Compositional Construction of Finite MDPs for Large-Scale Stochastic Switched Systems: A Dissipativity Approach ⋆

Abolfazl Lavaei [*], **Majid Zamani** [**,***]

[*] *Department of Electrical and Computer Engineering, Technical University of Munich, Germany (e-mail: lavaei@tum.de)*
[**] *Department of Computer Science, University of Colorado Boulder, USA (email: majid.zamani@colorado.edu)*
[***] *Department of Computer Science, Ludwig Maximilian University of Munich, Germany*

Abstract: In this paper, we provide a compositional technique for constructing finite abstractions (a.k.a. finite Markov decision processes) for networks of discrete-time stochastic switched systems. The proposed framework is based on the notion of *stochastic simulation functions*, using which one can employ a finite MDP as a substitution of the original one in the controller design process with guaranteed error bounds on their output trajectories. In this respect, we first leverage dissipativity-type compositional conditions for quantifying the error between the interconnection of stochastic switched subsystems and that of their finite abstractions. We then propose an approach to construct finite MDPs together with their corresponding stochastic simulation functions for a particular class of *nonlinear* stochastic switched systems. To demonstrate the effectiveness of our proposed results, we apply our approaches to two different case studies.

Keywords: Large-Scale Stochastic Switched Systems, Finite Markov Decision Processes, Multiple Storage Functions, Dwell-Time, Dissipativity Reasoning, Compositionality.

1. INTRODUCTION

Switched systems are an important modeling framework describing many engineering systems and play significant roles in many real-life applications including power grids, traffic networks, and so on. Since one may render a switched system unstable by fast switching between even stable modes, this issue motivated many researchers to mainly investigate which classes of switching strategies preserve stability (Liberzon (2003)).

In the past few years, there have been many works on the synthesis of controllers rendering switched systems stable. However, there is only a limited work on the construction of controllers for such systems with respect to complex logic properties. In fact, automated controller synthesis for complex switched systems to achieve some high-level specifications, e.g. those expressed as linear temporal logic (LTL) formulae (Pnueli (1977)), is inherently very challenging. To reduce this complexity, one promising approach is to employ finite abstractions of the given systems as a replacement in the controller synthesis procedure. In this regard, one can first abstract the original system by a simpler one (with finite-state set), perform analysis and synthesis over the abstract model (using algorithmic techniques from computer science (Baier and Katoen (2008))), and finally carry the results back over the concrete system, by providing guaranteed error bounds in this detour process.

Unfortunately, construction of finite abstractions for large-scale complex systems in a monolithic manner suffers severely from the so-called *curse of dimensionality*: the complexity exponentially grows as the number of state variables increases. To mitigate this issue, one promising solution is to consider the large-scale switched system as an interconnected system composed of several smaller subsystems, and provide a compositional framework for the construction of finite abstraction for the given system using the abstractions of smaller subsystems.

There have been several results, proposed in the past few years, on the construction of (in)finite abstractions for stochastic systems. Construction of finite abstractions for formal verification and synthesis is initially proposed in (Abate et al. (2008)). Finite bisimilar abstractions for randomly switched stochastic systems are presented in (Zamani and Abate (2014)). Moreover, finite bisimilar abstractions for incrementally stable stochastic switched systems are discussed in (Zamani et al. (2015)). Compositional construction of infinite abstractions (reduced-order models) for jump-diffusion systems using small-gain conditions is discussed in (Zamani et al. (2017)). Compositional construction of infinite abstractions is also proposed in (Lavaei et al. (2017)) and (Lavaei et al. (2018)) using small-gain type conditions, and dissipativity conditions, respectively. Compositional construction of finite abstractions is presented in (Soudjani et al. (2017)), and (Lavaei et al. (2018)) using dynamic Bayesian networks, and dissipativity conditions, respectively. Compositional (in)finite abstractions for stochastic systems using small-gain conditions are proposed in (Lavaei et al. (2018)). Recently, compositional synthesis of large-scale stochastic systems using a relaxed dissipativity approach is proposed in (Lavaei et al. (2019)).

Our main contribution here is to provide for the first time a compositional methodology for the construction of finite MDPs for networks of stochastic switched systems. The proposed technique leverages dissipativity-type conditions to establish the compositionality results which rely on relations between subsystems and their abstractions using stochastic simulation functions. Moreover, the proposed compositionality conditions can enjoy the structure of the interconnection topology and be potentially fulfilled *independently* of the number or gains of the subsystems (cf. the first case study). As our second contribution, we propose an approach to construct finite MDPs together

⋆ This work was supported in part by the H2020 ERC Starting Grant AutoCPS and the German Research Foundation (DFG) through the grant ZA 873/1-1.

with their corresponding stochastic simulation functions for a particular class of *nonlinear* stochastic switched systems. We demonstrate the effectiveness of the proposed results by applying our techniques to two different case studies. Proofs of all statements are omitted in this work due to space limitations.

2. DISCRETE-TIME STOCHASTIC SWITCHED SYSTEMS

2.1 Notation

We denote the sets of nonnegative and positive integers by $\mathbb{N} := \{0,1,2,\ldots\}$ and $\mathbb{N}_{\geq 1} := \{1,2,3,\ldots\}$, respectively. Moreover, the symbols $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{R}_{\geq 0}$ denote, respectively, the sets of real, positive and nonnegative real numbers. Given N vectors $x_i \in \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}_{\geq 1}$, and $i \in \{1,\ldots,N\}$, we use $x = [x_1;\ldots;x_N]$ to denote the corresponding vector of dimension $\sum_i n_i$. Given a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm of x. Symbols $\mathbb{I}_n$, $\mathbf{0}_n$, and $\mathbb{1}_n$ denote the identity matrix in $\mathbb{R}^{n\times n}$ and the column vector in $\mathbb{R}^{n\times 1}$ with all elements equal to zero and one, respectively. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, is said to be a class $\mathcal{K}$ function if it is continuous, strictly increasing, and $\gamma(0) = 0$. A class $\mathcal{K}$ function $\gamma(\cdot)$ is said to be a class $\mathcal{K}_\infty$ if $\gamma(r) \to \infty$ as $r \to \infty$.

2.2 Discrete-Time Stochastic Switched Systems

Definition 1. A discrete-time stochastic switched system (dt-SS) is characterized here by the tuple

$$\Sigma = (X, P, \mathcal{P}, W, \varsigma, F, Y_1, Y_2, h_1, h_2), \tag{1}$$

where:

- $X \subseteq \mathbb{R}^n$ is a Borel space as the state space of the system. We denote by $(X, \mathcal{B}(X))$ the measurable space with $\mathcal{B}(X)$ being the Borel sigma-algebra on the state space;
- $P = \{1,\ldots,m\}$ is the finite set of modes;
- $\mathcal{P}$ is a subset of $\mathcal{S}(\mathbb{N}, P)$ which denotes the set of functions from $\mathbb{N}$ to P;
- $W \subseteq \mathbb{R}^{\bar{p}}$ is a Borel space as the *internal* input space of the system;
- ς is a sequence of independent and identically distributed (i.i.d.) random variables on a set V_ς
$$\varsigma := \{\varsigma(k) : \Omega \to V_\varsigma,\ k \in \mathbb{N}\},$$
- $F = \{f_1,\ldots,f_m\}$ is a collection of vector fields indexed by p. For all $p \in P$, the map $f_p : X \times W \times V_\varsigma \to X$ is a measurable function characterizing the state evolution of the system;
- $Y_1 \subseteq \mathbb{R}^{q_1}$ is a Borel space as the *external* output space of the system;
- $Y_2 \subseteq \mathbb{R}^{q_2}$ is a Borel space as the *internal* output space of the system;
- $h_1 : X \to Y_1$ is a measurable function as the *external* output map that maps a state $x \in X$ to its output $y_1 = h_1(x)$;
- $h_2 : X \to Y_2$ is a measurable function as the *internal* output map that maps a state $x \in X$ to its output $y_2 = h_2(x)$.

The evolution of the state of Σ, for a given initial state $x(0) \in X$, input sequence $w(\cdot) : \mathbb{N} \to W$ and switching signal $\mathbf{p}(k) : \mathbb{N} \to P$, is described by

$$\Sigma : \begin{cases} x(k+1) = f_{\mathbf{p}(k)}(x(k), w(k), \varsigma(k)), \\ y_1(k) = h_1(x(k)), \\ y_2(k) = h_2(x(k)), \end{cases} \quad k \in \mathbb{N}. \tag{2}$$

We assume that signal $\mathbf{p}$ satisfies a *dwell-time* condition (Morse (1996)) as defined in the following definition.

Definition 2. Consider a switching signal $\mathbf{p} : \mathbb{N} \to P$ and define its switching time instants as

$$\mathfrak{S}_{\mathbf{p}} := \{k \in \mathbb{N}_{\geq 1} \mid \mathbf{p}(k) \neq \mathbf{p}(k-1)\}.$$

Then, $\mathbf{p} : \mathbb{N} \to P$ has *dwell-time* $k_d \in \mathbb{N}$ if elements of $\mathfrak{S}_{\mathbf{p}}$ ordered as $k_1 \leq k_2 \leq k_3 \leq \ldots$ satisfy $k_1 \geq k_d$ and $k_i - k_{i-1} \geq k_d, \forall i \geq 2$.

We use $\Sigma_p, \forall p \in P$, to refer to system (2) with constant switching signal $\mathbf{p}(k) = p$ for all $k \in \mathbb{N}$. System Σ is called finite if X, W are finite sets, and infinite otherwise.

Given the dt-SS in (1), we are interested in *Markov policies* to control the system as defined in the following definition.

Definition 3. A Markov policy for the dt-SS Σ in (1) is a sequence $\rho = (\rho_0, \rho_1, \rho_2, \ldots)$ of universally measurable stochastic kernels ρ_n (Bertsekas and Shreve (1996)), each defined on $P = \{1,\ldots,m\}$ given $X \times W$. The class of all such Markov policies is denoted by $\bar{\Pi}_{\bar{M}}$.

The interconnected dt-SS without internal inputs and outputs, results from the interconnection of dt-SS having both internal and external signals, is indicated by the simplified tuple $(X, P, \mathcal{P}, \varsigma, F, Y, h)$ with $f_p : X \times V_\varsigma \to X$, $\forall p \in P$.

2.3 Global Markov Decision Processes

A dt-SS Σ in (1) can be *equivalently* represented as a Markov decision process (MDP) (Puterman (2014))

$$\Sigma = (X, P, \mathcal{P}, W, T_{\mathsf{x}}, Y_1, Y_2, h_1, h_2),$$

where $T_{\mathsf{x}} : \mathcal{B}(X) \times X \times P \times W \to [0,1]$, is a conditional stochastic kernel that assigns to any $x \in X$, $p \in P$, and $w \in W$ a probability measure $T_{\mathsf{x}}(\cdot|x,p,w)$ on the measurable space $(X, \mathcal{B}(X))$ so that for any set $\mathcal{A} \in \mathcal{B}(X)$,

$$\mathbb{P}(x(k+1) \in \mathcal{A} \mid x(k), \mathbf{p}(k), w(k)) = \int_{\mathcal{A}} T_{\mathsf{x}}(d\bar{x}|x(k), \mathbf{p}(k), w(k)).$$

For given $\mathbf{p}(\cdot), w(\cdot)$, the stochastic kernel T_{x} captures the evolution of the state of Σ which can be uniquely determined by the pair (ς, f) employing (2).

In this paper, we consider $\Sigma_p, \forall p \in P$, as *local* MDPs and introduce the notion of *global* Markov decision processes as in the next definition. This provides an alternative description of switched systems enabling us to represent a switched system and its finite MDP in a common framework. Note that this notion is adapted from the definition of labeled transition systems defined in (Baier and Katoen (2008)) and modified to capture the stochastic nature of the system.

Definition 4. Given a dt-SS $\Sigma = (X, P, \mathcal{P}, W, \varsigma, F, Y_1, Y_2, h_1, h_2)$, we define the associated global MDP $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}_1, \mathbb{Y}_2, \mathbb{H}_1, \mathbb{H}_2)$, where:

- $\mathbb{X} = X \times P \times \{0,\ldots,k_d - 1\}$ is the set of states. A state $(x,p,l) \in \mathbb{X}$ means that the current state of Σ is x, the current value of the switching signal is p, and the time elapsed since the latest switching time instant saturated by k_d is l;
- $\mathbb{U} = P$ is the set of *external* inputs;
- $\mathbb{W} = W$ is the set of internal inputs;
- ς is a sequence of i.i.d. random variables;
- $\mathbb{F} : \mathbb{X} \times \mathbb{U} \times \mathbb{W} \times V_\varsigma \to \mathbb{X}$ is the one-step transition function given by $(x', p', l') = \mathbb{F}((x,p,l), \nu, w, \varsigma)$ if and only if $x' = f_p(x, w, \varsigma)$, $\nu = p$, and the following scenarios hold:
  - $l < k_d - 1, p' = p$, and $l' = l + 1$: switching is not allowed because the time elapsed since the latest switch is strictly smaller than the dwell-time;
  - $l = k_d - 1, p' = p$, and $l' = k_d - 1$: switching is allowed but no switch occurs;
  - $l = k_d - 1, p' \neq p$, and $l' = 0$: switching is allowed and a switch occurs;
- $\mathbb{Y}_1 = Y_1$ is the *external* output set;
- $\mathbb{Y}_2 = Y_2$ is the *internal* output set;
- $\mathbb{H}_1 : \mathbb{X} \to \mathbb{Y}_1$ is the *external* output map defined as $\mathbb{H}_1(x,p,l) = h_1(x)$;

- $\mathbb{H}_2 : \mathbb{X} \to \mathbb{Y}_2$ is the *internal* output map defined as $\mathbb{H}_2(x,p,l) = h_2(x)$.

We associate respectively to $\mathbb{U}$ and $\mathbb{W}$ the sets $\mathcal{U}$ and $\mathcal{W}$ to be collections of sequences $\{\nu(k) : \Omega \to \mathbb{U},\ k \in \mathbb{N}\}$ and $\{w(k) : \Omega \to \mathbb{W},\ k \in \mathbb{N}\}$, in which $\nu(k)$ and $w(k)$ are independent of $\varsigma(t)$ for any $k, t \in \mathbb{N}$ and $t \geq k$. We also denote the initial conditions of p and l by p_0 and $l_0 = 0$.

Remark 1. Note that in the global MDP $\mathbb{G}(\Sigma)$ in Definition 4, we added two additional variables p and l to the state tuple of the system Σ, in which l is a counter that depending on its value allows or prevents the system from switching, and p acts as a memory to record the input.

Proposition 1. Global MDP $\mathbb{G}(\Sigma)$ in Definition 4 is itself an MDP and includes all behaviours of the switched system Σ defined in (2).

2.4 Finite Markov Decision Processes

In this subsection, we approximate a dt-SS Σ with a *finite* $\widehat{\Sigma}$ using an abstraction algorithm. This algorithm first constructs a finite partition of state set $X = \cup_i \mathsf{X}_i$ and internal input set $W = \cup_i \mathsf{W}_i$. Then representative points $\bar{x}_i \in \mathsf{X}_i$, and $\bar{w}_i \in \mathsf{W}_i$ are selected as abstract states and internal inputs.

Given a dt-SS $\Sigma = (X, P, \mathcal{P}, W, \varsigma, F, Y_1, Y_2, h_1, h_2)$ with $F = \{f_1, \ldots, f_m\}$, the constructed finite MDP $\widehat{\Sigma}$ can be represented as

$$\widehat{\Sigma} = (\hat{X}, P, \mathcal{P}, \hat{W}, \varsigma, \hat{F}, \hat{Y}_1, \hat{Y}_2, \hat{h}_1, \hat{h}_2), \tag{3}$$

with $\hat{X} = \{\bar{x}_i, i = 1, \ldots, n_x\}, \hat{W} = \{\bar{w}_i, i = 1, \ldots, n_w\}$, and $\hat{F} = \{\hat{f}_1, \ldots, \hat{f}_m\}$, where $\hat{f}_p : \hat{X} \times \hat{W} \times V_\varsigma \to \hat{X}, \forall p \in P$, is defined as

$$\hat{f}_p(\hat{x}, \hat{w}, \varsigma) = \Pi(f_p(\hat{x}, \hat{w}, \varsigma)), \tag{4}$$

and $\Pi : X \to \hat{X}$ is the map that assigns to any $x \in X$, the representative point $\bar{x} \in \hat{X}$ of the corresponding partition set containing x. The initial state of $\widehat{\Sigma}$ is also selected according to $\hat{x}_0 := \Pi(x_0)$ with x_0 being the initial state of Σ. The output maps $\hat{h}_1, \hat{h}_2$ are the same as h_1, h_2 with their domain restricted to finite state set $\hat{X}$, and the output sets $\hat{Y}_1, \hat{Y}_2$ are just image of $\hat{X}$ under h_1, h_2, respectively.

Dynamical representation provided by (4) employs the map $\Pi : X \to \hat{X}$ satisfying the inequality

$$\|\Pi(x) - x\| \leq \delta, \quad \forall x \in X, \tag{5}$$

where $\delta := \sup\{\|x - x'\|,\ x, x' \in \mathsf{X}_i,\ i = 1, 2, \ldots, n_x\}$ is the state discretization parameter. Now all the ingredients are ready to formally define the finite abstractions of global MDPs as in the following definition.

Definition 5. Given a global MDP $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}_1, \mathbb{Y}_2, \mathbb{H}_1, \mathbb{H}_2)$ associated with Σ as in the Definition 4, one can construct its finite abstraction as a finite global MDP $\widehat{\mathbb{G}}(\widehat{\Sigma}) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \hat{\mathbb{W}}, \varsigma, \hat{\mathbb{F}}, \hat{\mathbb{Y}}_1, \hat{\mathbb{Y}}_2, \hat{\mathbb{H}}_1, \hat{\mathbb{H}}_2)$, where:

- $\hat{\mathbb{X}} = \hat{X} \times P \times \{0, \ldots, k_d - 1\}$ is the set of states;
- $\hat{\mathbb{U}} = \mathbb{U} = P$ is the set of *external* inputs that remains the same as in the global MDP;
- $\hat{\mathbb{W}} = \hat{W}$ is the set of *internal* inputs;
- ς is a sequence of i.i.d. random variables;
- $\hat{\mathbb{F}} : \hat{\mathbb{X}} \times \hat{\mathbb{U}} \times \hat{\mathbb{W}} \times V_\varsigma \to \hat{\mathbb{X}}$ is the one-step transition function given by $(\hat{x}', p', l') = \hat{\mathbb{F}}((\hat{x}, p, l), \hat{\nu}, \hat{w}, \varsigma)$ if and only if $\hat{x}' = \hat{f}_p(\hat{x}, \hat{w}, \varsigma)$ as defined in (4), $\hat{\nu} = p$, and the following scenarios hold:
  - $l < k_d - 1$, $p' = p$, and $l' = l + 1$;
  - $l = k_d - 1$, $p' = p$, and $l' = k_d - 1$;
  - $l = k_d - 1, p' \neq p$, and $l' = 0$;
- $\hat{\mathbb{Y}}_1 = \{\mathbb{H}_1(\hat{x}, p, l) \,|\, (\hat{x}, p, l) \in \hat{\mathbb{X}}\}$ is the *external* output set;
- $\hat{\mathbb{Y}}_2 = \{\mathbb{H}_2(\hat{x}, p, l) \,|\, (\hat{x}, p, l) \in \hat{\mathbb{X}}\}$ is the *internal* output set;
- $\hat{\mathbb{H}}_1 : \hat{\mathbb{X}} \to \hat{\mathbb{Y}}_1$ is the *external* output map defined as $\hat{\mathbb{H}}_1(\hat{x}, p, l) = \mathbb{H}_1(\hat{x}, p, l) = h_1(\hat{x})$;
- $\hat{\mathbb{H}}_2 : \hat{\mathbb{X}} \to \hat{\mathbb{Y}}_2$ is the *internal* output map defined as $\hat{\mathbb{H}}_2(\hat{x}, p, l) = \mathbb{H}_2(\hat{x}, p, l) = h_2(\hat{x})$.

In the next section, we define the notions of stochastic pseudo-storage and simulation functions to provide an approach for compositional synthesis of interconnected dt-SS.

3. STOCHASTIC PSEUDO-STORAGE AND SIMULATION FUNCTIONS

We first introduce a notion of stochastic pseudo-storage functions for dt-SS with internal inputs and outputs. We then define a notion of stochastic simulation functions for switched systems without internal inputs and outputs. We employ these definitions mainly to quantify closeness of global MDP and its finite abstraction.

Definition 6. Consider two global MDPs $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \mathbb{W}, \varsigma, \mathbb{F}, \mathbb{Y}_1, \mathbb{Y}_2, \mathbb{H}_1, \mathbb{H}_2)$ and $\widehat{\mathbb{G}}(\widehat{\Sigma}) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \hat{\mathbb{W}}, \varsigma, \hat{\mathbb{F}}, \hat{\mathbb{Y}}_1, \hat{\mathbb{Y}}_2, \hat{\mathbb{H}}_1, \hat{\mathbb{H}}_2)$. A function $V : \mathbb{X} \times \hat{\mathbb{X}} \to \mathbb{R}_{\geq 0}$ is called a stochastic pseudo-storage function (SPStF) from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$ if there exist $\alpha \in \mathcal{K}_\infty$, $0 < \kappa < 1$, $\psi \in \mathbb{R}_{\geq 0}$, and symmetric matrix $\bar{X}$ with conformal block partitions $\bar{X}^{ij}$, $i, j \in \{1, 2\}$, such that

- $\forall (x, p, l) \in \mathbb{X}, \forall (\hat{x}, p, l) \in \hat{\mathbb{X}}$,
$$\alpha(\|\mathbb{H}_1(x, p, l) - \hat{\mathbb{H}}_1(\hat{x}, p, l)\|) \leq V((x, p, l), (\hat{x}, p, l)), \tag{6}$$
- $\forall (x, p, l) \in \mathbb{X}, \forall (\hat{x}, p, l) \in \hat{\mathbb{X}}, \forall \hat{\nu} \in \hat{\mathbb{U}}, \forall w \in \mathbb{W}, \forall \hat{w} \in \hat{\mathbb{W}}$,
$$\mathbb{E}\Big[V((x', p', l'), (\hat{x}', p', l')) | x, \hat{x}, p, l, w, \hat{w}\Big] \leq \kappa V((x, p, l), (\hat{x}, p, l)) + z^T \bar{X} z + \psi, \tag{7}$$
  where
$$z = \begin{bmatrix} w - \hat{w} \\ \mathbb{H}_2(x, p, l) - \hat{\mathbb{H}}_2(\hat{x}, p, l) \end{bmatrix}, \quad \bar{X} = \begin{bmatrix} \bar{X}^{11} & \bar{X}^{12} \\ \bar{X}^{21} & \bar{X}^{22} \end{bmatrix},$$
  and the expectation operator $\mathbb{E}$ is with respect to ς under the one-step transition of both global MDPs with $\nu = \hat{\nu}$, i.e., $(x', p', l') = \mathbb{F}((x, p, l), \hat{\nu}, w, \varsigma)$ and $(\hat{x}', p', l') = \hat{\mathbb{F}}((\hat{x}, p, l), \hat{\nu}, \hat{w}, \varsigma)$.

If there exists an SPStF V from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, this is denoted by $\widehat{\mathbb{G}}(\widehat{\Sigma}) \preceq_{\mathcal{PS}} \mathbb{G}(\Sigma)$, and the system $\widehat{\mathbb{G}}(\widehat{\Sigma})$ is called an abstraction of concrete (original) global MDP $\mathbb{G}(\Sigma)$.

Now, we modify the above notion for global MDPs without internal inputs and outputs by eliminating all the terms related to $w, \hat{w}$ which will be employed in Theorem 8 for relating interconnected systems.

Definition 7. Consider two global MDPs $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$ and $\widehat{\mathbb{G}}(\widehat{\Sigma}) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \varsigma, \hat{\mathbb{F}}, \hat{\mathbb{Y}}, \hat{\mathbb{H}})$ without internal inputs and outputs. A function $V : \mathbb{X} \times \hat{\mathbb{X}} \to \mathbb{R}_{\geq 0}$ is called a stochastic simulation function (SSF) from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$ if

- there exists $\alpha \in \mathcal{K}_\infty$ such that $\forall (x, p, l) \in \mathbb{X}, \forall (\hat{x}, p, l) \in \hat{\mathbb{X}}$,
$$\alpha(\|\mathbb{H}(x, p, l) - \hat{\mathbb{H}}(\hat{x}, p, l)\|) \leq V((x, p, l), (\hat{x}, p, l)), \tag{8}$$
- $\forall (x, p, l) \in \mathbb{X}, \forall (\hat{x}, p, l) \in \hat{\mathbb{X}}, \forall \hat{\nu} \in \hat{\mathbb{U}}$,
$$\mathbb{E}\Big[V((x', p', l'), (\hat{x}', p', l')) \,|\, x, \hat{x}, p, l\Big] \leq \kappa V((x, p, l), (\hat{x}, p, l)) + \psi, \tag{9}$$

for some $0 < \kappa < 1$, and $\psi \in \mathbb{R}_{\geq 0}$, where the expectation operator $\mathbb{E}$ is with respect to ς under the one-step transition of both global MDPs with $\nu = \hat{\nu}$, i.e., $(x', p', l') = \mathbb{F}((x, p, l), \hat{\nu}, \varsigma)$ and $(\hat{x}', p', l') = \hat{\mathbb{F}}((\hat{x}, p, l), \hat{\nu}, \varsigma)$.

If there exists an SSF V from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, this is denoted by $\widehat{\mathbb{G}}(\widehat{\Sigma}) \preceq \mathbb{G}(\Sigma)$.

The next theorem shows how SSF can be employed to compare output trajectories of two global MDPs (without internal inputs and outputs) in a probabilistic sense.

Theorem 8. Let $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$ and $\widehat{\mathbb{G}}(\widehat{\Sigma}) = (\hat{\mathbb{X}}, \hat{\mathbb{U}}, \varsigma, \hat{\mathbb{F}}, \hat{\mathbb{Y}}, \hat{\mathbb{H}})$ be two global MDPs without internal inputs and outputs. Suppose V is an SSF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$. For any random variables a and $\hat{a}$ as the initial states of the two dt-SS, any initial mode p_0, and for any external input trajectory $\hat{\nu}(\cdot) \in \hat{\mathcal{U}}$ that preserves Markov property for the closed-loop $\widehat{\mathbb{G}}(\widehat{\Sigma})$, the following inequality holds:

$$\mathbb{P}\left\{\sup_{0\leq k\leq T_d} \|y_{a\hat{\nu}}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \geq \varepsilon \,|\, [a; \hat{a}; p_0]\right\} \tag{10}$$

$$\leq \begin{cases} 1-(1-\frac{V((a, p_0, l_0), (\hat{a}, p_0, l_0))}{\alpha(\varepsilon)})(1-\frac{\psi}{\alpha(\varepsilon)})^{T_d}; \text{ if } \alpha(\varepsilon) \geq \frac{\psi}{\kappa}, \\ \frac{V((a,p_0,l_0),(\hat{a},p_0,l_0))}{\alpha(\varepsilon)}(1-\kappa)^{T_d} + \frac{\psi}{\kappa\alpha(\varepsilon)}(1-(1-\kappa)^{T_d}); \text{otherwise.} \end{cases}$$

4. COMPOSITIONAL ABSTRACTIONS FOR INTERCONNECTED SWITCHED SYSTEMS

4.1 Interconnected Stochastic Switched Systems

Suppose we are given N *concrete* stochastic switched subsystems

$$\Sigma_i = (X_i, P_i, \mathcal{P}_i, W_i, \varsigma_i, F_i, Y_{1i}, Y_{2i}, h_{1i}, h_{2i}), i \in \{1, \ldots, N\}, \tag{11}$$

with its equivalent global MDP $\mathbb{G}(\Sigma_i) = (\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i, \varsigma_i, \mathbb{F}_i, \mathbb{Y}_{1i}, \mathbb{Y}_{2i}, \mathbb{H}_{1i}, \mathbb{H}_{2i})$. Now we provide a formal definition of *interconnection* of concrete dt-SS $\Sigma_i, \forall i \in \{1, \ldots, N\}$.

Definition 9. Consider $N \in \mathbb{N}_{\geq 1}$ dt-SS $\Sigma_i = (X_i, P_i, \mathcal{P}_i, W_i, \varsigma_i, F_i, Y_{1i}, Y_{2i}, h_{1i}, h_{2i})$, and a matrix M defining the coupling between these subsystems. We require the condition $M \prod_{i=1}^N Y_{2i} \subseteq \prod_{i=1}^N W_i$ to have a well-posed interconnection. The interconnection of Σ_i, $\forall i \in \{1, \ldots, N\}$, is the concrete interconnected dt-SS $\Sigma = (X, P, \mathcal{P}, \varsigma, F, Y, h)$, denoted by $\mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$, such that $X := \prod_{i=1}^N X_i$, $P := \prod_{i=1}^N P_i$, $\mathcal{P} := \prod_{i=1}^N \mathcal{P}_i$, $F := \prod_{i=1}^N F_i$, $Y := \prod_{i=1}^N Y_{1i}$, and function $h = \prod_{i=1}^N h_{1i}$, with the internal inputs constrained according to

$$[w_1; \ldots; w_N] = M[h_{21}(x_1); \ldots; h_{2N}(x_N)].$$

Similarly, given global MDPs $\mathbb{G}(\Sigma_i) = (\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i, \varsigma_i, \mathbb{F}_i, \mathbb{Y}_{1i}, \mathbb{Y}_{2i}, \mathbb{H}_{1i}, \mathbb{H}_{2i}), i \in \{1, \ldots, N\}$, one can also define the interconnection of $\mathbb{G}(\Sigma_i)$ as $\mathcal{I}(\mathbb{G}(\Sigma_1), \ldots, \mathbb{G}(\Sigma_N))$.

4.2 Compositional Abstractions for Interconnected Global MDPs

Assume that any global MDP $\mathbb{G}(\Sigma_i) = (\mathbb{X}_i, \mathbb{U}_i, \mathbb{W}_i, \varsigma_i, \mathbb{F}_i, \mathbb{Y}_{1i}, \mathbb{Y}_{2i}, \mathbb{H}_{1i}, \mathbb{H}_{2i}), i \in \{1, \ldots, N\}$, admits its *abstract* global MDP $\widehat{\mathbb{G}}(\widehat{\Sigma}_i) = (\hat{\mathbb{X}}_i, \hat{\mathbb{U}}_i, \hat{\mathbb{W}}_i, \varsigma_i, \hat{\mathbb{F}}_i, \hat{\mathbb{Y}}_{1i}, \hat{\mathbb{Y}}_{2i}, \hat{\mathbb{H}}_{1i}, \hat{\mathbb{H}}_{2i})$ together with an SPStF V_i from $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$ to $\mathbb{G}(\Sigma_i)$ with the corresponding function, constants and the conformal block partitions denoted by α_i, κ_i, ψ_i, $\bar{X}_i$, $\bar{X}_i^{11}$, $\bar{X}_i^{12}$, $\bar{X}_i^{21}$, and $\bar{X}_i^{22}$ as in Definition 6. In the next theorem, we provide sufficient conditions to quantify the error between the interconnection of global MDPs and that of their finite abstractions in a compositional manner.

Theorem 10. Consider the interconnected global MDP $\mathbb{G}(\Sigma) = (\mathbb{X}, \mathbb{U}, \varsigma, \mathbb{F}, \mathbb{Y}, \mathbb{H})$ induced by $N \in \mathbb{N}_{\geq 1}$ global MDPs $\mathbb{G}(\Sigma_i)$. Suppose that each $\mathbb{G}(\Sigma_i)$ admits a finite abstraction $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$ together with an SPStF V_i. Then function $V((x, p, l), (\hat{x}, p, l))$ defined as

$$V((x, p, l), (\hat{x}, p, l)) := \sum_{i=1}^N \mu_i V_i((x_i, p_i, l_i), (\hat{x}_i, p_i, l_i)), \tag{12}$$

is an SSF function from $\mathcal{I}(\widehat{\mathbb{G}}(\widehat{\Sigma}_1), \ldots, \widehat{\mathbb{G}}(\widehat{\Sigma}_N))$ with coupling matrix $\hat{M}$, to $\mathcal{I}(\mathbb{G}(\Sigma_1), \ldots, \mathbb{G}(\Sigma_N))$, if $\mu_i > 0$, $i \in \{1, \ldots, N\}$, and the following matrix (in)equality and inclusion hold:

$$\begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix}^T \bar{X}_{cmp} \begin{bmatrix} M \\ \mathbb{I}_n \end{bmatrix} \preceq 0, \tag{13}$$

$$M = \hat{M}, \tag{14}$$

$$\hat{M} \prod_{i=1}^N \hat{Y}_{2i} \subseteq \prod_{i=1}^N \hat{W}_i, \tag{15}$$

where

$$\bar{X}_{cmp} := \begin{bmatrix} \mu_1 \bar{X}_1^{11} & & & \mu_1 \bar{X}_1^{12} & & \\ & \ddots & & & \ddots & \\ & & \mu_N \bar{X}_N^{11} & & & \mu_N \bar{X}_N^{12} \\ \mu_1 \bar{X}_1^{21} & & & \mu_1 \bar{X}_1^{22} & & \\ & \ddots & & & \ddots & \\ & & \mu_N \bar{X}_N^{21} & & & \mu_N \bar{X}_N^{22} \end{bmatrix}.$$

5. CONSTRUCTION OF FINITE MARKOV DECISION PROCESSES

In this section, we impose conditions on the concrete dt-SS Σ enabling us to find an SPStF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$. The required conditions are represented via some matrix inequality for a class of *nonlinear* stochastic switched systems in the next subsection.

5.1 Stochastic Switched Systems with Slope Restrictions on Nonlinearity

The class of nonlinear switched systems is given by

$$\Sigma : \begin{cases} x(k+1) = A_{\mathbf{p}(k)} x(k) + E_{\mathbf{p}(k)} \varphi_{\mathbf{p}(k)}(F_{\mathbf{p}(k)} x(k)) \\ \qquad\qquad + B_{\mathbf{p}(k)} + D_{\mathbf{p}(k)} w(k) + R_{\mathbf{p}(k)} \varsigma(k), \\ y_1(k) = C_1 x(k), \\ y_2(k) = C_2 x(k), \end{cases} \tag{16}$$

where the additive noise $\varsigma(k)$ is a sequence of independent random vectors with multivariate standard normal distributions, and $\varphi_p : \mathbb{R} \to \mathbb{R}$ satisfies

$$0 \leq \frac{\varphi_p(c) - \varphi_p(d)}{c - d} \leq \bar{a}_p, \quad \forall c, d \in \mathbb{R}, c \neq d, \tag{17}$$

for some $\bar{a}_p \in \mathbb{R}_{>0} \cup \{\infty\}$.

We use the tuple $\Sigma = (A, B, C_1, C_2, D, E, F, R, \varphi)$ to refer to the class of nonlinear switched systems of the form (16), where $A = \{A_1, \ldots, A_m\}, B = \{B_1, \ldots, B_m\}, D = \{D_1, \ldots, D_m\}, E = \{E_1, \ldots, E_m\}, F = \{F_1, \ldots, F_m\}, R = \{R_1, \ldots, R_m\}, \varphi = \{\varphi_1, \ldots, \varphi_m\}$, for the finite set of $P = \{1, \ldots, m\}$.

Here, we employ quadratic storage function of the form

$$V_p(x, \hat{x}) = (x - \hat{x})^T \tilde{M}_p (x - \hat{x}), \quad \forall p \in P, \tag{18}$$

where $\tilde{M}_p \succ 0$ is a positive-definite matrix of appropriate dimension. In order to show that a nominated V employing V_p in (18) is an SPStF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$, we raise the following assumptions. These assumptions are essential to show the main result of this section as in Theorem 11.

Assumption 1. Assume that there exists $\mu \geq 1$ such that

$$\forall x, x' \in X, \ \forall p, p' \in P : \quad V_p(x, x') \leq \mu V_{p'}(x, x'). \tag{19}$$

Remark 2. Note that Assumption 1 is a standard assumption in switched systems accepting *multiple* storage functions with *dwell-time* similar to the one appeared in (Liberzon, 2003, equation (3.6)). Given the quadratic forms of V_p, $\forall p \in P$, in (18), one can always choose $\mu \geq 1$ satisfying this assumption.

Assumption 2. Assume that for some constants $0 < \bar{\kappa}_p < 1$, and $\pi_p \in \mathbb{R}_{>0}$, there exist $\tilde{M}_p \succ 0$, and common supply rate matrix $\bar{S}$ with partitions $\bar{S}^{ij}$, $i,j \in \{1,2\}$, such that the following inequality holds:

$$\begin{bmatrix}(1+\pi_p)A_p^T\tilde{M}_pA_p & A_p^T\tilde{M}_pD_p & A_p^T\tilde{M}_pE_p\\ * & (1+\pi_p)D_p^T\tilde{M}_pD_p & D_p^T\tilde{M}_pE_p\\ * & * & (1+\pi_p)E_p^T\tilde{M}_pE_p\end{bmatrix}$$
$$\preceq \begin{bmatrix}\bar{\kappa}_p\tilde{M}_p + C_2^T\bar{S}^{22}C_2 & C_2^T\bar{S}^{21} & -F_p^T\\ \bar{S}^{12}C_2 & \bar{S}^{11} & 0\\ -F_p & 0 & 2/\bar{a}_p\end{bmatrix}. \quad (20)$$

Assumption 3. Assume that for constants μ as appeared in Assumption 1, $\bar{\kappa}_p$ as in Assumption 2, $\epsilon > 1$, and $\forall l \in \{0,\dots,k_d-2\}$, where $k_d \geq \epsilon\frac{\ln(\mu)}{\ln(1/\bar{\kappa}_p)}+1, \forall p \in P$, there exist matrices $\bar{X}^{11}$, $\bar{X}^{12}$, $\bar{X}^{21}$, and $\bar{X}^{22}$ of appropriate dimensions such that the following inequality holds:

$$\frac{1}{\bar{\kappa}_p^{(1+l)/\epsilon}}\begin{bmatrix}\bar{S}^{11} & \bar{S}^{12}\\ \bar{S}^{21} & \bar{S}^{22}\end{bmatrix} \leq \begin{bmatrix}\bar{X}^{11} & \bar{X}^{12}\\ \bar{X}^{21} & \bar{X}^{22}\end{bmatrix}.$$

Now, we provide another main result of this paper showing that under which conditions a nominated V using V_p in (18) is an SPStF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$.

Theorem 11. Consider global MDP $\mathbb{G}(\Sigma)$ associated with $\Sigma = (A,B,C_1,C_2,D,E,F,R,\varphi)$ and $\widehat{\mathbb{G}}(\widehat{\Sigma})$ as its finite abstraction with state discretization parameter δ. If Assumptions 1, 2 and 3 hold, then

$$V((x,p,l),(\hat{x},p,l)) = \frac{1}{\bar{\kappa}_p^{l/\epsilon}}V_p(x,\hat{x}), \quad (21)$$

with V_p nominated in (18) is an SPStF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$.

Remark 3. Note that if there exists a common storage function $V : X \times X \to \mathbb{R}_{\geq 0}$ between all switching modes $p \in P$ satisfying Assumptions 1, 2 and 3, then $V((x,p,l),(\hat{x},p,l)) = V(x,\hat{x})$ and Definitions 6 and 7 reduce to Definitions 3.1 and 3.2 in (Lavaei et al. (2018)) (cf. the first case study).

6. CASE STUDY

6.1 Road Traffic Network

We first apply our results to a road traffic network in a circular cascade ring which is composed of 50 identical cells, each of which has the length of 500 meters with 1 entry and 1 way out, as schematically depicted in Figure 1 left. We construct compositionally a *finite* MDP of the network such that the compositionality condition does not require any constraint on the number or gains of the subsystems. The model of this case study is borrowed from (Le Corronc et al. (2013)) by including stochasticity in the model as an additive noise. The entry is controlled by a traffic light, that enables (green light) or not (red light) the vehicles to pass. In this model the length of a cell (denoted by L) is in kilometers ($0.5\ km$), and the flow speed of the vehicles is 100 kilometers per hour (km/h). Moreover, during the sampling time interval $\tau = 6.48$ seconds, it is assumed that 8 vehicles pass the entry controlled by the green light, and one quarter of vehicles goes out on the exit of each cell (ratio denoted by q). We want to observe the density of traffic x_i, given in vehicles per cell, for each cell i of the road. The set of modes is $P_i = \{1,2\}, i \in \{1,\dots,n\}$ such that

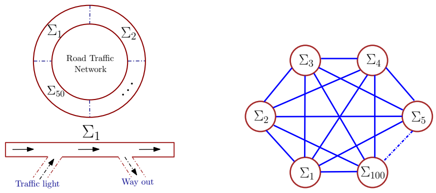

Fig. 1. Left: Model of a road traffic network of 50 identical cells. Right: A fully interconnected network of 100 nonlinear components (totally 200 dimensions).

- mode 1 means traffic light is red;
- mode 2 means traffic light is green.

The dynamic of the interconnected system is described by:

$$\Sigma : \begin{cases} x(k+1) = A\,x(k) + B_{\mathbf{p}(k)} + \varsigma(k),\\ y(k) = x(k),\end{cases}$$

where A is a matrix with diagonal elements $a_{ii} = (1 - \frac{\tau\nu_i}{L_i} - q)$, $i \in \{1,\dots,n\}$, off-diagonal elements $a_{i+1,i} = \frac{\tau\nu_i}{L_i}$, $i \in \{1,\dots,n-1\}$, $a_{1,n} = \frac{\tau\nu_n}{L_n}$, and all other elements are identically zero. Moreover, $B_p = [b_{1p_1};\dots;b_{np_n}]$, $x(k) = [x_1(k);\dots;x_n(k)]$, $\varsigma(k) = [\varsigma_1(k);\dots;\varsigma_n(k)]$, and

$$b_{ip_i} = \begin{cases}0, & \text{if} \quad p_i = 1,\\ 8, & \text{if} \quad p_i = 2.\end{cases}$$

Now, by introducing the individual cells Σ_i described as

$$\Sigma_i : \begin{cases} x_i(k+1) = (1 - \dfrac{\tau\nu_i}{L_i} - q)\,x_i(k) + D_iw_i(k) + b_{i\mathbf{p}_i(k)} + \varsigma_i(k),\\ y_{1i}(k) = x_i(k),\\ y_{2i}(k) = x_i(k),\end{cases}$$

where $D_i = \frac{\tau\nu_{i-1}}{L_{i-1}}$ (with $\nu_0 = \nu_n$, $L_0 = L_n$), one can readily verify that $\Sigma = \mathcal{I}(\Sigma_1,\dots,\Sigma_N)$, equivalently $\mathbb{G}(\Sigma) = \mathcal{I}(\mathbb{G}(\Sigma_1),\dots,\mathbb{G}(\Sigma_N))$, where the coupling matrix M is with elements $m_{i+1,i} = 1, i \in \{1,\dots,n-1\}$, $m_{1,n} = 1$, and all other elements are identically zero. Note that in this example $V_p = V_{p'}, \forall p,p' \in P$. Then one can readily verify that condition (20) (applied to linear systems with $E_p = F_p = 0, \forall p \in P$, and $\bar{S}^{ij} = \bar{X}^{ij}$, $i,j \in \{1,2\}$) is satisfied with $\tilde{M}_i = 1$, $\pi_i = 1.48$, $\bar{\kappa}_i = 0.99\ \forall i \in \{1,\dots,n\}$, and

$$\bar{X}_i = \begin{bmatrix}(\dfrac{\tau\nu_i}{l_i})^2(1+\pi_i) & (1-\dfrac{\tau\nu_i}{l_i}-q)\dfrac{\tau\nu_i}{l_i}\\ (1-\dfrac{\tau\nu_i}{l_i}-q)\dfrac{\tau\nu_i}{l_i} & -1.9(\dfrac{\tau\nu_i}{l_i})^2(1+\pi_i)\end{bmatrix}. \quad (22)$$

Then function $V_i(x_i,\hat{x}_i) = (x_i - \hat{x}_i)^2$ is an SPStF from $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$ to $\mathbb{G}(\Sigma_i)$ satisfying condition (6) with $\alpha_i(s) = s^2$ and condition (7) with $\kappa_i = 0.99$, and $\psi_i = 2.34\,\delta_i^2$.

Now, we look at $\widehat{\Sigma} = \mathcal{I}(\widehat{\Sigma}_1,\dots,\widehat{\Sigma}_N)$ with a coupling matrix $\hat{M}$ satisfying condition (14) as $\hat{M} = M$. By taking $\mu_1 = \dots = \mu_N = 1$, and using $\bar{X}_i$ as in (22), condition (13) is satisfied as

$$\begin{bmatrix}M\\ \mathbb{I}_n\end{bmatrix}^T\bar{X}_{cmp}\begin{bmatrix}M\\ \mathbb{I}_n\end{bmatrix} = (\frac{\tau\nu_i}{l_i})^2(1+\pi_i)M^TM + (1-\frac{\tau\nu_i}{l_i}-q)\frac{\tau\nu_i}{l_i}(M^T + M) - 1.9(\frac{\tau\nu_i}{l_i})^2(1+\pi_i)\mathbb{I}_n \leq 0,$$

without requiring any restrictions on the number or gains of the subsystems. Note that M^TM is an identity matrix and M^T+M is a matrix with $\bar{m}_{i,i+1} = \bar{m}_{i+1,i} = \bar{m}_{1,n} = \bar{m}_{n,1} = 1$, $i \in \{1,\dots,n-1\}$, and all other elements are identically zero. In order to show the above inequality, we used, $i \in \{1,\dots,n\}$,

$$2(1-\frac{\tau\nu_i}{l_i}-q)(\frac{\tau\nu_i}{l_i}) - 0.9(\frac{\tau\nu_i}{l_i})^2(1+\pi_i) \preceq 0,$$

employing Gershgorin circle theorem (Bell (1965)). By choosing finite internal input sets $\hat{W}_i$ of $\widehat{\Sigma}$ such that

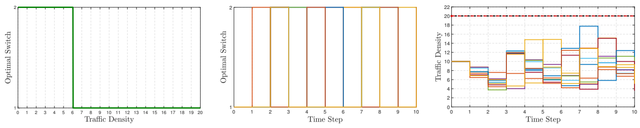

Fig. 2. Left: Optimal switch for a representative cell in a network of 50 cells. Middle: Optimal switch w.r.t. time for a representative cell. Right: Closed-loop state trajectories of a representative cell.

$\prod_{i=1}^{n} \hat{W}_i = \hat{M} \prod_{i=1}^{n} \hat{X}_i$, condition (15) is also satisfied. Hence, $V(x, \hat{x}) = \sum_{i=1}^{50} (x_i - \hat{x}_i)^2$ is an SSF from $\widehat{\mathbb{G}}(\widehat{\Sigma})$ to $\mathbb{G}(\Sigma)$ satisfying conditions (8) and (9) with $\alpha(s) = s^2$, $\kappa = 0.99$, and $\psi = \sum_{i=1}^{50} \psi_i = 117\,\delta^2$.

By taking the state discretization parameter $\delta_i = 0.02$, and choosing the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ as $10\mathbb{1}_{50}$, we guarantee that the distance between trajectories of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon = 1$ during the time horizon $T_d = 10$ with probability at least 90%, i.e.

$$\mathbb{P}(\|y_{a\nu}(k) - \hat{y}_{\hat{a}\hat{\nu}}(k)\| \leq 1, \ \forall k \in [0, 10]) \geq 0.9.$$

Let us now synthesize a controller for Σ via the abstraction $\widehat{\mathbb{G}}(\widehat{\Sigma})$ such that the *safety* controller maintains the density of traffic lower than 20 vehicles per cell. The idea here is to first design a local controller for abstraction $\widehat{\mathbb{G}}(\widehat{\Sigma}_i)$, and then refine it to system Σ_i. Consequently, controller for the interconnected system Σ would be a vector such that each of its components is the controller for subsystems Σ_i. We employ here software tool FAUST^2 (Soudjani et al. (2015)) by doing some modifications to accept internal inputs as disturbance, and synthesize a controller for Σ by choosing the standard deviation of the noise $\sigma_i = 0.83$, $\forall i \in \{1, \ldots, n\}$. Optimal switch for a representative cell in a network of 50 cells is plotted in Figure 2 left. Optimal switch w.r.t. time for a representative cell with different noise realizations is also illustrated in Figure 2 middle, with 10 realizations. Moreover, closed-loop state trajectories of the representative cell with different noise realizations are illustrated in Figure 2 right.

6.2 Switched Systems Accepting Multiple Storage Functions

In order to show applicability of our results to switched systems accepting *multiple* storage functions with *dwell-time*, we apply our proposed techniques to a fully interconnected network of 100 nonlinear subsystems (totally 200 dimensions), as illustrated in Figure 1 right, and construct their finite MDPs with guaranteed error bounds on their probabilistic output trajectories. The model of the system does not have a common storage function because it exhibits unstable behaviors for different switching signals (Liberzon (2003))(i.e., if one periodically switches between different modes, the trajectory goes to infinity). By selecting $\delta_i = 0.001$, we guarantee that the distance between trajectories of Σ and of $\widehat{\Sigma}$ will not exceed 1 during the time horizon 10 with probability at least 88%.

7. DISCUSSION

In this paper, we provided a compositional approach for the construction of finite MDPs for networks of discrete-time stochastic switched systems. First, we leveraged dissipativity-type compositional conditions for the compositional quantification of the probabilistic distance between the interconnection of stochastic switched subsystems and that of their finite abstractions. Then, we proposed an approach to construct finite MDPs together with their corresponding stochastic pseudo-storage functions for a particular class of discrete-time nonlinear stochastic switched systems. Finally, we applied our approaches to two different case studies. Compositional construction of finite MDPs for stochastic switched systems with *multiple supply rates* is under investigation as a future work.

REFERENCES

Abate, A., Prandini, M., Lygeros, J., and Sastry, S. (2008). Probabilistic reachability and safety for controlled discrete time stochastic hybrid systems. *Automatica*, 44(11), 2724–2734.

Baier, C. and Katoen, J.P. (2008). *Principles of model checking*. MIT press.

Bell, H.E. (1965). Gershgorin's theorem and the zeros of polynomials. *The American Mathematical Monthly*, 72(3), 292–295.

Bertsekas, D.P. and Shreve, S.E. (1996). *Stochastic Optimal Control: The Discrete-Time Case*. Athena Scientific.

Lavaei, A., Soudjani, S., Majumdar, R., and Zamani, M. (2017). Compositional abstractions of interconnected discrete-time stochastic control systems. In *Proceedings of the 56th IEEE Conference on Decision and Control*, 3551–3556.

Lavaei, A., Soudjani, S., and Zamani, M. (2018). Compositional construction of infinite abstractions for networks of stochastic control systems. *arXiv: 1801.10505*.

Lavaei, A., Soudjani, S., and Zamani, M. (2018). Compositional (in)finite abstractions for large-scale interconnected stochastic systems. *arXiv: 1808.00893*.

Lavaei, A., Soudjani, S., and Zamani, M. (2018). From dissipativity theory to compositional construction of finite Markov decision processes. In *Proceedings of the 21st ACM International Conference on Hybrid Systems: Computation and Control*, 21–30.

Lavaei, A., Soudjani, S., and Zamani, M. (2019). Compositional synthesis of large-scale stochastic systems: A relaxed dissipativity approach. *arXiv:1902.01223v2*.

Le Corronc, E., Girard, A., and Goessler, G. (2013). Mode sequences as symbolic states in abstractions of incrementally stable switched systems. In *Proceedings of the 52th IEEE Conference on Decision and Control*, 3225–3230.

Liberzon, D. (2003). *Switching in systems and control*. Springer Science & Business Media.

Morse, A.S. (1996). Supervisory control of families of linear set-point controllers-part i. exact matching. *IEEE transactions on Automatic Control*, 41(10), 1413–1431.

Pnueli, A. (1977). The temporal logic of programs. In *Proceedings of the 18th Annual Symposium on Foundations of Computer Science*, 46–57. IEEE.

Puterman, M.L. (2014). *Markov decision processes: discrete stochastic dynamic programming*. John Wiley & Sons.

Soudjani, S., Abate, A., and Majumdar, R. (2017). Dynamic Bayesian networks for formal verification of structured stochastic processes. *Acta Informatica*, 54(2), 217–242.

Soudjani, S., Gevaerts, C., and Abate, A. (2015). FAUST^2: Formal abstractions of uncountable-state stochastic processes. In *TACAS'15*, volume 9035 of *Lecture Notes in Computer Science*, 272–286.

Zamani, M. and Abate, A. (2014). Approximately bisimilar symbolic models for randomly switched stochastic systems. *Systems & Control Letters*, 69, 38–46.

Zamani, M., Abate, A., and Girard, A. (2015). Symbolic models for stochastic switched systems: A discretization and a discretization-free approach. *Automatica*, 55, 183–196.

Zamani, M., Rungger, M., and Mohajerin Esfahani, P. (2017). Approximations of stochastic hybrid systems: A compositional approach. *IEEE Transactions on Automatic Control*, 62(6), 2838–2853.