Some properties of full heaps

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The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.
A full heap is a labeled infinite partially ordered set with labeling taken from the vertices of an underlying Dynkin diagram, satisfying certain conditions intended to capture the structure of that diagram. The notion of full heaps was introduced by R. Green as an affine extension of the minuscule heaps of J. Stembridge. Both authors applied these constructions to make observations of the Lie algebras associated to the underlying Dynkin diagrams. The main result of this thesis, Theorem 4.7.1, is a complete classification of all full heaps over Dynkin diagrams with a finite number of vertices, using only the general notion of Dynkin diagrams and entirely elementary methods that rely very little on the associated Lie theory. The second main result of the thesis, Theorem 5.1.7, is an extension of the Fundamental Theorem of Finite Distributive Lattices to locally finite posets, using a novel analogue of order ideal posets. We apply this construction in an analysis of full heaps to find our third main result, Theorem 5.5.1, an $ADE$ classification of the full heaps over simply laced affine Dynkin diagrams.
Dedication

To Moxie:

Sometimes a person has to go a very long distance out of his way to come back a short distance correctly.

(Edward Albee, The Zoo Story, 1958)
Acknowledgements

I would like to say that, as Athena from Zeus, this dissertation sprang fully formed from my, and only my, mind. But I cannot. It is a small stretch, indeed, to call this dissertation fully formed even now, and anyone who has witnessed any part of my writing process knows nothing in it “sprang” forth so much as loomed imposingly. A much worse deception, however, is in the claim that I am solely responsible for this work. Somehow mine is the only name that appears on the title page, but I certainly did not accomplish this alone. So now, not for the sake of tradition, but for the irrepressible need to express my gratitude, I do now my devoir and thank some of those whose names are missing from the title page. I say “some” because I know I will forget too many. Please forgive any oversights; they come only from my own errancy and not my lack of appreciation.

First, I thank the University of Colorado Boulder Mathematics Department and staff for their support, both financially and educationally, over the past eight years. They accepted me as a student despite my somewhat checkered academic past and allowed me several academic indulgences along the way before settling firmly into the combinatorial algebra world. Even after missing every deadline the Graduate School has ever thought to impose, they kept me around and stood by my side to fix those mistakes. I also received

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1 As penance for writing a mathematics dissertation that uses just two Greek letters—and one just a variant at that—I thought a reference to their gods might be a wise precaution.
2 While I prefer not to lie, I am not above the occasional fib, in general. Yet after going through this document $n+1$ times (for any choice of $n$) and making sure that every single line is absolutely true, it seems silly to break that record here.
3 At times, this dissertation was to me as the Death Star II was to the forest moon of Endor: always there, but perpetually incomplete and equipped with a superlaser.
4 Sole responsibility for the work means sole responsibility for any errors. No way!
5 Fine. Certainly some of the errors are mine, but all of them? Surely someone can cop to a few.
6 Come on, I’m sure it was just an accident and you meant nothing by it. Anyone?
7 …
8 Ok, ok, here goes: All and any mistakes herein are my own. This includes the mistake of putting this declaration in a footnote.
9 How? There are advantages to being in charge of the source files.
10 Happy now, Chaucer?
11 [sic]
12 According to the count of the rings on my coffee cup.
13 They even put up with my brief foray into the field of evolutionary biology, where I learned about zombie ants and how I really should be doing math.
14 I should have gone to school in time management, it seems.
generous support from the Frances Stribic and Wolf Thron Fellowships, which kept me afloat in the leaner summer months,\textsuperscript{15} giving me time to research the material contained in this document; thank you.

I owe an incredible debt of gratitude to J. Matthew Douglass, Stephen Doty, Martin Walter and Nathaniel Thiem, who all served on my defense committee, accepting late drafts\textsuperscript{16} and many typos without complaint. Matt and Steve had not even met me personally before the day of my defense, yet they came to it engaged and supportive with insightful comments, even helping out with last minute scheduling issues. Marty, even though our mathematical interests are somewhat divergent, has been an incredible help throughout my career in Boulder, since I first served as his TA in calc my second semester. Always available with a smile, he has given me the chance to participate, several times, in strange and wonderful teaching opportunities that I could not remember with more fondness. Nat became my second reader without me even properly asking\textsuperscript{17} for his help. His thoughtful comments have guided this document to its present form\textsuperscript{18} and have been extremely helpful to my own understanding of the meaning of this work as whole. He also put up with and trenchantly critiqued several—sometimes marathon-length\textsuperscript{19}—talks I have given in developing the ideas contained herein; in particular, he co-ran an algebraic Lie theory seminar with my advisor that has been invaluable.

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Along with my advisor, I thank all those mentioned in my bibliography for creating this beautiful world in which I have played for the past few years. Their creations are my friends and enemies, but make me full. I also thank my mathematical siblings—Dana Ernst, Hugh Denoncourt, Brent Pohlman, Jacob Harper and Tyson Gern—for making me look better by

\begin{itemize}
\item \textsuperscript{15} But it’s harvest time now, baby!
\item \textsuperscript{16} Ibid.
\item \textsuperscript{17} Thankfully, he concluded his role resulted from a “symmetrical” argument to Tyson’s defense.
\item \textsuperscript{18} Any mistakes, though, are still my own.
\item \textsuperscript{19} Given the present document, this is probably not surprising.
\item \textsuperscript{20} Ok, always-painful
\item \textsuperscript{21} Even when, really, the communication problem was me not communicating.
\item \textsuperscript{22} Which I also thank.
\item \textsuperscript{23} Note that the present acknowledgements sections is the only section he’s had no hand in. So don’t blame him for all this.
\item \textsuperscript{24} Such as “Footnotes should be avoided.”
\end{itemize}
being in their company.\textsuperscript{25}

My family always deserves my thanks, no matter what I’m doing, for they are always there for me. My parent-in-laws, Bob and Teresa Lee, have been a source of strength for me since I have met them. They have given up their home and time to me, shown great interest into my studies and not once, unbelievably, asked, ”What is taking you so long?” Thank you. My parents, Ron and Carmen Dorsey, have always just wanted me to be happy, and have supported everything I do. The only time they could not quite hide their disappointment was when I quit graduate school the first time. They were correct; this was the right thing for me to do. Mom and Dad, thanks for your love and guidance; I’m happy.

I thank all of my friends for putting up with my constant level of distraction in this process. In particular, I thank Laura Holt and Jonathan Kish. Laura has listened to my every complaint and frustration in this process with empathy and kindness.\textsuperscript{26} I must have been incredibly boring and boorish at times, but she has never let me know. Both she and Jonathan have welcomed me, without restriction, into their home, as poor a guest as I was, in order that I might complete this process. They helped me do all the things I forgot to do, and let me be when I needed to just keep working. Jonathan, while not my mathematical sibling,\textsuperscript{27} has become, over these few years, as close to a real sibling as I will ever have. His support and inspiration has been invaluable; were he not in the math program, I would not be. Thank you, Jonathan; I promise to never ask to run another form anywhere ever again.\textsuperscript{28}

Finally, I thank my best friend and wife, Jessica Lee. There are so many reasons that I could not have done this without her,\textsuperscript{29} I can barely begin to list them. Were it not for her, I would have never even begun the program, let alone finished. She has taken me around the world and back, putting up with my every gripe and faux lost ticket along the way, and she has stayed with me when I couldn’t bring myself to leave the house. She has taken up miles and miles of my slack for too many years, and still happily encourages me on. I love you. I can’t wait to get home tonight.

\textsuperscript{25} I mean their awesomeness rubs off on me, not that I look better by comparison. However you read it, this is not to imply that they are ugly.

\textsuperscript{26} And an ever-ready ”Arrested Development” quote.

\textsuperscript{27} Analysis? Really?

\textsuperscript{28} But I will need some more of your espresso.

\textsuperscript{29} Actually, I cannot think of anything I can do without her.
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A heap is a partially ordered set (poset) with elements labeled by the vertices of some graph, satisfying certain conditions intended to capture the structure of that graph within the poset. Introduced by Viennot [27], this construction has been used in many fields, from computer science [3] to mathematical physics [26]. Our present interest is in the application to Lie theory via algebraic combinatorics. Stembridge applied heaps to Dynkin diagrams of simple Lie algebras, adding restrictions to their structure that relied on the specific properties of Dynkin diagrams. Called minuscule heaps, these structures classify minuscule elements of Weyl groups [25], an important subset of fully commutative elements and closely related to minuscule representations of Lie algebras [24].

In [8], Green modified and extended Stembridge’s work to the affine case, developing the notion of full heaps, which are infinite heaps that occur over Dynkin diagrams of affine type. He has since applied full heaps, using raising and lowering operators on the elements of the heap, among other applications, to develop new combinatorial constructions of all minuscule representations of simple Lie algebras [10]. These constructions are useful not just in providing alternative descriptions of minuscule representations, but also in rendering some attributes more clearly than previous constructions, by explicitly giving a Chevalley basis through elementary methods for those Lie algebras that admit a full heap. Green also showed that the examination of certain subheaps of full heaps gives a whole slew of applications to bitangents, weight polytopes, Gaussian posets and plane partitions, to name
just a few.

Within [8], [9] and [14], the full heaps over Dynkin diagrams of type $\tilde{A}_n$ ($n \geq 1$), $\tilde{B}_n$ ($n \geq 3$), $\tilde{A}_{2n-1}$ ($n \geq 3$), $\tilde{C}_n$ ($n \geq 2$), $\tilde{D}_{n+1}$ ($n \geq 2$), $\tilde{D}_n$ ($n \geq 4$), $\tilde{E}_6$ and $\tilde{E}_7$ were given. In [10], Green showed that this list is complete among affine Dynkin diagrams using an argument from representation theory. However, it was previously unknown if these were all the Dynkin diagrams admitting full heaps among all Dynkin diagrams with a finite number of vertices. Given the applications already known for full heaps, any additional full heaps could be quite useful, while their lack would indicate a near complete classification of simple Lie algebras from a direction separate from Lie algebra theory, further evidence of the fundamental status of simple Lie algebras in nature, as well as motivation to their continued study.

In this thesis, we prove, in Theorem 4.7.1, that the list above is complete among the larger set of all Dynkin diagrams with a finite number of vertices. (We do not address the infinite case, though we mention that three infinite families of full heaps are known in cases of the path-connected countably infinite graphs of type $A$ in both directions, $B$ and $D$.) Note that any finite graph without multiple edges and loops can be thought of as a Dynkin diagram, so the fact that this list is so easily described is remarkable. The techniques from representation theory used for affine Dynkin diagrams were not available to us due to the generality of our set, so the arguments contained herein are completely combinatorial in nature. After showing that full heaps can only occur over Dynkin diagrams of the type on this list, we use Green’s previous completeness result to show that Appendix C is, in fact, a complete list of full heaps over Dynkin diagrams with a finite number of vertices.

After establishing we have a list of all full heaps, we then establish relations between members of the list. The underlying posets of (certain) maximal minuscule heaps are the minuscule posets of Proctor, which have many interesting properties [17, 19]. As full heaps are analogues of maximal minuscule heaps, it is natural to examine their underlying posets. To do so, we extend in a novel way the definition of order ideal posets to locally finite posets in Section 2.5. Order ideal posets are well-known constructions that produce posets from
other posets; this is the content of the Fundamental Theorem of Finite Distributive Lattices [21]. We preserve important properties of this original construction, while allowing for new applications. In particular, Theorem 5.1.7 is a new version of the Fundamental Theorem of Finite Distributive Lattices for application to locally finite distributive lattices under a certain constraint. We then apply this novel definition to the underlying posets of full heaps, which are definitionally locally finite, to get a method for moving between the full heaps over simply laced affine Dynkin diagrams, resulting in the $ADE$ classification of Theorem 5.5.1, as well as some other relations between doubly laced affine Dynkin diagrams.

We conclude with the following resources for the reader: a list of all finite and affine Dynkin diagrams in Appendix A, an atlas of all minuscule posets in Appendix B and an atlas of all full heaps over Dynkin diagrams with a finite number of vertices in Appendix C.
Chapter 2

Preliminary terminology

Heaps are posets with certain characteristics dependent on an underlying graph. It is convenient for us to display these posets as graphs with a particular orientation, called Hasse diagrams. Our main construction of interest, the full heap, is associated with an underlying Dynkin diagram, another particular kind of graph. Thus, our discussion will require many of the tools and objects used in the study of graphs and posets, as well as the specific requirements of Dynkin diagrams. Before we introduce heaps, we set up the terminology and notation.

2.1 Graph and poset definitions

We assume the reader is somewhat familiar with graphs, posets and associated terminology. The purpose of this section is to be a quick summary of the terms we use, as well as introduce a few non-standard terms and notation.

Graphs

Definition 2.1.1. A graph consists of a set of elements we call vertices together with a set consisting of pairs of vertices we call edges. Two vertices are considered adjacent if they share an edge, i.e., are in a pair together in the set of edges. An edge is undirected if the pair of vertices defining it is unordered and directed if the pair is ordered. A graph is undirected when every edge is undirected and directed if at least one edge is directed.
We allow our graphs to have **multiple edges**, meaning that two vertices may have more than one edge between them, i.e., the set of edges is a multiset. We do not allow our graphs to contain any **loops**, i.e., an edge between a vertex and itself. An **induced subgraph** is a subset of the vertices of a graph together with any edges whose defining pair of vertices are both in this subset.

**Remark 2.1.2 (Some notational pliancy).** We will often refer to a vertex of a graph $\Gamma$ as simply an element of $\Gamma$. That is, when we say $x \in \Gamma$, we mean $x$ to be a vertex of $\Gamma$. We will refer to an edge of $\Gamma$ by its vertices.

We will also often refer to vertices as **labels** and will use the terms interchangeably. The reason for this will be apparent in the sequel.

Note that directed graphs may have some or all edges undirected. Indeed, all Dynkin diagrams below should be considered directed graphs, but most will have no undirected edges. This will not result in confusion because, graphically, these edges will be distinguished by an arrow when they are directed.

**Definition 2.1.3.** If $a$ is a vertex of a graph with at most one vertex adjacent to $a$, we then call $a$ a **terminal vertex** or simply **terminal**. Note that $a$ may share a multiple edge with an adjacent vertex and still be terminal.

**Definition 2.1.4.** A **path** of a undirected graph is a set $\{a_i\}_{i=1}^n$ of vertices such that every pair $\{a_i, a_{i+1}\}$ is connected by an edge. A **circuit** is a path with $a_1 = a_n$, $n \geq 4$ such that when $i \neq j$ with $i, j \in \{1, \ldots, n - 1\}$, we have $a_i \neq a_j$.

Every directed graph has an obvious underlying undirected graph in which each vertex pair defining an edge are taken to be unordered. When we use the terms “path” and “circuit” in application to directed graphs, we will mean the path or circuit in the underlying undirected graph. This is a departure from the common usage, so the reader should beware.
Posets

Definition 2.1.5. Let $E$ be a set and $a,b,c$ arbitrary elements in $E$.

(1) A **partial order** on $E$ is a relation $\leq$ that is reflexive (i.e. $a \leq a$), antisymmetric (i.e. if $a \leq b$ and $b \leq a$, then $a = b$) and transitive (i.e. if $a \leq b$ and $b \leq c$, then $a \leq c$).

(2) A **partially ordered set** (or **poset**) is a set equipped with a partial order. We write a poset as a pair $(E, \leq)$ or just $E$ when the relation is understood.

(3) The **dual** $(E', \leq')$ of a poset $(E, \leq)$ is the poset that has the properties that $E' = E$ and $a \leq' b$ in $E'$ if and only if $b \leq a$ in $E$.

(4) A **subposet** $(E', \leq')$ of a poset $(E, \leq)$ is a poset where $E' \subseteq E$ and, for any $a, b \in E'$, $a \leq' b$ if only if $a \leq b$.

Remark 2.1.6. It is easy to show from the definitions that dual posets and subposets are also posets.

The relations between elements of a poset will be of particular importance to our discussion, so we now develop a vocabulary for them.

Definition 2.1.7. Let $(E, \leq)$ be a poset and $a, b, c$ be arbitrary elements in $E$.

(1) The elements $a, b$ are **comparable** if $a \leq b$ or $b \leq a$ or both. Otherwise, we say they are **incomparable**.

(2) We write $a < b$ whenever $a \leq b$ and $a \neq b$. Additionally, we may write $b \geq a$ in place of $a \leq b$ and $b > a$ in place of $a < b$.

(3) A **closed** (respectively, **open**) **interval** $[a,b]$ (respectively, $(a,b)$) is the set $\{x \in E \mid a \leq x \leq b\}$ (respectively, $\{x \in E \mid a < x < b\}$). The intervals $[a,b]$ and $(a,b]$ are defined as expected.
(4) If \(a \in E\) has the property that, for all \(x \in E\), either \(x \leq a\) (respectively, \(x \geq a\)) or \(x\) and \(a\) are incomparable, then we say \(a\) is a maximal element (respectively, minimal element) of \(E\).

(5) If \(a < b\) and \((a, b) = \emptyset\), then we say that \(b\) covers \(a\) and write \(a \rightarrow b\) or \(b \leftarrow a\).

(6) If either \(a\) covers \(b\) or \(b\) covers \(a\), then we say that \(a\) and \(b\) are a covering pair.

(7) Suppose that \(b \leq a\) and \(c \leq a\) and, for any \(d \in E\) with \(b \leq d\) and \(c \leq d\), we have \(a \leq d\). We then call \(a\) the join of \(b\) and \(c\). (If it exists, the join of two elements is unique by antisymmetry.)

(8) If \(a\) is not minimal and not the join of two elements distinct from \(a\), then we call \(a\) join-irreducible.

Remark 2.1.8. It will often be necessary to use covering relations and the fact no elements exist between elements in a covering relation in our proofs below. As a shorthand, if an argument requires existence of an element \(c\) such that \(a < c < b\) when \(a\) is covered by \(b\), we will say \(c\) breaks the covering relation, usually giving a contradiction needed for the argument.

Definition 2.1.9.

(1) A poset or interval is finite when its set of elements is finite.

(2) A poset is locally finite if every interval is finite.

(3) A totally ordered set is a poset \(P\) in which every pair of elements is comparable. When a subposet \(F\) of a poset is totally ordered, we call \(F\) a chain. When \(P\) is locally finite, we also have the following:

(a) For a given chain \(\{a_i\}\), we assume the elements are indexed by the integers and use the convention that \(a_i < a_j\) if and only if \(i < j\).
(b) A **saturated chain** is a chain \(\{a_i\}_{i \in \mathbb{Z}}\) in which every consecutive pair in the chain is a covering pair in the original poset, i.e., \(a_i \rightarrow a_{i+1}\) in \(P\) for all \(i\).

(c) An arbitrary finite chain containing \(n\) elements is written as \([n]\).

(d) An arbitrary infinite but countable chain is written as \(\mathbb{Z}\).

(4) A **poset morphism** is a map between the underlying sets that preserves order. That is, for posets \((E, \leq)\) and \((E', \leq')\), the map \(\phi : E \rightarrow E'\) satisfies \(\phi(a) \leq' \phi(b)\) in \(E'\) whenever \(a \leq b\) in \(E\). If \(\phi\) is also a bijection, then \(\phi\) is a **poset isomorphism**.

(5) A subset of a poset in which all elements of the subset are pairwise incomparable is called an **antichain**. An antichain may be empty.

(6) A poset \(E\) has **constrained antichains** if for each \(a \in E\), the set of antichains containing \(a\) is finite.

**Remark 2.1.10.** It is necessarily the case that a poset with constrained antichains only has antichains of finite length, since any element in an infinite antichain is in all antichains consisting of that element and each other element of the antichain. However, it is possible for all antichains to be finite and the poset not have constrained antichains. Consider, for example, the poset consisting of \(\mathbb{Z}\) and one other element incomparable to all of \(\mathbb{Z}\). All antichains in this poset have at most two elements, but the set of pairs containing the special element is infinite.

To be clear, the property of having constrained antichains is a property of the poset; we cannot say that an antichain is constrained. We also note that the definition for constrained antichains is, to our knowledge, new to the literature.

### 2.2 Locally finite posets

We also need the following technical lemmas for the sequel.
Lemma 2.2.1. Let $E$ be a locally finite poset. For $a$ and $b$ in $E$ with $a < b$, there exists at least one element in the interval $[a, b]$ that covers $a$ and at least one that is covered by $b$. In particular, if an element of $E$ is not minimal (respectively, not maximal), then it covers (respectively, is covered by) at least one element.

Proof. By the local finiteness of $E$, the interval $[a, b]$ is finite. If $(a, b)$ is empty, then $b \rightarrow a$, and we are done. Otherwise, we note that the finiteness of $[a, b]$ and the antisymmetry of $E$ require $(a, b)$ to have at least one maximal element. Choose $c \in (a, b)$ to be maximal in the interval. The interval $(c, b)$ must be empty, as any element within it would be greater than $c$ and less than $a$, thus in $(a, b)$, contradicting the maximality of $c$. So, by definition, $c \rightarrow b$, as needed.

A symmetric argument shows that $a$ is covered by an element in the interval.

Finally, if $b$ is not minimal, then there exists some element $a \in E$ such that $a < b$, i.e., there is an interval $[a, b]$ satisfying the hypothesis. Similarly, if $a$ is not maximal, there is an interval $[a, b]$ satisfying the hypothesis. \qed

Note that it is not strictly necessary that $E$ be locally finite in Lemma 2.2.1 to guarantee the conclusion. That is, there are some posets that are not locally finite with this property. However, the conclusion does fail for some posets, e.g., the real number line, and looser restrictions tend to be cumbersome, so we include the result here.

The next lemma is well-known.

Lemma 2.2.2. A locally finite partial order is completely determined by the reflexive transitive closure of its covering relations.

Proof. It is immediate from the definitions that any partial order must contain the reflexive transitive closure of its covering relations.

Suppose there are two elements $a < b$ in the partial order. If they form a covering pair, then this relation is already in the set of covering relations. If they do not form a
covering pair, then, by local finiteness, there is some finite chain \( c_1 \to c_2 \to \cdots \to c_n \) such that \( a \to c_1 \to c_2 \to \cdots \to c_n \to b \). (The existence of this chain is obtained via repeated application of Lemma 2.2.1.) Thus \( a < b \) is in the transitive closure of covering relations. The reflexivity requirement for a partial order is satisfied by the reflexive closure. Anti-symmetry is not violated because the original set of relations were taken from an antisymmetric relation, namely the original poset.

Lemma 2.2.3. Let \( E \) be a locally finite poset in which a join exists for every pair of elements. Then \( a \in E \) is join-irreducible if and only if \( a \) covers exactly one element.

Proof. If \( a \) is join-irreducible, it cannot be minimal by definition, so there exists an element \( b \in E \) that is less than \( a \). By local finiteness, the interval \([b, a]\) is finite. It \( b \) is covered by \( a \), then \( a \) covers at least one element. Otherwise, \((b, a)\) is non-empty, so there is an an element \( b_1 \in (b, a) \) such that \([b_1, a]\) contains strictly fewer elements than \([b, a]\). By induction and the finiteness of \([b, a]\), we conclude that \( a \) covers at least one element.

Suppose \( a \) covers more than one element; say \( b \) and \( c \) are two such elements. Then \( a \) is clearly greater than both \( b \) and \( c \). By hypothesis, the join \( a' \) of \( b \) and \( c \) exists, so \( b \leq a' \leq a \) and \( c \leq a' \leq a \). We must then have \( a' = b \) or \( a' = a \), lest we break the covering relation \( b \to a \). If the former holds, we find that \( c < a' = b < a \), since \( c \neq b \), breaking the covering relation \( c \to a \). This is impossible, so \( a' = a \) and \( a \) is the join of two elements not equal to \( a \), contradicting the assumption that \( a \) is join-irreducible. So \( a \) covers exactly one element.

Now we examine the other direction. Suppose that \( a \) covers exactly one element, say \( b \). Then clearly \( a \) is not minimal. If \( a \) is the join of two elements distinct from \( a \), say \( c \) and \( d \), then \( a \geq c \). By local finiteness, the interval \([c, a]\) is finite, so it contains the only element \( a \) covers. In particular, \( b \geq c \). Similarly, \( b \geq d \). By definition of join, we then have \( a \leq b \), clearly a contradiction since \( a \) covers \( b \). So \( a \) is not a join and therefore join-irreducible. \( \square \)
**Hasse diagrams**

A convenient way to convey the structure of a locally finite poset is via a type of graph called a Hasse diagram where, loosely, the relation of two elements is given by relative position in the diagram (e.g., see Figure 2.1).

\begin{figure}[h]
  \centering
  \includegraphics[width=0.5\textwidth]{hasse_diagram.png}
  \caption{An example of a Hasse diagram. Here, the poset is defined on the subsets of the set \{a, b, c\} by the inclusion relation. The direction “up”, indicated by the arrow, indicates which elements are greater. Throughout this thesis, greater elements should be assumed to be towards the top of the page.}
  \label{fig:hasse_diagram}
\end{figure}

**Definition 2.2.4.** A **Hasse diagram** is a visual representation of a locally finite poset created with the following rules:

1. an orientation is given defining “up”;
2. each element of the poset is represented by a vertex;
3. an edge connects two vertices exactly when the corresponding elements are a covering pair;
4. If \( a \leq b \), there is a path connecting \( a \) and \( b \) on which one can move from \( a \) to \( b \) while always moving in the “up” direction.
Products of posets

A standard operator on posets is the product \((\times)\) operator which produces a new poset from two posets. While it is not necessary the posets be locally finite, the concept is much easier to visualize when the posets are.

**Definition 2.2.5.** Let \((E_1, \leq_1)\) and \((E_2, \leq_2)\) be posets. Then the **product**

\[
(E, \leq) = (E_1, \leq_1) \times (E_2, \leq_2)
\]

is the poset where \(E = E_1 \times E_2\) is a cartesian product of sets and \((a_1, a_2) \leq (b_1, b_2)\) if and only if both \(a_1 \leq_1 b_1\) and \(a_2 \leq_2 b_2\).

It is immediate from the definitions that the product of two posets is also a poset.

**Example 2.2.6.** A loose way of thinking about the product is to replace each element of one of the posets in the product with a copy of the other poset and then establishing relations between those copies that are the same as those between the original elements. We will be using products of chains in the sequel, so let us consider the product \([2] \times [3]\). See Figure 2.2 for the accompanying diagrams.

First, we draw a Hasse diagram of one of the posets, say \([3]\), as in 2.2(1). We then replace each element in \([3]\) with a copy of \([2]\), as in 2.2(2). The dashed lines here are to
remind us of the original relation in [3]. Finally, using that original relation, we connect the elements in each copy of [2] to their doppelgängers in the other copies, as in 2.2(3).

2.3 Dynkin diagrams

There are a few slightly different definitions of the Dynkin diagram; we will follow [2] here.

Definition 2.3.1. Let $A = (a_{ij})$ be a square matrix with integer values. We call $A$ a generalized Cartan matrix if

1. all diagonal entries are 2,
2. all off-diagonal entries are non-positive, and
3. the entry $a_{ij} = 0$ if and only if $a_{ji} = 0$.

When referring to a generalized Cartan matrix $A$, we will assume it has entries $(a_{ij})$ unless otherwise noted.

The information contained in any $n \times n$ generalized Cartan matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ can be encoded in a directed graph $\Gamma = \Gamma(A)$ on $n$ vertices as follows. Label each vertex with a unique label taken from $\{1, 2, \ldots, n\}$, chosen to correspond to the rows of $A$. (It will sometimes be convenient to take vertex labels from $\{0, 1, \ldots, n - 1\}$ and treat $A$ as the matrix $(a_{ij})_{0 \leq i, j \leq n-1}$, as should be clear from the context.) Consider a pair of vertices $i, j$ where $i \neq j$ and assume that $|a_{ij}| \geq |a_{ji}|$.

1. If $a_{ij}a_{ji} = 1$, then connect $i$ and $j$ with a single edge,
2. If $1 < a_{ij}a_{ji} \leq 4$, then connect $i$ and $j$ with $|a_{ij}|$ edges with an arrow pointing towards $i$,
3. If $a_{ij}a_{ji} > 4$, then connect $i$ and $j$ with an edge labeled $|a_{ij}|, |a_{ji}|$. 
Remark 2.3.2. An edge may have as many as two arrows, pointing in opposite directions. For example, when \( a_{ij} = a_{ji} = -2 \), the two edges connecting \( i \) and \( j \) have two arrows, one pointing towards \( i \) and one towards \( j \), due to symmetry in the construction.

Definition 2.3.3. The graph \( \Gamma = \Gamma(A) \) described above is the Dynkin diagram associated with the generalized Cartan matrix \( A \). If all pairs of vertices of \( \Gamma \) are connected with at most one (respectively, two) unlabeled edge(s), we call \( \Gamma \) simply laced (respectively, doubly laced). If \( \Gamma \) is connected, we say \( A \) (and \( \Gamma \)) is indecomposable.

Remark 2.3.4. We note that the Dynkin diagram \( \Gamma = \Gamma(A) \) is uniquely determined (up to relabeling of vertices) by its associated generalized Cartan matrix \( A \). Additionally, one can recover the generalized Cartan matrix \( A \) from the Dynkin diagram \( \Gamma(A) \), so we may write \( A = A(\Gamma) \).

In this thesis, we almost exclusively restrict ourselves to simply laced and doubly laced Dynkin diagrams, so we may ignore the third rule in our construction above, which is included only for completeness. We also assume Dynkin diagrams are connected (indecomposable) unless otherwise stated.

Definition 2.3.5. Let \( \Gamma \) be a Dynkin diagram and \( b \) be a vertex of \( \Gamma \). We say that the tally of \( b \) is equal to \( \left| \sum_{p \in \Gamma \setminus \{b\}} a_{b,p} \right| \). We will also say that an adjacent vertex \( c \) contributes \( |a_{b,c}| \) to the tally of \( b \).

Note all the summands of the sum in Definition 2.3.5 are all nonpositive, so this terminology makes sense. In the simply laced case, the tally of a vertex is simply the commonly used degree of the vertex, but a slight modification is required to account for arrows in Dynkin diagrams that are not simply laced.

Example 2.3.6. An example of a generalized Cartan matrix with corresponding Dynkin diagram is given in Figure 2.3.
Figure 2.3: Example of a generalized Cartan matrix and its Dynkin diagram. Here, the columns and rows are indexed by \( \{0, 1, 2, 3, 4, 5\} \). In the Kac notation, this Dynkin diagram is of type \( \tilde{B}_5 \).

\[
A = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 & 0 & 0 \\
-1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -2 & 2
\end{pmatrix}
\]

generalized Cartan matrix

\[
\Gamma(A)
\]

Dynkin diagram
We explicitly list a few corresponding properties of the two structures. Letting \( A = (a_{i,j})_{0 \leq i,j \leq 5} \), we see that \( a_{0,2} = a_{1,2} = a_{3,2} = -1 \) in the generalized Cartan matrix. This corresponds to vertex 2 in the Dynkin diagram sharing an edge with vertices 0, 1 and 3. There are two edges between vertices 4 and 5 in the Dynkin diagram, so we expect that \( a_{4,5} \cdot a_{5,4} = 2 \) in \( A \), which is easily verified. The arrow points towards 5 because \( a_{5,4} = -2 \) is the larger, in absolute value, of the two. Finally, \( a_{2,4} = 0 \) and vertices 2 and 4 share no edge, as expected.

We also note the tallies of the vertices of \( \Gamma(A) \) in Table 2.1.

Table 2.1: Tallies for vertices of Dynkin diagram of type \( \tilde{B}_5 \). The calculation of the tally for vertex \( p \) is \( |a_{p,0} + a_{p,1} + \cdots + \hat{a}_{p,p} + \cdots + a_{p,5}| \), where the caret indicates the omission of that term.

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Calculation of tally</th>
<th>Tally of vertex</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( 0 + (-1) + 0 + 0 + 0 )</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( 0 + (-1) + 0 + 0 + 0 )</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( (-1) + (-1) + (-1) + 0 + 0 )</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>( 0 + 0 + (-1) + (-1) + 0 )</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>( 0 + 0 + 0 + (-1) + (-1) )</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>( 0 + 0 + 0 + 0 + (-2) )</td>
<td>2</td>
</tr>
</tbody>
</table>

While we will be implicitly addressing all possible Dynkin diagrams in our text, we only need to refer to a small subset of them taken from the so-called finite and affine Dynkin diagrams; see Appendix A. We follow Carter’s version of Kac’s naming conventions for Dynkin diagrams [2].

Although the definition of a generalized Cartan matrix does not require the matrix and hence Dynkin diagram to be of finite size, we will assume it is unless otherwise stated.

2.4 Order ideals of posets

One of the most important tools we use in our analysis of full heaps is the order ideal.
**Definition 2.4.1.** Let $E$ be a poset with $I \subseteq E$ a subset. Suppose $I$ has the property that, for all $y \in I$, whenever $x \in E$ and $x \leq y$, we have $x \in I$. Then $I$ is an order ideal (or just ideal) of $E$.

It should be noted that elsewhere in the literature order ideals are sometimes referred to as down-sets or decreasing subsets. Additionally, some require that an order ideal $I$ also be a directed set, i.e., every pair of elements has an upper bound in $I$. This requirement occurs in lattice theory, but we do not adopt it; instead, we follow [21]. We also note that we consider the empty set and entire set to be order ideals.

**Lemma 2.4.2.** Let $I$ and $K$ be order ideals of a poset $(E, \leq)$ and let $\leq'$ be defined as the restriction of $\leq$ to $I$. Then

1. the poset $(I, \leq')$ is a subposet of $(E, \leq)$;
2. the sets $I \cup K$ and $I \cap K$ are order ideals of $E$.

**Proof.** That $(I, \leq')$ is a poset and hence subposet is immediate. In general, we will treat $I$ as this obvious subposet below.

For $y \in I \cup K$, we must have $y \in I$ or $y \in K$; assume the former. If $x \in E$ and $x \leq y$, the definition of order ideal requires $x \in I$ and thus $x \in I \cup K$. The case that $y \in K$ is similar. Therefore $I \cup K$ is an order ideal. The proof for $I \cap K$ is similar, but with just one case. 

If we have a set $S \in E$, we define $\langle S \rangle$ to be the set of all elements of $E$ that are less than or equal to at least one element of $S$. It is clear from the transitivity of posets that $\langle S \rangle$ is an ideal, which we call the ideal generated by $S$. (If $S = \{s_1, \ldots, s_n\}$, we abuse notation and write $\langle S \rangle = \langle s_1, \ldots, s_n \rangle$.)

**Definition 2.4.3.** We call an ideal of the form $\langle a \rangle$ a principal ideal.
We now show that every ideal of a finite poset is generated by its maximal elements. This observation will be helpful for the bookkeeping of ideals in the sequel and will guide us in a generalization to infinite posets.

The following lemma is well-known (see [10, Lemma A.1.8] for a more general version), so we omit the proof.

**Lemma 2.4.4.** Let $E$ be a finite poset. Then the set of order ideals of $E$ is in a bijection with the set of antichains of $E$. 

Lemma 2.4.4 is not generally applicable in the case of an infinite poset, because order ideals do not always contain a maximal element. For example, the set of rationals less than 1 is an ideal in $\mathbb{Q}$, but no element of this ideal is maximal. However, we will be analyzing infinite posets that are locally finite and avoid the problem illustrated by $\mathbb{Q}$. While locally finite posets can still have ideals without maximal elements, these ideals are, in some sense, extreme examples that will be viewed as exceptions.

The set of order ideals of a poset will be a major object of study of this thesis. This set has a natural partial order from inclusion, resulting in another poset, which we define here. The concepts here are standard, and we use the notation presented in [21].

**Definition 2.4.5.** Let $E$ be a poset. We define the poset $J(E)$ to be the set of all order ideals of $E$ ordered by inclusion. We call $J(E)$ the **order ideal poset of $E$**.

**Example 2.4.6.** From Lemma 2.4.4, we know the order ideals of a finite poset $E$ are in bijection with the antichains of $E$, so $J(E)$ can, in this case, be viewed as a poset on the set of antichains. We will use this bijection as a notational aid when presenting order ideal posets, as seen in Figure 2.4 where we compute the product $[2] \times [3]$.

Note that the elements of $J([2] \times [3])$ in the figure are all antichains in $[2] \times [3]$. Additionally, note that all principal ideals of $J([2] \times [3])$ cover exactly one element, i.e., are join-irreducible, while ideals that are not principal cover either no elements or more than one element. We formalize this below in Theorem 5.1.7.
Figure 2.4: A poset of the form $[2] \times [3]$ and its order ideal poset. Element labels included for clarity.

$[2] \times [3] \xrightarrow{J} J([2] \times [3])$
It is worth noting that Lemma 2.4.4 allows us to think of \( J(E) \) as a poset on antichains of \( E \) when \( E \) is finite. However, this is not generally the case when \( E \) is infinite, because order ideals do not always contain maximal elements that bound the other elements of the ideal, a key requirement for the proof.

### 2.5 Clipped Order Ideal Poset

We now restrict ourselves to the examination of locally finite posets, which are easier to work with than general posets. However, locally finite posets can still pose some difficulties. For example, the locally finite poset of integers \( \mathbb{Z} \) has no maximal elements, so the order ideal that consists of \( \mathbb{Z} \) itself has no corresponding antichain. Note, however, that this is the only order ideal of the poset \( \mathbb{Z} \) with no corresponding antichain. In general, it turns out that “most” order ideals of locally finite posets do have a corresponding antichain, inspiring the following novel definition.

**Definition 2.5.1.** Let \( E \) be a poset. Then the clipped order ideal poset of \( E \), denoted \( \tilde{J}(E) \), is the set of all order ideals \( \langle S \rangle \) where \( S \) is a nonempty antichain of \( E \), ordered by inclusion. Additionally, the order ideal \( \emptyset \) is included in \( \tilde{J}(E) \) precisely when each element \( x \) of \( E \) is comparable to some minimal element, dependent on \( x \), of \( E \).

**Remark 2.5.2.** The specific requirement for inclusion of \( \emptyset \) is a technical one. Just as the tying the order ideals to antichains addresses ideals that are somehow lacking in maximal elements, this requirement addresses those that lack minimal elements. By doing this, we make the the clipped order ideal poset of \( E \) dual to the clipped order ideal poset of the dual of \( E \).

**Example 2.5.3.** By construction, \( \tilde{J} \) is meant to reflect \( J \) as an operator. So it is unsurprising that, by Lemma 2.4.4, all order ideal posets of finite posets are examples of clipped order posets.
For a less trivial example, we return to the poset $\mathbb{Z}$ with the standard ordering. The only nonempty antichains of $\mathbb{Z}$ are singleton sets, since $\mathbb{Z}$ is totally ordered. Additionally, $\mathbb{Z}$ has no minimal element, so the property required for the inclusion of $\emptyset$ in $\tilde{J}(\mathbb{Z})$ is not satisfied. Thus, $\tilde{J}(\mathbb{Z})$ consists only of the principal ideals of $\mathbb{Z}$. It is clear that $\tilde{J}(\mathbb{Z})$ and $\mathbb{Z}$ are isomorphic as posets, with a bijection being between the elements of $\mathbb{Z}$ and the ideals they generate.

On the other hand, $J(\mathbb{Z})$ is not the same as $\tilde{J}(\mathbb{Z})$. In particular, both $\emptyset$ and $\mathbb{Z}$ are order ideals of $\mathbb{Z}$ and thus in $J(\mathbb{Z})$, but neither is generated by a nonempty antichain, so is not in $\tilde{J}(\mathbb{Z})$. The two posets are not even isomorphic since the empty set is minimal in $J(\mathbb{Z})$, but there is no minimal element of $\mathbb{Z}$ and hence $\tilde{J}(\mathbb{Z})$.

The above is a good example of how $J$ and $\tilde{J}$ differ when applied to infinite posets. However, they need not differ. Consider the example of an infinite poset $E$ in which all elements are incomparable. In that case, every element is minimal, and every element is comparable to itself, so $\tilde{J}(E)$ contains $\emptyset$. Also, the order ideal $E$ is one big antichain, so it too is in $\tilde{J}(E)$. It is easy to see that all other order ideals correspond to a nonempty antichain, so $J(E)$ and $\tilde{J}(E)$ are the same.

**Lemma 2.5.4.** The clipped order ideal poset of $E$ is contained in the the order ideal poset of $E$, i.e., $\tilde{J}(E) \subseteq J(E)$. When $E$ is locally finite, this inclusion is strict if and only if $E$ contains an infinite chain.

*Proof.* Because $\tilde{J}(E)$ contains only order ideals, the inclusion is clear.

Suppose $E$ is locally finite. If $E$ contains an infinite chain, either the infinite chain has no maximum, which excludes the order ideal $E$, or no minimum, which excludes the order ideal $\emptyset$. If $E$ has no infinite chains, then every chain has a maximum, so every ideal is generated by the maximal elements of the chains it contains. Additionally, every chain has a minimum, so every element is comparable to minimal element.

This definition is not without precedent. Let $E$ be a poset with the property that
every principal ideal is finite. In [21, 3.4.3], Stanley defines \( J_f(E) \) as the poset, ordered by inclusion, of all finite order ideals of \( E \). Under the right constraints, we see that the poset \( J_f(E) \) is the same as \( \tilde{J}(E) \).

**Lemma 2.5.5.** Let \( E \) be a poset with the properties that all principal ideals are finite and all antichains are finite. Then the posets \( J_f(E) \) and \( \tilde{J}(E) \) coincide.

*Proof.* First we note that these requirements on \( E \) give us that \( \emptyset \in \tilde{J}(E) \). Because all principal ideals are finite, they clearly must contain minimal elements, and any minimal element of an order ideal is also a minimal element of \( E \), by definition of an ideal. Every element of \( E \) generates a principal ideal, so is thus comparable to some minimal element, as needed to include \( \emptyset \) in \( \tilde{J}(E) \). We can now consider \( \tilde{J}(E) \) to be the set of all order ideals \( \langle S \rangle \) where \( S \) is a antichain, possibly empty, of \( E \).

We now show containment both ways, beginning with \( \tilde{J}(E) \subseteq J_f(E) \). That is, we must show that all ideals in \( \tilde{J}(E) \) are finite. By hypothesis, every antichain of \( E \) is finite. It is clear from the definition of of order ideal that the ideal generated by a finite antichain is the union of principal ideals generated by the elements in the antichain. By hypothesis, these principal ideals are finite, so, since the finite union of finite sets is finite, we get that the original ideal is finite. Since \( \emptyset \) is finite, its inclusion causes no problems, so we have \( \tilde{J}(E) \subseteq J_f(E) \).

Now we show \( J_f(E) \subseteq \tilde{J}(E) \), i.e., that every finite ideal of \( E \) is in \( \tilde{J}(E) \). Let \( I \) be a finite ideal and \( \text{max}(I) \) be the set of all elements maximal in \( I \). By the definition of maximality and the finiteness of \( I \), we know \( \text{max}(I) \) is a finite antichain of \( E \) and thus \( \langle \text{max}(I) \rangle \in \tilde{J}(E) \). By definition of order ideals, we know that \( \text{max}(I) \subseteq I \) implies \( \langle \text{max}(I) \rangle \subseteq I \). We now show the latter containment is, in fact, equality, because every element of \( I \) is less than or equal to element of \( \text{max}(I) \).

Suppose, for contradiction, that \( x_1 \in I \) is not less than or equal any element of \( \text{max}(I) \). Clearly, \( x_1 \) is not maximal in \( I \) since \( x_1 \notin \text{max}(I) \), so there is an element \( x_2 \in I \) greater than
This element too is not less than or equal to any element of $\max(I)$, else transitivity would require that $x_1$ was as well. By induction, we then get an increasing infinite chain $x_1 < x_2 < x_3 < \cdots$ of elements not comparable to any element of $\max(I)$. This is clearly a contradiction to the finiteness of $I$. So $I = \langle \max(I) \rangle \in \widetilde{J}(E)$. As $I$ was chosen arbitrarily, we have $J_f(E) \subseteq \widetilde{J}(E)$.

Remark 2.5.6. The requirement that all antichains be finite is necessary for this lemma, even under the assumption that all principal ideals be finite, which is required by the definition of $J_f(E)$. We return to Example 2.5.3 and the case when $E$ consists of an infinite set of incomparable elements. Here, all principal ideals of $E$ are just single elements, so they are finite. Thus $J_f(E)$ makes sense, consisting of all finite order ideals of $E$. The clipped order ideal poset $\widetilde{J}(E)$, however, includes the order ideal of $E$ itself, which is not finite by assumption. So we see that $J_f(E)$ and $\widetilde{J}(E)$ do not coincide here. Note that, because $\widetilde{J}(E)$ and $J(E)$ do coincide for this $E$, by Lemma 2.5.4, we also have that $J$ and $J_f$ do not always coincide.

We have expanded Stanley’s $J_f$ operator because we are most interested in situations where the poset is locally finite, a looser restriction than that of principal ideals being finite. The latter implies the former because any interval in a poset is a subset of the principal ideal generated by the maximum of the interval. Therefore, when all principal ideals are finite, it is clear that all intervals must be as well, resulting in local finiteness.

We close this section with a property common to both order ideal posets and clipped order ideal posets that will be useful in the sequel.

Lemma 2.5.7. Let $E$ be a poset. Then the posets $J(E)$ and $\widetilde{J}(E)$ each have the property that any pair of elements has a join in the poset.

Proof. For $J(E)$, we note that the join of two order ideals is simply its union, since the union clearly contains both ideals and any other ideal containing both ideals must contain its union. By Lemma 2.4.2, we are done.
For $\widetilde{J}(E)$, the join of two order ideals $\langle S \rangle$ and $\langle T \rangle$, where $S$ and $T$ are antichains, is $\langle S \cup T \rangle$: both ideals are clearly contained within $\langle S \cup T \rangle$; and any order ideal containing $\langle S \rangle$ and $\langle T \rangle$ must contain $S \cup T$, and thus contain $\langle S \cup T \rangle$, by definition of ideal. While $S \cup T$ may not be an antichain, the subset in which we remove all elements less than some other element in $S \cup T$ is, and defines the same ideal. So $\langle S \cup T \rangle$ is in $\widetilde{J}(E)$, as needed.

Note that if $\emptyset \in \widetilde{J}(E)$, we may consider $S$ to be the empty set and the argument still holds. Since the join of two nonempty ideals is never empty, we do not need $\emptyset$ in $\widetilde{J}(E)$ unless it is already there. So we are done.
Chapter 3

Heaps

Our chief combinatorial object of interest in this thesis is that of heaps. In short, a heap is a type of poset with elements associated to the vertices of an underlying graph and relations restricted by the edges of that graph. We will impose additional restrictions to develop the notions of minuscule and full heaps, whose underlying graphs will be Dynkin diagrams. Here, we follow [8] with some modifications useful for the present work.

3.1 Definition of heaps

Definition 3.1.1. Let $\Gamma$ be a graph. Let $E = (E, \leq)$ be a poset and $\varepsilon : E \rightarrow \Gamma$ be a function that assigns to each element of $E$ a vertex of $\Gamma$. We call $\varepsilon$ a labeling function. For $x \in E$, we refer to $\varepsilon(x)$ as the label of $x$. We use the terms vertex and label interchangeably in this setting.

Definition 3.1.2. Let $\Gamma$ be a graph without loops, $E = (E, \leq)$ a poset and $\varepsilon : E \rightarrow \Gamma$ a labeling function. Then we call $E$ together with $\varepsilon$ a heap over $\Gamma$ when the following properties hold:

(H1) for $x, y \in E$, if $\varepsilon(x)$ and $\varepsilon(y)$ are the same or adjacent vertices in $\Gamma$, then $x$ and $y$ are comparable in $E$;

(H2) the partial order $\leq$ is the minimal partial order on $E$ for which (H1) holds, i.e., the reflexive and transitive closure of the relations required by (H1).
**Definition 3.1.3.** Let \( \varepsilon : E \rightarrow \Gamma \) be a heap. Let \( E' \) be a subposet of \( E \) and \( \varepsilon' \) be the restriction of \( \varepsilon \) to \( E' \). If \( \varepsilon' : E' \rightarrow \Gamma \) satisfies (H1) and (H2), then \( \varepsilon' : E' \rightarrow \Gamma \) is a subheap of \( \varepsilon : E \rightarrow \Gamma \).

Heaps also inherit the following notion from posets:

**Definition 3.1.4.** Let \( \varepsilon : E \rightarrow \Gamma \) be a heap. Let \( (E', \leq') \) be the dual poset of \( (E, \leq) \). Then the heap \( \varepsilon : E' \rightarrow \Gamma \) is the dual of the heap \( \varepsilon : E \rightarrow \Gamma \). That is, \( \varepsilon : E' \rightarrow \Gamma \) is \( \varepsilon : E \rightarrow \Gamma \) with its poset relations reversed.

The definition of the dual of a heap is well-defined because \( E' = E \), by definition of poset dual, and because reversing the relation between two elements has no effect on the comparability of the two elements, (H1) and (H2) are satisfied.

**Remark 3.1.5.** For convenience, we often refer to the heap \( \varepsilon : E \rightarrow \Gamma \) as “the heap \( E \) over \( \Gamma \)” when \( \varepsilon \) is understood, or simply “the heap \( E \)” when \( \varepsilon \) and \( \Gamma \) are understood. Additionally, for notational ease, we shorten the references 3.1.2(H1) and 3.1.2(H2) to (H1) and (H2), respectively, without confusion.

Let \( \varepsilon : E \rightarrow \Gamma \) be a heap. The heap will inherit terminology from its underlying poset. For example, the heap is considered finite or locally finite if \( E \) is finite or locally finite, respectively. We may refer to covering relation, a chain or an antichain in the heap, and an order ideal of \( E \) is also considered an order ideal of the heap.

The following lemma gives a more useful way to interpret (H2) in the definition of heap when \( E \) is locally finite. When referring to (H2) in an argument, we often will be relying on this equivalent characterization.

**Lemma 3.1.6.** Let \( \varepsilon : E \rightarrow \Gamma \) be a locally finite heap. Then (H2) is equivalent to the following property:

(h2) If \( x, y \in E \) are in a covering relation, then \( \varepsilon(x) \) and \( \varepsilon(y) \) are adjacent or equal vertices in \( \Gamma \).
Proof. Assume that $E$ satisfies (H1). By Lemma 2.2.2, we know that a locally finite poset is completely determined by the reflexive transitive closure of its covering relations. We note that the reflexive transitive closure of a set of covering relations can never add a covering relation so, in fact, a locally finite poset is completely determined by its covering relations.

If there is a covering relation in $E$ that does not satisfy (h2), the labels in the relation are not adjacent or equal in $\Gamma$ and thus not subject to the requirement in (H1). Therefore, if we delete this relation, the poset determined by the remaining covering relations still satisfies (H1). This poset has fewer relations than $E$, so $E$ is not the minimal partial order satisfying (H1), i.e., $E$ does not satisfy (H2). So, by contraposition, we have that (H2) implies (h2).

On the other hand, if $E$ satisfies (h2), then every covering relation in $E$ is as required by (H1). Thus, again by Lemma 2.2.2, the covering relations of any poset satisfying (H1) must include all covering relations of $E$, so $E$ is minimal. So (h2) implies (H2) as needed.

Remark 3.1.7. In the sequel, we will always be working with locally finite posets, so, in proofs requiring the use of (H2), we will often directly use Lemma 3.1.6(h2), but reference (H2).

Definition 3.1.8. For a heap $\varepsilon : E \rightarrow \Gamma$, we call the interval $[x, y]$ a closed $p$-interval when $x < y$, $\varepsilon(x) = \varepsilon(y) = p$ and there is no element $z \in (x, y)$ such that $\varepsilon(z) = p$. We also call $(x, y)$ a open $p$-interval.

Remark 3.1.9. When it is clear from the context, we will often omit the modifiers “closed” and “open” and simply refer to the interval as a $p$-interval.

Example 3.1.10. In Figure 3.1, we give an example of a heap $E$ (as a Hasse diagram) over the graph $\Gamma$, which is the Dynkin diagram $D_4$. The elements of $E$ are represented by their respective labels in $\Gamma$, with subscripts used to distinguish different elements with the same label. Note that all elements with the same label (e.g., $2_1$ and $2_2$) are comparable in $E$, as
are elements with adjacent labels (e.g., 2\textsubscript{1} and 3\textsubscript{1}). The arrows just indicate the labeling function \( \varepsilon \).

The only incomparable elements, 1\textsubscript{1} and 3\textsubscript{1} have labels 1 and 3 which are not the same or adjacent in \( \Gamma \), so we verify that (H1) is satisfied. It is clear the \( E \) is locally finite, so to verify that (H2) is satisfied, we can, by Lemma 3.1.6, just examine each covering relation (e.g., between 1\textsubscript{1} and 2\textsubscript{2}) and verify that the labels are adjacent or equal (e.g., 1 and 2).

### 3.2 Minuscule Heaps

Minuscule heaps are a type of heap over a Dynkin diagram that uses the associated generalized Cartan matrix of the Dynkin diagram to place further limitations on its structure. Introduced by Stembridge [25], they are a restatement of the requirements of \( d \)-complete posets introduced by Proctor [20] in the language of heaps, with some modification.

We remind the reader that Dynkin diagrams do not contain loops, so we assume all graphs here and in the sequel do not as well.

**Definition 3.2.1.** Let \( \varepsilon : E \to \Gamma \) be a heap and \( \Gamma \) a Dynkin diagram associated to a generalized Cartan matrix \( A = (a_{ij}) \). We call a closed \( p \)-interval \([x, y] \subset E\) a **full interval**, or simply **full**, if

\[
\sum_{z \in [x, y]} a_{p, \varepsilon(z)} = 2.
\]

We call the \( p \)-interval **overfull** if

\[
\sum_{z \in [x, y]} a_{p, \varepsilon(z)} < 2
\]

and **underfull** if

\[
\sum_{z \in [x, y]} a_{p, \varepsilon(z)} > 2.
\]

An open \( p \)-interval \((x, y)\) is full, overfull or underfull if its corresponding closed \( p \)-interval \([x, y] \) is.
Figure 3.1: An example of a heap $\epsilon : E \rightarrow \Gamma$.

\[
\begin{align*}
E & : & 4_2 & & 2_2 & & 3_1 & & 1_1 & & 2_1 & & 4_1 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Gamma = D_4 & : & & 3 & & 1 & & 2 & & 4
\end{align*}
\]
Remark 3.2.2. If the $p$-interval $(x, y)$ is full, then it is immediate from the definition of generalized Cartan matrices that

$$
\sum_{z \in (x, y)} a_{p, \varepsilon(z)} = \sum_{z \in [x, y]} a_{p, \varepsilon(z)} - a_{p, p} - a_{p, p} = 2 - 2 - 2 = -2.
$$

Remark 3.2.3. The property of being full is thought of more naturally as a limitation on the number of elements in a $p$-interval with labels adjacent to $p$ in $\Gamma$. By the properties of generalized Cartan matrices, the value of an entry $a_{pq}$ is a negative integer when $q$ is adjacent to $p$ in $\Gamma$, 2 when $p = q$ and 0 otherwise. Removing the 0 terms that appear for elements will labels not adjacent and not equal to $p$ and noting that $x$ and $y$ are the only elements labeled $p$ in the interval, we see that the sum in the definition can only be $2 + 2 + (-1) + (-1)$ or $2 + 2 + (-2)$ when the interval is full. In other words, the interval can only contain two elements with labels adjacent to $p$ when both elements have a corresponding entry of $-1$, or one element with a label adjacent to $p$ with a corresponding entry of $-2$.

Definition 3.2.4. Let $\varepsilon : E \to \Gamma$ be a finite heap and $\Gamma$ a Dynkin diagram associated to a generalized Cartan matrix $A = (a_{ij})$. We call $\varepsilon : E \to \Gamma$ a minuscule heap when the following property holds:

(M3) if $[x, y] \subset E$ is a closed $p$-interval, then $[x, y]$ is full.

Remark 3.2.5. We refer to this property as (M3) to indicate we view it as subsequent to the two properties defining properties of heaps, (H1) and (H2); the “M” indicates “minuscule”. In [25], the property (M3) is referred to as (H2), as Stembridge states the properties required for a heap differently.

While the minuscule heap construction is not the main object of study in this thesis, we introduce it here for several reasons. Mainly, minuscule heaps serve as introduction and inspiration for full heaps, the construction in which we are most interested. In fact, full heaps would be a type of minuscule heap if we dropped the requirement that the heap be finite. Several results in the sequel will be applicable to minuscule heaps, though used in the
context of full heaps, and we state them in their full generality by referring to heaps that satisfy (M3).

Example 3.2.6. The heap in Figure 3.1 is a minuscule heap. To check this, it is helpful to recall the generalized Cartan matrix for $D_4$:

$$
\begin{pmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & -1 \\
0 & -1 & 2 & 0 \\
0 & -1 & 0 & 2
\end{pmatrix}.
$$

We need to examine all the $p$-intervals where $p$ is a vertex in $D_4$, i.e., where $p \in \{1, 2, 3, 4\}$. It is evident that the intervals $[2, 1, 2]$ and $[4, 1, 4, 2]$ are the only such intervals. (Recall the elements are presented by their labels with subscripts to distinguish different elements of the same label.) The former has two elements in it not labeled 2: elements $1_1$ and $3_1$. From the matrix, we know that both $a_{2,1}$ and $a_{2,3}$ equal $-1$, and $a_{2,2} = 2$. Therefore we have

$$
\sum_{z \in [2, 1, 2]} a_{2, e(z)} = a_{2,2} + a_{2,1} + a_{2,3} + a_{2,2} = 2 + (-1) + (-1) + 2 = 2,
$$

so the interval is full. For the interval $[4, 1, 4, 2]$, we use the matrix and find

$$
\sum_{z \in [4, 1, 4, 2]} a_{4, e(z)} = a_{4,4} + a_{4,2} + a_{4,1} + a_{4,3} + a_{4,2} + a_{4,4} = 2 + (-1) + 0 + 0 + (-1) + 2 = 2,
$$

so this interval is also full. We have checked all relevant intervals, so we see that the heap is minuscule.
Figure 3.2: An example of a heap $\varepsilon : E \to \Gamma$ with both overfull and underfull intervals.

$E :$

\[
\begin{array}{c}
1_1 \\
\ \ \ \ \\
2_1 \\
\ \ \ \\
3_1 \\
\end{array}
\quad \begin{array}{c}
2_2 \\
\ \ \ \\
3_2 \\
\end{array}
\]

$\Gamma = C_3 :$

\[
\begin{array}{c}
1 \\
\ \ \ \\
2 \\
\ \ \ \\
3 \\
\end{array}
\]

$\varepsilon$ $\varepsilon$ $\varepsilon$
The heap $\varepsilon : E \to C_3$ in Figure 3.2 provides examples of both overfull and underfull intervals. To check this, we recall the generalized Cartan matrix for $C_3$:

$$
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -2 \\
0 & -1 & 2
\end{pmatrix}
$$

First consider $[2_1, 2_2]$. While this appears similar to the interval in Figure 3.1, we must remember that the underlying Dynkin diagram determines if an interval is full, and, in $C_3$, we find that $a_{2,3} = -2$. So now our sum for the interval is

$$
\sum_{z \in [2_1, 2_2]} a_{2,\varepsilon(z)} = a_{2,2} + a_{2,1} + a_{2,3} + a_{2,2} \\
= 2 + (-1) + (-2) + 2 \\
< 2,
$$

so the interval is overfull. For the interval $[3_1, 3_2]$, we note that $a_{3,2} = -1$. Then we have

$$
\sum_{z \in [3_1, 3_2]} a_{3,\varepsilon(z)} = a_{3,3} + a_{3,2} + a_{3,3} \\
= 2 + (-1) + 2 \\
> 2,
$$

so the interval is underfull. Clearly, this heap is not minuscule.

**Minuscule Posets**

A structure related to minuscule heaps, unsurprisingly, is the minuscule poset. Minuscule posets are defined as posets whose order ideal poset is the weight poset of a particular kind of representation of a Lie algebra. (A rigorous definition, which also makes clear from where the term “minuscule” comes, requires an extended digression into Lie algebra theory that is tangential to this thesis, so we omit it here; see [18].) It turns out that all minuscule posets are underlying posets for certain minuscule heaps.
Figure 3.3: An example of a minuscule heap $\varepsilon : E \to \Gamma$ that is maximal and degenerate.

The limitation imposed on minuscule heaps by (M3) allows us to consider the concept of a maximal minuscule heap, i.e., a minuscule heap that is not a proper convex subheap of any other minuscule heap over the same Dynkin diagram. The heap in Figure 3.1 is maximal in this sense; if we attempt to add any other element of any label to it, we will violate (M3). If we restrict ourselves to consider only finite Dynkin diagrams, these maximal minuscule heaps are usually have minuscule posets as their underlying poset.

We say “usually” above because there are some degenerate maximal minuscule heaps where this is not the case. For example, Figure 3.3 shows a minuscule heap (trivially so) that is maximal but not underlain by a minuscule poset. While we do not formally verify its maximally here, $C_3$ is small enough that the reader can verify this by hand. Instead, we use this heap as an example of what causes degeneracy. While no elements can be added to this heap without violating (M3), the removal of a single element results in a heap that we can build up (i.e., add elements) into a much larger minuscule heap. In this case, we can remove $1_1$ and build up the larger (maximal) minuscule heap shown in Figure 3.4. We can exclude degenerate examples like this by eliminating those heaps in which the removal of an element allows for the construction of a minuscule heap strictly larger than the original.
Figure 3.4: An example of a minuscule heap $\varepsilon : E \to \Gamma$ that is maximal and not degenerate.
Minuscule posets have several significant properties; for example, they are the only known Gaussian posets [18], Section 9. Of most interest to us is the fact that minuscule posets provide a so-called ADE classification, a set of analogous links between the simply laced finite Dynkin diagrams [23]. This classification relies on order ideal posets to form the links. We will use this ADE classification in Chapter 5 as an analogy to develop our own “A\[overline{A}\]D\[overline{E}\] classification” for simply laced affine Dynkin diagrams, this time using clipped ordeal posets. To develop this analogy, it will be helpful to be familiar with minuscule posets, so we give an atlas of all minuscule posets (and examples of minuscule heaps that produce them) in Appendix B.

3.3 Full Heaps

We finally introduce our central object of study, the full heap. We follow Green [8] in our treatment, with minor notational modifications.

**Definition 3.3.1.** Let \( \varepsilon : E \to \Gamma \) be a nonempty locally finite heap and \( \Gamma \) a Dynkin diagram associated to a generalized Cartan matrix \( A = (a_{ij}) \). We call \( \varepsilon : E \to \Gamma \) a full heap if it satisfies the properties

(F3) if \([x, y] \subset E\) is a closed \( p \)-interval, then \([x, y]\) is full.

(F4) for every vertex \( p \in \Gamma \), the chain consisting of all elements labeled \( p \) is isomorphic as a poset to \( \mathbb{Z} \);

(F5) for every edge \((p, q)\) in \( \Gamma \) and every element \( x \in E \) labeled \( p \), there is an element \( y \in E \) labeled \( q \) such that \( x \) and \( y \) are a covering pair.

**Remark 3.3.2.** Again, the labels (F3), (F4) and (F5) are chosen so that these properties are considered in conjunction with properties (H1), (H2) and (M3). One can see that (F3) and (M3) are the same property, but the expansion from finite to locally finite in the hypotheses
require that it be restated. In the sequel, we refer to (M/F3) when both minuscule and full heaps are being considered.

Recall that all elements with the same label are comparable in $E$ by (H1), so they form a chain. Property (F4) guarantees there is a countably infinite number of these elements for any particular label. In particular, every possible label occurs in a full heap.

**Example 3.3.3.** Figure 3.5 gives our first example of a full heap. This is one of the full heaps that occur over the Dynkin diagram $\tilde{D}_5$. We have highlighted some elements in the heap that form a motif repeated throughout the heap that fully defines the heap structure. The actual full heap continues indefinitely above and below the figure, as the motif must be repeated infinitely in both directions, as a consequence of (F4).

As before, the elements of the heap are denoted by their label with a subscript to differentiate elements with the same label. It is not difficult to see that this structure is a heap. Property (H1) is visually verifiable, and property (h2) (and hence, (H2)) is checked by looking at the covering relations and verifying that the labels are adjacent.

For property (F3), there are really only two cases, as all others are analogous. In the first case, we can see that between any consecutive pair of elements labeled 0, there are two elements labeled 2, one element labeled 1 and one labeled 3. Because 1 and 3 are not adjacent to 0 in $\tilde{D}_5$, they contribute nothing to the relevant sum. The Dynkin diagram $\tilde{D}_5$ is simply laced, so each 2, being adjacent to 0, contributes $-1$ to the relevant sum, so the 0-intervals are full. In the second case, we see a 2-interval contains one element labeled 1 and one labeled 3. Both 1 and 3 are adjacent to 2, so each contributes a $-1$ to the relevant sum. Thus the 2-intervals are full.

Property (F4) is visually verified, since the subscripts give the map to $\mathbb{Z}$. Property (F5) is also easily verified. For example, each element labeled 3 is in covering relations with elements labeled 2, 4, and 5. This is the entire set of labels adjacent to 3, as needed for (F5). So we see this is a full heap.
Figure 3.5: An example of a full heap $\varepsilon : E \rightarrow \Gamma$. 

\[
\begin{array}{c}
\varepsilon \\
\downarrow \\
\Gamma = \tilde{D}_5 \\
0 \quad 2 \quad 3 \\
\downarrow \\
1 \quad 2 \quad 3 \\
\downarrow \\
0 \\
\end{array}
\]
An atlas of all known full heaps is given in Appendix C. One goal of this thesis is to show this is a complete list of all full heaps over Dynkin diagrams with a finite number of vertices.

**Notational conventions for elements of heaps**

We conclude this section with a notational convention, which we have already been using to a small extent.

Let $\varepsilon : E \rightarrow \Gamma$ be a heap. For an element of $E$ labeled by a vertex of $\Gamma$, we will usually refer that element by the label indexed with a subscript to differentiate elements with the same label. If we do not know the actual label, but wish to anchor it so that we can refer to other elements with the same label, we use the variables $a, b, c, d, p$ and $q$. For example, the elements $a_1$ and $a_2$ of $E$ refer to two elements with the same unspecified label.

By (H1), we can use real numbers for our subscripts and use the order of the subscripts to reflect the order of the poset. That is, if $i < j$ in $\mathbb{R}$, then $a_i < a_j$ in $E$. For example, if we wish to introduce an element of $E$ that has the label 1, we may simply say there is a $1_1 \in E$, assuming $1_1$ has not already been introduced.

In the case of minuscule and full heaps, we index these subscripts with integers, which we can do because minuscule heaps are finite and full heaps satisfy (F4). Note this means that $(a_i, a_{i+1})$ is an $a$-interval of $E$ when $E$ is minuscule or full.

If we need to refer to general elements of $E$ for which we do not know the label nor whether that label is the same as another, we will generally use $x, y, z, v$ or $w$ without subscript. If we need a series of elements in $E$, we will indicate that series with left superscripts to avoid confusion with the above. For example, we may say there is a saturated chain $1_x \rightarrow 2_x \rightarrow \cdots \rightarrow k_x$ in $E$ to refer to a saturated chain of $k$ elements for which we do not know the labels. In particular, $1_x$ does not necessarily have the same label as $2_x$.

There will be occasions in which we wish to anchor the label, but we need a long series of elements with different anchored labels. In this case, we use both left superscripts and
right subscripts. For example, there may be a series of \( k \) adjacent labels in \( \Gamma \), which we will refer to as \( ^1c, ^2c, \ldots, ^k c \). Then, we refer to elements of \( E \) that have these labels by placing right subscripts on them. For example, \( ^1c_1 \) and \( ^1c_2 \) are elements of \( E \) with the same label \( ^1c \). Elements \( ^1c_1 \) and \( ^2c_1 \) of \( E \) have possibly different labels, \( ^1c \) and \( ^2c \). (Note that the labels are adjacent, though, since \( ^1c \) and \( ^2c \) are adjacent in \( \Gamma \).)

Finally, we will now stop explicitly showing the map from a heap to its underlying Dynkin diagram in our figures. Also, unless explicitly needed, we will not show the Dynkin diagram, as they can be easily found in Appendix A.

### 3.4 Local structure of full and minuscule heaps

The results in this and the following section often refer to a heap that satisfies (M/F3). Recall this means that the heap either satisfies (M3), if the heap is finite, or (F3), if the heap is infinite and locally finite. Recall also that, by definition, all full heaps satisfy (F3), so all these results apply to full heaps. Implicitly, heaps that satisfy (M/F3) are locally finite, so using Lemma 3.1.6, we know (H2) and (h2) are equivalent. For consistency, we will appeal to (H2) below, but often use the property (h2) explicitly in the argument.

To fix notation, recall also that \( \Gamma = \Gamma( A ) \) is associated to a generalized Cartan matrix \( A = ( a_{ij} ) \). We assume that \( \Gamma \) is connected and has a finite number of vertices. The reader may find it useful to refer to Appendix C for examples of the phenomena described below.

**Lemma 3.4.1.** Let \( \varepsilon : E \to \Gamma \) be a heap satisfying (M/F3). If \( x, y \in E \) are in a covering relation and \( x \) is labeled \( a \in \Gamma \), then \( y \) is not labeled \( a \).

**Proof.** Suppose that \( y \) is labeled \( a \). Clearly, \( (x, y) \) or \( (y, x) \) is then an open \( a \)-interval, since no elements exist between \( x \) and \( y \), let alone elements labeled \( a \). Without loss of generality, assume \( (x, y) \) is the \( a \)-interval. Then \( \sum_{x < z < y} a_{a, \varepsilon(z)} = 0 \) because the sum is empty. This violates (M/F3), so we have a contradiction. \( \square \)

**Lemma 3.4.2.** Let \( \varepsilon : E \to \Gamma \) be a heap satisfying (M/F3). Let \( a_1 \in E \) have label \( a \in \Gamma \).
(1) If $a_1$ is less than another element labeled $a$, then

$$-2 \leq \sum_{x \leftarrow a_1} a_{a,\varepsilon(x)} \leq -1$$

and $a_1$ is covered by one or two elements only.

(2) If $a_1$ is greater than another element labeled $a$, then

$$-2 \leq \sum_{x \rightarrow a_1} a_{a,\varepsilon(x)} \leq -1$$

and $a_1$ covers one or two elements only.

(3) If $a_1$ is both less than and greater than other elements labeled $a$, then

$$-4 \leq \sum_{x \leftarrow a_1 \text{ or } x \rightarrow a_1} a_{a,\varepsilon(x)} \leq -2$$

and $a_1$ is in covering relations with two, three or four elements only.

Proof. The proofs of (1) and (2) are entirely symmetric, so we will only examine (1). If there is an element labeled $a$ greater than $a_1$, we use local finiteness to find the least such element, $a_2 \in E$. Then $(a_1, a_2)$ is an $a$-interval. By (M/F3) and Remark 3.2.2, we know that $\sum_{x \in (a_1, a_2)} a_{a,\varepsilon(x)} = -2$. By (H2), all elements covering $a_1$ have labels adjacent or equal to $a$, so must be comparable to $a_2$ by (H1). These elements cannot have labels equal to $a$ by Lemma 3.4.1.

It is impossible for the elements covering $a_1$ to be greater than $a_2$ without breaking their covering relations with $a_1$, so they must be in $(a_1, a_2)$. By definition of generalized Cartan matrices, $a_{a,\varepsilon(z)} \leq 0$ for all $z \in (a_1, a_2)$ since none are labeled $a$. Thus we must have

$$-2 = \sum_{x \in (a_1, a_2)} a_{a,\varepsilon(x)} \leq \sum_{x \leftarrow a_1} a_{a,\varepsilon(x)},$$

because the terms, all nonpositive, in the rightmost sum are a subset of those in the middle sum.
By local finiteness, we know the set of elements covering $a_1$ is not empty by Lemmas 2.2.1 and 3.4.1. If $x$ covers $a_1$ and thus has an adjacent label, then $a_{a,x}(x) < 0$. This gives us that

$$\sum_{x \prec a_1} a_{a,x}(x) \leq -1.$$ 

Finally, since each integer $a_{a,x}(x)$ in the sum is less than 0 but their sum is, at least, $-2$, there can be, at most, two such integers.

Statement (3) is obtained by simply summing the results of (1) and (2), noting that an element cannot both cover and be covered by $a_1$ by the antisymmetry axiom of posets.

**Remark 3.4.3.** Note that if $E$ is a full heap, we can drop the hypothesis specific to each statement in Lemma 3.4.2, since, by (F1), there are always elements labeled $a$ above and below $a_1$, no matter the choice of $a_1$.

**Lemma 3.4.4.** Let $\varepsilon : E \rightarrow \Gamma$ be a heap satisfying (M/F3). Suppose that $\Gamma$ is neither simply nor doubly laced. Then there are $a$ and $b$ adjacent in $\Gamma$ such that $E$ contains no $a$-intervals containing an element labeled $b$. In particular, if $E$ is a full heap, then $\Gamma$ is simply or doubly laced.

**Proof.** Recall that, if $\Gamma$ is neither simply nor doubly laced, there are two labels $a, b \in \Gamma$ such that $a_{a,b} < -2$. Let the $a$ and $b$ in the statement have this property.

Say $(a_1, a_2)$ is an $a$-interval in $E$. By the definition of $a$-interval, there is no $z \in (a_1, a_2)$ with the label $a$, so all $a_{a,x}(z)$ occur off the diagonal of the generalized Cartan matrix $A$ and must be nonpositive. Suppose now that there is at least one $b_1$ (labeled $b$) in $(a_1, a_2)$. Because $a_{a,b} < -2$, the sum $\sum_{a_1 < z < a_1} a_{a,x}(z)$ must also be strictly less than $-2$. This is a direct violation of (M/F3), so this $a$-interval cannot contain an element labeled $b$. As $(a_1, a_2)$ was chosen arbitrarily, we have the required result.

If $E$ is a full heap, then (F4) requires there to exist in $E$ an infinite chain isomorphic to $\mathbb{Z}$ of elements labeled $a$. Let $a_1$ and $b_1$ be two elements of $E$. (Recall the notational
convention that $a_1$ is labeled $a$ and $b_1$ is labeled $b$.) Without loss of generality, assume $a_1 < b_1$. By local finiteness, we know that $(a_1, b_1)$ is finite. However, there is an infinite number of elements labeled $a$ that are greater than $a_1$, so there must be an element labeled $a$ greater than $b_1$. Again by local finiteness, we see there must be an $a$-interval containing $b_1$. This is impossible when $a_{a,b} < -2$ by (M/F3), so we must have $\Gamma$ simply or doubly laced.

For adjacent vertices $a$ and $b$, there are certain conditions in a heap satisfying (M/F3) when an $a$-interval $(a_1, a_2)$ implies the existence of a $b$-interval containing it. Indeed, the $b$ consists only of the $a$-interval and two additional elements labeled $b$. This property inspires the following definition, useful for describing how full heaps are built.

**Definition 3.4.5.** Let $a$ and $b$ be adjacent vertices in $\Gamma$ and $(a_1, a_2)$ be an $a$-interval in $E$ satisfying the hypotheses of Corollary 3.4.7. We then say that we can **expand** $(a_1, a_2)$ **to** $(b_1, b_2)$, meaning that the we know $(b_1, b_2)$ exists as a $b$-interval and is equal to $[a_1, a_2]$.

We justify this definition with the following lemma and corollary.

**Lemma 3.4.6.** Let $\varepsilon : E \rightarrow \Gamma$ be a heap satisfying (M/F3). Suppose $a, b$ are adjacent vertices in $\Gamma$ and that there exists in $E$ an open $a$-interval $(a_1, a_2)$ contained within an open $b$-interval $(b_1, b_2)$. Then $(b_1, b_2) = [a_1, a_2]$.

**Proof.** Note that $\Gamma$ contains no loops, so $a \neq b$. Additionally, since $(a_1, a_2)$ is in $(b_1, b_2)$, then $[a_1, a_2]$ must also be for the following reason. As $a$ and $b$ are adjacent, $a_1$ and $a_2$ are each comparable to $b_1$ and $b_2$. We cannot have $a_1 > b_2$ or $a_2 < b_1$, else the containment of the hypothesis is clearly impossible. (Note, by Lemma 3.4.1, these intervals are not empty.) If $a_1 < b_1$, then $b_1 \in (a_1, a_2) \subseteq (b_1, b_2)$, which is absurd. So $a_1$, and similarly $a_2$, are in $(b_1, b_2)$ as needed.

Recall first that, by (M/F3), the interval $(b_1, b_2)$ is full. Thus, from Remark 3.2.3, we know that $(b_1, b_2)$ can contain, at most, two elements with labels adjacent to $b$. (None
can have labels equal to $b$ by the definition of $b$-interval.) By hypothesis and the above paragraph, the elements $a_1$ and $a_2$ are those two elements, so no other element of $(b_1, b_2)$ has label adjacent (or equal) to $b$.

Suppose $b_1$ is covered by an element $c$. By (H2), we know that any element covering $b_1$ must be have a label adjacent or equal to $b$, so $c$ is adjacent or equal to $b$. By (H1), we must then have $c$ comparable to $b_2$ as well. Were $b_2 < c$, this would break the covering relation between $b_1$ and $c$. Furthermore, $c$ cannot have label $b$ by Lemma 3.4.1. So $c$ is in $(b_1, b_2)$.

However, we have already shown that the only elements in $(b_1, b_2)$ with labels adjacent to $b$ are $a_1$ and $a_2$. Since $b_1 < a_1 < a_2$, we cannot have $c$ equal to $a_2$ with breaking the covering relation. Therefore $c = a_1$.

Because $c$ was arbitrarily chosen, it follows that $a_1$ is the only element covering $b_1$. By symmetry, $a_2$ is the only element covered by $b_2$. By local finiteness, any element in $(b_1, b_2)$ is part of a saturated chain from $b_1$ to $b_2$ that, by the above, has the form

$$b_1 \rightarrow a_1 \rightarrow \cdots \rightarrow \cdots \rightarrow a_2 \rightarrow b_2.$$ 

Thus we must have $(b_1, b_2) = [a_1, a_2]$.

\[ \square \]

**Corollary 3.4.7.** Let $\varepsilon : E \rightarrow \Gamma$ be a full heap. Suppose $a, b$ are adjacent vertices in $\Gamma$ and that there exists in $E$ an $a$-interval $(a_1, a_2)$ that contains no element labeled $b$. Then there exist $b_1$ and $b_2$ in $E$, both labeled $b$, such that $(b_1, b_2) = [a_1, a_2]$.

**Proof.** If $(a_1, a_2)$ contains no element labeled $b$, then $a_1$ is not covered by an element labeled $b$ by (H1). Therefore, by (F5), $a_1$ must cover an element labeled $b$, say $b_1$. Similarly, $a_2$ must be covered by an element labeled $b$, say $b_2$.

There can be no element $x$ in $(b_1, b_2)$ labeled $b$, otherwise, by (H1), $x$ would be comparable to both $a_1$ and $a_2$. By hypothesis, $x$ cannot be in $(a_1, a_2)$, so $x$ must either be less than $a_1$, breaking the $b_1 \rightarrow a_1$ covering relation, or greater than $a_2$, breaking the $b_2 \leftarrow a_2$ relation.
covering relation. So we see that the hypotheses of Lemma 3.4.6 are satisfied and obtain our conclusion.

\[\square\]

3.5 Diamonds

We now introduce and examine a structure in full heaps that will be integral in our discussion below.

**Definition 3.5.1.** In a locally finite heap \( \varepsilon : E \to \Gamma \) with \( p \in \Gamma \), a \( p \)-diamond is any \( p \)-interval \([p_i, p_{i+1}]\) in \( E \) in which, for all \( z \in (p_i, p_{i+1}) \), the element \( z \) covers \( p_i \) and is covered by \( p_{i+1} \).

See Figure 3.6 for a few examples of diamonds in a locally finite heap.

Figure 3.6: Three examples of \( p \)-diamonds in a locally finite heap \( \varepsilon : E \to \Gamma \). Here, we assume that \( p \in \Gamma \) is adjacent to each of \( a, b \) and \( c \) in \( \Gamma \). Here, the implication is that there are no other elements in the \( p \)-diamond other than those shown. We use the subscripts just to indicate that the elements are part of a heap; their value is otherwise meaningless.

\[\begin{array}{ccc}
(a) & p_2 & (b) & p_2 & (c) & p_2 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
a_1 & a_1 & b_1 & b_1 & c_1 \\
p_1 & p_1 & p_1 & p_1 \\
\end{array}\]

**Remark 3.5.2.** When the label is clear from the context, we will often refer to a \( p \)-diamond as simply a diamond. As with \( p \)-intervals, we refer to open intervals as diamonds if their respective closures are diamonds. From Lemma 3.4.2, we know that any \( p \)-diamond in heap satisfying (M/F3) consists of three or four elements, two of which are the minimum \( p_i \) and maximum \( p_{i+1} \). Examples (a) and (b) in Figure 3.6 show the only possible \( p \)-diamond forms in a heap satisfying (M/F3).
Lemma 3.5.3. Let \( \varepsilon : E \to \Gamma \) be a heap satisfying (M/F3). Let \( a \) be a vertex in \( \Gamma \) and \([a_1, a_2]\) be an \( a \)-interval in \( E \). Suppose there is a saturated chain \( a_1 \to z \to a_2 \) in \( E \). Then \([a_1, a_2]\) is an \( a \)-diamond.

Proof. First note that (M/F3) requires that \((a_1, a_2)\) be full. Recall from Remark 3.2.3, that, to satisfy the sum in Definition 3.2.1, the interval either contains exactly two elements with labels adjacent to \( a \), each sharing a single edge with \( a \), or exactly one element with label adjacent to \( a \) that shares a double edge with arrow pointing to \( a \).

Let \( b = \varepsilon(z) \). From Lemma 3.4.4 we know that \( a_{a,b} = -2 \) or \( a_{a,b} = -1 \).

Suppose first that \( a_{a,b} = -2 \). This translates into a double edge with arrow pointing from \( b \) to \( a \) in \( \Gamma \), so \( z \) is the only element in the \( a \)-interval with label adjacent to \( a \). Suppose \( v \) is an element distinct from \( z \) also in the \( a \)-interval. Then there is a finite saturated chain from \( a_1 \) to \( v \) in \( E \) by local finiteness. The element covering \( a_1 \) in this chain cannot be \( z \), otherwise \( v \) would break the covering relation \( z \to a_2 \), since \( v < a_2 \), so we can assume without loss of generality that \( v \) covers \( a_1 \).

By (H2), \( v \) must have a label equal to or adjacent to \( a \). The former is impossible by the definition of \( a \)-interval. The latter contradicts the fact that \( z \) is the only element in the interval with label adjacent to \( a \). Thus, \( v \) cannot exist and \((a_1, a_2) = \{z\}\). Since \( z \) covers \( a_1 \) and is covered by \( a_2 \), the requirement for an \( a \)-diamond is satisfied.

Now suppose \( a_{a,b} = -1 \), which translates into a single edge between \( a \) and \( b \), so there exists exactly one element other than \( z \), say \( x \), in the interval \((a_1, a_2)\) with label adjacent to \( a \). If \( x \) also has label \( b \), then it is comparable to \( z \) by the definition of a heap. Were \( x < z \), we would have \( a_1 < x < z \), breaking the covering relation \( a_1 \to z \). Similarly, we cannot have \( x > z \). Thus \( x \) is not comparable to \( z \) and has a label other than \( b \).

As above, because \( x \) is not comparable to \( z \), there is an element covering \( a_1 \) other than \( z \). This element has a label adjacent to \( a \), so, as \( x \) is the only element other than \( z \) with label adjacent to \( a \), we must have \( x \) cover \( a_1 \). Similarly, \( a_2 \) covers \( x \). The \( a \)-interval can
clearly contain no other elements without breaking covering relations or being overfull, so it is a diamond.

**Corollary 3.5.4.** Let \( \varepsilon : E \rightarrow \Gamma \) be a heap satisfying (M/F3). Let \( a \) and \( b \) be distinct vertices in \( \Gamma \) and \([a_1, a_2]\) be an \( a \)-interval in \( E \). Suppose that \( b \) shares a double edge with \( a \) in \( \Gamma \) with an arrow pointing towards \( a \). If \( b_1 \in (a_1, a_2) \) has label \( b \), then \((a_1, a_2) = \{b_1\}\), and \([a_1, a_2]\) is an \( a \)-diamond.

**Proof.** Because \((a_1, a_2)\) is full by (M/F3) and contains an element \((b_1)\) with label that shares a double edge with arrow pointing to \( a \), we recall from Remark 3.2.3 that \( b_1 \) is the only element with label adjacent to \( a \) in the interval. By (H1), (H2) and the definition of \( a \)-interval, the element \( b_1 \) is thus the only element covering \( a_1 \) and the only element covered by \( a_2 \).

The resulting chain \( a_1 \rightarrow b_1 \rightarrow a_2 \) in \( E \) gives, by Lemma 3.5.3, that \((a_1, a_2)\) is an \( a \)-diamond. By definition of diamond, any other element in \((a_1, a_2)\) must cover \( a_1 \) (and be covered by \( a_2 \)), contradicting the fact that \( b_1 \) is the only such element. So \((a_1, a_2) = \{b_1\}\) and we are done.

**Lemma 3.5.5.** Let \( \varepsilon : E \rightarrow \Gamma \) be a heap satisfying (M/F3). Let \( a, b, c \) be distinct vertices in \( \Gamma \), with \( a \) adjacent to both \( b \) and \( c \), and \([a_1, a_2]\) be an \( a \)-interval in \( E \). If \( b_1, c_1 \in (a_1, a_2) \) have labels \( b \) and \( c \) respectively, then either

1. \((a_1, a_2) = \{b_1, c_1\}\) and \([a_1, a_2]\) is an \( a \)-diamond, or

2. there is a path from \( b \) to \( c \) in \( \Gamma \) that does not contain \( a \) and the elements \( b_1 \) and \( c_1 \) are comparable.

Furthermore, \( b \) and \( c \) each share a single edge with \( a \) in \( \Gamma \).

**Proof.** The property (M/F3) requires that \((a_1, a_2)\) be full, so the interval either contains exactly two elements with labels adjacent to \( a \), each sharing a single edge with \( a \), or exactly
one element with label adjacent to $a$ that shares a double edge with arrow pointing to $a$, as in Remark 3.2.3. Note in particular that the interval can contain at most two elements with labels adjacent to $a$.

Since $b$ and $c$ are both adjacent to $a$, there are already two elements in the interval $(a_1, a_2)$ with labels adjacent to $a$. Therefore, $b$ and $c$ each share a single edge with $a$ in $\Gamma$.

Let $i^v \in (a_1, a_2)$ be an element with any label (other than $a$, due to the definition of an $a$-interval). There must exist some saturated chain

$$a_1 \rightarrow ^1v \rightarrow \cdots \rightarrow ^i v \rightarrow \cdots \rightarrow ^n v \rightarrow a_2,$$

by local finiteness.

If $n = 1$, then we have a saturated chain $a_1 \rightarrow ^1 v \rightarrow a_2$. By Lemma 3.5.3, the $a$-interval $[a_1, a_2]$ is an $a$-diamond. By the definition of $a$-diamond, all elements in $(a_1, a_2)$ are in a covering relation with both $a_1$ and $a_2$ and thus, by (H2) and the definition of $a$-interval, have labels adjacent to $a$. Because $(a_1, a_2)$ is full, there can be at most two elements in the interval with labels adjacent to $a$, so $b_1$ and $c_2$ are the only such elements, i.e., $(a_1, a_2) = \{ b_1, c_1 \}$, obtaining the consequence (1).

Now suppose that $n > 1$. By (H2) and Lemma 3.4.1, we must have $^1 v = b_1$ or $^1 v = c_1$. Similarly, $^n v = b_1$ or $^n v = c_1$. We can assume without loss of generality that $^1 v = b_1$ and $^n v = c_1$. Then our chain of covering relations becomes

$$a_1 \rightarrow b_1 \rightarrow ^2 v \rightarrow \cdots \rightarrow ^i v \rightarrow \cdots \rightarrow ^{n-1} v \rightarrow c_1 \rightarrow a_2.$$

Again by (H2) and Lemma 3.4.1, elements in a covering relation have adjacent labels in the underlying graph, so $\Gamma$ has, as an induced subgraph,
By the definition of \(a\)-interval, we know that \(\varepsilon^i(v) \neq a\) for all \(i \in \{2, n - 1\}\), so we have a path from \(b\) to \(c\) in \(\Gamma\) that does not include \(a\). As we already know that \(b_1\) and \(c_1\) are comparable, we have obtained consequence (2). All cases have been addressed, so the proof is complete.

Remark 3.5.6. Note that, if the hypotheses of Lemma 3.5.5 are satisfied and, additionally, \(\Gamma\) contains no circuits, then we always have consequence (1), since a path from \(b\) to \(c\) not containing \(a\) would result in a circuit in \(\Gamma\).

**Corollary 3.5.7.** Let \(\varepsilon : E \rightarrow \Gamma\) be a full heap. Let \(a, b, c\) be distinct vertices in \(\Gamma\) with \(a_1 \in E\) labeled \(a\).

(1) Suppose \(b_1, c_1 \in E\) have labels \(b\) and \(c\) respectively. If \(a_1\) is covered by (respectively, covers) \(b_1\) and \(c_1\), then there exists an element \(a_2 \in E\) such that \((a_1, a_2)\) (respectively, \((a_2, a_1)\)) is an open \(a\)-diamond equal to \(\{b_1, c_1\}\).

(2) Suppose that \(b\) shares a double edge with \(a\) in \(\Gamma\) with an arrow pointing towards \(a\), and \(b_1 \in E\) has label \(b\). If \(a_1\) is covered by (respectively, covers) \(b_1\), then there exists an element \(a_2 \in E\) such that \((a_1, a_2)\) (respectively, \((a_2, a_1)\)) is an open \(a\)-diamond equal to \(\{b_1\}\).

Proof. For (1), we prove the case when \(a_1\) is covered by \(b_1\) and \(c_1\), as the other case is symmetric. First note that (H2) and Lemma 3.4.1 require \(b\) and \(c\) to be adjacent to \(a\) in \(\Gamma\). Using (F4) and local finiteness, we obtain the needed element \(a_2\) for the \(a\)-interval \([a_1, a_2]\). Thus the hypotheses of Lemma 3.5.5 are satisfied. Furthermore, we cannot have \(b_1\) and \(c_1\) comparable, otherwise the lesser would break the covering relation between the greater and \(a_1\). Thus consequence (2) of Lemma 3.5.5 is impossible, so \((a_1, a_2)\) is an open \(a\)-diamond equal to \(\{b_1, c_1\}\).

For (2), we only need the existence of \(a_2\), which follows from the same argument as above, and the rest follows from Corollary 3.5.4.
Definition 3.5.8. Whenever an element satisfies either set of hypotheses from Corollary 3.5.7 required to force a diamond, we say $a_1$ is **fully covered** (respectively, **fully covering**).

Note that this corollary implies that, whenever $a, b \in \Gamma$ satisfy $a_{a,b} = -2$, there is always a closed $a$-diamond in $E$ consisting of exactly two elements labeled $a$ and one element labeled $b$. This is because (F5) requires that any element labeled $b$ be in a covering relation with an element labeled $a$, and Corollary 3.5.7 (2) applies either way.

Remark 3.5.9. There are many uses for Corollary 3.5.7, but the following observation is particularly useful when building full heaps. We also develop terminology to reference the observation easily. (Follow Figure 3.7 for examples of what is described.)

Suppose that $^0x \to ^1x \to \cdots \to ^{n-1}x \to ^nx$ is a saturated chain in a full heap such that $\varepsilon(^ix) \neq \varepsilon(^{i+2}x)$ for all $0 \leq i \leq n - 2$. Suppose further that $^0x$ is fully covered. (If $^0x$ is fully covered because it is covered by an element other than $^1x$, we say $^0x$ is covered outside the chain, where the chain is clear from the context.) See Figure 3.7(1).

Corollary 3.5.7 tells us there must be a $^0y \in E$ such that $(^0x, ^0y)$ is a diamond containing $^1x$. See Figure 3.7(2). However, then $^1x$ is covered by $^0y$ and $^0y \neq ^2x$ since $\varepsilon(^0y) = \varepsilon(^0x) \neq \varepsilon(^2x)$ by assumption. So again we can use Corollary 3.5.7 to find an element $^1y$ such that $(^1x, ^1y)$ is a diamond containing $^2x$. See Figure 3.7(3).

Repeatedly applying Corollary 3.5.7, we find that there is some element $^{i-1}y$ such that $(^{i-1}x, ^{i-1}y)$ is a diamond containing $^ix$ for each $0 < i \leq n$. See Figure 3.7(4) for the case in which $n = 4$. Whenever we use this observation, we say that we fill upwards with diamonds to $^nx$. Note in particular that, by the definition of diamonds, we know that, for each $i \in \{0, 1, 2\}$, the elements $^ix$ and $^iy$ have the same label.

There is an entirely symmetric situation using the same saturated chain, but now assuming that $^nx$ is fully covering. (If $^nx$ is fully covering, it covers by an element other than $^{n-1}x$, we say $^nx$ covers outside the chain.) We then build diamonds downwardly to obtain an element $^{i+1}y$ such that $(^{i+1}y, ^{i+1}x)$ is a diamond containing $^ix$ for $0 \leq i < n$. 
Figure 3.7: (1)–(4) An illustration of filling upwards with diamonds to $^3x$ from $^0x$. Note that in (1), $^0x$ is covered outside the chain by $y$. (We do not illustrate this, but if there is a double edge from between the labels of $^0x$ and $^1x$, with arrow pointing to the label of $^0x$, then $^0x$ is also covered fully, so we can still fill with diamonds. The only difference in the illustration in this case is that there would be no element $y$.) (5)–(6) An illustration of filling downwards with diamonds to $^0x$ from $^3x$. 

(1) 

(2) 

(3) 

(4) 

(5) 

(6)
Whenever we use this trick, we say that we fill downwards with diamonds to $^0x$. See Figure 3.7(5) and (6). Again, the elements $^ix$ and $^iy$ have the same label, for $i \in \{1, 2, 3\}$.

**Lemma 3.5.10.** Let $\varepsilon : E \to \Gamma$ be a heap satisfying (M/F3). If $\Gamma$ does not contain a circuit then any $a$-interval in $E$ must contain a diamond.

**Proof.** Suppose $(a_1, a_2)$ is an $a$-interval. If $(a_1, a_2)$ is a diamond, we are done. If not, we can use local finiteness to get a saturated chain

$$a_1 \to ^1x \to \cdots \to ^kx \to a_2.$$

If $^1x$ and $^kx$ do not have the same label, then the labels of this chain with give a circuit in $\Gamma$ from $a$ to itself, violating our hypothesis, so they must have the same label, say $b$. Now we may rewrite our chain as

$$a_1 \to b_1 \to \cdots \to b_2 \to a_2.$$

Since $b$ must be adjacent to $a$ by (H2), we can have no other $^ix$ with label $b$ without violating (M/F3), so $(b_1, b_2)$ is a $b$-interval. By Lemma 3.4.6, we find that $(a_1, a_2) = [b_1, b_2]$.

Thus $(b_1, b_2)$ is a $b$-interval that has two fewer elements than $(a_1, a_2)$. If $(b_1, b_2)$ is a diamond, we are done. Otherwise, we continue this process. Since each new interval has two fewer elements than the one preceding and local finiteness guarantees that $(a_1, a_2)$ has a finite number of elements, we must eventually reach a open $c$-interval, for some label $c$, that contains either one or two elements.

If the open $c$-interval contains just one element, it must be a diamond by Lemma 3.5.3. If there are two elements in the interval, they cannot have the same label by Lemma 3.4.1. Since there are only two elements, both must have labels adjacent to $c$ by (H2). Under the hypothesis that there are no circuits, we find that the $c$-interval is a diamond by Lemma 3.5.5, so we are done. $\square$
Chapter 4

On graphs admitting full heaps

The requirements of a full heap and, to a lesser extent, minuscule heaps, can be quite restrictive. Thus, most graphs do not admit a full heap, as we shall see below. Here, we catalog a few of these restrictions, which will be useful in our complete classification of full heaps. Our approach in general is to exclude various classes of graphs from those that possibly admit a full heap, so that we restrict ourselves to a manageable set we can examine directly.

Let $\varepsilon : E \rightarrow \Gamma$ be a full heap. All full heaps are defined only over graphs that are Dynkin diagrams (and do not make sense otherwise), so our first implication on a graph $\Gamma$ is that it is a Dynkin diagram. We also assume that $\Gamma$ is connected with a finite number of vertices. (This thesis does not address Dynkin diagrams with an infinite number of vertices.) We again fix the notation of a generalized Cartan matrix $A = (a_{ij})$ for a Dynkin diagram $\Gamma = \Gamma(A)$. By Lemma 3.4.4, we know that $\Gamma$ is simply or doubly laced and will assume so throughout.

We assume nothing else about the underlying Dynkin diagrams. In particular, we use nothing from Lie theory or the Lie algebras with which they are associated; the following is intended to be a completely elementary treatment. (See [10] for a treatment of affine Kac-Moody algebras, which uses representation theory.) As we proceed to explore the ramifications of admitting a full heap, we will find that the only Dynkin diagrams that admit a full heap are a subset of the affine Dynkin diagrams listed in Appendix A, as shown in
Theorem 4.7.1.

We remind the reader that we appeal only to (H2) below in place of distinguishing (H2) and (h2), which is justified by Lemma 3.1.6 because all full heaps are locally finite.

4.1 Some general implications

Connected graphs

While we are assuming that $\Gamma$ is connected, it is worth taking a moment to explain why. Essentially, we do so because it simplifies our discussion and the generalization to disconnected graphs is straightforward, as shown in the following lemma.

Lemma 4.1.1. A heap over a disconnected graph $\Gamma$ consists of the disjoint union of subheaps over each connected component of $\Gamma$.

Proof. If $\Gamma$ is not connected, elements of a heap over $\Gamma$ with labels from different connected components of $\Gamma$ will necessarily be incomparable by (H2) because we can delete any such relations between these elements without violating (H1) or the axioms for a poset. So the subposet consisting of all elements and their relations of the heap with labels from one connected component is disjoint from the subposet of all elements with labels from a different connected component. It is clear, then, that the heap is the disjoint union of these subposets, i.e., no additional relations are needed.

Now consider one such subposet of elements and their relations over a connected component of $\Gamma$. Because the same and adjacent vertices are much all be in the same connected component, we immediately have that this subposet satisfies (H1), since the heap does. That this subposet satisfies (H2) is also inherited from the heap, as any extraneous relation the subposet would be extraneous in the heap. So the subposet is a subheap and are done.

Lemma 4.1.1 shows that any heap over a graph will have mutually disconnected components (possibly empty or disconnected themselves) that correspond to the connected com-
ponents of the graph. These components of the heap can be seems as heaps over their respective components of $\Gamma$, so a classification of full heaps over connected Dynkin diagrams immediately generalizes to a classification over all Dynkin diagrams.

**Terminal vertices**

We now examine the implications on the terminal vertices of graphs that admit a full heap. In doing so, we greatly reduce the number of doubly laced graphs in our set of graphs that admit a full heap.

We first show that all vertices in $\Gamma$ have at least one adjacent vertex. Recall that a tally of a vertex in a Dynkin diagram is a count of all edges containing that vertex, weighted by the absolute value of entry in the generalized Cartan matrix corresponding to the two vertices of the edge. Explicitly, the tally is given by the sum $\left| \sum_{p \in \Gamma \setminus \{b\}} a_{b,p} \right|$.

**Lemma 4.1.2.** Any vertex of $\Gamma$ has a tally of at least one.

*Proof.* For a vertex, say $a$, to not have a tally of at least one, it must have a tally of zero, meaning it has no adjacent vertices. Since $\Gamma$ is connected, we see that $\Gamma$ is simply $\{a\}$.

Recall that (F4) requires a full heap over $\Gamma$ to have an infinite chain of $a$-labeled elements. Because there are no other vertices, this infinite chain entirely comprises $E$, so an $a$-labeled element covers another $a$-labeled element. This is impossible due to Lemma 3.4.1, so no such heap exists. \hfill \Box

Lemma 4.1.2 shows that if a vertex is terminal in $\Gamma$, it has exactly one adjacent vertex. We now look at some situations that force a vertex of $\Gamma$ to have only one adjacent vertex, which helps characterize our possible terminal vertices.

**Lemma 4.1.3.** If $\Gamma$ contains vertices $a$ and $b$ connected by more than one edge, then either $a$ or $b$ is a terminal vertex.

*Proof.* Recall from Lemma 3.4.4 that any graph admitting a heap satisfying $(M/F3)$ must be
simply or doubly laced. In particular, any vertices in a graph admitting a full heap connected by more than one edge are necessarily connected by a double edge; we assume so below.

By definition of a Dynkin diagram, the double edge between \(a\) and \(b\) tells us that either \(a_{a,b} = -2\) or \(a_{b,a} = -2\). Assume the former without loss of generality, meaning the double arrow points towards \(a\). Suppose that \(a\) and \(b\) are not terminal, so there exists in \(\Gamma\) a vertex \(c \neq b\) sharing an edge with \(a\) and a vertex \(d \neq a\) sharing an edge with \(b\).

Figure 4.1: An illustration of the proof of Lemma 4.1.3

See Figure 4.1 for an illustration of the next three paragraphs. Choose \(a_1 \in E\). By (F5), \(a_1\) is in a covering relation with an element \(b_1\). Applying Corollary 3.5.7(2), we find that \(E\) must contain a diamond \((a_0, a_1) = \{b_1\}\) or \((a_1, a_2) = \{b_1\}\). Assume the latter without loss of generality.

Because \((a_1, a_2)\) does not contain an element labeled \(c\), we can expand \((a_1, a_2)\) to \((c_1, c_2)\) by Corollary 3.4.7. We then have a saturated chain \(b_1 \to a_2 \to c_2\).

If \(b_1\) is covered by an element \(d_1\) labeled \(d\), then \(b_1\) is covered outside the chain since \(d \neq a\), so we can fill the chain upwards with diamonds to find an element \(b_2\) covering \(a_2\). In particular, there must be a diamond \((a_2, a_3)\) containing \(c_2\) and \(b_2\). Because \(a_{a,b} = -2\) and \(a_{a,c} \leq -1\), this \(a\)-interval is overfull, a contradiction to (F3). So \(b_1\) cannot be covered by an element labeled \(d\).
A symmetric argument filling the chain $b_1 \leftarrow a_1 \leftarrow c_1$ downwards with diamonds shows that $b_1$ cannot cover an element labeled $d$. So $b_1$ is not in a covering relation with an element labeled $d$, violating (F5). This contradiction tells us that our supposition that $a$ and $b$ are not terminal is impossible, so we are done.

\[ \square \]

**Corollary 4.1.4.** If $\Gamma$ contains vertices $a$ and $b$ connected by a double edge with just one arrow, then exactly one of $a$ and $b$ is a terminal vertex.

**Proof.** We know that at least one of $a$ and $b$ is terminal by Lemma 4.1.3, so we just need to show that both cannot be. Assume they are; since $\Gamma$ is connected, $a$ and $b$ must be the only vertices in $\Gamma$.

If $a$ is terminal, then its only adjacent vertex is $b$. By (F4), we know any element $a_1$ must be covered by and cover an element, and that element must have a label adjacent to $a$, so our only option is for $b_1 \rightarrow a_1 \rightarrow b_2$. The same argument applies for $b_1$, so we get the saturated chain $a_0 \rightarrow b_1 \rightarrow a_1 \rightarrow b_2$.

Note that $(a_0, a_1) = \{b_1\}$. If there were another element in the interval, it could not be labeled $a$ or $b$ without breaking one of the covering relations. However, these are the only vertices in $\Gamma$, so $\{b_1\}$ must be the entire interval. Similarly, $(b_1, b_2) = \{a_1\}$.

Since there is only one arrow, one of $a_{a,b}$ and $a_{b,a}$ is $-1$. If the former, then

$$\sum_{a_0 < z < a_1} a_{a,z}(z) = a_{a,z}(b_1) = a_{a,b} = -1 \neq -2,$$

a violation of (F3). Similarly, if the latter, then

$$\sum_{b_1 < z < b_2} a_{b,z}(z) = a_{b,z}(a_1) = a_{b,a} = -1 \neq -2,$$

a violation of (F3). These are the only possibilities, so we have contradicted our assumption that both vertices are terminal. Thus, exactly one of the vertices $a$ and $b$ is terminal. \[ \square \]

**Lemma 4.1.5.** Suppose $\Gamma$ contains $\tilde{A}_1$ as a subgraph. Then $\Gamma$ is $\tilde{A}_1$, up to relabeling of vertices.
Proof. Recall that $\widetilde{A}_1$ consists of two vertices, say $a$ and $b$, such that $a_{a,b} = a_{b,a} = -2$. As we are in a full heap, $\Gamma$ must be doubly laced by Lemma 3.4.4, so any Dynkin diagram containing but not equal to $\widetilde{A}_1$ must have an additional vertex.

Any element $a_1 \in E$ must be in a covering relation with an element labeled $b$; say $b_1$ covers $a_1$ without loss of generality. By Corollary 3.5.7 and the fact $a_{a,b} = -2$, there is diamond $(a_1, a_2)$ that consists of just $b_1$. Also by Corollary 3.5.7 and the fact $a_{b,a} = -2$, there is a diamond $(b_0, b_1)$ consisting of just $a_1$. Applying Lemma 3.4.2, we see that neither $a_1$ and $b_1$ can be covered or cover any other elements, since, for example, $\sum_{x \in \mathbb{E}} a_{a,b}(x) = a_{a,b} = -2$.

Were there an additional vertex in $\Gamma$, there would need to be another vertex adjacent to either $a$ or $b$, since $\Gamma$ is connected. However, any such vertex must be the label of an element in a covering relation with $a_1$ or $b_1$ by (F5), which we know is impossible. Thus $\Gamma$ cannot have an additional vertex and must equal $\widetilde{A}_1$.

4.2 Graphs with circuits

Here we discuss the few cases in which the Dynkin diagram $\Gamma$ underlying a full heap $\varepsilon : E \rightarrow \Gamma$ contains a circuit. In short, we find that, for $\Gamma$ to contain a circuit, it cannot contain any vertices outside of that circuit, greatly reducing the number of graphs without terminal vertices in our set of graphs that admit a full heap.

As a reminder to the reader, Dynkin diagrams of type $\widetilde{A}_n$ for some integer $n > 1$ are, as graphs, simply laced circuits on $n + 1$ vertices. For example, $\widetilde{A}_5$ is

![Diagram of $\widetilde{A}_5$]

Let us now introduce some temporary terminology. If $\Gamma$ contains a circuit, we label it so that the circuit consist of elements from $\{0, 1, \ldots, n\}$ with the convention that $i$ and $i + 1$ are connected by an edge in the circuit. (Recall, by our definition of circuit, that $n \geq 2$, so there are always at least three elements in the circuit.) For ease of exposition, we consider
any arithmetic done on the indices of these vertices to be done modulo $n + 1$. Without loss of generality, we can assume that an edge exists between $i$ and $j$, when $0 \leq i, j \leq n$, if and only if $j \in \{i - 1, i + 1\}$. If not, we can remove the vertices $\{i + 1, i + 2, \ldots, j - 1\}$ and get another circuit.

Suppose that $y \in E$ has label $i$ from the circuit. From (F4), we know there exists $x, z \in E$ such that $x \rightarrow y \rightarrow z$. If $\varepsilon(x) = i - 1$, $\varepsilon(y) = i$, and $\varepsilon(z) = i + 1$, we say “$E$ ascends at $y$.” If $\varepsilon(x) = i + 1$, $\varepsilon(y) = i$, and $\varepsilon(z) = i - 1$, we say “$E$ descends at $y$.“ (Here, the circuit relative which $E$ ascends or descends is always the one described above, so we do not make it explicit.)

**Lemma 4.2.1.** If $\Gamma$ contains a circuit, the circuit, as a subgraph, is simply laced.

**Proof.** By definition, no vertex $i \in \{0, 1, \ldots, n\}$ in the circuit can be terminal because each has two adjacent vertices in the circuit, namely $i - 1$ and $i + 1$. By Lemma 4.1.3, all edges between $i$ and $i + 1$ are then single edges. Thus, the circuit is simply laced. 

**Lemma 4.2.2.** Suppose that $\Gamma$ contain a circuit. For any $y \in E$ with a label from the circuit, $E$ either ascends or descends at $y$. Furthermore, if $y$ covers or is covered by two elements with labels from the circuit, then $E$ both ascends and descends at $y$.

**Proof.** As usual, assume the labels of the circuit are $\{0, 1, \ldots, n\}$. Without loss of generality, assume that $y$ is labeled 1, and call it $1_1$. By (F5), we know that $1_1$ is in covering relations with elements labeled 0 and 2, which are distinct since $n \geq 2$. Call these elements $0_1$ and $2_1$.

If $0_1 \rightarrow 1_1 \rightarrow 2_1$, then $E$ ascends at $1_1$. If $2_1 \rightarrow 1_1 \rightarrow 0_1$, then $E$ descends at $1_1$. The only other possible configurations are when $1_1$ covers both $0_1$ and $2_1$ or when $1_1$ is covered by both $0_1$ and $2_1$. Assume the former, as the other argument is symmetric. We will show that $E$ both ascends and descends at $1_1$. First we show that it ascends.

If $2_1$ is covered by any element other than $1_1$, then $2_1 \rightarrow 1_1$ is a chain with bottom element covered outside the chain, so we can fill upwards with diamonds and get that $1_1$ is covered by an element $2_2$ labeled 2. Thus $E$ ascends at $1_1$, by $0_1 \rightarrow 1_1 \rightarrow 2_2$. 

Otherwise, $2_1$ is covered only by $1_1$, so $2_1$ must be in a covering relation with an element labeled $3$, say $3_1$, and since $3$ and $1$ cannot be the same label (modulo $n + 1 \geq 3$), we cannot have $1_1 = 3_1$. So if $1_1$ is the only element covering $2_1$, it must be that $2_1$ covers $3_1$.

If $3_1$ is covered by any element other than $2_1$, again fill the chain with diamonds to get an element labeled $2$ covering $1_1$ as needed. Otherwise, if $3_1$ is covered only by $2_1$, note that $3_1$ is not comparable to $0_1$ as follows. It could not be greater than $0_1$ with breaking the covering relation between $0_1$ and $1_1$. It could not be less than $0_1$ without being covered by an element other than $2_1$, since $2_1$ is incomparable to $0_1$.

We continue in this fashion until we find an element $j_1$, $j \in \{2, \ldots, n\}$, that is covered by an element other than $(j - 1)_1$. Again, note that $(j - 1)_1$ is not comparable to $0_1$, by induction. Eventually, we must find such an element, because we get to $n_1$. By (H1), $n_1$ and $0_1$ must be comparable and, by the above argument, must be covered by an element outside the chain as we needed.

An entirely symmetric argument, using $0_1$ and counting backwards, shows that $E$ descends at $1_1$. The proof is complete. 

**Corollary 4.2.3.** Suppose that $\Gamma$ contains a circuit, and suppose $E$ ascends or descends, respectively, at $y \in E$. Then $y$ is an element in an infinite saturated chain on which $E$ ascends or descends, respectively, at every element in the chain.

**Proof.** We prove the case where $E$ ascends at $y$, as the other case is entirely symmetric. As usual, assume the labels of the circuit are $\{0, 1, \ldots, n\}$. Without loss of generality, assume that $y$ is labeled $1$, and call it $1_1$. As $E$ ascends at $1_1$, there is a chain $0_1 \rightarrow 1_1 \rightarrow 2_1$ in $E$. We claim that $E$ ascends at $0_1$ and $2_1$. By induction, this will complete the proof. (The chain is saturated because we are describing a covering relation at each step.)

We show that $E$ ascends at $2_1$; the case for $0_1$ is symmetric. Suppose $E$ does not ascend at $2_1$. By Lemma 4.2.2, we know that $E$ descends at $2_1$, so there is a chain $3_1 \rightarrow 2_1 \rightarrow 1_2$ in $E$. But then $2_1$ covers both $1_1$ and $3_1$, so again by Lemma 4.2.2, $E$ must ascend (and
descend) at 2₁ as we needed.

Note that in any of the infinite saturated chains described in Corollary 4.2.3, the labels of elements of the chain cycle through the labels of the circuit, in the order of the circuit. This is clear from the construction of the chains.

**Corollary 4.2.4.** Suppose that Γ contains a circuit, and suppose E is not totally ordered. Then E both ascends and descends at every element in E.

**Proof.** Let x be an element of E. By Corollary 4.2.3, we know x is an element of an infinite saturated chain on which E either ascends or descends at every element in the chain; assume the former without loss of generality. We show that E also descends at x.

It is certainly the case that E must descend on some element in the chain containing x. If not, no element of the chain can be covered by two elements or cover two elements, by Lemma 4.2.2. Since each element is already covered by and covers one element, no element of the chain is comparable to an element outside the chain. As the chain cycles through every label in Γ, (H1) implies that there must be no other elements in E, otherwise they would be compare to some element of the chain. But then E is a chain and totally ordered, contradicting our hypothesis.

Finally, if E descends on some element of the chain, we can fill upwards and downwards with diamonds from that element so that every element in the chain covers or is covered by two elements, including x. Applying Lemma 4.2.2, we find that E descends at x. We chose the element x arbitrarily, so E ascends and descends on every element of E.

**Proposition 4.2.5.** Let ε : E → Γ be a full heap. If Γ contains a circuit then Γ is \(\tilde{A}_n\) for some integer \(n > 1\).

**Proof.** As usual, assume the labels of the circuit are \{0, 1, ..., n\}, where \(i \in \{0, 1, ..., n\}\) shares an edge with \(i - 1\) and \(i + 1\), and where the arithmetic is performed modulo \(n + 1\). We can assume that our circuit contains no smaller circuits, i.e. the edges described are the
only edges between vertices in the circuit. If this is not the case, we simply use one of the smaller circuits as our circuit.

By Lemma 4.2.1, we know that the circuit is simply laced. Thus, the subgraph consisting of just the circuit is isomorphic to $\tilde{A}_n$. To finish our proof, it now suffices to show that $\Gamma$ contains no other vertices.

Because $\Gamma$ is connected, if $\Gamma$ did contain a vertex not in the circuit, at least one of the vertices in the circuit would be adjacent to a vertex not in the circuit. Without loss of generality, assume 2 is adjacent to a vertex $b$ not in the circuit. Let $2_1$ be an element of $E$ labeled 2.

By Lemma 4.2.2, we know that $E$ either ascends or descends at $2_1$. Without loss of generality, assume $E$ ascends at $2_1$, so, by Corollary 4.2.3, we have an infinite chain

$$\cdots \rightarrow 0_1 \rightarrow 1_1 \rightarrow 2_1 \rightarrow 3_1 \rightarrow 4_1 \rightarrow \cdots$$

in $E$.

We now use (F5) to find an element $b_1$ in a covering relation with $2_1$. Suppose for the remainder that $b_1$ covers $2_1$. Then we have the chain $2_1 \rightarrow 3_1 \rightarrow 4_1 \rightarrow \cdots$ with the bottom covered outside the chain, so we can fill upwards with diamonds. In particular, there is an element label 2 covering $3_1$. Since $3_1$ is also covered by $4_1$ and 2 cannot be the same as 4 modulo $n+1 \geq 3$, we know $3_1$ is covered by two distinct vertices with labels from the circuit. We use Lemma 4.2.2 to find that $E$ both ascends and descends at $3_1$.

However, if $E$ descends at $3_1$, then $3_1$ covers an element labeled 4, which must be distinct from $2_1$ by comparing labels. That is, we have a chain $\cdots \rightarrow 0_1 \rightarrow 1_1 \rightarrow 2_1 \rightarrow 3_1$ with the top element covering an element outside the chain, so we can fill downwards with diamonds. In particular, we get that $2_1$ covers an element labeled 3. Since $2_1$ also covers $1_1$, a distinct element by comparing labels, we again use Lemma 4.2.2 to find that $E$ both ascends and descends at $2_1$.

Thus we have that $2_1$ is covered by $b_1$, $3_1$ and (because $E$ descends at $2_1$) an element
labeled 1. By assumption, \( b \) is not in the circuit, so is not 1 or 3. Also, 1 and 3 are not equal modulo \( n + 1 \geq 3 \), so \( 2_1 \) must be covered by three distinct elements. By Lemma 3.4.2(1), this is impossible. With this contradiction, we see that \( 2_1 \) cannot be covered by an element labeled \( b \).

A symmetric argument shows that \( 2_1 \) cannot cover an element labeled \( b \). There can then be no vertex adjacent to 2 and outside the circuit. The choice of 2 as label was arbitrary, so the proof is complete.

\( \square \)

4.3 Vertices with a tally of four or more

In the remaining sections in this chapter, we examine the restrictions imposed on a \( \Gamma \) that admits a full heap by assuming certain tallies for its vertices. We will be able to then completely characterize all viable Dynkin diagrams using this information. We assume that \( \Gamma \) contains no circuits, because, if it does, we already know its form by Proposition 4.2.5. We remind the reader that we assume \( \Gamma \) is connected without losing general application, by Lemma 4.1.1.

**Lemma 4.3.1.** Suppose \( \Gamma \) admits a full heap \( E \), and let \( a \) be a vertex of \( \Gamma \) with \( a_1 \in E \) labeled \( a \). Let \( s \) be the number of labels that occur more than once as the label of an element in a covering relation with \( a_1 \). Then \( a \) has tally of at most \( 4 - s \). In particular, any vertex of \( \Gamma \) has a tally of at most 4.

**Proof.** Let \( t \) be the tally of \( a \). Let \( C = \{ x \in E \mid x \leftarrow a_1 \text{ or } x \rightarrow a_1 \} \) be the set of all elements of \( E \) in a covering relation with \( a_1 \) and \( D = \{ p \in \Gamma \mid p \text{ is adjacent to } a \} \). By Lemma 3.4.1, it is never the case that \( \varepsilon(x) = a \) when \( x \in C \), so \( a_{a,\varepsilon(x)} \leq 0 \) for all \( x \in C \). By definition of generalized Cartan matrices, we also get \( a_{a,p} \leq 0 \) for all \( p \in D \).

By (F5), every label adjacent to \( a \) occurs as a label of an element in a covering relation with \( a_1 \), so \( D \subseteq \varepsilon(C) \). Let \( S \) be the set of elements in \( C \) with labels that occur more than
once as labels of elements in $C$ and write

$$\sum_{x \in C} a_{a, \varepsilon}(x) = \sum_{x \in S} a_{a, \varepsilon}(x) + \sum_{p \in D} a_{a, p}.$$  

Now, since each $a_{a, \varepsilon}(x) \leq -1$ when $x \in S$, we find that $\sum_{x \in S} a_{a, \varepsilon}(x) \leq -|S|$.

Additionally, if $p \in \Gamma$ is not adjacent to $a$, we have $a_{a, p} = 0$, so

$$\sum_{p \in D} a_{a, p} = \sum_{p \in \Gamma \setminus \{a\}} a_{a, p} = -t,$$

where the second equality results from the fact that $a_{a, b}$ is only positive when $b = a$. Combining this relation with the above gives us that

$$\sum_{x \in C} a_{a, \varepsilon}(x) \leq -t - |S|.$$  

Now we make the following key observation: by (H1), the element $a_1$ cannot cover two distinct elements with the same label, so a given label can occur at most twice as a label of an element in $C$. Therefore, every element of $S$ has a distinct label, so the $s$ of our hypothesis is equal to $|S|$. We also recall Lemma 3.4.2(3) and find that

$$-4 \leq \sum_{x \in C} a_{a, \varepsilon}(x) \leq -t - s.$$  

Multiplying both sides by $-1$ and subtracting $s$ from both sides gives us our result, i.e., $t \leq 4 - s$.

Finally, since $s \geq 0$, we see that $t \leq 4$, i.e., $a$ has a tally of at most 4.

Now we look to the case that $\Gamma$ contains a vertex with a tally of 4. We assume that $\Gamma$ admits a full heap. As a side note, $\Gamma$ cannot contain any circuits in this case because, by Proposition 4.2.5, if it did, it would be $\tilde{A}_n$ for some $n \geq 2$, so each vertex would have a tally of 2.

**Lemma 4.3.2.** Let $d$ be any vertex of $\Gamma$ that has a tally of exactly 4 and $d_1 \in E$ be labeled $d$. Then each vertex adjacent to $d$ in $\Gamma$ occurs exactly once as a label of an element in a covering relation with $d_1$. Additionally, $d_1$ is fully covered and fully covering.
Proof. First, from Lemma 4.3.1, we know that the sum of the tally of $d$ and the number of elements that occur more than once as a label in a covering relation with $d_1$ is no more than 4. Because the tally of $d$ is already 4, there can be no elements occurring more than once as a label in a covering relation with $d_1$, so the first part of the lemma is proved.

To see that $d_1$ is fully covered and fully covering, we recall from Lemma 3.4.2(1) and (2) that $-2 \leq \sum_{x \leftarrow d_1} a_{d,\varepsilon(x)}$ and $-2 \leq \sum_{x \rightarrow d_1} a_{d,\varepsilon(x)}$. Since every label occurs exactly once as a label of an element in a covering relation with $d_1$, the set $\{\varepsilon(x) \mid x \leftarrow d_1 \text{ or } x \rightarrow d_1\}$ is the same as the set of labels adjacent to $d$. Recall that the tally of $d$ is just the sum of the absolute value of entries $a_{d,b}$ where $b \in \Gamma \setminus \{d\}$, with only the vertices adjacent to $d$ contributing nontrivially. Thus, since all relevant entries are nonpositive, we have $\sum_{x \leftarrow d_1} a_{d,\varepsilon(x)} + \sum_{x \rightarrow d_1} a_{d,\varepsilon(x)} = -4$. Putting these relations together, we find $-2 = \sum_{x \leftarrow d_1} a_{d,\varepsilon(x)}$ and $-2 = \sum_{x \rightarrow d_1} a_{d,\varepsilon(x)}$, so $d_1$ is fully covered and fully covering.

Lemma 4.3.3. Let $d$ be any vertex of $\Gamma$ that has a tally of exactly 4, and let $a$ be a vertex of $\Gamma$ adjacent to $d$. Then $a$ has a tally of exactly 1, contributed by $d$. In particular, $a$ is a terminal vertex.

Proof. Let $a$ be a vertex adjacent to $d$. As $a$ is adjacent to $d$, it clearly does not have a tally of 0, so we are only concerned that $a$ has a tally greater than one. There are two ways that can happen: either there is a double edge between $d$ and $a$ with an arrow pointing towards $a$, or $a$ is adjacent to a vertex $b \neq d$. (These are not necessarily exclusive cases.) The former is easily seen to be impossible. Consider any open $a$-interval containing an element $d_1$ labeled $d$; one must exist by (F4) and (F5). By Corollary 3.5.4 and the hypothesis, we know that $d_1$ is the only element in the $a$-interval. Thus $d_1$ is both covered by and covers an element labeled $a$. However, by Lemma 4.3.2, the set of elements in covering relations with $d_1$ share no labels. So this case is rendered impossible.

Now we look at the case that $a$ has other adjacent vertices. First let $(d_1, d_2)$ be an
open $d$-interval. Lemma 3.5.5 and Corollary 3.5.4 now prove that $(d_1, d_2)$ consists only of those elements that cover $d_1$, since $d_1$ is fully covered by Lemma 4.3.2. Similarly, there is an open $d$-interval $(d_0, d_1)$ that consists only of those elements covered by $d_1$.

Without loss of generality, suppose $a_1 \in (d_1, d_2)$ and let $b \neq d$ be a vertex adjacent to $a$. By (F2), there is an element $b_1$ in a covering relation with $a_1$; say $a_1$ is covered by $b_1$. Then $a_1$ is covered by $b_1$ and $d_2$, so Corollary 3.5.7 requires an element $a_2$ such that $(a_1, a_2) = \{b_1, d_2\}$ is a diamond. But then $d_2$ is covered by $a_2$ and covers $a_1$, two elements labeled $a$, which is impossible by Lemma 4.3.2. The case where $a_1 \in (d_0, d_1)$ is symmetric, so we are done.

Proposition 4.3.4. Suppose $\Gamma$ admits a full heap and has a vertex with a tally of exactly 4. Then $\Gamma$ is isomorphic to one of $\tilde{D}_4$, $\tilde{A}_5$ or $\tilde{C}_2$.

Proof. Suppose $\Gamma$ is a (connected) Dynkin diagram that admits a full heap and contains a vertex, say $d$, with a tally of exactly 4. By Lemma 4.3.3, all vertices adjacent to $d$ are terminal, so $\Gamma$ consists only of $d$ and its adjacent vertices. Additionally, Lemma 4.3.3 requires that any edges in $\Gamma$ to be either a single edge or a double edge with arrow pointing towards $d$. (Recall, $\Gamma$ is simply or doubly laced by Lemma 3.4.4.) So we quickly see only three configurations (up to relabeling of vertices) are possible; see Figure 4.2.

The following lemma addresses a specific case when a vertex of $\Gamma$ has three adjacent vertices. Here, the tally of the vertex could be three or four, but either way, the form of $\Gamma$ is very restricted.

Corollary 4.3.5. Suppose $\Gamma$ admits a full heap and that $d \in \Gamma$ has three adjacent, distinct vertices $a, b$ and $c$ such that $d$ is connected to $a$ with a double edge, and $d$ is connected to both $c$ and $b$ with a single edge. Then $a, b$ and $c$ are terminal. Additionally, $\Gamma$ is either $\tilde{B}_3$ or $\tilde{A}_5$, up to relabeling of vertices.
Figure 4.2: The only Dynkin diagrams containing a vertex with a tally of four that admit a full heap.

\[
\begin{align*}
\widetilde{D}_4 &: \\
&\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \\
&\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
4 \\
\end{array}
\end{align*}
\]

\[
\begin{align*}
\widetilde{A}_5^2 &: \\
&\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \\
&\begin{array}{c}
0 \\
1 \\
2 \\
3 \\
\end{array}
\end{align*}
\]

\[
\widetilde{C}_2 \\
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\end{array} \\
\begin{array}{c}
0 \\
1 \\
2 \\
\end{array}
\]

Figure 4.3: The Dynkin diagram of type $\tilde{B}_3$.

\[ \begin{array}{c}
\tilde{B}_3 : \\
0 \quad \circ \\
1 \quad \circ \\
2 \quad \circ \\
3 
\end{array} \]

Proof. Consider first the double edge between $a$ and $d$. If there are two arrows, one pointing to $a$ and one to $d$, $\Gamma$ contains $\tilde{A}_1$ and other vertices. We know this is impossible by Lemma 4.1.5.

If there is a single arrow pointing to $d$, then the tally of $d$ is 4, so by Proposition 4.3.4, we know, by visual inspection, that $\Gamma$ must be $\tilde{2}\tilde{A}_5$ up to relabeling of vertices. (See the second diagram of Figure 4.2.)

Our only other possibility is that there is a single arrow pointing from $d$ to $a$. This configuration corresponds to Figure 4.5(2). It is easily visually verified that this configuration by itself is $\tilde{B}_3$ up to relabeling of vertices (see Figure 4.3), so we only need to show the other vertices to be terminal. It may be helpful to refer to Figure 4.4 for the following.

Because $a_{a,d} = -2$, Corollary 3.5.7 gives us that $E$ contains a diamond $(a_1, a_2) = \{d_1\}$. By (F5), $d_1$ must be in covering relations with elements labeled $b$ and $c$. Since $d_1$ is both covered and covers an element labeled $a$, Lemma 3.4.2(1) and (2) show that the elements labeled $b$ and $c$ cannot both be greater than or both be less than $d_1$. Without loss of generality, say that $b_1 \to d_1 \to c_1$ in $E$. Then $d_1$ is covered by two elements, so Corollary 3.5.7 gives us a diamond $(d_1, d_2) = \{a_2, c_1\}$ in $E$. Since $(d_1, d_2)$ contains no element labeled $b$, we must expand it to a $b$-interval $(b_1, b_2) = [d_1, d_2]$ by Corollary 3.4.7. Additionally, $a_2$ is covered by $d_2$, so there is an $a_3$ covering $d_2$, by Corollary 3.5.7. Thus $d_2$ is covered by two elements, $b_2$ and $a_3$, and, by Lemma 3.4.2, can be covered by no more.

If $c$ is not terminal, then there is an element other than $d_1$ and $d_2$ in a covering relation with $c_1$. Without loss of generality, suppose this element covers $c_1$. Then $c_1$ is covered by
two elements, one being $d_2$, so there is a diamond $(c_1, c_2)$ containing $d_2$. That is, $c_2$ covers $d_2$, a contradiction to the fact $d_2$ can be covered by no other elements. (Having different labels, $c_2$, $b_2$ and $a_3$ are clearly distinct.) So $c_1$ cannot have an element other than $d_2$ covering it. A similar argument shows $c_1$ cannot cover an element other than $d_1$. By (F5), the only vertex adjacent to $c$ is $d$; i.e., $c$ is terminal. Symmetry implies that $b$ is terminal and Lemma 4.1.3 tells us that $a$ is terminal.

4.4 Two vertices with a tally of three

Now we look at the case where $\Gamma$ has at least two vertices with a tally of three. This is a less restrictive requirement than that of having a vertex with a tally of four, but it will still serve us well. We will have to examine several cases.

As usual, we assume $\Gamma$ is connected and admits a full heap. By Lemma 3.4.4, we also know that $\Gamma$ is simply or doubly laced. By Proposition 4.2.5 and the fact that every vertex in $\tilde{A}_n$ has a tally of two, we also know $\Gamma$ contains no circuits.

If a vertex $d$ of $\Gamma$ has a tally of three, then locally $\Gamma$ has one of the structures in Figure 4.5. This list is exhaustive as follows. Because $\Gamma$ is simply or doubly laced, no vertex adjacent to $d$ can contribute more than two to the tally of $d$, so the only partitions of 3 that are relevant are $1 + 1 + 1$ and $2 + 1$. To contribute one, the edge between $d$ and the adjacent
vertex must either be single or double with no arrow pointing towards $d$. To contribute two, the edge must be double with an arrow pointing towards $d$. The structures corresponding to the $1 + 1 + 1$ partition are labeled (1) through (4), and those corresponding to $2 + 1$ are labeled (5) through (8).

We will refer to these different structures by their label in the figure. When we say $\Gamma$ contains (1), e.g., we mean that configuration (1) is a subgraph $\Gamma$ and there are no additional edges between the vertices shown in (1). Note that, since $\Gamma$ contains no circuits, so we are not restricting ourselves meaningfully by not allowing edges between those vertices adjacent to $d$.

**Lemma 4.4.1.** Suppose $\Gamma$ has a vertex $d$ with a tally of three. Then $\Gamma$ either contains (1) or (5) from Figure 4.5 as an induced subgraph, or $\Gamma$ is isomorphic to $\tilde{B}_3$.

**Proof.** First, note that structure (2), by itself, is isomorphic to $\tilde{B}_3$. So, if $\Gamma$ contains (2), then Corollary 4.3.5 shows us that $\Gamma$ is isomorphic to $\tilde{B}_3$.

For $\Gamma$ to admit a full heap, we know from Lemma 4.1.5 that both (7) and (8) are impossible configurations. If $\Gamma$ contains (3) or (4), then $a_{a,d} = a_{b,d} = -2$, so Corollary 3.5.7 gives us that $E$ contains an $a$-diamond $(a_1, a_2) = \{d_1\}$. By (F5), $d$ is in a covering relation with an element labeled $b$, so we again use Corollary 3.5.7 to get that $d_1$ is in a $b$-diamond $(b_1, b_2)$. Then $d_1$ is covered by two elements and covers two elements, so by Lemma 3.4.2(3), it can be in a covering relation with no other element. Yet (F5) requires $d$ to be in a covering relation with an element labeled $c$, so we see that (3) and (4) are impossible configurations.

Similarly, if $\Gamma$ contains (6), then $E$ contains a $c$-diamond $(c_1, c_2) = \{d_1\}$. By (F5), $d_1$ is also in a covering relation with an element $a_1$. So $d$ either covers $c_1$ and $a_1$ or is covered by $c_2$ and $a_1$. Either way, since $a_{d,a} = -2$, we see this is impossible by Lemma 3.4.2. Thus (6) is an impossible configuration.

**Lemma 4.4.2.** Suppose $\Gamma$ has a vertex $d$ with a tally of three and $d_1$ is in $E$. Then either the labels of all elements in a covering relation with $d_1$ are distinct, or $d_1$ is fully covered and
Figure 4.5: All the possible local structures around a vertex $d$ of Dynkin diagram $\Gamma$ with a tally of three, assuming $\Gamma$ admits a full heap. Note that $a, b$ and $c$ are distinct in each configuration, else $d$ would have a smaller tally. Additionally, $d$ has no adjacent vertices other than those shown, else $d$ would have a larger tally. The vertices $a, b$ and $c$ may have additional adjacent vertices, but cannot share an edge with each other, as explained in the text.

(1) : 

\[
\begin{array}{c}
\text{a} & \text{d} & \text{c} \\
\text{b} & \text{d} & \text{c}
\end{array}
\]

(5) : 

\[
\begin{array}{c}
\text{a} & \text{d} & \text{c} \\
\text{b} & \text{d} & \text{c}
\end{array}
\]

(2) : 

\[
\begin{array}{c}
\text{a} & \text{d} & \text{c} \\
\text{b} & \text{d} & \text{c}
\end{array}
\]

(6) : 

\[
\begin{array}{c}
\text{a} & \text{d} & \text{c} \\
\text{b} & \text{d} & \text{c}
\end{array}
\]

(3) : 

\[
\begin{array}{c}
\text{a} & \text{d} & \text{c} \\
\text{b} & \text{d} & \text{c}
\end{array}
\]

(7) : 

\[
\begin{array}{c}
\text{a} & \text{d} & \text{c} \\
\text{b} & \text{d} & \text{c}
\end{array}
\]

(4) : 

\[
\begin{array}{c}
\text{a} & \text{d} & \text{c} \\
\text{b} & \text{d} & \text{c}
\end{array}
\]

(8) : 

\[
\begin{array}{c}
\text{a} & \text{d} & \text{c} \\
\text{b} & \text{d} & \text{c}
\end{array}
\]
Proof. Recall from Lemma 4.4.1 that only the forms (1), (2) and (5) of Figure 4.5 are possible as an induced subgraph in $\Gamma$ (containing $d$ as a vertex).

Assume the labels of elements in a covering relation with $d_1$ are not distinct, so $d_1$ covers and is covered by elements with the same label, by (H1). Lemma 3.4.2 requires that $d_1$ cover at most one other element and be covered by at most one other element. If $d$ has structure (1) or (2), it has three adjacent vertices, so must be in covering relations with at least two elements other than the two with the same label. Thus $d_1$ must cover and be covered by exactly one other element. Either that is impossible or $d_1$ is fully covered and fully covering as needed.

If $d$ has structure (5), suppose $d_1$ covers an element labeled $a$. Then, because there is a double edge between $a$ and $d$ with arrow pointing to $d$, the element $d_1$ is fully covering. Similarly, if $d_1$ is covered by an element labeled $a$, it is covered fully. Since $d_1$ must also be in a covering relation with an element labeled $c$ by (F5), we cannot have $d_1$ covered and covering an element labeled $a$.

If $d_1$ covers an element labeled $a$, it then must be covered by an element labeled $c$. It cannot also be covered by an element labeled $a$ by the above, and it cannot also be covered by an element labeled $c$ by (H1). As $d$ has no other adjacent vertices, we see that all labels of elements in a covering relation with $d_1$ are distinct. A symmetric argument holds if $d_1$ is covered by an element labeled $a$.

We now examine the local structure of a full heap over a Dynkin diagram $\Gamma$ containing (1) with some restrictions. In this calculation, we assume that $\Gamma$ admits a full heap, so, in each step, we assume the situation that allows for us to continue building the structure. In this case, we are restricted enough that we do not have make any choices that are not trivial or symmetric.
Calculation 4.4.3. Suppose $\Gamma$ has a vertex $d$ that has structure (1) and that at most one of the vertices adjacent to $d$ is terminal. We use the labeling of (1) in Figure 4.5, so $d$ is adjacent to $a, b$ and $c$. Let $d_1$ be labeled $d$.

Since there are three vertices adjacent to $d$, clearly $d_1$ must cover or be covered by two elements with labels taken from $\{a, b, c\}$, by (F5). Assume $d_1$ is covered by two elements labeled $a$ and $c$ without loss of generality. (The other cases are symmetric.) Thus, $d_1$ is covered fully, so by Corollary 3.5.7, there is a $d$-diamond $(d_1, d_2) = \{a_1, c_1\}$ in $E$. Since no element labeled $b$ appears in this $d$-diamond, we can expand the diamond to get a $b$-diamond $(b_1, b_2) = [d_1, d_2]$, by Lemma 3.4.6. See Figure 4.6(1) for the situation described.

Because only one of $a, b$ and $c$ is terminal, we know at least one of $a$ and $c$ is not, so suppose $\overline{a}$ is a vertex adjacent to $a$ distinct from $d$. By (F5), $a_1$ is in a covering relation with an element labeled $\overline{a}$; by symmetry, we can assume $\overline{a}_1$ covers $a_1$. Then $a_1 \to d_2 \to b_2$ is a chain with the least element covered outside the chain, we can fill upwards with diamonds, obtaining $a_2$ and $d_3$ in Figure 4.6(2). In that same figure, we see that $c_2$ must exist by (F5), since $d_3$ is fully covering.

Note that, by Lemma 3.4.6, the $d$-diamond $[d_2, d_3]$ expands to a $c$-diamond $(c_1, c_2) = [d_2, d_3]$, so $c_1$ is covered only by $d_2$. Similarly, $b_2$ covers only $d_2$.

Now notice that $b_2$ and $c_1$ are in symmetric situations, so, without loss of generality, assume $c$ is not terminal, since one of them must be. If $\overline{c}$ distinct from $d$ is adjacent to $c$, then $c_1$ must cover (since it cannot be covered by) an element $\overline{c}_1$. So, like above, we have a chain $c_1 \leftarrow d_1 \leftarrow b_1$ with the greatest element covering outside the chain, so we can fill downwards with diamonds, obtaining elements $c_0$ and $d_0$ in Figure 4.6(3). Property (F5) and Lemma 3.4.6 gives the existence of $a_0$.

Finally, we apply Lemma 3.4.6 twice more, once to $[a_0, a_1]$ to obtain $(\overline{a}_0, \overline{a}_1)$ and once to $[c_1, c_2]$ to obtain $(\overline{c}_1, \overline{c}_2)$.

We emphasize that, in this diagram, all apparent $p$-intervals are actually $p$-intervals. By that, we mean, e.g., that $(a_0, a_1) = [d_0, d_1]$, so no elements not depicted appear in this
Figure 4.6: Figures accompanying Calculation 4.4.3. The dashed lines in (4) indicate the only elements and direction where new elements could appear. Recall also that \( c \) is not a terminal vertex, so it has an adjacent label other than \( d \).
interval, no matter the form of $\Gamma$. This is either because Lemma 3.4.6 requires it, as in that case, or more trivially to avoid breaking covering relations, as in all the $d$-intervals. Practically, this tells us that we can only continue building this heap in the directions indicated by the dashed lines of Figure 4.6(4).

The only choices we made were regarding labeling and the direction of the covering relations. The result in (4) is, as a poset, dual to itself, so the direction of the covering relations are irrelevant. Thus, any full heap over a graph with a vertex that has structure (1) such that at most one adjacent vertex is terminal will have the local structure depicted in Figure 4.6(4).

**Lemma 4.4.4.** Suppose $\Gamma$ has two distinct vertices $d$ and $\hat{d}$ that have a tally of three. Then all but one vertex adjacent to $d$ is terminal, and the same is true of $\hat{d}$.

*Proof.* If $\Gamma$ is isomorphic to $\tilde{B}_3$, it is impossible for $\Gamma$ to have two distinct vertices with a tally of three, so, by Lemma 4.4.1, we only need consider when $d$ and $\hat{d}$ have structure (1) or (5) of Figure 4.5, where $d$ and $\hat{d}$ correspond to the vertex labeled $d$ in the figure. We label the vertices adjacent to $d$ and $\hat{d}$ as in the figure, with the additional use of $\hat{\ }$ for vertices adjacent to $\hat{d}$.

By Lemma 4.1.3, we know that one of vertices sharing the double edge in (5) is terminal. Since $d$ (respectively, $\hat{d}$) is clearly not terminal, we must have that $a$ (respectively, $\hat{a}$) is terminal. As $d$ (respectively, $\hat{d}$) has only one other adjacent vertex, the lemma holds in this case.

Suppose for the remainder that $d$ has structure (1). The vertex $\hat{d}$ can have either structure (1) or (5).

Since $\Gamma$ is connected, we know there is a path from $d$ to $\hat{d}$ that contains no vertex twice. This path must clearly contain exactly one of $a, b$ and $c$ and one of $\hat{a}, \hat{b}$ and $\hat{c}$, because $\Gamma$ contains no circuit. If $\hat{d}$ is in configuration (5), then the path cannot contain $\hat{a}$ because $\hat{a}$ must be terminal by Lemma 4.1.3. In configuration (1), the vertices are indistinguishable,
so we can thus assume that the path contains $c$ and $\hat{c}$. That is, there is a path

$$
\begin{array}{cccc}
& d & c & \hat{c} \\
\circ & \circ_{1c} & \circ_{2c} & \circ_{k-1c} & \circ_{k_c} & \hat{d} \\
\end{array}
$$

in $\Gamma$, with relabeling indicated below each node and with $k \geq 1$. Additionally, no $i_c$ is terminal, so all edges depicted are single edges in $\Gamma$, again by Lemma 4.1.3.

Since $a$ and $b$ are indistinguishable, we can assume without loss of generality that $a$ is not terminal with adjacent vertex $\overline{a} \neq d$. Note too that, because there are no circuits in $\Gamma$, the vertex $\overline{a}$ is also distinct from $b$, $\hat{a}$, $\hat{b}$, and $i_c$ for $1 \leq i \leq k$.

Consider $d_1$ in $E$. By Lemma 3.4.2, we know that $d_1$ covers or is covered by two elements with labels taken from $\{a, b, 1^c\}$. We only address the latter case, as the former is symmetric.

By Calculation 4.4.3, we can assume $d_1$ is as in the local structure shown in Figure 4.6(4). (Other cases are symmetric.) Here, label $c$ corresponds to label $1^c$ and $\overline{a}$ corresponds to $2^c$ when $k > 1$ or $\hat{a}$ when $k = 1$. Recall that $1^c$ cannot be covered by any element other than $d_2$.

Consider the chain in $E$

$$
1^c_1 \leftarrow 2^c_1 \leftarrow \cdots \leftarrow i^c_1,
$$

which must exist for some $1 \leq i \leq k$, as $i$ could be 1. However, we can take $i$ to be the first in this chain that is covered outside the chain, i.e., by an element other than $i-1^c_1$, or $i = k$ if no such element exists. We can do this because, if the chain cannot continue, then $j^c_1$ does not cover an element labeled $j+1^c$. By (F5), the element $j^c_1$ must be in a covering relation with an element with label $j+1^c$ for $1 \leq j < k$, so $j^c_1$ then must be covered by an element with that label, meaning $j^c_1$ is covered outside the chain.

Assume $k > i$. If $i^c_1$ is covered outside the chain, we fill upwards with diamonds to see that $1^c_1$ is covered by an element labeled $2^c$, i.e., an element other than $d_2$. This is a contradiction to the structure shown in Figure 4.6(4), so we must have $i = k$, since $i \leq k$. 
If \( i = k \), then the last element of the chain is actually labeled \( \hat{d} \), so we know the label of the last element has a tally of three. Consider the same chain as above. Since \(^1c_1\) also covers \( d_1 \), we can fill downwards with diamonds to get that \(^k c_1\) covers an element labeled \(^{k-1}c\). Because \(^k c_1\) is also covered by an element with that label in the chain and \(^k c_1\) has a tally of three, we apply Lemma 4.4.2 to get that \(^k c_1\) is covered fully. Since there is a single edge between \(^k c_1 = \hat{d}\) and \(^{k-1}c_1 = \hat{c}\), it follows that \(^k c_1\) is covered outside the chain. Thus, as above, this is a contradiction.

The original assumption we have contradicted is that \( a \) is not terminal, so it must be. Symmetry gives us that \( b \) is also terminal. So \( c \) is the only vertex adjacent to \( d \) that is not terminal and the proof is complete.

**Proposition 4.4.5.** Suppose \( \Gamma \) admits a full heap and has two distinct vertices with a tally of three. Then these are the only vertices with a tally of three, and \( \Gamma \) is one of \( \tilde{\mathbb{A}}_{2n-1} \) with \( n \geq 4 \), \( \tilde{\mathbb{C}}_n \) with \( n \geq 3 \), or \( \tilde{\mathbb{D}}_n \) with \( n \geq 5 \), up to relabeling of vertices.

**Proof.** See Figure 4.8 for a list of the Dynkin diagrams in the statement of this proposition.

Let \( d \) and \( \hat{d} \) in \( \Gamma \) be the two distinct vertices with a tally of three. Clearly, \( \tilde{\mathbb{B}}_3 \) does not have two distinct vertices with a tally of three, so by Lemma 4.4.1, \( d \) and \( \hat{d} \) have either the structure (1) or (5) of Figure 4.5, possibly one of each. Let the vertices adjacent to \( d \) and \( \hat{d} \) be labeled as in the figure, with the convention that \( \hat{\cdot} \) is used for the vertices adjacent to \( \hat{d} \).

By Lemma 4.4.4, we know that each of \( d \) and \( \hat{d} \) has only one adjacent vertex that is not terminal. Suppose there were a third vertex \( f \) with a tally of three. Because \( \Gamma \) is connected, there is a non repeating path from \( f \) to \( d \) and from \( f \) to \( \hat{d} \). As \( d \neq \hat{d} \), these paths must diverge at some vertex. That vertex then would have a tally of three, one tally from the path from \( f \), one from the path to \( d \) and one from the path to \( \hat{d} \). (This vertex cannot have a tally greater than three by Lemma 4.3.1 and Proposition 4.3.4.) But then the new vertex has only one adjacent vertex that is not terminal, by Lemma 4.4.4, which is clearly impossible since \( d \), \( \hat{d} \) and \( f \) are not terminal. So there must be exactly two distinct vertices...
with a tally of three.

As the adjacent vertices in (1) are indistinguishable and the vertex $a$ (or $\hat{a}$) in (5) must be terminal by Lemma 4.1.3, we can assume that $c$ and $\hat{c}$ are not terminal.

First we deal with the trivial case that $c = \hat{d}$. By construction, we know $\hat{c} = d$, since $d$ in this case is adjacent to $\hat{d}$ and not terminal. As all other adjacent vertices to $d$ and $\hat{d}$ are terminal, this completely determines the form of $\Gamma$. By going through the possibilities of $d$ and $\hat{d}$ that have structure (1) or (5), it is easy to see $\Gamma$ is one of $\tilde{A}_7$, $\tilde{C}_3$, or $\tilde{D}_5$, up to relabeling of vertices. (Note that $\tilde{A}_7$ covers both cases where $d$ and $\hat{d}$ are not of the same structure.) See Figure 4.7.

Figure 4.7: The smallest Dynkin diagrams with two vertices with a tally of three.

Now assume that $c$ is not $\hat{d}$. Since $\Gamma$ is connected, there is path (possibly of a single vertex) from $c$ to $\hat{c}$ in $\Gamma$. As $\Gamma$ contains no circuits, this path does not contain any of the vertices $d$, $\hat{d}$, $a$, $\hat{a}$, $b$ and $\hat{b}$. Assume also that the path has no repeated labels, which is possible because we can always delete the section of a path between repeated labels and still have a path.

By virtue of being on a path and not at the end, no elements between $c$ and $\hat{c}$ on this path are terminal. Furthermore, $c$ and $\hat{c}$ are not terminal by assumption, so all edges of the path are single edges, by Lemma 4.1.3. That is, the label of every element of the path has a
tally of two or greater. By Lemma 4.3.1, no element of the path can have a tally of greater than four. Since none of $\tilde{D}_4$, $\tilde{A}_5$, and $\tilde{C}_2$ have any vertices with a tally of three, Proposition 4.3.4 shows that no element of the path can have a tally of four.

If a label $t$ between $c$ and $\hat{c}$ on the path has an adjacent vertex that is not on the path, it has a tally greater than two, since the path itself contributes two to the tally, so $t$ would then have a tally of three. However, by Lemma 4.4.4, $t$ then must have only one adjacent vertex that is not terminal, since other vertices with a tally of three exist. No labels are repeated on the path, so this is clearly impossible, since we would have to arrive at and leave the vertex through its one nonterminal adjacent vertex on our path. Thus, every element between $c$ and $\hat{c}$ has a tally of two, i.e., the only vertices adjacent to elements between $c$ and $\hat{c}$ are on the path.

Finally, if $c$ has a tally of three and $c \neq \hat{c}$, then, because $d$ is adjacent to $c$ and not terminal, we must have that the next vertex on the path is terminal, which is impossible by the above. On the other hand, if $c = \hat{c}$, then $\hat{d}$ is terminal, which is also impossible. So $c$ and similarly $\hat{c}$ have a tally of two. That is, $c$ has $d$ and the next vertex on the path (or $\hat{d}$) as adjacent vertices.

We have now limited the form of $\Gamma$ as far as possible. We have shown that $\Gamma$ consists only of $d$ and $\hat{d}$, their respective adjacent vertices, and the simply laced path connecting $c$ and $\hat{c}$. Assuming $c$ is not $\hat{d}$, we get the following results. If both $d$ and $\hat{d}$ have structure (1), we see $\Gamma$ is isomorphic to $\tilde{D}_n$ for $n \geq 6$. If both $d$ and $\hat{d}$ have structure (5), we see $\Gamma$ is isomorphic to $\tilde{C}_n$ for $n \geq 6$. If $d$ and $\hat{d}$ have different structures, we see $\Gamma$ is isomorphic to $\tilde{A}_{2n-1}$ for $n \geq 5$. Combining these results with the case when $c = \hat{d}$, we obtain our result; see Figure 4.8.

\begin{flushright}
\square
\end{flushright}

\textbf{Lemma 4.4.6.} Suppose there is a vertex $d$ in $\Gamma$ with a tally of three for which there is a $d$-interval $(d_1, d_2)$ in $E$ that is not a diamond. Then $\Gamma$ is one of $\tilde{A}_{2n-1}$ with $n \geq 4$, $\tilde{C}_n$ with $n \geq 3$, or $\tilde{D}_n$ with $n \geq 5$, up to relabeling of vertices.
Figure 4.8: All possible Dynkin diagrams with two vertices with a tally of three.

\[ 2\tilde{\mathcal{A}}_{2n-1} \ (n \geq 4) : \]
\[
\begin{array}{c}
0 \\
1
\end{array}
\begin{array}{c}
\circ \\
\circ
\end{array}
\begin{array}{c}
2 \\
3
\end{array}
\begin{array}{c}
\circ \\
\circ
\end{array}
\begin{array}{c}
n-2 \\
n-1
\end{array}
\begin{array}{c}
n \end{array}
\]

\[ \tilde{\mathcal{C}}_n \ (n \geq 3) : \]
\[
\begin{array}{c}
0 \\
1 \\
2
\end{array}
\begin{array}{c}
\circ \\
\circ
\end{array}
\begin{array}{c}
\circ \\
\circ
\end{array}
\begin{array}{c}
n-2 \\
n-1
\end{array}
\begin{array}{c}
n \\
\circ
\end{array}
\]

\[ \tilde{\mathcal{D}}_n \ (n \geq 5) : \]
\[
\begin{array}{c}
0 \\
1
\end{array}
\begin{array}{c}
\circ \\
\circ
\end{array}
\begin{array}{c}
2 \\
3
\end{array}
\begin{array}{c}
\circ \\
\circ
\end{array}
\begin{array}{c}
n-3 \\
n-2
\end{array}
\begin{array}{c}
n \end{array}
\]

\[ n-1 \]
Proof. By Proposition 4.4.5, we only need to show that there is a vertex in \( \Gamma \) other than \( d \) that has a tally of three.

Note first that, because \( \Gamma \) contains a vertex with a tally of three, it cannot contain any circuits by Proposition 4.2.5, because all vertices in \( \tilde{A}_n \) for \( n \geq 2 \) have a tally of two.

Any vertex with label adjacent to \( d \) that is terminal must be covered by and cover an element labeled \( d \) by (H1) and (F4), so therefore must be in a \( d \)-diamond by Lemma 3.5.3. Therefore, by assumption, at least one vertex adjacent to \( d \) is not terminal. In particular, \( \Gamma \) is not isomorphic to \( \tilde{B}_3 \), so \( d \) has structure (1) or (5) of Figure 4.5, by Lemma 4.4.1. As before, without loss of generality, assume that \( c \) in the figure is not terminal.

Because \( d \) has a tally of three, an element labeled \( d \) is either fully covered or fully covering, by Lemma 3.4.2, so is either the minimum or maximum, respectively, of a \( d \)-diamond by Corollary 3.5.7. Without loss of generality, assume that \( (d_0, d_1) \) and \( (d_2, d_3) \) are \( d \)-diamonds. The labels adjacent to \( d \) that are not \( c \) are terminal by Lemma 4.4.4, so they must be covered by and cover elements labeled \( d \) (as there is no other choice). By Lemma 3.5.3, we see these elements can only be in diamonds. Thus, the only elements in the \( d \)-interval \( (d_1, d_2) \) with labels adjacent to \( d \) must have label \( c \), since the \( d \)-interval is not a diamond by hypothesis. Because \( c \) shares a single edge with \( d \), there must be two such elements in the \( d \)-interval by (F3). Assume \( c_1 \) covers \( d_1 \) and \( c_2 \) is covered by \( d_2 \).

By Lemma 3.5.10, we know there must be a diamond in \( (d_1, d_2) \). If \( (c_1, c_2) \) is a \( c \)-diamond, then the labels of the elements covering \( c_1 \) must contribute two to the tally of \( c \). Since there is no element labeled \( d \) in \( (d_1, d_2) \) and hence the \( c \)-diamond, we see that \( d \) contributes one more to the tally of \( c \), so \( c \) has a tally of three or greater. By Lemma 4.3.3 and Lemma 4.3.1, we see that \( c \neq d \) has a tally of exactly three and we are done.

If \( (c_1, c_2) \) is not a \( c \)-diamond, we notice that \( (c_1, c_2) \) is in a similar situation to \( (d_1, d_2) \). That is, by Lemma 3.5.5 and the fact \( \Gamma \) contains no circuits, the \( c \)-interval contains two elements of the same label that is adjacent to \( c \). This element cannot be labeled \( d \) since it is in a \( d \)-interval. Additionally, we again know it must contain a diamond by Lemma 3.5.10.
Thus, we can proceed by induction. As \( E \) is locally finite, we must eventually find a diamond and thus a vertex with a tally of three, so the proof is complete.

**Corollary 4.4.7.** Suppose \( \Gamma \) contains a vertex \( d \) that has structure (5) of Figure 4.5. Then \( \Gamma \) is one of \( 2 \tilde{A}_{2n-1} \) with \( n \geq 4 \) or \( \tilde{C}_n \) with \( n \geq 3 \), up to relabeling of vertices.

**Proof.** Again, we need to show that \( \Gamma \) has another vertex with a tally of three and apply Proposition 4.4.5, noting that \( \tilde{D}_n \) does not contain (5).

Let the vertices adjacent to \( d \) be labeled as in (5). By Corollary 3.5.7(2), we know there is a \( d \)-diamond \((d_1, d_2) = \{a_1\} \) in \( E \). By (F5), we then know \( d_1 \) covers an element \( c_1 \). Because there is no arrow pointing towards \( a \) on the edge between \( a \) and \( d \), and because \( a \) is terminal, we know \( d_1 \) cannot cover an element labeled \( a \), say \( a_0 \), because \((a_0, a_1)\) would be underfull. Thus, since \( d \) has no other adjacent vertices, \( c_1 \) is the only element \( d_1 \) covers. Since there is a single edge between \( c \) and \( d \), we have that \( d_1 \) does not cover fully, so the \( d \)-interval \((d_0, d_1)\) is not a diamond. As \( d \) has a tally of three, we done by Corollary 4.4.6.

### 4.5 At most one vertex with a tally of three

In this section, we look at the cases where at most one vertex of \( \Gamma \) has a tally of three and the remaining have a tally of one or two. This will complete the preliminaries we need for our final proof in Section 4.7. As usual, we assume \( \Gamma \) is connected and admits a full heap. By Lemma 3.4.4, we also know that \( \Gamma \) is simply or doubly laced. We also assume \( \Gamma \) contains no circuits.

We need the following technical lemma several times.

**Lemma 4.5.1.** Suppose \( \Gamma \) contains a vertex \( b \) with a tally of two, having two distinct adjacent vertices \( a \) and \( c \). Suppose a full heap \( E \) over \( \Gamma \) exists and contains a saturated chain

\[ b_1 \to a_1 \to b_2 \to a_2 \to \cdots \to b_{n-1} \to a_{n-1} \to b_n. \]
Then $E$ also contains a saturated chain

$$b_1 \to c_1 \to b_2 \to c_2 \to \cdots \to b_{n-1} \to c_{n-1} \to b_n.$$  

Proof. This proof is essentially a repeated application of Lemma 3.4.2(3). For example, this lemma applied to $b_1 \to a_1 \to b_2$ gives that $(b_1, b_2)$ is a $b$-diamond. Since $b$ has a tally of two, the vertex $a$ can only contribute one to the tally, as $c$ must also contribute a positive integer to the tally. Therefore, property (F3) requires that there be an element other than $a_1$ in the diamond. As this element cannot be labeled $a$ by (H1) and $c$ is the only other adjacent vertex to $b$, it must be labeled $c$. An element $c_1$ in the diamond immediately tells us $b_1 \to c_1 \to b_2$ is a saturated chain. Applying the same argument to every $b$-interval in the first chain gives the second chain. 

Now we look at the case where $\Gamma$ is doubly laced, beginning with another technical lemma. To aid in understanding, the reader should refer to Figure C.8 as an example of the phenomenon. The elements $^i a$ of the lemma correspond to the elements labeled $i - 1$ in the figure.

**Lemma 4.5.2.** Suppose $\Gamma$ contains a path $\{^1 a, ^2 a, \ldots, ^k a\}$, with no vertex repeated and $k > 1$, such that the tally of each $^i a$ for $1 \leq i < k$ is two and $^1 a$ is terminal. Suppose also that a full heap exists over $\Gamma$. Then

1. for each $^i a$ with $1 \leq i \leq k$, $E$ contains an infinite chain

$$\{\ldots, ^i a_{-1}, ^i a_0, ^i a_1, ^i a_2, \ldots\}$$

in which every $[^i a_j, ^i a_{j+1}]$ is an $^i a$-diamond;

2. if $1 < i \leq k$, each $^i a$-diamond contains an element labeled $^{i-1} a$;

3. if $1 \leq i < k$, then each $^i a$-diamond contains an element labeled $^{i+1} a$. 
Proof. We prove all three parts at the same time.

Since $1a$ is terminal, it can only be in covering relations with $2a$, so any element labeled $1a$ covers and is covered by elements labeled $2a$ by (F4). On the other hand, the fact that $1a$ has a tally of two means there is a double edge between $1a$ and $2a$ with arrow pointing to the former. By Corollary 3.5.7(2), any element labeled $2a$ in a covering relation with an element labeled $1a$ is the only element in an $1a$-diamond. That is, $E$ contains an infinite saturated chain

$$
\cdots \rightarrow 1_{a-1} \rightarrow 2_{a-1} \rightarrow 1_0 \rightarrow 2_0 \rightarrow 1_a \rightarrow 2_a \rightarrow 1_{a-1} \rightarrow 2_{a-1} \rightarrow \cdots
$$

in which each $(1a, 1a) = \{2a\}$ is a diamond. So (1) and (3) hold for $i = 1$.

Now suppose $1 < i < k$ and that $E$ contains an infinite saturated chain

$$
\cdots \rightarrow i^{-1}a_{a-1} \rightarrow ia_{a-1} \rightarrow i^{-1}a_0 \rightarrow ia_0 \rightarrow i^{-1}a_1 \rightarrow ia_1 \rightarrow i^{-1}a_2 \rightarrow ia_2 \rightarrow \cdots
$$

Then, by Lemma 4.5.1, $E$ must contain an infinite saturated chain

$$
\cdots \rightarrow i+1a_{a-1} \rightarrow ia_{a-1} \rightarrow i+1a_0 \rightarrow ia_0 \rightarrow i+1a_1 \rightarrow ia_1 \rightarrow i+1a_2 \rightarrow ia_2 \rightarrow \cdots
$$

Thus, each $(i^a, i_{a+1})$ is a diamond containing elements labeled $i^{-1}a$ and $i+1a$. By induction, (1), (2) and (3) are shown for $1 < i < k$.

Finally, for $i = k$, we already know that the chain

$$
\cdots \rightarrow ka_{a-1} \rightarrow k^{-1}a_{a-1} \rightarrow ka_0 \rightarrow k^{-1}a_0 \rightarrow ka_1 \rightarrow k^{-1}a_1 \rightarrow ka_2 \rightarrow k^{-1}a_2 \rightarrow \cdots
$$

exists in $E$ by the case $i = k - 1$. Repeated application of Lemma 3.5.3 tells us that each $(ka, ka_{a+1})$ is a diamond containing an element labeled $k^{-1}a$, finishing our proof for (1) and (2). (We implicitly use the assumption that $\Gamma$ admits a full heap, which then forces $ka$ to have a tally of at least two; we just do not care which vertices contribute to that tally.)

We now present our main proposition for the case that $\Gamma$ is doubly laced and has at most one vertex with a tally of three.
Proposition 4.5.3. Suppose $\Gamma$ contains a path $\{1a, 2a, \ldots, k a\}$, with no vertex repeated and $k > 1$, such that the tally of each $i a$ for $1 \leq i < k$ is two and $1a$ is terminal. Then $\Gamma$ is isomorphic to one of $\tilde{A}_1$, $2\tilde{D}_{n+1}$ for $n \geq 2$ or $\tilde{B}_n$ for $n \geq 3$.

Proof. First note that our hypothesis implies that no $i a$ for $1 < i < k$ can be adjacent to any vertex outside the path described, because each has two adjacent vertices in the path and a tally of two. Additionally, vertex $1a$ is terminal, so it can have no adjacent vertices outside the path and there must be an arrow pointing from $2a$ to $1a$ in order to have a tally of two. Thus we only need to examine $ka$.

Suppose $k = 2$ and $2a$ is terminal. Then Corollary 4.1.4 requires there to be an arrow pointing to $2a$, so Lemma 4.1.5 tells us $\Gamma$ is isomorphic to $\tilde{A}_1$.

For $k > 2$, we know by Lemma 4.5.2 that $E$ contains an infinite chain

$$\{\ldots, k a_{-1}, k a_0, k a_1, k a_2, \ldots\}$$

for which every $(k a_j, k a_{j+1})$ is an $ka$-diamond, each containing one element labeled $k^{-1}a$. Note that this accounts for all elements in $E$ with the label $ka$, by (H2) and the definition of diamond.

We have two cases to consider. The first is that at least one $ka$-diamond contains only an element labeled $k^{-1}a$. In order for the diamond to be full, there must be a double edge with arrow pointing from $k^{-1}a$ to $ka$, as any other possibility in a simply or doubly laced $\Gamma$ would result in an underfull interval. But then every $ka$-diamond in $E$ is full, so $ka$ can have no other adjacent vertex. That is, $\Gamma$ is isomorphic to $2\tilde{D}_{n+1}$ where $n = k - 1$.

Now suppose that at least one $ka$-diamond contains an element other than the one labeled $k^{-1}a$. By Lemma 3.4.2, there can only be one other element in the diamond; say it has label $b$. By (H2), $b$ is adjacent to $ka$. Without loss of generality, we now assume $ka$ does not have tally of two; otherwise we could extend the path to include it, since it is adjacent to another element. By Proposition 4.3.4, we know $ka$ cannot have a tally of four, since none
of Dynkin diagrams in Figure 4.2 has a terminal vertex with arrow pointing towards it, as \( ^1a \) requires. Thus \( ^ka \) has a tally of three.

If \( k = 2 \), then \( ^1a \) contributes one to the tally of \( ^ka \), by Lemma 4.1.5. If \( k > 2 \), we know that \( ^{k-1}a \) contributes one to the tally of \( ^ka \) by Lemma 4.1.3, because neither vertex is terminal.

If \( b \) contributes the remaining two to the tally of \( ^ka \), then our \( ^ka \)-diamond is overfull, having one element with a label that contributes one and another label with an element that contributes two. So there must be another vertex, say \( c \), adjacent to \( ^ka \), with both \( b \) and \( c \) contributing one to its tally. By Lemma 4.4.1, we see that \( b \) and \( c \) must have a single edge with \( ^ka \).

Let \( ( ^ka_1, ^ka_2 ) \) be the diamond containing elements \( ^{k-1}a_1 \) and \( b_1 \). Then the \( ^ka \)-diamonds \( ( ^ka_0, ^ka_1 ) \) and \( ( ^ka_2, ^ka_3 ) \) must contain elements labeled \( c \) by (F5). We also already know these diamonds contain elements labeled \( ^{k-1}a \), so they are full.

If \( b \) is not terminal, then (F5) requires that \( b_1 \) be in a covering relation with an element other than \( ^ka_1 \). In other words, \( b_1 \) is fully covering or is covered fully. Then, by Corollary 3.5.7, we would find an element labeled \( b \) in one of the \( ^ka \)-diamonds \( ( ^ka_0, ^ka_1 ) \) and \( ( ^ka_2, ^ka_3 ) \). This is impossible, since the diamonds are already full. Thus \( b \) is terminal. Similarly, \( c \) must be terminal. So there can be no other vertices in \( \Gamma \) other than the path of the hypothesis, \( b \) and \( c \). That is, \( \Gamma \) is isomorphic to \( \tilde{B}_n \) where \( n = k + 1 \).

Now we examine the situation when \( \Gamma \) is simply laced.

**Definition 4.5.4.** Suppose \( \Gamma \) contains a path \( \{ ^1a, ^2a, \ldots, ^ka \} \) where \( ^1a \) has a tally of one and each \( ^ia \) for \( 1 < i < k \) has a tally of two. If \( ^ia \neq ^{i+1}a \) for all \( 1 \leq i < k \), we call that path (and induced subgraph) a **wing** of \( \Gamma \). We call \( ^ka \) the **last element of the wing**.

**Lemma 4.5.5.** Suppose \( \Gamma \) is finite, connected, and simply laced and contains no circuits, and let \( \Gamma' \) be a connected subgraph of \( \Gamma \) such that \( \Gamma \setminus \Gamma' \) is also connected. If all the vertices of \( \Gamma' \) have a tally of one or two in \( \Gamma \), then \( \Gamma' \) is a wing of \( \Gamma \).
Proof. Let \( P \) be the longest path without repeated vertices in \( \Gamma' \). Any vertex of \( \Gamma \) with at tally of one or two must have at most two adjacent vertices in the subgraph, since \( \Gamma \) is simply laced. Thus, if a vertex \( t \) is in \( P \), any vertices adjacent to \( t \) in \( \Gamma' \) must also be in \( P \). Otherwise, the path would terminate at \( t \) because no vertices are repeated in \( P \), and we could then extend \( P \) to include the adjacent vertex not in \( P \), violating the assumption \( P \) is the longest such path. (Because the adjacent vertex was not on the path before, it is clear this new path also has no repeated vertices.) Since \( \Gamma' \) is connected and all vertices adjacent to vertices in \( P \) are also in \( P \), we see that \( \Gamma' = P \) as vertex sets.

Because \( \Gamma \) contains no circuits, \( \Gamma' \) cannot contain a circuit, so no nonconsecutive pair of vertices in \( P \) share an edge in \( \Gamma' \). Thus, the first and last vertices of \( P \) are terminal. Clearly, no other vertex is terminal, since \( P \) is a path. If neither terminal vertex of \( \Gamma' \) were terminal in \( \Gamma \), then \( \Gamma \setminus \Gamma' \) would not be connected, so we see that at least one terminal vertex in \( \Gamma' \) has a tally of one in \( \Gamma \). The nonterminal vertices all have a tally of two and the other terminal vertex may have a tally of one or two. We now see that \( P \) (and hence \( \Gamma' \)) is a wing.

A wing in \( \Gamma \) results in a particular structure in \( E \) which we examine here.

**Lemma 4.5.6.** Suppose \( \Gamma \) admits a full heap \( E \) and contains a wing \( \{i^a\}_{i=1}^k \) with \( k > 1 \). Then every element labeled \( i^a \) for \( 1 < i \leq k \) occurs in a saturated chain of the form

\[
i^a_1 \rightarrow i^{-1}a_1 \rightarrow i^a_2 \rightarrow i^{-1}a_2 \rightarrow \cdots \rightarrow i^{-1}a_{i-1} \rightarrow i^{-1}a_{i-1} \rightarrow i^a_i,
\]

up to reindexing, and this chain is the longest it can be. That is, \( i^a_1 \) does not cover an element labeled \( i^{-1}a \) and \( i^a_i \) is not covered by an element labeled \( i^{-1}a \). Additionally, there are no \( 1^a \)-diamonds.

**Proof.** The lemma is trivial for \( i = 1 \), so we look first at the case \( i = 2 \). Since \( 1^a \) is terminal, it is covered by and covers elements labeled \( 2^a \), giving us the chain \( 2a_1 \rightarrow 1^a_1 \rightarrow 2a_2 \) we need. Now, if \( 2a_1 \) covers an element labeled \( 1^a \), we would have an \( 1^a \)-interval consisting only of the chain \( 1^a_0 \rightarrow 2a_1 \rightarrow 1^a_1 \). Because \( 1^a \) has a tally of one, this interval is clearly
underfull and there no other elements to put in the interval to fill it. So this is impossible. Similarly, \(2a_2\) cannot be covered by an element labeled \(1a\). Note that this also shows there can be no \(1a\)-diamonds, since every element labeled \(1a\) is locally in the structure described.

We proceed by induction, assuming the appropriate chain exists for \(i<k\) and showing this implies the case for \(i+1\). Suppose we have a chain

\[
i a_1 \rightarrow i^{-1} a_1 \rightarrow i a_2 \rightarrow i^{-1} a_2 \rightarrow \cdots \rightarrow i a_{i-1} \rightarrow i^{-1} a_{i-1} \rightarrow i a_i
\]

in \(E\) that is longest it can be. Then, by Lemma 4.5.1, we know there is a chain

\[
i a_1 \rightarrow i+1 a_1 \rightarrow i a_2 \rightarrow i+1 a_2 \rightarrow \cdots \rightarrow i a_{i-1} \rightarrow i+1 a_{i-1} \rightarrow i a_i
\]

in \(E\). (Note that, when \(i=k\), Lemma 4.5.1 no longer applies, as we do not know the tally of \(k\), so induction fails.) Furthermore, since \(i a_1\) does not cover an element labeled \(i^{-1} a\) and its only other adjacent vertex is \(i+1 a\), we must have that \(i a_1\) covers an element labeled \(i+1 a\), by (H2) and (F4). Similarly, \(i a_i\) is covered by an element labeled \(i+1 a\). So we get that the chain

\[
i+1 a_0 \rightarrow i a_1 \rightarrow i+1 a_1 \rightarrow i a_2 \rightarrow i+1 a_2 \rightarrow \cdots \rightarrow i a_{i-1} \rightarrow i+1 a_{i-1} \rightarrow i a_i \rightarrow i+1 a_i
\]

in \(E\). Reindexing this chain gives us the chain we want.

Finally, we know that \(i+1 a_0\) cannot cover an element labeled \(i a\), otherwise the resulting \(i a\)-interval would be underfull. Since \(\Gamma\) is simply laced and \(i a\) has only two adjacent vertices \(i^{-1} a\) and \(i+1 a\), we would get that \(i a_i\) covers an element labeled the former, which is impossible by assumption. Similarly, \(i+1 a_i\) cannot be covered by an element labeled \(i a\), so this chain is the longest it can be.

Note that the chains described in Lemma 4.5.6 share elements, so we can view them all together as a single structure. This structure is easy to imagine: a single element labeled \(1a\) is in covering relations with two elements labeled \(2a\), which in turn are in covering relations with three elements labeled \(3a\) and so on. An example when \(k=4\) is given in Figure 4.9.
Definition 4.5.7. Suppose $\Gamma$ contains a wing and $E$ is a full heap over $\Gamma$. A single instance of the structure resulting from Lemma 4.5.6 is called a wing of $E$.

There should be no confusion in this terminology, as it will be clear from the context if we are discussing the wing of a Dynkin diagram or of a full heap.

Lemma 4.5.8. Suppose $\Gamma$ is simply laced and all vertices have a tally of one or two. Then $\Gamma$ does not admit a full heap.

Proof. It is easy to see, since $\Gamma$ is simply laced and connected, that $\Gamma$ is now just a single path $\{1a, 2a, \ldots, ka\}$, where $1a$ and $ka$ have a tally of one and all $ia$ for $1 < i < k$ have a tally of two. (Note that $k > 1$, otherwise $1a$ would have a tally of zero.) So now $\Gamma$ can be viewed as a wing with last element $ka$ or a wing with last element $1a$.

In the former case, Lemma 4.5.6 requires that, in any full heap over $\Gamma$, any element labeled $1a$ is not part of a diamond, since $k > 1$. In the latter case, every element labeled $1a$ is in a chain of $1a_1 < 1a_2 < \cdots < 1a_k$, up to indexing, where each $(1a_i, 1a_{i+1})$ is a diamond. Since $k > 1$, there is at least one diamond in this chain. Thus we have a contradiction. So $\Gamma$ cannot admit a full heap.

We now present the following technical lemma that, while seemingly unrelated, is the backbone of our final result of the section.

Lemma 4.5.9. Let $k, j, l$ be positive integers. Consider the sets $K = \{nk \mid n \in \mathbb{Z}\}$, $J = \{nj + 1 \mid n \in \mathbb{Z}\}$ and $L = \{nl + 2 \mid n \in \mathbb{Z}\}$. If every integer occurs in exactly one of these three sets, i.e. the sets are pairwise disjoint and have union $\mathbb{Z}$, then either $k = j = l = 3$ or $k = l = 4$ and $j = 2$.

Proof. Note first that none of $k, j$ and $l$ can be $1$; otherwise $K$, $J$ or $L$, respectively, would be all of $\mathbb{Z}$, which clearly impossible. Letting $n = 0$, we see that no matter the values of $k, j$
Figure 4.9: This is an example of a wing in a full heap resulting from a wing (path) of four vertices in which $a_1$ has a tally of 1 and each $a_i$ for $1 < i < 4$ has a tally of 2. The dashed lines indicate the only places where other covering relations can occur.
and $l$, we have $0 \in K$, $1 \in J$ and $2 \in L$. Therefore we cannot have $nk = 2$ for any $n$, so $k \neq 2$. Similarly, $nl + 2 \neq 0$, so $l \neq 2$. (Other similar observations are not useful.)

Suppose $k = 3$, so $K$ consists of all multiples of 3. If $j$ is any integer other than 3, it is coprime to 3, so by Euclid’s algorithm, we can find an $n$ such that $nj + 1$ is a multiple of 3. Thus $K \cap J \neq \emptyset$, a contradiction. So we must have $j = 3$. Similarly, $l = 3$. So, in this case, we have $k = j = l = 3$.

Suppose $k = 4$. Then we must have $3 \in J$ or $3 \in L$. In the latter case, we have $nl + 2 = 3$, so $l = 1$, an impossibility. So $3 \in J$, and we have $nj + 1 = 3$, so $j = 2$. So $K$ is the set of all integers divisible by 4 and $J$ is all odd integers. Thus $L$ must be all integers that are 2 modulo 4, i.e., $l = 4$. In this case, we have $k = l = 4$ and $j = 2$.

Finally, we need to make sure no other valid solutions exist. Suppose that $k > 4$. Then we need 3 and 4 in either $J$ or $L$. As above, we know we cannot have 3 in $L$, so 3 is in $J$ and, again, $j = 2$. Clearly 4 is not of the form $n(2) + 1$, so we must have an $n$ so that $nl + 2 = 4$. But then $l = 2$, so, when $n = -1$, we have $0 \in L$. This is impossible, since $0 \in K$, so $k$ cannot be greater than 4. We have examined all the possible values of $k$, so we are done.

**Lemma 4.5.10.** Suppose $\Gamma$ admits a full heap $E$, is simply laced, is connected and contains exactly one vertex $d$ with a tally of three. Then

(1) All vertices of $\Gamma$ other than $d$ have a tally of one or two,

(2) There is an infinite chain $\{\ldots, d_{-1}, d_0, d_1, d_2, \ldots\}$ in $E$ for which every $(d_i, d_{i+1})$ is a $d$-diamond, and

(3) $\Gamma$ consists of three distinct wings sharing the vertex $d$.

**Proof.** For (1), note first that, because $\Gamma$ contains a vertex with a tally of three, it cannot contain any circuits by Proposition 4.2.5, because all vertices in $\tilde{A}_n$ for $n \geq 2$ have a tally of two. Additionally, we know from Proposition 4.3.4 that, if $\Gamma$ is simply laced, it can only contain a vertex with tally of four if all other vertices have a tally of one, so $\Gamma$ cannot contain
a vertex with a tally of four. By Lemma 4.3.1, we see that all vertices of \( \Gamma \) that are not \( d \) have a tally of one or two.

For (2), we know from Lemma 4.4.6 that any \( d \)-interval in \( E \) that is not a \( d \)-diamond would imply that \( \Gamma \) has two two vertices with at tally of three. Because \( d \) is the only vertex of \( \Gamma \) with a tally of three, all \( d \)-intervals must be \( d \)-diamonds. That is, \( E \) contains an infinite chain \( \{\ldots, d_{-1}, d_0, d_1, d_2, \ldots\} \) for which every \((d_i, d_{i+1})\) is a \( d \)-diamond, by (F3).

For (3), as \( \Gamma \) is simply laced, Lemma 4.4.1 tells us that \( d \) has three adjacent vertices, say \( a, b \) and \( c \). Since all vertices other than \( d \) have a tally of one or two and \( \Gamma \) has no circuits, each of \( a, b \) and \( c \) is in a wing of \( \Gamma \) by Corollary 4.5.5. Each of \( a, b \) and \( c \) is in a different wing, since \( d \) has a tally of three, so \( d \) can only be the last element of a wing. In fact, we extend each wing towards \( d \) to include \( d \) as the last element of the wing. There are no elements outside the wings, since \( \Gamma \) is connected.

\[\square\]

**Lemma 4.5.11.** Suppose \( \Gamma \) admits a full heap \( E \), is simply laced, is connected and contains exactly one vertex \( d \) with a tally of three. Let \( a \) be a vertex adjacent to \( d \) in \( \Gamma \). Then every \( d \)-diamond of \( E \) contains an element labeled \( a \) except those of the form \((d_{nk}, d_{nk+1})\), up to reindexing, for \( n \in \mathbb{Z} \), where \( k \) is the number of vertices (including \( d \)) of the largest wing of \( \Gamma \) containing \( a \).

**Proof.** Recall from Lemma 4.5.10(3) that \( a \) is in a wing of \( \Gamma \) with last element equal to \( d \). Suppose this wing contains \( k \) vertices. Because \( d \) has a tally of three, this is clearly the largest wing of \( \Gamma \) containing \( a \). By Lemma 4.5.6, we must have a saturated chain in \( E \) that consists of alternating labels \( d \) and \( a \) that has \( k-1 \) elements labeled \( a \) and \( k \) elements labeled \( d \). We know from Lemma 4.5.10(2) that these \( d \)-intervals are \( d \)-diamonds, so there are \( k-1 \) \( d \)-diamonds in this saturated chain containing elements labeled \( a \). Without loss of generality, assume \( d_1 \) is the minimal element of the saturated chain and \( d_k \) the maximal element.

Now, also by Lemma 4.5.6, there can be no element labeled \( a \) covering \( d_k \). Again, since all \( d \)-intervals are \( d \)-diamonds, we see that \((d_k, d_{k+1})\) cannot contain an element labeled \( a \).
Therefore, by (F5), the element $d_{k+1}$ must be covered by an element labeled $a$. Then this element labeled $a$ must be part of a wing of $E$ resulting from the wing in $\Gamma$ containing $a$. Clearly, it is the least element labeled $a$ in the wing, since $(d_k, d_{k+1})$ does not contain an element labeled $a$. Thus, starting with $d_{k+1}$ as the least element labeled $d$ in the wing, we get $k - 1$ more diamonds containing an element labeled $a$, from $d_{k+1}$ to $d_k$.

We can continue in this way to conclude that every $d$-diamond of $E$ contains an element labeled $a$ except for those of the form $(d_{nk}, d_{nk+1})$ for $n \in \mathbb{Z}$.

\[\square\]

**Remark 4.5.12.** We wish to clarify here what we mean by reindexing in the statement of Lemma 4.5.11. Simply put, we are indexing the elements labeled $d$ such that the $d$-diamond $(d_0, d_1)$ does not contain an element labeled $a$. The assumption is only for ease and is not necessary; the key point is that the $d$-diamonds that do not contain an element labeled $a$ are separated by exactly $k - 1$ $d$-diamonds, so as soon as we know one such $d$-diamond, we know them all. Another way of writing this is to say that there is some constant $s \in \mathbb{Z}$ such that the only $d$-diamonds in $E$ not containing an element labeled $a$ are those of the form $(d_{nk+s}, d_{nk+1+s})$.

We now present our second main result of the section and the last result needed for our first main theorem.

**Proposition 4.5.13.** Suppose $\Gamma$ admits a full heap, is simply laced, is connected and contains exactly one vertex $d$ with a tally of three. Then $\Gamma$ is either $\tilde{E}_6$ or $\tilde{E}_7$, up to relabeling of vertices.

**Proof.** Let $a$, $b$ and $c$ be the three vertices adjacent to $d$. By Lemma 4.5.10(3) and the fact $\Gamma$ is finite, $\Gamma$ consists of the following three wings sharing only the vertex $d$:

1. a wing containing $a$ and last element $d$ and having $k$ vertices,
2. a wing containing $b$ and last element $d$ and having $j$ vertices, and
3. a wing containing $c$ and last element $d$ and having $l$ vertices.
Because $a$, $b$ and $c$ exist distinct from $d$, we know that $k, j, l > 1$. Without loss of generality, we assume that $k \geq j, l$.

We know all $d$-intervals in $E$ are $d$-diamonds by Lemma 4.5.10(2); assume the $d$-diamond $(d_0, d_1)$ does not contain an element labeled $a$. By Lemma 4.5.11, we know that the only $d$-diamonds not containing an element labeled $a$ are those of the form $(d_{nk}, d_{nk+1})$.

Since $\Gamma$ is simply laced, each diamond contains two elements with distinct labels taken from $\{a, b, c\}$, so $(d_0, d_1)$ contains elements labeled $b$ and $c$. Furthermore, since $k > 1$, the $d$-diamond $(d_1, d_2)$ contains an element labeled $a$, so it cannot contain two more elements labeled $b$ and $c$ respectively. Without loss of generality, assume $(d_1, d_2)$ does not contain an element labeled $b$. By Lemma 4.5.11, we now know that the only $d$-diamonds not containing an element labeled $b$ are those of the form $(d_{nj+1}, d_{nk+2})$.

We apply the same argument to the $d$-diamond $(d_2, d_3)$, with a small modification. As $j > 1$, we know that $(d_2, d_3)$ contains an element labeled $b$. If there is no element labeled $a$ in $(d_2, d_3)$, we must have $nk = 2$ for some $n$. Since $k > 1$, we can only have $n = 1$ and $k = 2$. However, each diamond contains two elements with distinct labels taken from $\{a, b, c\}$, so we know that $(d_0, d_1)$, $(d_1, d_2)$ and $(d_2, d_3)$ each contain elements with the label $c$, since they do not contain elements labeled $a$, $b$ and $a$, respectively.

By Lemma 4.5.11, the $d$-diamonds that does not contain an element labeled $c$ are separated by $l - 1$ diamonds, so we have $l > 3$, a contradiction to our assumption that $k \geq l$. Thus $(d_2, d_3)$ must contain an element labeled $a$ as well as $b$, so it does not contain an element labeled $c$. By Lemma 4.5.11, we now know that the only $d$-diamonds not containing an element labeled $c$ are those of the form $(d_{nj+2}, d_{nk+3})$.

Every $d$-diamond must contain exactly two elements with labels from $\{a, b, c\}$ and thus not contain exactly one element of the other label. We can reinterpret this as every integer must occur in exactly one of the forms $nk$, $nj + 1$ and $nl + 2$, corresponding to the least index of the diamonds that do not contain $a$, $b$ or $c$, respectively. By Lemma 4.5.9, we know that either $k = j = l = 3$ or $k = l = 4$ and $j = 2$. 
If $k = j = l = 3$, we visually verify the resulting Dynkin diagram is $\tilde{E}_6$, using Figure A.3. Notice there are three wings in $\tilde{E}_6$, each with three vertices. If $k = l = 4$ and $j = 2$, we visually verify the resulting Dynkin diagram is $\tilde{E}_7$, using Figure A.3. Notice there are three wings in $\tilde{E}_7$, two with four vertices and one with two. We have found all possible forms and the proof is complete.

While Proposition 4.5.13 shows that $\tilde{E}_6$ and $\tilde{E}_7$ are candidates for admitting full heaps, it is useful to remark that they actually do; see Figures C.11 and C.12. We can also verify that a full heap exists in the form described in this proof.

In Figure C.11, the vertex $d$ of the proof corresponds to 3. Consider the subset of the heap on the left of the figure consisting of $1_1, 2_2, 2_3, 3_3, 3_4$ and $3_5$. This clearly forms a wing of the heap. We can see a wing with the same labels starting at $3_6$ and ending at $3_8$, as expected.

In Figure C.12, the vertex $d$ again corresponds to 3. Consider the subset of the heap on the left of the figure consisting of $0_1, 1_2, 1_3, 2_3, 2_4, 2_5, 3_4, 3_5, 3_6$ and $3_7$. This clearly forms a wing of the heap. We can see a wing with the same labels starting at $3_8$ and ending at $3_{11}$, as expected.
4.6 Full Heaps over affine Dynkin diagrams

Before we present our main theorem, we give two results from work preceding this thesis. The first of these, in conjunction with our main theorem, shows that the list of full heaps in Appendix C is complete. It occurs as Theorem 6.6.2 in [10] and is reworded here slightly to match the notation of this thesis.

Theorem 4.6.1 (Green). Any full heap over an affine Dynkin diagram $\Gamma$ is isomorphic to one of the following full heaps.

<table>
<thead>
<tr>
<th>Full heaps</th>
<th>See Figures:</th>
</tr>
</thead>
<tbody>
<tr>
<td>The $n$ full heaps over $\tilde{A}_n$, $n \geq 1$</td>
<td>C.1 – C.4</td>
</tr>
<tr>
<td>The full heap over $\tilde{B}_n$, $n \geq 3$</td>
<td>C.5</td>
</tr>
<tr>
<td>The full heap over $\tilde{A}_{2n-1}$, $n \geq 3$</td>
<td>C.6</td>
</tr>
<tr>
<td>The full heap over $\tilde{C}_n$, $n \geq 2$</td>
<td>C.7</td>
</tr>
<tr>
<td>The full heap over $\tilde{D}_{n+1}$, $n \geq 2$</td>
<td>C.8</td>
</tr>
<tr>
<td>The full heap over $\tilde{D}_n$, $n \geq 4$, in the defining representation</td>
<td>C.9</td>
</tr>
<tr>
<td>The two full heaps over $\tilde{D}_n$, $n \geq 4$, in the spin representations</td>
<td>C.10</td>
</tr>
<tr>
<td>The two full heaps over $\tilde{E}_6$</td>
<td>C.11</td>
</tr>
<tr>
<td>The full heap over $\tilde{E}_7$</td>
<td>C.12</td>
</tr>
</tbody>
</table>

We also state Theorem 3.3.1 from [14], also proven in an entirely different way as Theorem 6.1.11 from [10]. While not necessary for our result as it is a specific case of Theorem 4.6.1, we provide a different elementary proof in the notation of this thesis.

Theorem 4.6.2. Any full heap over a Dynkin diagram $\Gamma$ of type $\tilde{A}_n$ for $n \geq 1$ is isomorphic to one of the $n$ full heaps over $\tilde{A}_n$ given in Figures C.1, C.2, C.3 and C.4.

Proof. Let $E$ be a full heap over $\Gamma$. 
For the case that $\Gamma$ is of type $\tilde{A}_1$, both vertices share a double edge with arrow pointing towards itself, so we repeatedly use Corollary 3.5.7(2) to see that $E$ can only have the form given in Figure C.1. (We could also use Lemma 3.4.1 to show the labels must alternate as in the figure, which gives the entire full heap by (F4) and the fact there are only two labels.)

Assume now that $n > 1$. First, we make a few observations. When $\Gamma$ is of type $\tilde{A}_n$ for $n > 1$, it contains a circuit, as is verified in Figure A.2, so we can apply our results in Section 4.2. As in that section, we use the label $\{0, 1, \ldots, n\}$ for our labels with $i$ and $j$ adjacent if and only if $j \in \{i - 1, i + 1\}$, where arithmetic is performed modulo $n + 1$ on these labels.

Suppose $E$ is totally ordered and $x \in E$. By Corollary 4.2.3, we know $x$ is an element of an infinite saturated chain on which $E$ either ascends or descends at every element in the chain. If $y$ is an element of $E$ not in this chain, it is comparable to every element of the chain since $E$ is totally ordered. But the chain is saturated, so then $y$ is either less than every element of the chain or greater than every element of the chain. However, the chain has no minimum or maximum, so the interval $[y, x]$ would be infinite, violating local finiteness. Thus, this chain is all of $E$. If $E$ ascends at every element, it is clearly isomorphic to the full heap in Figure C.2. If $E$ descends at every element, it is clearly isomorphic to the full heap in Figure C.4.

Suppose now that $E$ is not totally ordered. By Corollary 4.2.4, $E$ ascends and descends at every element of $E$. In particular, every element of $E$ is an element of an infinite saturated chain on which $E$ ascends at every element. Chains of this kind either share all elements or no elements, since no element can be covered by two elements of the same label, by (H1). So we can think of $E$ as made up of some number of these chains. Clearly, since $E$ is not totally ordered, there are at least two of these chains. We will see shortly that there cannot be $n$ or more these chains. It will turn out that, if there $k$ of these chains, then $E$ is the $k$th full heap over $\tilde{A}_n$, as given in Figure C.3.

Assume there are $k$ distinct chains on which $E$ ascends at every element, and label each chain by an integer from 1 to $k$. For ease of notation, we change our conventions slightly
so that the element is referred to by its label with two subscripts, the first used to indicate where on the chain the element is and the second to indicate which chain it is on. For example, consider the element \(0_{1,1}\). This element is on the first chain, so \(n_{0,1} \rightarrow 0_{1,1} \rightarrow 1_{1,1}\) is the local structure of that chain at that element. The element \(0_{2,1}\) is the next greatest element on that chain labeled 0. The element \(0_{1,2}\) is not on that chain, but still labeled 0.

Since \(E\) also descends at \(0_{1,1}\), it must cover an element labeled 1. This element cannot be on the same chain, since \(n\) and 1 cannot be equal modulo \(n+1 \geq 3\) and \(n\) is the label of the element on the chain \(0_{1,1}\) covers, by construction of the chain. There are no other restrictions, so without loss of generality, say \(0_{1,1}\) covers \(1_{1,2}\). By Lemma 3.4.2(1) and Corollary 3.5.7, we immediately find that \(1_{1,1}\) covers \(2_{1,2}\), since \(1_{1,2} \rightarrow 0_{1,1} \rightarrow 1_{1,1}\) is in \(E\). By induction, we find that \(i_{j,1}\) covers \(i+1_{j,2}\) for all \(i \in \{0,1,\ldots,n-1\}\) and \(j \in \mathbb{Z}\), and \(n_{j,1}\) covers \(0_{j+1,2}\) for all \(j \in \mathbb{Z}\). Abusing terminology, we say that the first chain covers the second chain in this situation. See Figure 4.10(a) for an example of this when \(n = 5\).

The above observations show that, if one element of a chain described above covers another element of a chain, all elements of the former chain are in covering relations with elements of the latter, i.e., the former chain covers the latter chain. Furthermore, since \(E\) ascends and descends at every element, every element is in covering relations with four elements, so every chain must cover exactly one chain (other than itself) and be covered by exactly one chain. Finally, because every label occurs in each chain, \(E\) is not a disjoint union of subheaps.

Thus, without loss of generality, we can assume that the \(l\)th chain covers the \(l+1\)st chain, for \(l \in \{1,2,\ldots,k-1\}\). In this case, only the first chain is not covered by another chain and only the \(k\)th does not cover another chain. That is, each element labeled \(i\) in the first chain needs an element labeled \(i-1\) to cover it, each element labeled \(i-1\) in the \(k\)th chain needs to cover an element labeled \(i\), and all elements in the the second through \(k-1\)st chain are fully covered and fully covering. The solution is to have the \(k\)th chain cover the first. This is only possible if \(i-1_{j,k}\) covers \(i_{j-1,1}\), because \(i-1_{j,k}\) must also be comparable
Figure 4.10: (a) Two chains with labels from $\tilde{A}_5$ where $E$ ascends at every element on the chain. Here, the first chain (second subscript is 1) covers the second chain (second subscript is 2). (b) The same two chains where each chain covers the other. This is the second full heap over $\tilde{A}_5$, with repeating motif highlighted. Note that the least element of the motif is labeled 2. (c) Four chains covering each other. This is the fourth full heap over $\tilde{A}_5$, with repeating motif highlighted. Note that the least element of the motif is labeled 4.
to $i - 1_{j-1,1}$ and $i - 1_{j,1}$. In Figure 4.10(c), the example is that $3_{1,4}$ covers $4_{0,1}$ (and not, say, $4_{1,1}$).

This completely describes the heap. See Figure 4.10(b) for an example of this when $n = 5$ and $k = 2$. See Figure 4.10(c) for an example when $n = 5$ and $k = 4$.

Finally, we follow the construction of one of the repeating motifs in this construction to see we have described the $k$th full heap over $\tilde{A}_n$. Starting at the element $0_{j,1}$ for some $j \in \mathbb{Z}$, we get the following two chains:

$$0_{j,1} \leftarrow 1_{j,2} \leftarrow \ldots k - 1_{j,k} \leftarrow k_{j-1,1}$$

and

$$0_{j,1} \leftarrow n_{j-1,1} \leftarrow \ldots k + 1_{j-1,1} \leftarrow k_{j-1,1},$$

so we see that an element labeled $k$ is the least in a repeating motif with an element labeled 0 as greatest. By our description in Figure C.3, this is the $k$th full heap over $\tilde{A}_n$. The proof is complete.

\[\square\]

**Corollary 4.6.3.** The full heaps described in Figures C.1, C.2, C.3 and C.4 are all full heaps over Dynkind diagrams with circuits.

**Remark 4.6.4.** Recall from Proposition 4.2.5 that, whenever $\Gamma$ contains a circuit, it must be isomorphic as a graph to $\tilde{A}_n$ for some $n > 1$. Thus, Theorem 4.6.2 constructs all full heaps over graphs with circuits.

### 4.7 A classification of full heaps over all Dynkin diagrams

We now present the first main result of the thesis. We note that while this theorem is similar to Theorem 4.6.1, this theorem applies to full heaps over any Dynkin diagram finitely many vertices, not just affine Dynkin diagrams.

**Theorem 4.7.1.** Let $\varepsilon : E \to \Gamma$ be a full heap over any Dynkin diagram $\Gamma$. Then each connected component of $\Gamma$ is isomorphic to one of
(1) $\tilde{A}_n$ for $n \geq 1$,

(2) $\tilde{B}_n$ for $n \geq 3$,

(3) $2\tilde{A}_{2n-1}$ for $n \geq 3$,

(4) $\tilde{C}_n$ for $n \geq 2$,

(5) $2\tilde{D}_{n+1}$ for $n \geq 2$,

(6) $\tilde{D}_n$ for $n \geq 4$,

(7) $\tilde{E}_n$ for $n \in \{6, 7\}$,

up to relabeling of vertices. Furthermore, every such Dynkin diagram admits a full heap.

Proof. By Lemma 4.1.1, we know that, when $\Gamma$ is not connected, the full heap over $\Gamma$ consists of disjoint subheaps over each connected component. These subheaps are also full in the sense they satisfy (F3), because any $p$-interval must consist of labels from a single connected component, (F4), trivially, and (F5), because adjacent elements are in the same connected component. So we just need to show that each connected component of $\Gamma$ is isomorphic to one of the Dynkin diagrams listed. That is, we now assume that $\Gamma$ is connected.

First, we know a full heap (possibly more than one) exists over all these Dynkin diagrams, by Theorem 4.6.1. These are given explicitly in Appendix C. All that remains is to show this list is complete.

From Lemma 3.4.4 that $\Gamma$ must be simply or doubly laced. By Lemmas 4.1.2 and 4.3.1, we know any vertex of $\Gamma$ has a tally of one, two, three or four.

If $\Gamma$ has a double edge with arrows pointing at both vertices, we know $\Gamma$ is isomorphic to $\tilde{A}_1$ by Lemma 4.1.5. If $\Gamma$ contains a circuit, we know from Proposition 4.2.5 that $\Gamma$ is $\tilde{A}_n$ for some $n \geq 2$. Now assume that no edge of $\Gamma$ is a double edge with arrows pointing at both vertices and that $\Gamma$ has no circuits.
If any vertex of $\Gamma$ has a tally of four, then Proposition 4.3.4 tells us that $\Gamma$ is $\widetilde{D}_4$, $\widetilde{A}_5$, or $\widetilde{C}_2$. So now assume all vertices of $\Gamma$ have a tally of one, two or three.

If $\Gamma$ has two distinct vertices with a tally of three, we know from Proposition 4.4.5 that these are the only vertices with a tally of three and that $\Gamma$ is one of $\widetilde{A}_{2n-1}$ for $n \geq 4$, $\widetilde{C}_n$ for $n \geq 3$, or $\widetilde{D}_n$ for $n \geq 5$. Now assume that $\Gamma$ has, at most, one vertex with at tally of three.

Now we specifically look at the case that $\Gamma$ has a double edge. Suppose $a$ and $d$ are vertices in $\Gamma$ that share a double edge. Recall from Lemma 4.1.3 that at least one of $a$ and $d$ is a terminal vertex. Assume $a$ is terminal. By Corollary 4.1.4, we know $d$ cannot be terminal.

Suppose first that the arrow points towards $d$. (Recall we are assuming that there cannot be two arrows on the double edge and, by definition of Dynkin diagram, there must be one arrow.) Since $d$ is not terminal, we must have that the tally of $d$ is greater than two, as $a$ by itself contributes two to the tally. Thus $d$ has a tally of three, so $d$ is adjacent to only one other vertex; let $c \neq a$ be this vertex. By Lemma 4.4.1, we know that $d$ has structure (5) in Figure 4.5. Then, by Corollary 4.4.7, The Dynkin diagram $\Gamma$ is one of $\widetilde{A}_{2n-1}$ for $n \geq 4$ or $\widetilde{C}_n$ for $n \geq 3$.

Now suppose that the arrow points towards $a$, so $a$ has a tally of two. We now have a path in $\Gamma$ consisting of at least $a$ and $d$ where the $a$ is terminal with a tally of two, so we use Proposition 4.5.3 and find that $\Gamma$ is isomorphic to one of $\widetilde{D}_{n+1}$ for $n \geq 2$ or $\widetilde{B}_n$ for $n \geq 3$. (It is not isomorphic to $\widetilde{A}_1$ since we have already disallowed the possibility of two arrows.)

Finally, we consider the case that $\Gamma$ is simply laced with at most one vertex with a tally of three. If $\Gamma$ does contain exactly one vertex with a tally of three, then Proposition 4.5.13 gives us that $\Gamma$ is $\widetilde{E}_6$ or $\widetilde{E}_7$. So now assume that $\Gamma$ only contains vertices with a tally of 1 or 2. (Note a tally of 0 is still impossible by Lemma 4.3.1.) By Lemma 4.5.8, we know $\Gamma$ does not admit a full heap.

We have considered all possible forms of $\Gamma$ and seen the only ones that admit a full heap are of the form of one of the Dynkin diagrams listed in the theorem, so we are done. □
An ADE classification in full heaps

The present chapter provides a method, using the notion of order ideals, to obtain the underlying poset of some full heaps from others. In particular, we can move between the underlying posets of certain full heaps over Dynkin diagrams of type \( \tilde{A} \), \( \tilde{D} \) and \( \tilde{E} \), giving us an ADE classification. We provide those calculations below, but first we need to explore our extension of the the order ideal poset, the clipped order ideal poset.

5.1 More results of clipped order ideal posets

Lemma 5.1.1. Suppose a locally finite poset \( E \) has constrained antichains. Then \( \tilde{J}(E) \) is also locally finite.

Proof. Assume \( E \) is nonempty, else the proof is trivial.

Let \( K \) and \( I \) be in \( \tilde{J}(E) \) with \( K < I \), and consider the interval \([K, I] \). As an nonempty element of \( \tilde{J}(E) \), we can write \( I = \langle i_1, i_2, \ldots, i_n \rangle \) where \( \{i_l\}_{l=1}^n \subset E \) is an antichain with \( n \geq 1 \). By hypothesis, \( n \) is finite.

We address the case that \( K \) is empty first. If the empty set is in \( \tilde{J}(E) \), recall the definition of a clipped order ideal poset requires that every element \( x \) of \( E \) be comparable to some minimal element of \( E \), dependent on \( x \). Any ideal in interval \([K, I] = (\emptyset, I] \) is generated by a nonempty antichain with elements less than or equal to one of \( \{i_l\}_{l=1}^n \). Because every element is comparable to a minimal element, the elements of this antichain are taken from the intervals between a minimal element and an \( i_l \), which are finite by the local finiteness of
E. Because antichains are finite and the set of minimal elements form an antichain, there
must be a finite number of such intervals. Thus each ideal in \((K, I)\) is determined by a finite
antichain with element taken from a finite number of finite intervals, so \((K, I)\) and thus \([K, I]\)
is finite, as needed.

For the remainder, suppose that \(K\) is not empty. We can write \(K = \langle k_1, k_2, \ldots, k_m \rangle\)
where \(\{k_j\}_{j=1}^m \subset E\) is an antichain with \(m \geq 1\). By hypothesis, \(m\) is finite.

Let \(X\) be an ideal in \([K, I] \subseteq \tilde{J}(E)\). The ideal \(X\) cannot be empty because it would
then by strictly less than \(K\). We know \(X\) is generated by an antichain, which is finite by
hypothesis, so suppose \(x\) is in the antichain generating \(X\). Because \(X \subseteq I\), we have \(x \in I\),
so \(x\) is less than or equal to one of the \(i_l\).

If \(x < k_j\) for some \(j\) in \(\{1, \ldots, m\}\), then \(k_j\) cannot be in \(X\) since \(x\) is maximal in \(X\).
This contradicts the assumption that \(K \leq X\). Thus either \(x\) is greater than or equal to
at least one of the \(k_j\), and therefore in one of the intervals \([k_j, i_l]\), for \(j \in \{1, \ldots, m\}\) and
\(l \in \{1, \ldots, n\}\), or \(x\) is incomparable to all elements of \(\{k_j\}_{j=1}^m\).

Because \(E\) is locally finite, we know each interval \([k_j, i_l]\) is finite when it exists. (Since
\(K < I\), it is never the case that a \(k_j\) is greater than an \(i_l\).) So there is a finite number of finite
intervals, thus the total number of elements in these intervals is finite. Additionally, because
\(E\) has constrained antichains, there is also a finite number of elements not comparable to
each of the \(k_j\) and thus to all the \(k_j\). So there is only a finite number of possibilities for
values of \(x\). As \(x\) was chosen arbitrarily, there is only a finite number of choices of antichains
that generate \(X\), an arbitrary ideal \([K, I]\). So \([K, I]\) is finite and we are done. \(\square\)

Remark 5.1.2. The hypothesis of constrained antichains is necessary for this lemma. For a
counterexample in the slightly weaker case where all antichains of \(E\) are finite, but \(E\) does
not have constrained antichains, refer to the example in Remark 2.1.10. There, we considered
the poset consisting \(Z\) and one other element \(x\) not comparable to all of \(Z\). Let \(I = \langle 0, x \rangle\)
and \(K = \langle x \rangle\). Then, there is an infinite chain of ideals \(\langle -1, x \rangle > \langle -2, x \rangle > \langle -3, x \rangle > \cdots\) in
the interval \([K, I]\), so \(\tilde{J}(E)\) is not locally finite.

**Corollary 5.1.3.** Suppose a locally finite poset \(E\) has constrained antichains. Then an element of \(\tilde{J}(E)\) is join-irreducible if and only if it covers exactly one element.

**Proof.** Lemma 5.1.1 gives us that \(\tilde{J}(E)\) is locally finite. Lemma 2.5.7 gives us that every pair of elements of \(\tilde{J}(E)\) has a join in \(\tilde{J}(E)\). Finally, from Lemma 2.2.3, one obtains the result.

For the next three lemmas (5.1.4, 5.1.5, and 5.1.6), suppose a locally finite poset \(E\) has constrained antichains.

**Lemma 5.1.4.** If \(I \in \tilde{J}(E)\) is join-irreducible, then \(I\) is a principal ideal.

**Proof.** Suppose first that \(I \in \tilde{J}(E)\) is join-irreducible. By Corollary 5.1.3, \(I\) covers exactly one other ideal. Call this element \(K\). Let \(L \neq K\) be some other ideal that is properly contained in \(I\). We wish to show that \(K\) contains \(L\).

Consider the interval \([L, I]\) in \(\tilde{J}(E)\). By Lemma 5.1.1, we know \(\tilde{J}(E)\) is locally finite, so by Lemma 2.2.1, there is an element in \([L, I]\) covered by \(I\). By assumption, \(K\) is the only ideal covered by \(I\), so \(K\) must be in \([L, I]\). Thus \(L < K\) as desired.

Now, because \(I\) and \(K\) are not equal, we know there is an element \(a \in E\) such that \(a \in I \setminus K\). Clearly \(\langle a \rangle \subseteq I\) since \(a \in I\), so, if \(I \neq \langle a \rangle\), we obtain \(\langle a \rangle \subseteq K\) by the above. However, this is clearly impossible since \(a \notin K\). Thus \(I = \langle a \rangle\) is a principal ideal as needed.

**Lemma 5.1.5.** If \(I \in \tilde{J}(E)\) is a principal ideal, then \(I\) covers at most one ideal in \(\tilde{J}(E)\).

**Proof.** Let \(I = \langle a \rangle\), and suppose \(I\) covers at least two ideals generated by a finite antichain, \(K\) and \(L\). Note that \(K\) and \(L\) must then be incomparable; otherwise, the greater would break the covering relation of the lesser. Recall also that \(K \cup L\) is an ideal by Lemma 2.4.2. In fact, it is clear \(K \cup L\) is generated by union of the finite antichains that generate \(K\) and \(L\) respectively, so \(K \cup L \in \tilde{J}(E)\).
Clearly, \( a \notin K \cup L \), since both \( K \) and \( L \) are properly contained in \( I \), so \( I = \langle a \rangle \) cannot in turn be contained in one of them. Additionally, as \( K \) and \( L \) are incomparable, \( K \) is properly contained in \( K \cup L \). Yet \( K \subsetneq K \cup L \subsetneq I \) would break the covering relation between \( K \) and \( I \). So we have a contradiction, and thus \( I \) covers at most one ideal.

\[\text{Lemma 5.1.6.} \quad \text{If } I \in \overline{J}(E) \text{ is a principal ideal, then } I \text{ covers at least one ideal in } \overline{J}(E).\]

\[\text{Proof.} \quad \text{Let } I = \langle a \rangle, \text{ and suppose, for contradiction, that } I \text{ does not cover an ideal.}\]

First, \( a \) must be a minimal element in \( E \), otherwise any element \( b \) less than \( a \) would generate an order ideal less than \( I \) in \( \overline{J}(E) \). By Lemma 5.1.1, we know \( \overline{J}(E) \) is locally finite, so by Lemma 2.2.1, there is an element in \([\langle b \rangle, I]\) covered by \( I \), a contradiction. So \( a \) is minimal in \( E \), and \( I = \{a\} \).

We now wish to show that, when \( E \) has constrained antichains and when \( I = \{a\} \) is an ideal in \( \overline{J}(E) \), so is the empty set. If the empty set is in \( \overline{J}(E) \), then \( I \) covers \( I \setminus \{a\} = \emptyset \) in \( \overline{J}(E) \), contradicting our assumption that \( I \) does not cover an ideal and proving our result.

By definition of the clipped order ideal poset, the empty set is included when each element of \( E \) is comparable to some minimal element of \( E \). When the empty set is not included in \( \overline{J}(E) \), there must be an element \( x_0 \) of \( E \) that is not comparable to any minimal element of \( E \). In particular, \( a \) is not comparable to \( x_0 \). So we have now reduced the proof to showing that \( a \) must be comparable to \( x_0 \).

Now, \( x_0 \) must be greater than some other element \( x_1 \) of \( E \), else it would be minimal, so by induction, we then have an infinite decreasing chain \( x_0 > x_1 > x_2 > \cdots \) in \( E \), all of which are not comparable to any minimal element and, in particular, not comparable to \( a \). Thus, there is an infinite set of antichains of the form \( \{a, x_i\} \) for \( i \) a nonnegative integer, i.e., an infinite set of antichains containing \( a \). This contradicts our hypothesis that \( E \) has constrained antichains. Thus, \( I \) covers at least one ideal, and we are done.

\[\text{The principal ideals of } E \text{ are clearly in bijection with the elements of } E, \text{ so, with the same restrictions on } E, \text{ we now obtain a way to recover the original poset } E \text{ from } \overline{J}(E),\]
giving us the following extension of the Fundamental Theorem of Finite Distributive Lattices.
(Note that all finite posets are locally finite and have constrained antichains.)

**Theorem 5.1.7.** Suppose a locally finite poset \( E \) has constrained antichains. Then \( I \in \tilde{J}(E) \) is join-irreducible if and only if \( I \) is a principal ideal. Additionally, the subposet of join-irreducible elements of \( \tilde{J}(E) \) is isomorphic to \( E \).

**Proof.** If an ideal \( I \) is join-irreducible, we know by Lemma 5.1.4 that \( I \) is principal. On the other hand, if \( I \) is principal, we invoke Lemmas 5.1.5 and 5.1.6 and find \( I \) covers exactly one ideal. Thus, using Lemmas 2.2.3 and 2.5.7, we obtain the result that \( I \) is join-irreducible, proving the biconditional.

Now that we have established that the join-irreducible elements of \( \tilde{J}(E) \) are precisely the principal ideals of \( E \), we use this to define a map from \( E \) to the join-irreducible elements of \( \tilde{J}(E) \) by sending each element of \( E \) to the principal ideal it generates. Since every element of \( E \) generates a distinct ideal and every join-irreducible element is a principal ideal, it is clear this map is bijective. Additionally, it is clear that \( a \leq b \) in \( E \) if and only if \( \langle a \rangle \subseteq \langle b \rangle \), so the partial order is also preserved, giving a poset morphism and the isomorphism we desire. \( \square \)

Inspired by Theorem 5.1.7, we define a final poset operator.

**Definition 5.1.8.** For a poset \( F \), we denote the subposet of \( F \) consisting of the join-irreducible elements of \( F \) as \( \tilde{J}^{-1}(F) \).

We could now restate Theorem 5.1.7 as simply \( \tilde{J}^{-1}(\tilde{J}(E)) \cong E \) when \( E \) satisfies the required properties. It should be noted that, despite the notation, it is not necessarily the case that \( \tilde{J}(\tilde{J}^{-1}(E)) \cong E \). For example, any poset \( P_n \) consisting only of \( n \) incomparable elements will contain no join-irreducible elements. (There are no covering relations at all, in fact, so we apply Lemma 2.2.3.) Therefore, the subposet of join-irreducible elements \( \tilde{J}^{-1}(P_n) \)
is empty for all $n$. So, for all $n$, we find

$$\tilde{J}(J^{-1}(P_n)) = \tilde{J}(\emptyset) \cong P_1,$$

i.e., the isomorphism does not hold for all $n > 1$.

This $\tilde{J}(J^{-1}(E)) \cong E$ isomorphism only occurs when $E$ is a distributive lattice [21] and is not needed for this thesis, so we do not examine it here.

The following lemma will be useful in the sequel.

**Lemma 5.1.9.** Let $\{a, b, c\}$ be an antichain of distinct elements in a poset $E$. Then there exists an element in the order ideal poset $J(E)$ that covers at least three elements. If we further assume that $E$ is locally finite and has constrained antichains, then there exists an element in $\tilde{J}(E)$ that covers at least three elements.

**Proof.** The reader may find it useful to refer to Figure 5.1 when reading the proof.

![Figure 5.1](image-url)

Figure 5.1: The general form of the part of order ideal poset resulting from three pairwise incomparable elements $a, b, c$. We depict the relations with dashed lines because it is not necessarily the case that the relations depicted are covering relations. For example, if there is an element $d \leq c$ that is incomparable to $a$ and $b$, the ideal $\langle a,b,d \rangle$ occurs between $\langle a,b \rangle$ and $\langle a,b,c \rangle$.

For $J(E)$, consider the ideal $\langle a,b,c \rangle$. As $c$ is a maximal element of this ideal, the set $\langle a,b,c \rangle \setminus \{c\}$ is also an ideal and is covered by $\langle a,b,c \rangle$. Similarly, $\langle a,b,c \rangle$ covers $\langle a,b,c \rangle \setminus \{a\}$ and $\langle a,b,c \rangle \setminus \{b\}$. Because $a, b$ and $c$ are distinct, the ideals are distinct, proving our result.
Because $\tilde{J}(E)$ does not contain all ideals, we have to be more careful. Consider the ideals $\langle a, b \rangle$, $\langle a, c \rangle$ and $\langle b, c \rangle$ in $\tilde{J}(E)$; these ideals must clearly be properly contained in $\langle a, b, c \rangle$.

We claim the ideal $\langle a, b, c \rangle$ covers at least three elements. It is clear that $\langle a, b \rangle$, $\langle a, c \rangle$ and $\langle b, c \rangle$ are all less than $\langle a, b, c \rangle$ in $\tilde{J}(E)$. By Lemma 5.1.1 and our hypothesis on $E$, the poset $\tilde{J}(E)$ is locally finite, so we apply Lemma 2.2.1 to the interval $[\langle a, b \rangle, \langle a, b, c \rangle]$ and find that $\langle a, b, c \rangle$ covers some ideal greater than or equal to $\langle a, b \rangle$. Similarly, $\langle a, b, c \rangle$ covers ideals greater than or equal to $\langle a, c \rangle$ and $\langle b, c \rangle$.

If $\langle a, b, c \rangle$ covers fewer than three elements, one of the ideals it covers that is greater than at least two of $\langle a, b \rangle$, $\langle a, c \rangle$ and $\langle b, c \rangle$. Call this ideal $I$. Without loss of generality, assume $I$ is greater than the first two. Then $a, b, c \in I$ so, by definition of ideal, $\langle a, b, c \rangle \subseteq I$. Since we assumed that $I$ is covered by $\langle a, b, c \rangle$, we have a contradiction.

5.2 An ADE classification in minuscule posets

We now present an ADE classification in minuscule heaps. An ADE classification is a complete list of of the simply laced finite Dynkin diagrams or related mathematical structures. In our case, we look at the related mathematical structures of minuscule posets of simply laced finite Dynkin diagrams and their order ideal posets. It turns out that, beginning with the minuscule posets for type $A_n$, repeated application of the $J$ operator produces all other minuscule posets of simply laced finite Dynkin diagrams.

In fact, we can also obtain minuscule posets over other finite Dynkin diagrams, and we record those results. However, if we restrict ourselves to only considering the simply laced diagrams, there is only one Dynkin diagram that corresponds to a given minuscule poset, which is necessary for a rigorous classification. For example, the doubly laced $B_n$ has the same minuscule poset as the simply laced $D_n$, in a spin representation. (See Figures B.2 and B.5.) Thus, a classification of all finite Dynkin diagrams by their minuscule posets would not distinguish these.
Table 5.1: Minuscule poset structures of finite Dynkin diagrams. Note that while there are two spin representations in type $D_n$, their minuscule posets are isomorphic, so we do not distinguish between the two. Similarly, we do not distinguish the two representations of $E_6$.

<table>
<thead>
<tr>
<th>Dynkin diagram type</th>
<th>Minuscule Posets isomorphism</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n$ $(n \geq 2)$</td>
<td>$[j] \times [n + 1 - j]$ for $1 \leq j \leq n$</td>
</tr>
<tr>
<td>$B_n$ $(n \geq 2)$</td>
<td>$J([2] \times [n - 1])$</td>
</tr>
<tr>
<td>$C_n$ $(n \geq 3)$</td>
<td>$[2n - 1]$</td>
</tr>
<tr>
<td>$D_n$ $(n \geq 4)$</td>
<td>$J^{n-3}([2] \times [2])$</td>
</tr>
<tr>
<td>$D_n$ (spin, $n \geq 4$)</td>
<td>$J([2] \times [n - 2])$</td>
</tr>
<tr>
<td>$E_6$</td>
<td>$J^2([2] \times [3])$</td>
</tr>
<tr>
<td>$E_7$</td>
<td>$J^3([2] \times [3])$</td>
</tr>
</tbody>
</table>

All results in this section are due to Proctor [18] and Stembridge [23]. We record them here to provide the basis for the analogy in Section 5.5. In [23], Stembridge lists the minuscule posets, which we summarize in Table 5.1. This table gives the $ADE$ classification fully.

The reader should note that there is some repetition in this table. For example, the minuscule poset of $D_4$ in the spin representation is isomorphic to the order ideal poset of the minuscule poset of $A_3$, when $j = 2$. (We refer to this as the second minuscule poset of $A_3$.) Additionally, the minuscule poset of $B_3$ is the same as the minuscule poset of $D_4$, so it too is isomorphic to an order ideal poset of the second minuscule poset of $A_3$. In fact, we can, using just the minuscule posets in type $A_n$ and the $J$ operator, describe all the other minuscule posets.

We summarize the relationships between the minuscule posets in the diagram in Figure 5.2. In this figure and the rest of this section, we denote the minuscule poset for the finite Dynkin diagram of type $X$ by $\downarrow X\downarrow$. For Dynkin diagrams with multiple minuscule posets, we add a second subscript as follows. For those of type $A_n$, the symbol $\downarrow A_{n,j}\downarrow$ denotes the $j$th minuscule poset, i.e., $[j] \times [n + 1 - j]$. For those of type $D_n$, the symbols $\downarrow D_{n,d}\downarrow$ and $\downarrow D_{n,s}\downarrow$
Figure 5.2: Minuscule poset structures of finite Dynkin diagrams using the minuscule posets in type $A_n$ and the $J$ operator. Each arrow here indicates an application of $J$. The dashed box is the backbone of the $ADE$ classification, as it contains all three types. See the text for a description of the notation.
denote the minuscule posets corresponding the defining representation and spin representation, respectively. Again, we conflate the two spin representations, as their minuscule posets are isomorphic. For \( n = 4 \), the defining representation minuscule poset is also isomorphic to the spin representation minuscule posets, so we use the symbol \( \downarrow D_{4,ds} \) for all three. (See Figures B.4 and B.5.) We also conflate the minuscule posets of the two representations of \( E_6 \), as they are isomorphic. (See Figure B.6.)

We use the trivial observation that \( [k] \cong J([k-1]) \cong J([1] \times [k-1]) \) for correspondences appearing near the top of the figure. The minuscule posets \( \downarrow D_{n,s} \) and \( \downarrow B_{n-1} \) are isomorphic, as are \( \downarrow A_{2n-1,1} \) and \( \downarrow C_n \), but we list them separately here. All relationships in the figure are derived from Table 5.1.

**Example 5.2.1.** As is clear from Figure 5.2, we see we can obtain the minuscule posets \( D_5 \) in the spin representation, \( E_6 \) and \( E_7 \) from the second minuscule posets of \( A_4 \). Here, we explicitly show these relations, as they are they form the \( ADE \) classification.

The labels of the elements here are chosen to reflect the fact that these posets are the underlying posets of minuscule heaps; see Appendix B. Here these labels are used primarily to keep track of the order ideals in the resulting order ideal poset. Recall, an order ideal is defined by its maximal elements, so we label the elements of the order ideal poset by the maximal elements in the ideal it represents. For example, in Figure 5.3, the element labeled \( \langle 1_1, 3_1 \rangle \) in \( J(\downarrow A_{4,2}) \) refers to the order ideal of \( \downarrow A_{4,2} \) with maximal elements \( 1_1 \) and \( 3_1 \), i.e., \( \langle 1_1, 3_1 \rangle \) is the subset \( \{1_1, 2_1, 3_1\} \) of the minuscule poset \( \downarrow A_{4,2} \).

From Figures 5.3 and 5.4, we can visually verify that the order ideal poset of \( \downarrow A_{4,2} \) is isomorphic to \( \downarrow D_{5,s} \). Similarly, Figures 5.4 and 5.5 show us that the order ideal poset of the minuscule poset \( \downarrow D_{5,s} \) is isomorphic to \( \downarrow E_6 \), and Figures 5.5 and B.7 show us that the order ideal poset of \( \downarrow E_6 \) is isomorphic to the minuscule poset \( \downarrow E_7 \).

We note here that, as a check, the reader may wish to look at the join-irreducible elements of the resulting order ideal posets. (Recall, the join-irreducible elements are those
Figure 5.3: The minuscule poset $\downarrow A_{4,2}$, in the form $[2] \times [3]$, and its order ideal poset, $J(\downarrow A_{4,2})$. The order ideal poset is isomorphic to the minuscule poset $\downarrow D_{5,1}$; see Figure B.5.
Figure 5.4: The minuscule poset $\downarrow D_{5,s}$ and its order ideal poset, which is isomorphic to the minuscule poset $\downarrow E_6$; see Figure B.6.

$\downarrow D_{5,s}$

Order ideal poset of $\downarrow D_{5,s}$
Figure 5.5: The minuscule poset $|E_6|$ and its order ideal poset, which is isomorphic to the minuscule poset $|E_7|$; see Figure B.7.
covering exactly one other element.) If all the other elements of the order ideal poset are deleted, the induced poset is isomorphic to the original, as we should expect from our discussion above. We will explicitly show this in several examples below.

5.3 Calculations for the $ADE$ classification in full heaps

Our $ADE$ classification in full heaps is more properly called a $\tilde{A}\tilde{D}\tilde{E}$ classification, as we will be describing a way of classifying the simply laced affine Dynkin diagrams. However, affine Dynkin diagrams are extensions of affine Dynkin diagrams (see [2] and [13]), and that extension is one-to-one among the simply laced diagrams, with the addition of $\tilde{A}_1$. So, a classification of the simply laced affine Dynkin diagrams is a classification of the simply laced finite Dynkin diagrams. Furthermore, in our particular context, Green shows a way to obtain minuscule posets of finite Dynkin diagrams from full heaps over their corresponding affine Dynkin diagram in [10]. Because we use an $ADE$ classification in minuscule posets from Section 5.2 as analogy for the classification in full heaps, it is natural to call this an $ADE$ classification.

We now make the necessary calculations to show the $ADE$ classification in full heaps. We show, similarly to Section 5.2, that the underlying posets of full heaps over affine Dynkin diagrams are obtainable from repeated application of the $\tilde{J}$ operator to the underlying posets of full heaps of type $\tilde{A}_n$. Recall that minuscule posets are underlying posets of certain maximal minuscule heaps and that full heaps are a sort of affine analog to maximal minuscule heap, so the analogy is natural. As such, we developing the background data for a figure similar to Figure 5.2 for full heaps.

It is important to note here that we must use $\tilde{J}$ instead of $J$. For example, if we used $J$, we would need to include the empty set in every order ideal poset, which would destroy our desired isomorphisms. Indeed, the very concept of the clipped order ideal poset is intended to avoid these complications.

The present section focuses on the calculations required for the $ADE$ classification
itself; the sequel addresses other relations obtained with clipped order ideal posets. Again, for ease of notation, we denote the underlying poset of a full heap over type $X$ as $\mathcal{X}$, throughout this section and the next. For Dynkin diagrams with multiple full heaps, we add a second subscript. For those of type $\tilde{A}_n$, the symbol $\mathcal{X}_{\tilde{A}_n,j}$ denotes the underlying poset of the $j$th full heap, as defined in Figures C.2, C.3 and C.4. For those of type $\tilde{D}_n$, the symbols $\mathcal{X}_{\tilde{D}_n,d}$ and $\mathcal{X}_{\tilde{D}_n,s}$ denote the underlying posets of the full heaps corresponding the defining representation and spin representation, respectively. Again, we conflate the two spin representations, as the posets underlying their full heaps are isomorphic. For $n = 4$, the posets underlying the full heaps of the defining representation and both spin representations are isomorphic, so we use the symbol $\mathcal{X}_{\tilde{D}_4,ds}$ for all three. (See Figures C.9 and C.10.) We also conflate the two dual representations of $\tilde{E}_6$, as they are isomorphic; see Figure C.11.

Each calculation uses the labels from the full heaps for bookkeeping, but the calculations occur entirely on the underlying poset. These calculations are most easily described graphically, and each is accompanied by a figure, but we do highlight important observations in the accompanying text. In the figures, we give a full heap and the clipped order ideal poset of that full heap. For clarity, the elements that appear in the latter are highlighted in the former. In some of the calculations, we also perform the $\tilde{J}^{-1}$ calculation by showing at the subposet of the join-irreducible elements of the clipped order ideal poset is isomorphic to the original underlying poset of the full heap. This is not strictly necessary, but can be enlightening and serves as a good check.

**Calculation 5.3.1.** Figure 5.6 gives the calculation of the clipped order ideal poset for $\mathcal{X}_{\tilde{A}_4,2}$. While the validity of this calculation is visibly verified, we wish to note a few general observations that may aid in this.

Recall that the elements in the clipped order ideal poset are given by ideals generated by antichains. In $\mathcal{X}_{\tilde{A}_4,2}$, it is easy to see any antichain contains at most two elements. Indeed, every vertex of the Dynkin diagram $\tilde{A}_4$ is adjacent to all but two elements, and those two
Figure 5.6: The second full heap of $A_4$ and its clipped order ideal poset, which is isomorphic to $|D_{5,4}|$. The highlighted labels in the full heap are those that appear in the section of the clipped order ideal poset shown.
elements are adjacent, so larger antichains are impossible by (H1). This is actually vital to its clipped order ideal poset being the underlying poset of a full heap, because Lemma 5.1.9 guarantees that three pairwise incomparable elements result in an element of the order ideal poset that covers three elements, which we know to be impossible in a full heap by Lemma 3.4.2(2).

Checking the calculation is, while possibly tedious, not hard. As an example, consider the element $0_4 \in \tilde{\mathcal{A}}_{4,2}$. It is comparable to all elements except $2_3$ and $3_3$, so the only ideals of $|\tilde{\mathcal{A}}_{4,2}|$ in which $0_4$ is maximal are $\langle 0_4 \rangle$, $\langle 2_3, 0_4 \rangle$ and $\langle 3_3, 0_4 \rangle = \langle 2_3, 3_3, 0_4 \rangle$. (The equality comes from the fact that $2_3 < 3_3$.) We see that all three ideals occur in the order ideal poset in the order expected and are the only ideals in which $0_4$ is maximal. We quickly see this to be the case with every element of $|\tilde{\mathcal{A}}_{4,2}|$.

Recalling from Theorem 5.1.7 that the poset consisting of only the join-irreducible elements of $\tilde{\mathcal{J}}(|\tilde{\mathcal{A}}_{4,2}|)$ and the induced relation is isomorphic the original $|\tilde{\mathcal{A}}_{4,2}|$, we can check the calculation in the other direction. For example, $\langle 0_4 \rangle$ is join-irreducible in $\tilde{\mathcal{J}}(|\tilde{\mathcal{A}}_{4,2}|)$. The only other join-irreducible elements in $\tilde{\mathcal{J}}(|\tilde{\mathcal{A}}_{4,2}|)$ that are not comparable to $\langle 0_4 \rangle$ are $\langle 2_3 \rangle$ and $\langle 3_3 \rangle$. Mapping these principal ideals to their generating element, we quickly see that the local structure around $0_4$ is as desired. We do not explicitly show this method here, but will for calculations below.

We calculate the ideals defined by elements in more than a single occurrence of the repeating motif that makes up $|\tilde{\mathcal{A}}_{4,2}|$, so we conclude this pattern continues uninterrupted. That is, the local structure of an element in $|\tilde{\mathcal{A}}_{4,2}|$ determines the local structure around all ideals in the order ideal poset in which that element is maximal. Since the local structure occurs in a repeating motif in $|\tilde{\mathcal{A}}_{4,2}|$ and between repeating motifs, it must also do so in $\tilde{\mathcal{J}}(|\tilde{\mathcal{A}}_{4,2}|)$. The finicky reader may wish to replace the indices with $n, n+1$, etc. for complete satisfaction.

Finally, we visually verify that the order ideal poset $\tilde{\mathcal{J}}(|\tilde{\mathcal{A}}_{4,2}|)$ is isomorphic to $|\tilde{\mathcal{D}}_{5,s}|$, depicted in Figure C.10. So the underlying poset of a full heap over $\tilde{D}_5$ in a spin represent-
tation is the clipped order ideal poset of the second full heap over $\tilde{A}_4$, i.e.,

$$\downarrow \tilde{D}_{5,s} \cong \tilde{J}(\downarrow \tilde{A}_{4,2}).$$

The calculation is complete. \hfill \square

**Calculation 5.3.2.** Figure 5.7 gives the calculation of the clipped order ideal poset for $\tilde{J}(\downarrow \tilde{D}_{5,s})$. The validity of this calculation is visibly verified. To aid in this verification, note that those elements labeled 0, 1, 4 or 5 are each incomparable with exactly three other elements in a chain. (For example, $0_3$ is incomparable to the elements in the chain $4_3 \to 3_6 \to 5_3$.) Thus, for each element with one of those labels, there are exactly four ideals in which they occur as a maximal element, namely the principal ideal generated by that element and the three ideals generated by that element and one other to which it is incomparable. There are no others because these three incomparable elements form a chain, so are all comparable.

Similarly, those elements labeled 2 or 3 are incomparable with exactly one other element, so they should occur as the maximal elements of two ideals, which they do. Note these observations also mean that antichains in $\tilde{J}(\downarrow \tilde{D}_{5,s})$ have at most two elements; as we noted in Calculation 5.3.1, this is required by Lemma 5.1.9 to be the underlying poset of a full heap.

An easier check, however, is to use Theorem 5.1.7 and examine the join-irreducible elements, which we do explicitly in Figure 5.8. The poset on the left is the induced poset on the join-irreducible elements from the poset we claim is $\tilde{J}(\downarrow \tilde{D}_{5,s})$. Here, we see this poset is isomorphic to $\downarrow \tilde{D}_{5,s}$, as expected.

We calculate the ideals defined by elements in more than a single occurrence of the repeating motif that makes up $\downarrow \tilde{D}_{5,s}$, so we conclude this pattern continues uninterrupted.

We can now visually verify that the right poset in Figure 5.7 is isomorphic to $\downarrow \tilde{E}_6$, shown in Figure C.11. So the underlying poset of a full heap over $\tilde{E}_6$ is the clipped order ideal poset of a full heap of type $\tilde{D}_5$ in the spin representation. That is, combining this with Calculation
Figure 5.7: One of the full heaps over $D_5$ in the spin representation and its clipped order ideal poset, which is isomorphic to $\mathcal{E}_6$. The highlighted labels are those that appear in the section of the clipped order ideal poset shown.

\[
\begin{array}{c}
15 \\
 2_{10} \\
 0_5 \\
2_9 \\
 0_4 \\
2_7 \\
0_3 \\
 2_6 \\
0_2 \\
2_1 \\
1_0 \\
0_0 \\
\end{array}
\quad
\begin{array}{c}
(0_4) \\
 (2_7, 4_4) \\
 (2_7) \\
 (1_3, 4_4) \\
 (1_3, 5_3) \\
 (3_7) \\
 (1_3) \\
 (2_6, 5_3) \\
 (2_6) \\
 (0_3, 5_3) \\
 (0_3, 4_3) \\
 (3_6) \\
 (0_3) \\
 (2_5, 4_3) \\
 (2_5) \\
 (1_2, 4_3) \\
 (1_2, 3_5) \\
 (4_3) \\
 (1_2, 5_2) \\
 (3_5) \\
 (1_2) \\
 (2_4, 5_2) \\
 (2_4) \\
 (0_2, 5_2) \\
 (0_2, 4_2) \\
 (3_4) \\
 (0_2) \\
 (2_3, 4_2) \\
 (2_3) \\
 (1_1, 4_2) \\
 (1_1, 3_3) \\
 (4_2) \\
 (1_1, 5_1) \\
 (3_3) \\
 (1_1) \\
 (2_2, 5_1) \\
 (2_2) \\
 (0_1, 5_1) \\
 (0_1, 3_2) \\
 (5_1) \\
\end{array}
\]

Clipped order ideal lattice
Figure 5.8: The subposet formed by the join-irreducible elements of $\tilde{J}(\tilde{D}_{5,s})$ and a full heap over $\tilde{D}_5$ in the spin representation. The poset isomorphism takes the generators of the ideal to themselves.
5.3.1, we have

$$|\tilde{E}_6| \cong \bar{J}(|\tilde{D}_{5,5}|) \cong \bar{J}(\bar{J}(|\tilde{A}_{4,2}|)).$$

The calculation is complete. \qed

**Calculation 5.3.3.** Figure 5.9 gives the calculation of the order ideal poset for $\bar{J}(|\tilde{E}_6|)$. The validity of this calculation is visibly verified. To aid in this verification, note that those elements labeled 0, 1 or 5 are incomparable with exactly five other elements in a chain, so they should occur as the maximal element of exactly six ideals, which they do in the order ideal poset. Similarly, those elements labeled 2, 4 or 6 are incomparable with exactly two other elements in a chain, so they should occur as the maximal elements of three ideals, which they do. Finally, those elements labeled 3 are incomparable with exactly one other element, so they should occur as the maximal elements of two ideals, which they do. Note these observations also mean that antichains in $\bar{J}(|\tilde{E}_6|)$ have at most two elements, which, as we noted in Calculations 5.3.1 and 5.3.2 is required for the order ideal poset to be the underlying poset of a full heap.

An easier check, however, is to examine the join-irreducible elements, which we do explicitly in Figure 5.10. The poset on the left is the induced poset on the join-irreducible elements from the poset $\bar{J}(|\tilde{E}_6|)$. Here, we see this poset is isomorphic to $|\tilde{E}_6|$, as expected.

We calculate the ideals defined by elements in more than a single occurrence of the repeating motif that makes up $|\tilde{E}_6|$, so we conclude this pattern continues uninterrupted. We can now visually verify that the right poset in Figure 5.9 is isomorphic to $|\tilde{E}_7|$, shown in Figure C.12. So the underlying poset of a full heap over $\tilde{E}_7$ is isomorphic to the clipped order ideal poset of a full heap of type $\tilde{E}_6$. The calculation is complete. \qed

Calculations 5.3.1, 5.3.2 and 5.3.3 give us the $ADE$ classification we noticed in the minuscule posets of the previous section. While we make further observations regarding the application of $\bar{J}$, we note this result in the following proposition.
Figure 5.9: A full heap over $\tilde{E}_6$ and its order ideal poset, which is isomorphic to $|\tilde{E}_7|$. The highlighted labels are those that appear in the section of the clipped order ideal poset shown.
Figure 5.10: The subposet formed by the join-irreducible elements of $\tilde{J}(\lceil E_6 \rceil)$ and a full heap over $\tilde{E}_6$. The poset isomorphism takes the generators of the ideal to themselves.
**Proposition 5.3.4.** In the notation of the section,

\[ |\tilde{E}_7| \cong \tilde{J}(|\tilde{E}_6|) \cong \tilde{J}(\tilde{J}(|\tilde{D}_{5,s}|)) \cong \tilde{J}(\tilde{J}(\tilde{J}(|\tilde{A}_{4,2}|))). \]

\[ \square \]

**Remark 5.3.5.** It is natural to wonder at this point what form the clipped order ideal poset of \(|\tilde{E}_7|\) has. While we do not present it here, we point the reader to Figure C.12. Notice that, for example, the elements 7, 4, and 6 form an antichain. Thus, by Lemma 5.1.9, there is an element of the clipped order ideal poset of \(|\tilde{E}_7|\) that covers three elements. Therefore, this clipped order ideal poset cannot be the underlying poset of a full heap by Lemma 3.4.2(2).

### 5.4 Other calculations of clipped order ideal posets of full heaps

The calculations preformed in the previous section recreate Example 5.2.1 in the full heap setting. We now perform a few more calculations to fill out the remaining cases not considered. We maintain the notation of Section 5.3.

**Calculation 5.4.1.** Note that this calculation is a generalization of Calculation 5.3.1, so it may be useful to refer back when the labels get cumbersome here. Figure 5.11 gives the calculation of the order ideal poset \(|\tilde{A}_{n,2}|\) for \(n \geq 3\). The verification of this calculation is basically the same as that in Calculation 5.3.1, so we only note that every element of \(|\tilde{A}_{n,2}|\) is incomparable to \(n - 2\) elements which form a chain. (The labels of these \(n - 2\) elements are the \(n - 2\) labels not adjacent to the reference element in \(\tilde{A}_n\).) Thus, for each element, there should be \(n - 1\) ideals in which it is maximal in the order ideal poset, which is the case. This is most clearly seen by examining the ideal \(\langle 0_3 \rangle\) and the chain of ideals above it containing 03 in their generating set.

To aid in understanding the clipped order ideal poset, see Figure 5.12 for an example when \(n = 6\).

We calculate the ideals defined by elements in more than a single occurrence of the repeating motif that makes up \(|\tilde{A}_{n,2}|\), so we conclude this pattern continues uninterrupted.
Figure 5.11: The second full heap over $\tilde{A}_n$, $n \geq 3$, and its order ideal poset, which is isomorphic to $|\tilde{D}_{n+1,s}|$. The highlighted labels are those that appear in the section of the clipped order ideal poset shown.
Figure 5.12: The clipped order ideal poset of the second full heap of $\tilde{A}_6$. This poset is isomorphic to $\tilde{D}_{7,s}$. 

\[
\begin{align*}
\{0_4, 2_4\} & \quad \{6_3, 3_6\} & \quad \{4_4\} \\
\{0_5\} & \quad \{6_3, 2_4\} & \quad \{5_3, 3_4\} \\
\{6_3, 1_4\} & \quad \{5_3, 2_6\} & \quad \{3_4\} \\
\{6_3\} & \quad \{5_3, 1_4\} & \quad \{4_3, 2_4\} \\
\{5_3, 0_4\} & \quad \{4_3, 1_4\} & \quad \{2_4\} \\
\{5_3\} & \quad \{4_3, 0_4\} & \quad \{3_3, 1_4\} \\
\{4_3, 6_2\} & \quad \{3_3, 0_4\} & \quad \{1_4\} \\
\{4_3\} & \quad \{3_3, 6_2\} & \quad \{2_3, 0_4\} \\
\{3_3, 5_2\} & \quad \{2_3, 6_2\} & \quad \{6_0\} \\
\{3_3\} & \quad \{2_3, 5_2\} & \quad \{1_3, 6_2\} \\
\{2_3\} & \quad \{1_3, 4_2\} & \quad \{0_3, 5_2\} \\
\{1_3, 3_2\} & \quad \{0_3, 4_2\} & \quad \{5_2\} \\
\{1_3\} & \quad \{0_3, 3_2\} & \quad \{6_1, 4_2\} \\
\{0_3, 2_2\} & \quad \{6_1, 3_2\} & \quad \{4_2\} \\
\{0_3\} & \quad \{6_1, 2_2\} & \quad \{5_1, 3_2\} \\
\{6_1, 1_2\} & \quad \{5_1, 2_2\} & \quad \{3_2\} \\
\{6_1\} & \quad \{5_1, 1_2\} & \quad \{4_1, 2_2\} \\
\{5_1, 0_2\} & \quad \{4_1, 1_2\} & \quad \{2_2\} \\
\{5_1\} & \quad \{4_1, 0_2\} & \quad \{3_1, 1_2\} \\
\{4_1, 6_0\} & \quad \{3_1, 0_2\} & \quad \{1_2\} \\
\{4_1\} & \quad \{3_1, 6_0\} & \quad \{2_1, 0_2\} \\
\{3_1, 5_0\} & \quad \{2_1, 6_0\} & \quad \{0_2\} \\
\end{align*}
\]
We can now visually verify that the right poset in Figure 5.11 is isomorphic to $|\tilde{D}_{n+1,s}|$, shown in Figure C.10. So the underlying poset of a full heap over $\tilde{D}_{n+1}$ in a spin representation is isomorphic to the clipped order ideal poset of $|\tilde{A}_{n,2}|$, i.e.,

$$|\tilde{D}_{n+1,s}| \cong \mathcal{J}(|\tilde{A}_{n,2}|).$$

We note additionally that the underlying poset of the full heap over $\tilde{B}_n$ is isomorphic to $|\tilde{D}_{n+1,s}|$ (see Figure C.5), so we also have

$$|\tilde{B}_n| \cong \mathcal{J}(|\tilde{A}_{n,2}|).$$

The calculation is complete.

Remark 5.4.2. We now wish to look at the posets underlying full heaps over $\tilde{D}_n$ in the defining representation. Recall that, for $\tilde{D}_4$, the spin and defining representations coincide, so, by Calculation 5.4.1, we know that the underlying poset of the full heap over $\tilde{D}_4$ is isomorphic to $\mathcal{J}(|\tilde{A}_{3,2}|)$. Our next calculation will show that, when $n \geq 4$, the clipped order ideal poset of $|\tilde{D}_{n,d}|$ is isomorphic to $|\tilde{D}_{n+1,d}|$, so the observation for $\tilde{D}_4$ will serve as a base case for this result.

Calculation 5.4.3. Figure 5.13 gives the calculation of the clipped order ideal poset for the poset underlying the full heap over $\tilde{D}_n$ in the regular representation, for $n \geq 4$. The verification of this calculation is relatively easy, as the only ideals in the poset that are not principal are in the form $\langle 0_i, 1_i \rangle$ and $\langle (n - 1)_i, n_i \rangle$ where $i \in \mathbb{Z}$. Thus, the resulting order ideal poset is similar in shape to the original poset, but with an additional element between each pair of diamonds.

We calculate the ideals defined by elements in more than a single occurrence of the repeating motif that makes up $|\tilde{D}_{n,d}|$, so we conclude this pattern continues uninterrupted. We can now visually verify that the right poset in Figure 5.13 is isomorphic to $|\tilde{D}_{n+1,d}|$, shown in Figure C.9. So the underlying poset of a full heap over $\tilde{D}_{n+1}$ in the defining
Figure 5.13: The full heap of $\tilde{D}_n$ in the defining representation and its order ideal poset, which is isomorphic to $|\tilde{D}_{n+1,d}|$. The highlighted labels are those that appear in the section of the clipped order ideal poset shown.
representation is isomorphic to the clipped order ideal poset of \(|\tilde{D}_{n,d}\| \) when \(n \geq 4\), i.e.,
\[ |\tilde{D}_{n+1,d}| \cong \tilde{J}(|\tilde{D}_{n,d}|). \]
By induction and Remark 5.4.2, we can rewrite this as

\[ |\tilde{D}_{n,d}| \cong \tilde{J}^{n-3}(|\tilde{A}_{3,2}|). \]

We note additionally that the underlying poset of the full heap over \(2\tilde{A}_{2n-1}\) is isomorphic to \(|\tilde{D}_{2n,d}|\) (see Figure C.6), so we also have
\[ |2\tilde{A}_{2n-1}| \cong \tilde{J}^{2n-3}(|\tilde{A}_{3,2}|). \]

The calculation is complete.

**Calculation 5.4.4.** Our final calculation is trivial, but we include it for completeness. Figure 5.14 gives the calculation of the clipped order ideal poset of \(\tilde{A}_{n,1}\) for \(n \geq 2\). Because this poset is a chain isomorphic to \(\mathbb{Z}\), there are no nontrivial sets of incomparable elements, so each ideal is principal. Thus, applying \(\tilde{J}\) just returns another poset isomorphic to \(\mathbb{Z}\).

So we have
\[ |\tilde{A}_{n+1,1}| \cong \tilde{J}(|\tilde{A}_{n,1}|). \]

In fact, we know that \(|\tilde{A}_1|\) and each \(|\tilde{C}_n|\) are also chains isomorphic to \(\mathbb{Z}\), so we trivially have that
\[ |\tilde{A}_{2,1}| \cong \tilde{J}(|\tilde{A}_1|) \]
and
\[ |\tilde{C}_n| \cong \tilde{J}(|\tilde{A}_{2n-2,1}|). \]

In each of these cases, we have chosen the indices to match up with those in the minuscule poset setting. The calculation is complete.

**Remark 5.4.5.** The underlying posets of the \(n-1\)st and \(n\)th full heaps over \(\tilde{A}_n\) are isomorphic to those under the second and first full heaps over \(\tilde{A}_n\), respectively. Thus, if we wished, we
Figure 5.14: The first full heap over $\tilde{A}_n$ and its order ideal poset, which is isomorphic to $\mathcal{I}(\tilde{A}_{n+1})$. \\

Clipped order ideal lattice
could include these in our calculations involving $|\tilde{A}_{n,1}|$ and $|\tilde{A}_{n,2}|$. We do not for simplicity of exposition, as there is little interesting in this observation.

Additionally, the reader may be curious about $\tilde{J}(|\tilde{A}_{n,k}|)$ when $2 < k < n - 1$. We do not present this clipped order ideal poset here, because there are antichains of length at least three. (In the language of Theorem 4.6.2, these antichains are made up of elements taken from distinct chains on which the full heap ascends at every element.) By Lemma 5.1.9, there is then an element of $\tilde{J}(|\tilde{A}_{n,k}|)$ that covers three elements. Therefore, this clipped order ideal poset cannot be the underlying poset of a full heap, by Lemma 3.4.2(2).

5.5 An ADE classification in full heaps

We now conclude by summarizing the results of the two previous sections 5.3 and 5.4 in a theorem. We continue the notation from these sections.

**Theorem 5.5.1.** The underlying posets of full heaps are related to each other via the clipped order ideal poset operator $\tilde{J}$ as shown in Figure 5.15. Among these relations is an ADE classification in full heaps.

**Proof.** First, we record the extraneous relationships via $\tilde{J}$ among full heaps. The maps given between $|\tilde{A}_{n,1}|$, $|\tilde{A}_{n+1,1}|$ and $|\tilde{C}_n|$ come from Calculation 5.4.4. The maps given between $|\tilde{A}_{n,2}|$, $|\tilde{B}_n|$ and $|\tilde{D}_{n+1,s}|$ come from Calculation 5.4.1. The maps given between $|\tilde{D}_{n,d}|$, $|\tilde{D}_{n+1,d}|$ and $|\tilde{A}_{2n-1}|$ come from Calculation 5.4.2.

Now we record the ADE classification in full heaps. If we restrict ourselves to the simply laced affine Dynkin diagrams $\tilde{A}_n$ for $n \geq 2$, $\tilde{D}_n$, $\tilde{E}_6$ and $\tilde{E}_7$, we move between the posets underlying the full heaps of these Dynkin diagrams using $\tilde{J}$ and $\tilde{J}^{-1}$. No two full heaps are the same, even when viewed as posets, for different simply laced affine Dynkin diagrams, so we have also distinguished these Dynkin diagrams in this process, give a complete and unambiguous listing of them. Thus, we have an ADE classification.
Figure 5.15: Full heap poset structures of affine Dynkin diagrams using the underlying posets of full heaps over type \( \tilde{A}_n \) and the \( \tilde{J} \) operator. Each arrow here indicates an application of \( \tilde{J} \). The dashed box is the backbone of the \( ADE \) classification, as it contains all three types. See the text for a description of the notation.
Of particular note is the dashed box in Figure 5.15, in which $\tilde{J}$ obtains at least one instance of all three general types in an $ADE$ classification. These maps are shown in Calculations 5.3.1, 5.3.2 and 5.3.3, given respective to the order in the figure.
Bibliography


Appendix A

Dynkin Diagrams

The following figures A.1, A.2, A.3 and A.4 present all finite and affine Dynkin diagrams. The modifier “finite” here refers to the dimension of the Lie algebras type those Dynkin diagrams describe, a notion outside the scope of this thesis. (In particular, “finite” does not refer the number of vertices the Dynkin diagram has.) “Affine” refers to infinite dimensional extensions of those Lie algebras. The terms “untwisted” and “twisted” also come from the underlying Lie algebra theory. Here, twisted affine Dynkin diagrams can be obtained by “folding”, i.e., identifying certain vertices of, other affine Dynkin diagrams.

Recall that we follow Carter’s version of Kac’s naming conventions for Dynkin diagrams [2]. In [13], Kac uses a similar notation, the only differences being the deletion of tildes and left superscripts, an additional right superscript of (1) in the untwisted case and (2) in the twisted case. For example, $\tilde{A}_2$ is written as $A_2^{(2)}$ in [13].
Figure A.1: A list of the finite Dynkin diagrams.

<table>
<thead>
<tr>
<th>Kac name</th>
<th>Dynkin diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_n \ (n \geq 1)$</td>
<td><img src="image" alt="A_n diagram" /></td>
</tr>
<tr>
<td>$B_n \ (n \geq 2)$</td>
<td><img src="image" alt="B_n diagram" /></td>
</tr>
<tr>
<td>$C_n \ (n \geq 3)$</td>
<td><img src="image" alt="C_n diagram" /></td>
</tr>
<tr>
<td>$D_n \ (n \geq 4)$</td>
<td><img src="image" alt="D_n diagram" /></td>
</tr>
<tr>
<td>$E_6$</td>
<td><img src="image" alt="E_6 diagram" /></td>
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<tr>
<td>$E_7$</td>
<td><img src="image" alt="E_7 diagram" /></td>
</tr>
<tr>
<td>$E_8$</td>
<td><img src="image" alt="E_8 diagram" /></td>
</tr>
<tr>
<td>$F_4$</td>
<td><img src="image" alt="F_4 diagram" /></td>
</tr>
<tr>
<td>$G_2$</td>
<td><img src="image" alt="G_2 diagram" /></td>
</tr>
</tbody>
</table>
Figure A.2: A list of the untwisted affine Dynkin diagrams, part I.

<table>
<thead>
<tr>
<th>Kac name</th>
<th>Dynkin diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \tilde{A}_1 )</td>
<td>[Diagram of ( \tilde{A}_1 )]</td>
</tr>
<tr>
<td>( \tilde{A}_n ) (( n \geq 2 ))</td>
<td>[Diagram of ( \tilde{A}_n ) for ( n \geq 2 )]</td>
</tr>
<tr>
<td>( \tilde{B}_n ) (( n \geq 3 ))</td>
<td>[Diagram of ( \tilde{B}_n ) for ( n \geq 3 )]</td>
</tr>
<tr>
<td>( \tilde{C}_n ) (( n \geq 2 ))</td>
<td>[Diagram of ( \tilde{C}_n ) for ( n \geq 2 )]</td>
</tr>
<tr>
<td>( \tilde{D}_n ) (( n \geq 4 ))</td>
<td>[Diagram of ( \tilde{D}_n ) for ( n \geq 4 )]</td>
</tr>
</tbody>
</table>
Figure A.3: A list of the untwisted affine Dynkin diagrams, part II.

<table>
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<th>Kac name</th>
<th>Dynkin diagram</th>
</tr>
</thead>
<tbody>
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<td><img src="E6.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$\tilde{E}_7$</td>
<td><img src="E7.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$\tilde{E}_8$</td>
<td><img src="E8.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$\tilde{F}_4$</td>
<td><img src="F4.png" alt="Diagram" /></td>
</tr>
<tr>
<td>$\tilde{G}_2$</td>
<td><img src="G2.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>
Figure A.4: A list of the twisted affine Dynkin diagrams.

<table>
<thead>
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<th>Kac name</th>
<th>Dynkin diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{A}_2$</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>$\tilde{A}_{2n}$ ($n \geq 2$)</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
<tr>
<td>$\tilde{A}_{2n-1}$ ($n \geq 3$)</td>
<td><img src="image3" alt="Diagram" /></td>
</tr>
<tr>
<td>$\tilde{D}_{n+1}$ ($n \geq 2$)</td>
<td><img src="image4" alt="Diagram" /></td>
</tr>
<tr>
<td>$\tilde{E}_6$</td>
<td><img src="image5" alt="Diagram" /></td>
</tr>
</tbody>
</table>
Appendix B

Atlas of Minuscule Posets

The following is an atlas of all minuscule posets, taken from [23]. Included with each minuscule poset is an example of a minuscule heap whose minuscule poset is the underlying poset. The labeling of the minuscule heaps comes from its underlying Dynkin diagram, which can be found in Appendix A. For the infinite families $A_n$, $B_n$, $C_n$ and $D_n$, we use the Dynkin diagram with five vertices as our example for the minuscule heaps displayed. It is not difficult to extrapolate the form for all $n$.

Up to relabeling of vertices, these minuscule heaps (and their extrapolations to different $n$) are the only minuscule heaps with a minuscule poset underlying them. The only exception is in the case of $E_6$ where the dual of the minuscule heap shown has the same underlying poset; this is not explicitly shown, but we do note it in the description. As always, elements of minuscule heaps are indicated by their label, taken from the underlying Dynkin diagram, together with a subscript to distinguish elements with the same label.

Also included with each minuscule poset is the chain, product of chains or order ideal poset of products of chains that the poset is isomorphic too. We include these to aid in the discussion in Chapter 5. We emphasize that this atlas is complete; no other connected, finite Dynkin diagrams have minuscule posets. In particular, the other finite Dynkin diagrams of $G_2$, $F_4$ and $E_8$ do not have minuscule posets.
Figure B.1: (a) The general form of the $n$ minuscule posets for type $A_n$. Each is a product of finite chains of the form $[j] \times [n+1-j]$ for $1 \leq j \leq n$. (b) The five minuscule heaps over the Dynkin diagram $A_5$ whose underlying posets are the five minuscule posets of type $A_5$. Below each is the underlying poset written as a product of chains.

\[ [j] \times [n+1-j] \]
\[ (1 \leq j \leq n) \]
Figure B.2: (a) The general form of the one minuscule poset for type $B_n$. It is isomorphic to the order ideal poset of $[2] \times [n - 1]$. (b) The minuscule heap over the Dynkin diagram $B_5$ whose underlying poset is the minuscule poset of type $B_5$. 

\[ J([2] \times [n - 1]) \]
Figure B.3: (a) The general form of the one minuscule poset for type $C_n$. This minuscule poset is a chain of length $2n - 1$. (b) The minuscule heap over the Dynkin diagram $C_5$ whose underlying poset is the minuscule poset of type $C_5$. 
Figure B.4: (a) The general form of the minuscule poset for the defining representation of type $D_n$. This minuscule poset is isomorphic to the poset $J^{n-3}([2] \times [2])$. (b) The minuscule heap over the Dynkin diagram $D_5$ whose underlying poset is the minuscule poset for the defining representation of type $D_5$. 
Figure B.5: (a) There are two spin representations in type $D_n$, and therefore two minuscule posets. However, as posets, they are isomorphic to each other and to the order ideal poset of $[2] \times [n - 2]$. (b) The minuscule heaps over the Dynkin diagram $D_5$ whose respective underlying posets are the two isomorphic minuscule posets of the two spin representations of type $D_5$. 

\[ J([2] \times [n - 2]) \]
Figure B.6: (a) The one minuscule poset for type $E_6$. It is isomorphic to $J^2([2] \times [3])$. (b) One minuscule heap over the Dynkin diagram $E_6$ whose underlying poset is the minuscule poset of type $E_6$. Note in this case that the dual of this minuscule heap is isomorphic to it as a poset, but not as a minuscule heap, since the labels change. This is consistent with the fact that there are two representations of $E_6$: the fundamental representation and its dual. There are thus two minuscule weights in $E_6$ which is expected due to the symmetry of the underlying Dynkin diagram.
Figure B.7: (a) The one minuscule poset for type $E_7$. It is isomorphic to $J^3([2] \times [3])$. (b) One minuscule heap over the Dynkin diagram $E_7$ whose underlying poset is the minuscule poset of type $E_7$. Note in this case that the dual of this minuscule heap is isomorphic to itself as a heap, unlike the $E_6$ case. This is consistent with the fact that there is only the fundamental representation of $E_7$, which is equivalent to its dual.
Appendix C

Atlas of Full Heaps

The following is an atlas of all full heaps over finite Dynkin diagrams. The completeness of this atlas is shown in Theorem 4.7.1. The labels for these full heaps are taken from the vertices of the relevant Dynkin diagram given in Appendix A. Elements of minuscule heaps are indicated by their label together with a subscript to distinguish elements with the same label. Each full heap consists of a repeating motif, one iteration of which is highlighted in each diagram, bounded by dotted lines for emphasis. Several iterations of the motif are shown from which the reader can infer the entire structure, as the motif repeats infinitely above and below the portion shown.
Figure C.1: Full heap over $\tilde{A}_1$
Figure C.2: The first full heap over $\tilde{A}_n$. There are $n$ full heaps over $\tilde{A}_n$; we distinguish them by the label of the least element in the repeating motif when the element labeled 0 is the greatest in the motif.
Figure C.3: The $k$th full heap over $\tilde{A}_n$ for $1 < k < n$. There are $n$ full heaps over $\tilde{A}_n$; we
distinguish them by the label of the least element in the repeating motif when the element
labeled 0 is the greatest in the motif. Figures C.2 and C.4 show the cases for $k = 1$ and
$k = n$, respectively. See Figure 5.6 for an example when $n = 4$ and $k = 2$. 
Figure C.4: The \( n \)th full heap over \( \tilde{A}_n \). There are \( n \) full heaps over \( \tilde{A}_n \); we distinguish them by the label of the least element in the repeating motif when the element labeled 0 is the greatest in the motif.
Figure C.5: Full heap over $\tilde{B}_n$. 
Figure C.6: Full heap over $\tilde{A}_{2n-1}$. 
Figure C.7: Full heap over $\tilde{C}_n$. 
Figure C.8: Full heap over $2\tilde{D}_{n+1}$.
Figure C.9: Full heap over $\tilde{D}_n$ in the defining representation.
Figure C.10: Full heaps over $\tilde{D}_n$ in the spin representations. There are two such full heaps, one for each spin representation; both are given below.

```
  2\_7   \_   \_   n\_3
    \_   3\_7   n - 2\_7
      \_   2\_6   n - 1\_3
        \_   0\_3   3\_6   n - 2\_6
          \_   2\_5   n\_2
            \_   1\_2   3\_5   n - 2\_5
              \_   2\_4   n - 1\_2
                \_   0\_2   3\_4   n - 2\_4
                  \_   2\_3   n\_1
                    \_   1\_1   3\_3   n - 2\_3
                      \_   2\_2   n - 1\_1
                        \_   0\_1   3\_2   n - 2\_2
                          \_   2\_1   n\_0
                            \_   1\_0   3\_1   n - 2\_1
                              \_   2\_0   n - 1\_0
                                \_   0\_0   3\_0   n - 2\_0
                                  \_   2\_-1   n\_-1
          0\_3   3\_7   n - 2\_7
    \_   2\_6   n - 1\_3
      \_   1\_3   3\_6   n - 2\_6
        \_   2\_5   n\_2
          \_   1\_2   3\_5   n - 2\_5
            \_   0\_2   3\_4   n - 2\_4
              \_   2\_3   n\_1
                \_   1\_1   3\_3   n - 2\_3
                  \_   2\_2   n - 1\_1
                    \_   0\_1   3\_2   n - 2\_2
                      \_   2\_1   n\_0
                        \_   1\_0   3\_1   n - 2\_1
                          \_   2\_0   n - 1\_0
                            \_   0\_0   3\_0   n - 2\_0
                              \_   2\_-1   n\_-1
```
Figure C.11: Full heaps over $\tilde{E}_6$. Here the repeating motif appears to twist; this is just an artifact of displaying the heap in two dimensions. Note that the two heaps differ only in their labels and, in fact, are dual to each other as heaps, as in Figure B.6.
Figure C.12: Full heap over $\tilde{E}_7$. 