Implementation of Guiding Center Rotation Drifts in Simulations of Tokamak Plasmas with Large Flows

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Abstract

Tokamak magnetic confinement experiments are the most researched prospect for controlled thermonuclear fusion energy production. The quasi-toroidal geometry along with the large, twisting toroidal magnetic field pose great engineering and physics challenges. Plasma physicists use large-scale simulations of Tokamak plasmas in order to further understand their physics and inform experimental design to strive towards the goal of reliable fusion energy [20]. One type of simulation uses a gyrokinetic formulation to study plasmas in strong magnetic fields. A fundamental concept in gyrokinetic plasma physics is that of the guiding center drifts, which influence the motion of the guiding center of a gyrating charged particle in electric and magnetic fields. Previously, the significant toroidal and poloidal equilibrium flows of a tokamak plasma were not fully included in simulation models. Experiments have suggested that these flows have noticeable effects on tokamak physics [5][17]. In this thesis, we discuss the formulation of a gyrokinetic model that includes these flows by shifting velocity space to the frame moving with the plasma flow [18]. Many of the previous calculations can be used with relatively small modifications of the standard lab frame model. However, in this new frame, the centrifugal and Coriolis forces, which are well established physics concepts in rotating reference frames, manifest themselves as new drift terms in the drift gyrokinetic equation [2].

In this thesis, we will step through the calculation of these terms in cylindrical coordinates starting with the Sugama-Horton model for drift velocity [18] to obtain the established result from the literature. We will also use the guiding center Hamiltonian formulation [15] in a tokamak plasma by explaining the previous work on this topic from a few references in detail to obtain an equivalent result [17][9]. The original work of this thesis is the implementation of the new drift terms in the simulation’s magnetic field-following coordinate system in a usable way for the purposes of large-scale tokamak simulations. We will examine the effect of the equilibrium flow by visualizing results for the simple test case of a linear eigenmode in a tokamak. We find that the fundamental structure of the mode is unchanged, but the $E \times B$ drift connected to the flow results in a tilt of the poloidal mode structure in accordance with our expectations. Finally, future work using the gyrokinetic model that includes large equilibrium flows is discussed.
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Chapter 1

Introduction

In this thesis, we step through the derivation of guiding center rotational drift terms and implement them in a large-scale tokamak fusion plasma simulation. These terms, called the Coriolis and centrifugal drifts in accordance with classical mechanics, arise when switching to the rotating frame of a tokamak plasma, and can be important for large plasma flows such as those in a tokamak. This change of frame is required to include the mean plasma flow and therefore develop a more accurate simulation model of electric fields in tokamak plasmas.

This thesis is outlined as follows. In the first chapter, we introduce thermonuclear fusion as a promising means of energy production and the role of plasma physics in this research to motivate our work and the use of numerical simulations in fusion plasma research. The second chapter includes a short introduction to the plasma physics required to understand the work in this thesis, namely gyrokinetic theory and drift kinetics. In the third chapter, we describe the overarching goal of building a more comprehensive model of the radial electric field in tokamaks using the comprehensive procedures of Sugama and Horton to include the equilibrium plasma flows in the gyrokinetic model. In the fourth chapter, we explain the derivation of the new rotation terms in the tokamak magnetic geometry from the Sugama-Horton gyrokinetic model of plasmas with large flows, and separately using the Hamiltonian formalism. Both of these calculations were accomplished by stepping through the work of previous authors. The original work of this thesis is the decomposition of the terms componentwise in the simulation’s magnetic field-following coordinates.
In the fifth chapter we discuss visualizations of the simulation results including flows on a test case of a linear eigenmode in a tokamak.

1.1 Fusion and Plasmas

1.1.1 Fusion Power

Nuclear fusion is the process by which nuclei collide and fuse to form a new nucleus. In general, the fusion of light nuclei converts a fraction of the mass to energy, while the fusion of heavier nuclei requires energy [8]. In practice, at energy scales currently accessible to laboratories on Earth, only the lightest nuclei such as Hydrogen or its isotopes (Deuterium and Tritium) are realistic reactants for fusion [8][3]. The release of energy in the reaction makes it an attractive prospect for energy production. However, creating controlled nuclear fusion reactions for energy production poses several difficult engineering and scientific problems. Researchers have been involved in solving these problems since the beginning of the 1950’s, when the idea was first proposed [8].

In laboratory fusion, the reactants are typically Deuterium and Tritium (D-T) [3]. The D-T reaction produces a Hydrogen nucleus, 14 MeV of energy, and an extra neutron, which in the case of laboratory fusion causes irradiation of the vessel walls. Furthermore, Tritium is radioactive with a half-life of about 12 years [8][3][20].

1.1.2 Advantages of Fusion

Laboratory research on controlled nuclear fusion for the purpose energy production began in the 1950’s. Controlled fusion as a means of energy production is desirable due to the small amount of relatively safe waste [8]. While it is not a totally clean reaction due to the irradiation of the reactor walls from the energetic neutron released in the reaction, the radioactivity of fusion is orders of magnitude safer than that of fission [3][8]. Furthermore, fusion requires only Deuterium and Tritium as fuel. Deuterium may be available in relatively large quantities on Earth, and
Tritium can be produced by the reactors themselves in many configurations [3]. The combination of a relatively abundant fuel and safe waste make it highly sought after as a sustainable energy producer. However, the positives of nuclear fusion as an energy source come at the cost of the difficulties in achieving fusion energy production.

1.1.3 Conditions for Fusion

Due to the Coulomb repulsion between nuclei of positive charge, fusion requires extremely high temperatures on the order of 100 million Kelvin in laboratory experiments for the nuclei to have enough kinetic energy to collide so that the strong nuclear force overcomes the Coulomb repulsion. At these temperatures, the reactants will be in the plasma state, in which the electrons are separated from the positively charged and significantly heavier nuclei [20]. The ionized nuclei are typically referred to as "ions" in plasma physics. Because the charged particles are separated, plasmas conduct electric and magnetic fields. Most fusion devices make use of these properties to confine the reactants, but plasmas can also self-assemble their own electric and magnetic fields, leading to complex behavior. Plasma physics is used to predict this behaviour in order to better confine the reactants in fusion [12][20].

Thus far, thermonuclear weapons have been the only man-made reaction to produce useful amounts of fusion energy. However, thermonuclear fusion reactions are undesirable for energy applications due to their uncontrollability [8]. Meanwhile, laboratory controlled fusion for use in civil energy production has not been fully realized.

1.2 Tokamaks

One of the most significant barriers facing fusion energy production is that the high temperatures required for fusion and the unpredictability of the plasma state make
confinement of the reactants difficult. The most popular and heavily researched fusion plasma confinement device is the tokamak, which uses strong magnetic fields to confine the plasma reactants [20].

A Tokamak confines the fusion plasma in a toroidal vessel. In principle, because the particles are charged, they tend to follow magnetic field lines. Therefore, a strong toroidal magnetic field created with external conducting coils is used to confine the plasma to the torus. However, charged particles may drift across magnetic field lines as well. Since the tokamak is toroidal, the magnetic field towards the center is stronger than at the edges. This gradient in magnetic field results in particles quickly drifting outwards; a toroidal field alone is not sufficient to confine the plasma. Research has shown that a relatively small poloidal magnetic field provides far more robust confinement, resulting in a twisting total magnetic field that is a hallmark of the tokamak. In a tokamak, the poloidal field is produced by allowing the plasma to carry an electric current in the toroidal direction [20].

1.2.1 Energy Losses and Difficulties

While the fusion of reactants produces energy, energy can also be lost to radiation and conduction of the plasma cloud [20][8][3]. In radiation, the plasma cloud loses energy in the form of light. In conduction, particles from the plasma collide with an external surface, for example the walls of the vessel, and lose kinetic energy. So far in laboratory fusion reactions, the power produced by fusion has not been able to overcome power losses due to conduction and radiation. Laboratory plasmas require external heating in order to maintain the temperature of the plasma against energy losses. Thus, more energy is put in to maintain fusion reactions than is produced by the fusion process [20]. In order to maintain self-sustaining fusion, the fusion must produce enough energy to maintain a high enough temperature against losses to maintain the reaction without significant external energy input; this is referred to as ignition [20]. A more stable tokamak plasma with better confinement will ultimately lose less energy to conduction, and will therefore achieve higher quality
The self-organizational properties of a plasma can result in long-range order or chaotic behaviour in particles and electric and magnetic fields. This results in a variety of activities including turbulence, energy transport, and waves [20]. This thesis will not discuss in-depth the classes of plasma behavior; rather, this discussion serves to provide context for the use of plasma physics as a tool for predicting the behavior of tokamak plasmas to ultimately better control them.

In particular, instabilities are growths of small perturbations that drive the plasma away from equilibrium, thereby degrading confinement, and resulting in energy transport and energy losses to conduction. These instabilities may also result in reduced confinement time, and therefore lower energy output. Plasma disruptions can also potentially damage the vessel and cause significant and costly damage to the experiments [20][3].

Fusion plasma stability and confinement is required for tokamak fusion to occur in equilibrium; therefore, an understanding of the plasma physics involved is vital. Plasma physicists are employed to predict the behavior of tokamak plasmas in order to inform experimental design for the ultimate goal of improved confinement and higher quality fusion [20]. The inherent complexity of tokamak plasmas makes a fully analytical description of the physical behaviour intractable. A useful tool in analyzing and predicting the physics of plasmas in a Tokamak is numerical simulation.

1.2.2 Current State of Tokamaks

While achieving stable fusion has proven to be difficult, significant progress has been made, and commercial fusion energy production seems more plausible every day due to relevant research in plasma physics and nuclear fusion.

ITER, which is an international experimental tokamak collaboration being constructed in France, is expected to achieve ignition for about 1000 seconds in 2027, producing 500 MW of fusion power. It has a proposed cost of $14 billion USD [13].
Importantly, it is expected to produce 10 times more fusion energy than is required to heat the reactants to fusion temperatures [7].
Chapter 2

Gyrokinetic Plasma Physics

Here, we will briefly discuss gyrokinetic theory and drift kinetic theory of plasmas, which is often used to model and simulate tokamak plasmas [10][11], and is relevant to the work in this thesis. Since this theory is often used in plasmas with strong magnetic fields, the motion of particles is discussed in relation to the magnetic field. Therefore, for the rest of this work we will use the following conventions: ”Parallel” and ”perpendicular” refer to a vector’s direction in relation to the magnetic field vector. $m$ is the mass of the particle. $e$ is the charge. $B$ is the magnetic field, and $\mathbf{b} \equiv \frac{\mathbf{B}}{|\mathbf{B}|}$ is the unit magnetic field vector.

2.1 Formulation of Gyrokinetic Theory in a Magnetized Plasma

2.1.1 Guiding center distribution

Charged particles in a magnetic field will gyrate around the magnetic field lines [12]. The gyroradius and gyrofrequency of a particle in a magnetic field are given by

\[
\begin{align*}
    r_g &= \frac{m v_\perp}{|e| B} \\
    w_g &= \frac{|e| B}{m}
\end{align*}
\]
The guiding center distribution function characterizes a plasma with strong magnetic fields where the gyroradius is much smaller than the characteristic length scales, as is the case in a tokamak [20][10][11].

\[ \delta = \frac{r_g}{L} \ll 1 \]

The total energy,

\[ \varepsilon = \frac{1}{2}mv^2 + e\Phi \]

and the magnetic moment,

\[ \mu = \frac{mv^2}{2B} \]

are approximate constants of motion in a magnetized plasma [12]. Therefore, we consider a 6-dimensional distribution function depending on \( \varepsilon \) and \( \mu \), and the gyrophase \( \theta \), as well as the guiding center position of the particle \( \mathbf{X} \), which is the instantaneous center of gyration of a particle; it is given by a time-average over a gyroperiod. \( \mathbf{X} \) is also a solution to the guiding center equation, which describes the guiding center velocity [12].

\[ \frac{d\mathbf{X}}{dt} = \mathbf{v}_{GC} = b\mathbf{v}_\parallel + \mathbf{v}_d \quad (2.1) \]

\( \mathbf{v}_d \) is the drift velocity of the guiding center perpendicular to the magnetic field. The typical drift velocity equation is given by (2.2).

\[ \mathbf{v}_d = \frac{\mu}{eB} (\nabla \times \mathbf{B}) \cdot \mathbf{bb} + \frac{1}{eB} \times \mu \nabla B + \frac{mv^2}{eB} \mathbf{b} \times (\mathbf{b} \cdot \nabla \mathbf{b}) + \frac{\mathbf{b}}{B} \times \nabla \Phi \quad (2.2) \]

The first three terms are the parallel drift, gradient of B drift, and the curvature drift, respectively. The last term is the \( \mathbf{E} \times \mathbf{B} \) drift [12]. These are guiding center drift terms that are commonly used in gyrokinetic plasma physics [12].
2.1.2 Drift-Kinetic Equation

We can write a kinetic equation [2] in terms of the guiding center variables, and the 6-dimensional distribution function $f$ [12].

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{d\varepsilon}{dt} \frac{\partial f}{\partial \varepsilon} + \frac{d\theta}{dt} \frac{\partial f}{\partial \theta} = C(f)$$

$C(f)$ is a collision operator. Since the gyromotion is much faster than the drift motion in this formulation, we average over the gyrophase $\theta$ in this equation to obtain the drift-kinetic equation, which describes the guiding center motion with the 5-dimensional distribution function after averaging over the gyrophase coordinate [12].

$$\frac{\partial f}{\partial t} + \mathbf{v}_{GC} \cdot \nabla f + \dot{\varepsilon}_{GC} \frac{\partial f}{\partial \varepsilon} = C \quad (2.3)$$

where

$$\dot{\varepsilon}_{GC} \equiv -\varepsilon \left( \frac{\partial \Phi}{\partial t} + \mathbf{v}_{GC} \cdot \frac{\partial \mathbf{A}}{\partial t} \right) - \mu \frac{\partial B}{\partial t}$$

Is the guiding center rate of energy change [12]. $f(X, \mu, \varepsilon, t)$ is the 5-dimensional guiding center distribution function at a certain time.

Guiding center gyrokinetics describes the motion of charged particles whose drift motion time scale is smaller than that of the gyromotion. Therefore, we average the motion over a gyroperiod to obtain the drift-kinetic equation that describes parallel and perpendicular drift velocities of guiding centers of particles. Simulations can use (2.3), along with Maxwell’s equations, to solve for the motion of charged particles in a tokamak geometry.
Chapter 3

Including Flows in the Co-moving Frame

Previously, the modeling of the electric fields in our drift-kinetic simulations was incomplete since toroidal and poloidal flows were not fully included. Experiments and recent research have suggested that the flow can be important [1][17][5]. In order to more accurately model the radial electric field, we must include the large equilibrium flows in our gyrokinetic simulation model. This can be done following the procedures of the Sugama-Horton model [18], which comprehensively describes electromagnetic gyrokinetic drift equations and drift velocities for plasmas with large flows in general geometry. Therefore, our main goal is to use the Sugama-Horton model to implement the flow in our simulations to more accurately describe the radial electric fields in tokamak plasma simulations.

3.1 The Co-moving Frame

The inclusion of flows on the order of the thermal velocity is problematic. The electric field associated with this large flow would be on the order of the thermal velocity, and hence the gyromotion. This breaks the standard hierarchy of gyrokinetics in which the drift velocity is much smaller than the thermal velocity. Furthermore, including the flow as a combination of the standard drift terms would be difficult or
impossible. This problem has been treated by several authors in drift-kinetic and gyrokinetic theory [18][17][9].

An elegant solution was proposed by Sugama-Horton, which shifts to the velocity frame moving with the equilibrium flow velocity, denoted by \( \mathbf{V}_0 \). According to the Sugama-Horton model, the new \( \mathbf{E} \) includes the component connected to the equilibrium flow.

\[
\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 = -\mathbf{V}_0 \times \mathbf{B} + \mathbf{E}_1
\]

We assume electrostatic fields \( \mathbf{E}_1 = -\nabla \Phi_1, \mathbf{E}_0 = -\nabla \Phi_0 \). \( \mathbf{E}_1 \) is the electric field resulting from the plasma density, and \( \mathbf{E}_0 \) is the electric field connected to the equilibrium flow [18].

The velocity in the locally co-moving frame is then given by \( \mathbf{v}' \equiv \mathbf{v} - \mathbf{V}_0 \), and \( \mu' \) and \( \varepsilon' \) can be calculated with \( \mathbf{v}', \Phi_1 \). In this frame, Sugama and Horton derive the electromagnetic gyrokinetic equations in general geometry along with the guiding center velocity for plasmas with strong flows using a recursive method for ballooning type fluctuations, which will not be outlined here (for more detail the reader is referred to Ref. [18]). The new drift-kinetic formulation is given by a straightforward replacement of \( \mathbf{v} \) with \( \mathbf{v}' \), and \( \Phi \) with \( \Phi_1 \) in the previous drift-kinetic equation (2.3), as well as a new guiding center velocity derived by Sugama-Horton.

\[
\mathbf{v}_{GC} = \mu' \parallel \mathbf{b} + \mathbf{V}_0 + \mathbf{v}_d
\]

\[
\mathbf{v}_d = \frac{\mu'}{eB} (\nabla \times \mathbf{B}) \cdot \mathbf{b} + \frac{1}{eB} \mathbf{b} \times [\mu'(\nabla B + m(\mathbf{v}' \parallel) B \cdot \nabla \mathbf{b} + e\nabla \Phi_1]
\]

\[
+ m\mathbf{V}_0 \cdot \nabla \mathbf{V}_0 + mv'_\parallel \mathbf{b} \cdot \nabla \mathbf{V}_0 + mv'_\parallel \mathbf{V}_0 \cdot \nabla \mathbf{b}]
\]

The first three terms of the drift velocity in (3.2) are the parallel drift, the \( \nabla B \) drift, and the curvature drift in the co-moving frame. The fourth term is the \( \mathbf{E} \times \mathbf{B} \) drift using \( \mathbf{E}_1 \). These are all included already in our simulation drift-kinetic model using the straightforward replacement of \( \mathbf{v} \) with \( \mathbf{v}' \), and the awareness that the electric field calculated from the plasma density is \( \Phi_1 \). We must also include \( \mathbf{V}_0 \) as an extra term in the guiding center velocity. However, the last three terms in the
drift velocity are due to the shift to the locally co-moving frame, and are therefore the centrifugal and Coriolis effects respectively in the rotating velocity frame [6][2]. These must be implemented as new drift terms in our simulation model, since they previously did not exist in the lab frame calculations.

3.2 Large Mean Flows in a Tokamak

Ultimately these rotation drifts may be significant when the plasma has a large equilibrium flow in the toroidal direction (large $V_0$) [6]. The Sugama-Horton model, for example, assumes toroidal flow on the order of the ion thermal velocity [18]. In a tokamak, the plasma often has a large toroidal flow, and therefore these effects may be important [20][1][16]. For example, Ref. [5] has studied the effects of these flows on kinetic ballooning modes in the DIII-D tokamak, and found they have a destabilizing effect on the growth rate especially when the flow velocity gradient is large, but do not significantly affect the real frequency. Since we so far have no way of knowing a priori in which cases the flow will have a relevant effect, it is necessary to implement the new model including the new rotational effects [5].

3.3 Axisymmetric Toroidal Configuration

We will use the axisymmetric toroidal configuration to model the tokamak’s geometry and magnetic field. In this configuration, the magnetic field is given in cylindrical geometry by (3.3) [20][17]. $I(\psi)$ is a flux function related to the poloidal plasma current. $\psi$ is a poloidal flux function related to the minor radius of a tokamak.

$$\mathbf{B} = \frac{I(\psi)}{R} \hat{\phi} + \nabla \phi \times \nabla \psi$$ (3.3)

The axisymmetric toroidal configuration is most easily described with cylindrical geometry. In this configuration, the lowest-order non-uniform equilibrium flow is purely toroidal [17][9][18], and is given by (3.4). Lowest-order indicates the lowest-
order term in an expansion of the flow about a small parameter, but can be interpreted as the dominant contribution to the equilibrium flow.

\[ \mathbf{V}_0 = u_t(\psi)\hat{\phi} \] (3.4)

Furthermore, this lowest-order equilibrium flow is incompressible \[17\][9],

\[ \nabla \cdot \mathbf{V}_0 = 0, \]

and is assumed to be a rigid body rotation at each flux function value \[18\][17][6].

\[ \mathbf{V}_0 = R\Omega(\psi)\hat{\phi} \]

\(\Omega(\psi)\) is the rotational frequency \[6\]. It is important to note that the rotational frequency, and thus the equilibrium flow, is not uniform in space. Hence, when we speak of shifting to the co-moving velocity frame, this is done \textit{locally}. Therefore, all of the following work will be performed at a given \(\psi\), and therefore at a given minor radius. In simulations, for example, there will be a radial profile that provides the equilibrium toroidal flow on a radial grid, and the simulation will shift to the locally co-moving velocity frame at each radius.
Chapter 4

Rotation Drift Effects in the Co-moving Frame

4.1 Derivation from Sugama-Horton Drift Equation

We will step through the derivation of the general geometry expressions for the Coriolis and centrifugal effects on the guiding-center drift velocity using the Sugama-Horton expression for guiding center drift velocity (4.1). The last two terms are the Coriolis drift, and the third to last term is the centrifugal drift. These are new drift terms that appear due to the shift to the locally moving frame, and must be implemented as new terms in the drift-kinetic simulation model. The form of these terms in the axisymmetric configuration has been derived previously by Refs. [17][5][18], and therefore the results of this chapter are not original, but we will explain how they are obtained.

\[
v_d = \frac{\mu}{eB}(\nabla \times B) \cdot b + \frac{1}{eB}b \times [\mu \nabla B + m(v'^2) b \cdot \nabla b] + e\nabla \Phi_1 + mV_0 \cdot \nabla V_0 + mv'_b \cdot V_0 + mv'_b V_0 \cdot \nabla b
\] (4.1)

Here, we explain and step through the reduction of the general magnetic geometry result of the Coriolis and centrifugal drift velocities from the Sugama-Horton drift-
kinetic formulation (4.1) to the form in the axisymmetric toroidal configuration used in this work [18]. We do this in cylindrical coordinates where $R$ is the major radius, $\phi$ is the toroidal angle, and $Z$ is the vertical direction orthogonal to $R$ and $\phi$. $\psi(r)$ is the poloidal flux function related to the minor radius of the tokamak.

It is first useful to derive an expression for the antisymmetric tensor $\nabla(R\hat{\phi})$ [19], which is equal to $\nabla R\hat{\phi} + R\nabla\hat{\phi}$ by application of chain rule. In cylindrical geometry, the second term is [2]

$$R\nabla\hat{\phi} = R\left(\hat{R}\frac{\partial}{\partial R}\hat{\phi} + \hat{\phi}\frac{\partial}{\partial \phi}\hat{\phi} + \hat{Z}\frac{\partial}{\partial Z}\hat{\phi}\right)$$

We substitute the expression $\hat{\phi} = -\sin(\phi)x + \cos(\phi)y$ for $\hat{\phi}$ in Cartesian coordinates [6]. The $\hat{R}$ and $\hat{Z}$ components both vanish, and we are left with the $\hat{\phi}$ component.

$$R\nabla\hat{\phi} = \hat{\phi}(-\cos(\phi)x - \sin(\phi)y)$$

$$= \hat{\phi}(-\hat{R})$$

$$= -\hat{\phi}\nabla R$$

Therefore, recalling that $\hat{\phi} = R\nabla\phi$, a useful form of this antisymmetric dyad [19] is

$$\nabla(R\hat{\phi}) = \nabla R\hat{\phi} - \hat{\phi}\nabla R$$

(4.2)

4.1.1 Centrifugal component

The centrifugal component of the guiding center drift [third to last term in (4.1)] is given by

$$\mathbf{v}_{CF} = \frac{1}{eB}\mathbf{b} \times [m\mathbf{V}_0 \cdot \nabla \mathbf{V}_0]$$

(4.3)

where $\mathbf{V}_0 = V_0\hat{\phi} = \Omega(\psi)R\hat{\phi}$ is the toroidal flow. $\Omega(\psi)$ is the rigid body rotational frequency [17][2]. The velocity gradient is given by application of chain rule [6][2].

$$\nabla \mathbf{V}_0 = \frac{\partial \Omega}{\partial \psi} R^2 \nabla \psi \nabla \phi + \Omega \nabla (R\hat{\phi})$$

(4.4)
Substituting (4.2),

$$\nabla V_0 = \frac{\partial \Omega}{\partial \psi} R^2 \nabla \psi \nabla \phi + \Omega R (\nabla R \nabla \phi - \nabla \phi \nabla R) \quad (4.5)$$

Finally, substituting (4.5) into the original form of the centrifugal drift velocity (4.4) provides the final expression for centrifugal drift. Recall $\nabla \phi \cdot \nabla \psi = \nabla \phi \cdot \nabla R = 0$, and $|\nabla \phi|^2 = \frac{1}{R^2}$ in cylindrical coordinates [6]. Therefore, the first two terms in (4.6) both vanish, and the expression for centrifugal drift can be simplified.

$$v_{CF} = \frac{1}{eB} b \times \{ m \Omega R^2 \nabla \phi \cdot [\frac{\partial \Omega}{\partial \psi} R^2 \nabla \psi \nabla \phi + \Omega R (\nabla R \nabla \phi - \nabla \phi \nabla R)] \}$$

$$= \frac{1}{eB} b \times (-m \Omega^2 R^3 |\nabla \phi|^2 \nabla R) \quad (4.6)$$

$$= -\frac{mR\Omega^2}{eB} b \times \nabla R$$

### 4.1.2 Coriolis component

The Coriolis component of the guiding center drift [last two terms in (4.1)] is

$$v_{CO} = \frac{b}{eB} \times \left[ mv'_{\parallel} b \cdot \nabla V_0 + mv'_{\parallel} V_0 \cdot \nabla b \right] \quad (4.7)$$

The first term in (4.7) can be simplified using (4.8).

$$b \cdot \nabla V_0 = V_0 \cdot \nabla b \quad (4.8)$$
To show the equality (4.8), we begin by computing \( \mathbf{B} \times \mathbf{V}_0 \) using the analytic expressions for each. Again, we use the fact that \( \nabla \phi \cdot \nabla \psi = 0 \), and \(|\nabla \phi|^2 = \frac{1}{R^2} \).

\[
\mathbf{B} \times \mathbf{V}_0 = [I(\psi)\nabla \phi + (\nabla \phi \times \nabla \psi)] \times R^2 \Omega \nabla \phi
\]

\[
= I(\psi) R^2 \Omega (\nabla \phi \times \nabla \phi) + R^2 \Omega (\nabla \phi \times \nabla \psi) \times \nabla \phi
\]

\[
= -R^2 \Omega [\nabla \phi \times (\nabla \phi \times \nabla \psi)]
\]

\[
= -R^2 \Omega [(\nabla \phi \cdot \nabla \phi) \nabla \phi - (\nabla \phi \cdot \nabla \phi) \nabla \psi]
\]

\[
= R^2 \Omega |\nabla \phi|^2 \nabla \psi
\]

\[
= \Omega \nabla \psi
\]

In order to obtain (4.8), we take the curl of the above expression, finding that it is equal to zero. For this, we use the fact that the curl of a gradient is zero [2].

\[
\nabla \times (\mathbf{B} \times \mathbf{V}_0) = \nabla \times (\Omega \nabla \psi)
\]

\[
= \nabla \Omega \times \nabla \psi + \Omega \nabla \times \nabla \psi
\]

\[
= \frac{\partial \Omega}{\partial \psi} \nabla \psi \times \nabla \psi + \Omega \nabla \times \nabla \psi = 0
\]

We then equate this to the form below (4.9) derived using vector calculus identities, using the fact that the lowest-order flow is incompressible (\( \nabla \cdot \mathbf{V}_0 \)), and Maxwell’s equations (\( \nabla \cdot \mathbf{B} = 0 \)).

\[
\nabla \times (\mathbf{B} \times \mathbf{V}_0) = \mathbf{B}(\nabla \cdot \mathbf{V}_0) - \mathbf{V}_0(\nabla \cdot \mathbf{B}) + (\mathbf{V}_0 \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{V}_0
\]

\[
0 = (\mathbf{V}_0 \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{V}_0
\]

\[
\mathbf{V}_0 \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{V}_0
\]

It remains to be shown that \( \mathbf{V}_0 \cdot \nabla \mathbf{B} = B \mathbf{V}_0 \cdot \nabla \mathbf{b} \), which may be accomplished by applying chain rule to the left-hand side of the last expression in (4.9), and using
the fact that $\nabla B$ has no component in the toroidal direction ($\nabla B \cdot V_0 = 0$) [20].

$$V_0 \cdot \nabla B = (V_0 \cdot \nabla B) b + BV_0 \cdot \nabla b$$

$$= BV_0 \cdot \nabla b$$

Substituting this expression into (4.9) leads to the desired result (4.8). We may then use (4.8) to rewrite the original expression for Coriolis drift (4.5).

$$v_{CO} = \frac{2mv'_\parallel}{eB} b \times \bcdot \nabla V_0$$ (4.10)

We then substitute our previously obtained expression for the velocity gradient (4.4), and use the fact that $\b \cdot \nabla \psi = 0$ [20][17], and the definition of rigid body rotational angular frequency, $\vec{\Omega} = (\nabla R \times \nabla \phi) R \Omega$ [2].

$$v_{CO} = \frac{2mv'_\parallel}{eB} b \times \left[ R^2 \frac{\partial \Omega}{\partial \psi} (\b \cdot \nabla \psi) (\nabla \phi) + \Omega R [(\b \cdot \nabla R) \nabla \phi - (\b \cdot \nabla \phi) \nabla R] \right]$$

$$= \frac{2mv'_\parallel}{eB} b \times \Omega R [b \times (\nabla \phi \times \nabla R)]$$

$$= \frac{2mv'_\parallel}{eB} b \times [\vec{\Omega} \times b]$$

Finally, we expand the above expression using vector identities to obtain the final result.

$$v_{CO} = \frac{2mv'_\parallel}{eB} \left[ (\b \cdot \b) \vec{\Omega} - (\b \cdot \vec{\Omega}) \b \right]$$

$$= \frac{2mv'_\parallel}{eB} (\vec{\Omega} - \Omega_\parallel \b)$$

$$= \frac{2mv'_\parallel}{eB} \vec{\Omega}_\perp$$ (4.11)

(4.6) for the centrifugal term and (4.11) for the Coriolis term are now in a suitable form to be implemented in the new gyrokinetic model as additional drift terms in the gyrokinetic drift equation, and these forms agree with those derived previously for the axisymmetric toroidal configuration [9][17][5].
4.2 Hamiltonian Formulation of First-order Drifts in the Co-moving Frame

In this section, we use the procedures of Refs. [9][15][17] to explicitly derive the form of the first-order guiding center drift terms in the axisymmetric toroidal configuration in the locally co-moving frame. This is done to ensure that Sugama-Horton correctly reduces to the toroidal configuration, as well as ensure that our final result for drift velocity is correct in the co-moving frame model (3.2). This calculation has also been previously done by Ref. [17][5], and we step through and explain their procedures here.

In gyrokinetic theory, the most convenient phase-space coordinates are \( z \equiv (X, \mu, \theta, v_\parallel) \), where \( X \) is the position of the guiding center, \( \mu = \frac{mv_\perp^2}{2B} \) is the lowest-order magnetic moment of the gyrating particle, \( v_\parallel \) is the parallel velocity, and \( \theta \) is the gyroangle [12]. However, these coordinates are not canonical, meaning we cannot simply write the Hamiltonian as a sum of kinetic and potential energy and apply the typical Hamilton’s equations to determine the equation of motion for \( X \) [15][2][6]. Therefore, in order to derive the equations of motion, we combine elements of previous work on this topic, as previous authors have created methods for studying non-canonical gyrokinetics, as well as the Hamiltonian formulation of gyrokinetic plasmas with large flows [17][9][15]. Here we describe and apply the non-canonical Poisson bracket method to the Hamiltonian in order to derive explicit guiding center drift equations of motion, borrowing and using similar techniques (although not totally identical) to the previous work.

We write the phase-space Lagrangian in the lab frame, defined by \( \gamma = \mathbf{p}_i \cdot \dot{\mathbf{q}}_i - h(z)dt \) [6][2][9], where \( \mathbf{p}, \mathbf{q} \) are the phase-space momentum and generalized coordinate respectively, and \( h(z) \) is the energy function [9][2]. Here, \( \mathbf{A} \) is the magnetic vector potential (\( \mathbf{B} = \nabla \times \mathbf{A} \)). All phase-space coordinates in this equation are lab frame coordinates, \( z \equiv (X, \mu, \theta, v_\parallel) \) [15][9]. This can be thought of as the differential
time element of the standard Lagrangian \([2][6]\).

\[
\gamma = (eA + mv) \cdot dX - (e\Phi + \frac{1}{2}mv^2)dt \tag{4.12}
\]

When we move to the frame locally co-moving with velocity \(V_0\), i.e. all phase-space coordinates transform to the moving frame coordinates, we take the transformations \((4.13)\) \([17]\), and we will use the convention that the new phase-space coordinates describing motion in the moving frame are the primed coordinates \(z' = (R', \mu', \theta', v'_\parallel)\).

As in the Sugama-Horton model, \(E_1, \Phi_1\) are the electric field and electrostatic potential in the moving frame \([18]\).

\[
dX = dR + V_0 dt \\
v = v' + V_0 \\
E = -\nabla \Phi = E_1 + V_0 \times B \\
\Phi = \Phi_1 + \mathbf{A} \cdot \mathbf{V}_0 \tag{4.13}
\]

We may substitute these equations in the original Lagrangian one-form \((4.12)\) to obtain the Lagrangian one-form of the moving frame \((4.14)\) in terms of the transformed phase-space coordinates of the moving frame.

\[
\gamma = [eA + m(v' + V_0)] \cdot dR - \left( e\Phi_1 + \frac{1}{2}mv'^2 - \frac{1}{2}m(V_0)^2 \right) dt \tag{4.14}
\]

The first-order Lagrangian one-form denoted by \((\Gamma)\), which is required to obtain the first-order drift equations, can be obtained using a Lie-transform perturbation approach developed by Littlejohn in Ref. [15] applied to the above equation, which will not be discussed in detail here. This approach effectively splits the parallel and perpendicular components of momentum up to first-order, which separates the fast gyromotion \((\mu'd\theta')\) from the parallel momentum \((mv'_\parallel \mathbf{b} \cdot d\mathbf{R})\) in the co-moving frame. This results in the first-order Lagrangian one-form in the co-moving frame.

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phase-space coordinates (4.15).

$$\Gamma = [qA + m(v'_\parallel b + V_0)] \cdot dR + \mu' d\theta - H dt$$ \hspace{1cm} (4.15)

where the guiding center Hamiltonian is

$$H = q\Phi_1 + \frac{1}{2}mv'^2_{\parallel} + \mu' B - \frac{1}{2}mV_0^2$$

We did not derive (4.15) rigorously, although it can be equivalently obtained by applying the transformations (4.13) to the first-order Lagrangian rigorously derived in the lab frame coordinates (4.16) from Ref. [9], in which the Lie-transformation perturbation algorithm of Ref. [15] is explicitly used.

$$\Gamma_{lab} = [eA + mv]\cdot dX + \mu d\theta - [e\Phi + \mu B + \frac{1}{2}mv^2]dt$$ \hspace{1cm} (4.16)

The guiding center non-canonical Poisson bracket can be obtained from the symplectic part ($\hat{\Gamma}$) of the Lagrangian one-form ($\Gamma = \hat{\Gamma} - H dt$), which applied to the Hamiltonian results in equations of motion [6][9][2][15]. The above differential form of the Lagrangian is commonly used in the literature because perturbation approaches can be applied to it that change the Poisson bracket and the Hamiltonian at the same time [9]. The derivation of the Poisson bracket involves a diagonalization of a $6 \times 6$ matrix of derivatives of the symplectic part of (4.15) [9][2][6]. Ref. [9] does this explicitly for the lab frame coordinate Lagrangian (4.16). We will replace our transformed phase-space coordinates into the Poisson bracket derived by Ref. [9] for their phase-space Lagrangian (4.16) written in lab frame coordinates to obtain our desired Poisson bracket in terms of moving frame coordinates (4.17).

$$\frac{dR}{dt} = \{R, H\} = \frac{\Omega}{B} \left( \frac{\partial R}{\partial \theta'} \frac{\partial H}{\partial \mu'} - \frac{\partial R}{\partial \mu'} \frac{\partial H}{\partial \theta'} \right) - \frac{b}{qB'} \cdot \nabla R \times \nabla H$$

$$+ \frac{B^*}{mB'} \left( \nabla R \frac{\partial H}{\partial v_{\parallel}} - \frac{\partial R}{\partial v_{\parallel}} \nabla H \right)$$ \hspace{1cm} (4.17)
The equation of motion of the guiding center position in the co-moving frame is then found by substituting the Hamiltonian from (4.15) into (4.17) \[2\][6][9][15].

\[
\frac{d\mathbf{R}}{dt} = \frac{\mathbf{b}}{eB^*} \times \nabla H + \frac{\mathbf{B}^*}{mB^*_||} \frac{\partial H}{\partial v'_||} \\
= \frac{\mathbf{b}}{eB^*} \times [e\nabla \Phi_1 + \mu' \nabla B - \frac{1}{2}m \nabla V_0] + \frac{\mathbf{B}^*}{B^*_||}v'_\parallel
\]

Here, \( \mathbf{B}^* \) is the generalized magnetic field, given by the generalized magnetic vector potential \( \mathbf{A}^* = \mathbf{A} + \frac{m}{e}(v'_|| \mathbf{b} + \mathbf{V}_0) \) \[20][10][11][9].

\[
\mathbf{B}^* = \nabla \times \mathbf{A}^* \\
= \nabla \times [\mathbf{A} + \frac{m}{q}(v'_|| \mathbf{b} + \mathbf{V}_0)] \\
= \mathbf{B} + \frac{m}{q} \nabla \times (v'_|| \mathbf{b} + \mathbf{V}_0)
\] 

We would like to write \( \mathbf{B}^* \) in terms of components of \( \mathbf{B}^* \), so we must write \( \mathbf{B} \) in a more usable form. To do this, we compute the component of \( \mathbf{B}^* \) parallel to the magnetic field, which is the dot product of the vector itself with the unit magnetic field vector.

\[
B^*_|| = \mathbf{b} \cdot \mathbf{B}^* \\
= \mathbf{b} \cdot \mathbf{B} + \frac{m}{e} \mathbf{b} \cdot [\nabla \times (v'_|| \mathbf{b} + \mathbf{V}_0)] \\
= B + \frac{m}{e} \mathbf{b} \cdot \mathbf{\xi}
\]

Where, for compactness, we define \( \mathbf{\xi} \equiv \nabla \times (v'_|| \mathbf{b} + \mathbf{V}_0) \). We can then write \( \mathbf{B} \) in terms of \( \mathbf{B}^* \).

\[
B^*_|| \mathbf{b} = B \mathbf{b} + \frac{m}{e} \mathbf{b} (\mathbf{b} \cdot \mathbf{\xi}) \\
= \mathbf{B} + \frac{m}{e} \mathbf{b} (\mathbf{b} \cdot \mathbf{\xi})
\]
Then,

\[ B = B^* \mathbf{b} - \frac{m}{e} \mathbf{b} (\mathbf{b} \cdot \hat{\xi}) \]  

(4.20)

We replace (4.20) in the original equation for \( B^* \) (4.19), once again using the fact that \( \mathbf{b} \cdot \mathbf{b} = 1 \), and vector identities [6].

\[ B^* = B^* \mathbf{b} - \frac{m}{q} \mathbf{b} (\mathbf{b} \cdot \hat{\xi}) + \frac{m}{q} \hat{\xi} \]

\[ = B^* \mathbf{b} - \frac{m}{q} [\mathbf{b} (\mathbf{b} \cdot \hat{\xi}) - (\mathbf{b} \cdot \mathbf{b}) \hat{\xi}] \]

\[ = B^* \mathbf{b} - \frac{m}{q} [\mathbf{b} \times (\mathbf{b} \times \hat{\xi})] \]

Then

\[ \frac{B^*}{B^*} = \mathbf{b} - \frac{m}{qB^*} [\mathbf{b} \times (\mathbf{b} \times \hat{\xi})] \]

Now we expand by replacing \( \hat{\xi}, \mathbf{b} \times (\mathbf{b} \times \hat{\xi}) = \mathbf{b} \times [\mathbf{b} \times (\nabla \times (v \mathbf{b} + V_0))] \), and separating the right-hand side into two separate terms.

\[ \mathbf{b} \times (\mathbf{b} \times \hat{\xi}) = \mathbf{b} \times [\mathbf{b} \times (\nabla \times (v \mathbf{b} + V_0))] = \mathbf{b} \times [\mathbf{b} \times (\nabla \times (v \mathbf{b} + V_0))] \]  

(4.21)

We will handle the two terms on the right-hand side of (4.21) separately using vector identities to reduce each to a more usable form.

The first term can be simplified only with vector calculus identities applied to \( \frac{1}{2} \nabla (\mathbf{b} \cdot \mathbf{b}) \). Since \( \mathbf{b} \cdot \mathbf{b} = b^2 = 1 \) by construction of the unit magnetic field vector, \( \nabla (b^2) = 0 \).

\[ \frac{1}{2} \nabla (\mathbf{b} \cdot \mathbf{b}) = \mathbf{b} \times (\nabla \times \mathbf{b}) + (\mathbf{b} \cdot \nabla) \mathbf{b} \]

\[ 0 = \mathbf{b} \times (\nabla \times \mathbf{b}) + (\mathbf{b} \cdot \nabla) \mathbf{b} \]

\[ \mathbf{b} \times (\nabla \times \mathbf{b}) = -(\mathbf{b} \cdot \nabla) \mathbf{b} \]
The left-hand side of this expression becomes the first term of (4.12) if we multiply both sides by $v'_\parallel$, and take a cross product with $b$.

$$b \times [b \times (\nabla \times v'_\parallel b)] = b \times v'_\parallel [-(b \cdot \nabla)b]$$

$$= -v'_\parallel [b \times (b \cdot \nabla)b]$$

The second term can be simplified by using the fact that vorticity is two times the angular velocity in rigid body rotation $\nabla \times V_0 = 2\vec{\Omega}$ [6].

$$b \times [b \times (\nabla \times V_0)] = b \times (b \times 2\vec{\Omega})$$

$$= -2[b \times (\vec{\Omega} \times b)]$$

$$= -2[(b \cdot b)\vec{\Omega} - (b \cdot \vec{\Omega})b]$$

$$= -2[\vec{\Omega} - \Omega_\parallel b]$$

$$= -2\vec{\Omega}_\perp$$

Combining the first and second terms,

$$b \times (b \times \vec{\xi}) = -v'_\parallel [b \times (b \cdot \nabla)b] - 2\vec{\Omega}_\perp$$

$$\frac{B^*}{B'^*} = b - \frac{m}{eB'^*}[b \times (b \times \vec{\xi})]$$

$$= b + \frac{mv'_\parallel}{eB'^*}[b \times (b \cdot \nabla)b] + \frac{2m}{eB'^*}\vec{\Omega}_\perp$$

Plugging this back into the equation of motion (4.18), we find the first-order equation of motion of the guiding center in the locally co-moving frame, i.e. the guiding center drift velocity in the co-moving frame (4.22). We once again use the rigid
body rotation, $\nabla V_0^2 = \nabla (\Omega R)^2 = 2\Omega^2 R$ \cite{2].

\[
\frac{d\mathbf{R}}{dt} = v' b + \frac{\mathbf{b} \times \nabla \Phi_1}{B^*_\parallel} + \frac{\mu' b \times \nabla B}{e B^*_\parallel} + \frac{m v'^2}{e B^*_\parallel} [\mathbf{b} \times (\mathbf{b} \cdot \nabla) \mathbf{b}] \\
- \frac{m \Omega^2 R}{e B^*_\parallel} \mathbf{b} \times \nabla R + \frac{2 m v'^*}{e B^*_\parallel} \Omega_\perp 
\] 

The first and second terms are the parallel drift and the $\mathbf{E}_i \times \mathbf{B}$ drift respectively. The third term is the $\nabla B$ drift, the fourth term is the curvature drift, and the last two terms are the centrifugal and Coriolis drifts respectively \cite{17}. These rotation terms coincide with those derived in the previous section using the Sugama-Horton drift velocity equation \cite{18}. Furthermore, we see that our model of the total guiding center drift velocity in the co-moving frame is correct (3.2), and this form of the total guiding center velocity can be used in the drift-kinetic equation. (4.23) agrees with the guiding center velocity from the Sugama-Horton model of the co-moving frame (3.2) \cite{18}, which includes the centrifugal and Coriolis components.

\[
\mathbf{V}_G = \frac{d\mathbf{X}}{dt} = \mathbf{V}_0 + \frac{d\mathbf{R}}{dt} \\
= \mathbf{V}_0 + \mathbf{v}_d + v'_b 
\]
Chapter 5

Implementation in Simulation

Coordinates and Visualizations of Results

5.1 Implementation in Simulation Coordinates

In (4.23), most of the terms in $v_d$, and $v'_\parallel$ are already included in our drift-kinetic simulation model [10][11]. We must now implement $V_0$ and $v_{CO}, v_{CF}$ as new terms in our simulation coordinates by decomposing them component-wise. While chapters 3 and 4 in this thesis have mainly dealt with explaining and deriving the results of previous authors, the work and results in this section are original to this thesis. The simulation coordinates are defined in terms of the toroidal coordinates $(r, \theta, \phi)$. These orthogonal magnetic field-following coordinates are given by the following expressions [10][11].

\begin{align*}
    x &= r - r_0 \\
    y &= \frac{r_0}{q_0}(\dot{q}(r)\theta - \phi) \\
    z &= q_0 R_0 \theta
\end{align*}

(5.1)
$r_0$ is a reference minor radius, $q_0 = \hat{q}(r_0)$ is a reference safety factor, and $
abla \phi$ is the safety factor that represents the amount of twist in the magnetic field lines. $R_0$ is a reference major radius. For this implementation, these are parameters which will, in general, depend on the specifications of the tokamak [20].

Decomposing each term in the three orthogonal simulation coordinates is, in principle, straightforward, but requires geometric arguments [6]. The general method for finding the component of some velocity vector $v$ along the simulation coordinate $x_i$, $i = x, y, z$, is

$$v_i = v \cdot \nabla x_i$$  \hspace{1cm} (5.2)

We must do this for the Coriolis, centrifugal, and equilibrium lowest-order flow velocity using the following expressions derived from (5.1)

$$\nabla x = \nabla r$$

$$\nabla y = \frac{\partial y}{\partial r} \nabla r + \frac{r_0 \hat{q}}{q_0} \nabla \theta - \frac{r_0}{q_0} \nabla \phi$$\hspace{1cm} (5.3)

$$\nabla z = R_0 \hat{q} \nabla \theta$$

5.1.1 Decomposition of $V_0$

We begin by decomposing the simplest of the three terms. Recall that $V_0 = u_t(\psi) \hat{\phi}$. Then the x-component is given by

$$V_0 \cdot \nabla x = V_0 \cdot \nabla r = 0$$

Similarly, the z-component is also zero

$$V_0 \cdot \nabla z = V_0 \cdot q_0 R_0 \nabla \theta = 0$$
by orthogonality of the toroidal coordinates. Therefore, \( \mathbf{V}_0 \) is purely in the \( y \)-direction.

\[
\begin{align*}
V_{0y} &= u_t(\psi)\hat{\phi} \cdot \nabla y \\
&= \frac{r_0}{q_0} u_t R \nabla \phi \cdot \left[ \frac{\partial \hat{\phi}}{\partial r} \theta \nabla r + \hat{q}(r) \nabla \theta - \nabla \phi \right] \\
&= \frac{r_0}{q_0} \frac{u_t}{R} \\
\end{align*}
\]

Equivalently, this can be interpreted as the \( \mathbf{E} \times \mathbf{B} \) drift using the electric field connected to the flow \( \mathbf{E}_0 \) [12].

\[
V_{0y} = \left( \frac{\mathbf{E}_0 \times \mathbf{b}}{B} \right) \cdot \nabla y \\
\]

### 5.1.2 Decomposition of \( \mathbf{v}_{CO} \)

The Coriolis drift in our tokamak configuration is written

\[
v_{CO} = \frac{2mv'_\parallel}{eB} \hat{\Omega}_\perp \\
\]

For the purposes of this geometric decomposition, we can factor out the magnitude, so we must write \( \hat{\Omega}_\perp \) in a more usable form in cylindrical coordinates. We use the fact that \( \hat{\Omega} = R\Omega (\nabla R \times \nabla \phi) \) in cylindrical rigid body rotation for flow in the \( \hat{\phi} \) direction only. Also recall the following geometric equalities in cylindrical geometry [6]: \( \nabla R = \hat{R}, \nabla \phi = \frac{1}{R} \hat{\phi}, \nabla Z = \hat{Z} \).

\[
\begin{align*}
\hat{\Omega}_\perp &= \mathbf{b} \times (\hat{\Omega} \times \mathbf{b}) \\
&= \mathbf{b} \times [(\nabla R \times \nabla \phi)R\Omega \times \mathbf{b}] \\
&= \mathbf{b} \times [(-\hat{Z} \times \mathbf{b})\Omega] \\
&= \Omega[(\mathbf{b} \cdot \nabla Z - (\mathbf{b} \cdot \nabla Z)b] \\
&= \Omega[(\mathbf{b} \cdot \nabla Z)b - \nabla Z] \\
\end{align*}
\]
We can then calculate the $x, y, z$ components of the Coriolis drift velocity using (5.7) as our expression for the vector direction of the Coriolis velocity (5.6) and the gradients of each coordinate (5.3). Here, we use the fact that $\mathbf{b} \cdot \nabla r = 0$ since the magnetic field has no radial component [20][10], as well as the fact that

$$\nabla Z \cdot \nabla r = \frac{\partial Z}{\partial r} |\nabla r|^2 + \frac{\partial Z}{\partial \theta} (\nabla r \cdot \nabla \theta).$$

For the $x$-component, we have

$$v_{COx} = \frac{2mv'}{eB} \Omega [(\mathbf{b} \cdot \nabla Z) \mathbf{b} - \nabla Z] \cdot \nabla r$$

$$= \frac{2mv'}{eB} \Omega [(\mathbf{b} \cdot \nabla Z)(\mathbf{b} \cdot \nabla r) - (\nabla Z \cdot \nabla r)]$$

$$= -\frac{2mv'}{eB} \Omega \left[ \frac{\partial Z}{\partial r} |\nabla r|^2 + \frac{\partial Z}{\partial \theta} (\nabla r \cdot \nabla \theta) \right]$$

(5.8)

For the $y$-component, it is first useful to calculate $\mathbf{b} \cdot \nabla y$ using the definition of $\hat{q} = \frac{\mathbf{b} \cdot \nabla \phi}{\nabla \phi}.$

$$\mathbf{b} \cdot \nabla y = \mathbf{b} \cdot \left( y' \nabla r + \frac{r_0 \hat{q}}{q_0} \nabla \theta - \frac{r_0}{q_0} \nabla \phi \right)$$

$$= \frac{r_0}{q_0} (\hat{q} \mathbf{b} \cdot \nabla \theta - \mathbf{b} \cdot \nabla \phi)$$

$$= 0$$

Then

$$v_{COy} = \frac{2mv'}{eB} \Omega (-\nabla z \cdot \nabla y)$$

$$= -\frac{2mv'}{eB} \Omega \left[ \frac{\partial Z}{\partial r} \frac{\partial y}{\partial r} |\nabla r|^2 + \left( \frac{r_0 \hat{q}}{q_0} \frac{\partial Z}{\partial r} + \frac{\partial Z}{\partial \theta} \frac{\partial y}{\partial \theta} \right) (\nabla r \cdot \nabla \theta) + \frac{\partial Z}{\partial \theta} \frac{r_0 \hat{q}}{q_0} |\nabla \theta|^2 \right]$$

(5.9)

Finally, the $z$-component can be calculated using the analytic expression for $\mathbf{b}$ [20].

$$v_{COz} = -\frac{2mv'}{eB} \Omega R_0 q_0 \left[ \frac{\partial Z}{\partial r} (\nabla r \cdot \nabla \theta) + \frac{\partial Z}{\partial \theta} |\nabla \theta|^2 \right]$$

(5.10)
5.1.3 Decomposition of $v_{CF}$

$v_{CF}$ can be decomposed in the same way, but now we must write $b \cdot \nabla R$ in a more usable form in cylindrical coordinates using the analytic expression for $b$ [20].

$$b \times \nabla R = \frac{1}{B} \left( I(\psi) \nabla \psi + \nabla \theta \times \nabla \psi \right) \times \left( \frac{\partial R}{\partial r} + \frac{\partial R}{\partial \theta} \nabla \theta \right)$$

$$= \frac{I(\psi)}{B} \frac{\partial R}{\partial r} (\nabla \phi \times \nabla r) + \frac{I}{B} \frac{\partial R}{\partial \theta} - \frac{\phi'}{B} \frac{\partial R}{\partial r} |\nabla r|^2 \nabla \phi - \frac{\psi'}{B} \frac{\partial R}{\partial \theta} (\nabla r \cdot \nabla \theta) \nabla \phi$$

(5.11)

We may then use (5.11) and (5.3) to calculate the $x, y, z$ components of the centrifugal drift velocity.

$$v_{CFx} = \frac{m \Omega^2 R}{eB} (b \times \nabla R) \cdot \nabla r$$

$$= \frac{m \Omega^2 R}{eB} \left[ I(\psi) \frac{\partial R}{\partial \theta} |\nabla r \times \nabla \theta| \right]$$

(5.12)

$$v_{CFy} = \frac{m \Omega^2 R}{eB} (b \times \nabla R) \cdot \nabla y$$

$$= -\frac{m \Omega^2 R}{eB} \left[ I(\psi) \left( -\frac{\partial y}{\partial r} \frac{\partial R}{\partial \theta} + \frac{r_0 q_0}{\partial r} \frac{\partial R}{\partial \theta} \right) |\nabla r \times \nabla \theta| \right]$$

(5.13)

$$v_{CFz} = \frac{m \Omega^2 R}{eB} (b \times \nabla R) \cdot \nabla z$$

$$= -\frac{m \Omega^2 R}{eB} \left[ R_0 q_0 I(\psi) \frac{\partial R}{\partial r} |\nabla r \times \nabla \theta| \right]$$

(5.14)

Thus, we have decomposed the components of each velocity. (5.4) for $V_0$, (5.8 - 5.10) for $v_{CO}$, and (5.12 - 5.14) for $v_{CF}$. Note that all of these expressions require a parametrization from toroidal coordinates $(r, \theta, \phi)$ to cylindrical coordinates $(R, Z, \phi)$, since the axisymmetric form of the drifts is written in cylindrical coordinates, but the simulation coordinates are written in terms of toroidal coordinates. This is represented in the equations by the partial derivatives of cylindrical coordinates with respect to toroidal coordinates (eg. $\frac{\partial R}{\partial \theta}$).
5.2 Toroidal to Cylindrical Parametrization

In general, this parametrization depends on the non-trivial specifications and geometry of each tokamak, and does not have an exact analytic expression. An example of a parametrization that approximates the geometry of a tokamak is the analytic Miller equilibrium parametrization (5.15), which uses two parameter profiles: the triangularity $\delta(r)$, and the elongation $\kappa(r)$ to shape the poloidal tokamak geometry [4].

$$
R = R_0(r) + r \cos[\theta + \sin^{-1}(\delta(r)) \sin \theta] \\
Z = \kappa(r) \sin \theta \\
\phi = \phi
$$

(5.15)

This issue is not unique to the decomposition of the terms in this thesis, and in general, will have been addressed before the inclusion of $V_0$, $v_{CO}$, and $v_{CF}$ [10][11]. However, this discussion provides context for how the terms are included in the simulation coordinates.

5.3 Linear Eigenmode in DIII-D Including Flows

Now that we have obtained a gyrokinetic simulation model including flows by working through the reduction of these terms starting with the Sugama-Horton formulation, and the derivation with the Hamiltonian formulation, as well as the implementation of these terms in the simulation, we may test the new simulation model. Below are visualizations of the electrostatic potential of a poloidal surface in a tokamak simulation. The simulations are of a linear eigenmode in the outer edge of the tokamak with a nonzero equilibrium flow, and with a zero equilibrium flow. The nonzero flow case has a radially increasing flow profile, so there is a flow velocity gradient. In this plots, the toroidal component of the magnetic field $B_t$ points out of the page, as does the equilibrium flow $V_0$. 

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Figure 5.3.1: Linear eigenmode simulation with radially increasing $V_0$. 
Figure 5.3.1: Linear eigenmode simulation with $V_0 = 0$

Qualitatively, the fundamental mode structure is unchanged when including an equilibrium flow as compared to without. However, when including a radially increasing $V_0$, there is a poloidal tilt in the structure. For a flow out of the page, the $E_0$ field connected to the flow is radially outwards, and therefore the $E \times B$ drift connected to the flow is in the clockwise poloidal direction in these images. As is the case in Fig. (5.3.1), a radially increasing flow results in a radially increasing $E \times B$ clockwise poloidal drift, which explains the tilt of the mode structure in Fig. (5.3.1) compared to the zero flow case in Fig. (5.3.2). Therefore, these simple visualiza-
tions of a test case with and without flows agree with our expectation considering the $E_0 \times B$ drift.

5.4 Summary and Future Work

We have constructed a new drift gyrokinetic simulation model including large equilibrium flows of a tokamak using the procedures of Ref. [18]. This was accomplished by first stepping through and explaining the derivation of these terms in the tokamak configuration to obtain the previously obtained results from the literature [18][17][5]. Finally, we implemented these terms in our simulation coordinates, which was the main original result of this thesis. We compared a test case using this simulation model with and without flows, and found that the visualizations qualitatively agree with our expectations.

We may now use this model to test the effects of flows on tokamak physics to determine their importance more quantitatively in terms of growth rates and frequencies of various plasma processes. Notably, in the future we may use this model to test the effects of radial velocity gradients on plasma physics.
Bibliography


