Introduction

This supplement to Scientific Report No. 11 (February, 1975) consists of the following:

i) a short list of errata;

ii) a revision of the beginning of section V;

iii) a revision of two portions of section VI;

iv) a revision of the entire Appendix B;

v) an additional appendix (Appendix G) dealing with an application of the theory to the Goubau line; and

vi) a number of additional references.

Items i) - iv) were made necessary by the discovery of an error in the derivation of the real (geometric) part of the correction to the propagation constant, which arose from an inconsistent use of the various coordinate systems involved. The corrected formula (20) has a very clear and suggestive physical interpretation.

The application of the theory to the Goubau line was motivated by current interest in the radiation properties of this structure, and by the apparent absence of any previous analysis of its radiation losses due to curvature.

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Errata

Equation (17), p. 17, should read

$$k_o(\nu - \nu_o) \simeq \frac{\Delta}{P} - i\,\frac{c}{P} \tag{17}$$

Equation (20), p. 18, should read

$$\Delta = \frac{\omega}{R}\int_S x[\mu_o \overline{H}_o \cdot \overline{H}_o + \varepsilon \overline{E}_o \cdot \overline{E}_o]\ dS \tag{20}$$

The comments made regarding Arnaud's [13] work do not apply if the proper interpretations of his quantities are made. A discussion of this point appears in [38] . Furthermore, Lewin has located an error of $\frac{1}{2}$ in his result and published a correction [39]. All three results for the fiber are thus now in agreement.

The first equation for Δ on p. 45 (two up from (E.1)) should read

$$\Delta = \frac{1}{R}\left\{\frac{1}{\omega\mu_o}\int_{-\infty}^{\infty}\hat{x}\left[k_o^2(\nu_o^2 + n^2)E_{o\hat{z}}^2 - \left(\frac{\partial E_{o\hat{z}}}{\partial\hat{x}}\right)^2\right]d\hat{x}\right\}$$

In the following two equations, the quantity i should be eliminated.

SECTION V*

GEOMETRIC CORRECTIONS TO THE PROPAGATION CONSTANT

Since the waveguide has been assumed lossless, we have seen in Appendix B that the correction Δ/P to the propagation constant is real,[2] and in fact suggests a geometrical interpretation as a shift of the phase velocity reference point to the center of gravity of the "energy" $\omega(\mu_o \bar{H}_o \cdot \bar{H}_o + \varepsilon \bar{E}_o \cdot \bar{E}_o)$ from the arbitrarily chosen reference point within the guide. This interpretation is reinforced by the fact, shown in Appendix C, that this $1/R$ correction vanishes for a mode on a symmetrical structure which itself possesses certain symmetry properties about the chosen origin of the local coordinate system. This is a result which is well-known in the case of certain closed waveguides [24] and for open slab waveguides [9], but not, apparently, in the general case.

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*This supersedes the first paragraph of section V (p.19) of the original report.

SECTION VI*

RADIATION CORRECTIONS TO THE PROPAGATION CONSTANT

The imaginary correction to the propagation constant is given by $-ic/P$. In contrast to the geometric correction of the previous section, it depends critically upon the perturbation fields $\bar{E}_p$ and $\bar{H}_p$. Before proceeding to the evaluation of c, it is interesting to note the similarity of the form of the term $-ic/P$ to a power balance relation [26], as well as to a modal coupling coefficient for surface waves [13,22]. These observations allow a number of possible physical interpretations of this quantity, for instance, the "power" lost (radiated) through the contour C, or Arnaud's [13] interpretation as coupling to a whispering gallery mode propagating along an artifically introduced perfectly conducting cylinder, which is allowed to approach infinity in a manner which circumvents the mathematical difficulty that such modes are not normalizable in the absence of the cylinder. The authors prefer to think of (18) as representing co-directional coupling to a second, image guide whose fields are $\bar{E}_{pr}$ and $\bar{H}_{pr}$, the portions of $\bar{E}_p$ and $\bar{H}_p$ reflected from the turning point. An investigation of crosstalk carried out by Arnaud [37] suggests that such fields could be produced by the (fictitious) image of the actual guide in a semi-infinite lossy medium, or by an (actual) second guide separated from the original one by a lossy layer. Comparison of the equations in [37] with the expression for radiation loss of a slab (see Appendix E) shows that for a suitable choice of parameters of the layer, this analogy is quite close, and the distance between the guide and its image can actually be identified (see section VII).

We now argue in a similar manner to Appendix B. We assume R_1 and R_2 are taken far enough away from the guide so that essentially all of the "power flow" is included in (19) (strictly speaking, of course, P is not a power since no complex conjugation is involved), but not so far that R_1 is near or past the WKB turning point. This may be done provided that $k_o\lambda(\nu_o,s)R >> 1$ for all s, or simply $k_o\lambda_oR >> 1$, where we have abbreviated

$$\lambda_o = \lambda(\nu_o,0)$$

Now, in equation (21), all terms from the dominant part of $\bar{E}_p^{\pm}$ and $\bar{H}_p^{\pm}$ (i.e., $\bar{E}_n^{\pm}$, $\bar{H}_n^{\pm}$ - see Appendix B) can be seen to be imaginary by examining (D.1). These, as was argued in Appendix B, are smaller than any order of $(k_oR)^{-1}$, and hence do not affect the real part of $\nu - \nu_o$. In the terms of $O(\sigma_o)$, however, not only is the contribution real, but the opposite exponential dependences on $\hat{x}$ cancel (i.e., the decaying fields $\bar{E}_o$, $\bar{H}_o$ and the locally growing parts of $\bar{E}_p,\bar{H}_p$), leaving a result independent of the choice of R_1. The main contribution, then, comes from C_1, and is given by:

This supersedes the last paragraph of section VI (pp. 23-24) of the original report, up to but not including eqn. (22).

Appendix B*

In order to derive an expression for $\nu-\nu_o$, it is most convenient to utilize a local coordinate system which is not distorted, as is the case with $(\hat{x}, \hat{y}, \hat{z})$. We thus choose a local coordinate system† (x,y,z) with $x = \rho = R$, $y = R\theta$ and $z = z$, in which x now is an actual distance from the curved axis of the waveguide. Since we can write

$$\hat{x} = x - \tfrac{1}{2}\frac{x^2}{R} + \frac{1}{3}\frac{x^3}{R^2} - \ldots \tag{B.1}$$

there will be no difficulty in using the results of section III, because the difference between $\hat{x}$ and x will not affect the accuracy retained in our calculations. Thus, from here on, these two coordinate systems will be considered interchangeable.

Now if $\overline{E}^+$, $\overline{H}^+$ and ν are the guide fields and propagation constant for the curved waveguide which satisfy (13), and $\overline{E}_o^-$, $\overline{H}_o^-$ and ν_o the fields and propagation constant for the corresponding straight waveguide which satisfy‡

$$\begin{aligned} \nabla_t \times \overline{E}_o^{\pm} \mp ik_o\nu_o\overline{a}_y \times \overline{E}_o^{\pm} &= -i\omega\mu_o\overline{H}_o^{\pm} \\ \nabla_t \times \overline{H}_o^{\pm} \mp ik_o\nu_o\overline{a}_y \times \overline{H}_o^{\pm} &= i\omega\varepsilon\overline{E}_o^{\pm} \end{aligned} \tag{B.2}$$

where $\nabla_t = \overline{a}_x\frac{\partial}{\partial x} + \overline{a}_z\frac{\partial}{\partial z}$ is the transverse del operator in the local <u>physical</u> system (x,y,z), and is related to the curl_t operator defined after equation (13) by

$$\text{curl}_t\, \overline{A} = \nabla_t \times \overline{A} + \overline{a}_z\frac{A_\theta}{\rho} \tag{B.3}$$

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*This Appendix supersedes Appendix B (pp.37-39) of the original report.

†Not to be confused with the global system (x,y,z) of Fig. 2.

‡Note that in eqn.(16) we must understand $\varepsilon = \varepsilon(\hat{x},\hat{z})$, whereas (13) and (B.2) contain $\varepsilon(x,z)$, i.e., in a physical coordinate system.

Applying (15) to the vector $\bar{F} = \bar{E}_o^- \times \bar{H}^+ - \bar{E}^+ \times \bar{H}_o^-$, we then find

$$\oint_c \frac{\rho}{R}[\bar{E}_o^- \times \bar{H}^+ - \bar{E}^+ \times \bar{H}_o^-]\cdot \bar{a}_n d\ell = \int_S \frac{\rho}{R}\, \mathrm{div}_t\, \bar{F} dS$$

$$= \int_S \{ik_o(\nu-\nu_o \frac{\rho}{R})\bar{a}_y \cdot [\bar{E}_o \times \bar{H} + \bar{E} \times \bar{H}_o] + \frac{1}{R}[E_z H_{oy} - E_{oy} H_z]\} dS \tag{B.4}$$

where fields without a superscript are understood to be "+".

For the moment we will not specify S or C. If we denote the perturbation fields by $\bar{E}_p = \bar{E} - \bar{E}_o$ and $\bar{H}_p = \bar{H} - \bar{H}_o$, we have

$$\oint_c \frac{\rho}{R} [\bar{E}_o^- \times \bar{H}_p^+ - \bar{E}_p^+ \times \bar{H}_o^-] \cdot \bar{a}_n d\ell = 2ik_o(\nu-\nu_o) \int_S \bar{a}_y \cdot \bar{E}_o \times \bar{H}_o dS$$

$$-2ik_o \frac{\nu_o}{R} \int_S x\bar{a}_y \cdot \bar{E}_o \times \bar{H}_o dS - \frac{1}{R}\int_S \bar{a}_x \cdot \bar{E}_o \times \bar{H}_o \, dS$$

$$+ \frac{1}{R}\int_S (E_{pz}H_{oy} - E_{oy}H_{pz}) dS + ik_o \int_S (\nu-\nu_o \frac{\rho}{R})\bar{a}_y \cdot [\bar{E}_o \times \bar{H}_p + \bar{E}_p \times \bar{H}_o] dS \tag{B.5}$$

We note that $\bar{a}_x = \nabla_t x$, so that

$$\nabla_t \cdot (x\bar{E}_o \times \bar{H}_o) = \bar{a}_x \cdot \bar{E}_o \times \bar{H}_o + x\nabla_t \cdot (\bar{E}_o \times \bar{H}_o) \tag{B.6}$$

and thus, using (B.2), (B.5) can be rewritten as

$$c = ik_o(\nu-\nu_o)P - i\Delta - \frac{1}{R} \oint_c x\, \bar{E}_o \times \bar{H}_o \cdot \bar{a}_n d\ell + \frac{1}{R} \int_S (E_{pz}H_{oy} - E_{oy}H_{pz}) dS$$

$$+ ik_o \int_S (\nu-\nu_o \frac{\rho}{R})\bar{a}_y \cdot [\bar{E}_o \times \bar{H}_p + \bar{E}_p \times \bar{H}_o] \, dS \tag{B.7}$$

where c, P and Δ are given by equations (18) - (20). To establish the validity of eqn. (17), then, it must be shown that all other real corrections to ν are smaller than $O(1/R)$, and that all other imaginary corrections are of smaller order than c/P.

On the basis of section III, we may assume that the perturbation fields and ν have asymptotic developments of the form*

$$\bar{E}_p^{\pm} \sim \frac{1}{k_o R}\bar{E}_1^{\pm} + \frac{1}{(k_o R)^2}\bar{E}_2^{\pm} + \ldots + O(\sigma_o) \qquad (|x|/R << 1) \tag{B.8}$$

$$\nu \sim \nu_o + \frac{\nu_1}{k_o R} + \frac{\nu_2}{(k_o R)^2} + \cdots + O(\sigma_o) \tag{B.9}$$

with a similar expression for $\bar{H}_p^{\pm}$. Here σ_o is given by (11) with $s=0$. By substitution into (13), and comparison of coefficients of like powers of $(k_o R)^{-1}$, we obtain

$$\nabla_t \times \bar{E}_{n+1}^{\pm} \mp ik_o\nu_o\bar{a}_\theta \times \bar{E}_{n+1}^{\pm} + i\omega\mu_o\bar{H}_{n+1}^{\pm} = -k_o x\, \nabla_t \times \bar{E}_n^{\pm} - k_o\bar{a}_z E_{n\theta}^{\pm} - i\omega\mu_o k_o x\bar{H}_n^{\pm}$$
$$\pm\, ik_o \sum_{j=0}^{n} \nu_{j+1}\,\bar{a}_\theta \times \bar{E}_{n-j}^{\pm}$$

$$\nabla_t \times \bar{H}_{n+1}^{\pm} \mp ik_o\nu_o\bar{a}_\theta \times \bar{H}_{n+1}^{\pm} - i\omega\varepsilon\bar{E}_{n+1}^{\pm} = -k_o x\nabla_t \times \bar{H}_n^{\pm} - k_o\bar{a}_z H_{n\theta}^{\pm} + i\omega\varepsilon k_o x\bar{E}_n^{\pm}$$
$$\pm\, ik_o \sum_{j=0}^{n} \nu_{j+1}\bar{a}_\theta \times \bar{H}_{n-j}^{\pm} \tag{B.10}$$

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*The validity of such an expansion implies that the curved guide mode is indeed merely a slight perturbation of the corresponding straight guide mode. This will not be the case, for example, with so-called "edge-guided" or "whispering gallery" modes which, due to a curvature-induced or -shifted caustic, are not well-approximated by any straight guide mode [14, 32-34]. In such cases, it would be necessary to know $\bar{E}_p$ and $\bar{H}_p$ independently in order to apply the present method.

for $n \geq 0$, while $\bar{E}_o^{\pm}$ and $\bar{H}_o^{\pm}$ satisfy (B.2). Manipulations of (B.2) and (B.10) similar to those which produced (B.7) yield

$$\oint_C [\bar{E}_o^- \times \bar{H}_{n+1}^+ - \bar{E}_{n+1}^+ \times \bar{H}_o^-] \cdot \bar{a}_n d\ell = k_o \int_S (E_{oz}^- H_{n\theta}^+ + H_{oz}^- E_{n\theta}^+)\, dS$$

$$+ k_o \int_S x\{\bar{E}_o^- \cdot [\nabla_t \times \bar{H}_n^+ - i\omega\varepsilon \bar{E}_n^+] + \bar{H}_o^- \cdot [\nabla_t \times \bar{E}_n^+ + i\omega\mu_o \bar{H}_n^+]\}dS$$

$$+ ik_o \sum_{j=0}^{n} \nu_{j+1} \int_S \bar{a}_\theta \cdot [\bar{E}_o^- \times \bar{H}_{n-j}^+ - \bar{E}_{n-j}^+ \times \bar{H}_o^-]\, dS \qquad \text{(B.11)}$$

We now proceed to argue by induction. It is known (see [22] , Appendix C therein) that for a general lossless waveguide, it is possible to choose the longitudinal fields $(E_\theta^{\pm}, H_\theta^{\pm})$ to be real, and consequently the transverse fields $(E_x^{\pm}, E_z^{\pm}, H_x^{\pm}, H_z^{\pm})$ are all imaginary. This will thus be the case for $\bar{E}_o$ and $\bar{H}_o$. Assume therefore, that this further holds for $\bar{E}_1^{\pm},\ldots, \bar{E}_n^{\pm}, \bar{H}_1^{\pm},\ldots, \bar{H}_n^{\pm}$, and that $\nu_o,\ldots,\nu_n$ are known to be real. By choosing the boundary C of the surface S sufficiently far from the guide (see also section VI), the contour integral in (B.11) can be made arbitrarily exponentially small by reason of the surface wave nature of the fields. It can then be concluded from our hypotheses that ν_{n+1} must be real. Similar considerations using (B.10) then allow us to conclude that $\bar{E}_{n+1}^{\pm}$ and $\bar{H}_{n+1}^{\pm}$ also possess the same phase relationships as $\bar{E}_o^{\pm}$ and $\bar{H}_o^{\pm}$. By induction therefore, these statements are true for any value of n.

Let us now examine the various terms on the right side of (B.7). Again the contour integral (which is imaginary) may be made smaller than any inverse order of k_oR by a suitable choice of C, and thus is negligible compared to $i\Delta$ (also imaginary). In the first surface integral, the imaginary part is seen to be $O(1/R^2)$ and so negligible, while the real part is $O(\sigma_o/R)$ and also negligible compared to c, which is (see section VII) at worst $O(\sigma_o/R^{\frac{1}{2}})$. The remaining surface integral can be disposed of in much the same fashion. Equation (17) has thus been established. Let us note in conclusion that the above stipulations regarding C and S will be made somewhat more quantitative when they are used again in section VI .

APPENDIX G*

As a further example, let us consider the lowest order (axially symmetric) TM mode on a Goubau line [35]. No formula for the bending loss of such lines seems to have appeared in the literature, and the only experimental results of which the authors are aware have been obtained for very small bend radii, to which the present theory is inapplicable [36]. For a Goubau line consisting of an inner conductor of circular cross-section with radius a, and an annular dielectric coating of outer radius b, and refractive index n, situated in free space, the fields for the mode under consideration are [35]:

$$E_y = A[J_o(k_o\kappa_o r) - Q_o(\nu_o)Y_o(k_o\kappa_o r)] \qquad a < r < b$$

$$= A[J_o(k_o\kappa_o b) - Q_o(\nu_o)Y_o(k_o\kappa_o b)]K_o(k_o\lambda_o r)/K_o(k_o\lambda_o b) \quad r > b$$

$$E_r = (\nu_o Z_o/\varepsilon_r)H_\phi \quad ; \quad H_\phi = (i\omega\mu_o)^{-1}[\varepsilon_r/(\varepsilon_r - \nu_o^2)]\ \partial E_y/\partial r$$

where

$$\kappa_o = (n^2 - \nu_o^2)^{\frac{1}{2}} \quad ; \quad Q_o(\nu_0) = J_o(k_o k_o a)/Y_o(k_o\kappa_o a)$$

$$\varepsilon_r = n^2 \quad (a < r < b) \quad \text{or} \quad 1 \quad (r > b)$$

$$Z_o = (\mu_o/\varepsilon_o)^{\frac{1}{2}}$$

A is some constant, and J_o, Y_o, K_o are the Bessel functions of the first kind, second kind, and modified Bessel functions of the first kind, respectively. Continuity of E_y already being satisfied in the above, continuity of H_ϕ at $r = b$ yields the eigenvalue equation

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*This Appendix is new.

$$-\frac{n^2}{\kappa_o}\ \frac{J_1(k_o\kappa_o b) - Q_o(\nu_0)Y_1(k_o\kappa_o b)}{J_o(k_o\kappa_o b) - Q_o(\nu_o)Y_o(k_o\kappa_o b)} = \frac{1}{\lambda_o}\ \frac{K_1(k_o\lambda_o b)}{K_o(k_o\lambda_o b)}$$

The P-integral (19) can be shown to be

$$P = 2\int_a^\infty \int_o^{2\pi} E_r H_\phi \, r dr d\phi$$

$$= -\frac{2\pi\nu_o A^2}{Z_o}\left\{ b^2(1-n^2)\left[\frac{n^2}{\kappa_o^2}F_1^2(\nu_o) + \frac{1}{\lambda_o^2}F_o^2(\nu_o) + \frac{n^2}{\lambda_o^2\kappa_o^2}F_o(\nu_0)F_2(\nu_0)\right]\right.$$

$$\left. - \frac{4n^2}{\pi^2 k_o^2\kappa_o^4 Y_o^2(k_o\kappa_o a)}\right\}$$

(Note that, though not evident, $P < 0$ if A is real)

where

$$F_j(\nu_0) = J_j(k_o\kappa_o b) - Q_o(\nu_0)Y_j(k_o\kappa_o b).$$

Now for $r > b$,

$$E_z = -E_r\sin(\phi-\phi_o) = i(\nu_o AF_o(\nu_o)/\lambda_o K_o(k_o\lambda_o b))K_1(k_o\lambda_o r)\sin(\phi-\phi_o)$$

$$H_z = -H\cos(\phi-\phi_o) = i(AF_o(\nu_0)/\lambda_o Z_o K_o(k_o\lambda_o b))K_1(k_o\lambda_o r)\cos(\phi-\phi_o)$$

where ϕ_o is the mode polarization angle, so that from Appendix F,

$$\tilde{E}(0) = -i(\nu_o AF_o(\nu_o)/2\lambda_o^2 K_o(k_o\lambda_o b))\ \sin\phi_o$$

$$\tilde{H}(0) = +i(AF_o(\nu_0)/2\lambda_o^2 Z_o K_o(k_o\lambda_o b))\ \cos\phi_o$$

so that

$$c = -\frac{\pi}{k_o Z_o}\left(\frac{\pi}{k_o R}\right)^{\frac{1}{2}} e^{-2\tau_o}\left(\frac{A^2F_o^2(\nu_o)}{2\lambda_o^{7/2}K_o^2(k_o\lambda_o b)}\right)(\nu_o^2\sin^2\phi_o + \cos^2\phi_o)$$

where τ_o is once more given by (26) with $\nu = \nu_o$ and $\lambda = \lambda_o$. Just as in the optical fiber case, the radiative attenuation c/P is nearly independent of the mode polarization if $\nu_o \simeq 1$, but becomes more polarization dependent for slower surface waves. Thus

$$c/P = \left(\frac{\pi}{k_o R}\right)^{\frac{1}{2}} e^{-2\tau_o} f(\phi_o) \frac{F_o^2(\nu_o)}{4k_o\nu_o\lambda_o^{7/2}K_o^2(k_o\lambda_o b)D(\nu_o)} \qquad (G.1)$$

where

$$f(\phi_o) = 1 + \lambda_o^2 \sin^2 \phi_o$$

$$D(\nu_o) = b^2(1-n^2)\left[\frac{n^2}{\kappa_o^2} F_1^2(\nu_o) + \frac{1}{\lambda_o^2} F_o^2(\nu_o) + \frac{n^2}{\lambda_o^2\kappa_o^2} F_o(\nu_o)F_2(\nu_o)\right] - \frac{4n^2}{\pi^2 k_o^2 \kappa_o^4 Y_o^2(k_o\kappa_o a)}$$

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