construct is the canonical sub-construct for most students and schooled adults (Lamon, 2007). Students are more successful when solving problems that involve the part-whole sub-construct than they are when solving problems involving the other sub-constructs, and mastery of the part-whole sub-construct is only very weakly correlated with mastery of the quotient sub-construct (Charalambous & Pitta-Pantazi, 2006; Clarke, Roche, & Mitchell, 2007). Furthermore, evidence suggests that overemphasis of the part-whole sub-construct can actually be detrimental to the development of the other sub-constructs (Pitkethly & Hunting, 1996).

These results suggest that the quotient sub-construct is qualitatively different than the part-whole sub-construct, and that students would benefit from more experience with problem
situations—such as fair-sharing multiple whole amongst multiple people—that lead to an understanding of fractions as quotients. By mathematizing such contexts, students can come to see that if \( a \) items are shared amongst \( b \) people, each person receives \( \frac{a}{b} \) items. Further abstraction and formalization leads to the general statement that \( a \div b = \frac{a}{b} \) (Confrey, 2012; Empson, 1999; Fosnot & Dolk, 2002; Streefland, 1993).

It is worth noting, however, that while the idea of sharing is natural for students, the process of finding the amount that each sharer receives is not. For one, sharing is *partitive division*, which is generally more difficult for students to construct than *quotative division* (Fosnot & Dolk, 2001b). This is because partitive division involves problems in which the dividend and the divisor do not have the same units and hence, repeated subtraction is not a valid strategy.

Consider the task of sharing 21 beads amongst three strings. As documented in Fosnot & Dolk (2001b), students first approach this problem through trial-and-error by guessing at the number of beads on each string and then checking to see if their guess results in all 21 beads being distributed equally. Eventually, students invent a “dealing-out” strategy, in which one bead is “dealt” to each strand, followed by a second bead to each strand, and so on until the beads are exhausted (cf., Wilson, Myers, Edgington, & Confrey, 2012). In the language of progressive formalization, this “dealing out” strategy has the potential to be a preformal strategy because although it does not take advantage of formal mathematics, it can be made general enough to be applied to any fair sharing problem (and can subsequently be made general enough to apply to any partitive division situation).

Applying the same strategy to situations in which each sharer receives a non-whole amount is yet more complicated, because the student has to (a) partition the wholes into a
number of pieces (unit fractions) that can be distributed equally, (b) reunitize each unit fraction in order to operate on it independently, (c) distribute the unit fractions equally amongst the sharers (possibly using the “dealing-out” strategy), and finally (d) iterate the distributed unit fractions to create a non-unit fraction, which describes the amount that each sharer receives (Lamon, 1996; Olive & Steffe, 2001; Olive, 1999). We call this the “partition-distribute-iterate” strategy.

Students construct a number of strategies when they solve fair-sharing problems (Charles & Nason, 2000; Empson, Junk, Dominguez, & Turner, 2006; Empson, 2002). For example, Empson et al. (2006) classified strategies according to how the students coordinated the number of items to be shared with the number of sharers. Precoordinating strategies are distinguished by the lack of coordination between the number of items to be shared and the number of sharers. Strategies in this category include giving unequal amounts to the sharers, or giving equal amounts without exhausting the quantity to be shared. Coordinating strategies are distinguished by partitions that create a number of parts that is either equal to or a multiple of the number of sharers. For example, each item to be shared may be partitioned into the number of sharers (e.g., sharing fours pizzas among six people by partitioning each pizza into sixths). Using the language of progressive formalization, we consider precoordinating strategies and to be informal strategies because they are not successful across a wide variety of problems. We consider coordinating strategies to be preformal strategies because they are general strategies that can be used successfully across problems.

In addition to strategies, researchers have explored various models that students use as the engage in equipartitioning activities. While the circle model seems to be the canonical model for fractions, many students find it difficult to partition circles (Ball, 1993; Confrey, 2012), and
many researchers have found that a “bar model” (i.e., a rectangle) is easier for students to work with (Connell & Peck, 1993; Keijzer & Terwel, 2001; Middleton et al., 1998; Moss & Case, 2011).

**Literature review: Teaching experiments on rational numbers**

Student learning of rational numbers has received an immense amount of research scrutiny (e.g., Pitkethly & Hunting, 1996). The research related to the quotient sub-construct is summarized above. In this section we broaden the lens, and summarize four seminal studies involving teaching experiments from the US, France, Russia, and the Netherlands. We then situate our study in this literature.

In the US, the Rational Number Project (RNP) has been conducting teaching experiments on rational numbers for over 30 years (Behr & Post, 1992; Cramer, Post, & DelMas, 2002). Researchers on the project have studied all aspects of rational number, and have distilled their findings into a two-year curriculum for rational numbers (Cramer, Behr, Post, & Lesh, 2009; Cramer, Wyberg, & Leavitt, 2009). Through their work, researchers on the RNP have convincingly demonstrated how “understanding is reflected in the ability to represent mathematical ideas in multiple ways, plus the ability to make connections among the different embodiments” (Cramer, 2003, p. 450).

In France, G. Brousseau and N. Brousseau (N. Brousseau & Brousseau, 1987; English translations in G. Brousseau, Brousseau, & Warfield, 2004, 2007, 2008, 2009) designed a curriculum for rational number as a proof-of-concept for G. Brousseau’s (1997) theory of didactical situations. Guided by theory, the curriculum includes 65 imaginative and well thought-out lessons which have been taught and refined in teaching experiments for over 15 years. Through engaging in meticulously designed situations, students “invent, understand and become
fluent with all the aspects of [rational numbers]” (G. Brousseau et al., 2007, p. 281). Students begin by inventing rational numbers as measures. By the end of the 65 lessons, students have invented all of the sub-constructs of rational number, fraction and decimal notation, formal operations on rational numbers and decimals, and the topology of rational numbers.

A somewhat similar sequence comes from Russia, in the work of Davydov and colleagues (Davydov, 1990; see also Schmittau & Morris, 2004). Davydov describes a curriculum for rational number that is also organized around measurement and which also results in students’ understanding the topology of the rational numbers. Davydov’s curriculum, however, has vastly different theoretical underpinnings. It is informed by activity theory (Schmittau, 2003), and takes an algebraic approach to number in which students learn rational numbers by ascending from the abstract to the concrete (Falmagne, 1995).

In the Netherlands, Streefland (1991, 1993) conducted teaching experiments over a 10-year period on a series of lessons for rational number designed using RME principles. In these lessons students engage in two types of activities: fair sharing and splitting the group of sharers. The activities were designed such that multiple sub-constructs of rational number were intertwined from the beginning, and as students engaged in the activities, they reinvented all of the sub-constructs of rational number.

Although each of the above research projects took place in different countries and had different foci and theoretical underpinnings, there are some common themes. First each research project explored students’ initial exposure to rational numbers. Second, the projects were quite comprehensive, exploring how students learned multiple sub-constructs of rational number, as well as formal operations on rational number, and—in some cases—the topology of rational number. The study that we describe in this paper differs from the above studies in both respects.
First, our study involves students who have vast prior experience with rational numbers. Second, our intention is not to understand how students come to a comprehensive understanding of rational numbers, but instead to explore how students come to understand fractions as they are used in algebra.

**Research questions**

Our study was guided by the following two research questions:

1. How do our students solve partitive division problems with integral and non-integral results?
2. How do our students reinvent the fraction-as-quotient sub-construct?

Notice that we are interested in *how* our students solve problems and reinvent the fraction-as-quotient sub-construct. As detailed in our conceptual framework, we believe that reinvention happens during mathematical activity, and thus we designed a sequence of activities to encourage students to reinvent the fraction-as-quotient sub-construct. In what follows, we present the sequence of activities along with our design rationale. We do so in order to communicate the conditions under which our study took place. We do not intend that the activities should serve as a “model” curriculum. Rather, the descriptions set the stage for our analysis of how students reinvented the fraction-as-quotient sub-construct.

**Materials and Methods of Analysis**

A two-person research team (the authors of this paper) conducted the design experiment: a teacher-researcher (FAP, the first author), and an observer (MM, the second author). The learning activities took place in a public high-school in a suburban area of the United States (US). In the US, all students are required to attend high school, and with rare exceptions students are not sorted into specific schools. The vast majority of students attend so-called
“comprehensive high schools,” which is the type of school in which our experiment took place. FAP was the teacher of the class in which the study took place (Cobb, 2000), and the entire class participated in the learning activities. The school served a predominantly white (approximately 60%) and Latino (approximately 30%) population. We do not have access to student-specific demographic data.

The course itself was a support class for ninth-grade students that were concurrently enrolled in Algebra I. Algebra I is a common course in US high schools, and is generally taken in ninth-grade (the first year of high school in the US; students in ninth-grade are about 14 years old). In the study high school, Algebra I is a required course for all students. At the time of the study, the school’s curriculum for Algebra I included solving single-variable linear equations and systems of two linear equations, and in-depth study of linear and quadratic functions.

Students were assigned to the support course based on the recommendation of their Algebra I teachers, as well as on the basis of their scores on the previous year’s state-level standardized test. With respect to teacher recommendation, Algebra I teachers were not given official criteria to use when recommending students. In general, teachers recommended students for the course based on the teacher’s subjective opinion that the student would benefit from having more time to explore Algebra I concepts in a small-class setting. This recommendation was cross-referenced with the student’s score on the state standardized test. The state categorized students into four proficiency levels based on their scores, and only students who were categorized in the lowest two proficiency levels were selected for the support course. In total, there were 12 students in the course.

Thus the students in our study are not representative of the Algebra I students at the school, or of some larger population of Algebra I students. However, our intention is not to make
a claim about the knowledge-level of some generalized population of Algebra I students, or about the effectiveness of a particular curriculum. Rather, as discussed above, we want to explore how students who are enrolled in high school Algebra I reinvent the fraction-as-quotient sub-construct.

In the course, students engaged primarily in activities that were correlated with the concurrent Algebra I topics, however, occasionally students also engaged in activities that were designed to increase familiarity with signed integers and rational numbers (for a description of such an approach, see Burris & Welner, 2005). FAP designed the entire course using RME principles. Students were accustomed to engaging in mathematical activity prior to being taught specific procedures, and they were accustomed to sharing strategies through “math congress” (Fosnot & Dolk, 2001b, 2002), in which a carefully sequenced subset of students presented their solution strategies to the class.

Prior to beginning the experiment, we developed a hypothetical learning trajectory (elaborated below). The experiment itself consisted of seven learning activities, each of which took a full (55-minute) class session. In order to keep the primary focus of the support course on the concurrent Algebra I topics, we distributed these learning activities throughout a two-month period, giving approximately one learning activity per week. After each learning activity, we met as a research team to examine student work, update our models of students’ mathematical realities, discuss our impressions of how the learning activity influenced those realities, and design subsequent learning activities. In this, our approach was cyclical: “What is invented behind the desk is immediately put into practice; what happens in the classroom is consequently analyzed, and the result of this analysis is used to continue the developmental work” (Gravemeijer, 1994b, p. 449).
**Data Sources**

During the experiment, we collected student work and MM recorded fieldnotes. When he felt it was appropriate, MM recorded classroom discourse in the fieldnotes. In addition, we kept a record of our meetings, and saved all analytical memos that we sent each other during the experiment. As discussed above, the initial analysis happened concurrently with the design experiment. We further analyzed the collected and created artifacts after the conclusion of the experiment.

**Hypothetical Learning Trajectory**

In a design experiment, the Hypothetical Learning Trajectory (HLT) is created *a priori*, and represents “a prediction as to the path by which learning might proceed” (Simon, 1995, p. 135). Because we define learning as the reinvention of progressively more-formal mathematical productions, our HLT describes a path of progressive formalization, as follows:

*Stage 1:* Students use informal models and strategies to solve problems involving fair sharing situations where multiple items are shared amongst multiple sharers.

*Stage 2:* Through social interaction (including math congress) and further experience with progressively more abstract fair-sharing situations, students develop the bar model and the “partition-distribute-iterate” strategy.

*Stage 3:* Students formalize the fractions-as-quotients sub-construct.

The HLT is only a prediction about what might happen, and the actual path of learning emerges in the design experiment itself as the research team interacts with student learning. In the next two sections we describe the path of learning that emerged in our design experiment.
Exploring Research Question 1, and Building an Instructional Starting Point

Our first research question was focused on how our students initially solved partitive division problems. As such, exploring this question provides us with the “instructional starting point” (Gravemeijer & Cobb, 2006) for the learning activities to follow. Based on the literature described above, we chose to begin with a problem designed to elicit student strategies and models in a fair-sharing context. This lead to the design of Learning Activity 1.

Learning Activity 1: The sub-sandwich problem

Motivation and design

In order to explore the ways that students solve partitive division problems, we began our experiment with a problem from a fair-sharing-based curriculum on rational number (Fosnot, 2008), henceforth called the “sub sandwich problem” (see Figure 3).

A class traveled on a field trip in four separate cars. The school provided a lunch of submarine sandwiches for each group. When they stopped for lunch, the subs were cut and shared as follows:

- The first group had 4 people and shared 3 subs equally.
- The second group had 5 people and shared 4 subs equally.
- The third group had 8 people and shared 7 subs equally.
- The last group had 5 people and shared 3 subs equally.

When they returned from the field trip, the children began to argue that the distribution of sandwiches had not been fair, that some children got more to eat than the others. Were they right? Or did everyone get the same amount?

From an RME standpoint this is a very good introductory problem for the following reasons:

- The context can be made real to students, which we did by having a class discussion about sub-sandwiches before giving the task. Furthermore, the question of fairness is motivating (Paley, 1986).
• Informal models of the situation (pictures of sub-sandwiches) are similar in shape to the bar model, and sub-sandwiches can be physically cut and distributed in a way that is similar to the partition-distribute-iterate strategy. Thus, the informal models and strategies that we predicted that students would use in the sub-sandwich problem can be mathematized to create the preformal bar model and partition-distribute-iterate strategy.

Because we used this activity to build our initial models of students’ mathematical realities, students worked on the problem individually. We asked questions to probe for students thinking, but we did not offer help or suggestions.

**Analysis**

In order to build our initial model of students’ mathematical realities, we coded the student work for level of formalization. We did this as a team, and came to a consensus for each student. Students used productions at all levels of formalization. For space purposes, we will limit our discussion below to the models and strategies that students used to find the quantity allotted to each person in Group 2 (four sandwiches shared amongst five people) as these productions are illustrative of the productions that students used for the other parts of the problem.

**Informal productions:** Two examples of students who used informal productions are shown in Figure 4. Figure 4a shows an example of a student who used informal models. These were models of the situation: pictures of sandwiches and people, with lines drawn to show how the sandwiches were distributed to the people. Both examples demonstrate informal precoordinating strategies, in which the partitions are based on benchmark fractions. None of the students who used informal strategies shared the sandwiches equally.
Preformal productions: Preformal models were models for the situation (e.g., bar model). Preformal strategies were coordinating strategies. Figure 5 shows an example of a student who used a pre-formal bar model and a coordinating strategy.

Formal productions: Students who used formal productions expressed quantities in symbolic fraction notation without any accompanying pictures or other work. When prompted, none of these students could draw pictures to connect their formal notation to the problem context, nor could they offer a mathematical justification as to why their solution was correct. For example, Justin and Joshua were sitting next to each other. As shown in Figure 6, they each wrote fractions that were reciprocals of each other. FAP asked Joshua and Justin to justify to try to convince each other why their solutions were correct. In response, Joshua drew a single bar model, cut into fifths, with four fifths shaded (demonstrating the part-whole sub-construct of fraction). However, he did not explain how this model was related to the problem situation of sharing four sub-sandwiches to five people. Justin justified his solution by pointing to the
numerals (5 and 4) in the problem. When FAP asked, neither Justin nor Joshua was convinced by the other to change their solution.

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<thead>
<tr>
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<tbody>
<tr>
<td><img src="image" alt="Fraction Symbol" /></td>
<td><img src="image" alt="Fraction Symbol" /></td>
</tr>
</tbody>
</table>

Figure 6. Two examples of students that used formal fraction symbols to represent the quantity

**Analysis:** When we created our HLT, we hypothesized that at the beginning of the experiment students would use informal models and strategies in fair-sharing situations. Through this learning activity, we learned that our students marshaled productions at all levels of formalization when solving fair-sharing problems. However, only the students who used preformal productions solved the problem correctly and justified their reasoning in the context of the problem. This suggested to us that preformal productions are a vital part of a student’s mathematical reality.

We also learned that students might not associate fair sharing with the division operation, as only three students mentioned division in their written work or verbal descriptions. This suggested that students might not recognize partitive division situations as opportunities to use the division operation. We explored this with our next learning activity.

**Learning activity 2: What operation?**

**Motivation and design**

In order to explore the operations that students associate with partitive division situations, we designed an online problem-solving environment in which students were presented with a series of problems and an on-screen calculator. Students could use the calculator to perform any arithmetic operation, and they were provided a space to enter their final answer. They were also
given the option of stating that a problem could not be solved. We used a screen-capture utility to record the students’ activities in this problem-solving environment. This way, we were able to capture the operations that students associated with each problem.

Students worked on the problems individually. During the activity, we monitored students to ensure that there were no technical problems, but we did not provide problem-solving assistance. As in Learning Activity 1, we made this decision because we used this activity primarily to help build our models of students’ initial mathematical realities.

After the activity we saved the screen capture videos, and later transcribed the students’ keystrokes. We each watched each video at least twice: once independently and once as a research group. Our primary focus during this analysis was on whether, when, and how students used the division operation. We therefore coded each (student, problem) instance in terms of whether the division operation was used correctly, and if so, whether the division operation was the first and/or only operation used. Occasionally, students entered a final answer without using the calculator; we coded this as “strategy unknown.”

Analysis

Nearly all of the partitive division problems were solved using the correct division operation. However, the correct division was the only operation just 70% of the time (this and other percentages presented in this paper are presented for summary purposes only, and are rounded to the nearest 5%). In other words, although many students ultimately settled on the answer provided by the correct division operation, many times students tried more than one operation.
We investigated these “multiple operation” instances further, and found that two students, Justin and Robert, consistently divided “both ways,” and then chose one of the answers. For example, Protocol A shows how Robert solved a problem by dividing both ways.

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<table>
<thead>
<tr>
<th>Protocol A</th>
<th>Robert’s “divide both ways” strategy for a problem that involved finding the weight of one chicken given that three chickens weigh 57 pounds</th>
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</thead>
<tbody>
<tr>
<td>Calculator: 3/57 = 0.05263…</td>
<td></td>
</tr>
<tr>
<td>Calculator: CLEAR</td>
<td></td>
</tr>
<tr>
<td>Calculator: 57/3 = 19</td>
<td></td>
</tr>
<tr>
<td>Types: 19</td>
<td></td>
</tr>
</tbody>
</table>

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This was not the first time that we had observed such reciprocal division. For example, recall Justin’s work on the sub sandwich problem (Figure 6b). Justin used formal notation to express the quantity of sub sandwiches that each person received, but his fraction \( \frac{5}{4} \) was the reciprocal of the correct fraction \( \frac{4}{5} \). At the time, we believed this mistake stemmed from a mathematical reality that did not include fractions as quotients (especially because Justin was unable to explain his solution). However, we were now confronted with a more complicated situation.

These two students recognized that division was the appropriate operation on all five problems. However, each of them divided both ways on four of the five problems. For the problems in which they divided both ways, they always chose the correct answer, even though there is no evidence that the correct direction was their first instinct (Robert divided in the correct direction first on 25% of his “both ways” solutions and Justin divided in the correct direction first of 50% of his “both ways” problems). This suggests that Justin and Robert recognized division situations from the structure of the problem, but that they were choosing the
direction of the division ex post based on the quotient, rather than on the structure of the problem.

Justin and Robert’s “divide both ways” method accounted for some of the “multiple operations” that we observed. However, we also observed that many students performed other operations, even though they usually settled on the correct division as their final answer. For example, Protocol B shows how two other students used multiple operations when solving a problem involving sharing 546 candies amongst 13 people.

<table>
<thead>
<tr>
<th>i.</th>
<th>ii.</th>
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<tbody>
<tr>
<td>Calculator: 546-13=533</td>
<td>Calculator: 13*546=7098</td>
</tr>
<tr>
<td>Calculator: CLEAR</td>
<td>Calculator: CLEAR</td>
</tr>
<tr>
<td>Calculator: 546/13=42</td>
<td>Calculator: 546/13=42</td>
</tr>
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<td>Types: 42</td>
<td>Types: 42</td>
</tr>
</tbody>
</table>

Protocol B. Two examples of how students used multiple operations to solve a problem that involved sharing 546 candies amongst 13 people

This suggests that although students almost always settled on the correct division, it was not necessarily their first instinct. For some students, the problem did not indicate division right away. For others, the problem indicated division, but not the direction of the division.

**Summary: Analysis of research question 1**

Our first research question was: “How do our students solve partitive division problems with integral and non-integral results?” Our analysis suggests that the mathematical realities of our students did not include strategies for recognizing division situations or for determining the direction of the division within a division situation. Furthermore, it suggests that pre-formal productions are key components of the mathematical realities of students who solve fair sharing problems correctly, but that such productions are not real for most students.
This analysis formed our “instructional starting point” for the learning path that emerged as we explored our second research question.

**Exploring research question 2**

The first two learning activities gave us a good sense of our students’ mathematical realities around partitive division problems and the fraction-as-quotient sub-construct. Our second research question involved an intervention to see how students reinvented new productions through mathematical activity. To explore this question, we engaged students in a series of activities, designing each learning activity only after analyzing the results of the previous activity. Figure 7 shows a schematic outline of the learning path that emerged from this process. As shown, Learning Activities 3-5 initially involved fair sharing situations, while Learning Activities 6 & 7 involved division, and linked division to fair sharing. These activities are described below.
Figure 7. A schematic of the path of learning. Each box represents a Learning Activity (LA), and the bulleted points within boxes represent the key outcomes of the learning activity. Learning Activities are arranged in temporal sequence from left to right. Bolded arrows represent ideational inheritance (i.e., LA 3 was informed by LA 1, whereas LA 6 was informed by LA 2 and LA 5).
Learning activity 3: Math congress and sub-sandwich follow-up problem.

Motivation and design

Our analysis of Learning Activity 1 suggested that pre-formal productions are key components of the mathematical realities of students who solve fair sharing problems correctly. We therefore set out to design a learning activity to help students reinvent these preformal productions.

In Learning Activity 1, two students had used preformal models and strategies to solve the sub-sandwich problem, and we wanted to create a learning activity in which these students could share their productions in such a way that they were rooted in the informal reality of the sub-sandwich problem. We therefore designed a “math congress” (Fosnot & Dolk, 2001b, 2002), in which a carefully sequenced subset of students presented their solution strategies to the class.

Specifically, we planned to ask Penny, then Julia, then Theo to present. We wanted to begin with Penny because her informal model of the situation (Figure 4a) was rooted in the reality of actually cutting and distributing sub sandwiches to a group of people. We chose Julia to present next because her preformal model (Figure 4b) signified this concept, with squares representing sandwiches and shapes representing people. However, Julia’s model still depicted the process of distributing pieces to people, thus rooting the abstraction in the informal reality of Penny’ model. Theo also used a preformal model with different designs corresponding to different people, but his model (Figure 5) was even more abstract because he did not show the process of distribution. Finally, while Penny and Julia both used benchmark fractions in their informal precoordinating strategies, Theo used a preformal coordinating strategy. Thus, we felt that this sequence of students would help the class construct a chain of signification beginning in the problem context and culminating in Theo’s preformal productions. Notice that Theo’s
strategy includes the first two pieces of the “partition-distribute-iterate” strategy, but Theo did not iterate the shares. A second goal of the math congress was therefore to reinvent this last piece of the strategy. During the math congress, we encouraged the use of iteration by asking the students questions about “how much each person received.”

After the math congress, we had students work on a follow-up problem (Figure 8), based on “Group 3” from the initial sub-sandwich problem. We specifically focused the question on the quantity each sharer received in order to encourage iteration (Wilson et al., 2012).

Eight people shared seven sub sandwiches equally. How much of a sandwich did each person get?

Figure 8. The "follow-up sub sandwich problem"

**Analysis**

During the math congress, Penny, Julia, and Theo each presented their work. Penny and Julia explained the partition and distribute strategy, and Theo explained his strategy for coordinating the partitions with the number of sharers. Following these presentations, Joshua explained how he could use iteration to quantify the share that each person received:

(1) FAP: Joshua, how much-
(2) Joshua: Four fifths
(3) FAP: Where do you see four fifths?
(4) Joshua: From each sandwich he is going to give him one fifth. One-fifth plus one-fifth plus one-fifth plus one-fifth

By sharing their productions, Penny, Julia, and Theo helped to create a chain of signification that included two preformal productions: bar models for fractions and a coordinating strategy for partitioning. These preformal productions then mediated the reinvention of new productions. Recall that in Learning Activity 1, Joshua wrote a formal fraction $\frac{4}{5}$ to quantify the amount that each sharer received, which he supported with a single
bar model with four-fifths shaded. As shown in the segment of talk above, once the bar model and coordinating strategy emerged in class, rooted to the informal reality of the sub-sandwich problem, Joshua’s thinking about $\frac{4}{5}$ changed from a quantity that represented 4 out of 5 pieces in a single bar to a quantity composed of four one-fifth pieces, scattered across multiple bars. Thus the preformal productions mediated Jonathan’s reinvention of iteration, the last piece of the partition-distribute-iterate strategy.

**Learning activities 4 and 5**

*Motivation and design*

These learning activities were motivated by stage 2 in our hypothetical learning trajectory. In this stage, we wanted students to mathematize progressively more formal and abstract fair-sharing situations. We hypothesized that such mathematization, along with social interaction organized around students sharing their productions with each other, would lead students to construct progressively more formal mathematical realities. As shown in Figure 9, the items to be shared in Learning Activities 3 and 4 became progressively more abstract as compared to the sub sandwiches that students initially shared. In Learning Activity 3, the items were bottles of liquid, which, although still shaped like a bar, cannot physically be cut like sub sandwiches. Hence applying the bar model to a bottle of liquid is more abstract than applying this model to a sub sandwich. In Learning Activity 4, the objects to be shared were “pounds of chicken food.” These are shapeless, and hence using a bar model to model a pound is still more abstract.
Learning Activity 3: Which party?
You have a choice between two parties. One party will have six bottles of coke for eight people. The other party will have four bottles for five people. If you want the most Coca-Cola in your cup, which party would you choose to go to?

Learning Activity 4: Chicken feed
Mr. Huang owns 7 chickens. He gives the same amount of food to each of them. Yesterday, he fed his chicken 4 pounds of food all together. How many pounds of food did each chicken get?

<table>
<thead>
<tr>
<th>Learning Activity 3: Which party?</th>
<th>Learning Activity 4: Chicken feed</th>
</tr>
</thead>
<tbody>
<tr>
<td>You have a choice between two parties. One party will have six bottles of coke for eight people. The other party will have four bottles for five people. If you want the most Coca-Cola in your cup, which party would you choose to go to?</td>
<td>Mr. Huang owns 7 chickens. He gives the same amount of food to each of them. Yesterday, he fed his chicken 4 pounds of food all together. How many pounds of food did each chicken get?</td>
</tr>
</tbody>
</table>

Figure 9. Learning Activities 3 and 4

Analysis

As in Learning Activity 1, students used productions from all levels as they worked on Learning Activities 3 and 4. However, by Learning Activity 4, all of the students used preformal productions. As we discuss below, these productions mediated the students’ activity and they also mediated the reinvention of more formal productions.

For example, in Learning Activity 4 a student initially used an informal precoordinating strategy, distributing $\frac{1}{2}$ of a pound to each chicken. She soon realized that she would have $\frac{1}{2}$ lb. left over. Importantly, however, when she experienced the perturbation (von Glasersfeld, 1989) caused by the remainder of $\frac{1}{2}$, she could access a more general strategy, namely a preformal coordinating strategy. This is an example of how preformal productions mediated activity.

In other cases, preformal productions mediated the reinvention of more formal mathematical productions, by helping to make the connection between fair-sharing and formal fractions real for students. For example, in Learning Activity 4, Joshua solved the problem correctly using formal notation ($\frac{4}{7}$) without any supporting work. This was very similar to his solution in Learning Activity 1, which was also a correct bald fraction. However, this time, when FAP asked him to explain his work, Joshua used the preformal bar model and the preformal partition-distribute-iterate strategy to show why each chicken got $\frac{4}{7}$ of a pound.
Thus, through mathematical activity and social interaction, students created a chain of signification, culminating in robust preformal productions for fair-sharing. These productions mediated activity and they also mediated the reinvention of more formal productions. Indeed, in Learning Activity 4, nearly half of the students initially used formal fraction notation to express their answers.

Even as these formal notions of fractions were being invented, at one point or another in Learning Activity 4, many of the students either wrote a reciprocated fraction (i.e. \( \frac{7}{4} \)), or modeled the problem “backwards” (i.e., they drew seven fraction bars and distributed them to four sharers). In all cases, students were able to catch their mistake by moving backwards through the chain of signification. For example, a group of students initially wrote the reciprocal of the correct fraction (i.e., \( \frac{7}{4} \)). When FAP asked them to explain their work, they explained that the seven represented the number of chickens and the four represented the number of pounds. From FAP’s perspective, they had essentially just explained that they had found the number of chickens per pound, rather than the number of pounds per chicken. Without indicating that their fraction was incorrect, FAP asked the students to draw a picture to justify their answer. Despite their reciprocated fractions, the students drew four bar models, which they explained represented the four pounds of food. They then proceeded to use the partition-distribute-iterate strategy to conclude that in fact, each chicken received \( \frac{4}{7} \) of a pound of food. Thus, the students were able to catch their mistake by reasoning from a formal fraction to its real-world referent via preformal productions.

Why were the preformal productions necessary? Recall that when the students initially justified their answer they referred to the referent of the numerator separately from that of the denominator, without linking the two units (for example, by using the word “per”). It therefore
seemed that these students perceived the units of the numerator and denominator as two extensive quantities, and not as two components of a single intensive quantity (e.g., “pounds per chicken;” see Nunes, Desli, & Bell, 2003; Schwartz, 1988). We realized that while the preformal bar model and partition-distribute-iterate strategy helped students reinvent formal fraction notation as the outcome of fair sharing (henceforth referred to as the “fraction-as-fair-sharing” sub-construct), these productions did not lead to a reality in which the fraction was perceived as an intensive quantity or as a result of a division operation.

We hypothesized that understanding the notion of an intensive quantity might help students (a) associate fair-sharing with division, and (b) perform the division in the correct direction. This is because intensive quantities are created by division. The nature of these quantities depends on two factors: (1) the units of the dividend and of the divisor, and (2) the direction of the division. For example, in the chicken feed problem, the two units are chickens and pounds. Through division, it is possible to create two different intensive quantities: “pounds per chicken,” and “chickens per pound.” The operation of division creates the intensive quantity, and the direction of the division determines the reference quantity (i.e., “chickens” in the intensive quantity “pounds per chicken”). Thus, we felt that if students could recognize a situation in which the final answer will be an intensive quantity, they could use this to recognize that division is an appropriate operation as well as the direction of the division. This led to the design of our next two learning activities.

**Learning activity 6: Scale up and down**

*Motivation and design*

Our goal for this learning activity was to help students reinvent the notion of an intensive quantity, and to associate this quantity with the division operation. One way that students can
reinvent these ideas is through engaging in missing value problems that involve proportional reasoning with rate pairs (e.g., if 3 apples cost $0.90, how much does 1 apple cost?) (M. R. Clark, 2005; Cramer, Bezuk, & Behr, 1989; Post, Behr, & Lesh, 1988). Students use a variety of strategies when solving such problems (M. R. Clark, 2005). Incorrect strategies are often additive, and involve adding or subtracting the same numerical value to both extensive quantities in the rate (e.g., adding “1” to both the apples and the cost in the example above). The most basic correct strategy is the build-up strategy, which also relies on repeated addition, but with equal quantities being added to both extensive quantities, rather than equal numerical values (e.g., adding 1 apple to the apples and $0.30 to the cost; Lesh, Post, & Behr, 1988).

More advanced strategies involve multiplicative reasoning that maintain constant ratios between units and within units. For example, in the apple problem given above, a within-unit strategy would involve students recognizing that the number of apples has decreased by a factor of three, and hence the price should decrease by a factor of three. A between-unit strategy would involve students recognizing that in the first rate pair, the numerical price is 0.30 times as large as the numerical apples, and hence in the second rate pair, the numerical price should also be 0.30 times as large as the numerical apples. This terminology is muddled, with some authors using the terms “within” and “between” to refer to operations within and between ratios, instead of within and between units, effectively switching the meanings of within and between. To maintain clarity, we will use the terms “within-units” and “between-units.” Under most circumstances, students prefer to use within-unit strategies (G. Brousseau et al., 2008; Karplus, Pulos, & Stage, 1983; Vergnaud, 1988).

Using the language of progressive formalization, we consider strategies that rely on additive reasoning (the incorrect additive strategy and the build-up strategy) to be informal
strategies because they are not successful across a wide variety of problems (a build-up strategy is not successful when the ratio between the rate pairs is non-integral; Kaput & West, 1994). We consider strategies that rely on multiplicative reasoning to be preformal strategies because they are general strategies that can be used successfully across problems. We felt that preformal multiplicative strategies would mediate the reinvention of division as a fair-sharing operation, so we wanted our learning activity to encourage these strategies. With this consideration in mind, we designed a problem string (Kindt, 2010) of missing value proportional reasoning problems (see Figure 10).
As shown in Figure 10, the problem string is composed of the following sequence, which encourages the use of within-unit multiplicative strategies:

- The *Coke problem* involves missing value problems that can be solved using a within-unit strategy by doubling the quantities in the rate pair directly above.
• The *marker problem* requires students to multiply by other factors, or to consider rate pairs that are not immediately above the particular missing value.

• The *t-shirt problem* requires division, first by two and then by other factors.

• The *pizza: make your own combinations* problem allows students to make their own combinations in a situation where the within ratio is not an integer.

• The *pizza: scale down to one* problem combines the proportional reasoning structure of the first four problems with a fair-sharing situation and a non-integral solution to the missing value.

We chose very familiar contexts for these problems in order to make the additive strategy unlikely (Karplus et al., 1983). Furthermore, we believed that the progression of the arithmetic required would encourage the progression from the informal build-up strategy to preformal multiplicative strategies, in particular the within-unit strategy.

This was important because the last problem in the string (*pizza: scale down to one*) combines the proportional reasoning structure of the first four problems with a fair-sharing situation and a non-integral solution to the missing value. We believed that if students used multiplicative reasoning (that is, division) to solve this problem, it would help them construct the association between division and fair-sharing, and the hence the association between the “fraction-as-fair-sharing” sub-construct and the fraction-as-quotient sub-construct.

*Analysis*

In the first problem (Coke), students used build-up and within-unit strategies. In the subsequent math congress, students presented both strategies, and the class seemed to coalesce around the within-unit strategy. For the second problem (markers), all students used the within-
unit strategy. During the math congress, a serendipitous moment occurred: Jody used the word “per.” FAP tried to capitalize on this moment, as shown below:

(1) Jody: I looked at the seven and the three and thought “21”. It is easier for me, a time saver… it’s just easier… if there is three packages and seven per package

(2) FAP: Oooh – that’s a good word, per. What does “per” mean? (writes “7 markers per package” on the board)

(3) Lori (overlapping): For every one package

(4) Joshua (overlapping): For every one

This was an important moment because, as discussed above, we hypothesized that an understanding of intensive units could mediate students’ activity in partitive division situations. This hypothesis relies on (a) students using the word “per” when stating the intensive units, and (b) understanding that “per” indicates “for every one.”

During the math congress for the third problem (t-shirts) a unit rate strategy (Cramer & Post, 1993) emerged, and two students explained how they used division to find the unit rate. At this point, FAP prompted the class use “Jody’s word” (per) to explain the unit rate. In the fourth problem (pizza: make your own combinations), only two students used a build-up strategy to make their own combinations, while the rest of the students used within-unit multiplicative strategies. Thus, the problem-string up to problem four, as well as social interaction encouraged reinvention of a within-unit strategy which then mediated subsequent activity.

In the final problem of the string (pizza: scale down to one), we were specifically interested in (a) whether students would associate the fair sharing situation with the division operation, and (b) the productions that students would use for fair sharing. All but one student indicated a division operation to solve this problem, although there was great diversity in the ways that students used division:
• **Within-unit division by seven**, which resulted in the correct answer. Students who used this strategy wrote formal fraction notation to write their final answer, and used the partition-distribute-iterate strategy to justify their solutions.

• **Repeated halving**, in which students initially tried to repeatedly divide both the people and pizzas by two (i.e., a within-unit strategy of repeated halving) in order to get down to one pizza. When this did not work, students used a within-unit strategy of dividing both extensive quantities by seven.

• **Different divisors**, in which students initially divided the pizzas by seven and the people by four, which resulted in the statement, “1 pizza can feed 1 person.” Students who used this strategy recognized the incongruence between this statement and the initial statement (4 pizzas can feed 7 people), and either used a within-unit strategy of dividing both extensive quantities by 7, or the partition-iterate-divide strategy.

• **Divide both directions.** As in Learning Activity 2, Robert divided in both directions and used the results to choose his answer. As shown in Figure 11, Robert indicated a correct division using a between strategy, but then used long division to divide seven by four and four by seven. Upon examining the decimal results, he chose the incorrect answer because “it looks easier” and rejected the correct answer of 0.57142 because one “can’t split a pizza into that.”

Thus, by the end of the sequence, nearly every student saw the division operation in a fair sharing problem. Furthermore, many students linked the division operation to the partition-distribute-iterate strategy. Finally, students continued to formalize the fraction-as-quotient sub-
construct: many students wrote the correct fraction without using the preformal partition-distribute-iterate strategy to find it (although some of these students used this strategy to justify their solutions, again showing the chain of signification in which the formal notion was built on the preformal strategy).

![Image of division problem]

Figure 11. Robert divided both ways, and chose the incorrect answer because it looked easier.

However, we also observed a number of mistakes in the way students used division on the final problem. Our next task was to design a learning activity that would continue to help students (a) associate fair sharing with the division operation, and (b) recognize how the units of the final answer can be used to determine the dividend and divisor.

**Learning activity 7: Reciprocal unit rates**

*Motivation and design*

At this point, students had created a chain of signification for fair-sharing, which included preformal productions and culminated in a “fraction-as-fair-sharing” construct. Furthermore, many students were linking fair-sharing to division, and hence formalizing the fraction-as-quotient sub-construct. However, this trajectory was obstructed for some students because either (a) they were making mistakes when using division to solve fair-sharing problems
(using incorrect divisors or reciprocating the dividend and the divisor), or (b) they had not-yet constructed the link between fair-sharing and division.

We felt that if students could recognize situations where an intensive quantity was the final answer, they could use this to recognize the necessity of dividing two extensive quantities, as well as the divided and divisor of this division operation. However, this relies on students reinventing the division operation as a referent-changing operation (Schwartz, 1988), namely one that transforms two extensive quantities into an intensive quantity. Our goal for this learning activity was to help facilitate this reinvention.

To do this, we designed a two-problem string to explicitly connect the division operation to the creation of an intensive quantity. As shown in Figure 12, the problems invoke a fair-sharing context and have the same structure as the problems in Learning Activity 6. This was to help students associate division with fair-sharing. Furthermore, both problems share the same antecedent, and both problems ask students to create an intensive quantity that can be interpreted as a unit rate. However, the two unit rates are reciprocals of each other. This was to help students associate the direction of the division with the compound units of the intensive quantity that is created. Our choice of dollars and pizza for the antecedents was purposeful for two reasons. First, we wanted to use quantities that were both “partitionable,” but which still invoked a fair-sharing context. Second, we wanted the intensive quantity (e.g., “pizza per dollar”) to feel new to students, as if they created it using division. We did not want to use familiar quantities such as speed or density because students do not always associate division operations with insensitive quantities that they are familiar with (e.g., Thompson, 1994b).
Analysis

Our analysis of this problem centered on three aspects: (1) How students used division, (2) how students used units, and (3) the continued formalization of the fraction-as-quotient sub-construct.

**How students used division:** All students used a within-units strategy and indicated division by three for problem one and division by 12 for problem two. None of the students divided both ways before choosing a final answer. This represented a significant change from Learning Activity 6. For example, Lori initially used repeated halving in Learning Activity 6, but she chose a more strategic divisor in Learning Activity 7 (Figure 13).
Furthermore, recall that Robert divided both ways in Learning Activity 6, and relied on the decimal representation of the division to guide him to an (incorrect) answer. In contrast, in Learning Activity 7, Robert did not divide both ways. Rather, he recognized the correct direction of the division on both problems, and he did so before dividing (Figure 14). He explained to FAP that he knew which direction to divide based on which quantity he was finding one of.

We had hypothesized that that the strategic use of units might help students understand how the direction of the division is inherent in the problem structure, namely “If students can state the units of the final answer as a compound unit using the word ‘per’ before they solve the
problem, they will be able to recognize partitive division situations and the direction of the division.” However, based on the student work, it seemed that students were not employing this “referent-transforming strategy.” Students were considering the units in the problem in a thoughtful way, but not in the way that we hypothesized.

**How students used units, and the emergence of new preformal productions:**

Consider problem one, which involves finding the missing dollar value that corresponds to one pizza. We hypothesized that students would recognize this as a situation in which they wanted to find the intensive quantity, “dollars per pizza,” and that, by stating the units in this way students would recognize the division of dollars by pizza.

As the learning activity progressed, however, it became clear that this is not how students thought of the missing value problem. Rather than using compound units to construct a division operation, students were choosing the dividend and divisor based on which of the extensive quantities they were trying to find one of. This “find-one” strategy turned out to be the dominant strategy. Rather than dividing dollars by pizzas to arrive at the intensive quantity “dollars per pizza,” students were dividing both dollars and pizzas by a dimensionless scalar that maintained the units in each extensive quantity, and then writing the “per” statement based on which quantity they had “found one” of. It was not clear that students recognized their result as a new quantity, nor was it clear that students had constructed the division operation as a referent-changing operation.

We believe that the visual design of the missing value problems supported students in their use of the “find-one” strategy. The vertical alignment of units helped students recognize the within-unit scalar needed to “find one,” and the white space between the lines encouraged students to draw arrows showing division as a dynamic operation that transformed a units into l
unit. With respect to the arrows themselves, consistent with G. Brousseau et al. (2009), we see the students’ use of arrows as a way of reasoning, “whose validity the student checks by reference to the actual meaning” (p. 109). We did not dwell on the arrows themselves in the classroom, but rather asked students to explain their mathematical reasoning in the context of the problem itself (e.g., the explanations in Figure 13 and Figure 14).

Students seemed to find the vertically-aligned structure of the problems so helpful that they even reproduced it themselves (see, e.g., the student work in Figure 13). Thus this structure soon became a preformal tool that students used to solve missing value proportional reasoning problems. As such, it is essentially a ratio table (Middleton & van den Heuvel-Panhuizen, 1995; Streefland, 1993). One key difference is that the units are made more salient in this tool, and hence, the division operation needed to “find one” is clearer. Because this tool keeps the units salient, we called it a “unit-salient ratio table.”

The continued formalization of the fraction-as-quotient sub-construct: When solving problem 2, students were confronted with a division problem with a non-integral solution. In this problem, all students recognized that $3 \div 12 = \frac{3}{12}$, indicating a formalization of the fraction-as-quotient sub construct (see Figure 13 and Figure 14 for examples of how two students demonstrated this formalization).

Summary: Analysis of research question 2, and the importance of preformal productions

Our second research question was, “how do beginning algebra students reinvent the fraction-as-quotient sub-construct?” Our hypothetical learning trajectory involved relatively unproblematic movement through a progressive-formalization sequence: students would initially use informal productions to solve fair sharing problems, and—through mathematizing progressively more abstract situations and social interaction—students would invent
progressively more formal mathematical productions culminating in the formal fraction-as-quotient sub-construct. What we found was that the instructional starting point was considerably more complicated, and that preformal productions played a larger role than we had initially expected.

First, we found that our students initially solved fair-sharing problems using productions at all levels formalization, but that only those students who used preformal productions could explain their reasoning and solve the problem correctly. This was our initial clue of the importance of preformal productions. As students shared their productions and mathematized more abstract situations, we found that preformal productions played two key meditational roles: (1) they mediated problem-solving activity, allowing, for example, students to solve the chicken feed problem in Learning Activity 4 after they realized that their informal strategy would not lead to equal shares; and (2) they mediated the reinvention of more formal productions, resulting in a chain of signification that made the fraction-as-quotient sub-construct real for students. Thus, preformal productions were important even for students who initially solved the sub-sandwich problem correctly using formal mathematics.

Second, we found that many students did not automatically associate partitive division situations with the division operation, and, even when students did use division, they sometimes performed the division operation both ways and compared the results to choose their final answer. Many times, this strategy led students to choose the correct answer (e.g., Robert and Justin in Learning Activity 2), but other times it did not (e.g. Robert in Learning Activity 6). Again, preformal productions—specifically the unit-salient ratio table and the find-one strategy—played a key role, mediating the reinvention of division as a “find one” operation.
In contrast to the (expected) preformal productions that emerged for fair sharing, the unit-salient ratio table and the find-one strategy for division emerged in class unexpectedly. We now recognize that the “referent-changing” strategy that we had hoped to design for is a between-unit strategy, and we designed for a within-unit strategy. In one sense, this is a failure of design, and we highlight it so as to provide an avenue for future design work. Despite this, students reinvented a robust strategy that they could use to recognize when and how to divide (G. Brousseau et al., 2004, describe a similar strategy used by students to find the thickness of a single sheet of paper). Thus, although students did not construct the strategy that we expected, we still consider outcome of the learning activity a success. In fact, we believe that these productions could play a powerful role in an algebra class. We did not have time to explore this in depth in our design experiment, but we have some evidence from the class. During a review day for the final exam at the end of the school year, we gave students the problem shown in Figure 15. In the figure, the student coordinates the “slope triangle” with a unit-salient ratio table and the find-one strategy. This leads us to conjecture that the unit-salient ratio table and find-one strategy could mediate students’ reinvention of slope as a unit rate. To be clear, the student whose work is shown in Figure 15 had already learned slope by the time we gave this problem. Thus, we present this work only as a suggestion of potential. We plan to explore this conjecture in our future work.
In the discussion above, we explained how preformal productions played two key roles for our students: (1) they mediated mathematical activity, and (2) they mediated the reinvention of more formal mathematical productions. We further showed how sometimes, preformal productions emerge in unexpected ways. In this section, we take a closer look at preformal productions, examining how they emerged in our classroom and reflecting on their epistemological and ontological status. We suggest that preformal productions can be designed for, and that that even when preformal productions emerge unexpectedly, they are not random. In both cases, preformal productions embody historic classroom activity and social interaction. As such, we define preformal productions ontologically as cultural artifacts.

**Preformal productions can be designed for**

Our goal for Learning Activities 3-5 was to design a sequence of activities and social interactions that would lead students to construct the preformal bar model and partition-distribute-iterate strategy. Following RME design principles (Gravemeijer, 1999), we explicitly designed math congresses and mathematical activity to support students’ reinvention of
progressively more formal mathematical productions. For example, we structured the math congress in Learning Activity 3 such that the student presentations followed a trajectory of progressive formalization. This notion of a progressive formalization trajectory also informed the design of Learning Activities 4 and 5. As students engaged in these activities, the preformal bar model and partition-distribute-iterate strategy came to embody the historic activity of fair sharing and the social interaction of the math congress. That students reinvented these productions “as hypothesized” suggests that preformal productions can be designed for by designing activity and social interaction around a progressive formalization trajectory.

**Preformal productions emerge unexpectedly, but they are not random**

In contrast to the preformal productions that emerged for fair sharing, the unit-salient ratio table and the find-one strategy for division that emerged in Learning Activities 6 & 7 were unexpected. However, this does not mean that they emerged randomly. Instead, we suggest that, like the designed-for productions discussed above, these preformal productions also embody historic classroom activity and social interaction. Below, we justify this claim for the find-one strategy.

*Preformal productions embody historic classroom activity*

Partitive division contexts can be sub-divided into the (non-mutually exclusive) contexts of fair-sharing on the one hand, and finding unit rates on the other. In a fair sharing problem, the quotient represents the extensive quantity that one sharer receives, and in a finding unit rate problem the quotient represents an intensive quantity (Confrey, Maloney, Nguyen, Mojica, & Myers, 2009 call the former a “many-as-one” conception, and the latter a “many-to-one” conception). Of these two sub-contexts, the fair-sharing sub-context is more focused on the concept of “finding one” (i.e., how much pizza one person receives) as opposed to a unit rate
problem (i.e., find the speed in miles per hour). To be sure, one can conceptualize the result of a fair-sharing problem as a unit rate (i.e., conceptualizing the aforementioned quantity of pizza as “pizza per person”): this, in fact, is how students wrote their final solutions in Learning Activity 7. The key here is the framing of the task: Are we looking for the (extensive) amount of pizza that corresponds to one person, or are we looking for a brand new (intensive) quantity, “pizza per person?” The former framing suggests a find-one strategy, while the latter farming suggests a “referent transforming” strategy. In the case of our experiment, all of our tasks were framed in the “find one” context. Thus, the students’ historic classroom activity was strongly suggestive of the find-one strategy.

*Preformal productions embody historic classroom social interaction*

Social interaction also played a role in shaping the find-one strategy. To explore how social interaction shaped the find-one strategy, we analyzed the classroom discourse that MM recorded in his fieldnotes. In analyzing discourse, we were guided by the notion that discourse is not just communication about action, it is itself action. That is, discourse communicates, but it also *does* (Gee, 2011; Jaworski & Coupand, 2006). These two uses of discourse are mutually-implicated in interactive sequences, in which “we produce action methodically to be recognized for what it is, and we recognize action because it is produced methodically in this way” (Heritage & Clayman, 2010, p. 10). Actions are produced and evaluated in *turns* at talk. Thus, we analyze the work of discourse by analyzing how turns are *produced*, *taken-up*, and *sequenced* in interaction. In what follows, we analyze two sequences to explore how social interaction shaped the find-one strategy.

First, recall the classroom discourse that occurred when Jody used the word “per” in Learning Activity 6:
In the first turn, Jody uses the word “per” to create an intensive quantity, “markers per package,” without fully articulating the units. FAP writes the full units of the quantity on the board, but rather than attend to this new quantity, FAP attends to the new word, “per,” and Lori and Joshua focus on the “oneness” of this word.

This focus on the oneness of “per” continued in the math congress held after students worked on the next problem in the string (t-shirts, see Figure 10). In the first part of this problem, students were given the cost of six shirts, and asked to find the cost of three shirts. We expected that students would recognize a within-unit ratio of two, and use this to find the cost of three shirts. However, Zane explained that he multiplied “eight times three” to find the cost of three shirts. We pick up the discourse as Toni attempts to explain Zane’s strategy to the class:

(1) Toni: He divided 48 by six, which is eight.
(2) FAP: That eight is an interesting number. What does the eight represent?
(3) Toni: How much one shirt costs
(4) FAP: What was Jody’s word?
(5) Many: Per
(6) Steve: It means one

Here, Toni starts by describing calculations on numbers, rather than calculations on extensive quantities. That is, she describes dividing the number 48 by the number 6, rather than on dividing the quantity “48 dollars” by the quantity “6 shirts.” Thus, while Toni has explained how Zane found the number 8, it is not yet clear how she conceptualizes of this number. When FAP asks about the meaning of her number, she focuses on the oneness of it, and the subsequent discourse continues to reify the oneness of “per.” Again, we can imagine a different situation where, instead of asking about the meaning of the “8” in turn 2, FAP asked about the meaning of
the 48 and the 6, and then asked what happens when you divide dollars by t-shirts. Perhaps the notion of division as a referent transforming operation may have come out of the subsequent classroom interaction.

Thus, the find-one strategy emerged unexpectedly in our experiment, but it was not random. Instead, this strategy embodies historic classroom activity (including solving fair sharing problems) and social interaction (including discourse that focused on the oneness of the word “per”).

**Preformal productions are cultural artifacts**

Having analyzed at some depth the role of preformal productions and their emergence in the classroom, we now turn to a more fundamental question: “what are preformal productions?” Drawing on the RME literature in our conceptual framework, as well as our expanded use of preformal productions in this paper, we define preformal productions as mathematical productions—such as models, tools, and strategies—that embody historic activity and social interaction. They are simultaneously general and specific, and as such they exist between students’ informal realities and formal mathematics. Through activity, preformal productions can be made general enough so as to be applicable to a wide variety of problems, but they retain contextual cues to specific situations.

In the definition above we referred to the “existence” of preformal productions. In what sense do these productions exist? One response is to think of preformal productions as having an epistemological existence. From this perspective, we might consider G. Brousseau’s distinction between two types of knowledge: *connaissance* and *savoir*. *Connaissances* are an individual’s internal ways of knowing within a situation, while *saviors* are the culturally-accepted ways of expressing and communicating these ways of knowing:
Events in class have the effect of provoking students to react, make declarations, reflect, and learn, all of these manifesting their intellectual activity. This activity reveals their *connaissances*: what they do, their intentions, their perceptions, their decisions, their beliefs, their language, their reasoning. Only one part of this set of *connaissances* is recognized as expressible, and expressed, whether by the student, by other students, by the teacher, or by society. These *connaissances* are recognized with the help of a repertory of *reference connaissances*: custom, language, rules of orthography, established definitions and theorems, logic, communal beliefs, culture, etc. These are the *savoirs*. *Savoirs* are the indispensable means of recognizing and expressing *connaissances* (G. Brousseau et al., 2009, p. 110).

Preformal productions seem to have elements of both *connaissances* and *saviors*. For example, the partition-distribute-iterate strategy may be considered a way of knowing, and thus a *connaissance*. On the other hand, the bar model may be considered a *savoir* because, within the classroom, it was the culturally-accepted external means through which students expressed their ways of knowing (in our case, the bar model become culturally-accepted within the classroom through the math congress described in Learning Activity 3). However, we feel that this does not quite capture the nature of preformal productions. First, notice that, tacit in the excerpt above is a definition of *culture* as a static entity that is established *a priori* and which exists to make the internal external. The implication is that the relationship between individuals and their culture is a one-way relationship: from internal to external, via culture. To capture the nature of preformal productions, we need a more dynamic definition of culture and a different conception of the relationship between individuals and their culture.
We define culture as both a process and a product. As a process, culture accumulates the prior accomplishment of a social group and propagates them into the present (Hutchins, 1995a). As a product, these accumulated accomplishments of history serve as resources for current activity (Cole, 2010). These resources are cultural artifacts, and they function as follows:

1. **The historical nature of artifacts can be understood on many timescales:** “Some artifacts are inherited (e.g., cultural tools, methods, signs, software tools) and practically do not change during activities. Others are in perpetual transformation. They have been freshly created as outcomes of previous actions and are used by the participants in further activities” (Schwarz & Hershkowitz, 2001, p. 251).

2. **Cultural artifacts mediate current activity:** “Higher mental functions are by definition culturally mediated. They involve not a direct action on the world, but an indirect action, one that takes a bit of material matter used previously and incorporates it as an aspect of action. Insofar as that matter itself has been shaped by prior human practice (e.g., it is an artifact) current action incorporates the mental work that produced the particular form of that matter” (Cole & Wertsch, 1996, p. 252).

3. **The relationship between persons and artifacts is bi-directional:** “[A]gents create meaning by drawing on cultural forms as they act in social and material contexts, and in so doing produce themselves as certain kinds of culturally located persons while at the same time reproducing and transforming the cultural formations in which they act” (O’Connor, 2003, pp. 61–62).

We suggest that the notion of a cultural artifact as defined above best-captures the nature of preformal productions. As we described above, preformal productions embody historic classroom activity and classroom interaction, and they were used by students in service of
current activity. Thus, they meet our initial definition of a cultural artifact. Preformal productions fit the remaining functions of artifacts as follows: (1) Students reinvented preformal productions on the timescale of the design experiment. (2) Preformal productions played two mediating roles: They mediated students’ mathematical activity, and they mediated students’ reinvention of more formal mathematical productions. (3) In the combination of (1) and (2) above, we see the bi-directional nature of the relationship between students and preformal productions: Students acted to produce preformal productions, but preformal productions acted back to produce students as cultural beings with particular mathematical realities.

Conclusion

We began this paper with a problem from our practice as algebra teachers and researchers, namely, that beginning Algebra I students had more trouble with the “division step” in an algebra equation when the quotient was non-integral than they did when the quotient was an integer. In our practice, we observed that upon encountering such a division step, students often either (a) abandon the problem, stating that the division “can’t be done” or (b) perform the division backwards. Furthermore, in our experience, nearly all students who attempt the division use the division symbol (÷) to represent the operation, and use the long division algorithm to express their final solution in decimal notation. Few students use the “fraction bar” (the line that separated $a$ from $b$ in the fraction $\frac{a}{b}$) to represent the division operation or use fractions to represent their solutions.

We hypothesized that this was because students did not have enough prior experience with the fraction-as-quotient sub-construct, and we conducted a design experiment organized around fair sharing in order to help students reinvent this sub-construct. In the beginning of the
design experiment, we found that none of our students’ initial realities included the fraction-as-quotient sub-construct. Furthermore, for many students the division operation was also problematic: students did not always associate partitive-division situations with the division operation, and, even when students did use division, they did not always see the how the problem situation dictated the direction of the division. By the end of the experiment, students had reinvented the fraction-as-quotient sub-construct, as well as the notion of using division to “find one” of a particular quantity. In describing how students reinvented these productions, we hope to have contributed to practice and to theory.

Our contributions to the practice of algebra teaching are (1) to highlight the importance of the fraction-as-quotient sub-construct for algebra students; (2) to suggest that this sub-construct, and indeed, the division operation itself, might not be a part of the mathematical realities for students entering Algebra I, and to have provided a detailed account of students’ mathematical realities around division and fractions; (3) to show that students might not construct the fraction-as-quotient sub-construct solely through experience with fair-sharing situations, and that explicit activities may be needed to help these students link the “fraction-as-fair-sharing” sub-construct to the fraction-as-quotient sub-construct; and (4) to have provided one possible sequence of activities through which Algebra I students might reinvent fractions and division as they are used in algebra. In the beginning of the paper we clarified that our goal was not to present a “model” curriculum, and we reiterate this now. That said, we have shown that students learned powerful mathematics as they engaged in the sequence of activities presented here. As such, our descriptions of this sequence and the design decisions that motivated it may prove useful for teachers to design their own sequences. Future work should explore the ways in the preformal
productions that students reinvented in this experiment mediate the reinvention of formal algebra.

Our contributions to theory include an expansion of the emergent modeling paradigm (Gravemeijer, 1999) to include all manner of mathematical productions, and an in-depth analysis of the role of preformal productions in (a) mediating students’ mathematical activity and (b) mediating the reinvention of more formal mathematics. Luria (Luria, 1928, p. 493) famously stated that “the tools used by man not only radically change his conditions of existence, they even react on him in that they effect a change in him and in his psychic condition.” This is precisely the role played by preformal productions in our study. Preformal productions changed the conditions of our students’ existence because they mediated students’ activity, rendering solvable problems that were previously not solvable. Preformal productions further effected a change in our students because they mediated the invention of more formal mathematics. Indeed, the formal mathematical realities that emerged were—to a large extent—dictated by the preformal productions that preceded them. Preformal productions are not “crutches for the weak” as one teacher with whom we have worked once described them. Instead, they are an integral and vital part of doing and learning mathematics.

Given the importance of preformal productions, it is important to understand how they emerge in the classroom. We suggest that the specific productions that emerged in our classroom embody historic classroom activity and social interaction. As such, they can be designed for. However, we have also shown that preformal productions emerged even when they were not explicitly designed for. This analysis has large implications for designers and teachers. Namely, it suggests that preformal productions should be a key part of any designed curriculum, and that teachers should be aware of the ways in which the activity and interactions within the classroom
are shaping the development of preformal productions, because it is on these productions that students create their formal mathematical reality.
A robust understanding of slope is vital for success in secondary and post-secondary mathematics (Thompson, 1994a). However, student understanding of slope is often formulaic and underdeveloped (Stump, 2001). In part this is because slope is a complicated concept, with multiple sub-constructs including “rate of change,” “physical property” (steepness), “geometric ratio” (rise over run), “algebraic ratio” (change in $y$ over change in $x$), and “parametric coefficient” (the $a$ in the equation, $y = ax + b$) (Stump, 1999).

While there has been extensive research on how students come to understand rate of change (Confrey & Smith, 1994; Cramer et al., 1989; Karplus et al., 1983; Nemirovsky, 1996; Nunes et al., 2003; Thompson, 1994b; Tierney & Monk, 2007; Yerushalmy, 1997), comparatively little research has been conducted on how students learn the remaining sub-constructs—especially the connections between the sub-constructs. To investigate how students
learn multiple sub-constructs of slope, we\textsuperscript{1} conducted a design experiment in a high school Algebra I classroom, in which we considered the following research questions:

**Research questions**

How do students invent, make-meaningful, and make connections between, five sub-constructs of slope such that the constructs become experientially real objects for students to *think with*?

- What sorts of tasks and problem contexts lead students to invent, make-meaningful, and make connections between the five sub-constructs of slope?
  - What mathematical artifacts—including models, tools, strategies, and representations—do students draw on as they engage in these activities?
  - What meanings do students put into the five sub-constructs, and how is this process mediated by tasks, problem contexts, and artifacts?
  - In what ways do students make connections between the five sub-constructs, and how is this process mediated by tasks, problem contexts, and artifacts?

In this paper, I will discuss the findings from this design experiment. I will show that learning emerged as a *cascade of artifacts*, which I describe as part of a *local instructional theory* (Gravemeijer, 1999, 2004).

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\textsuperscript{1} In this paper I occasionally use the first-person plural. This is because, as I explain below, the design and implementation of the design experiment was conducted by a research team, whereas I conducted the bulk of the analysis and produced this paper. Thus, when I report on a collaborative effort, I use the plural. When I report on my own effort, I use the singular.
The paper is organized as follows: In the next section I present a conceptual framework that explains how artifacts are central to learning. In Section 2 I discuss the instructional design, and the research and analysis methods, including the prior work on slope that informed the design. In Section 3 I present the local instructional theory, including the cascade of artifacts. Finally I conclude with implications and directions for future work.

1. Conceptual framework: How artifacts play a central role in learning

In this section I focus on four key concepts, culture, mediation, history, and objectification, to examine how artifacts play a central role in learning.

I take a process and product view towards culture. Cultural processes are those that “accumulate partial solutions to frequently encountered problems” (Hutchins, 1995a, pp. 354–355). The residua of these processes—the partial solutions themselves—exist as cultural artifacts (Cole, 1996). For example, a function table is a common cultural artifact in secondary mathematics. It is a partial solution to the frequently encountered problem of working with two quantities that are in a functional relationship. Function tables have a long history in mathematics and schooling (Campbell-Kelly, Croarken, Flood, & Robson, 2003). At least as far back as 2600 BCE, Sumerians used tables to record square areas for a given length, and indeed there are more than 100 known function tables from this time period. In ancient schools, students were presented with similar arithmetic tables, which they were required to memorize by copying (Robson, 2003). Moving forward in time, function tables were used to record values of elementary trigonometric functions and, later, John Napier used function tables to record values of perhaps the most consequential function ever invented, the logarithm (Maor, 1994). I will return to the logarithmic table below, and again in Section 3. For now, it suffices to say that
“Tables are quintessential cultural objects of the civilizations that created them, improved and perfected by each succeeding generation” (Campbell-Kelly et al., 2003, p. 14)

Artifacts, such as function tables, serve to propagate historical achievements into the present, and the set of these artifacts constitutes culture-as-product: “the species-specific medium of human life” (Cole, 2010, p. 462). Notice in this quote that Cole refers to the “medium” constituted by culture. Human actions take place in a cultural milieu and as such they are mediated by culture. What this means is that human actions “involve not a direct action on the world but an indirect action, one that takes a bit of material matter used previously and incorporates it as an aspect of action” (Cole & Wertsch, 1996, p. 252). Here, the “bit of material matter” is a cultural artifact and the set of cultural artifacts present in a society creates the cultural milieu: “the species-specific medium of human life.”

The above discussion of the cultural milieu and its use by humans serves to introduce the difference between a cultural perspective and a purely social perspective. To see this difference further, consider the difference between human and non-human simian societies. Simian societies are highly social in that all primates have to interact with other primates in order to accomplish their goals. But non-human simians differ in that they have little-to-no way to preserve their social accomplishments, and thus the vast majority of these accomplishments must be reenacted anew in each interaction (Latour, 1996). This is manifestly not the case for humans. Why not? Because of culture. Cultural processes capture and preserve accomplishments in the form of artifacts (cultural products) so that humans do not have to recreate these accomplishments in every interaction or activity. Humans don’t always have to go through other humans to accomplish a task, but they always have to go through culture. In doing so, humans delegate work to a non-human artifact. Above, I suggested that logarithms were perhaps the most
consequential function ever invented. This is because the logarithmic function reduces exponentiation to multiplication and multiplication to addition. Logarithmic tables were the vehicle through which this reduction took place. Thus, by coordinating a row and column in Napier’s logarithm table, an astronomer could execute a complex exponentiation task simply by adding. In doing so, the astronomer delegates some of the work of exponentiation to the logarithmic table. In return, the table requires certain behaviors of the astronomer, namely behaviors that coordinate rows with columns, and that compute a sum. Notice that these behaviors that simply would not have the same effect in the absence of the artifact (Johnson & Latour, 1998; Latour, 1992). Together then, the astronomer and the table form a system and the solution to the multiplication problem is an accomplishment of that system (A. Clark, 1998; Hutchins, 1995a, 1995b; Ingold, 1999).

The same is true for all human activity. When we use artifacts to mediate our activity we are participating in a system that marshals historic accomplishments to serve purposes in the present. Because artifacts are also participants in this system, they shape our behaviors such that we behave in ways that history would not recognize in order to create forms of activity that were historically not possible. In this way, mediating artifacts do more than simply facilitate or amplify an action that would otherwise exist. Rather, they enable new forms of human actions. In the case of the logarithm, it is no stretch to say that much of the scientific accomplishments of the Enlightenment were made possible by the logarithmic table (and its later embodiment in the slide rule).

These actions are enabled by culture, and that is what separates humans from other social animals. The focus on the role of culture is what separates a cultural perspective from a purely social perspective. We can describe the activity of non-human primates by appealing to their
social world. But the social world is simply not sufficient to account for human activity. We cannot describe human activity without accounting for the ways that the activity is shaped by the past in the form of mediating artifacts, and the ways in which the activity is contributing to future in the form of new artifacts.

So far, I have used the word “history” rather loosely. History can be understood on many timescales (Wertsch, 1985). Above, I described how function tables, like mathematics itself, developed over human sociocultural history. Humans learn mathematics over the course of their lives—ontogenetic history. Classrooms too have a history, and often artifacts emerge in classrooms that embody the history of activity in that classroom (Schwarz & Hershkowitz, 2001). Regardless of the timescale, culture is present twice in any human activity. Cultural processes accumulate partial solutions to problems that are frequently encountered in activity, and cultural products—artifacts—are literally “in the middle” (Cole, 1996, p. 116), mediating the activity.

Artifacts are imbued with history, and this history is manifested as affordances and constraints that shape the contours of current activity. However, artifacts are not deterministic and they can often afford many types of actions. This is the key insight exploited by Vygotsky and colleagues in their *double stimulation* experiments (Engeström, 2007; Vygotsky, 1978). The method involves giving a subject (in experimental conditions) a task to solve that is “beyond his [sic] present capabilities” (Vygotsky, 1978, p. 74). The task is the first stimulus. The experimental setting also contains a neutral object—a second stimulus—which the subject often incorporates into the task:

> Frequently we are able to observe how the neutral stimulus is drawn into the situation and takes on the function of a sign. Thus, the child actively incorporates these neutral
objects into the task of problem solving. We might say that when difficulties arise, neutral stimuli take on the function of a sign and from that point on the operation's structure assumes an essentially different character (Vygotsky, 1978, p. 74).

Thus the second (neutral) stimulus becomes a mediating artifact as the subject pours meaning into the artifact and then uses the artifact to accomplish the task. This is an apt description of my definition of learning in mathematics. More specifically, I define learning as a process of reinvention (Freudenthal, 1973, 1991; Gravemeijer, 1999) and objectification (Radford, 2008b) in which people reinvent mathematical artifacts and endow them with meaning. Artifacts acquire particular meanings through a social process in which perception is disciplined (Stevens & Hall, 1998), so that particular features and affordances of artifacts become salient. In this way, people learn to “see” artifacts in disciplinary ways (Radford, 2002; cf. Wittgenstein, 1958). This, in turn, happens through the use of focusing phenomena: “regularities in the ways that teachers, students, artifacts, and curricular materials act together to direct attention toward certain mathematical properties over others” (Lobato, Ellis, & Muñoz, 2003, p. 1). Often this is accomplished through discourse and gesture (Gee, 2011; Hutchins & Palen, 1997; Radford, 2008c; Streeck, 2009). So defined, learning is a cultural process in the sense described above.

Taken together, this conceptual framework suggests particular research and analysis methods, which I describe below.

2. Research, design, and analysis methods

The goal of our research was to develop a local instruction theory for slope. Local instructional theories (LITs) are theory-guided frameworks for the design of instructional
sequences for a particular topic in mathematics. The word “local” is used to denote the locality of the theory within a single topic in mathematics and to distinguish an LIT from a larger theory of learning. LITs are composed of three parts: (1) a description of how learning happens over time, (2) principles that guide the design of activities that support this learning, and (3) the rationale for how the activities support learning (Gravemeijer, 1999, 2004).

This highlights the difference between a local instructional theory and a learning trajectory. A learning trajectory represents a specific instructional sequence, and as such can be seen as an instantiation of a LIT (or of a portion of an LIT). Whereas many learning trajectories in mathematics education are designed by researchers (Daro, Mosher, & Corcoran, 2011) LITs enable learning trajectories to be designed by teachers. The benefit of this is that teachers can craft learning trajectories that are tailored to their classrooms and students. To facilitate this, an LIT acts as something of a bridge between a domain-level instructional theory (like RME) that exists at the level of “mathematics,” and the teacher’s particular situation (Gravemeijer & Cobb, 2006).

Local instructional theories are developed using design research. This a cyclical method composed of macro-cycles and micro-cycles. At a macro-level there are three phases: preparation, implementation, and retrospective analysis (Gravemeijer & Cobb, 2006; Gravemeijer, 1994b, 2004). In this paper, I report on one macro cycle. In the sections that follow, I elaborate on each phase of the macro cycle, including providing more detail about the setting and the roles of the members of the research team.

**Research team**

Four researchers participated in various parts of this study. The primary research team consisted of myself and a colleague, Michael Matassa. During the design phase we were joined
by a third colleague, Raymond Johnson. Raymond did not participate in the implementation phase. During the implementation phase, we were joined by David Webb. I was the classroom teacher for the entire year, and served as the teacher for this study. I conducted the retrospective analysis independently, but, as I detail below, I shared my emerging analyses with the other members of the group and we discussed and resolved disagreements as I moved through the analysis.

Setting

Our study took place in two sections of high-school Algebra I. The school was located in a suburban area of the United States, and served predominantly white (approximately 60%) and Latino (approximately 30%) students.\(^2\)

Preparation

The preparation phase lasted approximately 20 weeks before and during the first semester of the 2011-2012 academic year. The research team met weekly during this time for approximately two hours per week. The goal of this phase was to design a conjectured local instructional theory. We began by reading as much as we could find about slope. We would read papers during the week and then discuss them during our weekly meetings, often ending with a new set of readings which we identified using strategies such as “footnote chasing,” “citation searching,” and “author searching” (Bates, 1989). After eight weeks, we began to reach saturation, meaning that the readings mostly contained references to papers that we had already

\(^2\) These figures are based on publicly available data shared on the school’s website. We did not have permission to access student demographic information, thus I cannot discuss the specific demographics of the class.
read. At this point we each wrote a “research brief,” a 1500-word summary of the literature as we understood it. We used these research briefs to being to design the conjectured LIT. At the same time, the school year had started and the students and I were beginning to develop a repertoire of artifacts that the research team took into account. Ultimately, in designing the conjectured LIT, we drew on four sources: (a) theories of learning that describe learning as a cultural process in which artifacts are reinvented and objectified, (b) design heuristics from Realistic Mathematics Education (RME), (c) research in math education related to slope (summarized in our research briefs), and (d) students’ repertoires of artifacts that developed during the design phase. Below I discuss the latter three categories, relating them back to the notion of learning as a cultural process.

**RME design heuristics**

RME provides three design heuristics (Gravemeijer & Terwel, 2000)—emergent modeling, guided reinvention, and didactical phenomenology—and we drew on all three in our design. As discussed above, we defined learning as a process of reinvention and objectification, through which students invent mathematical artifacts and make them meaningful. From a design perspective, the “final” artifacts are known. In our case these artifacts are the five sub-constructs of slope, discussed in the literature review below. Our task as designers was to create a sequence of activities in which students were *guided to reinvent* these artifacts as partial solutions to frequently encountered problems.

This reinvention happens through a process of *emergent modeling*. The general idea is that students create mathematical artifacts and make them meaningful as they engage in mathematical activity. Similar to Vygotsky’s double stimulation experiments, the activities are structured so that they are just “beyond [the students’] present capabilities.” Students solve these
problems by incorporating artifacts into their activity: either by creating new artifacts (a slight difference from double stimulation) or, as in double stimulation, giving an existing artifact new meaning. The task for the designer is to sequence the problems so that artifacts emerge in a meaningful sequence: at any given time a particular artifact signifies the artifacts that came before it, and later it will be signified by a more general artifact. In this way, the “final” artifacts (the five sub-constructs of slope in our case) emerge as just one step in this sequence. This one-to-one sequence of hierarchical signification is called a “chain of signification” (Gravemeijer, 1999; Whitson, 1997).

In designing activities, we were guided by the RME principle of *didactical phenomenology*. The idea is that students should be presented with rich contexts that (a) from the students’ perspective “are begging to be organized” (Gravemeijer & Terwel, 2000, p. 787) and (b) can be organized by the artifact that is meant to be invented. In addition, we were guided by the literature from math education related to slope, as I describe below.

*Review of literature in math education related to slope*

Slope is composed of seven sub-constructs: (1) rate of change, (2) physical property (steepness), (3) geometric ratio (rise over run), (4) algebraic ratio \( \frac{y_2-y_1}{x_2-x_1} \), (5) parametric coefficient (the \( a \) in the equation, \( y = ax + b \)), (6) trigonometric ratio (the tangent of the angle that a graphed line makes with the \( x \)-axis), and (7) derivative of a function (Stump, 1999). We focused our study on the first five of these, as only these five were part of the Algebra I curriculum of the school (the latter two sub-constructs were introduced in later courses).

Of the first five sub-constructs, *rate of change* has received the most scrutiny from researchers. Much of the work on rate has explored how students come to understand rate through covariation (Confrey & Smith, 1994; Lobato et al., 2003; Thompson, 1994b).
Covariation can be constituted through tables. Working with tables, students coordinate changes in one variable with changes in another by moving up or down a well-ordered table (Confrey & Smith, 1994). This representation can make salient the changes from row to row, but it can also afford outcomes in which covariation is cast as differences rather than a ratio (Lobato et al., 2003; Schliemann & Carraher, 2000). This is mediated by particular focusing phenomena, including the form of the representation and the language used to describe changes within the representation. Specifically, well-ordered tables where the independent variable increases by one in each subsequent row can lead to “goes up by” language (i.e., “it goes up by three” as a way to describe the covariation present in a well-ordered table where the independent variable increases by 1 and the dependent variable increases by 3 in each subsequent row). When students and teachers use this language, they are attending to only one of the two variables that are changing, which is problematic because understanding covariation as the coordinated change of two variables is at the heart of understanding rates of change (Lobato et al., 2003).

Others have explored how students come to understand rates as measures of intensive quantities, for example of steepness, speed, or intensity of taste (Karplus et al., 1983; Lobato & Thanheiser, 2002; Nunes et al., 2003; Simon & Blume, 1994). This understanding is built as students engage proportional reasoning with ratio pairs, including missing value problems (e.g., if 3 apples cost $0.90, how much does 1 apple cost?) and comparison problems (e.g., given two mixtures of lemonade concentrate and water, which will taste stronger?) (M. R. Clark, 2005; Cramer et al., 1989; Fosnot & Dolk, 2002; Lamon, 2012; Post et al., 1988), and can be mediated by tools such as ratio tables (Middleton & van den Heuvel-Panhuizen, 1995). From a covariational perspective, rates describe dynamic phenomena: two quantities “accrue simultaneously and continuously, and accruals of quantities stand in the same proportional
relationship with their respective total accumulations” (Thompson, 1994a, p. 232). In contrast, from a measurement perspective a rate is a relatively static object. For example, when one creates a measure of unit price in order to answer the question about apples above, unit price is reduced to a single number; dynamic notions of simultaneous accruals are not at the fore.

Both interpretations of rate are important, and indeed, they can inform each other. For example, Lobato et al. (2003) speculated that understanding rates as measures of intensive quantities would help students understand rates as covariation. Specifically, they speculated that students who understand rates as measures would be more likely to attend to covariation rather than focusing on changes in only one quantity. Thus, these authors recommend that students should have experiences in which they create rates as measures of intensive quantities using division, and that students should use the language of intensive quantities to describe these rates. As I discuss below, Lobato et al.’s conjecture formed the foundation of our conjectured LIT.

Much of the research on the other sub-constructs of slope involves linking two sub-constructs together (Herbert & Pierce, 2005; Herbert, 2008; Lobato & Ellis, 2002; Tierney & Monk, 2007). Often, this involves multiple functions in a single representation or multiple representations of a single function. As an example of the former, graphs that show multiple linear functions at once can help constitute the connection between slope-as-steepness and slope-as-rate (Tierney & Monk, 2007).

With respect to the latter, many authors have argued that different representations (e.g., tables, graphs, and algebraic equations) make salient different aspects of functions, and this is certainly the case for slope—especially because certain sub-constructs are only manifest in a particular representation (for example, slope-as-parametric-coefficient is only manifest in algebraic equations, whereas slope-as-geometric-ratio is only manifest in graphs in the Cartesian
plane). However, simply providing multiple representations is not sufficient to link interpretations of slope across representations. Instead, “explicit connections between [representations] are required to enable students to transfer understandings of rate from one representation to another” (Herbert, 2008, p. 34). This is consistent with the concepts that I outlined earlier. The design principle of emergent modeling provides a heuristic for designing activities such that the meaning of one artifact is built on of the meanings of others. In this way, the “explicit connections between [artifacts]” is inherent in the process through which students make artifacts meaningful.

Repetoire of artifacts developed in class prior to the study

During the first semester of school, the students and I developed a repertoire of artifacts in class. Table 1 lists these artifacts, along with the meanings that we put into these artifacts. Evidence that these artifacts had the meanings that I attribute to them in the table came from: (1) my class records from the first semester, and (2) the ways that students took up these artifacts during the study.

Designing the conjectured LIT

Drawing on the RME design heuristic of didactical phenomenology, we asked ourselves, “what sort of contexts are (A) begging to be organized (from a student’s perspective), and (B) can be organized by the five sub-constructs of slope?” The canonical context for slope is steepness. However, we rejected this context because it seemed to fail both criteria. With respect to criterion A, we did not feel that finding steepness would be motivating enough to sustain an entire unit of study. With respect to criterion B it was not clear to us how steepness could be used to organize the five sub-constructs of slope. Steepness is clearly connected to the physical property (it is the physical property). And it’s not too big of a leap to see how one measures
steepness using the geometric ratio. Beyond these geometric sub-constructs however, steepness falls short. For example, consider the parametric coefficient. In the equation $y = ax + b$, why would someone multiply steepness by $x$? What problem is this solving? While it is certainly possible to find a scenario where this might be reasonable, it seemed to us to be implausible at best.

Table 1. Repertoire of artifacts developed in class prior to the study

<table>
<thead>
<tr>
<th>Artifact</th>
<th>Meaning(s) ascribed to artifact</th>
</tr>
</thead>
</table>
| 1. The fraction-as-quotient sub-construct | • Fractions are simultaneously a division problem and the numerical solution to the division problem  
• The “fraction bar” as a division operator |
| 2. “Find-one” strategy             | Using division to “find-one” (viz. a 1-to-many relationship) given a many-to-many relationship  |
| 3. Ratio tables                    | A tool to represent and create equivalent ratios                                                |
| 4. Intensive units                 | A way to describe a one-to-many relationship using the word “per” (e.g., 2/3 pounds per tomato) |
| 5. Rate of change                  | A measure of an intensive quantity                                                               |
| 6. Two-variable algebraic equations | • A rule to find the output for any particular input in a pattern or function                   
• “Point checkers” to determine if a given point would be on a graphed curve               |
| 7. Function tables                 | • A means of collecting and organizing input-output pairs in a pattern or functional relationship |
| 8. Graphs in the coordinate plane  | • A visual representation of the collection of points that makes a particular equation true.     |
Another option is a purely formal treatment, predicated on the connection between coordinate graphs and algebraic equations. This too seemed to fail criterion A, in that for students, formal mathematics is generally not something that is begging to be organized.

Ultimately, we decided to focus the unit on making predictions using mathematics. This seemed to meet both criteria. With respect to criterion A, making predictions about the future seemed to us to be a fairly motivating context for students. With respect to criterion B, predictions are not tied to a single representation or a single sub-construct of slope. In addition, organizing the unit around predictions allowed us to center the unit on the key concept of rate of change, as recommended by researchers and professional organizations (Confrey & Smith, 1994; NCTM, 2000; NGA, 2010; Stroup, 2002; Stump, 2001; Thompson, 1994a).

As discussed above, Lobato et al. (2003) speculated that understanding rates as measures of intensive quantities would help students understand rates as covariation. They recommended that students should have experiences in which they create rates as measures of intensive quantities using division, and that students should use the language of intensive quantities to describe these rates. This recommendation played a large role in classroom activities during the first semester, and the result is the first five artifacts in the students’ repertoire above.

Given Lobato et al.’s conjecture, we speculated that these five artifacts would form a productive instructional starting point. We next considered the principles of guided reinvention and emergent modeling to design the conjectured LIT presented in Table 2 (oriented in ascending order from beginning to end). The conjectured LIT begins with the students understanding rates as measures of intensive quantities. From there, students reinvent the remaining four sub-constructs through an emergent modeling process in which new sub-constructs emerge as differentiations of rates through activity. At each phase, then, students are
reinventing and objectifying new sub-constructs and building a web-like network of artifacts that connects the sub-constructs to one another.

<table>
<thead>
<tr>
<th>Phase</th>
<th>Artifacts reinvented, objectified, and connected</th>
<th>Means of support:</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>The <strong>physical property</strong>: (PP): slope as a measure of steepness.</td>
<td>Description of activities</td>
</tr>
<tr>
<td></td>
<td>Connection between <strong>ROC</strong> and <strong>PP</strong></td>
<td>• Compare rates given two intersecting linear functions graphed in a coordinate plane.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Measure and compare the steepness of objects</td>
</tr>
<tr>
<td>2</td>
<td>The <strong>algebraic ratio</strong> (AR) as a strategy to find a rate given two data points.</td>
<td>Make predictions in non-proportional linear situations given:</td>
</tr>
<tr>
<td></td>
<td>The <strong>geometric ratio</strong> (GR) as a strategy to find a rate given a graph</td>
<td>• Two or more data points (e.g., in a table or in words) where the change in the independent variable is not one.</td>
</tr>
<tr>
<td></td>
<td>Connection between <strong>ROC</strong>, <strong>AR</strong>, and <strong>GR</strong></td>
<td>• A graph of a linear function, such that the value for which we want to make a prediction is not shown in the graph.</td>
</tr>
<tr>
<td>1</td>
<td><strong>Rate of change</strong> (ROC) as a description of the covariation between two variables</td>
<td>Situations that involve making predictions in linear situations, given either:</td>
</tr>
<tr>
<td></td>
<td>The <strong>Parametric coefficient</strong> (PC) as the number of times that a rate is accumulated to make a prediction</td>
<td>• The rate of change and starting value</td>
</tr>
<tr>
<td></td>
<td>Connection between <strong>ROC</strong> and <strong>PC</strong></td>
<td>• Multiple data points (e.g., in a table), where the independent variable increases by one.</td>
</tr>
<tr>
<td>0:</td>
<td>Accessible idea: Students understand rates of change as measures of intensive quantities. In addition, students have the repertoire of artifacts described in Table 1.</td>
<td></td>
</tr>
</tbody>
</table>
Note that units on slope often start with graphs of linear functions in a coordinate plane. However, we delayed the introduction of graphs for two reasons. First, we felt that graphs are often poor representations for making predictions. Second, we knew that many students had learned slope as a rule for finding steepness of graphed lines in middle school. We wanted students to reinvent slope in a new way, thus we decided to delay the introduction of this familiar representation.

**Implementation phase**

The implementation phase lasted for 14 class periods over three weeks (810 minutes total) in the beginning of the second semester of the 2011-2012 academic year. Because the study took place halfway through the school year, the classes had developed some general routines and norms prior to the study. In general, there were three forms of activity in the class. (1) Students solved problems in four-person teams that were mostly the same for the entire year. (2) After team-based problem solving, students shared their results in a “math congress” (Fosnot & Dolk, 2001a, 2001b, 2002; Fosnot & Jacob, 2010). (3) Following the math congress, I conducted large-group discussions in which we summarized big results. For full-class activities, desks were arranged in pairs, with each desk facing the front of the room. During group discussions, I often asked students to discuss a topic in pairs before we discussed it as a class. For small group work, students moved their desks to be in pods of four, with all four desks facing the center of the group. This sequence of classroom activities is consistent with RME design principles, and with recommendations by authors who do RME-based work in classrooms (Cobb et al., 1997; Cobb, McClain, & Gravemeijer, 2003; Fosnot & Dolk, 2001a, 2001b, 2002; Fosnot & Jacob, 2010; Yackel, Rasmussen, & King, 2000). Furthermore, these different physical
organizations informed our data collection methods, as described in the “data collection” section below.

In the implementation phase, the research team implemented the conjectured LIT with students. The conjectured LIT served as a guide for classroom activities, but it was not “frozen,” to be implemented rigidly. Instead we engaged in daily micro-cycles of implementation, analysis, and design as described below:

*Implementation of day n:* During class I was in charge of teaching, and Michael and David took on the role of observer. Michael was present for 12 of the 14 days (he missed two days due to illness) and David joined us when his schedule allowed (5 days). Michael and David captured jottings on the fly and talked with students about the students’ problem-solving activity. For example, they often asked students “how did you solve this?” or “explain what this number means.” During small group work, Michael and David stayed with one group each for an entire problem-solving session, while I circulated amongst all of the groups.

*Analysis of day n, design for day n+1:* Immediately after class on day n, we (myself and Michael, and David when he was available) met to analyze the class. In this analysis we discussed our observations of the class, especially student discourse, and examined student work. In particular we considered how we saw students reinventing and/or objectifying the five sub-constructs of slope and our impressions of how the day’s learning activity influenced those reinventions and objectifications. We then used this analysis to revise our conjectured LIT and used the updated LIT to design subsequent
learning activities. After our meetings, I translated the design into a lesson plan and material artifacts for the next class, including handouts and PowerPoint slides. On day \( n+1 \), we met before class to examine and modify these material artifacts and finalize our plan for the day.

As I explain in Section 3, the actual path of learning emerged as the result of these daily micro-cycles. The conjectured LIT guided, but did not dictate, the activities and path of learning that was realized in the classroom. At the same time, the insights that we gained during these micro-cycles fed back into the LIT, which we modified throughout the implementation phase. For example, as I will explain in detail in Section 3, we initially conjectured that students would reinvent the geometric ratio (rise over run) through solving problems in which data were presented in a coordinate graph. We conjectured that simply presenting data in a coordinate graph would lead students to connect vertical and horizontal distances in a graph with numerical changes in variables. We found that students did not automatically make this connection. Instead, most students initially used the graph as a sign for a table, extracting ordered pairs from the graphs and using algebraic methods to find slope given the ordered pairs. We therefore modified our conjectured LIT to include activities that disciplined students’ perception to see change graphically in a coordinate graph.

Data collection

As a research team, we collected student work (from 19 students over 14 days), observer fieldnotes (from two observers: David, who was present for 5 days, and Michael, who was present for 12 days), and audio and video recordings of full-class and small-group work (over 14 days). In addition, I kept a daily reflective journal in which I recorded my thoughts and
observations before and after each class. We used two video cameras and three audio recorders during the study. The placement of these recording devices varied depending on the type of classroom activity. For all activities, I wore a lapel mike connected to an audio recorder. This is reflective of the central role that I played as the teacher, organizing the class such that mathematical productions emerged from students’ activities (Cobb et al., 2008). During small group work, we positioned video cameras such that the faces, hands, and desktops (and therefore inscriptions) of all students in the group were generally visible in the screen at the same time. In order to collect data from multiple groups given the constraints of our equipment, we rotated the cameras between groups each day. Classrooms are noisy places, and our video cameras were prone to picking up background noise at the expense of the group’s audio. We therefore placed wide-angle microphones in the center of each recorded group; these microphones were connected to audio recorders. During math congresses and full group discussions, we moved one audio recorder to the front of the room to capture students’ presentations. We placed the other audio recorder with a pair of students to capture their interactions during partner talk time. We rotated this recorder between pairs each day in order to collect data from multiple pairs.

We experimented with camera placement during the full-class activities. One camera was always pointed at the front board, where students presented their work during math congresses, and where I wrote summaries of key ideas during group discussions. At first we pointed the second camera towards a side-board that the class used occasionally. In this configuration, both cameras were pointed at the back of students’ heads. After about a week, we realized that we were missing important gestures that students were making during full-group discussions, so we moved the “side-board” camera to the front of the room and pointed it towards the students.
Retrospective analysis methods

As I described in the conceptual framework, I’m interested in how students reinvent and objectify mathematical artifacts. Because artifacts are inexorably bound to activity, my unit of analysis is activity, which I define as:

[T]he mediated actions and interconnected sequences of actions (i.e. operations) that individuals carry out in the attainment of a goal. This sense of activity, better captured by the German term *Tätigkeit* (as something related to the creative transformation and understanding of reality), [entails a…] fundamental epistemological claim according to which, in the course of the activity, individuals relate not only to the world of objects (the subject-object plane) but also to other individuals (the subject-subject plane or plane of social interaction) and acquire, in the joint pursuit of the goal and in the social use of signs and tools, human experience (Radford, Bardini, & Sabena, 2007, p. 512)

Notice in the above definition that activity is inherently social and interactive. Individuals-with-tools may engage in *actions* (Wertsch, 1998), but these actions can only be understood insofar as they come together in a constellation called “activity” (Engeström, 1987; Leont’ev, 1981). For example, a student using a calculator to divide 115 by 5 constitutes an action, which can only be understood in the larger context of a group’s joint work in solving a problem for which the mathematical operation of 115/5 is meaningful. Even this larger unit of group work is not activity as defined above. Activity is broader: “doing mathematics” is activity. Solving a school problem is (at best) a moment of this activity (Radford et al., 2007). These moments are my units of analysis, and my analysis task was to understand how artifacts emerged
and became meaningful within and across these moments. That is, my task was to capture cultural *processes*, by examining cultural *products* and how they are used.

To do so, I relied primarily on analyses of classroom video, which I coordinated with student work. I used observer fieldnotes and my own research journal primarily to triangulate my video analysis. For video analysis, I followed the video analysis method described by Powell, Francisco, and Maher (2003). This method is similar to the method described by Cobb and Whitenack (1996) in that both methods were developed to analyze a large corpus of video data to track mathematical learning over time, and both methods are cyclical, employing increasingly focused cycles of analysis. Whereas Cobb and Whitenack’s method is particularly useful for a particular interpretive framework (Cobb & Yackel, 1996), Powell et al.’s method is more flexible and amenable to various conceptual frameworks.

The method is composed of seven tasks: “1. Viewing attentively the video data 2. Describing the video data 3. Identifying critical events 4. Transcribing 5. Coding [and analysis – FP] 6. Constructing storyline 7. Composing narrative” (Powell et al., 2003, p. 413). Although this list is presented in a particular sequence, in practice these tasks are often interlocking and may appear in many different orders. In my analysis, I engaged in these seven tasks as follows.

First, I watched and described the video corpus (tasks 1 and 2). To do so, I watched the entire corpus in chronological order. As I watched the video I created *content logs*, which describe the activity in “meaningful chunks” (Maxwell, 2013) of approximately 3-5 minutes. My content logs have three columns: time, goings-on, and notes. In the goings-on column, I describe what’s happening on the video, including speakers and what is said. Much like ethnographic fieldnotes (Emerson, Fretz, & Shaw, 1995), the purpose is to capture the goings-on with a minimum of inference or judgment. When I wanted to make a tentative inference or judgment, or mark
something for later, I wrote a note in the notes section (much like an “observer comment” in ethnographic fieldnotes). Table 3 shows an example of a content log segment. As I created the content logs, I coordinated them with student work.

Table 3. Sample content log, from Jan 09 2012

<table>
<thead>
<tr>
<th>Time</th>
<th>Goings-on</th>
<th>Notes</th>
</tr>
</thead>
</table>
| 15:45| **Student articulates prediction:** Apple will produce more than 10 million iPhones over the course of 12 months  
F: We have all of the data we need to make this prediction for ourselves. With you partner, talk about how the author made this prediction.  
Class discussion: How did they make this prediction? Students: they multiplied 800,000 by 52 weeks. F: What does that 800,000 represent? S: iPhones made every week. F: Why is that so important? S: If you add 800,000 to the number of weeks in the year, you can make the prediction. F restates and writes on board. S: No, if you multiply. F: Why multiply? S: to make a prediction. F: How does weeks turn into iPhones? S: Find the prediction. S: Every week you have a certain amount of iPhones, so if you multiply that by the number of weeks, the number of iPhones will go up. That’s important for investors. F: As the weeks go up, the iPhones go up. One number (800,000) tells us how two variables change together. Weeks go up by one, iPhones go up by 800,000…| **Role of rates to make prediction.  
**Does F make a ratio table on the front board? Need to look at other camera to find out.  
**|

Second, I analyzed and coded the content logs and the coordinated student work (task 5). Given my conceptual framework I focused my coding on artifacts. I coded for the particular artifacts present in a given segment (e.g., function table), as well as the role the artifact was playing (e.g., reinvented, objectified, mediating problem solving). My coding scheme is therefore a combination of deductive (a-priori) and inductive (emergent) codes (Miles, Huberman, & Saldaña, 2014). Deductive codes are those that come directly from my conceptual framework and the conjectured LIT; these include codes for the role of artifacts, and for the artifacts in the conjectured LIT (including the five sub-constructs of slope, and the students’ repertoire of artifacts given in Table 1). Inductive codes are those that emerge from the data;
these include codes for the other artifacts that emerged during the design experiment. To
determine the reliability of the artifact codes, Raymond served as a second rater to check inter-
rater reliability. I first trained Raymond using 10% of the data corpus. Then, he coded a different
25% of the corpus. Overall, we agreed on 95% of the codes (Cohen’s Kappa=0.79). Using the
coded content logs, I identified “critical events” (task 3) where artifacts were reinvented or
objectified.

Third, I transcribed the segments of video that I had identified as critical events (task 4). Because objectification is often a sensuous process that is distributed across people and
inscriptions via talk and gesture (Radford, 2003), I transcribed these segments for talk and
gesture (Ochs, 1999) and coordinated the transcripts with inscriptions produced by students.

Fourth, I analyzed the transcripts to develop a more detailed picture of how artifacts were
reinvented and objectified (task 5). In this phase I drew on techniques that can broadly be
categorized as discourse analysis (Gee, 2011; Heritage & Clayman, 2010; Potter, 1996), which I
define as including both talk and gesture (Streeck, Goodwin, & LeBaron, 2011; Streeck, 2009).
In analyzing interaction, I was guided by the notion that discourse is not just communication
about action, it is itself action. That is, discourse communicates, but it also does (Gee, 2011;
Jaworski & Coupand, 2006). A key question that I asked during the analysis is, “what work is x
doing?” where x might refer to a single word or phrase, a turn at talk, a gesture, an artifact, a
math problem, or a feature of the situation (e.g., the imaginary context within which a particular
problem is set). Evidence for this work is found in subsequent interaction.

By now, I was able to start developing a storyline (task 6). As described by Powell et al.
(2003, p. 430): “Constructing a storyline requires the researcher to come up with insightful and
coherent organizations of the critical events, often involving complex flowcharting.” In my case,
I produced two data displays (Miles et al., 2014): a matrix of artifacts and days, and a flowchart of the ways that artifacts emerged in the process of reinvention and objectification.

As I created and worked with these displays, it became clear to me that a chain of signification was not sufficient to explain the process of learning that I saw in my data. Rather than the one-to-one process of signification implied by a chain of signification, my displays showed how students reinvented and objectified artifacts by creating coordinated assemblies of existing artifacts. Further, I found that existing artifacts acquired new meanings in this process, which belies the hierarchical process of signification suggested by the chain of signification. To describe this process I turned to science studies, and, in particular, to Bruno Latour’s (1986) notion of a cascade of inscriptions. Latour describes how scientists produce new inscriptions by coordinating and assembling existing inscriptions. As this was closer to the process that I saw in my data, I used this notion, along with my data flowchart, to produce the cascade of artifacts that I describe in Section 3.

For me, the process of constructing a storyline (task 6) and developing a written narrative (task 7) are inseparable, and both occurred during this portion of my analysis. As I constructed the storyline, I wrote about it in narrative form, and as I wrote the narrative I constructed the storyline.

As for my own positionality, it is important to keep in mind that I was the teacher of the course. I have fond memories of the class and of my students, and it would be folly to imagine that these memories do not enter into my analysis. At the same time, my participation as the teacher is a key strength of this work. Because I was in the class every day for the entire year, I have a unique perspective on the history of the class and the students. As such, my analysis should be understood as a narrative approach, in which the researcher “makes his or her
perspectives as transparent as possible. There is no pretense of objectivity because one main tool the researcher uses is his or her ability to learn and represent the culture” (Derry et al., 2010, p. 11).

Although I make no pretense of objectivity, I did take steps to reduce the effect of my biases on the interpretation of the data. First, as I described above, a second member of the research team checked the reliability of my codes (Agreement=95%, Cohen’s Kappa=0.79). Second, I shared my interpretations with Michael during the analysis process. I presented to him my analysis, and asked if he agreed with my interpretations, given my conceptual framework. We discussed all disagreements and generally came to a consensual resolution that is consistent with my conceptual framework. In the text, I present the analysis in its “resolved” state.

That said, I do not suggest that this process resulted in bias-free interpretations. The validity of my results is not located in the neutrality of the data or analysis, but rather in the reader’s assessment of the reasonableness and justifiability of my claims given my unique subjectivity, my data, and my conceptual framework (Carspecken, 1996; Gravemeijer & Cobb, 2006). To facilitate the reader in judging the reasonableness of my claims, I include detailed examples of data and analysis in the paper.

Providing detailed analyses serves an additional purpose. As discussed above, my goal for this analysis is to produce a well-warranted LIT for how students reinvent and objectify the five sub-constructs of slope. LITs are not meant to describe “what works” but rather to “offer teachers an empirically grounded theory on how a certain set of instructional activities can work” (Gravemeijer, 2004, p. 105 my italics). Understanding how the sequence works is key because the sequence is not meant to be adopted verbatim by teachers. Rather, the goal is to provide a framework that teachers can use to design their own learning sequences, consistent with the LIT
but customized by the teacher to match her local circumstances. By describing the LIT and the
decisions and analyses that led to it, I provide a basis for others to create deliberate adjustments
(Gravemeijer & Cobb, 2006; Gravemeijer, 2004).

3. A local instructional theory for slope

In this section, I discuss the local instructional theory that emerged as we conducted the
design experiment. Recall that LITs are composed of three parts: (1) a description of how
learning happens over time, (2) principles that guide the design of activities that support this
learning, and (3) the rationale for how the activities support learning (Gravemeijer, 1999, 2004).

I present the description of how learning happened over time in the form of a cascade of
artifacts (Figure 16), which shows how new artifacts were reinvented and objectified as
coordinated assemblies of existing artifacts. I represent the cascade as a directed graph. As such,
it shares some surface similarities with “cognitive models” (Gierl, Wang, & Zhou, 2008),
however the cascade is not a representation of an internal cognitive structure. Instead, it is a
representation of how artifacts (which, recall, exist in the cultural world, not in the head) inform
and are informed by other artifacts. Further, the cascade is not a representation of artifacts from a
disciplinary perspective. Rather, it is a representation of artifacts from a learning perspective.
Thus it is not built from a disciplinary (or expert) perspective but instead shows how students in
our study structured their mathematical world by coordinating and assembling mathematical
artifacts.

The cascade is organized such that the vertical dimension is somewhat meaningful but
the horizontal dimension is not. In general, the cascade flows downward with artifacts that are
higher in the cascade contributing to those that are lower. Thus the cascade can be “read” from
top to bottom, as shown by the downward facing arrows in Figure 16. However, artifacts don’t just “push down,” they also push up and push laterally, as shown by the many double-facing arrows in Figure 16. Thus objectification is a syncretic and symmetric process in which artifacts objectify and are objectified by each other, all at the same time.

The cascade was constituted in six ordered phases. This is shown on the bottom of Figure 16, where the artifacts that are invented in each phase are highlighted in red, and the artifacts that are assembled and coordinated in the phase are shown in yellow. In introducing the notion of order, I am introducing a tension. On the one hand, there is a purposeful order to the phases. On the other hand, because artifacts push up and sideways, later phases influence the results of earlier phases just as earlier phases influence later phases. While there is a general downward push in the cascade, it would be a mistake to conclude that any given artifact is a pre-requisite for another, because earlier artifacts gain meaning as they are used later. Rather than minimize the tension I embrace it and suggest that the best way to conceptualize a given phase is to think of it as informing, simultaneously, the present, the future, and the past.

The six phases of the LIT, along with the activities through which the phases are constituted, are summarized in Table 4. In the remainder of this section I elaborate each phase, focusing on how activities and artifacts support reinvention and objectification at that phase.
Figure 16. Learning progressed in a cascade of artifacts (top), which was constituted in six phases (bottom). The dotted lines represent conjectured relationships. In the bottom figures, the artifacts that were assembled and coordinated in each phase are shown in yellow, and those that were reinvented at each phase are shown in red.
Table 4. Overview of the LIT

<table>
<thead>
<tr>
<th>Phase</th>
<th>Artifacts</th>
<th>Characteristics of tasks</th>
</tr>
</thead>
</table>
| 1     | Reinvented & objectified:  
- Ratio table  
- “find one” strategy  
- Intensive units  
- Fraction-as-quotient  

Tasks that involve the activity of partitive division, including:  
- finding fair shares  
- finding unit values |
| 2     | Assembled and coordinated:  
- Intensive units  

Reinvented & objectified:  
- Algebraic equations  
- Function tables  
- Graphs in coord. plane  
- Rate of change  

Tasks that involve:  
- Finding and continuing patterns in geometric figures and tables of values, where there is a “starting value” and the independent variable increases by 1  
- Converting between multiple representations of functions (focusing on table rows and points in the plane as solutions to two-variable equations) |
| 3     | Assembled and coordinated:  
- Algebraic equations  
- Rate of change  

Reinvented & objectified:  
- Parametric coefficient  

Objectified  
- Rate of change  
- Function tables  

Making predictions in linear situations, given:  
- The rate of change and starting value  
- Multiple data points (e.g., in a table), where the independent variable increases by one. |
<table>
<thead>
<tr>
<th>Phase</th>
<th>Artifacts</th>
<th>Characteristics of tasks</th>
</tr>
</thead>
</table>
| 4     | Assembled and coordinated:  
• “Find one” strategy  
• Ratio table  
• Fraction as quotient  
• Function tables  
• Rate of change  
Reinvented & objectified:  
• Unit rate strategy  
• Algebraic ratio  
Objectified:  
• Rate of change | Make predictions in linear situations given:  
• A single data point, for situations where the values of the variables are proportional  
• Two data points, for situations where there is a starting value.  
Problem contexts should be chosen to make clear the distinction between changes and values. |
| 5     | Assembled and coordinated:  
• Number line  
• Graphs in coord. plane  
• Rate of change  
• Algebraic ratio  
Reinvented & objectified:  
• Geometric ratio  
Objectified:  
• Graphs in coord. plane |  
• Show change on number-line diagrams.  
• Make predictions in linear situations where there is a starting value, given a graph of a function in a coordinate plane. |
| 6     | Assembled and coordinated:  
• Rate of change  
• Graphs in coord. plane  
Reinvented & objectified  
• Physical property |  
• Compare rates given two intersecting linear functions graphed in a coordinate plane.  
• Measure and compare the steepness of objects |
Phases 1 and 2

Phases 1 and 2 occurred before our design experiment started, and they led to the repertoire of artifacts that I described in Table 1. I discuss these phases here because the artifacts that were reinvented and objectified in these sessions formed the instructional starting point for the artifacts in the design experiment. I don’t have access to student-level data to draw on for analysis of these phases. However, because I was the teacher of the course I do have access to my course records, and I draw on these records to explicate these phases. Evidence that the artifacts had the meanings that I attribute to them in the sequel comes from: (1) my class records from the first semester, and (2) the ways that students took up these artifacts during the study.

Phase 1

Phase 1 took place in the beginning of the year during a short unit on fractions-as-quotients (see Peck & Matassa, in press, Chapter 2, this dissertation, for a detailed summary of a similar unit). In this phase, students solved problems that involved finding unit values given a many-to-many relationship (the sort of problems that a mathematician might classify as involving partitive division). For example, students solved fair-sharing problems using equipartitioning (Empson, 1999; Streefland, 1993; Wilson, Edgington, Nguyen, Pescosolido, & Confrey, 2011), as well as other problems such as finding the weight of a single tomato given the weight of multiple tomatoes (see Error! Reference source not found. Figure 17). In situations like these, there are two ways to conceptualize the “final answer.” A many-as-one conceptualization attends to only one of the dimensions in the many-to-many relationship, while a many-to-one conceptualization attends to both dimensions. (Confrey et al., 2009). In the example shown in Error! Reference source not found. Figure 17, a student who used a many-as-one conceptualization would express the answer as “3/7 pounds,” whereas a student who used
a many-to-one conceptualization would express the answer as “3/7 pounds per tomato.” As shown in Figure 17, we encouraged the latter conceptualization. We did so because the many-to-one conceptualization more readily lends itself to the notion of covariance (Confrey et al., 2009; Lobato et al., 2003). Even though covariance was not a focus of this unit, we knew that it would be a focus of our design experiment. In this way the unit was future-oriented with respect to the curriculum, preparing students for future phases, months in the future, in which covariation would come to the fore.

Through these problems, the students invented and/or objectified the following mathematical artifacts, which later served as the foundation for objectifying rates:

1. The fraction-as-quotient sub-construct of rational number (Kieren, 1980), as well as the notion that the “fraction bar” can serve as a division operator. Thus students’ perception was disciplined such that they could see the symbol ¾ as simultaneously the operation of three divided by four, and the numerical result of the operation, three-fourths.
2. The “find-one” strategy. This strategy was invented and named by the class during the course of the unit. It links the division operation to situations in which the goal is to find the value of one object.

3. The ratio table (Middleton & van den Heuvel-Panhuizen, 1995). Notice that the tomato problem in Figure 17 is presented such that within-unit pairs are aligned vertically, and between-unit pairs are aligned horizontally. Students used this structure as a ratio table, and eventually reproduced it as they solved new problems.

4. Intensive units using the word “per” (e.g., pounds per tomato). The word “per” became synonymous with the notion of “one-ness” and, as discussed above, we used intensive units to maintain both of the original dimensions in many-to-one situations (Confrey et al., 2009). Using intensive units also follows from Lobato et al.’s (2003) conjecture about creating rates-as-measures, and using intensive units to describe those measures.

Phase 2

Phase 2 took place during a unit on functions, including vocabulary and concepts (function, independent and dependent variables, inputs and outputs) and common representations (including tables, graphs, equations, words, and “arrow chains”, which were models of function machines). In this phase, students engaged in the following activities: (1) finding and continuing patterns in geometric figures and function tables, and (2) representing patterns using multiple representations of functions. Although the main goal of the unit was to introduce the concept of a function and its associated vocabulary and representations, a secondary goal was to discipline students’ perception such that they would see how a linear function’s output could be composed
of a constant part and a changing part. As shown in Figure 18, we incorporated visual focusing phenomena to make these separate components salient (inspired by the use of tables in Brenner et al., 1997). All of the patterns were presented such that the independent variable increased by one for each subsequent iteration. We were mindful of Lobato et al.’s (2003) caution that such well-ordered representations can lead to students’ perceptions being disciplined to changes in the dependent variable only (i.e., without considering the simultaneous change in the independent variable). We were therefore very careful to use focusing phenomena that called attention to both variables, such as quantifying covariation using intensive units (e.g., cost per square meter).

Through these problems, the students reinvented and/or objectified the following mathematical artifacts:

1. Function tables. This was the primary representation used in the unit. Students used tables to collect numbers extracted from geometric patterns, and tables served as bridges between other representations of functions. For example, students used tables to convert from algebraic equations to graphs by using the table to collect solution pairs for the equation, and then plotting the pairs on a graph.
2. Rate of change. This was the first of the five sub-constructs of slope that students reinvented and objectified. Above I discussed how we used representations to discipline students’ perception to the change in the dependent variable for every unit change in the independent variable. As students called attention to this covariation, I defined “rate of change” as “the amount that the output changes by when the input goes up by one” (class records, 11/16/2011). Students used intensive units to quantify rates, and we discussed how rates play a role as an “exchanger” (class records 11/16/2011), converting the independent variable to the dependent variable through multiplication. As shown in Figure 16, and discussed earlier, rate of change is the central artifact in the cascade, and it informed all of the other sub-constructs of slope.

3. Algebraic equations. Equations were used as “rules” (class records 11/04/2011) to find the output for any input in a particular pattern. We discussed how there were infinite solutions to these equations, each representing a different possible input, output pair.

4. Graphs in the coordinate plane. Graphs were objectified as the collection of points that made a particular equation true. We introduced graphs as a visual representation of the set of the infinite solutions to two-variable equations. As discussed above, students created graphs from equations by first finding solution pairs to the equation, and then plotting them. Similarly, equations were used as “point-checkers” (class records 12/06/2011) to determine if a given point would be on a graphed line.

In summary, eight key artifacts, including one sub-construct of slope, were invented and/or objectified in Phases 1 and 2. In both phases, the notion of many-to-one was pervasive. In
Phase 1, the entire unit was oriented around “finding one.” In Phase 2, students explored patterns in which changes in dependent variables were associated with unit changes in independent variables, leading ultimately to a “many-per-one” definition of rate of change.

Phase 3

Phase 3 marked the beginning of the design experiment. In this phase we introduced the theme of the unit, “using math to make predictions.” Students solved two different categories of problems, both of which involved making predictions in linear situations: (1) making predictions where the rate of change and starting value are given; and (2) making predictions where multiple data points are given (e.g., in a table), and the independent variable increases by one (this second category is similar to the problems that students encountered in phase 2, but this time there was an explicit focus on making predictions rather than finding rules).

As students solved these problems, they reinvented the parametric coefficient by coordinating rate of change with algebraic equations. Recall that the parametric coefficient is the \(a\) in the linear equation, \(y = ax + b\). We wanted to discipline students’ perception such that they saw the \(ax\) term as one in which the rate is being multiplied by the independent variable to make a prediction. We conjectured two different ways that students might objectify rate in order to see the \(ax\) term in this way, both related to the ways that rate of change was objectified in phase 2. One way is to objectify rate as an “exchanger” that works though multiplication to exchange the independent variable for the dependent variable. A second way is to objectify rate as a many-per-one relationship, which can then be accumulated through multiplication. Overwhelmingly, as I discuss below, students objectified rate as the latter.

In the initial activity, students read news articles and blog posts in which authors presented rate and made predictions based on those rates. We then discussed how the authors
made their predictions, and disciplined students’ perception to see that rates were being multiplied by an independent variable to make predictions. For example, in an article about Apple iPhones (Lane, 2008) the author describes that Apple is manufacturing 800,000 iPhones per week, and later suggests that “[a]t the current rate, Apple stands to produce more than 40 million iPhone 3Gs over the course of twelve months.” After students read the article we had a discussion about the prediction. In the discussion, students identified “800,000 iPhones per week” as a rate of change and students explained that the author could make the prediction by multiplying 800,000 by 52.

The students were explaining a straightforward multiplication situation, the sort that they had probably seen since early elementary school. We wanted to use this simple task as a way to help students see something new in the multiplication, namely the way that rates can be used to make predictions through multiplication. Thus I asked students why we would multiply. This interrogative is a request for detailed information, and works to reframe the activity away from the calculation. Randy explained:

Segment [1]: Why multiplication?
1 FAP: Randy why is that [multiplication] going to get us a prediction for the number of iPhones in a year? How does weeks turn into iPhones?
2 Randy: Because for every week you have, you produce a certain amount of iPhones, so if you multiply it by a certain amount of weeks, the amount of iPhones will go up. [The reason-
3 FAP:   [For every –
4 Randy:            -that might be important is for {investors to know}]

I begin turn 1 with the why interrogative. I ended my turn with a question about how “weeks turn into iPhones.” In doing so, I offered the “exchanger” objectification of rate discussed above. However Randy provides an explanation that puts the many-per-one meaning

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3 All student names are pseudonyms.
into rate of change. This can be seen in turn 2. In this turn, the phrase “for every” plays the role of “per”, associating the singular “week” (one) with the plural “iPhones” (many). The determiner “every” makes clear that the week in question is not a one-off event, but rather can be repeated (consider how the meaning would be different if Randy had used a different determiner, say, “the”, instead of “every”).

Having objectified rate as a many-per-one relationship, Randy further objectifies rate when he says: “if you multiply it by a certain amount of weeks, the amount of iPhones will go up.” Here, Randy is doing more than explaining how he sees the multiplication problem. He is objectifying rate—publicly—as a number that can be accumulated through multiplication to make predictions. In subsequent problems, students continued to objectify rates in this way, and coordinated this objectification with algebraic equations to reinvent and objectify the parametric coefficient. For example, when analyzing the table in the “X-box problem” (Figure 19A), students identified the rate using intensive units as “2 dollars per game.” Stacy explained how she saw the rate in the table (Segment [2]):

Segment [2]: Seeing rate in a table
1 FAP: Stacy, where do you see that rate of change in the table?
2 Stacy: Um, for every number of games, the total cost goes up by two
3 FAP: Every what about the number of games?
4 Stacy: The, each time the number of games increases by one, the total cost increases by two.
5 FAP: (draws arrows on table, one on each side, pointing from one row to the subsequent row. On the left-side arrow, writes “+1” and on the right-side arrow writes “+2”; See Figure 19)
In this exchange, Stacy and I work together to objectify both function tables and rate of change. The sequence in turns 2-4 works to objectify rate as many-per-one. In turn 2 Stacy uses similar “for every” language as Randy did in [1]. However, Stacy uses a mass noun: for every number of games,” thus making it ambiguous whether she is making a many-per-many association or a many-per-one. In turns 3 and 4, she and I clarify that she is making a many-per-one association. In turn 5, I bring this objectification into coordination with the function table, drawing arrows to discipline perception such that rate becomes salient in the movement between table rows. Thus, function tables helped to objectify rate, and rate “pushed up” in the cascade to further objectify function tables. While Stacy and I accomplished this objectification in moment-by-moment social interaction, we drew on historical artifacts, like function tables, that have histories on various time scales. Consider, once again, the logarithmic table. Napier built his logarithmic tables row-by-row, through a process of repeated subtraction (Maor, 1994). In doing so, he availed himself of a particularly important feature of a function table, namely, that function tables betray patterns in covariation, as one moves down successive rows. In the literature review, I explained how math educators have availed themselves of this same feature
of function tables to help constitute covariation for students. In our moment-by-moment interaction, Stacy and I availed ourselves of this feature to constitute the covariation of games and dollars for the class. In the moment, Stacy encoded this covariation in spoken words, and I encoded it with inscribed arrows. In doing so, Stacy and I were engaged not only in a momentary social process, but also a historical cultural process, in which we incorporated a product of human history—an artifact—into our objectification of rate.

The objectification of rate continued as a different student (Melissa) wrote \( y = 2x + 4 \) as an equation for the table. She explained:

\[
\begin{align*}
\text{Segment [3]: Seeing rate in an equation} \\
1 \quad &\text{Melissa: Okay, um Y is like the final, cost, and,} \\
2 \quad &\text{two is the one time fee times how many games you have- or not the one time fee} \\
2 \quad &\text{like, how much dollars it is per game, and four is the one time fee.}
\end{align*}
\]

In this turn, Melissa explains how she sees quantities in the symbols that form the equation. Of interest to us are the two arrowed passages, as it is in these passages that Melissa objectifies the parametric coefficient. In the first arrowed passage Melissa makes a mistake, referring to the 2 as the “one-time fee” She soon initiates a self-repair, but not immediately. The repair comes after she explains that the one-time fee is multiplied by “how many games you have.” The timing of this repair suggests that it is brought on by Melissa’s recognition that the role she has ascribed to the “2” doesn’t fit with its behavior in the equation: a “one-time fee” would not be accumulated through multiplication. The presence and timing of the repair therefore suggests that Melissa has objectified the \( 2x \) term, and that her explanation does not fit her objectification.
Having initiated the repair, in the second arrowed passage Melissa explains (correctly) that “how many dollars it is per game” is being multiplied by the number of games. In doing so, she draws on the objectification of rate as many-per-one and coordinates this objectification with its role in the algebraic expression. In this way, Melissa assembles and coordinates rate of change with algebraic equations to objectify the parametric coefficient.

The above shows how students assembled rates, function tables, and algebraic equations to reinvent and objectify the parametric coefficient. In addition, the above discussion shows how students continued to objectify rates of change. Students put a “many-per-one” meaning into rate of change, and objectified rates as numbers that can be accumulated through multiplication. As rates became meaningful in this way, they were objectified as a tool to make predictions.

Phase 4

In Phase 4 students reinvented and objectified the unit-rate strategy and the algebraic ratio \( \frac{y_2-y_1}{x_2-x_1} \), the third sub-construct of slope to be reinvented and objectified. I will describe each below.

*Reinventing and objectifying the unit rate strategy*

The unit rate strategy is a strategy for solving proportional reasoning problems, which involves taking a ratio that represents a many-to-many relationship, scaling it down to a unit rate (a many-per-one relationship) using division, and then scaling the unit rate up to make a new many-per-many relationship using multiplication (Cramer & Post, 1993). Notice that this is very similar to the process of finding slope using the algebraic ratio (using division to scale a many-to-many relationship down to many-per-one) and then using slope in the equation \( y = ax + b \) where the \( ax \) scales-up the many-per-one relationship represented by \( a \) via multiplication to a new many-per-many relationship. The distinction, of course, is the existence of the constant, \( b \),
in the generalized linear function \( y = ax + b \). I discuss this later. For now, let’s return to the unit rate strategy.

Students reinvented and objectified this strategy while making predictions in proportional relationships (i.e., missing value proportional reasoning problems). The key in designing these problems is that the within-unit pairs should be relatively prime: this is what necessitates the unit rate. For example, as shown in Figure 20, the within-unit pair (6 miles, 11 miles) is relatively prime.

To reinvent and objectify the strategy, students assembled and coordinated ratio tables, the “find one” strategy, and rates of change (objectified as many-per-one). For example, notice how the student work in Figure 20 demonstrates the assembly and coordination discussed above. On the left side, the student coordinates a ratio table with the “find one” strategy. The result is a rate in the form of a many-per-one relationship. In the center of Figure 20, the student brings this assembly into coordination with the objectification of rate as a number that can be accumulated via multiplication to make predictions. This is shown by the multiplication in the center, and the interpretation of the result on the right.

Ms. Magro runs 6 miles every day. On average, she can run six miles in 54 minutes. At this rate, how long would it take Ms. Magro to run an 11-mile race?

\[
\begin{align*}
\frac{6 \text{ mile}}{1 \text{ mile}} &= \frac{54 \text{ min}}{9 \text{ min}} \\
\text{Takes 99 minutes}
\end{align*}
\]

Figure 20. Students assembled and coordinated ratio tables, the “find one” strategy and rates of change to reinvent the unit rate strategy.
Reinventing and objectifying the algebraic ratio

The algebraic ratio is a general formula for finding the rate of change, often expressed as $\frac{y_2 - y_1}{x_2 - x_1}$. In this general formula, one finds differences in the dependent and independent variables, and then divides to create a rate. Up to this point, students had invented and objectified two less-general strategies for finding a rate of change. One strategy involved looking at differences in well-ordered tables when the independent variable increased by 1 (e.g., Figure 19). Division was not necessary here because the independent variable increased by one. A second strategy (part of the unit-rate strategy) involved using division to find a rate for situations where the variables were proportional. Subtraction was not necessary here, as there is an implied (0, 0) value (see Figure 20).

For students to reinvent the more general strategy that is captured by the algebraic ratio, they needed to solve problems “just beyond [their] current abilities” where their current strategies would not work, but where a new artifact (the algebraic ratio) would. These problems have five key features: (1) the students are asked to make a prediction, in (2) a situation where two variables have a linear relationship but (3) are not proportional. Students (4) are given only two points, such that (5) the difference in the independent variable is greater than one. The students’ first encounter with such a problem was the “window problem” shown in Figure 21. While the window problem meets the criteria above, the problem context led students to objectify rates of change in ways that we didn’t intend. I will return to this point below. First, however, I will show how, while solving this problem, students assembled and coordinated fraction as quotient, the “find one” strategy, the unit rate strategy, rate of change, and function tables, to reinvent and objectify the algebraic ratio.
Figure 21 shows two examples of how two groups of students reinvented the algebraic ratio by assembling and coordinating artifacts while solving “the window problem,” shown at the top of the figure. Group I assembled and coordinated function tables and the “find one” strategy. Group II, in the upper portion of the inscription that is crossed-out, coordinated fraction as quotient, the unit rate strategy, and a ratio table into a single material assembly. By bringing two of these assemblies into coordination, the group realized a contradiction. This motivated them to invent a new strategy, shown in the bottom half of the inscription. For both groups, the strategy involved finding differences in cost (115 dollars) and windows (5 windows) between the given data points, and dividing these differences to find the rate of change (23 dollars per window). As discussed above, this is the strategy captured by the algebraic ratio.

Problem:
Leslie is a window installer. On Friday, she installed two windows, and charged 402 dollars. Last week, on another job, she charged 517 dollars to install seven windows. A new customer has asked Leslie to install five windows. How much will this cost?

Figure 21. Students assembled and coordinated fraction as quotient, the find one strategy, the unit rate strategy, rate of change, and function tables to reinvent the algebraic ratio.
A detailed analysis of these groups’ work and discourse is beyond the limits of this paper (see Peck, 2015a, 2015c, Chapter 4 this dissertation, for detailed analyses of Group II). However, it is worth noting the importance of the material instantiation of the artifacts, afforded by the inscriptions each group created. For Group I, the inscription of the function table structured the spatial arrangement of the data, which afforded a previously-disciplined way of perceiving differences between data points (notice the similarities in the arrows drawn by Group I in Figure 21 and those drawn by me in Figure 19, and incidentally, notice that Group II also used similar arrows on their table on the bottom left of their inscription). For Group II, the material inscription enabled the group to bring two assemblies of artifacts into coordination, revealing a contradiction and necessitating the reinvention of the algebraic ratio.

The above discusses how students reinvented the algebraic ratio by assembling and coordinating fraction as quotient, the find one strategy, the unit rate strategy, rate of change, and function tables through mathematical activity as they worked on the window problem shown in Figure 21. At the same time, the context of the window problem led students to objectify rates of change in a way that, while perfectly consistent with the window context, deviated from disciplinary ways of understanding rates of change. Mathematically, the differences in the dependent and independent variables are understood as changes in those variables, and the rate of change is understood as a measure of the change in the dependent variable with respect to a unit change in the independent variable. From this perspective, the differences in cost and windows would be interpreted as changes in cost associated with changes in windows (i.e., when the windows change by 5, the cost changes by 115 dollars). Similarly, the rate of change would be interpreted as the change in cost associated with a unit change in windows.
However, rather than objectifying the differences and rates in terms of *change*, students overwhelmingly objectified them as different kinds of *values*. For example, after Group II (Andrew, Blake, Kelly and Theo) solved the problem, Michael asked them to explain their solution. First, Andrew stated “the rate of change is 23 dollars per window.” Blake explained that to find this, they “minused” to find 115 dollars and 5 windows, and divided. Michael then asked the group to “make sense of the 5 and 115.” This led to the talk depicted in Segment [4]:

**Segment [4]: A different kind of value?**

<table>
<thead>
<tr>
<th>Turn</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>B: What relation does 5 and 115 have? Because it’s not 5 windows for 115 dollars.</td>
</tr>
<tr>
<td>2</td>
<td>A: No it’s not that.</td>
</tr>
<tr>
<td>3</td>
<td>M: It’s not 5 windows for 115 dollars?</td>
</tr>
<tr>
<td>4</td>
<td>B: No</td>
</tr>
<tr>
<td>5</td>
<td>A: It may be, without-</td>
</tr>
<tr>
<td>6</td>
<td>K: Wait do 23-</td>
</tr>
<tr>
<td>7</td>
<td>A: Wait try- do 23 times 5.</td>
</tr>
<tr>
<td>8</td>
<td>B: 5 times 23 is 115 dollars</td>
</tr>
<tr>
<td>9</td>
<td>A: So that’s it, without delivery cost.</td>
</tr>
<tr>
<td>10</td>
<td>B: So that’s it without delivery cost. So 115 is the cost of five windows, minus the delivery cost. So you divide by that to get the (rate)</td>
</tr>
<tr>
<td>11</td>
<td>A: Ohhh.</td>
</tr>
<tr>
<td>12</td>
<td>B: Bang! That was way too much work</td>
</tr>
<tr>
<td>13</td>
<td>A: So, in theory, five windows does cost 115, without-</td>
</tr>
<tr>
<td>14</td>
<td>B: Without the delivery cost.</td>
</tr>
</tbody>
</table>

In turns 1 and 2, Blake and Andrew begin by rejecting the notion that the 5 and the 115 represent values of windows and dollars, respectively. After Michael’s prompt in turn 3, Blake again rejects the “values” interpretation, but Andrew, in turn 5, is starting to reconsider. At this point he is speculating, as indicated by his evocation of the subjunctive mood (“it may be”), but he offers a key modifier, “without.” As we soon see in turn 9, Andrew is speculating that the 5 and 115 are particular kinds of values, that is, values “without the delivery cost.” By turn 9, Andrew’s speculation has turned to fact (Latour & Woolgar, 1987), as indicated by Andrew and
Blake’s evocation of the indicative mood with no modality (“that’s it”; contrast this with
Andrew’s speculative framing of this statement in turn 5, or with a modal framing like, “I think
that’s it”).

The conversion of Andrew’s speculation into fact is accomplished in turns 6-8. In these
turns, Kelly, Andrew, and Blake recreate the 115, not by subtracting as they explained earlier,
but instead by multiplying the rate of change (23) by 5. From a mathematical perspective there is
nothing interesting in this reconstruction. Earlier the group explained that they found 23 as
115/5, and now they are finding 115 as 23*5. The latter is simply a consequence of the former.
However, for this group, the reconstruction has pedagogical and rhetorical power. Pedagogically,
it enables the students to objectify the 5 and the 115 as particular kinds of values, namely, values
without delivery cost. Rhetorically, the reconstruction does the work of converting a speculation
into a fact.

Thus the group converted 5 and 115 into particular kinds of values—values without a
delivery cost—rather than changes. They similarly objectified the rate of change as: “one
window equals 23 without the delivery cost.” This objectification, of differences as different
kinds of values rather than changes, turned out to be the dominant interpretation in the class. This
is a perfectly reasonable interpretation given the context of the problem. In fact, in retrospect it is
a more sensible interpretation, because rarely does an order for windows change. Ordering
windows is not really a dynamic situation. Thus interpreting the differences as changes is
somewhat awkward in this context. Still, we wanted students to understand the numerator and
denominator in the algebraic ratio as changes, and the resulting rate as a measure of covariation.
Thus, we needed to discipline their perception to see the quantities in terms of changes.
In the next class, something fortuitous happened. A student was explaining how he solved an isomorphic window problem that we had given for homework. As shown in Figure 22, the student found negative differences in windows and dollars.

What could these negative numbers mean? At first, students tentatively interpreted the negative values in terms of “missing three windows” or “in debt three windows” but that didn’t seem to make sense in this context. Next, a student suggested that the –3 meant that you returned three windows. Ultimately, we arrived at a scenario in which a person changes their window order by decreasing the number of windows by three, and hence the cost of the order decreases by 63 dollars. Both of these capture a change in windows, and I capitalized on them in a subsequent discussion to help discipline students’ perception to see the differences as changes. And while the static nature of a window order doesn’t easily afford thinking about differences in terms of changes, we realized that, even in a context that doesn’t easily afford dynamic interpretations, negative changes can bring about these interpretations. In my journal that evening, I wrote:
Ultimately, I think we stumbled onto something here -- we should have more problems with negative changes. I think that this will encourage students to think about the differences as changes and not amounts. (Journal reflection, 1/13/2012)

From this point forward, we endeavored to use more dynamic contexts—contexts where changes actually make sense—and to incorporate negative changes into contexts where negative values would not make sense. For example, we used a context of snow melting, such that the change in snow depth is negative, but the values of the snow depth must always be non-negative. In addition, I focused on the notion of “change” in nearly every group discussion. Taken together, these modifications helped students objectify differences and rates in terms of changes.

**Phase 5**

In Phase 5 we introduced the word “slope” and students reinvented and objectified the geometric ratio (“rise over run”; the fourth sub-construct of slope to be reinvented and objectified).
In one sense, the geometric ratio is simply the algebraic ratio in a graph, with \( y_2 - y_1 \) and \( x_2 - x_1 \) replaced their geometric equivalents ("rise" and "run" respectively). Thus, we conjectured that students would reinvent the geometric ratio if they solved problems in which they made predictions given a graph. In other words, we conjectured that students would coordinate the algebraic ratio with graphs in the coordinate plane to create the geometric ratio. What we found when students made predictions given a graph was that students did indeed coordinate the algebraic ratio with graphs, just not how we expected.

Rather than reinventing the geometric ratio, every group used the graph to extract points from which they either made a table or used the algebraic ratio. For students, the graph signified a collection of points. In retrospect, this is not surprising as "graph as a collection of points" is the way that graphs were objectified in phase 2. Furthermore, as described above, the geometric ratio is just the algebraic ratio with numerical differences replaced by their geometric equivalents. Because students already had a means to (a) translate graphed points into numbers, and (b) find a rate of change from those numbers, students already had a strategy for finding a rate given a graph. Contrary to our initial hypothesis, simply presenting data in a graph was not sufficient to cause students to reinvent a new strategy for finding a rate of change.

Ultimately, we view the geometric ratio as nothing more than a shortcut for the algebraic ratio when given a graph. Because the students’ existing strategies will always work in any situation amenable to the geometric ratio, we felt that it would not be possible to develop a problem that *necessitated* reinventing the geometric ratio. Rather, we realized that, just like students’ perception needed to be disciplined so that they could see change in a table, so too did their perception need to be disciplined so that they could see change in a graph. To do this, we introduced a new artifact into the environment, namely a number line. We designed a set of tasks
that involved showing change on horizontal and vertical number lines with arrows. This activity and artifact helped to discipline students’ perception to see change in a graph. As shown in Figure 23A, students brought number lines into coordination with coordinate graphs by drawing arrows on the $x$- and $y$-axes (shown by the red and blue arrows in Figure 23A). As inscriptions, these arrows could be mobilized to show the traditional “slope triangle,” and brought into coordination with other inscriptions to show the connection between slope and rate of change (see Figure 23B).

![Figure 23. (A) Students brought number lines into coordination with coordinate graphs to show change by drawing arrows on the $x$- and $y$-axes. (B) These arrows could be mobilized to form the traditional slope triangle, and brought into coordination with other inscriptions to make the connection between slope and rate of change.](image_url)

In this way, students objectified coordinate graphs by coordinating them with number lines to make horizontal and vertical changes meaningful. In addition, students reinvented the geometric ratio by coordinating number lines, coordinate graphs, and the algebraic ratio. In subsequent activity students further objectified the geometric ratio including work to distinguish change (represented by arrows and the traditional slope triangle), from values (represented by points).
Phase 6

Due to time constraints at the school, we did not have time to implement Phase 6 of the conjectured LIT. Thus, I only reiterate our conjectures, and offer a small amount of supporting evidence from the study and from prior research.

In phase 6, we planned that students would reinvent and objectify the physical property (steepness). We conjecture that students can reinvent the physical property by engaging in activities in which they compare rates for multiple functions graphed in the same coordinate plane. Through these activities, students will reinvent the physical property by coordinating rate of change with graphs in a coordinate plane, and students’ perception can be disciplined such that they “see” steepness corresponding to rate of change.

There is some evidence from the research literature and from our study that this reinvention will happen, but that it requires disciplining perception. In the research literature, Tierney and Monk (2007) describe how a class began to reinvent the physical property as the students engaged with a graph of two linear functions that represented the height of two different plants over time. Some students associated the height of the graphed functions with the rate of plant growth, while other students associated the steepness of the graphed functions with the rate. In a class discussion, students debated these different interpretations and disciplined each other’s perception so that the association between rate and steepness became salient.

In our design experiment, we saw similar evidence of students associating both the height and the steepness of a graphed line with the rate. We gave students the problem shown in Figure 24 (inspired by a problem in McDermott, Rosenquist, & Van Zee, 1987) on an individual assessment at the end of the unit. The student response shown in the figure is representative. Many students appealed to steepness to explain that Linus was running faster than Charlie at
$t = 2$, however many also suggested that Linus and Charlie were running at the same speed at $t = 4$ (associating rates with height). If we had more time, we would have followed the example in Tierney and Monk, and conducted a discussion that would be purposefully orchestrated to reveal the contradicting interpretations, and through which students could discipline each other’s perception to see rates as steepness in a graph. Because we were not able to do so, we leave it as an open question as to whether and how students would have disciplined each other’s perception to see rate as steepness, as they did in the discussion reported by Tierney and Monk. One key question concerns the role of the teacher in such a discussion. Teachers play a vital role in facilitating classroom argumentation (Yackel, 2002), but Tierney and Monk attend little to the role of the teacher.

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8. Charlie and Linus are running along a straight track. A position vs. time graph for both runners is shown below.

- **a.** At the instant, $t = 2$ sec, who is running faster, Charlie or Linus?  
  Explain your reasoning  
  *Linus.*  
  *Linus’s line is steeper so he is running faster.*

- **b.** Do Linus and Charlie ever have the same speed? If so, at what time?  
  Explain your reasoning.  
  *Yes at 4 seconds they are going at the same speed.*
Having reinvented the physical property by coordinating rate of change with coordinate graphs, students can further objectify the physical property by engaging with tasks that involve measuring and comparing the steepness of various objects (similar to those described in Lobato & Thanheiser, 2002).

**Conclusion**

In this paper I presented a local instructional theory that describes how students reinvented and objectified the sub-constructs of slope as they engaged in activities organized around making predictions. I conceptualized these sub-constructs as mathematical artifacts, and I introduced the notion of a *cascade of artifacts* to explain how students reinvented and objectified new artifacts by assembling and coordinating other artifacts in mathematical activity. Further, I explained that a considerable amount of objectification involved disciplining perception, that is, learning to “see” particular features in material instantiations of artifacts.

This is the first comprehensive study of how students learn all five sub-constructs of slope, which is a fundamental concept in secondary mathematics. As such, the work makes important contributions to theory and practice by providing a LIT that shows how students can be guided to reinvent, objectify, and connect all five sub-constructs of slope by engaging in mathematical activity and social interaction organized around finding rates of change and making predictions. Teachers can use the LIT as a framework to design instructional sequences...
that are tailored to their own unique circumstances, and which leverage their students’ unique repertoires.

Additionally, this work builds on and contributes to cultural theories of learning by introducing the notion of a *cascade of artifacts*. This construct has the potential to describe both the *process* and *product* of learning in a variety of settings. In depicting the process of learning, the cascade shows how students reinvent and objectify artifacts by assembling and coordinating existing artifacts in activity and social interaction. In addition, the cascade shows how learning is a syncretic process, as artifacts “push up” as well as down. In depicting the product of learning, the cascade shows a web-like mathematical world in which artifacts are understood in relation to other artifacts. Taken together, the LIT and the cascade of artifacts accomplish something profound. They show how a mathematical world can be relational—rather than linear and hierarchical—even as learning takes place in linear time.

In this paper, I have demonstrated how learning mathematics is an active, social, and cultural process, as shown in Figure 25. New artifacts are reinvented and take on meaning based on mathematical activity, social interaction, and the students’ cultural worlds.

![Figure 25. Learning mathematics is an active (left), social (right), and cultural (top) process.](image)
If learning mathematics is an active, social, and cultural process, then learning is an accomplishment of a system rather than an individual mind (in the paragraph that follows, I give three examples from the paper to justify this claim). For researchers, attending to the way that learning is done by a system helps to avoid attribution errors, as Hutchins (1995a, p. 173) reminds us:

> If we ascribe to individual minds in isolation the properties of systems that are actually composed of individuals manipulating systems of cultural artifacts, then we have attributed to individual minds a process they do not necessarily have.

For teachers, understanding that learning is the property of a system can be empowering and liberating. If learning is a property of an “individual mind,” then, if learning is not happening, teachers have only recourse: to “remedy” the student. Such a perspective leads to traditional notions of “remediation”. However, when learning is understood as a property of a system, then teachers have multiple means to manipulate learning by manipulating different parts of the system. That is teachers can re-mediate the system (Cole & Griffin, 1983; Gutiérrez, Hunter, & Arzubiaga, 2009). In particular, if learning is an active, social, and cultural process, then teachers can manipulate learning by manipulating the activity, the cultural environment, and the social environment. In this paper, I provided examples of all three. In my analysis of phase 4, I described how we modified the types of activities (from predicting in “static” situations like window prices to dynamic situations like snow melt) to change the ways that students were objectifying rates of change. In my description of phase 5, I described how we changed the conditions under which the geometric ratio was reinvented by introducing a new cultural
artifact—number lines—into the system. Finally, in my discussion of phase 3 and elsewhere, I described how particular social interactions—namely, disciplining perception—played an important role in objectifying rates of change and the parametric coefficient.

A major limitation of this work is that the classroom walls bound everything that I have described. As designers, we took into account the local history of the classroom (e.g., in considering students’ repertoire of artifacts), as well as assumed out-of-school histories (e.g., with capitalism, which was necessary for students to experience the iPhone problem, X-box problem, and the window problem as real). However we did not make any attempt to understand the students’ locally meaningful out-of-school histories or practices. While we attempted to create conditions under which students had epistemic agency, we did not position students as design agents. Of course, students were indirect agents in our study to the extent that they resisted our designs and forced us to accommodate those resistances (cf. Pickering, 1995), but they were not positioned as designers, nor did we involve their out-of-school histories in our design. In other words, students were designed for.

This may have weakened the cascade of artifacts that emerged in the study. There is a long literature on the “encapsulation” of school learning (Engeström, 1991; Whitehead, 1929), which often happens because schools privilege building “vertical” expertise within narrow academic domains. One way to overcome this encapsulation is to purposefully design for “horizontal” movement across contexts, not just from school out, but from outside in as well (Engeström, 1996; Gutiérrez, 2011). While our study describes a powerful case of building vertical expertise within school, we ignored horizontal movement across contexts to our students’ detriment. Future work should involve students as collaborators, to design for cascades
of artifacts that (a) mobilize students’ everyday world into schools and (b) “reach out” beyond the school walls to strengthen students’ local mathematical practices.
Beginning in the late 1960s, the Dutch mathematician Hans Freudenthal (1968, 1983, 1987, 1991) began to sketch a vision for mathematics education based on the radical notion of mathematics as an activity, rather than a pre-existing structure or body of knowledge. This vision led Freudenthal to conclude that mathematics education should not be concerned with the transmission (or even the discovery) of mathematical structure, but rather with engaging students in the activity of structuring the world mathematically, which Freudenthal (1968, 1983) called “mathematizing.” As Freudenthal and his colleagues worked out the implications of this vision of mathematics as the activity of mathematizing, they created a domain-specific instructional theory for mathematics known as Realistic Mathematics Education (RME; Gravemeijer, 1994a; Treffers, 1987, 1993; van den Heuvel-Panhuizen & Wijers, 2005). RME has always been a work in progress:

RME started out as a vision, or as a philosophy of mathematics education that still had to be worked out. This is being done in developmental research projects, in which each time the research question is: What would mathematics education, which fulfills the initial points of departure, look like for a given topic? (Gravemeijer, 1999, p. 159)
This is the spirit in which I approach this paper. I respect the initial points of departure, while offering a new vision of RME based on cultural psychology. I am not the first to explore such concepts. Perhaps most prominently, Paul Cobb and colleagues (Cobb & Bowers, 1998; Cobb et al., 1997, 2008; Cobb & Yackel, 1996; Cobb, 1998, 2002) explored the ways in which RME was compatible with cultural psychology. Ultimately, although these researchers found a “possible point of contact” (Cobb et al., 2008, p. 109) between RME and cultural psychology, they rejected the notion that RME ought to be considered a cultural theory.

I see this group’s thought-provoking work as the start of a conversation amongst RME researchers. In the years since, this conversation has been taken up in mathematics education generally (e.g., McDonald, Le, Higgins, & Podmore, 2005; Radford et al., 2007; Radford, 2008a, 2008c; Stevens & Hall, 1998) but it has not been taken up in the RME literature. In this paper, I aim to reignite the conversation in the RME community. I will present a different conclusion that that of Cobb and his collaborators, namely, I argue that a cultural perspective is a necessary consequence of the first principles of RME. I start by summarizing some key aspects of RME. I then discuss how some of these aspects are in tension with each other, and show how a cultural perspective resolves these tensions. I next turn to an empirical example to illustrate these ideas in a classroom. Then, I return to Cobb’s research group, and re-interpret their reasons for rejecting a cultural perspective in light of my analysis. Finally, I conclude with the implications of my argument for RME.

1. **Summary of RME, and a summary of cultural perspectives on learning**

   RME is rooted in Hans Freudenthal’s notion that mathematics is the human activity of mathematizing the world. In 1987, Adrian Treffers (1987) described “five instructional
principles” consistent with this approach. In the three decades since, these five principles have been adapted into six (van den Heuvel-Panhuizen & Wijers, 2005):

1. Activity principle: Mathematics is, first and foremost, the human activity of mathematizing. Mathematics education should involve students in mathematizing, and through this activity, students should create their “own [mathematical] productions” (Treffers, 1987, p. 249) as they engage in mathematical activity. That is, “mathematics can and should be learned on one’s own authority and through one’s own mental activities” (Gravemeijer, 2004, p. 109), such that students “come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible” (Gravemeijer, 1999, p. 158).

2. Reality principle: Freudenthal recognized that in order to be meaningful, mathematical activity had to be experientially real for students (Freudenthal, 1987). However, this did not mean that Freudenthal rejected formal and abstract mathematics. On the contrary, he observed that mathematicians use and discuss abstract mathematical productions as if they were real objects. Indeed, for the mathematician, these imaginary productions are real objects, and formal mathematics is experientially real. Learning mathematics should be a reality-expanding endeavor, such that “formal mathematics comes to the fore as a natural extension of the student's experiential reality” (Gravemeijer, 1999, p. 156).

3. Level principle (sometimes called “emergent modeling,” Gravemeijer, 1999; or “progressive formalization,” Webb et al., 2008): Mathematical activity and productions exist at various levels of abstraction. At first, mathematizing takes place in a particular context, and productions exist as models of this activity in context.
These models can themselves be mathematized into models for future, more general activity. Finally, at the level of formal activity, students use culturally-accepted formal algorithms and means of formal symbolizing (Gravemeijer, 1999; Treffers, 1987; van den Heuvel-Panhuizen & Wijers, 2005; Webb et al., 2008).

4. Intertwinement principle: The various domains of mathematics should not be treated as silos, but rather intertwined. Students should engage in rich contextual activity that calls upon multiple domains at once (Treffers, 1987; van den Heuvel-Panhuizen & Wijers, 2005).

5. Interaction principle: Students should “share their strategies and inventions with each other” so that they can “get ideas for improving their strategies” (van den Heuvel-Panhuizen & Wijers, 2005, p. 290).

6. Guided reinvention principle: Learning is a process of reinvention (Freudenthal, 1991). The role of the teacher is to map a leaning route, “along which the students can find the intended mathematics for themselves… [and] students should be given the opportunity to build their own mathematical knowledge store on the basis of such a learning process” (Gravemeijer, 1999, p. 158).

As summarized above, there are two tacit aspects of RME in the six instructional principles. First, in RME mathematics is theorized as both an activity and a product (Gravemeijer & Terwel, 2000). Second, there is a prominent role attributed to the individual in RME: the individual is seen as the reinventor of mathematics and the possessor of private mathematical knowledge that results from this reinvention (see descriptions of Principles 1 and 6; Gravemeijer, 1999, 2004; Treffers, 1987). In the next section, I will show that these aspects are in tension with
one-another and with the first principles of RME. To resolve this tension, I will draw on cultural theories of learning, often called sociocultural (Wertsch, 1994, 1998) or cultural-historical (Cole, 1996) psychology, or situated (Lave, 2008) or distributed cognition (Hutchins, 1995a; Pea, 1993). In recruiting these theories, I will draw primarily on the notions of “culture” and “mediation,” which I describe below.

I take a process and product view towards culture. Cultural processes are those that “accumulate partial solutions to frequently encountered problems” (Hutchins, 1995a, pp. 354–355). The residua of these processes – the partial solutions themselves – exist in material and ideal form as cultural artifacts. These artifacts serve to propagate the achievements of past generations into the present. Further, the set of these artifacts constitutes culture-as-product: “the species-specific medium of human life” (Cole, 2010, p. 462).

Notice in the above quote that Cole refers to the “medium” constituted by culture-as-product. Human actions take place in a cultural milieu and as such they are mediated by culture. What this means is that human actions “involve not a direct action on the world but an indirect action, one that takes a bit of material matter used previously and incorporates it as an aspect of action” (Cole & Wertsch, 1996, p. 252). Here, the “bit of material matter” is a cultural artifact. Mediating artifacts do more than simply facilitate or amplify an action that would otherwise exist. Rather, they enable new forms of human actions, and further, they “act back” on the human actors, such that humans are constituted by culture. Thus artifacts and activity are

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4 These diverse traditions have different histories and make slightly different theoretical commitments (see, e.g., Nardi, 1996). However, one common thread is that all of the approaches reject dualist notions of the separation between mind and world, and pay particular attention to the meditational role of culture in human activity (O’Connor & Glenberg, 2003). As cultural mediation and non-dualism are my primary focus in this paper, I attend little to the distinctions between theories, and instead batch the theories together under the terms, “cultural perspective” or “cultural theories of learning.”
**productively entangled:** In activity, humans produce artifacts as “partial solutions to frequently encountered problems,” and these artifacts go on to play productive roles in future activity.

This productive entanglement is easily seen in physical actions. For example, consider the activity of pole vaulting (cf., Wertsch, 1998). Here, the pole is an artifact that exists as a partial solution to the problem of getting one’s body over a high bar. It is a product of human activity. In addition, it mediates future activity. The mediating role of the pole is obvious: the pole-vaulter does not jump over the bar directly but rather she does so indirectly by incorporating the pole into the action. As such, the pole does not make humans better jumpers (i.e., amplifying an action that would otherwise exist), but rather it enables a completely new action that could not otherwise exist.\(^5\) In addition, the pole acts back on the human actor: because it affords particular ways of being used, it shapes the actor in particular ways both in microgenetic and ontogenetic time. In microgenetic time, the pole shapes the movements of the pole-vaulter as she engages in the activity of pole vaulting. That is, the way she holds her hands and moves her body only makes sense if there is a pole. Across ontogenetic time, the pole shapes her body, callusing her hands in particular places and making certain muscles more prominent (Wertsch, 1998). It also shapes her identity: as she comes to know and use the pole fluently, she *becomes* a pole-vaulter (cf., Bowker & Star, 1999).

\(^5\) The moment when an action becomes a “completely new” action is never clear, and is often contentious. For example, it is quite reasonable to say that pole vaulting is qualitatively different from jumping, or from any other human action that exists without a mediating artifact. But what about pole vaulting with a bamboo pole vs. a fiberglass pole? As Wertsch (1998) documents, pole vaulting with a fiberglass pole is very different from pole vaulting with a bamboo pole. Are they the same action? The answer was historically contested, with people setting new records with the fiberglass poles asserting *continuity* of activity, but critics arguing that it is in fact a *new form of action*. Similar struggles can be seen in mathematics education, for example, in the historical and contemporary struggles over the nature of the effect of calculators on school mathematics.
Cultural mediation plays a key role in psychological processes as well. Indeed, the key tenant of cultural psychology is that all higher psychological functions are culturally mediated (Cole & Wertsch, 1996; Vygotsky, 1978). What this means is that culture is not a background against which an independent mind creates knowledge through presumed biologically-universal processes such as assimilation and accommodation (cf., Piaget, 1970). Rather, the human mind is constituted by and through culture. As described by Vygotsky (1997b, p. 138), “the central fact about our psychology is the fact of mediation.”

A paradigmatic example psychological mediation comes from Vygotsky’s double stimulation experiments (Vygotsky, 1978). The method involves giving a subject (in experimental conditions) a cognitive task to solve that is “beyond his [sic] present capabilities” (Vygotsky, 1978, p. 74). This task is the first stimulus. The experimental setting also contains a neutral object—a second stimulus:

[F]requently we are able to observe how the neutral stimulus is drawn into the situation and takes on the function of a sign. Thus, the child actively incorporates these neutral objects into the task of problem solving. We might say that when difficulties arise, neutral stimuli take on the function of a sign and from that point on the operation's structure assumes an essentially different character (Vygotsky, 1978, p. 74).

Thus the second (neutral) stimulus becomes a mediating artifact as the subject creatively pours meaning into the neutral artifact and then uses the artifact to accomplish the task. Because the artifact fundamentally changes the character of the problem-solving operation, it is not sufficient to claim that the problem is solved by the individual acting alone. Rather, the artifact is
a constituent part of the problem-solving activity, and therefore the problem is solved by a system that includes both persons and artifacts.

Cognitive processes, then, do not happen solely “in the head” (Cole & Wertsch, 1996, p. 253). Rather they are distributed—that is, “stretched over, not divided among” (Lave, 1988, p. 1)—persons and their cultural environment (A. Clark, 1998; Cole & Engeström, 1993; Cole, 1995; Hutchins, 1995a). The wording in this quote is important. A common misunderstanding of so-called “distributed cognition” is that the claim is that cognition is something that can be partitioned such that separate partitions can be attributed to persons and artifacts respectively. What Lave’s quote makes clear is that this quest is folly. Just as we cannot attribute different parts of walking to the right and left legs—walking only happens when the legs work together in a system, and we don’t get a “half walk” if we take away one leg (cf., Rogoff, 2003 who makes a similar argument about “nature” and “nurture”)—neither can we attribute different parts of cognitive activity to different elements of the person-culture system. Rather, persons and culture work together to create cognitive activity that is qualitatively different than the sum of the individual parts.

To recognize that artifacts are constituent parts of human cognitive activity does not reduce humans to simple stimulus-response creatures. This is because artifacts do more than simply store and transmit meaning. As the example of double stimulation makes clear, human creativity and agency are fundamental components of mediated activity. For Vygotsky, human agency is always mediated by artifacts: we control ourselves through artifacts of our own creation (Engeström, 2007; Wertsch & Rupert, 1993), and become particular kinds of people as we work with artifacts (Bowker & Star, 1999). This is how culture “acts back” and constitutes the actor.
To sum up, a cultural perspective puts culture “in the middle” (Cole, 1996, p. 116) of all human activity. Activity and culture become intertwined: humans produce culture, in the form of artifacts, as they engage in activity, and those artifacts mediate and transform future activity. In the process, culture “acts back” and produces people in particular ways as they engage in activity. Thus, a cultural perspective is non-dualist and relational. People, culture, and activity can only be understood in relation to each other. In the next section, I will show how this cultural perspective is not only compatible with RME, it is a necessary consequence of the three key aspects of RME that I highlighted above.

2. **A cultural perspective is a necessary consequence of the first principles of RME**

I will make this claim in two parts. First, beginning with the first principles of RME, I will show that mathematizing is a mediated activity. Second, I will show that mathematical productions are cultural artifacts. The upshot of these two claims is the following larger claim: Mathematizing is always a culturally mediated activity and mathematics is a cultural artifact.

**Claim 1: Mathematizing is a mediated activity**

Let’s start from the initial point of departure in RME, Freudenthal’s adage that mathematics is the human activity of mathematizing—that is, structuring the world mathematically. To warrant the claim that this is a mediated activity, I will first make a softer claim, that mathematizing is *sometimes* a mediated activity. One example is sufficient to make this softer claim. The example that I will use comes from Jean Lave and colleagues (Lave, 1988), and involves a weight-watcher in his kitchen trying to figure out how to make a serving of cottage cheese that is \( \frac{3}{4} \) of \( \frac{2}{3} \) of a cup. Rather than multiplying \( \frac{2}{3} \) cups by \( \frac{3}{4} \), the weight-watcher uses a measuring cup to measure out \( \frac{2}{3} \) of a cup of cottage cheese, pours this measured serving on the table, shapes it into a circle, cuts the circle into quarters by making perpendicular
radial cuts with his finger, and then scoops one of these quarters back in to the cottage cheese container. He is left with \( \frac{3}{4} \) of \( \frac{2}{3} \) of a cup.

Comparing the end-state (a precise measurement of cottage cheese, shaped into a \( \frac{3}{4} \) circle) with the beginning state (a container of unmeasured, shapeless cottage cheese) it’s clear that the end is more mathematically-structured than was the beginning, and thus the weight watcher mathematized (i.e., he structured his environment mathematically). Examining the activity of mathematization, it is clear that this was a mediated activity—at the very least, it was mediated by the measuring cup in the same way that the pole mediates pole vaulting.

Thus mathematizing is—at least sometimes—a mediated activity. To make the stronger claim, that mathematizing is always a mediated activity, I remind the reader that within RME, mathematics is an activity and a product. In itself, this is not a controversial statement within RME. Even though Freudenthal critiqued educational practices that focused on “the transmission of mathematics as a pre-formed system” (Gravemeijer & Terwel, 2000, p. 779), he also recognized that mathematics does exist as a product. For example, in his China Lectures (Freudenthal, 1991, p. 16), he argued that mathematics is a human activity, but clarified, “if I were to continue in the same way, that is, by focusing on the mathematical process, I would imprudently be neglecting the medium in which this process takes place.” For Freudenthal, this “medium” is the world of mathematical productions.

The existence of mathematics-as-product is essential to understanding Freudenthal’s notion of mathematizing: “a form of organizing that also incorporate[s] mathematical matter” (Gravemeijer & Terwel, 2000, p. 781). Note the similarity between these quotes about mathematics, and the definitions of culture and mediation given earlier. Below, I will have more to say about the relationship between mathematics and culture. For now, it is sufficient to note
that if mediation involves the incorporation of pre-existing matter into activity, then mathematizing is *always* a mediated activity. Indeed, the mediation of mathematics-as-product is what separates mathematical activity (i.e., mathematizing) from non-mathematical activity.

We can see this in the weight-watcher example. The reason that I could claim that the weight watcher was structuring his world *mathematically* is because his activity was mediated by mathematical productions. As one example of many, consider how the mathematical production of *circle* mediates the activity. On the one hand, we can say that the Weight Watcher uses “circle” because he *circlitizes* the cottage cheese. On the other hand, we can say that “circle” uses the Weight Watcher, because “circle” structures his hand movements in a particular way. Just as the pole acts back on the pole vaulter, so too does “circle” act-back on the Weight Watcher in microgenetic time. The activity of structuring the cottage cheese mathematically is stretched across the Weight Watcher and mathematical productions, and indeed all mathematizing has this character.

Thus I have warranted my first claim, that mathematizing is a mediated activity. It is mediated by material artifacts in the environment and by mathematical productions that constitute the “medium” in which the activity takes place. But just what are mathematical productions? Within RME, they are commonly referred to using individualistic terms such as “mental objects” (Freudenthal, 1983, p. 33), and “private knowledge” (Gravemeijer, 1999, p. 158). Below I argue that, in contrast to these individual-centric notions, mathematical productions are cultural artifacts.

**Claim 2: Mathematical productions are cultural artifacts.**

To make this claim, I will draw on the reinvention principle. As described by Gravemeijer (Gravemeijer, 2004, p. 114), this means that “students should be given the
opportunity to experience a process similar to the process by which a given piece of mathematics was invented.” There are two notable features of this description. First, the notion that mathematical productions are invented rather than discovered suggests that they are the products of human activity as opposed to being a part of a Platonic structure that exists independent of humans. Freudenthal (1968, p. 6) also located the origins of mathematics-as-product in human activity, stating, “arithmetic and geometry have sprung from mathematizing part of reality.”

Second, the “re” in reinvention along with the past tense in Gravemeijer’s phrase “a given piece of mathematics was invented” communicates that the productions that a given student is to reinvent pre-exist the student. This is not simply a linguistic accident. Freudenthal was very careful in his choice of the term “guided reinvention,” as opposed to “construction” or “discovery” or even “invention”:

Guiding reinvention means striking a subtle balance between the freedom of inventing and the force of guiding (...) [T]he learner’s free choice is already restricted by the “re” of “reinvention.” The learner shall invent something that is new to him [sic] but well-known to the guide (Freudenthal, 1991, p. 48).

None of this is controversial within RME. But once we understand that productions pre-exist individual students, it becomes impossible to maintain the notion that productions are private and internal. So just what is the ontological nature of mathematical productions? As documented above, the notion of guided reinvention imposes strict criteria on the ontology of mathematical productions within RME: First, productions are the product of human activity so we have to reject the idea that they exist in some metaphysical Platonic structure, independent of
humans. Second, they are durable and shared, existing across time and outside of any single individual. This means we have to reject the idea that productions are individual possessions. The only object that meets these criteria is a cultural artifact. Thus, this warrants my claim that mathematical productions are cultural artifacts (see also, Peck & Matassa, in press, Chapter 2 this dissertation).6

**Summary before moving on**

Thus far I have discussed how, within RME, mathematics has been theorized as a collection of “mental objects” and learning has been taken up as the acquisition of “private knowledge.” I have demonstrated that such a position is in tension with the first principles of RME (specifically the activity principle and the guided reinvention principle). I have shown that instead, a cultural perspective is a necessary consequence of these principles. Specifically I have claimed that, if one follows the first principles of RME, mathematical productions must be considered cultural artifacts, and that mathematizing must to be understood as a culturally-mediated activity. In other words, culture—in the form of mediating artifacts that are produced through, and recruited into, activity—is at the very heart of RME. To illustrate this idea in practice, I next turn to an empirical example.

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6 Following the definition of culture given earlier, when I say that mathematical productions are *cultural artifacts*, I mean that they are the collected product of human mathematical history, a “symbolic and materially constituted social inheritance” (Cole, 2010, p. 462). In this way, mathematical productions exist in the same way that languages exist, or symphonies or literature.

To say that these artifacts pre-exist a given individual is to recognize that “we live from birth to death in a world of persons and things which in large measure is what it is because of what has been done and transmitted from previous human activities” (Dewey, 1938, pp. 39–40). However, this is not to say that all humans are born *aware* of language, symphonies, literature, or mathematics, nor is it to suggest that the cultural world into which we are born is static. As cultural objects, mathematical productions can be *objectified* (Radford, 2008b, 2010, 2013; Roth & Radford, 2011) through social processes such that they become real to individuals. Further, they can be manipulated and extended by humans to create new culture. That is, just as Shakespeare manipulated existing language to create new works, so too can humans manipulate existing mathematics to produce new mathematics.
3. Mathematizing as a mediated activity: Empirical example

The example for this paper comes from a design study (Cobb, Confrey, et al., 2003; Gravemeijer & Cobb, 2006) in an activity-based Algebra I classroom in which I was the teacher (Peck, 2015b, Chapter 3 this dissertation). Throughout the school year, my colleagues and I designed activities and activity sequences using RME principles, and RME guided the design of the study under consideration in this paper. For this study, our goal was to create a local instructional theory (Gravemeijer & Cobb, 2006; Gravemeijer & van Eerde, 2009; Gravemeijer, 1994b) for how students could be guided to reinvent and connect five sub-constructs of slope in a high-school Algebra I course. The five sub-constructs of slope that we focused on were: (1) Rate of change; (2) the parametric coefficient (the $a$ in $y = ax + b$); (3) the algebraic ratio ($\frac{y_2 - y_1}{x_2 - x_1}$); (4) the geometric ratio (“rise over run”); and (5) the physical property (steepness) (Stump, 1999).

In this paper, I will focus on a single problem-solving episode, in which a group of students reinvents the algebraic ratio. I focus on reinvention because it is perhaps the quintessential mathematical activity in RME. As I discussed in Section 1, reinvention is mathematization, and it is also the mechanism through which learning is hypothesized to occur in RME (Freudenthal, 1991).

Prior to the episode in this paper, the class had reinvented and/or made meaningful two of the sub-constructs of slope: rate of change and the parametric coefficient. As I document elsewhere (Peck, 2015b, Chapter 3 this dissertation), the class understood rate of change as a both a measure of an intensive quantity (see below) and a measure of linear covariation between two quantities, which can be iterated and accumulated via multiplication to make predictions. Students coordinated this latter understanding of rates of change with algebraic equations to reinvent the parametric coefficient.
In addition, the class had reinvented and/or used a number of other mathematical productions in class, some of which became consequential in the problem-solving episode. In particular, ratio tables (Middleton & van den Heuvel-Panhuizen, 1995), fractions-as-quotients (Kieren, 1980), the “find one” strategy (Peck & Matassa, in press, Chapter 2 this dissertation), and a unit rate strategy (Cramer et al., 1989). The first three of these productions emerged in the classroom as “partial solutions to frequently encountered problems” before the design study in a unit that involved finding unit values given a many-to-many relationship (the sort of problems that a mathematician might classify as involving partitive division). For example, as shown in Figure 26, students solved equal-sharing problems using equipartitioning (Empson, 1999; Streefland, 1993; Wilson et al., 2011), as well as other missing value proportional reasoning problems (Kaput & West, 1994).

Figure 26 shows an example of the first three mathematical productions above. On the left side, the student has constructed a ratio table, which she coordinated with the “find one” strategy to find the equal share. The “find one” strategy was invented and named by the class during the course of this early unit. It links the division operation to situations in which the goal is to find the value of one object. The student’s use of this strategy is indicated by the downward curving arrows coordinated with the division operation, which together link the initial situation on the first row of the ratio table to the equal share on the second row of the ratio table. The student expresses the equal share using a fraction-as-quotient \( \frac{2}{5} = \frac{2}{5} \). Her understanding of the link between the fraction and the sharing activity is shown in the diagram on the right side, which depicts the operation of sharing using equipartitioning and distributing (see Peck and Matassa, in press, Chapter 2 this dissertation, for a detailed description of how these artifacts emerged in the classroom as “partial solutions to frequently encountered problems”).
While the “find one” strategy is sufficient for missing value proportional reasoning problems in which the missing value corresponds to a unit value in the given variable (e.g., the amount of water for one person), it will not suffice for problems in which the missing value corresponds to a non-unit value in the given variable (e.g., how much water for three people?). For problems such as this, students used a unit rate strategy, which involves using the “find one” strategy to create a unit rate given a many-to-many relationship, and then scaling that rate using multiplication to make a new many-to-many relationship (Cramer et al., 1989). Notice that, while this strategy is more general than the “find one” strategy, it is still limited to situations in which the two variables are proportional to each other. For more general linear situations, in which changes between variables are proportional but values may not be, the unit rate strategy is not sufficient to make predictions. For these situations, a more general strategy involves subtracting values of the independent and dependent variables to find changes in each, and then dividing these the latter by the former to create a unit rate. This is the strategy depicted by the algebraic ratio, $(y_2 - y_1)/(x_2 - x_1)$.

Thus, to guide students to reinvent the algebraic ratio, we presented them with linear, but not proportional, situations and asked student to make predictions given two value pairs. The
class’ first experience with such a situation was the “window problem” shown in Figure 27. As shown, the window problem asks students to make a prediction in a functional situation in which the dependent variable (price) varies linearly, but not proportionally, with the independent variable (number of windows). As will become clear, the students in the focal group do not have a strategy to solve such a problem in the beginning of the episode, but by the end they have created the “subtract and divide” strategy depicted by the algebraic ratio. Thus it is appropriate to refer to their activity as “reinvention.”

To trace the process of reinvention, I will analyze the problem-solving activity of four students (referred to using the pseudonyms David, Stacy, Melissa, and Tyler) as they work on the window problem. As shown in Figure 28, the students are seated in a cluster of four individual desks, with David and Stacy facing Tyler and Melissa. Melissa has a large piece of paper in front of her with the problem written at the top. The problem is also projected onto a whiteboard that is located behind Melissa and Tyler and in sight of David and Stacy. The physical layout of these inscriptions will become consequential, thus Figure 27 shows both inscriptions.

Figure 27. The window problem as it is represented on the white board (A) and on the group’s paper (B).
I used ethnographic microanalysis (Erickson, 1992, 1995) to analyze the moment-by-moment multi-modal mathematical activity of this group. As recommended by Erickson (1992) and others (Arzarello, Paola, Robutti, & Sabena, 2008), I collected video and audio recordings to facilitate the microanalysis. The video camera was positioned such that the faces, hands, and desktops of all four students were generally visible in the screen at the same time. A wide-angle microphone, placed in the center of the group, captured audio. The group produced a single inscription on their poster-sized piece of paper, which I collected. The group also made use of the projected inscription of the problem statement. This projected inscription is off camera. However, I have the PowerPoint slide that was projected (Figure 27A), as well as a diagram of the classroom that shows how students were positioned relative to the white board (depicted in Figure 28).
My unit of analysis is the *semiotic bundle* (Arzarello et al., 2008; Arzarello, 2006), which can be explained as follows: As students engage in mathematical activity, they marshal multiple *semiotic means of objectification* (Radford, 2003), including talk, gesture, body position, gaze, inscriptions, and other artifacts. The semiotic bundle is the coordinated ensemble of these semiotic means that students recruit within and across moments of activity. For example, in the episode that follows, I will show how Stacy coordinated talk, the group’s written inscription, and multiple mathematical artifacts in a key turn at talk that created the need to reinvent the algebraic ratio. Notice that this bundle includes cultural products such as inscriptions and mathematical artifacts. Thus, using a semiotic bundle as my unit of analysis is appropriate for my task of showing how mathematical activity is distributed across cultural artifacts.

I analyzed these bundles synchronically and diachronically. Synchronic analysis examines how the semiotic resources are coordinated in a single moment. Diachronic analysis explores how the semiotic resources change over time (Arzarello et al., 2008; Arzarello, 2006). For example, I will show how David employed talk, gesture, and the projected inscription to map values of the numeric words that were projected onto the physical space that these words occupied in the inscription (synchronic analysis). I will also show how this gesture itself became an artifact for the group across time, even as it was coordinated with different inscriptions and talk (diachronic analysis).

To facilitate analysis, I created multiple data sources. First, I created a video that places the group video captured by the camera alongside a scanned copy of the students’ poster. This is important because the poster is generally too small to read in the overview video. Placing the two side-by-side helps to reveal the coordination between talk, gesture, and the inscription. From this video, I created a transcript of talk and gesture, using a transcription style based on the format
described by Ochs (1999) for capturing both verbal and non-verbal behavior. Specifically, this means that I have two separate columns in my transcript: one for talk and one for non-verbal action.

Reinvention as a mediated activity

The episode begins with Melissa reading the problem out loud. Immediately after Melissa reads the problem, two things happen. First, the group begins to solve the problem by recruiting mediating artifacts:

Segment 1: Recruiting artifacts

13: S: A new customer asks (.). Leslie to install five windows how much is ( )? So what we should do is [set it up ((D looks down towards paper))
   ((D looks up at screen))
   ((S looks toward paper, D continues to look toward screen))

14: M: [is do tw- divided by two1 divided by two2 to find the one, the price of one window? ((M points at the “2” on paper))
   ((M points at the “402” on paper))

As will become apparent soon (see Figure 30 below), Stacy recruits a ratio table into the activity when she calls for the group to “set it up” in turn 13. In turn 14, Melissa recruits a find-one strategy.

Second, the group fractures into two ensembles of persons and artifacts. I have analyzed the social dynamics that lead to, and resulted from, this fracturing elsewhere (Peck, 2015a). For the analysis and arguments presented here, the primary focus is the nature of the ensembles themselves. As shown in Figure 29, one ensemble consists of Stacy, Melissa, Tyler, and the group’s paper. The other ensemble consists of David and the white board.
Figure 29. The social organization after 13 turns at talk.

The intra-ensemble interactions are exemplified in Segment 2 below:

Segment 2: Two kinds of interactions

31  S: This rate is four oh two over two
32  M: Divide first {{
33  S: [That's the rate, yeah
34-36  ((Crosstalk))
37  S: [And then equals-
38  M: [Five seventeen over seven
39  S: =equals one\(^1\) (. window. And then\(^2\) also four oh two\(^3\) (...)\(^4\) over two equals one window.

Note the different sorts of interactions that are happening within the triad (Melissa, Stacy, Tyler, and the group’s shared paper) on the one hand, and David and the white board on the other. Melissa and Stacy talk about the problem using overlapping and latched speech. Their turns build on each other, and they coordinate their talk with their shared artifact. They are
continuing the strategy that Melissa and Stacy set out in Segment 1, using a find-one strategy to find the rate associated with each of the two (window, cost) pairs given in the problem.

In Segment 2, turn 39 (arrowed), David is doing something very different. He moves his fingers rhythmically. He points to the board. His movements are not coordinated with the talk or work of the triad.

Over the next thirteen minutes, David continues to gesture at the white board. We’ll return to David in a moment. For now, let’s follow the work of the triad. Over these 13 minutes, the triad produces the inscription shown in Figure 30. This inscription coordinates ratio tables, the find one strategy, fractions-as-quotients, and the unit rate strategy into two material assemblies, one for each of the (window, cost) pairs in the problem. One assembly is in the lower left of the figure, and the other is in the upper right. Each assembly can be read as a 2x2 vertically-aligned ratio table (similar to the ratio table depicted in Figure 26), with columns separated by the equals sign. Each column contains like-units (cost in the left column and windows in the right column), and each row contains a ratio of cost to windows. At the lower left of each assembly is the prediction for five windows, which the triad arrived at via a unit rate strategy (recall from above that a unit rate strategy involves finding a unit rate via the find one strategy—which occupies the first row of each assembly—and then scaling that unit rate to make a prediction—in this case, shown via downward curving arrows coordinated with a multiplication operation).
Thus far, the triad’s problem-solving process has been mediated by multiple mathematical artifacts: rates of change, fractions-as-quotients, ratio tables, the find-one strategy, and the unit rate strategy. That the students are engaged in mathematical structuring activity (mathematizing) should be obvious: the material assemblies that the triad has produced by bringing the various artifacts into coordination with each other and with the problem statement constitute a more-structured state of the problem context than that with which they began.

But the structuring activity is not located entirely within or even between the students in the triad. The artifacts too, structured the activity. For example, the “find one” strategy structured much of the triad’s activity. Earlier, I stated that Melissa recruited the find one strategy into the activity in turn 14. This is accurate. But it is also accurate to say that the find one strategy recruited the triad into doing a particular kind of activity. Indeed, the triad mentions the need to “find one” using the find one strategy eight separate times during the first thirteen minutes. Similarly, the triad uses ratio tables to structure the problem, but also, the ratio table uses the students to structure their activity around the particular organization that the ratio table demands. This can be seen, for example, in the “blanks” drawn as horizontal underlines in the lower left
corner of each assembly in Figure 30. Initially, these “blanks” were empty, and they were only later filled in by the students as they completed the arithmetic to make each prediction. Drawing the “blank,” then, is an action demanded by the ratio table to maintain its row and column organization. Without the ratio table, such an action is nonsensical. Thus, the students use artifacts and the artifacts use the students, such that the students’ mathematizing activity is distributed—stretched over, not divided among—the artifacts that co-constitute it.

Up to now, the triad’s talk has looked very similar to that depicted in Segment 2. The students in the triad build on each other’s turns, they are hunched over the shared paper, and they maintain a solution-oriented affect. In other words, they are proceeding as if they are comfortable that they know how to solve the problem. After the production of each assembly, however, things change. Stacy exclaims, “what the heck?” (turn 181) and Tyler moans, “Oh God!” (turn 194). We uncover the source of their displeasure when Stacy asks, in turn 198, “are these supposed to, like, relate?”

Stacy’s use of the deictic, “these” is a reference to the two predictions, and in asking if they are “supposed to like, relate,” Stacy’s talk works to bring the two predictions into coordination. Stacy’s question is a sense-making question—a *semiotic* question—that marshals together the bundle of semiotic resources the group has assembled so far. The distress evident in Stacy and Tyler’s earlier exclamations suggests that something in this bundle doesn’t conform to their expectations.

The trouble, of course, is that the forms of activity that are enabled by the artifacts that the students have recruited are not sufficient to solve this this problem. As co-constituents of the triad’s activity, the artifacts enabled certain forms of activity, but they also constrained that activity (cf., Pickering, 1995). In the window problem, these constraints manifest themselves as
two different predictions for what ought to be the same price. The students’ exclamations, and
Stacy’s question, are recognitions of the contradiction produced by this constraint. For the triad,
the problem of finding the price of five windows has now become a dilemma of two different
predictions. The students become visibly and audibly frustrated.

What we have, then, is a task that is, to borrow Vygotsky’s phrase, “beyond the present
capabilities” of the triad and the artifacts that help to constitute their possibilities for action. But
even as human activities are enabled and constrained by artifacts, humans maintain a very
particular kind of agency: the potential for “intentional collective and individual actions aimed at
transforming the activity” (Engeström, 2007, p. 381, italics added; see also Kaptelinin & Nardi,
2005, Chapter 10). This transformative agency, like all human agency, is exerted through
artifacts.

Here we rejoin David. Recall that when we left David, he was gesturing at the white
board. For the past 13 minutes, his activity has largely considered of such gestures. At times, he
makes bids to rejoin the group, but up until now, those bids have been denied. Now, however,
the triad is in a very different state than they had been up to this point. In Segment 3 below,
David introduces the key insight needed for the group to reinvent the algebraic ratio. As we will
see, this insight is itself distributed across artifacts.

Segment 3: A key insight

253  D: What's uh, what's a hundred and fifteen divided by

254  S: Where are you getting a hundred and fifteen?
David initiates the event with a question (turn 253). Stacy’s uptake, in the form of another question, works to position David as a contributing member of the group, at least temporarily. In turn 255 (arrowed), David explains his strategy. As shown in Figure 31 this explanation involves the coordination of talk and gesture.

![Figure 31. David's gesture in turn 255](image)

On first glance, the gestures shown in Figure 31A and Figure 31B seem to be deictic. As David says the words “four hundred and two,” he points to the number 402 on the white board.
(Figure 31A), and similarly for “five hundred seventeen” (Figure 31B; recall from Figure 27 that the problem statement is written such that the numbers are on different lines, with 402 above 517). These deictic gestures bring the problem statement into coordination with David’s speech and in this respect they facilitate the problem statement being used as a warrant for David’s use of the numbers 402 and 517 (notice that whereas David was questioned about where the number 115 came from [turn 254], no one questions him on where he got the numbers 402 and 517).

However, these gestures do more than simply index the problem statement to provide a warrant for a claim. In addition, they coordinate the written numerals with physical locations in space, enabling a phenomenon that Wittgenstein (1958) refers to as *seeing-as*. In this case the gestures facilitate *seeing* the written numerals *as* locations in space. Melissa’s uptake in turn 256 supports this claim. Notice that as she mimics the gesture (more on this below) she uses the word “here” while pointing to numbers on the paper. The word “here” is a spatial pronoun, whose antecedents are points in space and not values of numbers. Thus, Melissa’s use of the word “here” to refer to numbers on a paper indicates that she is seeing the numbers on the paper as points in space.

Now let’s return to turn 255 and consider David’s next move, shown in Figure 31C. In this move, David says, “so that’s the price between the two” while he moves his hand rapidly up and down between the locations that he pointed to in Figure 31A and Figure 31B. Like the gestures in Figure 31A and Figure 31B, this gesture does both deictic and creative work. As a deictic, it works to provide a referent for the pronoun, “the two.” As a creative act, it works in coordination with David’s talk to make the physical space between the written numerals meaningful.
First, consider the talk: David says, “the price between the two.” The word “between” is a spatial word. It is sometimes used to refer to an intermediate point, and sometimes to an intermediate space. The former is the most likely use in the context of prices (as in, “the price for five windows ought to be between the price for four windows and the price for seven windows”). However, that isn’t how the word is used here – David uses “the price between the two” to refer to the number 115, which is not an intermediate price between 402 and 517. To say that 115 is “the price between” 402 and 517 is to use “between” to refer to intermediate space. Of course, talk is not acting alone here, it is coupled with gesture. In this case, the gesture also works to put meaning into the physical space between the two written numerals. This is because the rapid movement in the gesture draws the eye to the space between the two endpoints rather than a specific intermediate point.

Overall, in turn 255 David’s talk and gesture work together to map the physical space occupied by the written numerals on the whiteboard with the value of those numbers. Talk and gesture then work together to make the physical space between the written numerals meaningful. The overall effect is that physical space can be seen as numerical values. To be sure, these moves are afforded by the way that the numerals occupy space on the white board (if the problem statement had been verbalized instead of written, this affordance would have been absent), as well as by cultural technologies that facilitate number-as-space mappings, such as the “mental number line” (Núñez, 2011). However, just because artifacts afforded this way of seeing does not mean that it was “inside” the whiteboard a priori. It certainly wasn’t designed that way; when we created the slide that was projected onto the whiteboard (Figure 27A) we never considered how the spatial locations of the numerals might be meaningful. If the numerals were separated in physical space it was only because this is a consequence of written language. So
where did this meaning come from? It came from David, from the whiteboard, and from other cultural technologies such as the mental number line. It is a distributed accomplishment, achieved through the coupling of gesture and talk.

But the gesture does even more. In addition to facilitating seeing as, the gesture itself becomes a representational artifact, signifying the white board. This can be seen in turn 256 (Figure 32), where Melissa revoices David’s talk and mimics the gesture on the group’s paper. In doing so, she brings the paper into coordination with the white board. This is key because David’s strategy is linked to the white board—in particular to the mapping between the physical location of the numbers on the white board and the value of those numbers. By coordinating the paper and the white board, Melissa mobilizes the strategy into the triad. So mobilized, the strategy can be taken up by other members of the group, which Tyler and Stacy do in turns 260 and 261, respectively. The gesture does the work of mobilization. In creating the gesture, David created the rails (Latour, 1983) on which the strategy travels.

So between here and here is a hundred and fifteen dollars

Figure 32. Melissa’s gesture
The whiteboard, then, began as a neutral object. As David interacted with it, he imbued it with meaning, and from then on the problem-solving episode took on a different character—the group repaired its fracture, and then went on to reinvent the algebraic ratio and solve the problem (Peck, 2015a). The white board thus served as a second stimulus, which transformed the group’s activity and mediated the group’s reinvention of the algebraic ratio. In this group, reinvention is a mediated activity—a sensuous, extra-psychological phenomenon that is distributed across persons and artifacts.

4. Returning to Cobb et al.’s argument

Thus far, I have presented and exemplified my argument that a cultural perspective is a necessary consequence of the first principles of RME. As I said in the introduction, I am not the first person to consider such a position. Most prominently, Paul Cobb and his collaborators considered this idea in the 1990s and early 2000s. However, Cobb and colleagues came to a different conclusion than I have. In this section, I return to their argument, and re-interpret it in light of the analysis that I have presented here.

Recall that Cobb and colleagues (2008, p. 109) noted a “possible point of contact” between RME and cultural psychology concerning the role of mediating artifacts in both RME and cultural theories of learning. Often they stress the importance of mediating artifacts to both cultural theories of learning and RME as more than simply a “possible point of contact.” For example, in a paper in which Cobb and colleagues describe an RME instructional sequence in a first-grade classroom, they “fully accept Vygotsky's fundamental insight that semiotic mediation is crucially involved in student conceptual development” (Cobb et al., 1997, p. 221). Moreover, they explain how their analysis of the classroom activity is consistent with the claim that “tools
are not merely amplifiers of human capabilities, but they lead to the reorganization and restructur
ing of activity” (Cobb et al., 1997, p. 222). In another example, Cobb and Bowers (1998, p. 111) explain how mediating artifacts are central to RME:

From [our] viewpoint, the use of tools is viewed as integral to mathematical activity rather than an external aid to internal cognitive processes located in the head (cf. Hutchins, 1995; Meira, 1998; Pea, 1993). From this perspective, it therefore makes sense to speak of students reasoning with physical materials, pictures, diagrams, and computer graphics as well as with conventional written symbols. This nondualist focus on tool use is, in fact, central to the RME design theory that guides our development of instructional sequences.

To a large extent, the arguments made by Cobb and colleagues above are those that I have made in this paper. However, while I claim that these arguments necessitate a cultural perspective within RME, for Cobb et al., the connection between RME and cultural theories of learning remains only partial. There are two reasons for this: First, Cobb and colleagues find a disparity in the way that learning is said to occur in the two approaches, and second, they find a disparity in the way that individuals are characterized. Below I address each of these perceived disparities, and argue that there is, in fact, no disparity in either case.

Disparity #1: How learning is said to occur: Cobb and colleagues interpret cultural theories of learning as implying that learning is “a process of transmitting mathematical meaning from one generation to the next” (Cobb et al., 2008, p. 110, italics added). They contrast this with RME, for which learning is a process of “emergence of mathematical meaning in the
classroom” (Cobb et al., 2008, p. 110, italics added). I certainly agree with Cobb et al. that RME theorizes learning as a process of emergence of mathematical meaning. On the other hand, I suggest that the association that Cobb et al. make between cultural theories of learning and transmission pedagogy is misguided. The association seems to stem from a somewhat common, but narrow, reading of Vygotsky’s writing on the social situation of development and internalization of cultural forms. While Vygotsky was interested in internalization, he rejected the notion that culture could simply be transmitted:

It is impossible to exert a direct influence on, to produce changes in, another individual… reducing the process of education and instruction to a passive apprehension by the student of a teacher’s lessons and outlines [is] the ultimate of psychological nonsense (Vygotsky, 1997a, pp. 47–48)

Furthermore, the notion that meaning is an emergent phenomenon that happens through activity is perfectly consistent with cultural theories of learning, as the quote below makes clear (see also, Radford, 2000, 2003).

From the time of their earliest publications in the late 1920's and early 1930's, the Russian cultural historical psychologists emphasized the tripartite nature of human mental processes. They represented the basic structure of consciousness as the emergent process involving an active subject, an object, and the cultural medium (Cole & Levitin, 2000, p. 65, italics added).
RME, with its focus on emergent modeling, offers a particularly refined theory of how mathematical meaning emerges through activity. Such an emergent process is perfectly compatible with a cultural perspective.

Disparity #2: The role of the individual. Cobb and colleagues interpret distributed accounts of cognition as “dismiss[ing] the individual from theoretical consideration” (Cobb et al., 1997, p. 227), and they argue for a perspective that preserves the “active individual” (Cobb, Stephan, McClain, & Gravemeijer, 2001, p. 105). In order to preserve the active individual, Cobb and colleagues argue that one must reject a purely sociocultural framework and must instead import ideas from psychological constructivism.

However, cultural perspectives have always included a place for active individuals (Cole & Wertsch, 1996). For example, Vygotsky (1997a, p. 48) explained that “the educational experience must be based on the student’s individual activity, and the art of education should involve nothing more than guiding and monitoring this activity.” Thus, as Cole and Wertsch (1996) make clear, the notion that cultural perspectives reject the active individual is misguided. One needn’t turn to psychological constructivism to preserve an active individual. However, the ontological nature of the “active individual” is different from a cultural perspective:

The knowing and learning individual is both active and acted on. When constructivism assumes that this activity is always intellectual and individual it fails to grasp the affective, relational, and cultural dimensions of activity. (...) [a cultural perspective] envisions a practical process of construction where people shape the social world, and in doing so are themselves transformed (Packer & Goicoechea, 2000, pp. 234–235).
I will return to the ontological differences in the next section. To sum up my argument in this section, Cobb and colleagues largely agree with my analysis of the role of mediation within RME. However, whereas I have argued that a cultural perspective is necessary within RME, Cobb et al. came to a different conclusion, noting only a “possible point of contact” between RME and cultural theories of learning. They did so because of two perceived disparities between RME and cultural theories of learning: (1) a disparity in how learning is theorized to occur, and (2) a disparity in the role of the individual. In this section I have shown that the disparities are ephemeral, and that a cultural perspective is fully compatible with RME. This is important, because a cultural perspective carries multiple implications for RME, which I summarize in the next section.

5. Implications for RME of adopting a cultural perspective

The main implication to RME of adopting a cultural perspective is a shift in perspective, away from a focus on individuals, and towards a focus on systems of persons and artifacts engaged in activity. Understanding the implications of this shift in perspective is a task for future work. However, below I sketch some possible implications for three of the principles of RME that I summarized in Section 1, and I introduce a new principle that I argue should be incorporated into RME. My exposition of these implications is brief. My goal here is not to settle anything, but rather to point to possible implications that should be studied in future research.

Implications to the activity principle

Freudenthal (1991) characterized mathematics as a “mental activity” (p. 2) and a “private activity” (p. 14). From a cultural perspective, such a view is much too narrow. As I have argued
in this paper, mathematizing is a mediated activity, and therefore cannot be located solely within an individual:

> Because what we call mind works through artifacts, it cannot be unconditionally bounded by the head or even by the body but must be seen as distributed in the artifacts that are woven together and that weave together individual human actions (Cole, 1995, p. 110).

At the very least, mathematizing is a “mediated action” (Wertsch, 1994, 1998)—that is, an “individual(s)-operating-with-mediational-means” (Wertsch, Tulviste, & Hagstrom, 1993, p. 343). Often, however, individual actions with mediational means cannot be understood except as part of a larger constellation of interactive activity. For example, earlier I described how David marshaled talk, gesture, and projected inscription into a semiotic bundle that mediated the mapping of numeric values onto physical space. I further described how the gesture itself became a representational artifact, which Stacy incorporated into her action to mobilize David’s strategy into the triad. Each of actions—David’s use of the semiotic bundle to map numbers onto space, and the Stacy’s subsequent use of the gesture—is a mediated action, but neither can be understood except as moment of the larger group’s activity. In these cases, a more expansive notion of activity is required:

> [T]he mediated actions and interconnected sequences of actions (i.e. operations) that individuals carry out in the attainment of a goal. (...) In the course of the activity, individuals relate not only to the world of objects (the subject-object plane) but also to other individuals (the subject-subject plane or plane of social interaction) and acquire, in
the joint pursuit of the goal and in the social use of signs and tools, human experience

(Radford et al., 2007, p. 512)

Such a definition seems to best capture the examples of mathematizing that I have given in this paper (the Weight Watcher and the students reinventing the algebraic ratio). Furthermore, this expansive definition seems to be most appropriate to educational contexts, where the limited notion of an individual operating with mediational means is likely insufficient to provide an adequate account of learning. As Dewey (1916, p. 19) explained, “things gain their meaning by being used in a shared experience or joint action.”

For RME researchers, the implication is that joint mediated activity is an appropriate unit of analysis for studying mathematizing. Researchers should pay particular attention to the constituting role of artifacts in human activity and human mental functioning. For RME designers, the implication is to create and study “artifact-saturated environments” (Gutiérrez, 2011, p. 32) for mathematizing.

**Implications to the reality principle**

Freudenthal (1983, 1991) took a phenomenological approach to reality, stating:

I prefer to apply the term “reality” to that which at a certain stage common sense experiences as real. (…) [Reality] is not bound to the space-time world. It includes mental objects and mental activities (Freudenthal, 1991, p. 17).
Thus, “in Freudenthal’s view, ‘common sense’ and ‘reality’ were construed from the viewpoint of the actor” (Gravemeijer & Terwel, 2000, p. 783). This perspective is problematic because it is easy to follow Freudenthal’s definition of an actor-construed reality all the way to solipsism. This, in turn, begs the question of why one should learn mathematics, if it is only to create an individual reality. Freudenthal’s (1968) answer was that mathematics is *useful*, “for the understanding and the technological control not only of the physical world but also of the social structure.” Of course, for mathematics to be useful to understand and control the physical and social world, it must have an existence beyond the individual.

One can understand the bind that Freudenthal was in. On the one hand, he was committed to the notion that mathematics could be experienced as real. On the other hand, he only considered two classes of objects: material and mental, and neither seem to capture the particular ontology of a useful mathematics, which I described in Section 2 as “durable and shared, existing across time and outside of any single individual.” In essence, Freudenthal’s bind is a version of the same dilemma that I introduced in Section 2 when discussing the nature of mathematical productions. The dilemma comes from dualist notions of persons and world, a position which states that “mathematical reality must lie either within us, or outside us” (White, 1947, p. 291).

Just as a cultural perspective resolved the dilemma in Section 2, so to does a cultural perspective resolve Freudenthal’s bind. From a cultural perspective, mathematics does have an objective existence: it exists objectively as a cultural artifact (Hersh, 1994; Radford, 2007; Roth & Radford, 2011; White, 1947). It is a special kind of artifact, which Wartofsky (1979) calls a “tertiary artifact.” Tertiary artifacts are those which have been “abstracted from their direct representational function” (p. 209) such that they:
come to constitute a relatively autonomous ‘world’, in which the rules, conventions and outcomes no longer appear directly practical, or which, indeed, seem to constitute an arena of non-practical, or ‘free’ play or game activity (…) derived from and related to a given historical mode of perception, [but] no longer bound to it (pp. 208-109).

Hence a cultural perspective enriches the reality principle by explaining how mathematics can constitute a world that is more than just “experienced as real,” but which is actually real, for a given time and within a given culture. As such it allows us avoid the solipsism inherent in Freudenthal’s phenomenological account of mathematical reality. In addition, Wartofsky’s notion of a tertiary artifact expands the usefulness of mathematics, beyond “the understanding and the control” of the physical and social world. Mathematics, like all teritary artifacts, is a vehicle of social change:

Once the visual picture can be ‘lived in’, perceptually, it can also come to color and change our perception of the ‘actual’ world, as envisioning possibilities in it not presently recognized.(…) The upshot [is] that the constructions of alternative imaginative perceptual modes, freed from the direct representation of ongoing forms of action, and relatively autonomous in this sense, feeds back into actual praxis, as a representation of possibilities which go beyond present actualities (Wartofsky, 1979, p. 209)

Because mathematics is means of social control and a vehicle of social change, learning mathematics is political. The implication for researchers and designers is to create and study
opportunities for students to develop sociocritical literacies (Gutiérrez et al., 2009) and critical agency (Gutstein, 2006) as students live mathematics as a liberatory experience (Radford, 2012).

**Implications to the interaction principle**

A particularly unfortunate consequence of the individualistic focus in RME and the corresponding commodification of knowledge is an impoverishment in the interaction principle. As I described in Section 1, the interaction principle describes interaction as largely transactional, existing solely to enrich the individuals who engage in it. For example, Treffers (1987, p. 249) introduced the interaction principle by stating: “This means that the pupils are also confronted with the constructions and productions of their fellows, which can stimulate them.”

Cobb and colleagues (Cobb & Yackel, 1996; Yackel & Cobb, 1996) greatly enriched the interaction principle when they introduced the notion of a normative classroom mathematical practice that becomes “taken as shared” by participants. Such normative practices exist at a social level, “above” any individual student. Through interaction, participants do more than transact, they also contribute to a “taken as shared” world of normative practices. Cobb and colleagues noted a reflexive relationship between individual and normative ways of understanding. As such, they declined to give primacy to either. Each informs the other, however individual understandings remain private and unknowable by others, and thus social understandings are merely “taken as shared.”

A cultural perspective enriches the interaction principle even more. Interaction is the mechanism by which consciousness is created and objectively real mathematical objects come to have a shared meaning—actually shared, not just “taken as shared.” Interaction has this power because it is, by its very definition, a shared event. In interaction, “we produce action methodically to be recognized for what it is, and we recognize action because it is produced
methodically in this way” (Heritage & Clayman, 2010, p. 10). In other words, interaction requires attunement to the Other. Interaction is a symmetric activity which participants, through successive production of—and reaction to—turns that are Other-oriented, create shared meaning. Here is where mathematical consciousness emerges. Consciousness doesn’t pre-exist, or stand in reflexive relation to the social, but rather it is produced in the shared social world:

“consciousness constitutes, according to the etymology of the word, ‘knowing’ (Lat. sciēre) ‘together’ (Lat. con-)” (Roth & Radford, 2011, p. 141). This is how mathematics can have an objective meaning. Participants work together in interaction to make culturally objective mathematical knowledge emerge in their shared consciousness (Roth & Radford, 2010, 2011).

In interaction, then, individuals are connected to the cultural world. Interaction is both the route by which the mathematical world becomes a place that can be “lived in” and the route by which this world “feeds back into actual praxis, as a representation of possibilities which go beyond present actualities” (Wartofsky, 1979, p. 209). Through interaction, people take their place as a “presence in the world” (Friere, 2004, p. 74; cf., Radford, 2012):

That is to say, to become individuals who are more than in the world, individuals who relate to each other, intervene, transform, dream, apprehend, and hope. Becoming a presence in the world is not a natural process; it occurs against the background of history and culture. (…) And presence in the world is not about fleeing from cultural forms of thinking because they are not ours, because others have formed them before us. On the contrary, presence in the world requires the critical encounter with, and immersion in, those always evolving cultural-historical forms of thinking (Radford, 2012, p. 110).
For designers, the implication is to design interactive experiences that are more than simply opportunities for students to share and confront mathematical productions. More important is that students labor together such that they come to understand themselves and others as cultural, historical, and political persons. Such an understanding “consists not only of the mathematics ideas that students express in speech and deeds. Understanding is certainly this, but it is much more too. It is the understanding of another presence, and as such goes beyond the cognitive realm” (Radford, 2012, p. 111).

A new principle: The producer principle

As this paper has made clear, from a cultural perspective learning is much more than an epistemic activity. Ontology is implicated as well. Artifacts act back on people, interaction produces a shared consciousness, and through interactive activity, students become a presence in the common world. The upshot is that people are shaped and produced as particular kinds of people as they engage in joint mediated activity, even as people produce and shape the activity.

From this standpoint, identity is multiply-implicated: who students are, and who students are becoming are primary concerns. Students come to school with particular repertoires of practice (Gutiérrez & Rogoff, 2003), and having been produced in particular ways by their previous experiences. Up to now, RME has incorporated students’ histories via Freudenthal’s called for designers focus on creating contexts that are “experientially real” (Gravemeijer & Doorman, 1999, p. 111) for students, but I suggest that we can and should go further to design experiences that are meaningful, not just in a classroom but in students’ lives. The implication is

7 I realize that the phrase “who student are” conveys a static notion of identity. In using this term, I don’t mean to convey such a notion, exactly. On the one hand, people have relatively durable subjectivities that have been produced across time (Bourdieu, 1972; Halperin, 2012). On the other hand, people actively create and recreate themselves—and are being created and recreated—as they participate in activity (Dreier, 1999, 2009).
that designers should understand both the social realities of their students as well as the history and valued practices of the local community (Martin, 2007), and incorporate students’ repertoires and histories meaningfully into classroom activities (Esmonde & Caswell, 2010).

We also need to consider teloi (endpoints) beyond epistemic objectives. This requires a new way of thinking about mathematics within RME, which has, up to this point, theorized mathematics as an activity and as a product. In addition, instructors, designers, and researchers in RME ought to also consider mathematics as a producer, and learning and doing mathematics as involving the production of persons (Bishop, 1991; Packer & Goicoechea, 2000; Radford, 2008b) The questions for instructors, designers, and researchers are, what sort of person is consistent with the vision that guides RME? What sort of people do we want our students to become (Packer, 2001)? I consider this question to be the most pressing question for future research in RME.

6. Conclusion

I opened this paper with a quote from Keono Gravemeijer (1999, p. 159), in which he states that, within RME-based research, “each time the research question is: What would mathematics education, which fulfills the initial points of departure, look like[?]” This paper has been my response to that question.

Within RME, mathematics-as-product has been theorized as a collection of “thought objects” and learning has been taken up as the acquisition of “private knowledge.” I have demonstrated that such a position is in tension with the first principles of RME, namely the activity principle and the reinvention principle. I resolved this tension by showing that mathematical objects are cultural objects, and by showing how a cultural perspective, which accounts for the constituent role of artifacts in human activity, is necessary if we are to “fulfill
the initial points of departure” of RME. I exemplified this position with an empirical example of
the quintessential form of mathematizing in RME: reinvention.

A cultural perspective enriches RME’s instructional principles, with implications to the
activity principle, the reality principle, and the interaction principle. I have also suggested that a
new principle ought to be incorporated into RME: the producer principle, which states that
learning mathematics involves the production of persons. Exploring and refining these
implications is the next frontier for research in Realistic Mathematics Education.
CHAPTER 5

CONCLUSION

This dissertation opened with a quote from Keoko Gravemeijer (1999, p. 159), explaining that, within RME-based research, “each time the research question is: What would mathematics education, which fulfills the initial points of departure, look like[?]” This dissertation has been my response to that question.

The initial point of departure in RME is that mathematics is the human activity of mathematizing—that is, of structuring the world mathematically. From this point of departure, researchers and designers have developed six core principles of RME. My overall argument in this dissertation is that a cultural perspective, which takes all human activity to be inexorably intertwined with cultural artifacts, is a necessary consequence of the point of departure and the six principles of RME.

I make the argument for the necessity of a cultural perspective in RME most forcefully in Chapter 4. In that chapter, I substantiate two key claims: First, I showed that mathematizing is always mediated by mathematical productions. Second, I showed that mathematical productions are cultural artifacts. Taken together, these claims put culture “in the middle” of mathematical activity, and point to the necessity of adopting a cultural perspective within RME. Below, I revisit each claim, summarizing the evidence that I marshaled in Chapter 4, and adding evidence and conceptual tools from Chapters 2 and 3.
Claim 1: mathematizing is a mediated activity

In Chapter 4, I drew upon the following definitions of mediated actions and mathematizing to make the claim that mathematizing is a mediated activity:

*Definition of mediated activity:* [Mediated actions] involve not a direct action on the world, but an indirect action, one that takes a bit of material matter used previously and incorporates it as an aspect of action. (Cole & Wertsch, 1996, p. 252).

*Definition of mathematizing:* a form of organizing [activity] that also incorporate[s] mathematical matter (Gravemeijer & Terwel, 2000, p. 781).

Despite the somewhat technical-sounding term, “mediation” is actually a very simple concept. As Cole and Wertsch make clear in the first quote, anytime an activity incorporates previously used matter, then that activity is a mediated activity. What about the activity of mathematizing? As Gravemeijer and Terwel explain, mathematizing is an activity that *incorporates* mathematical matter. Thus mathematizing is, *by definition*, a mediated activity. To be blunt: mathematizing is always mediated by mathematical productions.

Chapters 2, 3, and 4 each provide empirical examples of how mathematizing is a mediated activity. Chapter 2 showed how a particular kind of mathematical production, a *preformal* production, played important meditational roles in student's mathematical activity. When students incorporated preformal productions into their activity, students were able to solve problems meaningfully in ways that they could not when incorporating informal or formal productions. For example, we described how students incorporated the preformal bar model and
partition-distribute-iterate strategy into fair sharing problems when informal strategies broke down, and when formal mathematics was not meaningful.

Chapter 3 gave many examples of how students incorporated mathematical productions into their activity. For example, I explained how students mathematized the Bluefin tuna problem by incorporating function tables and the algebraic ratio into their activity. Finally, Chapter 4 described in detail how students incorporated the find-one strategy, ratio tables, rate of change, and the fraction-as-quotient sub-construct into their activity as they mathematized the window problem.

Thus, across chapters 2-4 I made a theoretical and empirical argument that mathematizing is mediated by mathematical productions. In addition, I provided evidence that preformal productions play a particularly powerful meditational role in students’ mathematical activity. Below I turn to my second claim, which addresses the ontological status of mathematical productions in RME.

**Claim 2: mathematical productions are cultural artifacts**

Within RME, mathematical productions have been theorized as “thought objects” (Freudenthal, 1983, p. 33), and “private knowledge” (Gravemeijer, 1999, p. 158). In Chapter 4, I demonstrated that such an ontological position is in tension with the activity principle and the reinvention principle in RME. I resolved this tension by showing that mathematical objects are cultural artifacts. In particular, I showed that, together, the activity principle and the reinvention principle place a number of restrictions on the ontological nature of mathematical productions. On the one hand, productions are the product of human activity so we have to reject the idea that they exist in some metaphysical Platonic structure, independent of humans. On the other hand they are durable and shared, existing across time and outside of any single individual. This
means we have to reject the idea that productions are individual possessions. The only objects that meet these criteria (products of human activity that are durable and shared across individuals and time) are cultural artifacts: “socially inherited (extragenetic) accomplishments of past human activities that serve as crucial resources for the current life of a social group” (Cole, 2010, p. 462).

Chapters 2 and 3 can be read as empirical case studies for how, in learning sequences designed using RME principles, mathematical productions emerge as products of human activity, and how they come to serve as resources for future activity. Chapter 2 documented in detail how the preformal find-one strategy embodied historic classroom activity oriented around “many-as-one” (Confrey et al., 2009) fair sharing problems, and social interaction that focused on the oneness of the word “per.” Similarly, in Chapter 3, I documented how “rate of change” came to embody classroom social interaction (e.g., in the way that rate of change acquired a particular meaning through a class discussion of the iPhone problem), and classroom activity (e.g., in the way that rate of change acquired new meanings as it participated in different activities such as the window problem).

Each of these chapters also described how artifacts serve as resources for the social group (the students in the classroom). Artifacts mediated problem-solving activity, as described above, and they also mediated the reinvention of new mathematical artifacts such that future artifacts embody prior artifacts. Chapter 2 described how formal artifacts embody the preformal artifacts that came before them. For example, the (formal) division operation came to embody the preformal find-one strategy. Chapter 3 examined the process through which existing artifacts mediate the reinvention of new artifacts in more detail. This process involves coordinating and assembling existing artifacts, and I introduced the notion of a cascade of artifacts to describe and
represent the general phenomenon of how new artifacts are reinvented as coordinated assemblies of existing artifacts, through activity and social interaction.

**Putting the claims together: The necessity of a cultural perspective in RME**

*Claim 1:* Mathematizing is always mediated by mathematical productions

*Claim 2:* Mathematical productions are cultural artifacts

Taken together, these claims put culture “in the middle” of mathematical activity. Culture is the medium in which mathematical activity takes place, and it is a constituent part of all mathematical activity. This matters for RME because the very point of departure in RME is that mathematics is a human activity. By showing that mathematics is a *culturally mediated* activity, I have shown that culture must necessarily be central to RME.

Chapters 2 and 3 provide empirical evidence for this position, as well as tools for researchers and designers in RME who adopt a cultural perspective. Chapter 2 orients researchers and designer to the constituent role of preformal productions in mathematical activity—including the reinvention of formal mathematics. Chapter 3 introduces the notion of a *cascade of artifacts*, which shows both the process and product of learning from a cultural perspective.

In Chapter 4 I sketch some potential implications of a cultural perspective on RME principles. In particular, I call on RME researchers to reject the individualist perspective that has permeated RME in favor of a more expansive perspective. Two decades ago, Hutchins (1995a, p. 169) made a similar call for cognitive science in general:
In watching people thinking in the wild, we may be learning more about their environment for thinking than about what is inside them (...) [But] we should not pack up and leave, concluding we cannot learn about cognition here.

In this dissertation I have shown that, similarly, as we watch people mathematize, we may be learning more about their cultural environment for mathematizing that about what is inside them. But RME researchers should not pack up and leave. Instead, this dissertation is an invitation to RME researchers to study “artifacts as integral and inseparable components” (Engeström, 1999, p. 29) of mathematizing.

Such a perspective dissolves dualist barriers between persons and their world. This leads to perhaps the most profound implication of this dissertation: Mathematics produces people even as people produce mathematics. Thus, adopting a cultural perspective in RME entails a shift from a purely epistemic focus to incorporate an ontological focus (Packer & Goicoechea, 2000; Radford, 2008b, 2012). Up to now, RME research to now has been guided by the question of what kind of education fulfills the initial point of departure in RME. I suggest that it is time for a new guiding question: “what kind of person fulfills the initial point of departure in RME?” Exploring this question is the next challenge for RME researchers.
References


