Abstract

The problem of the reflection of current at the end of a semi-infinite wire is solved using Wiener-Hopf techniques. It is found that the end effect can be expressed as a single reflection parameter which is a function of the height of the wire, the electrical properties of the earth, and the frequency of the excitation. This simplification yields an accurate representation for the reflected current for points not very close to the end of the wire. Representation of the end effect in this manner leads to a simple theory for long finite horizontal antennas similar to that found in lumped parameter transmission line problems. Input conductance changes as a function of earth parameter for a resonant and an anti-resonant antenna are then presented.
PART III

ANALYSIS OF SEMI-INFINITE AND FINITE-THIN WIRE ANTENNAS

3.1. Introduction

The object of this part of the research is to find the input conductance and the current distribution of a finite horizontal wire in the presence of a conducting earth. The problem has been approached by several methods. Berry and Chrisman[1] have solved for the radiation resistance of a resonant half-wavelength antenna near the earth. In their technique, the earth is replaced by an image source weighted by a Fresnel reflection coefficient and a sinusoidal current distribution is assumed. Thus, their results are not applicable if the wire is close to the earth or if the earth is not highly conducting. Miller et al.[2] have solved the exact integral equation for the current on a finite horizontal antenna over a conducting earth. Since the solution of this exact equation is a very tedious numerical process, they used an approximate kernel for the integral equation which is valid over a wider range than Berry and Chrisman's solution but still requiring that the antenna not be too close to the earth. The integral equation approach is a useful one for antennas less than a wavelength long.

In the following section an approximate solution to the long horizontal antenna problem is found which is valid for much smaller heights than Miller's approximate solution. The approach is basically the same as used by Shen[3] in his analysis of long antennas in free
space. The theory can be shown, in the case of antennas close to the
earth, to have a lower limit of applicability much smaller than free
space antenna theory.

3.2. Development of the Wiener-Hopf Equation
for the Semi-Infinite Antenna

An approximate integral equation for the current on a thin
horizontal antenna excited by a delta function voltage source near a
conducting earth was given in Part II as Equation 2.4a. Its solu-
tion was found in closed form for the case of an infinitely long
antenna. The current distribution obtained for that antenna can now
be used to solve for the current reflected from the end of a semi-
infinite antenna via the Wiener-Hopf technique. Knowledge of the
current reduces the long finite antenna problem to a simple boundary
value problem such as those found in lumped parameter transmission
line problems. The input conductance and current distribution of the
finite antenna, obtained in this manner, can be expressed in closed
form.

Consider an antenna of finite length $2\lambda$ ($0 \leq x \leq 2\lambda$) excited
at $x = \lambda$ with a delta function generator of magnitude $V$ (Fig. 3.1).
The integral equation for the current on this antenna was given in
Eq. 2.4.

$$\int_{0}^{2\lambda} I(x') M(x - x') \, dx' = \frac{4}{c_0} \frac{V}{k} \delta(x - \lambda) \quad 0 \leq x \leq 2\lambda \quad (3.1)$$

$I(x) = 0$ outside the range $0 \leq x \leq 2\lambda$. It is assumed that all
approximations made previously are retained. The expression for
$M(x - x')$ in terms of its Fourier transform has been previously
Fig. 3.1
Geometry of the Displaced Finite Antenna
in Eq. 1.23. In general I(x) can be written as

\[ I(x) = I_\infty(x) + \phi(x); \quad -\infty < x \leq 2\lambda \]  

(3.2a)

where

\[ \phi(x) = \begin{cases} 
\phi_+(x) & ; 2\lambda \geq x > 0 \\
\phi_-(x) = -I_\infty(x); & x < 0 
\end{cases} \]  

(3.2b)

and \( I_\infty(x) \) is the current on an infinitely long antenna with its source at \( x = \lambda \). With this definition the integral equation can be written

\[ \frac{\epsilon_0 k^2}{4} \int_{-\infty}^{2\lambda} [I_\infty(x') + \phi(x')] M(x - x') \, dx' \]

\[ = \begin{cases} 
V \delta(x - \lambda) & 0 \leq x \leq 2\lambda \\
g_-(x) & -\infty < x \leq 0 
\end{cases} \]  

(3.3)

The lower limit can be extended to \( -\infty \) because the integrand is zero for \( x < 0 \). \( g_-(x) \) is unknown and represents the axial electric field for \( x < 0 \) on a fictitious cylindrical surface collinear with and of the same radius as the antenna. At this time \( \lambda \) and \( V \) are allowed to increase without bound in such a way that the magnitude of \( I_\infty(x) \) at \( x = 0 \) is unity. \( \phi_+(x) \) can be identified as a single reflected current wave.

\[ \phi_+(x) = I_R(x) \quad x > 0 \]  

(3.4)
The integral equation for current on an infinitely long antenna can be subtracted from Eq. 3.3 leaving

\[ \int_{-\infty}^{\infty} \phi(x') M(x-x') \, dx' = \begin{cases} 0; & x > 0 \\ g_-(x); & x < 0 \end{cases} \quad (3.5) \]

the constants having been absorbed into \( g_-(x) \).

The convolution theorem\(^{[4]}\) can now be applied to Eq. 3.3 to obtain

\[ \{\phi_+(\alpha) + \phi_-(\alpha)\} M(\alpha) = G_-(\alpha) \quad (3.6) \]

where

\[ \phi_+(\alpha) = \frac{1}{2\pi} \int_{0}^{\infty} \phi_+(x) \exp(-ik_1\alpha x) \, dx; \]

\[ \phi_-(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{0} \phi_-(x) \exp(-ik_1\alpha x) \, dx; \]

\[ G_-(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{0} g_-(x) \exp(-ik_1\alpha x) \, dx; \quad (3.7) \]

and \( M(\alpha) \) is given in Eq. 1.23. According to Eq. 3.7, it is not difficult to see that the functions with subscripts \( +(-) \) are regular in the lower (upper) half of the complex \( \alpha \)-plane.

3.3. Solution of the Wiener-Hopf Equation

The first step in solving the Wiener-Hopf equation as given in Eq. 3.6 is to factorize \( M(\alpha) \) such that
\[ M(\alpha) = M_+(\alpha) M_-(\alpha) \]  

where \( M_+(\alpha) \) is regular in the lower half of the complex \( \alpha \) plane and has no zeros there, and \( M_-(\alpha) \) is regular in the upper half of the complex \( \alpha \) plane and has no zeros there. A region of the complex \( \alpha \) plane in which both \( M_+(\alpha) \) and \( M_-(\alpha) \) are regular is necessary when analytic continuation arguments are used. This region can be obtained by mathematically allowing the air to become slightly lossy. \( (k_1 \) acquires a small imaginary part.)

Dividing through Eq. 3.6 by \( M_-(\alpha) \) gives

\[ \phi_+(\alpha) M_+(\alpha) + \phi_-(\alpha) M_+(\alpha) = \frac{G_-(\alpha)}{M_-(\alpha)} \]  

(3.9)

The product \( \phi_-(\alpha) M_+(\alpha) \) can be decomposed such that

\[ \phi_-(\alpha) M_+(\alpha) = F(\alpha) = F_+(\alpha) + F_-(\alpha) \]  

(3.10)

where \( F_+(\alpha) \) is regular in the lower half of the complex \( \alpha \) plane, and \( F_-(\alpha) \) is regular in the upper half plane. With the substitution of Eq. 3.10, Eq. 3.9 can be rearranged so that all terms regular in the upper half plane are on one side of the equation and all terms regular in the lower half plane are on the other side. Since both sides are regular in a common region, they must be equal to an entire function by analytic continuation arguments. Thus

\[ \phi_+(\alpha) M_+(\alpha) + F_+(\alpha) = \frac{G_-(\alpha)}{M_-(\alpha)} - F_-(\alpha) = P(\alpha) \]  

(3.11)
Equation 3.11 is then valid over the entire complex \( \alpha \) plane. \( P(\alpha) \) is an arbitrary entire function which can be evaluated by using information derived from the asymptotic expansions of the unknowns. These asymptotic expansions can be obtained from the electromagnetic edge conditions which require the axial electric field or \( g_-(x) \) to \( O(|x|^{-1/2}) \) as \( |x| \to 0 \)\(^{[5]} \) and \( \phi_+(x) \) be bounded as \( x \to 0 \).

From the fact that \( \phi_+(x) \) is bounded as \( x \to 0 \), it follows that \( \phi_+(\alpha) \) is \( O(\alpha^{-1}) \) as \( |\alpha| \to \infty \) in the lower half plane. On the other hand, when \( |\alpha| \to \infty \), the dominant term of \( M(\alpha) \) is \( \zeta^2 H_0^{(1)}(A\alpha)J_0(A\alpha) \) so that the asymptotic values of \( M_+(\alpha) \) can be evaluated as \( O(\alpha^{1/2})^{[6]} \). It is shown in Appendix D that \( F_\pm(\alpha) \) are \( O(\alpha^{-1/2}) \). Using Eq. 3.10, the asymptotic expansions just mentioned and the extended form of Louisvilles' Theorem\(^{[7]} \), it readily follows that \( P(\alpha) = 0 \) and

\[
\Phi_+(\alpha) = -\frac{F_+(\alpha)}{M_+(\alpha)} \tag{3.12}
\]

The inverse Fourier transform of Eq. 3.12 then gives an approximate expression for the reflected current.

The function \( M(\alpha) \) has been factorized into \( M_+(\alpha) \) and \( M_-(\alpha) \) using an adaptation of the method given by Mittra and Lee\(^{[5]} \). The details of the method are discussed in Appendix E.

The decomposition of \( F(\alpha) \) will now be discussed. From Eq. 3.10 and the results of Appendix D, we have

\[
F(\alpha) = \Phi_-(\alpha) M_+(\alpha) = \frac{4}{\Sigma_{S=1}^{\infty}} \frac{1}{2\pi k_1} \frac{M_+(\alpha)}{\alpha + \alpha_S} \tag{3.13}
\]
For ease of presentation, the notation $\alpha_s \ s = 1, 4$ has been adopted to represent the four singularities of $M(\alpha)$ ($\alpha_1 = \alpha_{p1}$, $\alpha_2 = \alpha_{p2}$, $\alpha_3 = 1$, $\alpha_A = \alpha_B$). $F(\alpha)$ is regular in the lower half plane except at each location $-\alpha_s$ where there is a pole. Therefore $F_+(\alpha)$ can be found by subtracting from $F(\alpha)$ the pole singularities. Thus

$$F_+(\alpha) = \frac{1}{2\pi ik_1} \sum_{s=1}^{4} \left\{ \frac{M_+(\alpha)}{\alpha + \alpha_s} - \frac{M_+(-\alpha_s)}{\alpha + \alpha_s} \right\}$$

$$F_-(\alpha) = \frac{1}{2\pi ik_1} \sum_{s=1}^{4} \frac{M_+(-\alpha_s)}{\alpha + \alpha_s}$$  \hspace{1cm} (3.14)

It can be seen then that $F_\pm(\alpha)$ are $O(\alpha^{-1/2})$ as stated previously. It should be noted that to compute the value of $M_+(\alpha_s)$ the following relationship can be used\[6\].

$$M_-(\alpha_s) = M_+(-\alpha_s)$$  \hspace{1cm} (3.15)

3.4. The Reflected Current

Using Eqs. 3.12 and 3.14, the inverse Fourier Transform and assuming a single component of the incident current, the reflected current can be written as

$$\phi_+(x) = -\frac{\varepsilon_0}{2} \frac{M_+(-\alpha_s)}{\pi \varepsilon_0} \int_{-\infty}^{\infty} \frac{M_-(\alpha) \exp(ik_1\alpha x)}{(\alpha + \alpha_s) M(\alpha)} d\alpha$$  \hspace{1cm} (3.16)

where the other term is zero since it is regular in the upper half plane. Equation 3.16 can be evaluated by deforming its contour of integration into the upper half plane as was done for the infinite
antenna analysis. The integration can then be expressed as a sum of four terms.

\[
\phi_{+s}(x) = \frac{-\varepsilon_0}{2} \frac{M_\perp(-\alpha_3)}{\varepsilon_0(\alpha_1 + \alpha_s)} \exp(ik_1\alpha_1x) \left[ \frac{-2 M_\perp(\alpha_1)}{M'(\alpha_1)} \right]
\]

\[
- \frac{2 M_\perp(\alpha_2)}{\varepsilon_0(\alpha_2 + \alpha_s)} \frac{\exp(ik_1\alpha_2x)}{M'(\alpha_2)}
\]

\[
+ \frac{i}{\pi \varepsilon_0} \int_{\Gamma_3} \frac{M_\perp(\alpha) \exp(ik_1\alpha x)}{(\alpha + \alpha_s) M(\alpha)} d\alpha
\]

\[
+ \frac{i}{\pi \varepsilon_0} \int_{\Gamma_4} \frac{M_\perp(\alpha) \exp(ik_1\alpha x)}{(\alpha + \alpha_s) M(\alpha)} d\alpha
\]

(3.17)

where \( \Gamma_3 \) and \( \Gamma_4 \) are respectively contours surrounding the branch cut ending at \( \alpha_3 = 1 \) and the branch cut ending at \( \alpha_4 = \alpha_B \) (Fig. 2.3).

For each of the branch cut integrations, the major contribution comes from near the branch point provided that \( k_1 \) is not small. Under this condition, the branch integrations can be approximated as

\[
\frac{i}{\pi \varepsilon_0} \frac{M_\perp(\alpha_3)}{(\alpha_3 + \alpha_s)} \int_{\Gamma_3} \frac{\exp(ik_1\alpha x)}{M(\alpha)} d\alpha
\]

and

\[
\frac{i}{\pi \varepsilon_0} \frac{M_\perp(\alpha_4)}{(\alpha_4 + \alpha_s)} \int_{\Gamma_4} \frac{\exp(ik_1\alpha x)}{M(\alpha)} d\alpha.
\]

(3.18)

Each of the four terms of Eq. 3.17 can now be recognized as some reflection parameter multiplied by one component of the current on
the infinitely long antenna. Thus, the use of Eqs. 3.15, 3.17, and 3.18 yields the following approximate solution:

\[ \phi_{+s}(x) = \sum_{r=1}^{4} R_{sr} I_{\infty r}(x) \]  \hspace{1cm} (3.19)

where

\[ R_{sr} = \frac{-\varepsilon_0 M_1^r(\alpha_s) M_2^r(\alpha_r)}{2(\alpha_r + \alpha_s)} \]  \hspace{1cm} (3.20)

and \( I_{\infty r}(x) \) is the \( r \)th component of the infinite antenna current.

The total reflected current is then

\[ \phi_+(x) \approx \sum_{s=1}^{4} \sum_{r=1}^{4} R_{sr} I_{\infty r}(x) \]  \hspace{1cm} (3.21)

Due to the approximation made in Eq. 3.17, Eq. 3.21 is evidently not valid for calculating reflected currents near the end of the antenna where currents \( I_{\infty 3}(x) \) and \( I_{\infty 4}(x) \) constitute a major part of the total current.

3.5. The Finite Antenna

The total current on a finite antenna can now be considered as the current from the source plus current generated by reflections at the ends of the antenna. In the following derivation it will be assumed that the source is located at the center of the antenna although this assumption is not necessary (Fig. 3.2). The total current can be written as
Fig. 3.2
Geometry of the Finite Antenna
\[ I(x) = \sum_{s=1}^{4} (I_{s,x} + c_s (I_{s,x} - l) + I_{s,x} + l)). \quad (3.22) \]

The \( c_s \) are to be determined from the reflection properties of the end of the antenna as follows. The current incident at \( x = l \) is

\[ I^{\text{inc}}(x = l) = \sum_{s=1}^{4} \{I_{s,x} + c_s I_{s,x} (2l)\} \quad (3.23) \]

The \( r \)th component of the reflected current is

\[ I^{\text{ref}}(x = l) = \sum_{s=1}^{4} R_{s,r} (I_{s,x} + c_s I_{s,x} (2l)) \quad (3.24) \]

The \( r \)th component of the reflected current is also equal to \( c_r \). Thus for each \( r \)

\[ c_r = \sum_{s=1}^{4} R_{s,r} (I_{s,x} + c_s I_{s,x} (2l)) \quad r = 1,4 \quad (3.25) \]

In matrix form Eq. 3.25 can be written

\[ [c] = [R] ([I_1] + [I_2][c]) \quad (3.26) \]

where \([I_1]\) is the column matrix

\[ [I_1] = \begin{bmatrix} I_{s,1}(l) \\ I_{s,2}(l) \\ I_{s,3}(l) \\ I_{s,4}(l) \end{bmatrix} \quad (3.27) \]
and \([I_{2\lambda}]\) is the diagonal matrix

\[
[I_{2\lambda}] = \begin{bmatrix}
I_{\infty_1(2\lambda)} & 0 & 0 & 0 \\
0 & I_{\infty_2(2\lambda)} & 0 & 0 \\
0 & 0 & I_{\infty_3(2\lambda)} & 0 \\
0 & 0 & 0 & I_{\infty_4(2\lambda)}
\end{bmatrix}.
\] (3.28)

An inversion of the square matrix \([[1] - [R][I_{2\lambda}]]) thus yields the solution

\[
[c] = ([1] - [R][I_{2\lambda}])^{-1} ([R][I_{\lambda}]).
\] (3.29)

The current distribution for a finite antenna is then determined by substituting Eq. 3.29 into Eq. 3.22. Also, the input conductance is readily given as the real part of the current at the excitation.

\[
G_{IN} = \text{Real } [I(x = 0)]
\] (3.30)

It should be noted that in the evaluation of the reflected current the value of \(M_-(\alpha)\) at \(\alpha_s\) and \(\alpha_r\) is needed. However, since \(M_-(\alpha)\) is regular in the upper half of the complex \(\alpha\) plane, it can be expanded in a Taylor series around \(\alpha_r\) and \(\alpha_s\). Thus for values of \(\alpha\) close to \(\alpha_{r,s}\), \(M_-(\alpha)\) will not differ significantly from \(M_-(\alpha_r,s)\) unless \(M'_-(\alpha_{r,s})\) is large. From an examination of Fig. 1.9, it can be seen that the locations at which \(M_-(\alpha)\) is evaluated are very closely spaced provided \(d\) is not too small. Thus, it is to be expected that \(M_-(\alpha)\) evaluated at the four values of \(\alpha\) will have
approximately the same value unless \( d \) is very small. When \( M_\ldots(\alpha) \) has this behavior it is possible to simplify Eq. 3.29 by making the substitution

\[
R_{sr} = R \quad s = 1-4 \\
r = 1-4
\]

(3.31)

The matrix \([c]\) becomes a scalar and is equal to

\[
c = \frac{R I_\infty(\lambda)}{1 - R I_\infty(2\lambda)}
\]

(3.32)

where \( I_\infty(\lambda) \) and \( I_\infty(2\lambda) \) are the total currents on an infinite antenna evaluated at \( x = \lambda \) and \( x = 2\lambda \) respectively. The current on the finite antenna under this approximation is then

\[
I(x) = I_\infty(x) + c (I_\infty(x - \lambda) + I_\infty(x + \lambda))
\]

(3.33)

3.6. Results

It is to be expected that as the earth becomes very highly conducting and as \( d \) becomes very small compared to a wavelength, the antenna system should behave as an excited, open circuited transmission line. It is known that for an open circuited two-wire transmission line consisting of infinitely thin wires, the ratio between the incident current and the reflected current at the end is \(-1^{[8]}\). The expressions for the reflected and incident currents will now be examined to verify their consistency with this fact.
The current incident at the end of the antenna has been assumed to be equal to unity. The reflected current is

\[ I_R(x) = R I_\infty(x) \]  \hspace{1cm} (3.34)

It was shown in Chapter 2 that for very large earth conductivities and for small \( d/\lambda \) essentially the only contribution to the current comes from the TEM pole \( \alpha_1 \) as would be expected from transmission line theory. From Eq. 2.8, the TEM contribution to the current at \( x = 0 \) has the value

\[ I_\infty(0) = \frac{4i}{\xi_0} \frac{1}{M'(\alpha_1)} \]  \hspace{1cm} (3.35)

where \( \alpha_1 \approx 1 \) under the condition \( n \) large, \( d/\lambda \) small according to Eq. 1.25. \( \alpha_2 \) and \( \alpha_4 \) are also approximately equal to 1.

It has been shown in Appendix E that under the assumed conditions

\[ R = -i\xi_0 \frac{M(\alpha = 0)}{2} \]  \hspace{1cm} (3.36)

Also,

\[ M(\alpha) \approx 2 \frac{(1 - \frac{\alpha^2}{1\pi})}{\frac{\pi}{\pi}} \ln 2D/A \]

and

\[ M'(\alpha) \approx \frac{i4\alpha}{\pi} \ln 2D/A. \]  \hspace{1cm} (3.37)

Substitution of Eqs. 3.35, 3.36, and 3.37 into Eq. 3.34 yields
\[ I_R(0) = - I_\infty(x) \]  \hspace{1cm} (3.38)

as expected.

Figure 3.3 shows an excellent agreement between the magnitude \((|R|)\) and phase \((\phi)\) of the reflected current (assuming a unity incident current) for an antenna above a near perfect conductor and the reflected current of a two wire open circuited transmission line given by Wu\[^9\] and others\[^10,11\]. For this curve the term \(Q(\alpha)\) of \(M(\alpha)\) was not included since, as indicated in Chapter 1, it is less important than the other terms for calculating the transmission line pole current contribution when the earth is very highly conducting and when \(d/\lambda\) is small.

The general relations for the reflection parameters and the input conductance of a finite antenna will now be evaluated. The first step is to examine the validity of the simplified expressions for the current distribution given in Eqs. 3.33, 3.32, and 3.31 under the assumption of a single reflection coefficient \((R = R_{r,s} \text{ for } r,s \in 1,2,3,4)\). Table 3.1 shows values of \([M_-(\alpha)]^2\) evaluated at \(\alpha_1, \alpha_2,\) and \(\alpha_3\) for the case \(n = 7.43 + i \ 6.73, a = 10^{-2}\lambda,\) and \(d = .15\lambda, .55\lambda.\) It is apparent from the table that it is possible to use the simplified expression for current in the range of parameters presented. Outside of this range (smaller \(d\)) the current has been calculated by assuming that \(M_-(\alpha)\) has one value at \(\alpha = \alpha_1\) and a second value at \(\alpha = \alpha_2, \alpha_3, \) and \(\alpha_4.\) This choice is made in light of the fact that, for small \(d, \alpha_2, \alpha_3,\) and \(\alpha_4\) are closely spaced in the complex \(\alpha\) plane while \(\alpha_1\) is far removed.
TEM REFLECTED CURRENT $I_R(0) = |R| \exp(i \phi)$

$a = 0.0005 \lambda$
$|n| = 10^{10}$

$\phi \times 10^2$

$(1 - |R|) \times 10^4$

---

Fig. 3.3
Transmission Mode Reflected Current
TABLE 3.1
VARIATION OF $M_2^2(\alpha_i)$ IN $\alpha$ PLANE

<table>
<thead>
<tr>
<th>$d/\lambda$</th>
<th>$i$</th>
<th>$\alpha_i$</th>
<th>$M_2^2(\alpha_i)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>.15</td>
<td>1</td>
<td>1.0104 + i .0146</td>
<td>2.832 + i 2.114</td>
</tr>
<tr>
<td>.15</td>
<td>2</td>
<td>0.9986 + i .0043</td>
<td>2.823 + i 2.092</td>
</tr>
<tr>
<td>.15</td>
<td>3</td>
<td>1.0 + i 0.0</td>
<td>2.832 + i 2.120</td>
</tr>
<tr>
<td>.55</td>
<td>1</td>
<td>1.0015 + i .0059</td>
<td>2.536 + i 1.874</td>
</tr>
<tr>
<td>.55</td>
<td>2</td>
<td>.99895+ i .0014</td>
<td>2.530 + i 1.885</td>
</tr>
<tr>
<td>.55</td>
<td>3</td>
<td>1.0 + i 0.0</td>
<td>2.528 + i 1.893</td>
</tr>
</tbody>
</table>

Figure 3.4 shows the value of $R$ (calculated at $\alpha = \alpha_2$) as a function of $d/\lambda$ for the parameters $n = 7.43 + i 6.73$ and $a/\lambda = 10^{-2}$. The straight lines are values of $R$ computed by Hallén [12] for an antenna in free space. As expected, $R$ oscillates about this free space value. The value of $R$ is quite insensitive to the refractive index of the earth. Table 3.2 illustrates the small difference in $R$ calculated for refractive indexes $n = 7.43 + i 6.73$ and $n = 5.52 + i 4.53$.

TABLE 3.2
REFLECTION PARAMETERS

<table>
<thead>
<tr>
<th>$d/\lambda$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>7.43 + i 6.73</td>
</tr>
<tr>
<td>.1</td>
<td>-277 - i 163</td>
</tr>
<tr>
<td>.25</td>
<td>-229 - i 207</td>
</tr>
<tr>
<td>.5</td>
<td>-232 - i 171</td>
</tr>
</tbody>
</table>
Figure 3.5 shows the input conductance of a finite resonant and anti-resonant antenna for two values of refractive index as a function of \(d/\lambda\). The straight lines are input conductances of antennas in free space of the same length\(^{[12]}\).

It is apparent that the input conductance of the resonant antenna is more sensitive to the refractive index of the earth than is the input conductance of the anti-resonant antenna. The reason for this behavior will be seen below.

For geologic remote sensing applications it is desirable to design an antenna which has an input conductance that is very sensitive to variations in the refractive index of the earth. To this end, it is important to examine the characteristics of wave propagation on the finite wire antenna to determine which ones are most influenced by variations in the earth's refractive index. The antenna can then be designed to maximize the influence of these characteristics on the input conductance.

Setting \(x = 0\) in Eq. 3.33

\[
G_{IN} = G_{\infty} + \frac{2R \tau_0^2(\xi)}{1 - R \tau_0(2\xi)}
\]  

(3.39)

where

\[
I_0(\xi) = G_{\infty 1} \exp(-i\alpha_1 k_1 \xi) + G_{\infty 2} \exp(-i\alpha_2 k_1 \xi)
\]

\[
+ I_{B1}(\xi) + I_{B2}(\xi)
\]  

(3.40)

According to Chapter 2, \(I_{B1}(x)\) is slowly varying except near \(x = 0\) and has the approximate phase variation \(\exp(-ik_1 x)\). \(I_{B2}(x)\) varies approximately as \(G_{\infty 4} \exp(-i\alpha_3 k_1 x)\).
Fig. 3.5
Input Conductances of Resonant and Anti-resonant Antennas
As illustrated in Chapter 2, the first term of Eq. 3.39 is quite insensitive to changes in the refractive index of the earth. Thus, the effect of the refractive index of the earth is mainly due to the second term of Eq. 3.39. As shown in Fig. 3.5, resonant antennas are more sensitive to refractive index than are anti-resonant antennas. This is evident also from the fact that the denominator of the second term of Eq. 3.39 is nearly zero for resonant antennas. $G_{IN}$ is then more sensitive to changes in the reflection parameters, the excitation factors, and propagation constants of the component currents and the length of the antenna. It should be noted that since the current on the wire has a phase factor different than $k_1$, the resonant lengths for the antenna will not occur at $N\lambda/4$. Generally the resonant lengths (defined as the length for which $G_{IN}$ is maximum) are shorter than $N\lambda/4$. As shown in Figs. 3.6 and 3.7, the sensitivity to the earth's refractive index is enhanced when the antenna is at a resonant length.

Earlier, it was shown that the reflection parameter $R$ is insensitive to changes in the earth's refractive index. Thus, the factors which influence the sensitivity are the excitation factors and propagation constants of the component currents. The propagation constant of the transmission current is more sensitive to the refractive index than is that of the fast wave. Thus, it is desirable to couple as much energy as possible into the transmission line wave. The maximum excitation of the transmission current occurs for small antenna heights as shown in Figs. 2.6 and 2.8. In addition, for small antenna heights, the propagation constant of the transmission current is more sensitive to changes in the earth's
Fig. 3.6
Input Conductance Near a Resonant Length
Fig. 3.7

Input Conductance Near a Resonant Length

\[ d = 0.1\lambda \]
\[ a = 0.01\lambda \]
refractive index. Thus, for remote sensing purposes, it is desirable to locate the antenna close to the earth. This corresponds to the physically intuitive idea that an antenna which is heavily coupled to the earth will be more sensitive to changes in its electrical properties.

It should be noted that the effect of differences in the propagation constant can be magnified by using longer antennas. The longer the antenna, the larger effect that the propagation constant has on the current and thus the input conductance due to the factors $\exp(-ikz)$. This procedure has an upper limit, however, because for very long antennas the resonance phenomena are less pronounced. Figures 3.6 and 3.7 illustrate this point. For the longer antenna of Fig. 3.7, the percentage change in input conductance is larger; however, its resonance is less pronounced.

3.7. Concluding Remarks

In this part of the research the current distribution and input conductance of long horizontal antennas over a conducting earth have been found. The Wiener-Hopf technique has been used to compute the end effects for a semi-infinite antenna. From this, the solution for the long antenna has been constructed via multi-reflection techniques.

It has been shown that, for geologic remote sensing applications, antennas should be located close to the ground in order to achieve maximum coupling. Resonant antennas have been shown to be most sensitive to the earth's electrical parameters. Lastly, it has been shown that longer resonant antennas are more sensitive than
short ones. There is an upper limit on this because the resonance phenomenon is less pronounced for longer antennas.

3.8. References


APPENDIX D

CALCULATION OF $\phi_-(\alpha)$

From Eq. 3.26, $\phi_-(x)$ is equal to $-I_\infty(x)$. Therefore it can be found from the known current on the infinitely long antenna. It is known from an earlier part of the work that the current on the infinite antenna is composed of four parts. Two are due to the zeros of the modal equation and have the form $\exp(-i\alpha_1 x)$. For these two currents $\phi_-(\alpha)$ is found to be

$$\phi_-(\alpha) = \frac{1}{2\pi ik_1} \frac{1}{(\alpha + \alpha_1)^2} \quad (D.1)$$

The other two parts of the infinite antenna current result from the branch cut integrations and do not have a simple form as a function of $x$. However, as is evident from Fig. 2.5 in Part II the current due to the branch cut associated with $\alpha_3 = 1$ varies rapidly near the source but becomes slowly varying as it moves away from the source. Its phase variation away from the source over a short distance is approximately that of $\exp(-ik_1 x)$, and its amplitude is approximately constant. Thus for the region near the end of a long antenna this current is taken to be $\exp(-ik_1 x)$. The technique of using an approximate current has been successfully exploited by Shen in his treatment of the long cylindrical antenna in free space. As he points out, however, this approximation does limit the theory to "long antennas"
(i.e., antennas such that \( I_\alpha(x) \) near \( x = \lambda \) is of approximately the same form as assumed in calculating \( \Phi_-(\alpha) \) where \( \lambda \) is the half length of the antenna. The long antenna criteria, however, are not as restrictive in the case of antennas near the earth as they are in free space. This is due to the fact that close to the earth the current on the infinite antenna is dominated by the discrete modal currents for which no approximation need be made to calculate \( \Phi_-(\alpha) \).

Thus, for the third portion of the current, we can assume the form of

\[
\Phi_-(\alpha) = \frac{1}{2\pi k_1} \frac{1}{(\alpha + 1)} \tag{D.2}
\]

An examination of the current due to the branch cut associated with \( \alpha_4 = \alpha_B \) shows that the form \( \exp(-i\beta k_1 x) \) can be assumed. Thus

\[
\Phi_-(\alpha_4) = \frac{1}{2\pi k_1} \frac{1}{(\alpha + \alpha_B)} \tag{D.3}
\]
APPENDIX E

FACTORIZATION OF $M(\alpha)$

$M(\alpha)$ can be written as

$$M(\alpha) = J_0(A_\zeta) \cdot G(\alpha) \quad (E.1)$$

where

$$G(\alpha) = \zeta^2 \left[ H_0^{(1)}(A_\zeta) - H_0^{(1)}(2D_\zeta) \right] + P(\alpha) - Q(\alpha)$$

Note that, in $G(\alpha)$, $J_0(A_\zeta)$ has been set equal to one, since it can be shown that the terms of $G(\alpha)$ which would contain $J_0(A_\zeta)$ are important only in regions of the complex alpha plane where the approximation is valid. The factorization can be carried out on each term separately. When the two terms are combined, the product must have the asymptotic behavior stated in Section 3.3 for $M_\pm(\alpha)$.

The Bessel function $J_0(A_\zeta)$ can be factorized by a method described by Mittra and Lee for meromorphic functions.

$$[J_0(A_\zeta)]_\pi = \sqrt{J_0(A)} \prod_{m=1}^{\infty} \left[ 1 + \frac{\alpha}{i_{Y_m}} \right] \exp(i\alpha A/m) \quad (E.2)$$

where
\[ \gamma_m = \left( \frac{e_m}{A} \right)^2 - 1 \right)^{1/2} \]

and \( e_m \) is the \( m \)th ordered zero of \( J_0(x) \). Equation E.2 can be written as

\[ [J_0(A\zeta)]_+ = \sqrt{J_0(A)} \prod_{m=1}^{\infty} \frac{[1 - \frac{i\alpha}{\gamma_m}]}{\frac{1 - \frac{i\alpha}{\gamma_m}}{\gamma_m}} \prod_{m=1}^{\infty} \frac{[1 - \frac{i\alpha}{\gamma_m}]}{\gamma_m} \exp(i\alpha A/m\tau) \]

\[ \gamma_m = \left( \frac{m - 1/4}{A} \right) \pi \] and is the asymptotic expression for \( \gamma_m \).

Also, the identity

\[ \prod_{m=1}^{\infty} \frac{[1 - \frac{i\alpha}{\gamma_m}]}{\frac{1 - \frac{i\alpha}{\gamma_m}}{\gamma_m}} \exp(i\alpha A/m\tau) = \frac{\exp(i c_e A\alpha/\tau) \Gamma(3/4)}{\Gamma(3/4 - iA\alpha/\tau)} , \]

where \( \Gamma(x) \) is the Gamma function and \( c_e = 0.57721 \) is Euler's constant, can be used. Equation E.3 can then be approximated by realizing that the ratio of the infinite products will, after a finite number of terms, be equal to \( 1 + \varepsilon(M) \) where \( \varepsilon(M) \rightarrow 0 \) as \( M \rightarrow \infty \). Therefore, the infinite product can be truncated with as small an error as desired. Equation E.3 can be written

\[ [J_0(A\zeta)]_+ = \sqrt{J_0(A)} \prod_{m=1}^{M} \left( 1 - \frac{i\alpha}{\gamma_m} \right) \frac{\exp(i c_e A\alpha/\tau) \Gamma(3/4)}{\Gamma(3/4 - iA\alpha/\tau)} \]

(E.5)

For the numerical results given in this work \( M = 2 \) is usually satisfactory since \( A \) is small.

The factorization of \( G(\alpha) \) is accomplished through the use of a modification of the factorization formula given by Mittra and Lee.
It was shown that for a function \( G(\alpha) \) which satisfies the following conditions

1. For slightly lossy free space \((k_l \text{ slightly complex})\), \( G(\alpha) \) is regular in a strip around the real axis \((\alpha = \sigma + i\tau)\)
2. \( G(\alpha) \) is even and non-zero in the strip \( G(-\alpha) = G(\alpha) \neq 0 \).
3. \( G(\alpha) \) is analytic
4. \( G(\alpha) \sim B \alpha^{v_1} \exp(-v_2|\alpha|) \) as \( \sigma \to \infty \) \((v_1, v_2 \text{ real constants})\)

the factorization of \( G(\alpha) = G_+(\alpha) \ G_- (\alpha) \) is known as \( G_+ (\alpha) = G_- (-\alpha) \) and

\[
G_-(\alpha) = \sqrt{G(0)} \ (1 + \alpha)^{v_1/2} \ \exp(A_-(\alpha) + \frac{ik_1v_2\gamma}{\pi} \ \ln(\alpha - \gamma))
\]

\[
- \frac{ik_1v_2}{2}
\]

(E.6)

where

\[
A_-(\alpha) = \int_{-\infty}^{\infty} Q(\beta) \ \ln \frac{\beta + \alpha}{\beta} \ d\beta
\]

\[
Q(\beta) = \frac{\beta k_1v_2}{2\pi(1 - \beta^2)^{1/2}} + \frac{\beta v_1}{2\pi i(1 - \beta^2)} + \frac{G'(\beta)}{2\pi i \ G(\beta)}
\]

\[
\gamma = i(\alpha^2 - 1)^{1/2} = (1 - \alpha^2)^{1/2} \ ; \ \text{Im}(1 - \alpha^2)^{1/2} > 0
\]

The integral \( A_-(\alpha) \) can be evaluated by deformation of its contour of integration into the upper half \( \beta \) plane. \( G(\beta) \) has the following singularities outside the strip of regularity

1. branch points at \( \beta = \pm 1 = \beta_3; \ \text{Re}(\beta^2 - 1)^{1/2} > 0 \)
2. branch points at \( \beta = \pm \beta_B = \beta_4; \ \lambda_B = (\beta_B^2 - \beta^2)^{1/2}; \ \text{Im} \ \lambda_B > 0 \).
3. zeros at ± $\beta_1$
4. zeros at ± $\beta_2$

In addition, it is assumed that $G'(\beta)/G(\beta)$ has a pole at $\beta_B$. No similar pole exists at $\beta = 1$. As depicted in Fig. E.1, the integration path can be deformed in the upper half plane into the branch cut integration along $T_1$, the branch cut integration along $T_2$, plus the residues of the three poles. Under the transformations $s = (\beta_B^2 - \beta^2)^{1/2}$ the integral $T_2(\alpha)$ becomes

![Diagram](image)

Fig. E.1 Complex Beta Plane
\[ T_2(\alpha) = \frac{-1}{2\pi i} \int_0^\infty \left\{ \frac{G_1^0[(\beta_B^2 - s^2)^{1/2}]}{G_0^0[(\beta_B^2 - s^2)^{1/2}]} - \frac{G_1^0[(\beta_B^2 - s^2)^{1/2}]}{G_0^0[(\beta_B^2 - s^2)^{1/2}]} \right\} \ln \left\{ \frac{\alpha + \sqrt{\beta_B^2 - s^2}}{\sqrt{\beta_B^2 - s^2}} \right\} \frac{s \, ds}{\sqrt{\beta_B^2 - s^2}} \]  \quad (E.7)

where \( G^0 \) is the derivative with respect to \( \beta \) and \( 0, \pi \) refers to the argument of \( s \) in evaluating \( G \). On the other hand, Mittra and Lee have shown that the integral \( T_1(\alpha) \) can be expressed as

\[ T_1(\alpha) = \int_0^\infty k(w) \ln \left[ 1 + \frac{\alpha}{(1 - w^2)^{1/2}} \right] \, dw + \ln (1 + \alpha)^{-1/2} \]  \quad (E.8)

\[ = T_{1A}(\alpha) + T_{1B}(\alpha) \]

where

\[ K(w) = \frac{k_1 v_2}{\pi} - \frac{1}{2\pi i} \left[ B(w) + B(W \exp(i\pi)) \right] \]

and

\[ B(w) = \frac{G_1^0(\sqrt{1 - w^2})}{G(\sqrt{1 - w^2})} \]

The residues at \( \beta = \beta_1 \) and \( \beta_2 \) can be evaluated as follows:

\[ \lim_{\beta \to \beta_1} \frac{G'_{1}(\beta)}{G(\beta)} \left( \beta - \beta_1 \right) \ln \frac{\beta + \alpha}{\beta} = \ln \left(1 + \frac{\alpha}{\beta_1} \right)^{1/2} \]  \quad (E.9)
The residues at \( \beta_B \) can be computed in the following way. Near \( \beta_B \), \( G(\beta) \) is approximately

\[
G(\beta) \approx \frac{-4i\beta^2}{n^3(1 - \beta^2 - 1/n^2)} \exp(-i2D/n) \tag{E.10}
\]

Thus, \( G'(\beta) \) can be computed approximately,

\[
G'(\beta) \approx \frac{-4i\beta^3}{n^3(1 - \beta^2 - 1/n^2)} \exp(-i2D/n) \tag{E.11}
\]

The ratio of the two is then

\[
\frac{G'(\beta)}{G(\beta)} = \frac{\beta}{(\beta_B + \beta)(\beta_B - \beta)} \tag{E.12}
\]

and the residue is

\[
\lim_{\beta \to \beta_B} (\beta - \beta_B) \frac{G'(\beta)}{G(\beta)} \ln \frac{\beta + \alpha}{\beta} = -\frac{1}{2} \ln \frac{\beta_B + \alpha}{\beta_B} \tag{E.13}
\]

An asymptotic expansion of \( G(\alpha) \) reveals that the constants \( v_1 \) and \( v_2 \) are equal to zero and \( A \) respectively. From this it follows that \( T_{1B}(\alpha) = 0 \). The complete factorization of \( G(\alpha) \) is then

\[
G_-(\alpha) = \sqrt{G(0)} \left(1 + \frac{\alpha}{\alpha_1}(1 + \frac{\alpha}{\alpha_2})(1 + \frac{\alpha}{\alpha_4})^{-1/2}\right)
\]

\[
\exp(T_{1A}(\alpha) + T_2(\alpha) + i \frac{Av}{\pi} \ln (\alpha - \gamma) - \frac{iA}{2}) \tag{E.14}
\]
From Eq. E.6 it can be shown that the asymptotic expansion of Eq. E.14 is

\[ G_{-}(\alpha) = \exp\left(\frac{-iA\alpha}{\pi} \ln 2\alpha\right) \quad (E.15) \]

From Eq. E.5 and the asymptotic expansion of the Gamma function the asymptotic behavior of \( J_{-}(\alpha) \) is

\[ J_{-}(\alpha) = \frac{\exp\left(i c_e A \alpha/\pi\right)}{\exp\left(i\alpha A/\pi\right) \exp\left[\left(-\frac{i\alpha A}{\pi}\right)\left(\ln \frac{2\pi}{A} - c_e + 1 + i \pi/2\right)\right]} \quad (E.16) \]

Thus the asymptotic behavior of \( M_{-}(\alpha) \) is

\[ \exp\left[\left(-\frac{iA\alpha}{\pi}\right)\left(\ln \frac{2\pi}{A} - c_e + 1 + i \pi/2\right)\right] \quad (E.17) \]

To ensure that \( M_{\pm}(\alpha) \) have the algebraic behavior stated in Section 3.3 and that \( M_{+}(\alpha) M_{-}(\alpha) = M(\alpha) \), \( M_{\pm}(\alpha) \) are multiplied by entire functions in the following manner.

\[ M_{-}(\alpha) = J_{0-}(\alpha) \ G_{-}(\alpha) \ \exp(-\chi(\alpha)) \quad (E.18) \]

\[ M_{+}(\alpha) = J_{0+}(\alpha) \ G_{+}(\alpha) \ \exp(\chi(\alpha)) \]

where

\[ \chi(\alpha) = \left(-\frac{iA\alpha}{\pi}\right)\left(\ln \frac{2\pi}{A} - c_e + 1 + i \pi/2\right) \quad (E.19) \]
Several comments on the integrands of $T_{1A}(\alpha)$ and $T_2(\alpha)$ can be made under the condition of a moderately conducting earth. First, it should be noted that the integrand of $T_{1A}(\alpha)$ has an integrable logarithmic singularity at $W = 1$. In addition, it has a sharp peak near $W = 0$ for large values of $d$ although it is bounded at $W = 0$ in all cases. Its behavior is oscillatory with a period of $1/2d$ and slowly decaying for large $W$. The main problem with the integrand of $T_2(\alpha)$ is that due to a singularity near the integration path (near $s = 0$) the integrand has a very large peak especially for large $n$.

An important limiting case in this theory is $n$ large, $d/\lambda$ small, $a/\lambda$ small. $M_-(\alpha)$ will now be evaluated under these conditions. It can be shown that in this case $\alpha_1$, $\alpha_2$, and $\alpha_4$ are all approximately equal to one. For small values of $A$, $[J_0(A\zeta)]_-$ is approximately one. A numerical investigation of $T_{1A}(\alpha)$ and $T_2(\alpha)$ reveals that $T_{1A}(\alpha) \approx 0$ and $T_2(\alpha) \approx -2n/2$. The remaining terms in the exponential factor are approximately zero since $A$ is small. The same is true of $\chi(\alpha)$. Thus, the factorization becomes

\[
M_-(\alpha) = 4 M(\alpha = 0)
\] (E.20)