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RADIATION LOSS FROM A CURVED DIELECTRIC CHANNEL WAVEGUIDE IN A DENSE SUBSTRATE

by

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CHAPTER 1
Prologue

1.1 Study Objective and Methodology

The primary objective of this investigation was to analytically determine the effect of a plane dielectric interface upon the radiation loss of a curved channel dielectric waveguide embedded in a dense substrate and bent in a plane parallel to that of the material interface. The radiation loss from bent dielectric waveguides is of interest for a variety of potential applications in integrated optics. Knowledge of the factors influencing the radiation from bend dielectric waveguides will allow the design of integrated optical component characteristics which, in one application, may result in decreased radiation to conserve signal energy or which, in another application, may achieve enhanced radiation effects in order to optimize the coupling between various optical waveguides. An obvious example for which the conservation of signal energy is of prime concern is the dielectric fiber transmission line used in simple point-to-point data transfer applications.

Examples of integrated optical components in which some specific level of radiation is desired include optical filters (e.g., ring resonator discriminators; cf. [1]) and optical directional couplers (cf. [2]). Figure 1.1 illustrates potential realizations of these latter components.

Arnaud (cf. [3]) has treated radiation loss from a slab-loaded, bent dielectric rod, and this structure is the one described in the literature that has the most in common with the configuration considered herein.
(a) A RING RESONATOR DISCRIMINATOR

(b) A DIRECTIONAL COUPLER

Figure 1.1 Dielectric Channel Waveguides in Potential Integrated Optics Applications
However, primarily because of the finite extent of the slab and the fact that the refractive indices of the slab and the rod are equal, Arnaud's problem differs significantly from that discussed in this report. Indeed, the differences are substantial enough to preclude extensive corroboration of either study's results by direct comparison.

Figure 1.2 illustrates the fundamental geometry of the problem and depicts the coordinate reference frames chosen for the analysis. The refractive indices of the channel waveguide, the substrate material, and the covering material have the values $n, n_2, \text{and } n_1$, respectively. These refractive indices are ordered according to the relation $n > n_2 \gg n_1$, the relation $n_2 \gg n_1$ expressing what is meant by "dense" as applied to the substrate material. A global, circular-cylindrical reference frame with spatial coordinates $r, \phi, \text{and } z$ has its origin at the point $0$, and a local cartesian reference frame with spatial coordinates $x, y, \text{and } s$ has its origin at the point $(R_0, 0, 0)$ relative to the global frame. The extent of the channel waveguide is defined by the radial boundaries at $r'''$ and $r''''$ and by the coordinates $z = 0$ and $z = z''$, which define horizontal planes between which the guide is confined. The quantity $R_0$ is given by the expression $R_0 = (r' + r'''')/2$ and is defined to be the radius of curvature of the channel waveguide.

A propagating mode with phase factor $\exp(\omega t - k_0 r R_0 \phi)$ is assumed to exist, where $k_0 = \omega \sqrt{\mu_0 \varepsilon_0}$ is the free space propagation constant and $\gamma$ is the normalized (to $k_0$), complex propagation constant of the mode in the curved channel waveguide. Given that the normalized propagation constant for the straight channel waveguide is $\gamma_0$, the analysis proceeds to compute a first order representation of $\gamma$ of the form

$$\gamma = \gamma_0 - i\alpha, \quad \alpha > 0$$  \hspace{1cm} (1.1)
Figure 1.2 A Curved Channel Waveguide Embedded in a Dense Substrate
where $\alpha$ is the normalized attenuation coefficient of the mode. The mode attenuation due to continuous radiation from the bend following propagation through an angular sector $\phi$ of the channel guide is then given by the factor $\exp(-\alpha k_0 R_0 \phi)$.

Once $\alpha$ has been determined, the study objective is attained via a comparison of $\alpha$ with the normalized attenuation constant of a bent channel waveguide embedded in a homogeneous medium.

1.2 Summary of Results

The normalized attenuation constant $\alpha'$ of a bent, rectangular dielectric waveguide immersed in a homogeneous medium has the general form [5]

$$\alpha' = \frac{C_h}{(k_0 R_0)^{3/2}} e^{k_0 R_0 Q(0)}$$

where $C_h > 0$ and $Q(0) < 0$ are parameters depending upon material characteristics of the guide (including physical dimensions) and its surrounding medium as well as the specific mode assumed to be propagating in the guide. As a result of this investigation, the normalized attenuation constant $\alpha$ for the bent, rectangular dielectric waveguide embedded in a dense substrate below a material interface is found to be of the form

$$\alpha = \frac{C_i}{(k_0 R_0)^{3/2}} e^{k_0 R_0 q_2(0)}$$

where $C_i > 0$ and $q_2(0) < 0$ depend upon the channel guide, substrate, and covering medium characteristics (including the physical dimensions of the guide) and upon the specific mode assumed to be propagating in the guide.
Three observations are of importance. First, the quantities $C_h$ and $C_i$ are in general unequal, as are $Q(0)$ and $q_2(0)$ (although a certain formal similarity between the latter two exists). Second, the fundamental exponential dependence of both $\alpha'$ and $\alpha$ on $R_o$ is the same. Third, the algebraic dependence of $\alpha'$ and $\alpha$ on $R_o$ is distinctly different. The $(\frac{1}{R_o})^{3/2}$ algebraic dependence of $\alpha$ upon $R_o$ arises as the leading term of the asymptotic expansion of $\alpha$ and thus imparts a lateral wave characteristic (cf. 4, Section 5.5) to the radiation loss of a rectangular dielectric guide in a dense substrate below a material interface. Indeed, the absence of a $(\frac{1}{R_o})^{3/2}$ leading term in the asymptotic expansion of $\alpha$ may be thought of as resulting from the first order cancellation of source and image field due to the presence of the interface.

As a means of comparing the influence of the factors $\frac{C_h}{(k_o R_o)^{3/2}}$ and $\frac{C_i}{(k_o R_o)^{3/2}}$ on the radiation loss of their associated waveguiding structures, $\alpha'$ and $\alpha$ are plotted versus $k_o R_o$ for bent, rectangular dielectric waveguides whose dimensions and/or refractive indices are chosen so that $Q(0) = q_2(0)$. This not only assures that the exponential variation with $R_o$ will be the same in each case, but it also corresponds to a situation in which propagation constant along the central axis of each guide is the same. For the cases considered, $\alpha$ is found to be always less than $\alpha'$ and is, in particular, less than an order of magnitude smaller than $\alpha'$ for $k_o R_o = 500$ and between one and two orders of magnitude smaller than $\alpha'$ for $k_o R_o = 5000$. This indicates that one may in some instances achieve a significant reduction in the radiation loss of a bent, rectangular dielectric waveguide by embedding it below a high contrast dielectric interface in the material of greater refractive index.
1.3 Analytical Preliminaries

This report will not present a derivation of the general form of \( \alpha \), but will merely quote the result as given in [5], Chapter 9. Thus

\[
\alpha = \frac{C}{k_0 P} \tag{1.2}
\]

where

\[
P = 2 \int \overline{a} \cdot \overline{E}_0 \times \overline{H}_0 \, ds \tag{1.3}
\]

\[
C = \left. \int \left( \overline{E}_0^- \times \overline{E}_0^- - \overline{E}_0^- \times \overline{H}_0^- \right) \cdot \overline{a} \, dy \, dz \right|_{y=y_0} \tag{1.4}
\]

In (1.3), the integration extends over an infinite plane containing a cross section of the channel waveguide, and \( y_0 \) is some point chosen outside the bend (beyond \( r'' \)) of the curved channel guide where the fields have decayed sufficiently (cf.[2]). The fields \( \overline{E}_0 \) and \( \overline{H}_0 \) are those of the straight channel guide, and \( \overline{E}_0^- \) and \( \overline{H}_0^- \) are the fields of the straight channel traveling in the "negative" \( \phi \) (or \( s \))-direction. The fields \( \overline{\delta E} \) and \( \overline{\delta H} \) constitute that portion of the fields outside the curved channel guide which have been reflected from the caustic or turning point beyond which the mode must radiate in a radially outward direction. The fundamental analytical task of this investigation was that of expressing the fields \( \overline{\delta E} \) and \( \overline{\delta H} \) in terms of known quantities, i.e., in terms of \( \overline{E}_0, \overline{H}_0, y_0, R_0 \), etc.

As a result of the circular-cylindrical symmetry of the global reference frame, our subsequent analysis will be facilitated by concentrating on the field components \( E_z \) and \( H_z \), although we must eventually consider all of the field components. Via Maxwell's equations and standard
analysis for circular-cylindrical geometry, we find that we may express the four remaining field components in terms of $E_z$ and $H_z$, whence

$$ (k^2 + \frac{\alpha^2}{\alpha z^2}) E_\phi = i \zeta_0 k_0 \frac{\partial H_z}{\partial r} - i \gamma k_0 \frac{R_0}{r} \frac{\partial E_z}{\partial z} $$  \hspace{1cm} (1.5a) 

$$ (k^2 + \frac{\alpha^2}{\alpha z^2}) E_r = - \zeta_0 \gamma k_0^2 \frac{R_0}{r} H_z + \frac{\alpha^2}{\partial r \partial z} E_z $$  \hspace{1cm} (1.5b) 

$$ (k^2 + \frac{\alpha^2}{\alpha z^2}) H_\phi = -i \frac{n^2}{\zeta_0} k_0 \frac{\partial E_z}{\partial r} - i \gamma k_0 \frac{R_0}{r} \frac{\partial H_z}{\partial z} $$  \hspace{1cm} (1.5c) 

$$ (k^2 + \frac{\alpha^2}{\alpha z^2}) H_r = \frac{n^2}{\zeta_0} \gamma k_0^2 \frac{R_0}{r} E_z + \frac{\alpha^2}{\partial r \partial z} H_z $$  \hspace{1cm} (1.5d) 

where $\zeta_0 = \sqrt{\mu_0 / \varepsilon_0}$, $k^2 = n_1^2 \gamma k_0^2$ for $z > 0$, $k^2 = n_2^2 k_0^2$ for $z < 0$.

We will show in Chapter 2 that both $E_z$ and $H_z$ have Fourier-Bessel representations of the form

$$ E_z(r, \phi, z) = e^{-i \nu \phi} \int_0^\infty E_z(\alpha, z) J_\nu(k_0 r \alpha) d\alpha $$  \hspace{1cm} (1.6a) 

$$ H_z(r, \phi, z) = e^{-i \nu \phi} \int_0^\infty H_z(\alpha, z) J_\nu(k_0 r \alpha) d\alpha $$  \hspace{1cm} (1.6b) 

All of the desired field components have similar representations, and the $z$-variation is such as to yeild the following equivalence of operators

$$ \left( \frac{\partial^2}{\partial z^2} + k^2 \right) = k_0^2 \alpha^2 $$  \hspace{1cm} (1.7) 

for the field transforms.

For example, consider the electric field component $E_\phi$ given by

$$ E_\phi = e^{-i \nu \phi} \int_0^\infty E_\phi(\alpha, z) J_\nu(k_0 r \alpha) d\alpha $$  \hspace{1cm} (1.8)
\( \tilde{E}_\phi(\alpha,z) \) is the field transform of \( E_\phi \) and \( \left( \frac{\partial^2}{\partial z^2} + k^2 \right) \tilde{E}_\phi(\alpha,z) = \frac{k_0^2 \alpha \tilde{E}_\phi}{\gamma} \).

Substitution of (1.6a), (1.6b), and (1.8) into (1.5a) and making use of (1.7) gives

\[
\alpha^2 \tilde{E}_\phi = i \zeta_o \alpha \frac{J'_v(k_o \alpha r)}{J_v(k_o \alpha r)} \tilde{H}_z - \frac{i \gamma R_o}{r k_o} \frac{\partial}{\partial z} \tilde{E}_z . \tag{1.9a}
\]

In a similar fashion we find the following as well:

\[
\alpha^2 \tilde{E}_r = - \frac{\zeta_o \gamma R_o}{r} \tilde{H}_z + \frac{\alpha}{k_o} \frac{J'_v(k_o \alpha r)}{J_v(k_o \alpha r)} \frac{\partial}{\partial z} \tilde{E}_z \tag{1.9b}
\]

\[
\alpha^2 \tilde{H}_\phi = - \frac{i \gamma R_o}{r k_o} \frac{\partial}{\partial z} \tilde{H}_z - i \frac{\alpha n_{1,2}^2}{\zeta_o} \frac{J'_v(k_o \alpha r)}{J_v(k_o \alpha r)} \tilde{E}_z \tag{1.9c}
\]

\[
\alpha^2 \tilde{H}_r = \frac{\gamma R_o n_{1,2}^2}{r \zeta_o} \tilde{E}_z + \frac{\alpha}{k_o} \frac{J'_v(k_o \alpha r)}{J_v(k_o \alpha r)} \frac{\partial}{\partial z} \tilde{H}_z \tag{1.9d}
\]

The relationships (1.10a) through (1.10d) between the field (Fourier-Bessel) transforms will be of use in Chapter 2.
CHAPTER 2

INTEGRAL REPRESENTATIONS OF THE
CARTESIAN FIELD COMPONENTS

2.1 Introduction

This chapter is devoted to finding integral representations for the Cartesian field components $E_z$ and $H_z$. As mentioned in Chapter 1, knowledge of these allows one to determine the remaining four field components via (1.5a) through (1.5d). The components $E_z$ and $H_z$ are preferred by the circular-cylindrical geometry of our problem to the extent that they both satisfy the homogeneous (in source free regions), scalar wave equation, i.e., the Helmholtz equation. This fact, together with the assumed form of the field variation with the spatial coordinate $\phi$, allows specification of the Fourier-Bessel transforms of $E_z$ and $H_z$ to within multiplicative factors which must be determined through the application of boundary and source conditions. Explicit evaluation of these is the subject of Section 2.3.

Although at first glance the final form of the integral representations of $E_z$ and $H_z$ seems too cumbersome for practical use due to the complexity of the required integrations, it will be seen in Chapter 3 that the integral representations derived in this chapter are amenable to asymptotic analysis when $k_0 R_0 \gg 1$, i.e., when the radius of curvature of the bend channel guide is large relative to the wavelength of the propagating fields.
2.2 Extended Fourier-Bessel Representations for Field Solutions of the
Helmholtz Equation

Let \( F(r,\phi,z) = \phi(r,z)e^{-i\nu \phi} \) be a scalar field satisfying the
Helmholtz equation for source-free regions,

\[
(\nabla^2 + k^2)F(r,\phi,z) = 0, \quad \text{then for } \nu = k_o \gamma R_o,
\]

\[
\left\{ \frac{1}{r} \frac{\partial}{\partial r} \left[ r \frac{\partial}{\partial r} \right] + \frac{\partial^2}{\partial z^2} + k^2 - \frac{\gamma^2 k_o^2 R_o^2}{r^2} \right\} \phi(r,z) = 0
\]  

(2.1)

where, as before, \( k^2 = n_1^2 k_o^2 \) for \( z > 0 \), and \( k^2 = n_2^2 k_o^2 \) for \( z < 0 \).

In this section, \( \phi(r,z) \) is a generic symbol representative of either
\( E_z(r,z) \) or \( H_z(r,z) \), since the mathematical manipulations required herein
do not require us to distinguish these two field components. Appropriate
associations between \( E_z(r,z) \) and \( H_z(r,z) \) and the results of this section
will be made as required in later sections.

From the form of \( F(r,\phi,z) \) we know that there is a Fourier-Bessel
transform representation (cf. [6]) for \( \phi(r,z) \), whence

\[
\phi(r,z) = \int_0^\infty \tilde{\phi}_\nu(\alpha,z)J_\nu(k_o \alpha r) \alpha d\alpha
\]  

(2.2)

\[
\tilde{\phi}(\alpha,z) = k_o^2 \int_0^\infty \phi(r,z)J_\nu(k_o \alpha r) r dr
\]  

(2.3)

where \( J_\nu \) is a Bessel function of order \( \nu \). We shall refer to \( \phi(r,z) \)
as the field and to \( \tilde{\phi}(r,z) \) as the field transform.

Substitution of (2.2) into (2.1) produces the result

\[
\left\{ \frac{\partial^2}{\partial z^2} + k_o^2(n_1^2 - \alpha^2) \right\} \tilde{\phi}_\nu(\alpha,z) = 0
\]  

(2.4)
where \( n' = n_1 \) for \( z > 0 \), and \( n' = n_2 \) for \( z < 0 \).

If we consider \( z = z' \), where \( z'' < z' < 0 \), as a plane in which point sources lie, and if we likewise require that the field should vanish for \( |z| \to \infty \), then (2.4) has solutions

\[
\tilde{\phi}_v(\alpha, z) = A(\alpha)e^{-k_0 u_1 z}, \quad z \geq 0
\]  

(2.5)

\[
\tilde{\phi}_v(\alpha, z) = B_p(\alpha)e^{-k_0 u_2 z} + B'_p(\alpha)e^{k_0 u_2 z} \quad z' \leq z \leq 0
\]  

(2.6)

\[
\tilde{\phi}_v(\alpha, z) = C(\alpha)e^{k_0 u_2 z}, \quad z \leq z'
\]  

(2.7)

where

\[
u_1 = (\alpha^2 - n_1^2)^{1/2}, \quad \text{Re} \, u_1 > 0
\]  

(2.8)

\[
u_2 = (\alpha^2 - n_2^2)^{1/2}, \quad \text{Re} \, u_2 > 0
\]  

(2.9)

With (2.5), (2.6), and (2.7) we have determined the field transforms except for a set of multiplicative factors dependent upon \( \alpha \). In the following sections we will apply source and boundary conditions to determine these factors. At that time it will be necessary to identify our field \( \phi(r, z) \) as either the electric field \( E_z(r, z) \) or the magnetic field \( H_z(r, z) \).

To facilitate the asymptotic analysis of Chapter 3, it is desirable to extend the range of integration in (2.2) to the negative \( \alpha \) axis. To do this we recall that

\[
J_v(\alpha) = \frac{1}{2}(H^1_v(\alpha) + H^2_v(\alpha))
\]

and employ the circuital relations for the Bessel functions so that (2.2) becomes
\[ \phi(r,z) = \frac{1}{y} \int_{\infty}^{\infty} \tilde{\phi}(\alpha,z) H^{(2)}_\nu(k_0 \alpha \alpha) d\alpha \quad (2.10) \]

Figure 2.1 illustrates the integration contour of \( \phi(r,z) \) in the complex \( \alpha \)-plane. The branch cuts associated with \( n_1 \) and \( n_2 \) are those which must not be crossed if we are to insure that \( \text{Re } u_1 > 0 \) and \( \text{Re } u_2 > 0 \). Some slight loss is assumed in both the covering medium and the substrate in order to impart a small imaginary component to both \( n_1 \) and \( n_2 \). The branch cut running along the negative real axis insures that \( H^{(2)}_\nu(k_0 \alpha r) \) in (2.10) is single-valued.

2.3 Evaluation of the Field Transforms

2.3.1 Equivalent Polarization Sources

The concept upon which the analysis of this section rests is that of equivalent polarization currents (cf. [7]) which can be used to convert the sourceless three medium problem into an equivalent two medium problem with sources below the dielectric interface. This approach attributes the fields existing outside the dielectric channel to the radiation from an array of polarization current sources contained within the geometrical boundaries of the channel waveguide but embedded in the substrate medium only. This circumstance is illustrated in Figure 2.2.

The justification for this procedure follows from a consideration of the differential form of Ampere's Law, i.e., the curl \( \mathbf{H} \) relation from Maxwell's equations. In the channel region, we have

\[ \nabla \times \mathbf{H} = i \omega n^2 \varepsilon_0 \mathbf{E} \quad (2.11) \]

and in the substrate region the fields must satisfy
$\alpha$-PLANE
$-\pi \leq \arg \alpha \leq \pi$

Figure 2.1 The Contour of Integration for $\Phi(r,z)$ in the Complex $\alpha$-Plane
Figure 2.2 Alternative, Equivalent Problem Representations
\[ \nabla \times \overline{H} = i w n_{20}^2 \varepsilon \overline{E} \quad (2.12) \]

In (2.11) we add and subtract \( j w n_{20}^2 \varepsilon \overline{E} \), then (2.11) becomes

\[ \nabla \times \overline{H} = i w (n_{20}^2 - n^2) \varepsilon_{o} \overline{E} + i w n_{20}^2 \varepsilon \overline{E} \quad (2.13) \]

We now identify the first term on the RHS of (2.13) as the equivalent polarization current density \( \overline{J}_p \), thus

\[ \overline{J}_p = i w (n_{20}^2 - n^2) \varepsilon_{o} \overline{E} \quad (2.14) \]

and (2.13) is now

\[ \nabla \times \overline{H} = \overline{J}_p + i w n_{20}^2 \varepsilon \overline{E} \quad (2.15) \]

Comparison of (2.15) with (2.12) shows that we may now characterize the fields below the dielectric interface through the use of (2.15) alone, as long as we keep in mind that the equivalent polarization currents exist only within the region occupied by the channel guide. Since the only refractive index that appears explicitly in (2.15) is \( n_2 \), our problem is seen to be equivalent to that of a two-medium interface problem with sources as illustrated in Figure 2.2b.

It should be emphasized again that to solve the problem discussed herein we assume that the fields \( \overline{E}_o \) and \( \overline{H}_o \) in the channel guide are known, and the implication of this assumption in the present discussion is that \( \overline{J}_p \) will be given explicitly as

\[ \overline{J}_p = i w (n_2^2 - n^2) \varepsilon_o \overline{E}_o \quad (2.16) \]

We have thus converted information about known channel fields into a known distribution of equivalent current sources. Since \( \overline{E}_o \) will in general be
capable of resolution along each of the three mutually perpendicular axes of the local coordinate system (cf. Figure 1.2) (2.16) reveals that we will in general have polarization currents directed along all three of the local coordinate axes. The remainder of this chapter is devoted to finding the field transform amplitudes associated with each unique component of the polarization currents. The total electric and magnetic fields outside the channel will ultimately be constructed via superposition of the results obtained for the special cases to be considered currently.

2.3.2 \textit{z-Directed Phased Arrays of Polarization Currents}

We consider a \textit{z-directed}, phased array of polarization currents located at a radius \( r' \) from the \( z \)-axis and a distance \( z' \) below the dielectric interface as illustrated in Figure 2.3. Analytically, polarization line current densities of unit amplitude along the \( z \)-direction are given by

\[
\hat{J}_p = a_z \delta (z-z') \delta (r-r') e^{-i \nu \phi}
\]  

(2.17)

hence reference to (2.16) gives

\[
\bar{J}_p = i w (n^2 - n_z^2) e_0 E_{0z} \hat{J}_p dr'dz'
\]

(2.18)

To facilitate the analysis of this section we shall consider a \textit{z-directed} electric type Hertzian potential of the form

\[
\Pi e_{(r,\phi,z)} = a_z e^{-i \nu \phi} \int_0^{\infty} \Pi e_{(\alpha,z)} J_{\nu}(k_0 r \alpha) d\alpha,
\]

so that

\[
\Pi e_{z}(r,z) = \int_0^{\infty} \Pi e_{(\alpha,z)} J_{\nu}(k_0 r \alpha) d\alpha
\]

(2.19a)
Figure 2.3 A z-Directed Phased Array of Polarization Currents
and

\[ \frac{\tilde{\Pi}^e_{z}(\alpha, z)}{z} = k_0^2 \int_0^\infty \frac{\tilde{\Pi}^e_z(r, z) J_0(k_0 r \alpha) r dr}{r} \]  \hspace{1cm} (2.19b)

In this section we will be dealing with the cartesian component of the electric field only, because \( H_z = 0 \) for a z-directed Hertzian potential.

The Hertzian potential satisfies the inhomogeneous wave equation (cf. [8])

\[ (\nabla^2 + k_{1,2}^2) \tilde{\Pi}^e_{z}(r, z) = \frac{-j}{iw \varepsilon_0 n_2^2} \]  \hspace{1cm} (2.20)

where \( k_{1,2} = n_{1,2} k_0 \), thus using (2.17) in (2.20)

\[ (\nabla^2 + k_{1,2}^2) \tilde{\Pi}^e_{z}(r, z) = -\left( \frac{1}{iw \varepsilon_0 n_2^2} \right) \delta(z-z') \delta(r-r') \]  \hspace{1cm} (2.21)

Substitution of (2.19a) into (2.21) and use of the Fourier-Bessel representation of \( \delta(r-r') \) yields

\[ \left\{ \frac{\partial^2}{\partial z^2} - k_{1,2}^2 \right\} \tilde{\Pi}^e_{z}(\alpha, z) = \left[ \frac{-k_0^2 r'}{iw \varepsilon_0 n_2^2} \right] J_0(k_0 r' \alpha) \delta(z-z') \]  \hspace{1cm} (2.22)

where, as in (2.8) and (2.9), \( u_{1,2} = (\alpha^2 - n_{1,2}^2)^{\frac{1}{2}} \).

Following the same line of reasoning as in Section 2.2, we have the following solutions for \( \tilde{\Pi}^e_{z}(\alpha, z) \) from (2.22):

\[ \tilde{\Pi}^e_{z}(\alpha, z) = A(\alpha) e^{-k_0 u_1^2 z}, \quad z > 0 \]  \hspace{1cm} (2.23a)

\[ \tilde{\Pi}^e_{z}(\alpha, z) = B_p(\alpha) e^{-k_0 u_2^2 z} + B_n(\alpha) e^{k_0 u_2^2 z}, \quad z' \leq z \leq 0 \]  \hspace{1cm} (2.23b)

\[ \tilde{\Pi}^e_{z}(\alpha, z) = C(\alpha) e^{-k_0 u_2^2 z}, \quad z \leq z' \]  \hspace{1cm} (2.23c)
We solve for the transform coefficients $A(\alpha)$, $B_p(\alpha)$, $B_n(\alpha)$, and $C(\alpha)$ via application of source conditions at $z = z'$ (two equations) and boundary conditions at $z = 0$ (two equations); this yields

\[
A(\alpha) = \frac{u_2 n_2^2}{u_1 n_2^2 + u_2 n_1^2} \left\{ \frac{k_0 J_0(k_0 r' \alpha)}{i u_2 w e_0 n_2^2} \right\} e_0^0 u_2 z'
\]

\[ (2.24a) \]

\[
B_p(\alpha) = \frac{k_0 J_0(k_0 r' \alpha) r'}{i 2 u_2 w e_0 n_2^2} e_0^0 u_2 z'
\]

\[ (2.24b) \]

\[
B_n(\alpha) = \frac{k_0 J_0(k_0 r' \alpha) r'}{i 2 u_2 w e_0 n_2^2} \left\{ \frac{u_2 n_1^2 - u_1 n_2^2}{u_2 n_1^2 + n_1 u_2} \right\} e_0^0 u_2 z'
\]

\[ (2.24c) \]

\[
C(\alpha) = \frac{k_0 J_0(k_0 r' \alpha) r'}{i 2 u_2 w e_0 n_2^2} \left\{ \frac{-k_0 u_2 z' + u_2 n_1^2 - u_1 n_2^2}{u_2 n_1^2 + u_1 n_2} \right\} e_0^0 u_2 z'
\]

\[ (2.24d) \]

We can now use (2.24a) through (2.24d) in (2.23a) through (2.23c) in order to produce the following expressions for the potential transforms

\[
\tilde{\Pi}_z^e(\alpha, z) = \frac{k_0 J_0(k_0 r' \alpha) r'}{i 2 u_2 w e_0 n_2^2} \left\{ \frac{2 u_2 n_2^2}{u_2 n_1^2 + u_1 n_2} \right\} e_0^0 (u_2 z' - u_1 z), \quad z \geq 0
\]

\[ (2.25a) \]

\[
\tilde{\Pi}_z^e(\alpha, z) = \frac{k_0 J_0(k_0 r' \alpha) r'}{i 2 u_2 w e_0 n_2^2} \left\{ \frac{-k_0 u_2 |z| z'|}{u_2 n_1^2 + u_1 n_2^2} + \frac{u_2 n_1^2 - u_1 n_2^2}{u_2 n_1^2 + u_1 n_2^2} e_0^0 (z+z') \right\} \quad z \leq 0
\]

\[ (2.25b) \]
Substitution of (2.25a) and (2.25b) into (2.19a) produces the desired integral representations for the Hertzian potential in the covering region \((z > 0)\) and the substrate region \((z \leq 0)\):

\[
\Pi_z^e(r, z) = -i\xi_0 r' \int_0^\infty \left( \frac{u_2}{u_1 n_2^2 + u_2 n_1^2} \right) J_v(k_0 r' \alpha) J_v(k_0 r \alpha) e^{k_0 (u_2 z' - u_1 z)} \frac{\alpha \alpha \alpha}{u_2} \cdot z > 0 \tag{2.26a}
\]

\[
\Pi_z^e(r, z) = \frac{-i\xi_0 r'}{2n_2^2} \int_0^\infty \left\{ e^{-k_0 u_2 |z-z'|} \left[ \frac{u_2 n_1^2 - u_1 n_2^2}{u_2 n_1^2 + u_1 n_2^2} \right] e^{k_0 u_2 (z+z')} \right\} J_v(k_0 r' \alpha) J_v(k_0 r \alpha) \frac{\alpha \alpha \alpha}{u_2}, \quad z \leq 0 \tag{2.26b}
\]

where we have used \(\frac{w u_0}{k_0} = \xi_0\), the intrinsic wave impedance of free-space.

Proceeding as was done to extend the range of integration of (2.2) to yield (2.10), we may extend the range of integration in (2.26a) and (2.26b) to give

\[
\Pi_z^e(r, z) = \frac{-i\xi_0 r'}{2} \int_0^\infty \left\{ e^{-k_0 u_2 |z-z'|} \right\} H_v^{(2)}(k_0 r' \alpha) J_v(k_0 r \alpha) e^{k_0 (u_2 z' - u_1 z)} \frac{\alpha \alpha \alpha}{u_2}, \quad z > 0 \tag{2.27a}
\]

\[
\Pi_z^e(r, z) = \frac{-i\xi_0 r'}{4n_2^2} \int_0^\infty \left\{ e^{-k_0 u_2 |z-z'|} \right\} H_v^{(2)}(k_0 r' \alpha) J_v(k_0 r \alpha) \frac{\alpha \alpha \alpha}{u_2}, \quad z \leq 0 \tag{2.27b}
\]
2.3.3 \textit{r-Directed Phased Arrays of Polarization Currents}

We consider an \( r \)-directed phased array of polarization currents located at a radius \( r' \) from the \( z \)-axis and at a distance \( z' \) below the dielectric interface as illustrated in Figure 2.4. Figure 2.4b is a cross-sectional view while Figure 2.4a is a view as seen from the covering medium looking down on the \( z \)-axis.

Analytically, the polarization current sources of unit amplitude are given by

\[
\hat{j}_p = \alpha \delta(z-z') \delta(r-r') e^{-i\nu \phi} \tag{2.28}
\]

hence

\[
\hat{J}_p = i\omega(n_2^2-n_2^2) \varepsilon_0 \varepsilon \hat{j}_p \, dr'dz' \tag{2.29}
\]

The approach taken in this section is to consider directly the cartesian components of the electric and magnetic fields. We have seen in Section 2.2 that \( E_z \) and \( H_z \) satisfy the Helmholtz equation in source-free regions and thus that their associated field (Fourier-Bessel) transforms satisfy equations (2.5), (2.6), and (2.7). In light of the results in Section 2.3.2, we will hypothesize from the start the following forms for the field transforms

\[
\tilde{\phi}^{e,m}_r(\alpha,z) = A^{e,m}_r(\alpha) e^{-k_0(u_zz-u_{z'}z')} \quad z \geq 0 \tag{2.30a}
\]

\[
\tilde{\phi}^{e,m}_r(\alpha,z) = B^{e,m}_r(\alpha) e^{k_0u_z(z+z')} + B^{e,m}_n(\alpha) e^{-k_0u_z(z-z')} \quad \text{for} \quad z' \leq z \leq 0 \tag{2.30b}
\]

\[
\tilde{\phi}^{e,m}_r(\alpha,z) = C^{e,m}_r(\alpha) e^{k_0u_zz} \quad z \leq z' \tag{2.30c}
\]

where we will make the associations \( \hat{\phi}^e_r \triangleq \hat{E}_z \) and \( \hat{\phi}^m_r \triangleq \hat{H}_z \), and the subscript
Figure 2.4 An r-Directed Phased Array of Polarization Currents
r indicates that these are field transforms associated with r-directed polarization currents.

In order to compute the eight, as yet unknown coefficients of the field transforms in (2.30a) through (2.30c), we must determine and apply the boundary and source conditions applicable to \( \Phi_r^e \) and \( \Phi_r^m \). This process is presented in detail in Appendix A and yields:

\[
A_r^m = \frac{2u_2}{u_1 + u_2} B_{pr}^m \tag{2.31a}
\]

\[
B_{nr}^m = \frac{u_2 - u_1}{u_2 + u_1} B_{pr}^m \tag{2.31b}
\]

\[
A_r^e = \frac{2u_2n_2^2}{\varepsilon_0 n_2^2 + u_1 n_2^2} B_{p}^e \tag{2.31c}
\]

\[
B_{nr}^e = \frac{u_2 n_1^2 - u_1 n_2^2}{u_2 n_1^2 + u_1 n_2^2} B_{p}^e \tag{2.31d}
\]

\[
D_r^m = \left\{ \frac{u_2 - u_1}{u_1 + u_2} \begin{pmatrix} k u_2 z' & -k u_2 z' \\ e^{k u_2 z'} & e^{-k u_2 z'} \end{pmatrix} \right\} B_{pr}^m \tag{2.31e}
\]

\[
D_r^e = \left\{ \frac{u_2 n_1^2 - u_1 n_2^2}{u_2 n_1^2 + u_1 n_2^2} \begin{pmatrix} k u_2 z' & -k u_2 z' \\ e^{k u_2 z'} & e^{-k u_2 z'} \end{pmatrix} \right\} B_{pr}^e \tag{2.31f}
\]

where \( B_{pr}^m \) and \( B_{pr}^e \) are given by

\[
B_{pr}^m = \frac{i \gamma R k_0^2}{2u_2} J_0(k_0 \alpha r') \tag{2.32a}
\]

\[
B_{pr}^e = \frac{\alpha r' \gamma k_0^2}{12 n_2^2} J_0'(k_0 \alpha r') \tag{2.32b}
\]

Substitution of (2.31a) through (2.31f) into (2.30a) through (2.30c) yields the following forms for the field transforms:
\[ \tilde{\phi}_r^{e}(\alpha, z) = \beta^{e}_{pr} \frac{2u_2n_2^2}{u_2n_1^2 + u_1n_2^2} e^{-k_0(u_1z-u_2z')} e^{\frac{-k_0(u_1z-u_2z')}{u_2n_1^2 + u_1n_2^2}} z \geq 0 \]  

\[ \tilde{\phi}_r^{m}(\alpha, z) = \beta^{m}_{pr} \left\{ \frac{u_2n_2^2 - u_1n_2^2}{u_2n_1^2 + u_1n_2^2} e^{k_0u_2(z+z')} \right\} z \leq 0 \]  

(2.33b)  

and  

\[ \tilde{\phi}_r^{m}(\alpha, z) = \beta^{m}_{pr} \frac{2u_2}{u_1u_2} e^{-k_0(u_1z-u_2z')} , z \geq 0 \]  

(2.34a)  

\[ \tilde{\phi}_r^{m}(\alpha, z) = \beta^{m}_{pr} \left\{ \frac{u_2u_1 - u_1}{u_2 + u_1} e^{k_0u_2(z+z')} + e^{\frac{-k_0u_2(z-z')}{u_2n_1^2 + u_1n_2^2}} \right\} , z \leq 0 \]  

(2.34b)  

where \( \text{sgn}[(z-z')] \) in (2.33b) is the numerical sign of \( (z-z') \).

With the field transforms thus specified, we may employ (2.10) directly in order to generate the desired integral representations for the cartesian components of the electric and magnetic fields in the covering and substrate regions, hence  

\[ \phi_r^{e}(r,z) = \frac{k_0^2 e^{-r}}{4\pi i} \int_{-\infty}^{\infty} J_1'(k_0r') H_2^{(2)}(k_0r') \alpha^2 d\alpha , z \geq 0 \]  

(2.35a)  

\[ \tilde{\phi}_r^{e}(r,z) = \frac{k_0^2 e^{-r}}{4\pi i} \int_{-\infty}^{\infty} J_1'(k_0r') H_2^{(2)}(k_0r') \alpha^2 d\alpha \]  

\[ \left\{ \frac{u_2n_2^2 - u_1n_2^2}{u_2n_1^2 + u_1n_2^2} e^{k_0u_2(z+z')} + \text{sgn}[(z-z')] e^{\frac{-k_0u_2(z-z')}{u_2n_1^2 + u_1n_2^2}} \right\} \alpha^2 d\alpha , z \geq 0 \]  

(2.35b)
\[ \phi^m_r(r,z) = \frac{i\gamma R_0 k_0^2}{4} \int_{-i\pi}^{i\pi} e^{-ik_0r' \frac{z-z'}{2}} \frac{\alpha dq}{u_2} \, , \quad z \geq 0 \]  
\[ \phi^m_r(r,z) = \frac{i\gamma R_0 k_0^2}{2} \int_{-i\pi}^{i\pi} e^{-ik_0r' \frac{z-z'}{2}} \frac{\alpha dq}{u_2} \, , \quad z \leq 0 \]  
(2.36a)  
(2.36b)

where (2.35a), (2.35b), (2.36a), and (2.36b) are integrated along the contour shown in Figure 2.1.

Finally, with reference to (2.27a) and (2.27b), we see that (2.35a) and (2.35b) can be related to \( \Pi^e_z(r,z) \) by

\[ \phi^e_r(r,z) = r' \frac{\partial^2}{\partial r' \partial z'} \frac{1}{r'} \Pi^e_z(r,z) \]  
(2.37)

2.3.4 \( \phi \)-Directed Phased Arrays of Polarization Current

As per Section 2.3.3, we consider the cartesian components

\[ \Phi^e_\phi \triangleq E_z \quad \text{and} \quad \Phi^m_\phi \triangleq H_z \]  
of the electric and magnetic fields in the covering and substrate media. The procedure for finding integral representations of these cartesian field components is very similar to that followed above. Indeed, the only formal difference between the case currently under consideration and that of the \( r \)-directed polarization currents is that the unit vector in the representation of the polarization current densities changes from \( \vec{a}_r \) to \( \vec{a}_\phi \), with the result that (A.3c) and (A.3d) [cf. Appendix A] are replaced, respectively, by
Figure 2.6 A φ-Directed Pigned Array of Polarization Currents
\[
\frac{\partial}{\partial z} \tilde{\phi}_m^m(\alpha, z) \bigg|_{z=z'+} - \frac{\partial}{\partial z} \tilde{\phi}_m^m(\alpha, z) \bigg|_{z=z'-} = k_o^3 r'_v J'_v(k_o ar') \quad (2.38a)
\]

\[
\tilde{\phi}_e^e(\alpha, z'+) - \tilde{\phi}_e^e(\alpha, z'-) = \frac{v k_o^2}{w e_o n_o^{-2}} J'_v(k_o ar') \quad (2.38b)
\]

Comparing (2.38a) with (A.3c) (recalling \(v = \gamma k_o R_o\)) we see

\[-i\gamma R_o k_o^3 J'_v(k_o ar') + k_o^3 ar' J'_v(k_o ar') \quad (2.39)\]

and comparison of (2.38b) with (A.3d) shows that

\[
\frac{k_o^2 \alpha o ar'}{in_o^{-2}} J'_v(k_o ar') + \frac{R_o k_o^2 \alpha o}{n_o^{-2}} J'_v(k_o ar') \quad (2.40)
\]

The changes indicated in (2.39) and (2.40) manifest themselves in the values of \(B_m^m\) and \(B_e^e\). Thus, (2.32a) and (2.32b) are replaced by

\[
B_m^m = \frac{-k_o^2 ar'}{2u_2} J'_v(k_o ar') \quad (2.41a)
\]

\[
B_e^e = \frac{\gamma R_o k_o^2 \alpha o}{2n_o^{-2}} J'_v(k_o ar') \quad (2.41b)
\]

For the case of \(\phi\)-directed polarization currents, equations (2.33a), (2.33b) and (2.34a), (2.34b) are reproduced except that the subscript \(r\) is replaced by the subscript \(\phi\). Using (2.33a) through (2.34b) modified for \(\phi\)-directed currents in (2.10) we have
\[ \phi^{e}_{\phi}(r,z) = \frac{\gamma R_0}{4n_2} \int_{-\pi}^{\pi} J_0(k_0 r') H_1^{(2)}(k_0 r) \, \alpha d\alpha, \quad z \geq 0 \]  

\[ \frac{2u_2n_2^2}{u_2n_2 + u_1n_2} e^{-k_0(u_1 z - u_2 z')} \alpha d\alpha, \quad z \geq 0 \quad (2.42a) \]

\[ \phi^{e}_{\phi}(r,z) = \frac{\gamma R_0}{4n_2} \int_{-\pi}^{\pi} J_0(k_0 r') H_1^{(2)}(k_0 r) \]

\[ \left\{ \begin{array}{l} \frac{u_2n_2^2 - u_1n_2^2}{u_2n_2 + u_1n_2} e^{k_0 u_2(z + z')} + \text{sgn}(z-z')e^{-k_0 u_2|z-z'|} \end{array} \right\} \alpha d\alpha, \quad z \leq 0 \]

\[ 2u_2 e^{-k_0 u_2(z-z')} \frac{u_2^2}{u_2} \alpha d\alpha \quad z \geq 0 \quad (2.42c) \]

\[ \phi^{m}_{\phi}(r,z) = \frac{-k_0^2 r'}{4} \int_{-\pi}^{\pi} J_1'((k_0 r') H_1^{(2)}(k_0 r) \]

\[ \left\{ \begin{array}{l} u_2 - u_1 \frac{k_0 u_2(z + z')}{u_2} + \frac{k_0 u_2|z-z'|}{u_2} \end{array} \right\} \frac{\alpha d\alpha}{u_2} \quad z \leq 0 \quad (2.42d) \]

Upon comparison of (2.42a) and (2.42c) with (2.27a) and (2.27b), respectively, we see

\[ \phi^{e}_{\phi}(r,z) = \frac{i \gamma R_0}{r'} \frac{\partial}{\partial r'} \Pi^e_z(r,z) \quad (2.43) \]

Finally, comparison of (2.42c) and (2.42d) with (2.36a) and (2.36b), respectively, yields

\[ \phi^{m}_{\phi}(r,z) = \frac{-r'}{i \gamma R_0} \frac{\partial}{\partial r'} \phi^m_r(r,z) \quad (2.44) \]
2.3.5 The Ancillary Potential Integrals $V_e, V_m$

In terms of the Hertzian potential $\Pi^e_z$ of Section 2.3.2, we can write the associated cartesian component of the electric field $\phi^e_z$ as

$$\phi^e_z(r,z) = (\frac{\partial^2}{\partial Z^2} + k_{1,2}^2) \Pi^e_z$$  \hspace{1cm} (2.45)

Now, considering (2.27a) and (2.27b) we define the scalar, electric, ancillary potential $V_e$ by

$$V_e(r,z) = \left(\frac{\text{i}Z}{4n_2}\right) \int_{\infty}^{\infty} \text{J}_\nu(k_0 r' \alpha) H^{(2)}_{\nu}(k_0 r \alpha) \frac{2u_2^2}{u_1 n_2^2 + u_2^2 n_1^2} e^{(u_2 z' - u_1 z)} \frac{\text{d} \alpha}{u_2}, \hspace{1cm} z \geq 0$$  \hspace{1cm} (2.46a)

$$V_e(r,z) = \left(\frac{\text{i}Z}{4n_2}\right) \int_{\infty}^{\infty} \text{J}_\nu(k_0 r' \alpha) H^{(2)}_{\nu}(k_0 r \alpha) \left\{ \frac{u_2 n_2^2 - u_1 n_2^2}{u_2 n_1^2 + u_1 n_2^2} e^{k_0 u_2 (z+z')} + e^{-k_0 u_2 |z-z'|} \right\} \frac{\text{d} \alpha}{u_2}, \hspace{1cm} z \leq 0$$  \hspace{1cm} (2.46b)

Having thus defined $V_e$, we see that

$$\phi^e_z(r,z) = r' \left\{ \frac{\partial^2}{\partial Z^2} + k_{1,2}^2 \right\} V_e(r,z)$$  \hspace{1cm} (2.47a)

In a like manner, considering (2.46a) and (2.46b) along with (2.35a) and (2.35b), we have

$$\phi^e_{r'}(r,z) = r' \frac{\partial^2}{\partial r' \partial Z'} V_e(r,z)$$  \hspace{1cm} (2.47b)

Consider now the definition of the scalar, magnetic, ancillary potential $V_m$ given as
\[ V_m(r,z) = \left( \frac{\kappa_0}{4} \right) \int_{-\pi}^{\pi} J_\nu(k_0 r \alpha) H_\nu^{(2)}(k_0 r \alpha) \left( \frac{2u_z}{u_1 + u_2} e^{-k_0 (u_1 z - u_2 z')} \frac{\alpha d \alpha}{u_2} \right), \quad z \geq 0 \]  

\[ V_m(r,z) = \left( \frac{\kappa_0}{4} \right) \int_{-\pi}^{\pi} J_\nu(k_0 r \alpha) H_\nu^{(2)}(k_0 r \alpha) \left( \frac{u_2 - u_1}{u_2 + u_1} e^{k_0 u_2 (z + z')} + e^{-k_0 u_2 |z - z'|} \right) \frac{\alpha d \alpha}{u_2}, \quad z \leq 0 \]  

Comparison of (2.48a) and (2.48b) with (2.36a) and (2.36b) gives

\[ \phi^m_r(r,z) = (i \gamma R_0 k_0) V_m \]  

and a similar comparison of (2.48a) and (2.48b) with (2.42c) and (2.42d) yields

\[ \phi^m_\theta(r,z) = -r' \frac{\partial}{\partial r'} V_m \]  

We can conveniently summarize the representation of the total cartesian field components \( \phi^e \) and \( \phi^m \) due to polarization currents in the channel region in terms of the potentials \( V_e \) and \( V_m \), i.e., since

\[ \phi^e = \phi^e_z + \phi^e_r + \phi^e_\theta \quad \text{and} \quad \phi^m = \phi^m_r + \phi^m_\theta, \]

we can write

\[
\begin{bmatrix}
\phi^e(r,z) \\
\phi^m(r,z)
\end{bmatrix} = i \omega \varepsilon_0 (n^2 + n_z^2) \int ds' \begin{bmatrix}
r' \frac{\partial^2 V_e}{\partial r'^2} \\
-i \gamma R_0 k_0 \frac{\partial e}{\partial z'}
\end{bmatrix} \begin{bmatrix}
r' \left( \frac{\partial^2 + k_0^2}{\partial z^2} \right) V_e \\
-\frac{\partial^2}{\partial r^2} V_m
\end{bmatrix} \begin{bmatrix}
E_{or} \\
E_{o\theta}
\end{bmatrix} + \begin{bmatrix}
0 \\
E_{oz}
\end{bmatrix}
\]

(2.49)
\[ \Gamma_c = \gamma_0 + \gamma_1 + \gamma_2 + \gamma_{R_1} + \gamma_{R_2} + \gamma_{R_3} \]

Figure 3.1 The Contour \( \Gamma_c \)
where we have symbolized the integrand of (3.1a) by (F) and the integrand of (3.1b) by (G,H).

If now we let \( R \to \infty \), we may invoke Jordan's lemma to justify the claim that both \( \int_{\gamma_{R_1}^+\gamma_{R_2}^+\gamma_{R_3}^+} (F) \) and \( \int_{\gamma_{R_1}^+\gamma_{R_2}^+\gamma_{R_3}^+} (G,H) \) vanish (recall \( z' < 0 \)).

Also, the uniform tendency toward zero of \( F(\alpha) \), \( G(\alpha) \), and \( H(\alpha) \) together with the exponential factors in \( (F) \) and \( (G,H) \) assure the convergence of the remaining integrals, whence (3.2a) and (3.2b) become

\[
\int_{e^{-i\pi}}^{\infty} (F) = - \int_{e^{-i\pi}}^{\infty} (F) - \int_{e^{-i\pi}}^{\infty} (F) \tag{3.3a}
\]

\[
\int_{e^{-i\pi}}^{\infty} (G,H) = - \int_{e^{-i\pi}}^{\infty} (G,H) - \int_{e^{-i\pi}}^{\infty} (G,H) \tag{3.3b}
\]

where \( \gamma_1 \) and \( \gamma_2 \) are now contours which "wrap-around" the full extent of their associated branch cuts.

Notice that both \( V_e \) and \( V_m \) take the form of (3.3a) for \( z > 0 \), and that both \( V_e \) and \( V_m \) take the form of (3.3b) for \( z < 0 \). Thus, we can now transfer our attention from an integration of the ancillary potentials \( V_e \) and \( V_m \) along the \( \text{Re} \alpha \) axis to a pair of integrals which, in each case, follow the branch cuts of \( u_1 \) and \( u_2 \) in the lower half of the \( \alpha \)-plane.

### 3.2.2 Application of the Debye Expansions

Our objective in this section is to find an asymptotic representation for the product \( J_{\nu}(k_0 r'\alpha)H_{\nu}^{(2)}(k_0 r\alpha) \) which appears in the integrands
of the integral representations of $V_e$ and $V_m$. Since we are interested in geometries for which $k_o R_o >> 1$, and since $\nu = \gamma k_o R_o$, we are certainly dealing with Bessel and Hankel functions of large order. Likewise, since we are interested in fields just beyond the outer radius of the channel guide, both $r'$ and $r$ are on the order of $R_o$; so, for $\alpha$ of sufficiently large modulus, the Bessel and Hankel functions we encounter are of large argument as well. The requirement that $\alpha$ be of sufficient modulus means that there is a region about the origin of the $\alpha$-plane of radius $O\left(\frac{1}{k_o R_o}\right)$ within which we cannot allow our integration contour to pass. This presents no significant difficulty since we are free to deform the contours $\gamma_1$ and $\gamma_2$ onto the improper Riemann sheet (if necessary) for a finite distance and hence to circumvent the forbidden region, whereupon we re-enter the proper sheet and continue integrating along the branch cuts. We will not consider this subtlety in greater depth since we will ultimately prosecute a saddle point evaluation of our integrals for which the saddle point is well removed from regions where the large argument assumption is invalid.

The Debye asymptotic expansions for large order and argument may be written in the form (cf.[5], [9])

\begin{equation}
(2\pi \nu)^{\frac{1}{2}} j_\nu(\nu z) \sim (1 - z^2)^{-\frac{1}{2}} e^{-\nu f(z)} \tag{3.4a}
\end{equation}

\begin{equation}
(2\pi \nu)^{\frac{1}{2}} y_\nu(\nu z) \sim -2(1 - z^2)^{-\frac{1}{2}} \nu e^{\nu f(z)} \tag{3.4b}
\end{equation}

where $0 < \text{Re } z < 1$, and

\begin{equation}
f(z) = \tanh^{-1}(1 - z^2)^{\frac{1}{2}} - (1 - z^2)^{\frac{1}{2}} \tag{3.4c}
\end{equation}

and where $y_\nu$ is the Neumann function of order $\nu$. 
Since \( H_{\nu}^{(2)} = j_\nu - i y_\nu \), (3.4a) and (3.4b) give

\[
H_{\nu}^{(2)}(vz) \sim \frac{i 2(1-z^2)^{-\frac{1}{2}}}{(2\pi v)^{\frac{1}{2}}} \left\{ e^{vf(z)} - \frac{i}{2} e^{-vf(z)} \right\} 
\]  
(3.5)

If we let \( vz = k_o \alpha (R_o + y) \), then

\[
z = \frac{\alpha}{\gamma} \left( 1 + \frac{y}{R_o} \right) 
\]  
(3.6)

and

\[
z_o \triangleq z \bigg|_{y=0} = \frac{\alpha}{\gamma} 
\]  
(3.7)

Recall that \( y \) is the radial coordinate of the local coordinate system situated at the center of the channel guide (cf. Figure 1.2). In an integrated optics environment we will have \( n > \gamma > n_2 \), thus inspection of the contour \( \gamma_2 \) reveals that \( \text{Re}(\frac{\alpha}{\gamma}) < 1 \) for all \( \alpha \) on \( \gamma_2 \).

A Taylor series expansion of (3.4c) in powers of \( y \) about \( z_o \) as given in (3.7) yields the result

\[
vf(z) \approx k_o R_o \gamma f(\frac{\alpha}{\gamma}) - y k_o \gamma (1 - \frac{\alpha^2}{\gamma^2}) \]  
(3.8)

Using (3.8) in (3.5) gives

\[
H_{\nu}^{(2)}(vz) \sim \frac{i 2}{(2\pi v)^{\frac{1}{2}}} \left[ 1 - \frac{2}{\gamma^2} \left( 1 + \frac{y}{R_o} \right) \right] \left\{ e^{\gamma k_o R_o f(\frac{\alpha}{\gamma})} e^{-k_o \lambda y (1 - \frac{\alpha^2}{\gamma^2})^{\frac{1}{2}}} - \frac{i}{2} e^{-\gamma k_o R_o f(\frac{\alpha}{\gamma})} - k_o \lambda y (1 - \frac{\alpha^2}{\gamma^2}) \right\} \]  
(3.9)

to \( O(\frac{1}{\gamma}) \).

With the definitions

$$\lambda \triangleq (\gamma^2 - \alpha^2)^{1/2} \tag{3.10a}$$

and

$$F(\lambda) \triangleq \gamma f(\frac{\alpha}{\gamma}) = \gamma \tanh^{-1}(\frac{\lambda}{\gamma}) - \lambda \tag{3.10b}$$

(3.9) gives (recall \( k_o R_o >> 1 \))

$$\frac{H_{2v}^{(2)}(k_o a r)}{H_{2v}^{(2)}(k_o a R_o)} \sim e^{-k_o \lambda y} - 2k_o R_o F(\lambda) - ie^{-\gamma k_o R_o f(\frac{\alpha}{\gamma})} \sinh (k_o \lambda y) \tag{3.10c}$$

where we have also used \( r = R_o + y \).

Using (3.4a) with (3.6), (3.7), and (3.8), we find

$$J_{v}(k_o a r) \sim \frac{[1 - \frac{\alpha^2}{2} \left(1 + \frac{y}{R_o}\right)^2]^{-1/2}}{2\pi} e^{k_o \lambda y (1 - \frac{\alpha^2}{2})} e^{-\gamma k_o R_o f(\frac{\alpha}{\gamma})} \tag{3.11}$$

to \( O(\frac{1}{v}) \). Employing (3.11) we find directly

$$\frac{J_{v}(k_o a r)}{J_{v}(k_o a R_o)} \sim e^{k_o \lambda y} \tag{3.12}$$

Also, using (3.9) and (3.11) we have

$$J_{v}(k_o a R_o)H_{v}^{(2)}(k_o a R_o) \sim \frac{1}{\pi k_o R_o \lambda} \left\{ 1 - 2 \gamma k_o R_o f(\frac{\alpha}{\gamma}) \right\} \tag{3.13}$$

We now have each of the elemental results necessary for finding an asymptotic representation for \( J_{v}(k_o a r')H_{v}^{(2)}(k_o a r') \). From (3.10c) and (3.12) we have
\[ J_{\nu}(k_oar')H^{(2)}_\nu(k_oar) \sim J_{\nu}(k_oar_o)H^{(2)}_\nu(k_oar_o)e^{y'k_o\lambda} \]

\[
\left\{ \begin{array}{c}
-k_o^2y + 2k_o^2R_oF(\lambda) \\
-\frac{k_o^2}{2}e^{y'k_o\lambda} \end{array} \right\} \sinh(k_o\lambda y) \]

Finally, substitution of (3.13) in (3.14) yields our objective,

\[
J_{\nu}(k_oar')H^{(2)}_\nu(k_oar) \sim \frac{1}{\pi k_o^2 R_o} \left\{ \begin{array}{c}
\frac{i}{\lambda} e^{-k_o^2y} \\
+ \frac{1}{2\lambda} e^{k_o^2(y+y')} -2k_o^2R_oF(\lambda) \\
-4k_o^2R_oF(\lambda)
\end{array} \right\} + O\left(\frac{1}{\nu^2}\right) \quad (3.15)
\]

where terms of \( O(\frac{1}{\nu^2}) \) have been neglected.

3.2.3 The Modified Integral Representations of \( V_e, V_m \)

Using results (3.3a), (3.3b) and (3.15) in (2.46a) and (2.46b) as well as (2.49a) and (2.48b), the ancillary potentials \( V_e \) and \( V_m \) seen to be separable into primary and secondary components; i.e., we may write \( V_e = V^p_e + V^s_e \), \( V_m = V^p_m + V^s_m \), where

\[
V^p_e(y,y',z,z') = \left( \frac{-e}{n^2} \right) \frac{1}{4\pi k_o^2 R_o} \int_{\gamma_{1,2}} \frac{e^{-k_o^2(y-y')}}{\lambda} \frac{\alpha d\alpha}{u_2} \{ - \} \quad (3.16a)
\]

\[
V^s_e(y,y',z,z') = \left( \frac{ie}{n^2} \right) \frac{1}{8\pi k_o^2 R_o} \int_{\gamma_{1,2}} \frac{k_o^2(y+y') - 2k_o^2R_oF(\lambda)}{\lambda} \frac{\alpha d\alpha}{u_2} \{ - \} \quad (3.16b)
\]

\[
V^p_m(y,y',z,z') = \left( \frac{-i}{4\pi R_o} \right) \int_{\gamma_{1,2}} \frac{-k_o\lambda(y+y')}{\lambda} \frac{\alpha d\alpha}{u_2} \{ - \} \quad (3.16c)
\]

\[
V^s_m(y,y',z,z') = \left( \frac{-1}{8\pi R_o} \right) \int_{\gamma_{1,2}} \frac{k_o\lambda(y+y') - 2k_o^2R_oF(\lambda)}{\lambda} \frac{\alpha d\alpha}{u_2} \{ - \} \quad (3.16d)
\]
For convenience we have omitted detailed representation of those portions of the integrands which remain unchanged as the result of applying the Debye expansions. Classification of terms as primary or secondary components is made on the basis of the fact that in the limit as \( R_0 \to \infty \), \( R_0 V_e^p \) and \( R_0 V_m^p \) approach the forms assumed by the ancillary potentials in the straight guide case. Because of the presence of the factor \( e^{-2k_0 R_0 F(\lambda)} \), choice of the proper branch of \( \lambda \) will in general yield \( V_e^S << V_e^p \) and \( V_m^S << V_m^p \); indeed, both \( R_0 V_e^S \) and \( R_0 V_m^S \) vanish as \( R_0 \to \infty \).

Despite the fact that the secondary ancillary potentials are much smaller in magnitude than their primary counterparts, \( V_e^S \) and \( V_m^S \) are the terms of interest for our purposes here. This follows from the fact that, in addition to their relatively small magnitudes when compared to the primary potentials, both \( V_e^S \) and \( V_m^S \) exhibit an exponentially increasing character for increasing \( y \), and each stands in quadrature to its associated primary potential. These features of \( V_e^S \) and \( V_m^S \) allow an association between these secondary potentials and the electric and magnetic fields exterior to the bent channel guide which have been reflected from the caustic; i.e., the electric and magnetic fields derivable from the secondary, ancillary potentials \( V_e^S \) and \( V_m^S \) are the fields \( \delta \vec{E} \) and \( \delta \vec{H} \), respectively.

We will henceforth focus attention on the ancillary potentials \( V_e^S \) of (3.16b) and \( V_m^S \) of (3.16d) in order to characterize the radiation from the bent channel guide. Because of the linearity of all operators involved, equation (2.49) still applies and the components produced are the z-components of \( \delta \vec{E} \) and \( \delta \vec{H} \).
3.3 The Asymptotic Form of $V_e^s$

3.3.1 A Change of Variables

To assure convergence of our fundamental integral representations of $V_e$ and $V_m$, we have chosen to employ the branches of $u_1 = (\alpha^2 - n_1^2)^{\frac{1}{2}}$ and $u_2 = (\alpha^2 - n_2^2)^{\frac{1}{2}}$ for which $\text{Re } u_1 > 0$, $\text{Re } u_2 > 0$. It is in connection with these branches of $u_1$ and $u_2$ that the branch cuts at $n_1$ and $n_2$ arise in the $\alpha$-plane. The contours $\gamma_1$ and $\gamma_2$ follow the branch cuts at $n_1$ and $n_2$, respectively, and it follows from the same analysis that specifies the branch cuts (not discussed herein) that $u_1$ is purely imaginary along $\gamma_1$ and $u_2$ is purely imaginary along $\gamma_2$. Indeed, if we consider $\gamma_2$ to be composed of segments $\gamma_{2e}$, $g_2$, $\gamma_{2u}$ (i.e., $\gamma_2 = \gamma_{2e} + g_2 + \gamma_{2u}$) it is possible to show that $\arg u_2 = -\frac{\pi}{2}$ on $\gamma_{2e}$ and $\arg u_2 = \frac{\pi}{2}$ on $\gamma_{2u}$ (cf. Figure 3.2). The segment $g_2$ is a circular segment centered on $n_2$ whose radius we allow to vanish, whereupon we find no contribution from $g_2$ to the integrals over $\gamma_2$ in (3.16b) and (3.16d). Analogous statements follow for the argument of $u_1$ on $\gamma_1$, with $\gamma_1 = \gamma_{1e} + g_1 + \gamma_{1u}$, and for the contribution from $g_1$ to the integrals over $\gamma_1$ in (3.16b) and (3.16d).

Analysis shows that we may select a branch of the function $(n_2^2 - \alpha^2)^{\frac{1}{2}}$ consistent with the extant branch cut at $n_2$ such that $\text{Im}(n_2^2 - \alpha^2)^{\frac{1}{2}} < 0$, thus we may write

$$u_2 = i s_2 \quad (3.17a)$$

where

$$s_2 = (n_2^2 - \alpha^2), \quad \text{Im}(n_2^2 - \alpha^2)^{\frac{1}{2}} < 0 \quad (3.17b)$$
Figure 3.2 A Close-Up of Contour $\gamma_2$
Since $u_2$ is purely imaginary along $\gamma_{2u}$, then $s_2$ must be purely real along these same segments. Indeed, the nature of $u_2$ along $\gamma_{2e}$ and $\gamma_{2u}$ along with (3.17a) requires $s_2$ real and $s_2 < 0$ on $\gamma_{2e}$, and $s_2$ real and $s_2 > 0$ along $\gamma_{2u}$ (cf. Figure 3.2). Thus, a change of variables from $\alpha$ to $s_2$ in (3.16b) and (3.16d) using (3.17a) will convert integrals along $\gamma_2$ in the $\alpha$-plane to integrals along the full extent of the real axis ($-\infty$ to $+\infty$) in the $s_2$-plane. In the same way, one may write

$$u_1 = is_1$$  \hspace{1cm} (3.18a)

where

$$s_1 = (n_1^2 - \alpha^2)^{1/2}, \quad \text{Im}(n_1^2 - \alpha^2)^{1/2} < 0$$ \hspace{1cm} (3.18b)

as a change of variables from $\alpha$ to $s_1$ in (3.16b) and (3.16d) to convert integrals over $\gamma_1$ in the $\alpha$-plane to integrals over the full extent of the real axis in the $s_1$-plane.

For the purpose of illustration, we will consider that portion of $V_e^S$ obtained from integration over the contour $\gamma_2$, while noting that an analogous procedure may be performed for that portion of $V_e^S$ obtained from integration over $\gamma_1$. At the end of this chapter we will see, in fact, that the contribution to $V_e^S$ due to the integration over $\gamma_1$ can be neglected because of its insignificant amplitude relative to the contribution arising from the integration over $\gamma_2$.

From (3.17a) and (3.17b) we have $\frac{ds_2}{d\alpha} = -i\alpha(\alpha^2 - n_2^2)^{-1/2}$, so

$$ds_2 = -i \frac{\alpha d\alpha}{u_2}$$ \hspace{1cm} (3.19)
Recalling $\lambda$ from (3.10a) and using (3.17a) and (3.17b), we may write

$$\lambda = (\lambda_2^2 + s_2^2)^{\frac{1}{2}}$$  \hspace{1cm} (3.20a)

where

$$\lambda_2^2 = \gamma^2 - n_2^2$$  \hspace{1cm} (3.20b)

Also, we may write

$$u_1 = (v_2^2 - s_2^2)^{\frac{1}{2}}$$  \hspace{1cm} (3.21a)

where

$$v_2^2 = n_2^2 - n_1^2$$  \hspace{1cm} (3.21b)

Using (3.19), (3.20a), (3.20b), (3.21a), and (3.21b) in (3.16b), we have

$$V^s(y, y', z, z') = \left(\frac{-\zeta_0}{2\pi n_2 k_0 \rho_0}\right) \int_{-\infty}^{\infty} \tau_e(y, y', z, z'; s_2) e^{k_0 R_0 q^2(s_2)} ds_2$$  \hspace{1cm} (3.22a)

where

$$\tau_e(y, y', z, z'; s_2) = e^{\frac{k_0 \lambda(y+y')}{\lambda}} \frac{2u_2 n_1^2}{u_1 n_2^2 + u_2 n_1^2} e^{k_0 (u_2 z' - u_1 z)}$$ \hspace{1cm} (3.22b)

and

$$\tau_e(y, y', z, z'; s_2) = e^{\frac{k_0 \lambda(y+y')}{\lambda}} \frac{u_2 n_1^2 - u_1 n_2^2}{u_1 n_2^2 + u_2 n_1^2} e^{k_0 u_2 (z+z')}$$ \hspace{1cm} (3.22c)

$$+ e^{-k_0 u_2 |z-z'|}$$ \hspace{1cm} (3.22d)

where $u_2 = is_2$, $u_1 = (v_2^2 - s_2^2)^{\frac{1}{2}}$, and

$$q^2(s_2) = 2 \left[\frac{\gamma}{2} \log \frac{\gamma + \lambda}{\gamma + \lambda} + \lambda\right]$$ \hspace{1cm} \(\lambda = (\lambda_2^2 + s_2^2)^{\frac{1}{2}}\)  \hspace{1cm} (3.22d)
3.3.2 Evaluation of $V_e^s$

Since we assume $k_o R_o >> 1$, the form of $V_e^s$ given in (3.22a) lends itself readily to a steepest descent type of asymptotic evaluation (cf. [7]).

Proceeding as outlined in Appendix B we find

$$V_e^s(y, y', z, z') \sim \frac{-\varepsilon_0}{16\pi^2 n_2^2 k_o R_o} \left( \frac{\pi n_2^2}{\lambda_2^2 k_o R_o} \right)^{3/2} e^{k_o R_o q_2(0)}$$

$$\frac{\lambda_1 \lambda_2 (y + y')}{\lambda_2} \left[ k_o (2 - \frac{n_1^2}{n_2^2} \frac{z - z'}{n_2^2(n_2^2 - n_1^2)}) \right]$$

(3.23)

A similar result is obtained for the component of $V_e^s$ derived by integration over $\gamma_1$. However, consider the series representation of $q_1(0)$ and $q_2(0)$ derived by expanding their logarithmic terms [cf. (3.22d)]

$$q_1(0) = -2\gamma \sum_{i=1}^{\infty} \frac{1}{i(2n+1)} \left( \frac{\lambda_i}{\gamma} \right)^{2n+1}$$

(3.24)

where

$$\frac{\lambda_i}{\gamma} = \left(1 - \frac{n_i^2}{\gamma^2} \right), \quad i = 1, 2$$

For $q_1(0)$, $\frac{n_1}{\gamma} << 1$, so $\frac{\lambda_1}{\gamma} \approx 1$.

whereas for $q_2(0)$, $\frac{n_2}{\gamma} \approx 1$, so $0 < \frac{\lambda_2}{\gamma} << 1$. For typical integrated optics material parameters then, (3.24) gives $|q_1(0)| >> |q_2(0)|$ so that

$$e^{k_o R_o q_1(0)} \approx e^{k_o R_o q_2(0)}$$

and we are justified in neglecting the contribution from the integration over contour $\gamma_1$. 
CHAPTER 4

RADIATION LOSS FROM A CURVED RECTANGULAR CHANNEL GUIDE FAR FROM CUTOFF

4.1 Introduction

As discussed briefly in Chapter 1, we will know the continuous radiation loss from a section of curved dielectric channel waveguide once we have found the normalized attenuation constant $\alpha$ as given by (1.2). Evaluation of $\alpha$ requires computation of the quantities $P$ and $c$ in (1.3) and (1.4), respectively. Our efforts in Chapters 2 and 3 have been to derive an expression from which we can determine the caustic-reflected fields $\delta E$ and $\delta H$ in terms of the straight channel fields. Equations (2.49) and (3.23) allow us now to do so. Thus, we are left with the task of specifying the straight channel fields $E_0$, $H_0$ for any case of practical interest and using these with (2.49), (3.23), (1.2), and (1.3) in order to find $\alpha$.

The general problem of determining the fields within a straight, rectangular dielectric channel waveguide embedded in a substrate has not been rigorously solved analytically. However, an approximate analytical method has been developed by Marcatili (cf. [2]) which yields straight channel field expressions which are valid for well confined channel modes, i.e., channel modes that are far from cutoff. We will employ Marcatili's results and thus generate an expression for the radiation from a curved dielectric channel waveguide with associated straight channel modes that are far from cutoff. In particular, we consider the straight, rectangular
dielectric channel waveguide configuration of Fig. 4.1 which illustrates
the general disposition of material regions to which Marcatili's
analytical procedure can be applied. The material parameters assigned
to the various regions are consistent with those of the associated
curved channel guide problem illustrated in Figure 1.2. The assumption
of propagating modes that are far from cutoff and therefore highly con-
 fined within the channel region is extended by the stipulation that the
fields in the shaded regions of Figure 4.1 are negligible, hence no
material parameters need be specified there.

It should also be noted that the origin of the \((x,w,s)\) coordinate
system is off-set from the local cartesian system \((x,y,s)\) in Figure 1.1.
To convert to the \((x,w,s)\) system from the \((x,y,s)\) system we need only
employ the transformation

\[
y = w - \frac{b}{2}
\]  

(4.1)

The structure of Figure 4.1 can support a set of modes that are
predominantly \(x\)-polarized and are designated \(E^x_{pq}\) modes. In addition,
such a structure can also support a set of modes that are predominantly
\(y\)-polarized and are designated \(E^y_{pq}\) modes. Although such a structure
can support both modes, we consider only radiation due to the \(E^x_{pq}\) modes
whose field distributions in the channel guide are given by Marcuse
(cf. [11])

\[
E_{0s} = A \cos k_x (x + \xi) \cos k_w (w + \eta) 
\]  

(4.2a)

\[
E_{ox} = \frac{iA}{\beta} \frac{n^2 k_0^2 - k_x^2}{k_x} \sin k_x (x + \xi) \cos k_w (w + \eta)
\]  

(4.2b)

\[E_{ow} \approx 0\]  

(4.2c)
Figure 4.1 The Straight, Rectangular Dielectric Channel Waveguide Configuration
\[
H_{0s} = -A^2(n_0^2 \frac{k_w}{k_x} \frac{\xi}{\omega} \sin k_x (x+\xi) \sin k_w (w+\eta)) \tag{4.2d}
\]

\[H_{0x} = 0 \tag{4.2e}\]

\[
H_{0w} = iA(n_0^2 \frac{k_w}{k_x} \sin k_x (x+\xi) \cos k_w (w+\eta)) \tag{4.2f}
\]

where \(k_0 = \frac{w}{\sqrt{\mu_0 \varepsilon_0}}\) is the free-space phase constant, \(\beta\) is the phase constant of the fields in the channel (propagation of the form \(e^{i(\omega t - \beta s)}\) is assumed), and \(\zeta_0 = \frac{\mu_0}{\varepsilon_0}\) is the free-space wave impedance. The parameters \(k_x\) and \(k_w\) are separation constants related by

\[
n^2 k_0^2 - \beta^2 = k_x^2 + k_y^2 \tag{4.3}\]

and are solutions of the eigenvalue equations

\[
\tan k_x d = \frac{n^2 k_x (n_1^2 k_2 + n_2^2 k_3)}{(n_1^2 k_2^2 - n_4^2 k_3)} \tag{4.4a}\]

\[
\tan k_w b = \frac{2k_w \gamma_4}{(k_w^2 - \gamma_4^2)} \tag{4.4b}\]

The parameters \(\gamma_j\) are given by

\[
\gamma_2 = [(n^2 - n_2^2) k_0^2 - k_x^2]^{\frac{1}{2}} \tag{4.5a}\]

\[
\gamma_3 = [(n^2 - n_1^2) k_0^2 - k_x^2]^{\frac{1}{2}} \tag{4.5b}\]

\[
\gamma_4 = [(n^2 - n_2^2) k_0^2 - k_w^2]^{\frac{1}{2}} \tag{4.5c}\]

(3.18a)
The parameters $\gamma_j$ are transverse decay constants of the fields in
the material regions below, above, and on either side of the region of
refractive index $n$ (the channel guide), respectively; e.g., the fields in
the material region below the channel guide region decay like
$e^{-\gamma_2(x+d)}$, $x \leq -d$. It follows that each of the parameters $\gamma_j$ satisfies $\gamma_j > 0$.

Once $k_x$ and $k_w$ are known, the phase parameters $\xi$ and $\eta$ are
determined from

\[ \tan k_x \xi = -\left(\frac{n_1}{n}\right)^2 \frac{k_x}{\gamma_3} \]

\[ \tan k_w \eta = -\frac{\gamma_4}{k_w} \]

Also, notice that once $k_x$ and $k_w$ are known, (4.2) may be solved to
yield $\beta$.

4.2 Expressions for $P$ and $C$

Application of the assumption of modes far from cutoff and employing
the local coordinate representations for the field components allows
(1.3) and (1.4) to be rewritten, respectively, as
\[ P = 2 \int_{-d}^{b} dx \int_{0}^{b} dy (E_{0x} H_{0w}) \]  \hspace{1cm} (4.7)

\[ C = \int_{-d}^{b} \left\{ \frac{E_{0} H_{0} - \delta E \times \delta H}{\delta E \times \delta H} \right\} \cdot \mathbf{a}_{w} dx \bigg|_{w=b} \]  \hspace{1cm} (4.8)

As discussed in detail in Appendix C, the straight channel field expressions (4.2) may be used with the ancillary potential \( V_{e}^S \) from (3.23) in (4.7), and (4.8) to evaluate \( P \) and \( C \) as

\[ P = -2A^{2} \left[ \frac{n_{2}^{2} k_{0}^{2} - k_{x}^{2}}{k_{x}^{2}} \right] \frac{k_{0} n_{2}^{2}}{\beta \tau_{0}} \left( -\frac{d}{2} + L_{x} \right) \left( \frac{b}{2} + L_{w} \right) \]  \hspace{1cm} (4.9)

\[ C = -A^{2} C_{1} C_{2}^{2} \text{sgn} [\cos (k_{w} b)] \text{sgn} [k_{w}^{2} - \gamma_{4}^{2}] \]

\[ \left[ \frac{k_{w} k_{0} (n_{2}^{2} k_{0}^{2} - k_{x}^{2})(n_{2}^{2} - n_{2}^{2})}{16 \pi^{2} k_{x}^{2} \lambda_{2}^{2} \tau_{0}} \right] \left[ \lambda_{2} (n_{2}^{2} k_{0}^{2} - k_{x}^{2}) + n_{2}^{2} k_{0}^{2} \gamma_{4} \right] \]

\[ \left( \frac{\pi n_{2}^{2}}{k_{w}^{2} \lambda_{2}^{2}} \right)^{3/2} e^{(k_{w} \lambda_{2} b)/2} k_{w}^{2} R_{o} q_{2}(0) \]  \hspace{1cm} (4.10)

where the constants \( L_{x}, L_{w}, C_{1}, \) and \( C_{2} \) are given in Appendix C as equations (C.5a), (C.5b), (C.22), and (C.25), respectively.

4.3 The Evaluation of \( \alpha \)

The normalized attenuation constant is given in (1.2). Before evaluating \( \alpha \) in terms of (4.9) and (4.10), it is advantageous to normalize \( C_{1}, C_{2}, \) and \( A_{e} \) by \( L_{x} \left( \frac{d}{2} + L_{w} \right) / k_{0} \). Doing this we have

\[ C_{1} = \frac{V_{w}}{k_{0}^{2} (n_{2}^{2} - n_{2}^{2})^{2}} \left\{ \lambda_{2} \left[ \text{sgn} [\cos (k_{w} b)] \text{sgn} [V_{w}^{2} - \gamma_{4}^{2}] \right] \right\} \]

\[ \cdot \left[ \text{sgn} [\cos (k_{w} b)] \text{sgn} [V_{w}^{2} - \gamma_{4}^{2}] \right] \]

\[ \cdot \left[ \text{sgn} [\cos (k_{w} b)] \text{sgn} [V_{w}^{2} - \gamma_{4}^{2}] \right] \]
\[ c_2 = \frac{1}{k_0 \nu_x} \left\{ \frac{\text{sgn}[\cos(k_0v_x)\nu_x]}{\text{sgn}\left[ (n_1^2n_2^2v_x^2 - n_4^2n_2^2v_x^2) \right]} \left[ \frac{(n_2^2(v_x^2 + n_1^2v_x^2) + n_2^2v_x^2)}{(n_1^2v_x^2 + n_4^2v_x^2)^2} \right] + \frac{n_2^2n_1^2}{n_2^2(n_2^2 - n_1^2)^2} \right\} \]

\[ \frac{\Gamma_3}{(n_2^2v_x^2 + n_4^2v_x^2)^2} \right\} \] (4.13)

\[ A_e = \frac{1}{4k_0} \left\{ (k_0d) + n^2 \left[ \frac{n_2^2\nu_x}{(n_1^2v_x^2 + n_4^2v_x^2)} + \frac{n_2^2\nu_x}{(n_2^2v_x^2 + n_4^2v_x^2)} \right] \right\} \left( k_0b \right) + \frac{2\Gamma_4}{(n_2^2 - n_1^2)^2} \) (4.14)

where

\[ \nu_i = \frac{k_i}{k_0} \] (4.15a)

\[ \Gamma_i = \frac{\gamma_i}{k_0} \] (4.15b)

With (4.12), (4.13), (4.14) and (4.9), (4.10) in (1.2) we find

\[ \alpha = -\text{sgn}[\cos(k_0b)\nu_w] \text{sgn}[\nu_w^2 - \Gamma_4] \left( \frac{F_N}{F_D} \right) \] (4.16a)

where
\[ F_N = \left\{ \frac{r_2^3}{n^2} \left( \frac{\nu_w}{\nu_x} \right)^2 \left[ \lambda_2 (\text{sgn}[\cos(k_o b)\nu_w]) \text{sgn}[\nu_w^2 - \nu_x^2] \right. \right. \]
\[ \left. \left. e^{(k_o b)\lambda_2/2} - e^{-(k_o b)\lambda_2/2} \right) + \Gamma_4 (\text{sgn}[\cos(k_o b)\nu_x]) \right\} \]
\[ \text{sgn}[\nu_w^2 - \nu_x^2] e^{(k_o b)\lambda_2/2} + e^{-(k_o b)\lambda_2/2} \right] \left[ \text{sgn}[\cos(k_o d)\nu_x] \right] \]
\[ \frac{\text{sgn}[n_1^2 n_2^2 \nu_x^2 - n_4^2 \Gamma_2 \Gamma_3]}{n_4^2 \nu_x^4 + n_4^2 \nu_x^2} - \frac{n_1^2}{n_4^2 \nu_x^4 + n_4^2 \nu_x^2} \]
\[ + \frac{n_2^2}{n_2(n_2^2 - n_1^2)} \left( \frac{\Gamma_3}{n_4^2 \nu_x^4 + n_4^2 \nu_x^2} + \text{sgn}[\cos(k_o d)\nu_x] \right) \]
\[ \text{sgn}[n_1^2 n_2^2 \nu_x^2 - n_4^2 \Gamma_2 \Gamma_3] \frac{\Gamma_2}{n_4^2 \nu_x^4 + n_4^2 \nu_x^2} \right) \right\}^2 \]
\[ \left[ \lambda_2 (n_x^2 - \nu_x^2) + n_4^2 \Gamma_4 \right] \left[ \frac{(k_o R_o) q_2(0)}{(k_o R_o)^{3/2}} \right] \right\} (4.16b) \]
\[ F_D = \left\{ \frac{8\pi\lambda_2^{5/2}}{\gamma_0 \Gamma_2} \left[ (k_o b) + \frac{2 \Gamma_4}{n^2 - n_2^2} \right] \right. \]
\[ \left[ (k_o d) + n^2 \left( \frac{n_4^2 \Gamma_3}{n_4^2 \nu_x^4 + n_4^2 \nu_x^2} + \frac{n_4^2 \Gamma_2}{n_4^2 \nu_x^4 + n_4^2 \nu_x^2} \right) \right] \right\} (4.16c) \]

and where

\[ \gamma_0 = \beta/k_o = \left[ n^2 - (\nu_x^2 + \nu_y^2) \right]^{\frac{1}{2}} \] (4.17a)
\[ \lambda_2 = (\gamma_0^2 - n_2^2)^{\frac{1}{2}} \] (4.17b)
In order to highlight the dependence of this cumbersome result on the radius of curvature of the channel we simplify (4.16a) through (4.16c) by writing

$$\alpha = \frac{C_i}{(k_0 R_0)^{3/2}} e^{k_0 R_0 q_2(0)}$$

(4.18)

where $C_i$ (the subscript $i$ designating that this result is associated with an "inhomogeneous" medium about the channel guide) is a parameter dependent upon the refractive indices of the channel guide and its surrounding media, the dimensions of the channel guide, and the specific $E_{pq}^x$ mode propagating in the channel.

Figure 4.2 is a plot of the normalized attenuation coefficient versus $k_0 R_0$ for $k_0 d = 5$ and $k_0 b = 10$. The refractive index of the channel guide is assigned to be 4.04, that of the substrate 4.00, and that of the covering medium 1.00.

4.4 Conclusions

As mentioned in Section 1.1, there are no analytical reports in the literature which address a problem similar enough to that considered here to allow comparisons with the force to effectively confirm or reject the principal result derived in this report, i.e., the form of the normalized attenuation coefficient $\alpha$ presented in Section 4.4. To this extent, then, the expression for $\alpha$ in (4.16) is a new result. To appreciate that the form of $\alpha$ constitutes a unique as well as a new result, we consider the form of the normalized attenuation coefficient $\alpha'$ (primes do not indicate differentiation here) for a bent, rectangular dielectric waveguide immersed in a homogeneous medium of lower refractive index than that of the guide (cf. [5], Section 9.8), i.e.
Figure 4.2 The Normalized Attenuation Coefficient $\alpha$ Versus $k_o R_o$ for a Bent, Rectangular Channel Guide in a Dense Substrate
\[ \alpha' = \frac{C_h}{(k_o R_o)^{3/2}} e^{k_o R_o Q(0)} \] (4.19)

where \( C_h \) and \( Q(0) \), like \( C_i \) and \( q_2(0) \) in (4.41), are parameters depending upon material characteristics and the specific mode assumed to be propagating in the guide.

Upon comparison of (4.19) with (4.18), two conclusions are immediate. First, even though \( Q(0) \) and \( q_2(0) \) are not necessarily equal, the form of the exponential dependence of \( \alpha \) and \( \alpha' \) on \( k_o R_o \) is the same. Second, the algebraic dependence of \( \alpha \) and \( \alpha' \) on \( R_o \) is not the same, being \( \left( \frac{1}{k_o R_o} \right)^{3/2} \) for \( \alpha \) and \( \left( \frac{1}{k_o R_o} \right)^{3/2} \) for \( \alpha' \). Certainly, then, for a pair of bent, rectangular dielectric waveguides, one in a dense substrate below a material interface and the other in a homogeneous medium, when \( q_2(0) = Q(0) \), one expects the rate of decrease of attenuation with increasing \( k_o R_o \) to be greater in the case of the guide in the substrate below a material interface. In order to understand how the overall attenuation behaves in such situations we consider Figure 4.3, which is a plot (versus \( k_o R_o \)) of the values of the normalized attenuation constant for each of three bent waveguide configurations.

In case I, the guide has a refractive index \( n = 4.04 \), dimensions of \( k_o d = 3, \ k_o b = 10 \), and is embedded in a homogeneous medium of refractive index \( n_1 = 4.00 \). In case II, the guide has a refractive index \( n = 4.028 \), dimensions of \( k_o d = 5, \ k_o b = 10 \), and is likewise embedded in a homogeneous medium of refractive index \( n_1 = 4.00 \). Case III has a guide of refractive index \( n = 4.04 \) and dimensions \( k_o d = 5, \ k_o b = 10 \) embedded in a substrate of refractive index \( n_2 = 4.00 \) below a material region refractive
Figure 4.3 The Normalized Attenuation Constant v.s. $k_0 R_0$ for Three Bent Waveguide Configurations with Identical Exponential Variation
index \( n_1 = 1.00 \). The dimensions in case I were altered relative to those of case III in order to provide \( q_2(0) = Q_1(0) \). Likewise, the refractive index of the guide in case II was altered relative to that of the guide in case III to again assure \( q_2(0) = Q_{II}(0) \). Thus, the exponential variation of the normalized attenuation coefficient is the same for the guides in each of cases I, II, and III.

From Figure 4.3 we see that in addition to a more rapid decrease in attenuation with increasing \( k_0 R_0 \), the guide in a dense substrate below a material interface actually exhibits a lesser absolute attenuation than do the guides surrounded by homogeneous media. Indeed, while the difference in values of normalized attenuation constant between cases I or II and III is less than an order of magnitude for \( k_0 R_0 = 500 \), it lies between one and two orders of magnitude for \( k_0 R_0 = 5000 \).

The divergent tendency of the curve for case III relative to those of cases I and II results from the distinctly different algebraic variation with \( k_0 R_0 \) in case III relative to cases I and II. However, this divergent tendency is ultimately overcome by the identical exponential decay terms present in each case. This fact is illustrated in Figure 4.4, which is a plot of the normalized attenuation constants for cases I and III over extended ranges in both the abscissa and the ordinate.

Recalling the inverse three-halves power algebraic dependence of \( \alpha \) on \( k_0 R_0 \), we may interpret this behavior as a kind of lateral-wave phenomenon due to the presence of the material interface. This interpretation implies that the direct radiation which accounts for losses of a curved guide in a homogeneous medium is, in the case of the guide in a dense substrate below a material interface, effectively cancelled by an image contribution from the material interface.
Figure 4.4 The Normalized Attenuation Constant v.s. $k_0R_0$ for Cases I & III
(Extended Range)
References


Appendices
Appendix A

Determination and Application
of Source and Boundary
Conditions for \( \Phi_\tau^m, \Phi_\rho^m \)

A.1. Introduction

Inspection of (2.30a) through (2.30c) reveals that we have eight unknown transform coefficients, hence we expect to be able to specify eight independent equations relating these coefficients as determined by boundary and source conditions on the field components. We begin by considering equations (1.9a) through (1.9b) which relate the transforms of the fields transverse to the \( z \)-direction to the transforms of the cartesian field components. Direct integration of (1.9a) through (1.9b) after dividing both sides by \( \alpha \) and multiplying through by \( J_v(k_0 \alpha) \) gives the following results:

\[
E_\phi(r, z) = i\xi_0 \int_0^\infty \phi^m_\tau(\alpha, z) J'_v(k_0 \alpha r) d\alpha \nonumber
\]

\[
= \frac{-i\gamma R_0}{k_0 r} \int_0^\infty \frac{\partial}{\partial z} \Phi^e_\tau(\alpha, z) J_v(k_0 \alpha r) d\alpha \tag{A.1a}
\]

\[
E_r(r, z) = \frac{-\xi_0 \gamma R_0}{r} \int_0^\infty \phi^m_\rho(\alpha, z) J_v(k_0 \alpha z) d\alpha \nonumber
\]

\[
+ \frac{1}{k_0} \int_0^\infty \frac{\partial}{\partial z} \phi^e_\rho(\alpha, z) J'_v(k_0 \alpha r) d\alpha \tag{A.1b}
\]
\[ H_\phi(r, z) = \frac{-i\gamma R_0}{rk_0^2} \int_0^\infty \frac{d\alpha}{\alpha} \frac{\bar{\phi}^m_r(\alpha, z) J_v(k_0 r) da}{\alpha} \]

\[ -i n_1^2 \int_0^{2\pi} \frac{\phi_e^r(\alpha, z) J_v^l(k_0 r) d\alpha}{c_0} \]  

\[ H_r(r, z) = \frac{\gamma n_1^2 \alpha R_0}{rk_0^2} \int_0^\infty \frac{d\alpha}{\alpha} \frac{\bar{\phi}^e_r(\alpha, z) J_v(k_0 r) da}{\alpha} \]

\[ + \frac{1}{k_0} \int_0^\infty \frac{d\alpha}{\alpha} \frac{n_1^2 \bar{\phi}^m_r(\alpha, z) J_v^l(k_0 r) d\alpha}{\alpha} \]

(A.1c)

(A.1d)

A.2 Boundary Conditions at the Dielectric Interface

Since \( E_\phi \) and \( E_r \) must be continuous at \( z = 0 \), inspection of (A.1a) and (A.1b) yields

\[ \tilde{\phi}^m_r(\alpha, 0+) = \tilde{\phi}^m_r(\alpha, 0-) \]

(A.2a)

\[ \frac{3}{\partial z} \tilde{\phi}^e_r(\alpha, z) \bigg|_{z=0^+} = \frac{3}{\partial z} \tilde{\phi}^e_r(\alpha, z) \bigg|_{z=0^-} \]

(A.2b)

Likewise, continuity of \( H_\phi \) and \( H_r \) at \( z = 0 \) and inspection of (A.1c) and (A.1d) give

\[ n_1^2 \tilde{\phi}^e_r(\alpha, 0+) = n_2^2 \tilde{\phi}^e_r(\alpha, 0-) \]

(A.2c)

\[ \frac{3}{\partial z} \tilde{\phi}^m_r(\alpha, z) \bigg|_{z=0^+} = \frac{3}{\partial z} \tilde{\phi}^m_r(\alpha, z) \bigg|_{z=0^-} \]

(A.2d)
A.3 Boundary and Source Conditions at the Source Plane

The continuity of \( E_{\phi} \) and \( E_r \) at \( z = z' \) and inspection of (A.1a) and (A.1b) yields

\[
\tilde{\Phi}_r^m(\alpha, z') = \tilde{\Phi}_r^m(\alpha, z')^{-} \quad (A.3a)
\]

\[
\frac{a}{a} \tilde{\Phi}_r^e(\alpha, z) \bigg|_{z = z'} = \frac{a}{a} \tilde{\Phi}_r^e(\alpha, z) \bigg|_{z = z'}^{-} \quad (A.3b)
\]

Using the fact that the magnetic intensity \( \tilde{\Phi} \) must be discontinuous as the result of surface currents, while the electric flux density \( \tilde{\Phi} \) must be discontinuous at points where surface charge accumulations exist, we find the additional expressions

\[
\frac{a}{a} \tilde{\Phi}_r^m(\alpha, z) \bigg|_{z = z'} = -\frac{a}{a} \tilde{\Phi}_r^m(\alpha, z) \bigg|_{z = z'}^{-} = -ivk_0J_v(k_0r') \quad (A.3c)
\]

\[
\tilde{\Phi}_r^e(\alpha, z') + \tilde{\Phi}_r^e(\alpha, z')^{-} = \frac{k_0^2\zeta^2}{in^2} J'_v(k_0r') \quad (A.3d)
\]

Equations (A.2a) through (A.2d) and equations (A.3a) through (A.3b) constitute the independent set of eight equations we need in order to solve for the transform coefficients. Indeed, using (2.30a) and (2.30b) with (A.2a) through (A.2b) we find

\[
A_r^m = \frac{2u_2}{u_1+u_2} B_r^m \quad (A.4a)
\]

\[
B_{nr}^m = \frac{u_2-u_1}{u_2+u_1} B_{pr}^m \quad (A.4b)
\]
\[ A^e_r = \frac{2u_2n_2^2}{u_2n_1^2+u_1n_2^2} B^e_p \quad (A.4c) \]

\[ B^e_{nr} = \frac{u_2n_1^2-u_1n_2^2}{u_2n_1^2+u_1n_2^2} B^e_p \quad (A.4d) \]

Likewise, using (2.30b) and (2.30c) with (A.3a) through (A.3d), we have

\[ B^m_{pr} \left\{ \frac{u_2-u_1}{u_1+u_2} e^{k_0 u_2 z'} + e^{-k_0 u_2 z'} \right\} B^m_{pr} \quad (A.5a) \]

\[ B^e_{pr} \left\{ \frac{u_2n_1^2-u_1n_2^2}{u_2n_1^2+u_1n_2^2} e^{k_0 u_2 z'} - e^{-k_0 u_2 z'} \right\} B^e_{pr} \quad (A.5b) \]

\[ B^m_{pr} = \frac{i\gamma R_0 e^{\frac{k_0^2}{2u_2}}}{u_2} J_\nu(k_0 a r') \quad (A.5c) \]

\[ B^e_{pr} = \frac{k_0^2 e^{\frac{k_0^2}{2u_2}}}{i2n_2} J_\nu(k_0 a r') \quad (A.5d) \]

Since (A.4a) through (A.4d) and (A.5a) and (A.5b) are given in terms of either \( B^m_{pr} \) or \( B^e_{pr} \), and since these are given explicitly by (A.5c) and (A.5d), respectively, the transform coefficients \( \phi^m_r \) and \( \phi^e_r \) are now known.

### A.4 Integral Representations for \( \phi^e_r, \phi^m_r \)

Upon substitution of (A.4a) through (A.4d) and (A.5a) and (A.5b) into (2.30a) through (2.30c) we find

\[ \phi^e_r(a,z) = B^e_{pr} \frac{2u_2n_2^2}{u_2n_1^2+u_1n_2^2} e^{-k_0(u_1 z-u_2 z')} , \quad z \geq 0 \quad (A.6a) \]
\[ \Phi^m_r(a, z) = B^m_{pr} \left[ \frac{2u_2}{u_1+u_2} \right] e^{-k_0(u_1z-u_2z')} e^{k_0u_2(z+z')} \delta_{m, 0}, \ z \geq 0 \] (A.6b)

\[ \Phi^e_r(a, z) = B^e_{pr} \left\{ \frac{u_2n_1^2-u_1n_2^2}{u_2n_1^2+u_1n_2^2} \right\} e^{k_0u_2(z+z')} e^{-k_0u_2|z-z'|} + \text{sgn}[(z-z')] e^{k_0u_2|z-z'|} \right\}, \ z \leq 0 \] (A.6c)

\[ \Phi^m_r(a, z) = B^m_{pr} \left\{ \frac{u_2-u_1}{u_2+u_1} \right\} e^{k_0u_2(z+z')} e^{-k_0u_2|z-z'|} \right\}, \ z \leq 0 \] (A.6d)

where \( \text{sgn}[(z-z')] \) in (A.6c) is the numerical sign of \((z-z')\).

With the field transforms thus specified, we may employ (2.10) directly in order to generate the required integral representations, thus

\[ \Phi^e_r(r, z) = \frac{k_0^2}{i4n_2^2} \int_{-i\pi}^{i\pi} J'_v(k_0ar')H_v^2(k_0ar) e^{-k_0(u_1z-u_2z')} e^{k_0u_2(z+z')} \right\}, \ z \geq 0 \] (A.7a)

\[ \Phi^m_r(r, z) = \frac{i\gamma R_0 k_0^2}{4} \int_{-i\pi}^{i\pi} J'_v(k_0ar')H_v(k_0ar) e^{-k_0u_2(z-z')} \right\}, \ z \geq 0 \] (A.7b)
\[
\varphi^e_r(r,z) = \frac{k_o r'}{i4n_2^2} \left[ \int \limits_{-\pi}^{\pi} J_v(k_o r') H_v^{(2)}(k_o r) \right.
\]
\[
\left\{ \frac{u_2 n_1^2 - u_1 n_2^2}{u_2 n_1^2 + u_1 n_2^2} e^{k_o u_2 z + z'} + \text{sgn}(z - z') e^{-k_o u_2 |z - z'|} \right\} \alpha^2 d\alpha
\]
\[z \geq 0 \quad (A.7c)\]

\[
\varphi^m_r(r,z) = \frac{iyR_o k_o^2}{2} \left[ \int \limits_{-\pi}^{\pi} J_v(k_o r') H_v^{(2)}(k_o r) \right.
\]
\[
\left\{ \frac{u_2 - u_1}{u_2 + u_1} e^{k_o u_2 (z + z')} + e^{-k_o u_2 |z - z'|} \right\} \frac{\alpha d\alpha}{u_2}, \quad z \leq 0 \quad (A.7d)\]
Appendix B

The Steepest Descent Evaluation of $V_e^5$

To find the saddle-points of $q_2(s_2)$ in the $s_2$-plane we must evaluate

$$\frac{dq_2}{ds_2},$$

and upon doing this we find

$$\frac{dq_2}{ds_2} = -\left[\frac{2\lambda s_2}{n_2^2-s_2^2}\right] \quad (B.1)$$

From (B.1) we conclude that $s_2 = 0$ is a saddle-point for $q_2(s_2)$. Further analysis shows that for $\text{Im } n_2 \ll 1$ (as we assume), the path of steepest descent is essentially along the $\text{Re } s_2$ axis.

Differentiation of (B.1) provides

$$\left.\frac{d^2q_2}{ds_2^2}\right|_{s_2=0} = \frac{-2\lambda_2}{n_2^2} \quad (B.2)$$

Notice that since $\lambda$ is an even function of $s_2$, then $q(s_2)$ is an even function of $s_2$, so the coefficient of the $s_2^3$ term in a Taylor series must vanish, thus

$$\frac{d^3q_2}{ds_2^3} = 0 \quad (B.3)$$

A Taylor series representation of $q(s_2)$ through the first four terms about $s_2 = 0$ is thus, using (B.1), (B.2), and (B.3),
\[ q_2(s_2) = q_2(0) - \frac{\lambda_2 n_2^2}{\gamma} s_2^2 \]  
(B.4a)

where

\[ q_2(0) = 2 \left\{ \frac{\lambda_2 + \frac{\gamma}{2} \log \frac{\gamma - \lambda_2}{\gamma + \lambda_2}}{\gamma} \right\} \]  
(B.4b)

Consider now \( \gamma_e \) for \( z < 0 \) as given by (3.22c). If we let \( u_2 = is_2 \), then

\[ \gamma_e(y, y', z, z'; s_2) = e^{\frac{k_0 \lambda (y + y')}{\gamma}} \left\{ \frac{is_2 n_1^2 - u_1 n_2^2}{is_2 n_1^2 + u_1 n_2^2} e^{ik_0(z + z')s_2} + e^{-ik_0|z-z'|s_2} \right\} \]  
(B.5)

Consideration of (B.5) for \( s_2 = 0 \) yields

\[ \gamma_e(y, y', z, z'; 0) = 0 \]  
(B.6)

so we must evaluate

\[ \left. \frac{\partial^2 \gamma_e}{\partial s_2^2} \right|_{s_2=0} \]

in order to generate the leading term in the asymptotic expansion of \( \gamma_e^S \). Doing this, we find
\[
\frac{3}{2} \gamma_e \left| \frac{s^2}{s_2^2} \right|_{s_2=0} = 4 e^{\frac{\gamma_e}{\left(\gamma^2 - n_2^2\right)^{1/2}}} \left[ k_0 z' - \frac{n_1^2}{n_2^2(n_2^2 - n_1^2)^{1/2}} \right]
\]

\[
\left[ k_0 z - \frac{n_1^2}{n_2^2(n_2^2 - n_1^2)^{1/2}} \right] \quad \text{(B.7)}
\]

The leading term of the asymptotic expansion of \( V_e^s \) has the form (cf. [10])

\[
V_e^s(y, y', z, z') \sim \frac{-c_0}{8n_2^2k_0R_0} e^{k_0R_0q_2(0)} \left| \frac{3}{2} \gamma_e \left| \frac{s^2}{s_2^2} \right|_{s_2=0} \right|
\]

\[
\int_0^\infty e^{-k_0R_0\left(\frac{\lambda_2}{n_2^2}\right)s_2} s_2^2 ds_2 \quad \text{(B.8)}
\]

Using

\[
\int_0^\infty e^{-k_0R_0\left(\frac{\lambda_2}{n_2^2}\right)s_2} s_2^2 ds_2 = \left(\frac{n_2^2}{\lambda_2k_0R_0}\right)^{3/2} \frac{\sqrt{\pi}}{8}
\]

and (B.7) in (B.8),

\[
V_e^s(y, y', z, z') \sim \frac{-c_0}{16n_2^2k_0R_0} \left(\frac{\pi n_2^2}{\lambda_2k_0R_0}\right)^{3/2} e^{k_0R_0q_2(0)}
\]

\[
e^{\frac{k_0\lambda_2(y+y')}{\lambda_2}} \left[ k_0 z' - \frac{n_1^2}{n_2^2(n_2^2 - n_1^2)^{1/2}} \right] \left[ k_0 z - \frac{n_1^2}{n_2^2(n_2^2 - n_1^2)^{1/2}} \right] \quad \text{(B.9)}
\]

which is (3.23) in Section 3.3.2.
Appendix C

The Evaluation of P and C

C.1. The Evaluation of P

Application of (1.3) to the channel guide cross section in Figure 4.1 gives

\[ p = 2 \int_0^b \int_{-d}^0 dy(E_{0x}H_{0y}) \]  \hspace{1cm} (C.1)

Our consideration of modes far from cutoff manifests itself in (C.1) in that the integration can be taken over the guide cross section only.

Substitution of (4.2b) and (4.2f) in (C.1) yields

\[ p = -2A^2 \frac{n_2k_x^2-k_0^2}{k_x^2} \frac{k_0n_2^2}{k_0} \int_{-d}^0 \sin^2[k_x(x+\xi)]dx \int_0^b \cos^2[k_w(w+n)]dw \]  \hspace{1cm} (C.2)

Expansion of the integrand in \( \int \sin^2[k_x(x+\xi)]dx \) and use of (D.7a), (D.7b), (D.9a), and (D.9b) [cf. Appendix D] gives

\[ \int_{-d}^0 \sin^2[k_x(x+\xi)]dx = \frac{d}{2} + \frac{n_2^2}{2} \left\{ \frac{n_1^2\gamma_3}{(n_22+n_41k_x^2)} + \frac{n_2^2\gamma_2}{(n_2^42+n_4^4k_x^2)} \right\} \]  \hspace{1cm} (C.3)

Similarly, expansion of the integrand of \( \cos^2[k_w(w+n)]dw \) and use of (D.4a), (D.4b), (D.6a), and (D.6b) gives

\[ \int_0^b \cos^2[k_w(w+n)]dw = \frac{b}{2} + \frac{\gamma_4}{k_0^2(n_2^2-n_4^2)} \]  \hspace{1cm} (C.4)
Making the definitions

\[ L_x = \frac{n^2}{2} \left\{ \frac{n_1^2 \gamma_3}{(n_1^2 \gamma_3 + n_1^2 k_x^2)} + \frac{n_2^2 \gamma_2}{(n_2^2 \gamma_2 + n_2^2 k_x^2)} \right\} \]  

(C.5a)

\[ L_w = \frac{\gamma_4}{k^2_0 (n^2 - n_w^2)} \]  

(C.5b)

and substituting (C.3) and (C.4) into (C.2) gives

\[ p = -2A^2 \left[ \frac{n_0^2 k_0^2 - k_x^2}{k_x^2} \right] \left[ \frac{k_0 n_0^2}{\beta \gamma_0} \right] \left( \frac{d}{2} + L_x \right) \left( \frac{b}{2} + L_w \right) \]  

(C.6)

C.2. The Evaluation of C

C.2.1 An Explicit Integral Representation of C

For the circumstance of modes far from cutoff, \( \beta = (n_0^2 k_0^2 - k_x^2)^{1/2} = n k_0 \), thus from (4.2a) and (4.2b) we have \( |E_{ox}/E_{os}| = \beta/k_x \gg 1 \), thus we will neglect \( E_{os} \) as a field source and consider our straight guide modes to have only x-directed components. Recalling (2.49), where \( E_{or} \rightarrow E_{ow} \) (i.e., \( E_{or} \) becomes \( E_{ow} \) in our local coordinate system for the straight guide), \( E_{oq} \rightarrow E_{os} \), and \( E_{oz} \rightarrow E_{ox} \), we conclude that for the special case we are now addressing \( \phi_m(x,w) = 0 \) or, in particular

\[ \phi_s(x,w) = 0 \]  

(C.7)

Rewriting (1.4) in terms of the local coordinates of the straight guide,

\[ C = \int_{-d}^{0} \left\{ E_0^\perp \delta H - \delta E_0^\perp \right\} \cdot a_w \, dx \left|_{x=b} \right. \]  

(C.8)
The straight guide fields $E^-$ and $H^-$ can be written in the form

$$E^- = E_{oX} \bar{a}_x$$

$$H^- = -H_{ow} \bar{a}_w + H_{os} \bar{a}_s$$

(C.9a)

(C.9b)

Because of the scalar product with $\bar{a}_w$ that occurs in (C.8) and the forms of $E_o^-$ and $H_o^-$, we conclude that we need only find the components $\delta E_x$ of $\delta E$ and $\delta H_s$ of $\delta H$ in order to evaluate $C$.

Using (C.7) in (1.5c) and recalling that we have defined $\Phi^e = E_x$ and $\phi^m = H_x$ (where the subscript $z$ becomes $x$ and the subscript $\phi$ becomes $s$ in our local coordinate system) we have

$$k_0^2 \beta^2 H^s = i \omega c \frac{\partial \Phi^e}{\partial r} - e^{i \phi} \int_0^\infty \left( \frac{\partial H^s}{\partial z} \right) J_0 (k_0 r) r dr$$

(C.10)

where we have inserted the Hankel Transform representation of $H^s$. Substituting $H^s$ from (1.10c) in (C.10) gives

$$k_0^2 \beta^2 H^s = i \omega c \frac{\partial \Phi^e}{\partial r} + i \frac{n_z^2}{c_0} \int_0^\infty \left( \frac{\partial \Phi^e}{\partial z} \right) J_0 (k_0 r) r dr$$

(C.11)

However, since from (3.23) we see that $V_e^s$ is linear in $z$, it follows from (2.49) that $\frac{\partial^2 \Phi^e}{\partial z^2} = 0$, thus $\frac{\partial^2 \Phi^e}{\partial z^2} = 0$, and (C.11) becomes

$$H^s = \frac{-i}{\tau_0 k_0} \frac{\partial}{\partial w} \Phi^e(x, w)$$

(C.12)

where we have used $\frac{\partial}{\partial r} = \frac{\partial}{\partial y} = \frac{\partial}{\partial w}$. 

If we now substitute (4.1) into (3.23) we see that

$$\frac{\partial e}{\partial \omega} = \frac{\lambda}{\tau} \phi(x, \omega)$$  \hspace{1cm} (C.13)

so use of (C.13) in (C.12) yields

$$\delta_{\omega} = \delta_{\omega} = \frac{-i\lambda}{\tau} \phi(x, \omega)$$ \hspace{1cm} (C.14)

With (C.9a), (C.9b), and (C.14) we find the following:

$$\delta_{\omega} = \frac{-i\lambda}{\tau} \phi(x, \omega)$$ \hspace{1cm} (C.15)

$$\delta_{\omega} = \phi(x, \omega) \delta_{\omega}$$ \hspace{1cm} (C.16)

Substitution of (C.15) and (C.16) in (C.8) yields

$$C = \int_{-d}^{b} \phi(x, \omega) \left( \frac{i\lambda}{\tau} \right) E_{\omega}(x, \omega) + \phi(x, \omega) \delta_{\omega} \right) \left. \right| \text{w=b}$$ \hspace{1cm} (C.17)

Finally, we can use (4.2b) and (4.2d) in (C.17) to generate the expression

$$C = - \int_{-d}^{b} \phi(x, \omega) \left\{ \frac{A}{\tau} \left[ \frac{n k^2 - k^2}{\beta} \left( \frac{\lambda^2}{k^2} \right) \cos k(x+\xi) \right] + \frac{n k^2}{\beta} \left( \frac{k^2}{k^2} \sin k(x+\xi) \right) \right. \right. \left. \right| \text{w=b}$$ \hspace{1cm} (C.18)
C.2.2 An Explicit Representation of $\Phi^e_s(x,w)$

Having (C.18), our objective becomes that of finding an explicit form of $\Phi^e_s(x,w)$ for the special case under consideration. We begin by rewriting (3.23) in the abbreviated form (in terms of local coordinates)

$$V^e_s(w,w',x,x') = C \cdot V^S_e(w,w') u^S_e(x,x')$$  \hspace{1cm} (C.19a)

where

$$C = \frac{-\epsilon}{16\pi n_2^2 k_o R_o} \left( \frac{\pi n_2}{\lambda_o k R_o} \right)^{3/2} \frac{k_o R_o q_2(0)}{\lambda_2}$$  \hspace{1cm} (C.19b)

$$V^S_e(w,w') \triangleq e^{-\frac{k_o \lambda_2 (w-b/2)}{e_o \lambda_2 (w'-b/2)}}$$  \hspace{1cm} (C.19c)

$$u^S_e(x,x') \triangleq (k_o x'-k) (k_o x-k)$$  \hspace{1cm} (C.19d)

where

$$K = \frac{n_2^2}{n_2^2 - n_1^2}$$  \hspace{1cm} (C.19e)

Upon substitution of (C.19a) and (4.2b) into (2.64) we find

$$\Phi^e_s(w,x) = -AC \cdot w(n_2-n_1)R_o (k_o^2 + \frac{\partial^2}{\partial x^2}) \frac{n_2^2 - k}{\partial k} \int_{-d}^{b} V^S_e(w,w') \cos k_w (w'+\eta) d\eta' \int_0^o u^S_e(x,x') \sin k_x (x'+\zeta) dx'$$  \hspace{1cm} (C.20)

The integral on $[0,b]$ in (C.20) becomes, upon inserting (C.19c),

$$\int_0^b V^S_e(w,w') \cos k_w (w'+\eta) d\eta' =$$

$$k_o \lambda_2 \frac{w-b}{2} \int_0^b k_o \lambda_2 \frac{w'-b/2}{2} \cos k_w (w'+\eta) d\eta'$$  \hspace{1cm} (C.21)
Carrying out the integration on the RHS of (C.21) and using (D-4a), (D-4b), (D-6a), and (D-6b) yields

\[
C_1 \Delta = \int_0^b e^{k_0 \lambda_2 (w'-b)/2} \cos k_w (w'+\eta) dw' = \\
\frac{k_w}{k_0 \gamma_2 (n_2^2 - n_2^2)^{1/2}} \left\{ \lambda_2 k_0 \left[ \text{sgn}[\cos(k_w \eta)] \frac{\text{sgn}[k_w^2 - \gamma_4^2]}{k_w^2} \right] \right. \\
\left. \left( k_o \beta_2^2 \right)/2 - e^{-\left( k_o \beta_2^2 \right)/2} \right\} + \gamma_4 \left[ \frac{\text{sgn}[\cos(k_w \eta)]}{k_w^2} \frac{\gamma_4^2}{k_w^2} \left( k_o \beta_2^2 / 2 \right) + e^{-\left( k_o \beta_2^2 \right)/2} \right] \right\} \quad (C.22)
\]

Combining (C.22) with (C.21) we have

\[
\int_0^b v_e^{s}(w,w') \cos k_w (w'+\eta) dw = C_1 e^{k_0 \lambda_2 (w'-b)/2} \quad (C.23)
\]

The integral on [-d,0] in (C.20) becomes, upon inserting (C.19d),

\[
\int_{-d}^0 u_e^{s}(w,w') \sin k_\xi (x'+\xi) dx' = \\
\left( k_o \frac{x}{\xi} \right) \int_{-d}^0 \frac{(k_o x' - \xi)}{\xi} \sin k_\xi (x'+\xi) dx' \quad (C.24)
\]

Evaluating the integral on the RHS of (C.24) and using (D-7a), (D-7b), (4.6a), and (4.6b) we find

\[
C_2 \Delta = \int_{-d}^0 (k_o x' - \xi) \sin k_\xi (x'+\xi) dx' = \left\{ \frac{k_o}{k_x} \right. \\
\left. \left[ \text{sgn}[\cos(k_d \xi)] \frac{\text{sgn}[n_2^2 n_2^2 k_x^2 - n_2^2 \gamma_2^2 \gamma_3]}{4 n_2^2 \gamma_2 d + n_2^2 \gamma_2^2 \gamma_3} \right] \frac{(n_2^2 \gamma_2 d + n_2^2)}{(n_2^2 \gamma_2 + n_2^2 \gamma_2^2)^{1/2}} \right\}
\]
\[ \begin{align*}
\frac{n_1^2}{(n_1^2 + n_1^2 x^{1/2})} + \frac{K_n^2}{K_x} \left[ \frac{\gamma_3}{(n_3^2 + n_3^2 x^{1/2})} \right] + \text{sgn} \left[ \cos(k_x d) \right] \text{sgn} \left[ \frac{2 n_2^2 n_1^2 x - n_2 \gamma_3}{(n_2^2 + n_2 \gamma_3 x^{1/2})} \right] \gamma_2 \left( \frac{x^{1/2}}{n_2^2 + n_2 \gamma_3 x^{1/2}} \right)^{1/2} \right] 
\end{align*} \] (C.25)

Combining (C.25) with (C.24) we have

\[ \int_{-d}^{0} u^S_{e}(x, x') \sin k_x (x' + x) \, dx' = C_2 \left( k x - k \right) = \frac{1}{k} (k x - k) \] (C.26)

We can now employ (C.19b), (C.23), and (C.26) with (C.20) to produce the result

\[ \phi_s^e(w, x) = \begin{cases} 
Ac_1 C_2^2 \frac{k_0^2 (n^2 - n_2^2) (n_2^2 + n_2^2 \gamma_3 x)}{16 \pi^2 \beta k_x \lambda_2} \\
\left( \frac{\pi n_2^2}{k_0 R \lambda_2} \right)^{3/2} e^{k_0 \lambda_2 (\frac{w}{2})} e^{k_0 R q_2(0)} (k_0 x - k) \end{cases} \] (C.27)

where we have also used \( w_0 = \frac{k_0}{\epsilon_0} \).

C.2.3 An Explicit Representation of C

Having (C.27), we need only use it with (C.18) to obtain

\[ C = \begin{cases} 
-A^2 c_1 c_2^2 \text{sgn} \left[ \cos(k b) \right] \text{sgn} \left[ \frac{k_0^2 - \gamma_4^2}{w} \right] \\
\frac{k_k}{k_0} \frac{(n_2^2 + n_2^2 \gamma_3 x^{1/2})}{16 \pi^2 \beta k_x \lambda_2^2 R_0} \left[ \frac{\lambda_2 (n_2^2 + n_2^2 \gamma_3 x^{1/2}) + n_2^2 \gamma_4}{\lambda_2 (n_2^2 + n_2^2 \gamma_3 x^{1/2}) + n_2^2 \gamma_4} \right] \\
\left( \frac{\pi n_2^2}{k_0 R \lambda_2} \right)^{3/2} e^{(k_0 \lambda_2) / 2} e^{k_0 R q_2(0)} \end{cases} \] (C.28)
APPENDIX D

THE EXPLICIT REPRESENTATION OF TRIGONOMETRIC FUNCTIONS IN TERMS OF PROBLEM PARAMETERS

During the course of the analysis in Appendix C we employ a set of identities which express trigonometric functions explicitly in terms of the eigenvalues $K_X, K_W$ and the material parameters of our problem. The development of these identities from equations (4.4a), (4.4b), (4.6a), and (4.6b) is a straightforward arithmetical procedure, but one must exercise care in accounting for a multitude of possible choices of numerical sign if ambiguous results are to be avoided.

Consideration of (4.4a) and (4.4b) reveals that the eigenvalues $K_X, K_W$ may be either less than or greater than zero. Indeed, review of the field expressions (4.2a) through (4.2f) shows that the numerical signs of $K_X$ and $K_W$ don't effect these formulae at all; i.e., the field expressions depend on $|K_X|$ and $|K_W|$. We are thus free to choose the numerical signs of $K_X$ and $K_W$ and will choose these to be the positive solutions of the eigenvalue equations (4.4a) and (4.6b).

Consider now equations (4.6a) and (4.6b). Since we choose $K_X > 0$ and $K_W > 0$, and since $\gamma_3 > 0$ and $\gamma_4 > 0$, we can always find solutions $\xi$ and $\eta$ such that

$$\frac{-\pi}{2K_X} < \xi < 0 \quad (D.1a)$$
\[
\frac{-\pi}{2K_w} < \eta < 0 \quad \text{(D.1b)}
\]

Choosing \( \xi \) and \( \eta \) as in (D.1a) and (D.1b), respectively we have

\[
\sin(K_w \eta) < 0 \quad \text{(D.2a)}
\]
\[
\cos(K_w \eta) > 0 \quad \text{(D.2b)}
\]
\[
\sin(K_x \xi) < 0 \quad \text{(D.2c)}
\]
\[
\cos(K_x \xi) > 0 \quad \text{(D.2d)}
\]

We can now proceed to outline the derivation of the identities we require in Appendix C.

Using the trigonometric identities

\[
\sin \theta = \frac{\text{sgn}[\cos \theta] \tan \theta}{(1 + \tan^2 \theta)^{1/2}} \quad \text{(D.3a)}
\]
\[
\cos \theta = -\frac{\text{sgn}[\cos \theta]}{(1 + \tan^2 \theta)^{1/2}} \quad \text{(D.3b)}
\]

with (4.6b) and (D.2a), (D.2b), we find

\[
\sin(K_w \eta) = \frac{-\gamma_4}{k_o(n^2 - n_2^2)^{1/2}} \quad \text{(D.4a)}
\]
\[
\cos(K_w \eta) = \frac{k_w}{k_o(n^2 - n_2^2)^{1/2}} \quad \text{(D.4b)}
\]

Similarly, use of (4.4b) with (D.3a), (D.3b) gives

\[
\sin(K_w b) = \frac{\text{sgn}[\cos(K_x b)] \text{sgn}[K_w^2 - \gamma_4^2](2K_w \gamma_4)}{k_o^2(n^2 - n_2^2)} \quad \text{(D.5a)}
\]
\[
\cos(K_w b) = \frac{\text{sgn}[\cos(K_w)]|K_w^2 - \gamma_4^2|}{K_0(n^2 - n_2^2)}
\]

(D.5b)

The use of both (D.4a), (D.4b) and (D.5a), (D.5b) with the standard trigonometric identities for sines and cosines of angular sums yields

\[
\sin[K_w(b+\eta)] = \text{sgn}[\cos(K_w)] \text{sgn}[K_w^2 - \gamma_4^2] \frac{\gamma_4}{K_0(n^2 - n_2^2)^{\frac{3}{2}}}
\]

(D.6a)

\[
\cos[K_w(b+\eta)] = \text{sgn}[\cos(K_w)] \text{sgn}[K_w^2 - \gamma_4^2] \frac{K_w}{K_0(n^2 - n_2^2)^{\frac{3}{2}}}
\]

(D.6b)

Now, proceeding in a similar way, one employs (4.6a) with (D.3a), (D.3b) and (D.2c), (D.2d) to produce

\[
\sin K_x \xi = \frac{-K_x n_1^2}{(n^2 \gamma_3 + n_1^2 \gamma_1^2)^{\frac{3}{2}}}
\]

(D.7a)

\[
\cos K_x \xi = \frac{\gamma_3 n_1^2}{(n^2 \gamma_3 + n_1^2 \gamma_1^2)}
\]

(D.7b)

Using (D.3a) and (D.3b) with (4.4a) one finds

\[
\sin(K_x d) = \frac{\text{sgn}[\cos(K_x)] \text{sgn}[n_1^2 n_2^2 K_x^2 - n^4 \gamma_2 \gamma_3] n_1^2 K_x (n_2^2 \gamma_2 + n_2^2 \gamma_3)}{(n_1^2 K_x + n_1^2 \gamma_2^2)(n_2^2 K_x + n_1^2 \gamma_2^2)^{\frac{3}{2}}}
\]

(D.8a)

\[
\cos(K_x d) = \frac{\text{sgn}[\cos(K_x)] |n_1^2 n_2^2 K_x^2 - n^4 \gamma_2 \gamma_3|}{(n_1^2 K_x + n_1^2 \gamma_2^2)(n_2^2 K_x + n_1^2 \gamma_2^2)^{\frac{3}{2}}}
\]

(D.8b)

Finally, use of (D.7a), (D.7b) and (D.8a), (D.8b) with the trigonometric formulae for angular sums and differences yields

\[
\sin(K_x(\xi - d)) = \frac{-\text{sgn}[\cos(K_x)] \text{sgn}[n_1^2 n_2^2 K_x^2 - n^4 \gamma_2 \gamma_3] n_1^2 K_x}{(n_2^2 K_x + n_1^2 \gamma_2^2)^{\frac{3}{2}}}
\]

(D.9a)
\[
\cos[K_x(\xi-d)] = -\frac{\text{sgn}[\cos(K_x)d] \text{sgn}[n_1 n_2^2 K_x^2 - n_4^2 \gamma_3 \gamma_2^3 n_2^2 \gamma_2^4]}{(n_2^2 K_x^2 + n_4^2 \gamma_2^4)^{1/2}}
\]

(D.9b)