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SIMPLE EXPRESSIONS FOR CURRENT ON A
THIN CYLINDRICAL RECEIVING ANTENNA

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1. INTRODUCTION

Simple expressions of reasonable accuracy describing the current on a cylindrical receiving antenna are not only of academic interest but also of practical value. In some system analyses, the current is but one parameter that needs to be calculated repeatedly a large number of times. An example occurs in the study of field penetration into a coaxial cylindrical enclosure through a small aperture. Following a direct approach described by Taylor and Harrison [1], the knowledge of current and charge distributions at the aperture location over a wide range of incident angles as well as frequencies is needed for the determination of the penetration field. A simple current expression should greatly reduce the computational effort in this example. Another advantage of simple expressions is that they usually allow clear physical interpretations which helps one to identify the "cause" and "effect" of the electromagnetic phenomenon.

The problem of the current distribution on a finite length cylinder due to an arbitrarily incident plane wave has been studied in various ways. A Wiener-Hopf technique was employed by Chen [1] to attack an integral equation for the current. Components identifiable with the induced and reflected currents were contained in his solution, thus allowing some physical insight into this rather complicated problem. Chen's solution is valid for rather long antennas and for angles of incidence away from grazing. King [3] deduced a solution from an integral equation of the current by utilizing the peaking qualities of the kernel. Although his final result is valid for all angles of incidence,
it is difficult to attach qualitative interpretations to the individual terms. Weinstein [4] used a Wiener-Hopf technique and obtained approximate current expressions for both semi-infinite, and finite cylinders. His result allows a clear physical interpretation and is valid for an arbitrarily incident plane wave. Shen [5] has provided what is perhaps the most readily understandable approach to the finite cylinder problem. By characterizing the end reflection of a semi-infinite cylinder, Shen was able to construct the current for the finite cylinder by summing the multiple reflections from both ends. His solution contains an induced current term, and two reflection related terms which arise from reflections at the two ends of the cylinder. However, Shen's solution is not valid when the incident plane wave is near grazing, as it predicts an infinite instead of (the correct) zero current at exactly grazing incidence. In the present paper, we will remove this difficulty in Shen's solution, and present a modified current expression that is uniformly valid for all incident angles.
2. CURRENT ON SEMI-INFINITE THIN CYLINDER

We will attack the problem of a finite receiving antenna by first studying the current reflection on a semi-infinite cylinder. Let the semi-infinite cylinder in Figure 1 be illuminated by an incident plane wave (the time factor exp-iwt is dropped throughout)

$$E^i(r) = -a^i_\theta E^i_\theta e^{i\mathbf{k} \cdot \mathbf{r}}$$

(1)

where $E^i_\theta$ is the amplitude of the plane wave, and $\mathbf{k}$ is the wave vector having a magnitude $|\mathbf{k}| = k = 2\pi/\lambda$ and pointing to the direction $(\theta, \phi)$. The radius, a, of the cylinder is taken to be sufficiently small ($ka << 1$) such that the azimuthal dependence of the induced current is negligible. The total (longitudinal) current induced on the cylinder, $I(z)$, produces a TM field which is derivable from the z-directed vector potential.

Following the so-called Jones method as discussed by Mittra and Lee [6], it is not difficult to show that the Fourier transforms of the total current on the semi-infinite cylinder, $I^+(a)$, and the longitudinal electric field away from the cylinder at $\rho = a$, $E_{z-}(a, \alpha)$, satisfy the Wiener-Hopf equation

$$\xi^2 K(\alpha) I^+ (\alpha) = i 4 \pi \omega \varepsilon \left\{ \frac{1}{\sqrt{2\pi}} E^i_\theta \sin \theta \left( \frac{1}{\alpha + k \cos \theta} \right) + E_{z-}(a, \alpha) \right\}$$

(2)

where

$$K(\alpha) = 2I_0 (\xi a) K_0 (\xi a) = i \pi J_0 (i \xi a) H_0^{(1)}(i \xi a)$$

(3a)

$$\xi = \sqrt{\alpha^2 - k^2}$$

and $\text{Re}(\gamma) > 0$ for all $\alpha$
Figure 1. A semi-infinite cylinder illuminated by an incident plane wave.
In (3), $J_0$, $H_0^{(1)}$, $I_0$, and $K_0$ are the usual Bessel functions as defined in [10]. The functions $I_+(\alpha)$ and $E_z-(\alpha,\alpha)$ are defined by:

$$I_+(\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^\infty I(z)e^{i\alpha z}dz$$  \hspace{1cm} (4)

$$E_z-(\alpha,\alpha) = \frac{1}{\sqrt{2\pi}} \int_0^\infty E_z(a,z)e^{i\alpha z}dz$$  \hspace{1cm} (5)

Hereafter, we will rewrite the current $I(z)$ as $I(\theta,z)$ to emphasize the dependence of the incident angle $\theta$. Solving (2) by the standard Wiener-Hopf technique, one obtains an integral representation for the total current on the semi-infinite cylinder, namely,

$$I(\theta,z) = \frac{E_{\theta}^1}{K_+(k \cos \theta)} \frac{2 \sin \theta}{\eta (1+\cos \theta)} \int_{C_0} \frac{e^{-i\alpha z}}{(\alpha+k)(\alpha+k \cos \theta)K_+(\alpha)} \, d\alpha \, ; \, \, z > 0$$ \hspace{1cm} (6)

where the integration contour $C_0$ is shown in Figure 2 and $\eta = \sqrt{\mu_0/\varepsilon_0} = 120 \, \pi$.

In (6), the function $K_+(\alpha)$ is the "plus part" of $K(\alpha)$ in the standard Wiener-Hopf notation. An exact formula for $K_+(\alpha)$ is given in Eq. (2.41), p. 236 of [6] (note that $K_+(\alpha)$ is equal to $\sqrt{2} L_+(\alpha)$ defined in [6]), which contains an infinite integral. For the present application, we use the approximate formula given by Hallén [8] valid for $ka << 1$, namely,

$$K_+(\alpha) \approx \sqrt{2C_0 + i\pi} \left[ 1 - \frac{1}{2C_0 + i\pi} \ln \frac{k + \alpha}{2k} \right]$$ \hspace{1cm} (7)

where

$$C_0 = -\ln(ka) - \gamma, \quad \gamma = 0.57712... \quad .$$ \hspace{1cm} (8)
We have compared (7) with the exact $K_+(\alpha)$ calculated from the formula given in [6]. Some typical results are presented in Table I, and they show that (7) is quite accurate for $ka < 0.01$.

**TABLE I**

VALUES OF $K_+(\alpha)$ FOR $ka = 0.01$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>Exact $K_+(\alpha)$</th>
<th>Approximate $K_+(\alpha)$</th>
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<tr>
<td>$k$</td>
<td>$2.8583 + i0.5765$</td>
<td>$2.8899 + i0.5436$</td>
</tr>
<tr>
<td>$k + \frac{i\alpha}{2}$</td>
<td>$2.8215 + i0.4780$</td>
<td>$2.8643 + i0.4636$</td>
</tr>
<tr>
<td>$k + i\alpha$</td>
<td>$2.7682 + i0.3937$</td>
<td>$2.8234 + i0.3956$</td>
</tr>
<tr>
<td>$k + \frac{i3\alpha}{2}$</td>
<td>$2.7066 + i0.3264$</td>
<td>$2.7748 + i0.3425$</td>
</tr>
<tr>
<td>$k + i2\alpha$</td>
<td>$2.6434 + i0.2744$</td>
<td>$2.7247 + i0.3029$</td>
</tr>
<tr>
<td>$k + \frac{i5\alpha}{2}$</td>
<td>$2.5823 + i0.2345$</td>
<td>$2.6763 + i0.2737$</td>
</tr>
<tr>
<td>$k + i3\alpha$</td>
<td>$2.5249 + i0.2036$</td>
<td>$2.6311 + i0.2521$</td>
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In order to separate the distinct components of the total current, the contour $C_0$ in (6) is deformed into $C_1$ with the result,

$$I_0 (\theta, z) = E^0_\theta \left[ I_{\infty} (\theta, z) + I_r (\theta, z) \right], \quad z > 0$$

(9)

where

$$I_{\infty} (\theta, z) = \frac{i 4\pi}{k\eta} \frac{e^{ikz \cos \theta}}{\sin \theta K(k \cos \theta)}$$

(10)
\[ I_r(\theta, z) = \frac{\cos^i \theta}{K_+(k \cos \theta)} \frac{2 \sin \theta}{\eta(1+\cos \theta)} \int_1^C e^{-i\alpha z} \frac{\eta(1+\cos \theta)}{(\alpha+k)(\alpha+k \cos \theta)K_+(\alpha)} d\alpha \] (11)

Here \( I_{sw}(\theta, z) \) can be identified as the current induced on an infinitely long cylinder by a unit incident plane wave, and \( I_r(\theta, z) \) as the reflected current from the end at \( z = 0 \). Our problem at hand is to evaluate \( I_r(\theta, z) \).
3. EVALUATION OF THE REFLECTED CURRENT

In his evaluation of the reflected current from (11), Shen [5] argued that the major contribution to the integral comes from the neighborhood of \( \alpha = -k \). Therefore, the term \((\alpha + k \cos \theta)\) was approximated by \(k(-1 + \cos \theta)\). This approximation is valid only for incident angles away from \( \theta = 0^\circ \). In order to obtain a result for \( I_r(\theta, z) \) valid for all angles of incidence, we depart from Shen's procedure and rewrite (11) in the form,

\[
I_r(\theta, z) = \frac{-E_\theta}{K_+(k \cos \theta)} \frac{2 \sin \theta}{\eta(1 + \cos \theta)} \left\{ \int e^{ikz \cos \theta} \int_0^\infty e^{-ik'z' \cos \theta} \int_{C_1} \left[ \frac{(\alpha - k)K_-(\alpha)}{(\alpha^2 - k^2)K(\alpha)} e^{-i\alpha z'} d\alpha \right] dz' \right\}
\]

(12)

The integration contour \( C_1 \) in (12) may be deformed into \( C_2 \) (Figure 2) which is wrapped around the branch cut with branch point at \( \alpha = -k \).

We note that the residue contributions due to the poles of \([K(\alpha)]^{-1}\) are related to evanescent waveguide modes in the interior of the cylinder, which will be studied in Section 5, and are ignored for now. Then, the major contribution to the integral in the \( \alpha \)-plane comes from the neighborhood of \( \alpha = -k \). The term \((\alpha - k)K_-(\alpha)\) may be approximated by the first term in its Taylor series expansion about \( \alpha = -k \). The integral may then be approximated by deforming the contour from \( C_2 \) back to \( C_1 \) and again ignoring the residue terms of \([K(\alpha)]^{-1}\) for values of \( z \) away from zero.

The exterior component of the reflected current then becomes
Figure 2. The complex $\alpha$-plane
\[ I_{\text{ext}}^{\text{R}}(\theta, z) = \frac{2kE_\theta^i}{K_+(k \cos \theta)} \frac{2 \sin \theta}{\eta(1 + \cos \theta)} \left\{ i e^{ikz \cos \theta} \int_{z'}^\infty e^{-ikz' \cos \theta} I_\infty(z')dz' \right\} \]

where,

\[ I_\infty(z) = \frac{2k}{i \eta} \int \frac{e^{-i\alpha z}}{C_1 (\alpha^2 - k^2) K(\alpha)} d\alpha \]

may be identified to the current on an infinitely long center-fed cylinder driven by a unit impulse voltage generator. Shen et al. [7] have shown that a suitable approximation for \( I_\infty(z) \) is

\[ I_\infty(z) \approx i \frac{e^{ik|z|}}{\eta} \ln \left\{ 1 - \frac{2\pi}{\ln \left[ k|z| + \sqrt{(kz)^2 + 0.3} + \frac{3\pi}{2} \right]} \right\} \]

where \( \Gamma = e^{\gamma} \approx 1.781 \ldots \) For sufficiently large \( kz \), (15) may be further approximated by,

\[ I_\infty(z) \approx \frac{2\pi}{\eta} e^{ik|z|} \frac{1}{2C_w + \gamma + i\pi/2 + \ln 2k|z|} \]

where \( C_w \) was defined in (8).

Now using (16) in (13) and applying the following transformation and definitions,

\[ v = k(1 - \cos \theta)(z' - z) \]

\[ v_\theta = kz(1 - \cos \theta) \]

\[ Q_\theta = 2C_w + i\pi + \gamma - 2 \ln \left( \sin \frac{\theta}{2} \right) \]

an alternative expression of (15) is derived,
\[
I_{r}^{\text{ext}}(\theta, z) = -\frac{2\pi}{\eta} \frac{2 \sin \theta}{1+\cos \theta} \frac{K_+(k)}{K_+(k \cos \theta)} \frac{\exp(ikz)}{k(1-\cos \theta)} \int_{0}^{\infty} \frac{\exp(iv)}{Q_0 + \ln(-i(v+i\gamma))} \, dv
\]

The approximate evaluation of the integral in (20) is detailed in Appendix A. This result used in conjunction with the small argument form of \(K_+(\alpha)\) from (7) yields for the reflected current,

\[
I_{r}^{\text{ext}}(\theta, z) \approx -\frac{4\pi}{ik\eta} \frac{1}{\sin \theta \left[2C_w + i\pi - 2 \ln \frac{\sin \theta}{2}\right]} \]

\[
= \frac{Q_0 - \gamma}{Q_0 + \ln(-iv_o) + e^{-iv_o}E_1(-iv_o)} \exp(ikz)
\]

(21)

It is noteworthy that the result in (21) is identical to the Weinstein's solution given in Eq. (62.63), p. 341 of [4], which is obtained through a much more complicated manipulation based on the variational calculation of an integral equation.

In order to obtain an expression similar to that of Shen's, we rewrite (21) as

\[
I_{r}^{\text{ext}}(\theta, z) = -I_{s\infty}(\theta, 0) [1 + \frac{M}{Q_0 - \gamma}]^{-1} \exp(ikz)
\]

(22a)

where

\[
M = \gamma + \ln(-iv_o) + e^{-iv_o}E_1(-iv_o)
\]

(22b)

Since \(M\) is always much less than \((Q_0 - \gamma)\), a Taylor series expansion may be performed for \([\cdot]^{-1}\) in (22a). This manipulation leads to

\[
I_{r}^{\text{ext}}(\theta, z) \approx -\left(\frac{n}{2\pi}\right) I_{s\infty}(\theta, 0) \left[2C_w + \gamma + \frac{i\pi}{2} + \ln 2kz\right]
\]

\[-(1 + N)M I_{\infty}(z)
\]

(23a)

where \(I_{\infty}(z)\) was defined in (15) and
\[ N = \left[ \gamma + \lambda n(-iv) \right]/(Q_0 - \gamma) \]  

(23b)

Provided that \( k_z \) is not extremely large, \( N \) in (23) is much less than 1 and can be neglected. It may be argued that this inequality does not hold for \( \theta \approx 0^\circ \). However, for such a case, \( M \) tends to zero faster than that \( N \) becomes comparable to 1. Therefore, the dropping of \( N \) is still justified. Now, (23a) becomes

\[ I_{\text{ext}}^0 (\theta, z) \approx -I_{\infty}^0 (\theta, 0) R_S(\theta, z) I_\infty (z) \]  

(24)

where

\[ R_S(\theta, z) = R_S^\text{Shen}(\theta) - \frac{n}{2\pi} e^{-iv_0} E_1(-iv_0) \]  

(25)

\[ R_S^\text{Shen}(\theta) = \frac{n}{2\pi} \left[ 2C_w + i\pi - 2 \ln(\sin \frac{\theta}{2}) \right] \]  

(26)

\( R_S^\text{Shen}(\theta) \) is the so-called "reflection coefficient" defined by Shen [5].

The present reflection coefficient given in (25) differs from Shen's only by the additional term \(-\frac{n}{2\pi} e^{-iv_0} E_1(-iv_0)\). This term is usually small except in the case of near grazing. The form of (24) suggests that away from the end, the reflected current can be viewed as equivalent to the current on an infinitely-long antenna driven by a voltage generator at \( z = 0 \), i.e. \( I_\infty (z) \). The amplitude of this voltage source being proportional to the product of the incident current at the end \( I_{\infty}^0 (\theta, 0) \), and the reflection coefficient \( R_S(\theta, z) \). It should be also remarked that the present reflection coefficient \( R_S(\theta, z) \) is a function of \( z \), and differs from the "reflection coefficient" that is ordinarily understood. Substituting (24) into (7), we obtain the final solution for the current induced on a semi-infinite cylinder due to the incident plane wave in (1), namely
\[ I^\text{ext}(\theta, z) = E_0^i (I^\infty_\infty(\theta, z) - I^\infty_\infty(\theta, 0) R_\text{ext}(\theta, z) I^\infty_\infty(z)), \quad z \geq 0 \]  \quad (27)

To demonstrate that (27) does behave properly for near grazing incident angles, we expand the various terms as \( \theta \to 0 \). From (7) and (20), we have

\[ I^\infty_\infty(\theta, z) \sim \frac{2\pi}{ik} \frac{e^{ikz}}{\theta \ln \theta} + O(\theta^2) \]  \quad (28)

The use of the series expansion for the exponential integral,

\[ E_1(z) = -\gamma - \ln z - \sum_{n=1}^{\infty} \frac{(-1)^n z^n}{n \ln^n \theta} \]  \quad (29)

in (25) leads to

\[ R_\text{ext}(\theta, z) \sim \frac{n}{2\pi} \left[ 2C_w + \gamma + i \frac{\pi}{2} + \ln(2kz) + O(\theta^2) \right] \]  \quad (30)

Substituting (28), (30), and (14) into (27), we have

\[ I^\text{ext}(\theta, z) \sim \frac{1}{ik} \frac{e^{ikz}}{\theta \ln \theta} \left\{ 1 - \frac{2C_w + \gamma + i \frac{\pi}{2} + \ln(2kz) + O(\theta^2)}{2C_w + \gamma + i \frac{\pi}{2} + \ln(2kz)} \right\}, \quad \theta \to 0 \]  \quad (31)

which immediately reduces to,

\[ I^\text{ext}(\theta, z) \sim \frac{1}{ik} \frac{e^{ikz}}{\theta \ln \theta} \left\{ \frac{O(\theta^2)}{2C_w + \gamma + i \frac{\pi}{2} + \ln(2kz)} \right\}, \quad \theta \to 0 \]  \quad (32)

It is now clear that

\[ I^\text{ext}(\theta, z) \to 0, \quad \text{as} \quad \theta \to 0 \]  \quad (33)

which is the physically expected behavior.

It is interesting to note that in the limit as \( z \to 0 \), the reflected current given by (24) tends to \(-I^\infty_\infty(\theta, 0)\). At \( z = 0 \), this reflected term cancels with the induced component \( I^\infty_\infty(\theta, z) \), resulting in a zero exterior current at the end of the cylinder. This agrees with the physically expected result that the external current should be small at the end of the thin cylinder. The point of interest here is that in the evaluation of \( I^\text{ext}(\theta, z) \) approximations were made which require that \( z \) is not close to zero. Thus...
it is indeed a pleasant surprise that even with such approximations, the
final result does yield a vanishing current at the end.

Figure 3 illustrates the reflected current calculated from (24) as
a function of \( z \), normalized by the incident current at the end, \( I_{\infty}(\theta,0) \).
For the small incident angle \( \theta = 5^\circ \), the normalized reflected current
is equal to approximately \(-1\) for all \( z \), which nearly cancels the induced
current \( I_{\infty}(\theta,z) \). To check the accuracy of (24), we have also calculated
the original expression for the reflected current in (11) by a "brute force"
numerical integration. The results are also presented in Figure 3 (circles),
and they are in good agreement with those calculated from (24).

4. CURRENT ON FINITE THIN CYLINDER

Equation (24) expresses the reflected current from the end of the
semi-infinite cylinder in terms of the current incident upon the end. With
this type of characterization it is possible to construct a solution for
the current on a finite cylinder (Figure 4) by summing the multiple
reflections from the ends. Such a construction can be found in the
literature [11]; we give here only the result. Due to the incident plane
wave in (1), the current existing on the exterior surface of the finite
cylinder is approximately given by

\[
I_{\text{ext}}(\theta,z) = E_\theta^i \left\{ I_{\infty}(\theta,z) \right. \\
+ \left[ I_{\infty}(\theta,h) \delta R_S(\pi-\theta,-z) + C_s(\pi-\theta) \right] I_\infty(h-z) \right. \\
+ \left[ I_{\infty}(\pi-\theta,h) \delta R_S(\theta,z) + C_s(\theta) \right] I_\infty(h+z) \right\} \\
\]  \hspace{1cm} (34)

where

\[
\delta R_S(\theta,z) = R_S(\theta,2h) - R_S(\theta,h+z), \hspace{1cm} (35)
\]

\[
R = \frac{n}{2\pi} \left[ 2C_\pi + i\pi \right] = R_\pi(\pi,2h) \hspace{1cm} (36)
\]
Figure 3. Magnitude and phase of the normalized reflected current on a semi-infinite cylinder with $ka = 0.01$. The solid curves are calculated from the approximate formula in (24), and the circles from the exact integral in (11).
Figure 4. A finite length cylinder illuminated by an incident plane wave.
and
\[
C_s(\theta) = \frac{[RI_\infty(2h)]R_s(\pi-\theta,2h)I_{\infty}(\theta,h) - R_s(\theta,2h)I_{\infty}(\pi - \theta,h)}{1 - [RI_\infty(2h)]^2}
\] (37)

Here \(C_s(\theta)\), and \(C_s(\pi-\theta)\) are similar to \(C_{s1}\), and \(C_{s2}\), respectively, used by Shen [5] with the exception that \(R_s(\theta,z)\) now replaces \(R_{Shen}(\theta)\). Equation (35) has the same form as the final solution of Shen [5] except for the additional term \(R_s\). For cases other than grazing, \(R_s \approx R_{Shen}\), and \(\delta R_s\) has the same order of \((kh)^{-1}\) as those terms which have been neglected earlier. Thus, in order to be consistent we modify (35) to yield

\[
I^{ext}(\theta,z) = E_0^i \{I_{\infty}(\theta,z) + [I_{\infty}(\theta,h) \delta R_s(\pi-\theta,-z) U(\theta-\pi/2) + C_s(\pi-\theta)]I_{\infty}(h-z) + [I_{\infty}(\pi-\theta,h) \delta R_s(\pi/2-\theta) U(\pi/2-\theta) + C_s(\theta)]I_{\infty}(h+z)\}
\] (38)

where the unit step function is defined as \(U(x) = 0\) for \(x < 0\), \(\frac{1}{2}\) for \(x = 0\), and 1 for \(x > 0\). Equation (38) is the final expression for the current on a finite cylinder due to the incident plane wave in (1). This expression is approximately valid for all \(\theta\) in the range \((0, \pi)\), and \(z\) in the range \((-h,h)\), provided that \(ka \ll 1\) and \(kh \gg 1\).

The behavior of the current in (38) for near-grazing incidence of the plane wave may be determined from the following expansions
\begin{align*}
R_s(\theta, h, z) &\sim \frac{n}{2\pi} \left[ 2C_w + \gamma + \frac{i\pi}{2} + \ln 2k(h+z) + O(\theta^2) \right] \quad (39) \\
R_s(\pi-\theta, h, z) &\sim \frac{n}{2\pi} \left[ 2C_w + i\pi + \frac{1}{2k(h-z)} + O(\theta^2) \right] \quad (40) \\
C_s(\theta) &\sim -i \frac{2\pi}{k\eta} \frac{1}{\theta \ln \theta} + O(\theta^2) \quad (41) \\
C_s(\pi-\theta) &\sim -i \frac{2\pi}{k\eta} \frac{1}{\theta \ln \theta} \frac{e^{ikh}}{I_{\infty}(2h)} + O(\theta^2) \quad (42)
\end{align*}

Substituting (28) and (39)-(42) in (38) yields

\[ I^{\text{ext}}(\theta, z) \sim \frac{\theta}{\theta \ln \theta} \quad (43) \]

It follows from (43) that

\[ I^{\text{ext}}(\theta, z) \rightarrow 0, \text{ as } \theta \rightarrow 0 \quad (44) \]

Because of the symmetrical relation of (38)

\[ I^{\text{ext}}(\theta, z) = I^{\text{ext}}(\pi - \theta, -z) \]

The current also goes to zero as \( \theta \rightarrow \pi \).

In Figure 5, we plot the magnitude of the current \( I^{\text{ext}}(\theta, z_0) \) as a function of \( \theta \) calculated from (38) with parameters \( z_0 = -0.039 \, \text{m} \), \( 2h = 1.30 \, \text{m} \), \( E^i_\theta = 1 \, \text{V/m} \), and \( \lambda = 1.0 \, \text{m} \). For comparison, the corresponding result calculated from Shen's formula [5] is also given. We note that these two results agree reasonably well except when \( \theta \) is close to zero. In the latter case, Shen's formula breaks down, while our result remains finite and predicts a zero current at \( \theta = 0 \). Figure 6 shows the current as a function...
Figure 5. Magnitude of current \( i_{\text{ext}}(\theta, z_o) \) on a finite cylinder at \( z = -0.039 \text{ m} \) as a function of the incident angle \( \theta \). The parameters are \( z_o = -0.039 \text{ m} \), \( 2h = 1.3 \text{ m} \), \( a = 0.05 \text{ m} \), \( E^i = 1 \text{ V/m} \), and \( \lambda = 1 \text{ m} \).
Figure 6. Magnitude and phase of the current $I_{\text{ext}}^\vartheta(z)$ as a function of $z$. The result is calculated from (38) with parameters $E^i = 1 \text{ V/m}$, $\lambda = 1.0 \text{ m}$, $a = .05 \text{ m}$, and $h = .65 \text{ m}$. 
of $z$ on the same cylinder for various angles of incidence of the plane wave. The overall decreasing magnitude of the current is apparent for the small value of $\theta$. Figure 7 is a similar illustration for a slightly longer cylinder ($2h = 1.39$ m). Although similar in appearance, a close examination of the relative magnitudes at particular values of $z$ shows significant differences. These differences can best be seen in the polar plot of $|I_{\text{ext}}^{\text{ext}}(\theta, z)|^2$ given in Figure 8. One observes that the major-to-side lobe ratio and the overall magnitude of the distribution is a sensitive function of the overall length of the cylinder.

The result of the present analysis, specifically (38), for $I_{\text{ext}}^{\text{ext}}(\theta, z)$, has been compared with the work of other authors. Figure 9 shows the current on a cylinder where $kh = \pi$ and $\Omega = 2 \ln(ah/a) = 15.0$ from King's integral equation approximation [12], Price's [13] numerical solution using the moment method and finally from (38) for $I_{\text{ext}}^{\text{ext}}(\theta, z)$. Agreement between all three solutions is good and extremely good between the numerical data and our present solution.

As earlier discussed, the derivation of $I_{\text{ext}}^{\text{ext}}(\theta, z)$ for the semi-infinite cylinder, (27), included the restriction that $z$ be not too small. It was seen, however, that this current behaved in a physically predictable manner and tended to zero as the end was approached. Still it should not be expected that (38) for $I_{\text{ext}}^{\text{ext}}(\theta, z)$ would yield an acceptable result for very short cylinders, since most of the points on the cylinder are not very far from either end. However, as shown in Figure (10), the current on a fairly short cylinder, $kh = \pi/2$ with $\Omega = 2 \ln(2h/a) = 15.0$, from (38) agrees quite well with the results of King [12] and Price [13]. Thus the expected limitations do not appear here.
Figure 7. Magnitude and phase of the current, $\mathbf{I}^{\text{ext}}(\theta, z)$, as a function of $z$. The result calculated from (38) with parameters $E_{0} = 1.0 \text{ v/m}$, $\lambda = 1.0 \text{ m}$, $a = .05 \text{ m}$, and $h = .695 \text{ m}$.
Figure 8. Squared magnitude of the exterior current, \( |I_{\text{ext}}^{\text{ext}}(\theta,z_0)|^2 \), at \( z_0 = -0.039 \) m, \( k = 2\pi \), \( a = 0.05 \) m and

1. \( 2h = 1.39 \) m.
2. \( 2h = 1.43 \) m.
3. \( 2h = 1.47 \) m.
Figure 9: Current distribution on a full-wave cylinder, \((kh = \pi, \Omega = 2 \ln 2h/a = 15.0)\) illuminated by a unit \((E^1 = 1.0 \text{ volt/meter})\) incident plane wave.
Figure 10: Current distribution on a half-wave cylinder, \((kh = \pi/2, \Omega = 2 \ln 2h/a = 15.0)\) illuminated by a unit \((E_0 = 1.0 \text{ volt/meter})\) incident plane wave.
Experimental confirmation of King's [12] work just presented has been given by Morita [14] which in turn confirms the present analysis. Also, related experimental findings were given in reference [9] and the experimental field pattern corresponding to the theoretical situation stated above has been repeated in Figure 11. The experimental cylinder was 1.39 meters in length but incorporated an end cap at each end, thereby increasing the overall effective length of the cylinder. Comparing the patterns of Figures 7 and 8 reveals a favorable agreement between the experimental result and the theoretical prediction for an overall length somewhat greater than 1.39 meters.
Figure 11. Experimental field pattern for \( |I^{ext}(\theta, -0.039 \text{ m})|^2 \)

\( k = 2\pi, \ a = 0.05 \text{ m}, \ 2h = 1.39 \text{ m}. \)

Dashed line - Dahlgren Lab. antenna range.
Solid line - Univ. of Colorado antenna range.
5. CURRENT PENETRATION INTO THE OPEN END OF A FINITE CYLINDER

In the evaluation of the reflected current, \( I_r(\theta, z) \), on the semi-infinite cylinder (Section 3), the deformation of the integration contour from \( C_1 \) to \( C_2 \) resulted in an infinite number of poles being enclosed. Since at that time only the exterior current was being sought, the residues of these poles were neglected because they were associated with the currents of evanescent circular waveguide modes on the interior cylinder wall. These interior currents will now be investigated.

The forementioned poles occur at the zeroes of \( K(\alpha) \), which for \( ka < 1 \) are located approximately at \( \alpha \approx -\frac{i\rho_n}{a} \), where \( \rho_n \) is the \( n^{th} \) ordered zero of \( J_0(x) \). Expressing these interior currents in terms of the incident exterior current, \( I_{s\infty}(\theta, 0) \), and the residue contributions of the poles in (9), we have,

\[
I_{\text{int}}(\theta, z) = -I_{s\infty}(\theta, 0) \sum_{n=1}^{\infty} T_n(\theta)e^{-\frac{i\rho_n}{a}z} \tag{45}
\]

where \( T_n(\theta) \) has the character of a transmission coefficient of the current from the exterior surface into the interior surface and is given by,

\[
T_n(\theta) = (1-\cos \theta)K_0(k \cos \theta) \frac{ka}{\pi \rho_n^2 J_1(\rho_n) H_1(1)(\rho_n)} K_+(\frac{i\rho_n}{a}) \tag{46}
\]

The form of the interior current given by (45) immediately suggests that these currents are associated with evanescent TM\(_{on}\) circular waveguide modes.

In a manner similar to the construction of the solution for the exterior currents on a finite length cylinder from the reflection characterized solution of the current on a semi-infinite cylinder, the current penetration into the end of the finite length cylinder can also
be obtained. This requires the calculation (and subsequent summation) of the interior current components each time an exterior current is incident upon the end. This procedure is most expeditiously performed by noting that the total incident current upon a particular end of the cylinder is available from the total exterior current expression, (38). For example, the incident current upon the end at \( z = -h \) is given by the terms containing \( I_{s,\infty}(\theta, z) \) and \( I_{s,\infty}(h+z) \). The related interior currents are then determined by applying the transmission characterization of (45) to these incident currents, the result being given by,

\[
I_{\text{int}}(\theta, z) = \sum_{n=1}^{\infty} [-I_{s,\infty}(\theta, -h)T_{n}(\theta) - C_{s}^{(n-\theta)}(h)T_{n}(\theta)]e^{-\frac{\rho_{n}(h+z)}{a}} \quad (47)
\]

A polar graph of \( |I_{\text{int}}(\theta, -h)| \) associated with the TM\(_{01}\) mode for a cylinder where, \( 2h = 1.30\text{m} \), \( a = 0.05\text{m} \) and \( \lambda = 1.0\text{m} \) is shown in Figure 12. The most important feature is that the major lobe occurs for an incident angle at which the open end of the cylinder \( (z = -h) \) is pointing away from the illumination. Figure 13 illustrates the value of \( |I_{\text{int}}(\theta, z)|^{2} \) for several overall lengths greater than \( 2h = 1.30 \text{ meters} \). This demonstrates that quite unlike \( |I_{\text{ext}}^{\theta}(\theta, z_{o})| \), the \( \theta \) variation of \( |I_{\text{int}}(\theta, -h)| \) is less sensitive to the change in the overall length, \( 2h \).

Related experimental results were given in reference [9] and the particular case corresponding to the above theoretical data has been repeated in Figure 14. An exact pattern correspondence is not obtained between the theory and experiment, possibly due to additional reflections in the interior region of the experimental model, although the dominant features of both are indeed very similar.
Figure 12. Theoretical internal current, \( |i_{\text{int}}(\theta,-h)| \), for the TM_{01} mode at the cylinder end, \( z = -h \).

\( k = 2\pi \), \( a = 0.05 \text{ m} \) and \( 2h = 1.39 \text{ m} \).
Figure 13. $| \int_{\theta}^{\text{int}} (\theta, -h) |^2$ for the TM$_{01}$ mode for $k = 2\pi$, $a = .05$ m and

$2h = 1.30$ m. \(1\)

$1.32$ m. \(2\)

$1.34$ m. \(3\)
Figure 14. Experimental field pattern for $|I_{\text{int}}(\theta, h)|^2$.

$k = 2\pi$, $a = 0.05 \text{ m}$, and $2h = 1.39 \text{ m}$.
VI Conclusion

The theory for the currents on a receiving antenna of Shen has been amply modified to include all angles of incidence of the uniform plane wave. The resulting general expression has revealed some very interesting features for the antenna currents. The major-to-side lobe ratio of the exterior current and the overall magnitude of both the exterior and interior currents have been found to depend quite sensitively upon the overall effective length of the cylinder. In addition it was found for a cylinder with an effective length near $3\lambda/2$ that the major lobe of the interior current occurred (theoretically predicted and experimentally verified) at an angle at which the open end was not directly illuminated.
APPENDIX A

Consider the integral,

\[ I = \int_0^\infty \frac{e^{iv}}{Q_o + \ln[-i(v+v_o)]} \, dv \]

and the complex v-plane illustration of Figure A1. The integration contour, \( C_a \), for \( 0 \leq v \leq \infty \) may be deformed into the contour designated \( C_b \) defined along the imaginary axis for \( 0 \leq v \leq i\infty \). Applying the transformation, \( t = -iv \), to the above integral yields,

\[ I = \frac{i}{Q_o} \int_0^\infty \frac{e^{-t}}{\ln(t-i\nu)} \, dt \]  \hspace{1cm} (A1)

For \( z \) not overly large the term, \( [1 + \frac{\ln(t-i\nu)}{Q_o}]^{-1} \), may be approximated by the first two terms of its Taylor series expansion. In general this approximation is not valid for large \( t \) or in the case where \( \theta \approx 0^\circ \) and \( t \approx 0 \). However, in the confines of the integral, A1, the exponential term sufficiently reduces the contribution for large \( t \) to a negligible value. And for \( \theta \approx 0^\circ \) and \( t \approx 0 \), the range in which the approximation is poor is insignificant compared to the meaningful integration range. In other words, the truncated Taylor series is an acceptable approximation over the most significant portion of the integration range. Thus we have;

\[ I \approx \frac{i}{Q_o} \int_0^\infty e^{-t} \left[ 1 - \frac{\ln(t-i\nu)}{Q_o} \right] dt \]  \hspace{1cm} (A2)

Applying the transformation \( \omega = t - i\nu \) followed by integration by parts yields;

\[ I \approx \frac{i}{Q_o} \left[ 1 - \frac{1}{Q_o} \left[ \ln(-i\nu) + e^{-i\nu}e_1(-i\nu) \right] \right] \] \hspace{1cm} (A3)

where \( E_1 \) is the exponential integral of the first kind, [10].
Figure A1. The complex $v$ plane.
Since \( \ln(-iv_o) + e^{iv_0} E_1(-iv_o) \ll q_o \), the R.H.S. of (A2) is the first two terms of a Taylor series expansion. This observation allows the original integral to be approximated by,

\[
I \approx \frac{i}{Q_o + \ln(-iv_o) + e^{iv_0} E_1(-iv_o)} \quad \text{(A4)}
\]
References


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