On the Descending Central Series of Higher Commutators

by

Steven Weinell

B.A., University of California, Berkeley, 2007

M.S., California State University, Long Beach, 2009

A thesis submitted to the

Faculty of the Graduate School of the

University of Colorado in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

2019

This thesis entitled:
On the Descending Central Series of Higher Commutators
written by Steven Weinell
has been approved for the Department of Mathematics

Professor Keith A. Kearnes

Professor Ágnes Szendrei

Date

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Weinell, Steven (Ph.D., Mathematics)

On the Descending Central Series of Higher Commutators

Thesis directed by Professor Keith A. Kearnes

This thesis characterizes the potential behaviors of higher commutators in a simple algebra.

Acknowledgements

This thesis was inspired by earlier work on representing the binary commutator in [5] by Keith A. Kearnes, William A. Lampe, and Ross Willard.

Contents

Figures

Figure

Chapter 1

Introduction

In [2], Bulatov defined a higher commutator for general algebraic structures (Definition 2.9). Using this higher commutator we may define **the descending central series of higher commutators**:

$$1_{\mathbf{A}}$$

$$[1_{\mathbf{A}}, 1_{\mathbf{A}}]$$

$$[1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}]$$

$$[1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}]$$

$$\vdots$$

This series is a generalization of the descending central series for groups. This is not the only generalization of descending central series for groups. For a general algebraic structure, the **descending central series**:

$$1_{\mathbf{A}}$$

$$[1_{\mathbf{A}}, 1_{\mathbf{A}}]$$

$$[1_{\mathbf{A}}, [1_{\mathbf{A}}, 1_{\mathbf{A}}]]$$

$$[1_{\mathbf{A}}, [1_{\mathbf{A}}, [1_{\mathbf{A}}, 1_{\mathbf{A}}]]]$$

$$\vdots$$

is a generalization of the descending central series for groups which more closely captures the iterative nature of the definition in groups. In particular, the descending central series uses the binary commutator operation which it recursively applies to the previous congruence in the series. The descending central series of higher commutators instead uses an n-ary commutator operation to generate the n^{th} congruence in the series.

These two generalizations of the descending central series for groups allow two generalizations of nilpotence for groups. An algebra $\mathbf{A}$ is said to be **nilpotent** if the descending central series is eventually $0_{\mathbf{A}}$. $\mathbf{A}$ is said to be **supernilpotent** if the descending central series of higher commutators is eventually $0_{\mathbf{A}}$. While nilpotence in an algebra may seem a more honest generalization of nilpotence in a group due to its iterative nature, it was shown in [1] by Aichinger and Mudrinski that in a Mal'cev variety supernilpotence implies nilpotence. In that paper, Aichinger and Mudrinski went on to show that in a Mal'cev variety with a finite language, a finite nilpotent algebra is the product of prime power algebras if and only if it is supernilpotent. This result shows that supernilpotence better generalizes the property that a group is nilpotent if and only if it is the product of its Sylow subgroups. The study of supernilpotence has driven much of the study of descending central series of higher commutators. In [6] Kearnes and Szendrei showed that in any finite algebra, supernilpotence inplies nilpotence. In [7] it was shown by Moore and Moorhead that supernilpotence does not always imply nilpotence. A corollary of Theorem 5.7 in this thesis is that there exist nonabelian simple algebras which are supernilpotent. Such algebras cannot be nilpotent.

The importance of supernilpotence leads us to desire more understanding of the descending central series of higher commutators. The goal of this thesis is to determine the order theoretic properties of the descending central series of higher commutators in the congruence lattice of a simple algebra. In general, representation theorems are used to show that a list of properties for some concept is comprehensive. This thesis will establish a representation theorem for descending central series of higher commutators. Higher commutators satisfy $[\alpha_0, \alpha_1, \dots, \alpha_n] \leq [\alpha_1, \dots, \alpha_n]$ for congruences $\alpha_0, \alpha_1, \dots, \alpha_n$ (see Proposition 2.10 (2)).

So the descending central series of higher commutators is weakly descending. More precisely, the descending central series of higher commutators forms a weakly descending chain in the lattice of congruences of $\mathbf{A}$. A simple algebra is an algebra whose congruence lattice is the two element lattice. We separate all weakly descending chains $\theta_1 \geq \theta_2 \geq \theta_3, \geq \dots$ in a two element lattice, $L = \langle \{0,1\}; \leq^L \rangle$, with $\theta_1 = 1$ into three types. We can have a weakly descending chain which never descends, so $\theta_m = 1$ for all m. We can have a chain which immediately descends, so $\theta_m = 0$ for all $m \geq 2$. And finally we can have the general case where there is some $n \geq 2$ with

$$\theta_1 = \theta_2 = \cdots = \theta_n = 1$$

and

$$\theta_{n+1} = \theta_{n+2} = \cdots = 0.$$

This thesis will establish that any of these possibilities may be represented as the descending central series of higher commutators in the congruence lattice of a simple algebra. Representing the first two possibilities is not difficult. Since the descending central series of higher commutators is a generalization of the descending central series for groups, representing a weakly descending chain in the two element lattice which never descends can be done by finding a simple group which is not abelian. We may use, for example, the alternating group on five elements. Representing a weakly descending chain in the two element lattice which immediately descends will be done by finding a simple group which is abelian. The two element group is one such group. Our main theorem will construct an algebra which represents the general case.

Chapter 2

Preliminaries

This chapter is devoted to presenting the basic definitions and properties needed to understand the rest of this thesis. The intent is to carefully define the basic notions to agree with their use throughout this thesis. Though an example from groups is used to make the definition of the commutator more accessible, this chapter does not attempt to give a comprehensive introduction to the study of binary and higher commutators in general algebra. For a more in depth exposition on the binary commutator see [3] and [4]. For more on higher commutators see [1], [8], and [9].

Remark 2.1. It will be convenient to now state standardized notation for tuples in the rest of this thesis. A tuple will be represented by a bold letter, e.g. $\mathbf{p}, \mathbf{x}, \mathbf{x}_1$. The components of a tuple will be represented by non-bold letters which match the tuple name and are subscripted by non-zero natural numbers. If a letter has two subscripts, we will separate them by commas. So given tuples $\mathbf{p}, \mathbf{x}, \mathbf{x}_1$ of lengths k, l, and m, respectively, our convention will be $\mathbf{p} = (p_1, p_2, \ldots, p_k)$, $\mathbf{x} = (x_1, x_2, \ldots, x_l)$, $\mathbf{x}_1 = (x_{1,1}, x_{1,2}, \ldots x_{1,m})$.

Definitions 2.2.

(1) A **language for an algebra** (or just a **language** in the rest of this thesis) is a set of function symbols $\mathcal{F}$ together with an arity function $\mathrm{Ar} : \mathcal{F} \to \omega$. For a function symbol $f \in \mathcal{F}$ we call the natural number $\mathrm{Ar}(f)$ the **arity** of f.

(2) A **partial algebra** $\mathbf{A} = \langle A; F \rangle$ in the language $\mathcal{F}$ is a nonempty set A together with a set of functions

$$F = \{\, f^{\mathbf{A}} : \mathrm{dmn}(f^{\mathbf{A}}) \to A \mid f \in \mathcal{F}, \mathrm{dmn}(f^{\mathbf{A}}) \subseteq A^{\mathrm{Ar}(f)}, \text{ and } \mathrm{dmn}(f^{\mathbf{A}}) \neq \emptyset \,\}.$$

We call the functions in F the **fundamental operations** of $\mathbf{A}$. We call the set A the **universe** of $\mathbf{A}$.

(3) For $c \in \mathcal{F}$ with $\mathrm{Ar}(c) = 0$ we call c a **constant symbol** and $c^{\mathbf{A}}$ a **constant** of $\mathbf{A}$. We identify constants with the one element of A which they output.

(4) An **algebra** in the language of $\mathcal{F}$ is a partial algebra $\mathbf{A}$ such that for all $f \in \mathcal{F}$, $f^{\mathbf{A}}$ has domain $\mathrm{dmn}(f^{\mathbf{A}}) = A^{\mathrm{Ar}(f)}$.

Definition 2.3. Let $\mathbf{A} = \langle A; F \rangle$ be a partial algebra. A **congruence** on $\mathbf{A}$ is an equivalence relation α on A such that for any function symbol $f \in \mathcal{F}$ and tuples $\mathbf{p}, \mathbf{q} \in \mathrm{dmn}(f^{\mathbf{A}})$,

$$\text{if } p_i \equiv_\alpha q_i \text{ for all } 1 \leq i \leq \mathrm{Ar(f)}, \text{then } f^{\mathbf{A}}(\mathbf{p}) \equiv_\alpha f^{\mathbf{A}}(\mathbf{q}).$$

If $\mathbf{A}$ is an algebra, the congruences of $\mathbf{A}$ form a lattice ordered under $\subseteq$. We call this lattice $\mathrm{Con}(\mathbf{A})$ the congruence lattice of $\mathbf{A}$.

Definition 2.4. Let $\mathcal{F}$ be a language and let X be a nonempty set. We define a **term** in the language $\mathcal{F}$ with variables from X recursively as follows.

(1) Any $x \in X$ is a term.

(2) Any constant symbol is a term.

(3) If f is a function symbol and $\tau_1, \ldots, \tau_{\mathrm{Ar}(f)}$ are terms, then $f(\tau_1, \ldots, \tau_{\mathrm{Ar}(f)})$ is a term.

If a term is generated by rule (3), we call the function symbol f in the definition the **outer function symbol** of τ. "$\tau(x_1, \ldots, x_m)$ **is a term in the language** $\mathcal{F}$" is shorthand for the statement, "τ is a term in the language $\mathcal{F}$ with variables from the set $\{\, x_1, \ldots, x_m \,\}$".

Note that a term τ is syntactical object, i.e. τ is just a formal string of symbols. The **length of a term** $|\tau|$ will just be the number of symbols in τ. Thus by the **shortest term** satisfying some property P, we mean a term τ satisfying P such that if σ is any other term satisfying P, we have $|\tau| \leq |\sigma|$. The next definition describes how an algebra may attribute a semantical interpretation to a term.

Definition 2.5. Let $\mathbf{A} = \langle A; F \rangle$ be an algebra and let $\tau(x_1, \ldots, x_m)$ be a term in the language of $\mathbf{A}$. We define the function

$$\tau^{\mathbf{A}} : A^m \to A$$

recursively as follows.

(1) If τ is the variable x_i for some $i \in \{ 1, \ldots, m \}$, $\tau^{\mathbf{A}}$ is the function

$$\tau^{\mathbf{A}} : (p_1, \ldots, p_m) \mapsto p_i.$$

(2) If τ is the constant symbol c, $\tau^{\mathbf{A}}$ is the constant function

$$\tau^{\mathbf{A}} : (p_1, \ldots, p_m) \mapsto c^{\mathbf{A}}.$$

(3) If there exists a function symbol f and terms $\tau_1, \ldots, \tau_{\mathrm{Ar}(f)}$ such that τ is the term $f(\tau_1, \ldots, \tau_{\mathrm{Ar}(f)})$, then $\tau^{\mathbf{A}}$ is the function

$$\tau^{\mathbf{A}} : (p_1, \ldots, p_m) \mapsto f^{\mathbf{A}}(\tau_1^{\mathbf{A}}(p_1, \ldots, p_m), \ldots, \tau_{\mathrm{Ar}(f)}^{\mathbf{A}}(p_1, \ldots, p_m))$$

Example 2.6.

- Any group G is an example of an algebra.
- The language of G is $\{\cdot, ^{-1}; e\}$
- For each normal subgroup N of G the equivalence relation whose equivalence classes are the cosets of N is a congruence of G. All congruences are of this form.

- The lattice of normal subgroups of G is isomorphic to the lattice of congruences of G.
  - $*$ 1_G corresponds to G.
  - $*$ 0_G correstponds to $\{e^G\}$.
- $\tau(x,y) = (((x^{-1} \cdot y^{-1}) \cdot x) \cdot y)$ is a term in the language of G.

$$\begin{aligned} \tau^G : G^2 &\to G \\ (a,b) &\mapsto a^{-1}b^{-1}ab \end{aligned}$$

Definitions 2.7. Let $\mathbf{A}$ be an algebra. Let $\alpha_1, \ldots, \alpha_n$ be congruences on $\mathbf{A}$.

(1) Let $\tau(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be a term in the language of $\mathbf{A}$. Let $\mathbf{p}_1^0, \mathbf{p}_1^1, \mathbf{p}_2^0, \mathbf{p}_2^1, \ldots, \mathbf{p}_n^0, \mathbf{p}_n^1$ be tuples of elements from A with $|\mathbf{p}_i^0| = |\mathbf{p}_i^1| = |\mathbf{x}_i|$ and $\mathbf{p}_i^0 \equiv_{\alpha_i} \mathbf{p}_i^1$ for $i \in \{1, \ldots, n\}$. The $(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$-**term cube** generated by $\tau^{\mathbf{A}}$ on tuples $\mathbf{p}_1^0, \mathbf{p}_1^1, \mathbf{p}_2^0, \mathbf{p}_2^1, \ldots, \mathbf{p}_n^0, \mathbf{p}_n^1$ is the 2^n tuple

$$(r_1, r_2, r_3, \ldots, r_{2^n})$$

with

$$r_i = \tau^{\mathbf{A}}(\mathbf{p}_1^{i_1}, \mathbf{p}_2^{i_2}, \ldots, \mathbf{p}_n^{i_n}) \qquad \text{where } i - 1 = \sum_{j=0}^{n-1} i_{n-j} 2^j.$$

Note i_j is the j^{th} digit of the number $i-1$ written in binary. We call r_i the $i^{\mathbf{th}}$ **vertex** of the $(\alpha_1, \ldots, \alpha_n)$-term cube.

(2) If

$$r = \tau^{\mathbf{A}}(\mathbf{p}_1^{i_1}, \mathbf{p}_2^{i_2}, \ldots, \mathbf{p}_k^{i_k}, \ldots, \mathbf{p}_n^{i_n})$$

is a vertex on the term cube above, we say we **move in the $k^{\mathbf{th}}$ dimension along the term cube** $(r_1, r_2, r_3, \ldots, r_{2^n})$ **from r to s** to indicate that s is the vertex

$$s = \tau^{\mathbf{A}}(\mathbf{p}_1^{i_1}, \mathbf{p}_2^{i_2}, \ldots, \mathbf{p}_k^{j_k}, \ldots, \mathbf{p}_n^{i_n})$$

where $j_k \equiv i_k + 1 (\text{mod } 2)$. We will often say, "move in the k^{th} dimension from r to s" if the term cube is understood from context. Note that if we move in the k^{th} dimension from r to s we have that $r \equiv_{\alpha_k} s$.

(3) Let δ be a congruence on $\mathbf{A}$. We say $\mathbf{C}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n; \delta)$ **holds** if every $(\alpha_1, \ldots, \alpha_n)$-term cube $(r_1, r_2, r_3, \ldots, r_{2^n})$ generated in $\mathbf{A}$ has the property that

$$r_{2i-1} \equiv_\delta r_{2i} \text{ for all } 1 \leq i \leq 2^{n-1} - 1 \text{ implies } r_{2^n-1} \equiv_\delta r_{2^n}.$$

Proposition 2.8. *Let* $\mathbf{A}$ *be an algebra. Let* $\alpha_1, \ldots, \alpha_n$ *be congruences on* $\mathbf{A}$. *For a collection of congruences* $\{ \delta_i \mid i \in I \}$ *such that* $\mathbf{C}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n; \delta_i)$ *holds for each* $i \in I$, *we have that* $\mathbf{C}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n; \bigwedge_{i\in I} \delta_i)$ *holds as well.*

Proof. Suppose $\{ \delta_i \mid i \in I \}$ is a collection of congruences such that $\mathbf{C}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n; \delta_i)$ holds for each $i \in I$. Set $\delta = \bigwedge_{i\in I} \delta_i$. Let $(r_1, r_2, r_3, \ldots, r_{2^n})$ be an $(\alpha_1, \ldots, \alpha_n)$-term cube. Suppose $r_{2j-1} \equiv_\delta r_{2j}$ for all $1 \leq j \leq 2^{n-1} - 1$. For all $i \in I$, we have that $\delta \leq \delta_i$. So $r_{2j-1} \equiv_{\delta_i} r_{2j}$ for all $1 \leq j \leq 2^{n-1} - 1$. Since $\mathbf{C}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n; \delta_i)$ holds, we get that $r_{2^n-1} \equiv_{\delta_i} r_{2^n}$. This was true for all $i \in I$, so $r_{2^n-1} \equiv_\delta r_{2^n}$, as desired. □

Proposition 2.8 allows us to make the following definition.

Definition 2.9. Let $\mathbf{A}$ be an algebra. Let $\alpha_1, \ldots, \alpha_n$ be congruences on $\mathbf{A}$. The n-commutator $[\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n]$ is the smallest congruence δ such that $\mathbf{C}(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n; \delta)$ holds.

Proposition 2.10. *Let* $\mathbf{A}$ *be an algebra. Let* $\alpha_0, \alpha_1, \ldots, \alpha_n$ *be congruences on* $\mathbf{A}$. *The following inequalities hold in* $\text{Con}(\mathbf{A})$.

(1) $[\alpha_1, \ldots, \alpha_n] \leq \bigwedge_{i=1}^n \alpha_i$

(2) $[\alpha_0, \ldots, \alpha_n, \alpha_n] \leq [\alpha_1, \ldots, \alpha_n]$.

Proof. To prove (1) we will show that $\mathbf{C}(\alpha_1, \ldots, \alpha_n; \alpha_i)$ holds for all $i \in \{1, \ldots, n\}$. Suppose $(r_1, r_2, r_3, \ldots, r_{2^n})$ is an $(\alpha_1, \ldots, \alpha_n)$-term cube with $r_{2j-1} \equiv_{\alpha_i} r_{2j}$ for all $1 \leq j \leq 2^{n-1} - 1$. Move in the i^{th} dimension from r_{2^n-1} to s. Then $r_{2^n-1} \equiv_{\alpha_i} s$. Now move in the n^{th} dimension from s to t. We will have $s \equiv_{\alpha_i} t$ from our assumption. Finally, move in the i^{th} dimension from t to u. We will have $t \equiv_{\alpha_i} u$. Further, we know $u = r_{2^n}$ by the definition of moving in the k^{th} direction. We then have $r_{2^n-1} \equiv_{\alpha_i} s \equiv_{\alpha_i} t \equiv_{\alpha_i} r_{2^n}$. So $r_{2^n-1} \equiv_{\alpha_i} r_{2^n}$ as desired.

To prove (2) we will show that $\mathbf{C}(\alpha_0, \alpha_1, \ldots, \alpha_n; [\alpha_1, \ldots, \alpha_n])$ holds. To show this we'll prove that if $\mathbf{C}(\alpha_1, \ldots, \alpha_n; \delta)$ holds, then $\mathbf{C}(\alpha_0, \alpha_1, \ldots, \alpha_n; \delta)$ holds. In fact, we'll prove the contrapositive of this statement. If $\mathbf{C}(\alpha_0, \alpha_1, \ldots, \alpha_n; \delta)$ does not hold then there exists an $(\alpha_0, \alpha_1, \ldots, \alpha_n)$-term cube $(r_1, r_2, r_3, \ldots, r_{2^{n+1}})$ such that $r_{2j-1} \equiv_\delta r_{2j}$ for all $1 \leq j \leq 2^n - 1$ and $r_{2^{n+1}-1} \not\equiv_\delta r_{2^{n+1}}$. Observe that the hyperface $(r_{2^n+1}, r_{2^n+2}, r_{2^n+3} \ldots, r_{2^{n+1}})$ is an $(\alpha_1, \ldots, \alpha_n)$-term cube with $r_{2^{n+1}-1} \not\equiv_\delta r_{2^{n+1}}$. Thus $\mathbf{C}(\alpha_1, \ldots, \alpha_n; \delta)$ does not hold.

$\square$

Chapter 3

Notation in Simple Algebras

In this thesis the algebra that we analyze, $\mathbf{A}$, will be **simple**, i.e. our algebra will only have two congruences, $1_{\mathbf{A}}$ and $0_{\mathbf{A}}$. We will take advantage of this fact to allow ourselves to simplify our arguments in two ways. First, we recall that in the congruence lattice of $\mathbf{A}$, $\mathrm{Con}(\mathbf{A})$, from Proposition 2.10 (1) we have $[\alpha_1, \alpha_2, \ldots, \alpha_n] \leq \alpha_i$ for all $1 \leq i \leq n$. So in a simple algebra, we have:

$$\text{If there exists an } i \text{ with } \alpha_i = 0_{\mathbf{A}} \text{ then } [\alpha_1, \alpha_2, \ldots, \alpha_n] = 0_{\mathbf{A}}.$$

Thus we will only be concerned with higher commutators of all $1_{\mathbf{A}}$'s. Second, we will simplify notation from what is normally needed to discuss higher commutators in a general algebra. This chapter is devoted to describing our simplified notation.

Definition 3.1. Let $\tau(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be a term in the language of $\mathbf{A}$. The **n-term cube** for $\tau^{\mathbf{A}}$ on tuples $\mathbf{p}_1^0, \mathbf{p}_1^1, \mathbf{p}_2^0, \mathbf{p}_2^1, \ldots, \mathbf{p}_n^0, \mathbf{p}_n^1$ is the 2^n tuple

$$C^n_{\tau^{\mathbf{A}}}(\mathbf{p}_1^0, \mathbf{p}_1^1; \mathbf{p}_2^0, \mathbf{p}_2^1; \ldots; \mathbf{p}_n^0, \mathbf{p}_n^1) = (r_1, r_2, r_3, \ldots, r_{2^n})$$

with

$$r_i = \tau^{\mathbf{A}}(\mathbf{p}_1^{i_1}, \mathbf{p}_2^{i_2}, \ldots, \mathbf{p}_n^{i_n}) \qquad \text{where } i - 1 = \sum_{j=0}^{n-1} i_{n-j} 2^j.$$

Note i_j is the j^{th} digit of the number $i-1$ written in binary. We call r_i the $i^{\mathbf{th}}$ **vertex** of the n-term cube. We will write $C^n_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$ for the term cube where $\mathbf{p}_i^0 = \mathbf{p}_i$ and $\mathbf{p}_i^1 = \mathbf{q}_i$ for $1 \leq i \leq n$. I.e.

$$C^n_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = C^n_{\tau^{\mathbf{A}}}(\mathbf{p}_1, \mathbf{q}_1; \mathbf{p}_2, \mathbf{q}_2; \ldots; \mathbf{p}_n, \mathbf{q}_n).$$

We will sometimes write $C^n_{\tau^{\mathbf{A}}}$ as the n-term cube for $\tau^{\mathbf{A}}$ if the tuples are understood from context. We will call $C^2_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$ the **term square** for $\tau^{\mathbf{A}}$ on $\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2$ and write $S_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$. We will display

$$S_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (r_1, r_2, r_3, r_4)$$

pictorially as in Figure 3.1.

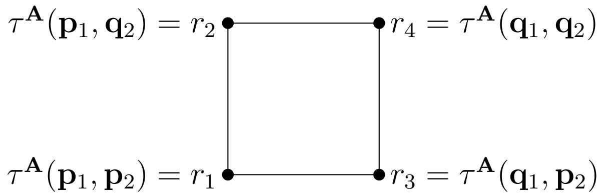

Figure 3.1: A pictorial representation of $S_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$.

We will call $C^3_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$ the **term cube** for $\tau^{\mathbf{A}}$ on $\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2, \mathbf{p}_3, \mathbf{q}_3$ and write $C_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$. We will display

$$C_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8)$$

pictorially as in Figure 3.2.

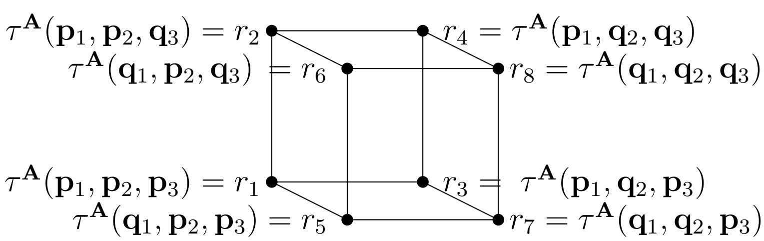

Figure 3.2: A pictorial representation of $C_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$.

Definition 3.2. We say that an algebra **A fails the** n**-dimensional term condition** if there exists a term in the language of $\mathbf{A}$, $\tau(\mathbf{x}_1, \ldots, \mathbf{x}_n)$, and tuples $\mathbf{p}_1, \ldots, \mathbf{p}_n$ and $\mathbf{q}_1, \ldots, \mathbf{q}_n$ such that

$$C^n_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (r_1, r_2, r_3, \ldots, r_{2^n})$$

has

$$r_{2i-1} = r_{2i} \qquad \text{for all } 1 \le i \le 2^{n-1} - 1$$

and

$$r_{2^n-1} \neq r_{2^n}.$$

We say the term τ above **witnesses** the failure of the n-dimensional term condition. An algebra **satisfies** the n-dimensional term condition if it does not fail the n-dimensional term condition. The 2-dimensional and 3-dimensional term condition may be displayed pictorially as in Figure 3.3. If there is a term whose n-term cube has equality along the bold vertical lines and inequality along the dashed vertical lines, then the algebra fails the n-dimensional term condition for $n = 2$ or 3, respectively. We call the dashed vertical edge the **critical edge** for this reason.

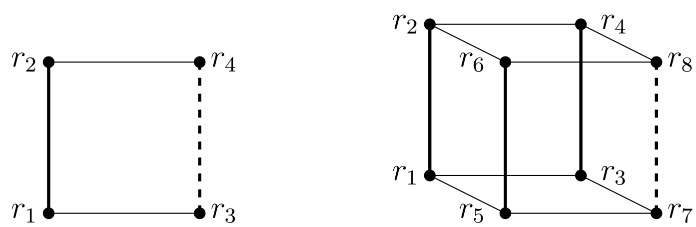

Figure 3.3: Pictorial representations of the 2-dimensional and 3-dimensional term conditions.

Remark 3.3. The n commutator $[\underbrace{1_{\mathbf{A}}, 1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}}_{n \text{ many}}]$ equals $0_{\mathbf{A}}$ if and only if $\mathbf{A}$ satisfies the n-dimensional term condition.

Definition 3.4. An algebra $\mathbf{A}$ is **abelian** if $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 0_{\mathbf{A}}$.

Example 3.5. As in Example 2.6, let G be a group. $\tau(x, y) = (((x^{-1} \cdot y^{-1}) \cdot x) \cdot y)$ is a term in the language of G. If $a, b \in G$ and $e^G = e$, one term square for τ is:

$$\begin{tikzcd}
\tau^G(e,b) = e \arrow[r, no head] \arrow[d, no head] & a^{-1}b^{-1}ab = \tau^G(a,b) \arrow[d, no head] \\
\tau^G(e,e) = e \arrow[r, no head] & e = \tau^G(a,e)
\end{tikzcd}$$

Note that if $[1_G, 1_G] = 0_G$, the equality along the left vertical edge implies equality along the right vertical edge. So we get

$$a^{-1}b^{-1}ab = e.$$

Therefore, if G is abelian in the sense of the general commutator, $[1_G, 1_G] = 0_G$, then for all $a, b \in G$ we have $ab = ba$. So G will be commutative, as we would hope.

Conversely, if G is commutative, then every group term operation reduces to a sum of unary functions, and this representation for term operations can be used to verify that $[1_G, 1_G] = 0_G$ for any commutative group.

Definition 3.6. Let $\tau(x_1, \ldots, x_n)$ be a term in the language of $\mathbf{A}$. For $x_i \in \{ x_1, \ldots, x_n \}$, we say $\tau^{\mathbf{A}}$ **depends on** x_i if there exist tuples $\mathbf{p}, \mathbf{q} \in A^n$ with $p_j = q_j$ for $j \in \{ 1, \ldots, i-1, i+1, \ldots, n \}$ and $p_i \neq q_i$ such that $\tau^{\mathbf{A}}(\mathbf{p}) \neq \tau^{\mathbf{A}}(\mathbf{q})$. $\tau^{\mathbf{A}}$ is **essentially unary** if $\tau^{\mathbf{A}}$ is constant or $\tau^{\mathbf{A}}$ depends on only one of its variables.

Lemma 3.7. *Let τ be a term in the language of $\mathbf{A}$. If $\tau^{\mathbf{A}}$ is essentially unary, then any term cube derived from $\tau^{\mathbf{A}}$ has two parallel, constant hyperfaces.*

Proof. Suppose that $\tau^{\mathbf{A}}$ depends on at most $\mathbf{x}_i$. Since $\tau^{\mathbf{A}}$ is then independent of all other variables, the vertices cannot change when moving in a dimension perpendicular to the i^{th} dimension. Thus any hyperface perpendicular to the i^{th} direction is constant. $\square$

Chapter 4

The Construction

Fix $n \geq 2$ for the remainder of this thesis. In this chapter we will define the algebra $\mathbf{A}$ which represents the general case of a weakly descending chain in a two element lattice as the descending central series of higher commutators. In particular, $\mathbf{A}$ will be a simple algebra satisfying $[\underbrace{1_{\mathbf{A}}, 1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}}_{n \text{ many}}] = 1_{\mathbf{A}}$ and $[\underbrace{1_{\mathbf{A}}, 1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}, 1_{\mathbf{A}}}_{n+1 \text{ many}}] = 1_{\mathbf{A}}$. Though not explicit in our notation, the constructed algebra $\mathbf{A}$ will depend on the fixed number n.

To define the algebra $\mathbf{A}$ we will first need to recursively define an ω-sequence of partial algebras. Each algebra in the sequence will extend the previous one. We will then take the union of the universes of these algebras to get the universe for our algebra $\mathbf{A}$ and we will union the fundamental operations of these algebras to form the only non-unary fundamental operation of our algebra $\mathbf{A}$. The first partial algebra in the sequence $\mathbf{A}_0$ will have a fundamental operation $f_0^{\mathbf{A}_0}$ which ensures that $\mathbf{A}$ will ultimately fail the n-dimensional term condition. The following partial algebras, $\mathbf{A}_{i+1}$ for $i \in \omega$, will be defined to freely extend $f_0^{\mathbf{A}_0}$ to a fundamental operation $f_{i+1}^{\mathbf{A}_{i+1}}$ which will be defined on all n-tuples of elements from the previous universe A_i. This will ensure that the union of the operations $f^{\mathbf{A}} = \bigcup f_i^{\mathbf{A}_i}$ will be defined on all n-tuples of elements from the universe $A = \bigcup A_i$ and $f^{\mathbf{A}}$ will witness the failure of the n-dimensional term condition.

Definition 4.1. To define the universe A_0 of our first algebra $\mathbf{A}_0$ we will first need to define the set

$$B = \{\, a_{i,j} \mid 1 \leq i \leq n \text{ and } j \in \omega \,\} \cup \{\, b_{i,j} \mid 1 \leq i \leq n \text{ and } j \in \omega \,\}.$$

For convenience we set $a_i = a_{i,0}$ and $b_i = b_{i,0}$ for $1 \leq i \leq n$ and

$$C = \{\, a_1, \ldots, a_n, b_1, \ldots, b_n \}.$$

We now define the first partial algebra $\mathbf{A}_0$. $\mathbf{A}_0$ will be constructed to ensure that $\mathbf{A}$ will fail the n-dimensional term condition. The universe of $\mathbf{A}_0$ will be

$$A_0 = B \cup \{\, d_i \mid 1 \leq i \leq 2^{n-1} + 1 \,\} \cup \{\, c \,\}.$$

The language of $\mathbf{A}_0$ will have one n-ary function symbol f_0. $\mathbf{A}_0$ will interpret this function symbol as the fundamental operation $f_0^{\mathbf{A}_0}$ with domain

$$\operatorname{dmn}(f_0^{\mathbf{A}_0}) = \{\, (x_1, x_2, \ldots, x_n) \mid x_i \in \{\, a_i, b_i \,\}\,\}$$

so that the term $f_0(x_1, \ldots, x_n)$ has n-term cube

$$C^n_{f_0^{\mathbf{A}_0}}(a_i, b_i) = (d_1, d_1, d_2, d_2, \ldots, d_{2^{n-1}-1}, d_{2^{n-1}-1}, d_{2^{n-1}}, d_{2^{n-1}+1}).$$

More precisely, $f_0^{\mathbf{A}_0}$ is defined as follows.

$$f_0^{\mathbf{A}_0}(a_1, a_2, a_3, \ldots, a_{n-2}, a_{n-1}, a_n) = d_1$$

$$f_0(^{\mathbf{A}_0}a_1, a_2, a_3, \ldots, a_{n-2}, a_{n-1}, b_n) = d_1$$

$$f_0^{\mathbf{A}_0}(a_1, a_2, a_3, \ldots, a_{n-2}, b_{n-1}, a_n) = d_2$$

$$f_0^{\mathbf{A}_0}(a_1, a_2, a_3, \ldots, a_{n-2}, b_{n-1}, b_n) = d_2$$

$$f_0^{\mathbf{A}_0}(a_1, a_2, a_3, \ldots, b_{n-2}, a_{n-1}, a_n) = d_3$$

$$f_0^{\mathbf{A}_0}(a_1, a_2, a_3, \ldots, b_{n-2}, a_{n-1}, b_n) = d_3$$

$$\vdots$$

$$f_0^{\mathbf{A}_0}(b_1, b_2, b_3, \ldots, b_{n-2}, a_{n-1}, a_n) = d_{2^{n-1}-1}$$

$$f_0^{\mathbf{A}_0}(b_1, b_2, b_3, \ldots, b_{n-2}, a_{n-1}, b_n) = d_{2^{n-1}-1}$$

$$f_0^{\mathbf{A}_0}(b_1, b_2, b_3, \ldots, b_{n-2}, b_{n-1}, a_n) = d_{2^{n-1}}$$

$$f_0^{\mathbf{A}_0}(b_1, b_2, b_3, \ldots, b_{n-2}, b_{n-1}, b_n) = d_{2^{n-1}+1}.$$

Next we define $\mathbf{A}_{i+1}$ for $i \in \omega$. Each $\mathbf{A}_{i+1}$ is constructed to extend the previous $\mathbf{A}_i$ as freely as possible, ensuring $f_{i+1}^{\mathbf{A}_{i+1}}$ extends $f_i^{\mathbf{A}_i}$ while not introducing any new failures of injectivity. I.e. $f_{i+1}^{\mathbf{A}_{i+1}}$ will be injective as a function restricted to the domain $\operatorname{dmn}(f_{i+1}^{\mathbf{A}_{i+1}}) \setminus \operatorname{dmn}(f_i^{\mathbf{A}_i})$. This is what ultimately allows $\mathbf{A}$ to satisfy the $(n+1)$-dimensional term condition. The universe of $\mathbf{A}_{i+1}$ will be

$$A_{i+1} = A_i \cup ((A_i^n \setminus \operatorname{dmn}(f_i)) \times \{ i \}).$$

The language of $\mathbf{A}_{i+1}$ will have one n-ary function symbol f_{i+1}. $\mathbf{A}_{i+1}$ will interpret this function symbol as the fundamental operation with domain

$$\operatorname{dmn}(f_{i+1}^{\mathbf{A}_{i+1}}) = A_i^n$$

and

$$f_{i+1}^{\mathbf{A}_{i+1}}(\mathbf{x}) = \begin{cases} f_i^{\mathbf{A}_i}(\mathbf{x}) & \text{for } \mathbf{x} \in \operatorname{dmn}(f_i^{\mathbf{A}_i}) \\ (\mathbf{x}, i) & \text{otherwise.} \end{cases}$$

We are now ready to define our desired algebra $\mathbf{A}$. The universe of $\mathbf{A}$ will be $A = \bigcup_{i\in\omega} A_i$. The language of $\mathbf{A}$ will have the following set of function symbols:

$$\mathrm{Fcn} = \{ f, u \} \cup \{ u_{pqr} \mid (p,q,r) \in (A \setminus B)^3 \text{ and } p,q,r \text{ are pairwise distinct} \}.$$

f will have arity n and the rest of the function symbols will be unary. We will set $f^{\mathbf{A}} = \bigcup_{i\in\omega} f_i^{\mathbf{A}_i}$. For any triple $(p,q,r) \in (A \setminus B)^3$ with p,q,r pairwise distinct we will set

$$u_{pqr}^{\mathbf{A}}(x) = \begin{cases} q & \text{if } x = p \\ r & \text{if } x = q \\ p & \text{if } x = r \\ a_{i,j+1} & \text{if } x = a_{i,j} \\ b_{i,j+1} & \text{if } x = b_{i,j} \\ x & \text{otherwise.} \end{cases}$$

Finally, $\mathbf{A}$ will interpret u as a permutation, written in cycle notation as

$$u^{\mathbf{A}} = (\, a_1\, b_1\, a_2\, b_2\, \ldots\, a_n\, b_n\, c\,).$$

Note that $f^{\mathbf{A}}$ is well defined since each $f_{i+1}^{\mathbf{A}_{i+1}}$ is an extension of $f_i^{\mathbf{A}_i}$. Further note that $f^{\mathbf{A}}$ has domain $\operatorname{dmn}(f^{\mathbf{A}}) = A^n$ and each unary function symbol υ has $\operatorname{dmn}(\upsilon^{\mathbf{A}}) = A$, so that $\mathbf{A}$ is in fact an algebra, not just a partial algebra.

Remark 4.2. We list some useful observations about $\mathbf{A}$ here.

(1) All of the unary fundamental operations of $\mathbf{A}$ are injective. This is because $u_{pqr}^{\mathbf{A}}$ is defined only on pairwise distinct p, q, and r to ensure $u_{pqr}^{\mathbf{A}}$ is injective and $u^{\mathbf{A}}$ is a permutation.

(2) The range of $f^{\mathbf{A}}$ does not intersect B. In fact, for any $\mathbf{p} \in A^n$ either $\mathbf{p} \in \operatorname{dmn}(f_0^{\mathbf{A}_0})$ in which case there is some $i \in \{\, 1, 2, \ldots, 2^{n-1}+1 \,\}$ with $f^{\mathbf{A}}(\mathbf{p}) = d_i$ or $\mathbf{p} \notin \operatorname{dmn}(f_0^{\mathbf{A}})$ and $f(\mathbf{p}) = (\mathbf{p}, j)$ where $j \in \omega$ is the smallest natural number with $\mathbf{p} \in \operatorname{dmn}(f_{j+1}^{\mathbf{A}_{j+1}})$.

(3) $u^{\mathbf{A}}$ is the only fundamental operation with any element of C in its range. Indeed, for $(p, q, r) \in (A \setminus B)^3$ with p, q, r pairwise distinct, $u_{pqr}^{\mathbf{A}}$ has range $\operatorname{rng}(u_{pqr}^{\mathbf{A}}) = A \setminus C$. By (2) above, $f^{\mathbf{A}}$ does not have any element of B in its range and $C \subseteq B$ so $f^{\mathbf{A}}$ does not have any element of C in its range.

Lemma 4.3. *If $f^{\mathbf{A}}(\mathbf{p}) = f^{\mathbf{A}}(\mathbf{q})$ and $\mathbf{p} \neq \mathbf{q}$, then $\mathbf{p}, \mathbf{q} \in \operatorname{dmn}(f_0^{\mathbf{A}_0})$.*

Proof. Assume $\mathbf{p}, \mathbf{q} \in A^n$ are tuples such that $f^{\mathbf{A}}(\mathbf{p}) = f^{\mathbf{A}}(\mathbf{q})$ and $\mathbf{p} \neq \mathbf{q}$. Let k be the smallest index such that $\mathbf{p} \in \operatorname{dmn}(f_k^{\mathbf{A}_k})$. Suppose for contradiction $k \neq 0$. Then

$$f^{\mathbf{A}}(\mathbf{p}) = f_k^{\mathbf{A}_k}(\mathbf{p}) = (\mathbf{p}, k-1).$$

Now as in Remark 4.2 (2) above, either there exists some $i \in \{\, 1, 2, \ldots, 2^{n-1}+1 \,\}$ with $f^{\mathbf{A}}(\mathbf{q}) = d_i$ or there exists some $j \in \omega$ with $f^{\mathbf{A}}(\mathbf{q}) = (\mathbf{q}, j)$. But in any of these cases

$$(\mathbf{p}, k-1) \neq d_i \qquad \text{and} \qquad (\mathbf{p}, k-1) \neq (\mathbf{q}, j).$$

Thus $f^{\mathbf{A}}(\mathbf{p}) \neq f^{\mathbf{A}}(\mathbf{q})$, a contradiction. So $k = 0$. Thus $\mathbf{p} \in \mathrm{dmn}(f_0^{\mathbf{A}_0})$. Switching the roles of $\mathbf{p}$ and $\mathbf{q}$ above, we get $\mathbf{q} \in \mathrm{dmn}(f_0^{\mathbf{A}_0})$ as well. $\square$

Chapter 5

The Theorem

In this chapter we prove that the algebra $\mathbf{A}$ constructed in Chapter 4 has the properties

(1) $\mathbf{A}$ is simple

(2) $[\underbrace{1_{\mathbf{A}}, 1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}}_{n \text{ many}}] = 1_{\mathbf{A}}$

(3) $[\underbrace{1_{\mathbf{A}}, 1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}, 1_{\mathbf{A}}}_{n+1 \text{ many}}] = 1_{\mathbf{A}}$

as desired in the beginning of Chapter 4. At the end of this chapter we summarize our main results in a theorem.

Proposition 5.1. $\mathbf{A}$ *is simple.*

Proof. Suppose $\theta \in \mathrm{Con}(\mathbf{A})$ has $(p, q) \in \theta$ with $p \neq q$. We will show that $\theta = 1_{\mathbf{A}}$. It will suffice to show that for all $r \in A$ we have $(q, r) \in \theta$.

It will be advantageous to have that p and q are not elements of B and that $q \neq c$. Observe that since θ is a congruence,

$$(f^{\mathbf{A}}(p, p, \ldots, p), f^{\mathbf{A}}(q, q, \ldots, q)) \in \theta.$$

By Lemma 4.3, since $\mathrm{dmn}(f_0^{\mathbf{A}_0})$ contains no tuples with all entries equal,

$$f^{\mathbf{A}}(p, p, \ldots, p) \neq f^{\mathbf{A}}(q, q, \ldots, q).$$

Finally, by Remark 4.2 (2),

$$f^{\mathbf{A}}(p,p,\ldots,p), f^{\mathbf{A}}(q,q,\ldots,q) \in A \setminus B$$

and further $f(q,q,\ldots,q) \neq c$. If either p or q is in B, or if $q = c$, replace p by $f^{\mathbf{A}}(p,p,\ldots,p)$ and replace q by $f^{\mathbf{A}}(q,q,\ldots,q)$. We may assume from now on that p and q are both elements of $A \setminus B$ and $q \neq c$.

Let $r \in A \setminus B$. If $r \in \{p,q\}$ then since θ is an equivalence relation, $(q,r) \in \theta$. If $r \notin \{p,q\}$, then $\mathbf{A}$ has a fundamental operation $u^{\mathbf{A}}_{pqr}$. Since $(p,q) \in \theta$, we have $(q,r) = (u^{\mathbf{A}}_{pqr}(p), u^{\mathbf{A}}_{pqr}(q)) \in \theta$.

To show that all elements of B are θ related to q, first note that $c \in A \setminus B$, so we have that $(q,c) \in \theta$ by the previous paragraph. Thus for all $k \in \omega$ we have $((u^{\mathbf{A}})^k(q), (u^{\mathbf{A}})^k(c)) \in \theta$. This gives us that (q,a_i) and (q,b_i) are in θ for all $1 \leq i \leq n$. We are left to show that $(q,a_{i,j})$ and $(q,b_{i,j})$ are in θ for all $1 \leq i \leq n$ and all $j \geq 1$. Let p_1,p_2,p_3 be pairwise distinct elements of $A \setminus (B \cup \{q\})$. Then for all $j \geq 1$, $(q,a_{i,j}) = ((u^{\mathbf{A}}_{p_1p_2p_3})^j(q), (u^{\mathbf{A}}_{p_1p_2p_3})^j(a_i)) \in \theta$. Similarly for all $j \geq 1$, $(q,b_{i,j})$ is in θ.

We have thus shown for all $r \in A$ that $(q,r) \in \theta$, as desired. □

Proposition 5.2. $[\underbrace{1_{\mathbf{A}}, 1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}}_{n\ many}] = 1_{\mathbf{A}}$.

We prove this proposition first for the case when $n = 2$. We do this so that the reader may more easily understand the core of the argument and have pictures for reference. The proof for arbitrary n follows a very similar structure to the $n = 2$ proof, and in fact works when $n = 2$.

Proof for $n = 2$. Consider the term $f(x,y)$. Observe in Figure 5.1 that the term square $S_{f^{\mathbf{A}}}(a_i,b_i)$ has an inequality on its critical edge. So $[1_{\mathbf{A}}, 1_{\mathbf{A}}] \neq 0_{\mathbf{A}}$. Since $\mathbf{A}$ is simple, $[1_{\mathbf{A}}, 1_{\mathbf{A}}] = 1_{\mathbf{A}}$ as desired.

□

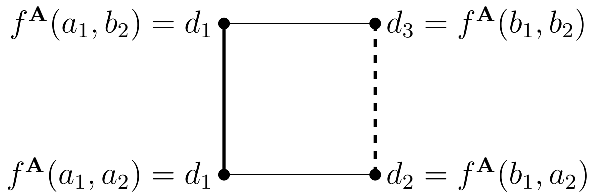

Figure 5.1: $S_{f^{\mathbf{A}}}(a_i, b_i)$.

Proof. For the term $f(x_1, \ldots, x_n)$ we have

$$C^n_{f^{\mathbf{A}}}(a_i, b_i) = (d_1, d_1, d_2, d_2, \ldots, d_{2^{n-1}-1}, d_{2^{n-1}-1}, d_{2^{n-1}}, d_{2^{n-1}+1}).$$

Note that

$$d_{2^{n-1}} \neq d_{2^{n-1}+1}.$$

So f fails the n-dimensional term condition. Thus

$$[\underbrace{1_{\mathbf{A}}, 1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}}_{n \text{ many}}] \neq 0_{\mathbf{A}}.$$

Since $\mathbf{A}$ is simple, we must have

$$[\underbrace{1_{\mathbf{A}}, 1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}}_{n \text{ many}}] = 1_{\mathbf{A}}.$$

$\square$

Definition 5.3. Let $\tau(x_1, \ldots, x_k)$ be a term in the language of $\mathbf{A}$. We will say $\tau^{\mathbf{A}}$ **is essentially a power of** $u^{\mathbf{A}}$ to indicate that there exists some $i \in \{1, \ldots, k\}$ and some $m \in \omega$ such that $\tau^{\mathbf{A}}(x_1, \ldots, x_k) = (u^{\mathbf{A}})^m(x_i)$. Note that we consider $(u^{\mathbf{A}})^0(x) = x$.

Lemma 5.4. *Let $\tau(\mathbf{x})$ be a term in the language of $\mathbf{A}$. If $\tau^{\mathbf{A}}(\mathbf{p}) \neq \tau^{\mathbf{A}}(\mathbf{q})$ and $\tau^{\mathbf{A}}(\mathbf{p}), \tau^{\mathbf{A}}(\mathbf{q}) \in C$, then $\tau^{\mathbf{A}}$ is essentially a power of $u^{\mathbf{A}}$.*

Proof. Suppose there is a counterexample to the claim. Let τ be the shortest such counterexample. I.e. let τ be a k-ary term such that $\tau^{\mathbf{A}}$ is not essentially a power of $u^{\mathbf{A}}$ and there are fixed $\mathbf{p}$ and $\mathbf{q}$ with $\tau^{\mathbf{A}}(\mathbf{p}) \neq \tau^{\mathbf{A}}(\mathbf{q})$ and $\tau^{\mathbf{A}}(\mathbf{p}), \tau^{\mathbf{A}}(\mathbf{q}) \in C$.

Note that we know $\tau^{\mathbf{A}}(\mathbf{x}) \neq x_i$ for $1 \leq i \leq k$ by assumption. We consider the outer function symbol of τ. By Remark 4.2 (3) we must have some term σ_1 such that $\tau(\mathbf{x}) = u(\sigma_1(\mathbf{x}))$.

Recall that $u^{\mathbf{A}}$ is the permutation

$$u^{\mathbf{A}} = (\, a_1\, b_1\, a_2\, b_2\, \ldots\, a_n\, b_n\, c\,).$$

So $u^{\mathbf{A}}(\sigma_1^{\mathbf{A}}(\mathbf{p})) \neq u^{\mathbf{A}}(\sigma_2^{\mathbf{A}}(\mathbf{q}))$ gives $\sigma_1^{\mathbf{A}}(\mathbf{p}) \neq \sigma_1^{\mathbf{A}}(\mathbf{q})$ and $\tau^{\mathbf{A}}(\mathbf{p}), \tau^{\mathbf{A}}(\mathbf{q}) \in C$ gives $\sigma_1^{\mathbf{A}}(\mathbf{p}), \sigma_1^{\mathbf{A}}(\mathbf{q}) \in C \cup \{\, c \,\}$. Since $\tau^{\mathbf{A}}$ is not essentially a power of $u^{\mathbf{A}}$, neither is $\sigma_1^{\mathbf{A}}$. Since τ is the shortest counterexample, we cannot have both $\sigma_1^{\mathbf{A}}(\mathbf{p})$ and $\sigma_1^{\mathbf{A}}(\mathbf{q})$ in C. Thus we must have one of $\sigma_1^{\mathbf{A}}(\mathbf{p})$ or $\sigma_1^{\mathbf{A}}(\mathbf{q})$ equal to c. Relabel $\mathbf{p}$ and $\mathbf{q}$ so that $\sigma_1^{\mathbf{A}}(\mathbf{p}) = c$. Then $\sigma_1^{\mathbf{A}}(\mathbf{q}) \in C$. By Remark 4.2 (3) again we must have some term σ_2 such that $\sigma_1(\mathbf{x}) = u(\sigma_2(\mathbf{x}))$.

Since $\sigma_1^{\mathbf{A}}(\mathbf{p}) = c$ we have $\sigma_2^{\mathbf{A}}(\mathbf{p}) = b_n$. Since $\sigma_1^{\mathbf{A}}(\mathbf{q}) \in C$ we have $\sigma_2^{\mathbf{A}}(\mathbf{q}) \in C \cup \{\, c \,\}$. Note that $\sigma_2^{\mathbf{A}}$ is not essentially a power of $u^{\mathbf{A}}$ since $\sigma_1^{\mathbf{A}}$ is not. Again, since τ is the shortest counterexample we must have $\sigma_2^{\mathbf{A}}(\mathbf{q}) = c$. By Remark 4.2 (3) once more, since $\sigma_2^{\mathbf{A}}(\mathbf{p}) \in C$, there must be some term σ_3 such that $\sigma_2(\mathbf{x}) = u(\sigma_3(\mathbf{x}))$.

Noting the definition of $u^{\mathbf{A}}$ again, we observe that $\sigma_3^{\mathbf{A}}(\mathbf{p}) = a_n$ and $\sigma_3^{\mathbf{A}}(\mathbf{q}) = b_n$. These are both elements of C, but $\sigma_3^{\mathbf{A}}$ is not essentially a power of $u^{\mathbf{A}}$ since $\sigma_2^{\mathbf{A}}$ isn't, so we have a shorter counterexample to the statement than τ. This is a contradiction, so no counterexample exists. □

Lemma 5.5. *Suppose $C^n_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (r_1, r_2, \ldots, r_{2^n})$ has its first vertex r_1 equal to all of its adjacent vertices. I.e. suppose that*

$$r_1 = r_{2^0+1} = r_{2^1+1} = r_{2^2+1} = \cdots = r_{2^{n-1}+1}.$$

Then the n-term cube is constant, that is

$$C^n_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (r_1, r_1, \ldots, r_1).$$

Proof. We will prove this lemma by contradiction. To that end, suppose τ is the shortest term in the language of $\mathbf{A}$ such that there exist tuples $\mathbf{p}_i, \mathbf{q}_i$ for $1 \leq i \leq n$ such that

$C^n_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (r_1, r_2, \ldots, r_{2^n})$ has its first vertex equal to all adjacent vertices but $C^n_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$ is not constant.

Suppose that $\tau^{\mathbf{A}}$ is essentially unary. Then since $C^n_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$ is not constant, there is an inequality along an edge. Since $\tau^{\mathbf{A}}$ is essentially unary, all parallel edges must then also have an inequality. This would mean that one of the vertices adjacent to r_1 could not be equal to r_1, a contradiction. Thus $\tau^{\mathbf{A}}$ is not essentially unary. From this we conclude that τ is not a variable.

Suppose τ has one of the unary function symbols as its outer function symbol, that is suppose $\tau = \upsilon(\sigma)$ for some fundamental unary function symbol υ and some term σ. Then as in Remark 4.2 (1), $\upsilon^{\mathbf{A}}$ is injective. Thus we must have that σ is a term which is shorter than τ such that $C_{\sigma^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (r_1, r_2, \ldots, r_{2^n})$ has its first vertex equal to all adjacent vertices but $C^n_{\sigma^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$ is not constant, contradicting the assumption that τ is the shortest such term.

Suppose τ has outer function symbol f, that is suppose $\tau = f(\sigma_1, \sigma_2, \ldots, \sigma_n)$ for terms $\sigma_1, \sigma_2, \ldots, \sigma_n$. Label

$$C^n_{\sigma_i^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (s_{i,1}, s_{i,2}, \ldots, s_{i,2^n}).$$

Since $C^n_{\tau^{\mathbf{A}}}$ is not constant, we know that there exists $j \in \{1, \ldots, n\}$ such that $C^n_{\sigma_j^{\mathbf{A}}}$ is not constant. Fix j such that $C^n_{\sigma_j^{\mathbf{A}}}$ is not constant. Since σ_j is shorter than τ, we know that $C^n_{\sigma_j^{\mathbf{A}}}$ cannot have its first vertex equal to all adjacent vertices. Thus we may fix k such that $s_{j,k}$ is adjacent to $s_{j,1}$ and $s_{j,k} \neq s_{j,1}$. Note that r_k is adjacent to r_1 so $r_1 = r_k$. The reader may want to refer to Figure 5.2 to see a potential configuration at this point in the proof.

We then have

$$f^{\mathbf{A}}(s_{1,1}, s_{2,1}, \ldots, s_{n,1}) = r_1 = r_k = f^{\mathbf{A}}(s_{1,k}, s_{2,k}, \ldots, s_{n,k})$$

but

$$(s_{1,1}, s_{2,1}, \ldots, s_{n,1}) \neq (s_{1,k}, s_{2,k}, \ldots, s_{n,k}).$$

By Lemma 4.3 we see that we must have

$$(s_{1,1}, s_{2,1}, \ldots, s_{n,1}), (s_{1,k}, s_{2,k}, \ldots, s_{n,k}) \in \operatorname{dmn}(f_0^{\mathbf{A}_0}).$$

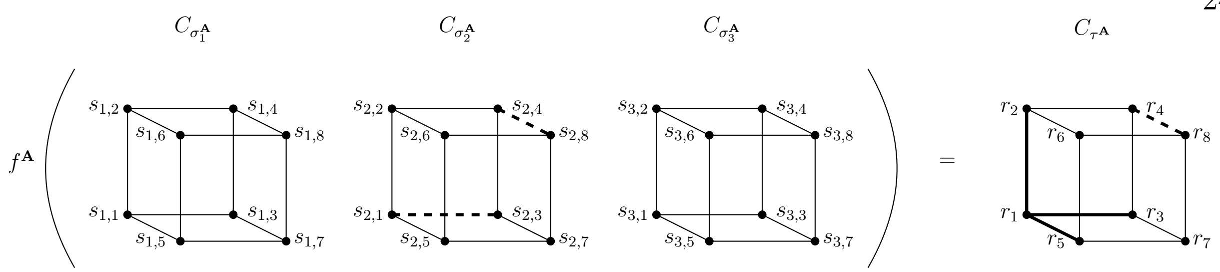

Figure 5.2: One possible configuration near the beginning of the proof of Lemma 5.5. In this picture we have $n = 3$, $j = 2$, $k = 3$. The inequality $r_4 \neq r_8$ in $C_{\tau^{\mathbf{A}}}$ is representing the assumption that $C_{\tau^{\mathbf{A}}}$ is not constant. The inequality $r_4 \neq r_8$ forces an inequality in one of the cubes. The cube that turned out to have this inequality was $C_{\sigma_2^{\mathbf{A}}}$. We may further observe that the forced inequality had to be along the same edge in $C_{\sigma_2^{\mathbf{A}}}$ as in $C_{\tau^{\mathbf{A}}}$, thus the inequality $s_{2,4} \neq s_{2,8}$. Since $\sigma_2^{\mathbf{A}}$ cannot be a shorter counterexample to the assumptions than τ, we had to have an inequality between $s_{2,1}$ and one of its adjacent vertices. In this configuration that inequality is $s_{2,1} \neq s_{2,3}$.

Note that $f_0^{\mathbf{A}_0}(\mathbf{x}) = f_0^{\mathbf{A}_0}(\mathbf{y})$ only when the tuples $\mathbf{x}$ and $\mathbf{y}$ agree on their first $n-1$ entries. So we must have $s_{i,1} = s_{i,k}$ for all $1 \leq i \leq n-1$. So the fact that $s_{j,k} \neq s_{j,1}$ tells us that $j = n$. Observe that since $(s_{1,1}, s_{2,1}, \dots, s_{n,1}) \in \operatorname{dmn}(f_0^{\mathbf{A}_0})$ there must be some l such that $r_1 = d_l$. We then know that all vertices adjacent to r_1 in $C^n_{\tau^{\mathbf{A}}}$ are also equal to d_l. Figure 5.3 gives an updated potential configuration at this point in the proof.

Again using that $f_0^{\mathbf{A}_0}(\mathbf{x}) = f_0^{\mathbf{A}_0}(\mathbf{y})$ only when the tuples $\mathbf{x}$ and $\mathbf{y}$ agree on their first $n-1$ entries, the fact that all vertices adjacent to r_1 are d_l gives that for all $1 \leq i \leq n-1$ any vertex adjacent to $s_{i,1}$ in $C^n_{\sigma_i^{\mathbf{A}}}$ is in fact equal to $s_{i,1}$. Since each σ_i is shorter than τ, we have that $C^n_{\sigma_i^{\mathbf{A}}}$ is constant for all $1 \leq i \leq n-1$. We consider $C^n_{\sigma_n^{\mathbf{A}}}$. We know that $s_{n,1} \neq s_{n,k}$ and since $(s_{1,1}, s_{2,1}, \dots, s_{n,1}), (s_{1,k}, s_{2,k}, \dots, s_{n,k}) \in \operatorname{dmn}(f_0^{\mathbf{A}_0})$ we further have that $s_{n,1}, s_{n,k} \in C$. From this Lemma 5.4 gives us that $\sigma_n^{\mathbf{A}}$ is essentially a power of $(u^{\mathbf{A}})^m$. Important in this fact is that $\sigma_n^{\mathbf{A}}$ is essentially unary and thus all vertices of $C^n_{\sigma_n^{\mathbf{A}}}$ are either $s_{n,1}$ or $s_{n,k}$. See Figure 5.4 for the configuration just before reaching our contradiction.

We are at a point where for $1 \leq i \leq n-1$, $C^n_{\sigma_i}$ is the constant cube $(s_{i,1}, s_{i,1}, \dots s_{i,1})$ and each corner of $C^n_{\sigma_n}$ is either $s_{n,1}$ or $s_{n,k}$. So each corner of $C^n_{\tau^{\mathbf{A}}}$ is either computed by $f(s_{1,1}, \dots, s_{n-1,1}, s_{n,1}) = d_l$ or $f(s_{1,1}, \dots, s_{n-1,1}, s_{n,k}) = f(s_{1,k}, \dots, s_{n-1,k}, s_{n,k}) = d_l$. Thus

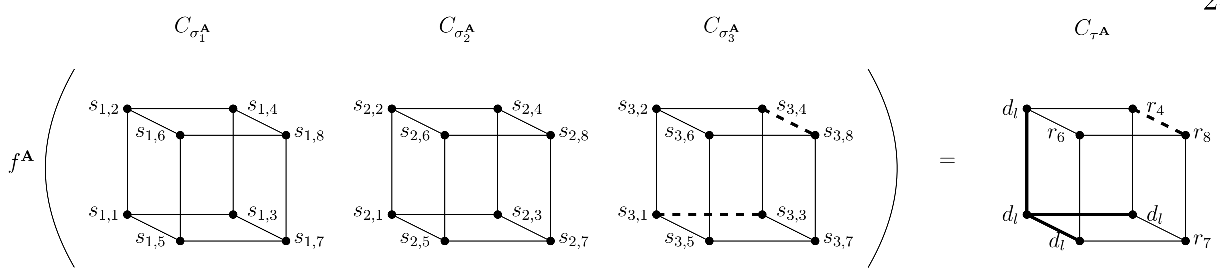

Figure 5.3: Further on in the proof of Lemma 5.5, we see that for $n = 3$, we must have $j = 3$. This picture still has $k = 3$.

$C^n_{\tau^{\mathbf{A}}}$ must be the constant cube with all vertices d_l, a contradiction.

We have thus found that τ may not be a variable, and τ may not have any symbol in the language of $\mathbf{A}$ as its outer function symbol. Thus no such τ may exist, as desired. $\square$

Proposition 5.6. $\underbrace{[1_{\mathbf{A}}, 1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}]}_{n+1 \textit{ many}} = 0_{\mathbf{A}}$.

For similar reasons to those in Proposition 5.2, we first prove this proposition when $n = 2$, then follow with the proof for the general case.

Proof for $n = 2$. We will prove this proposition by contradiction. To that end, suppose

$$[1_{\mathbf{A}}, 1_{\mathbf{A}}, 1_{\mathbf{A}}] \neq 0_{\mathbf{A}}.$$

Let $\tau(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ be the shortest term witnessing the failure of the 3-dimensional term condition.

If $\tau^{\mathbf{A}}$ were essentially unary, then the inequality along the critical edge of $C_{\tau^{\mathbf{A}}}$ would imply inequality along the three bold vertical edges (see Figure 5.5) contradicting τ being a witness to the failure of the 3-dimensional term condition. Thus $\tau^{\mathbf{A}}$ cannot be essentially unary. So τ is not a variable.

Suppose there is a unary function symbol υ and term σ such that

$$\tau(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \upsilon(\sigma(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)).$$

Noting that $\upsilon^{\mathbf{A}}$ is injective, as in Remark 4.2 (1), we see that σ will be a shorter term witnessing the failure of the 3-dimensional term condition, a contradiction.

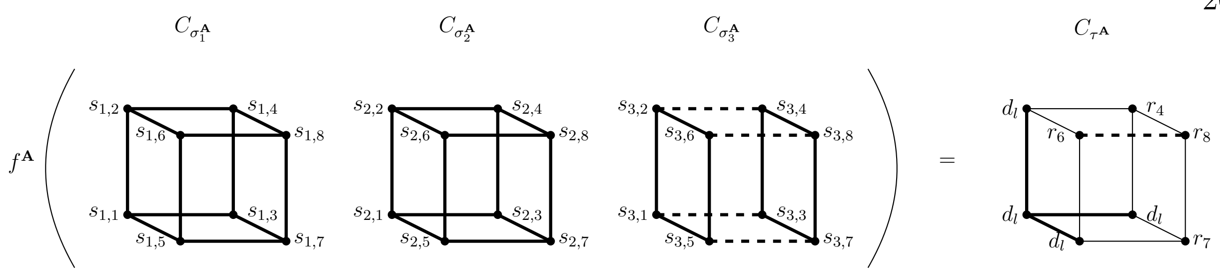

Figure 5.4: One possible configuration near the end of the proof of Lemma 5.5. We have $j = 3$ as required when $n = 3$. This picture still has $k = 3$. We've seen that the first two cubes had to be constant and the two inequalities in $C_{\sigma_3^{\mathbf{A}}}$ had to be parallel because $\sigma_3^{\mathbf{A}}$ is essentially unary. This meant the direction of the inequality in $C_{\tau^{\mathbf{A}}}$ had to be switched as well.

Suppose there are terms σ_1, σ_2 such that

$$\tau(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = f(\sigma_1(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3), \sigma_2(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)).$$

Let $C_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (r_1, r_2, \ldots, r_8)$ witness the failure of the 3-dimensional term condition, as in Figure 5.5.

Label the term cubes $C_{\sigma_1^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (s_1, s_2, \ldots, s_8)$ and $C_{\sigma_2^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (t_1, t_2, \ldots, t_8)$ as in Figure 5.6. Since

$$f(s_7, t_7) = r_7 \neq r_8 = f(s_8, t_8)$$

we know that

$$(s_7, t_7) \neq (s_8, t_8).$$

So $s_7 \neq s_8$ or $t_7 \neq t_8$. Since σ_1 and σ_2 are both shorter terms than τ, we know that neither witnesses the failure of the 3-dimensional term condition. Thus there must be a failure in equality along a non-critical vertical edge in $C_{\sigma_1^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$ or $C_{\sigma_2^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$, respectively. I.e. there is some j with $1 \leq j \leq 3$ such that

$$s_{2j-1} \neq s_{2j} \text{ or } t_{2j-1} \neq t_{2j}.$$

In either case

$$(s_{2j-1}, t_{2j-1}) \neq (s_{2j}, t_{2j}).$$

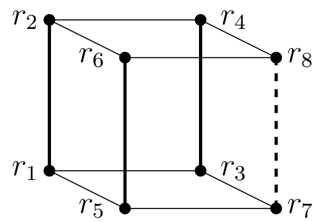

Figure 5.5: $C_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$

Observe that

$$f^{\mathbf{A}}(s_{2j-1}, t_{2j-1}) = r_{2j-1} = r_{2j} = f^{\mathbf{A}}(s_{2j}, t_{2j}).$$

By Lemma 4.3 we get that

$$(s_{2j-1}, t_{2j-1}), (s_{2j}, t_{2j}) \in \mathrm{dmn}(f_0^{\mathbf{A}_0}).$$

Thus

$$s_{2j-1}, s_{2j}, t_{2j-1}, t_{2j} \in C.$$

and, in fact, $s_{2j-1} = s_{2j}$, while $t_{2j-1} \neq t_{2j}$. So t_{2j-1} and t_{2j} are distinct elements of C in the range of $\sigma_2^{\mathbf{A}}$. By Lemma 5.4, $\sigma_2^{\mathbf{A}}$ is essentially a power of $u^{\mathbf{A}}$. So $\sigma_2^{\mathbf{A}}$ is essentially unary. From this we may conclude that

$$C_{\sigma_2^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (t_1, t_2, t_1, t_2, t_1, t_2, t_1, t_2)$$

with t_1, t_2 distinct elements in $\{a_2, b_2\}$. We now have

$$f(s_1, t_1) = r_1 = r_2 = f(s_2, t_2)$$

$$f(s_3, t_1) = r_3 = r_4 = f(s_4, t_2)$$

$$f(s_5, t_1) = r_5 = r_6 = f(s_6, t_2)$$

So Lemma 4.3 gives

$$s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = a_1.$$

Note that both (s_1, s_3, s_5, s_7) and (s_2, s_4, s_6, s_8) are term squares for σ_1. By Lemma 5.5 we get that both term squares are constant. So $s_7 = s_8 = a_1$ as well. We may now explicitly

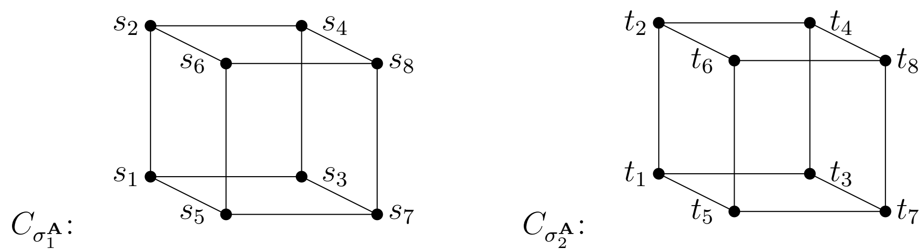

Figure 5.6: $C_{\sigma_1^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$ on the left. $C_{\sigma_2^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$ on the right.

compute that $C_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i)$ is a constant cube:

$$\begin{array}{lcl}
r_1 = f(a_1, a_2) = d_1 & \text{or} & r_1 = f(a_1, b_2) = d_1, \\
r_2 = f(a_1, a_2) = d_1 & \text{or} & r_2 = f(a_1, b_2) = d_1, \\
r_3 = f(a_1, a_2) = d_1 & \text{or} & r_3 = f(a_1, b_2) = d_1, \\
r_4 = f(a_1, a_2) = d_1 & \text{or} & r_4 = f(a_1, b_2) = d_1, \\
r_5 = f(a_1, a_2) = d_1 & \text{or} & r_5 = f(a_1, b_2) = d_1, \\
r_6 = f(a_1, a_2) = d_1 & \text{or} & r_6 = f(a_1, b_2) = d_1, \\
r_7 = f(a_1, a_2) = d_1 & \text{or} & r_7 = f(a_1, b_2) = d_1, \\
r_8 = f(a_1, a_2) = d_1 & \text{or} & r_8 = f(a_1, b_2) = d_1.
\end{array}$$

This contradicts our assumptions and we have completed the proof. $\square$

Proof. We will prove this proposition by contradiction. To that end, suppose

$$\underbrace{[1_{\mathbf{A}}, 1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}]}_{n+1 \text{ many}} \neq 0_{\mathbf{A}}.$$

Let $\tau(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})$ be the shortest term witnessing the failure of the $(n+1)$-dimensional term condition.

If $\tau^{\mathbf{A}}$ were essentially unary, then the inequality along the critical edge of $C^{n+1}_{\tau^{\mathbf{A}}}$ would imply inequality along all other vertical edges, contradicting τ being a witness to the failure of the $(n+1)$-dimensional term condition. Thus $\tau^{\mathbf{A}}$ cannot be essentially unary. So τ is not a variable.

Suppose there is a unary function symbol υ and term σ such that

$$\tau(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}) = \upsilon(\sigma(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})).$$

Noting that $\upsilon^{\mathbf{A}}$ is injective, as in Remark 4.2 (1), we see that σ will be a shorter term witnessing the failure of the $(n+1)$-dimensional term condition, a contradiction.

Suppose there are terms $\sigma_1, \ldots, \sigma_n$ such that

$$\tau(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}) = f(\sigma_1(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}), \ldots, \sigma_n(\mathbf{x}_1, \ldots, \mathbf{x}_{n+1})).$$

Let

$$C^{n+1}_{\tau^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (r_1, r_2, \ldots, r_{2^{n+1}})$$

witness the failure of the $(n+1)$-dimensional term condition, so that

$$r_{2j-1} = r_{2j} \qquad \text{for all } 1 \leq j \leq 2^n - 1$$

and

$$r_{2^{n+1}-1} \neq r_{2^{n+1}}.$$

For each $1 \leq i \leq n$, label the $(n+1)$-term cubes as

$$C^{n+1}_{\sigma_i^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (s_{i,1}, s_{i,2}, \ldots, s_{i,2^{n+1}}).$$

Since $r_{2^{n+1}-1} \neq r_{2^{n+1}}$, there exists k with $1 \leq k \leq n$ such that $s_{k,2^{n+1}-1} \neq s_{k,2^{n+1}}$. Let $k \in \{1, \ldots, n\}$ be such that $s_{k,2^{n+1}-1} \neq s_{k,2^{n+1}}$. Since σ_k is a shorter term than τ, we know that σ_k does not witness the failure of the $(n+1)$-dimensional term condition. So there exists l with $1 \leq l \leq 2^n - 1$ such that $s_{k,2l-1} \neq s_{k,2l}$. Fix $l \in \{1, \ldots, 2^n - 1$ such that

$$s_{k,2l-1} \neq s_{k,2l}.$$

Thus

$$(s_{1,2l-1}, s_{2,2l-1}, \ldots, s_{n,2l-1}) \neq (s_{1,2l}, s_{2,2l}, \ldots, s_{n,2l}).$$

Observe that

$$f^{\mathbf{A}}(s_{1,2l-1}, s_{2,2l-1}, \ldots, s_{n,2l-1}) = r_{2l-1} = r_{2l} = f^{\mathbf{A}}(s_{1,2l}, s_{2,2l}, \ldots, s_{n,2l}).$$

By Lemma 4.3 we get that

$$(s_{1,2l-1}, s_{2,2l-1}, \ldots, s_{n,2l-1}), (s_{1,2l}, s_{2,2l}, \ldots, s_{n,2l}) \in \operatorname{dmn}(f_0^{\mathbf{A}_0}).$$

Thus for each $1 \leq i \leq n$ we have

$$s_{i,2l-1}, s_{i,2l} \in C$$

and, in fact, for each $1 \leq i \leq n-1$, we have $s_{i,2l-1} = s_{i,2l}$, while $s_{n,2l-1} \neq s_{n,2l}$. So $s_{n,2l-1}$ and $s_{n,2l}$ are distinct elements of C in the range of $\sigma_n^{\mathbf{A}}$. Recall that $s_{n,2l-1}$ and $s_{n,2l}$ are in the range of $\sigma_n^{\mathbf{A}}$. Lemma 5.4 then tells us that $\sigma_n^{\mathbf{A}}$ is essentially a power of $u^{\mathbf{A}}$. So $\sigma_n^{\mathbf{A}}$ is essentially unary. Noting that we move in the n^{th} dimension from $s_{n,2l-1}$ to $s_{n,2l}$, we may follow the reasoning in the proof of Lemma 3.7 to conclude that

$$C^{n+1}_{\sigma_n^{\mathbf{A}}}(\mathbf{p}_i, \mathbf{q}_i) = (s_{n,1}, s_{n,2}, s_{n,1}, s_{n,2}, \ldots, s_{n,1}, s_{n,2})$$

with $s_{n,1}, s_{n,2}$. Since $s_{n,1} = s_{n,2l-1} \neq s_{n,2l} = s_{n,2}$, we further have that $s_{n,1}, s_{n,2}$ are distinct elements in $\{a_n, b_n\}$. For all $1 \leq j \leq 2^n - 1$, since we know $r_{2j-1} = r_{2j}$, we now have

$$f(s_{1,2j-1}, s_{2,2j-1}, \ldots, s_{n-1,2j-1}, s_{n,1}) = r_{2j-1} = r_{2j} = f(s_{1,2j}, s_{2,2j}, \ldots, s_{n-1,2j}, s_{n,2}).$$

So Lemma 4.3 gives

$$s_{i,2j-1} = s_{i,2j} \in \{a_i, b_i\} \qquad \text{for all } 1 \leq j \leq 2^n - 1 \text{ and all } 1 \leq i \leq n-1.$$

For each cube $C^{n+1}_{\sigma_i^{\mathbf{A}}}$ we have two possible cases.

Case 1. For all j_1 and j_2 with $1 \leq j_1 \leq j_2 \leq 2^{n+1} - 2$ we have $s_{i,j_1} = s_{i,j_2}$.

In this case we see that Lemma 5.5 applies and yields that $C^{n+1}_{\sigma_i^{\mathbf{A}}}$ is the constant cube with vertices all a_i or all b_i.

Case 2. There are j_1 and j_2 with $1 \leq j_1 \leq j_2 \leq 2^{n+1} - 2$ such that $s_{i,j_1} \neq s_{i,j_2}$.

In this case we have that $\sigma_i^{\mathbf{A}}$ outputs two distinct elements of C, so Lemma 5.4 tells us that $\sigma_i^{\mathbf{A}}$ is essentially a power of $u^{\mathbf{A}}$, so is essentially unary. From this and the fact that $s_{i,2j-1} = s_{i,2j} \in \{a_i, b_i\}$ for all $1 \leq j \leq 2^n - 1$, Lemma 3.7 yields that $s_{i,2^{n+1}-1} = s_{i,2^{n+1}} \in \{a_i, b_i\}$.

Note that in either case,

$$s_{i,2^{n+1}-1} = s_{i,2^{n+1}} \in \{a_i, b_i\}.$$

Recall we already have

$$s_{i,2j-1} = s_{i,2j} \in \{a_i, b_i\} \qquad \text{for all } 1 \leq j \leq 2^n - 1 \text{ and all } 1 \leq i \leq n-1.$$

We may now explicitly compute that for all $1 \leq l \leq 2^{n+1}$,

$$r_l = f(s_{1,1}, s_{2,1}, \ldots, s_{n-1,1}, a_n) \quad \text{or} \quad r_l = f(s_{1,1}, s_{2,1}, \ldots, s_{n-1,1}, b_n).$$

with $s_{i,1} \in \{\, a_i, b_i \,\}$ for all $1 \leq i \leq n-1$. From this we see that it does not matter if the last entry in the input tuple is a_n or b_n since

$$f(s_{1,1}, s_{2,1}, \ldots, s_{n-1,1}, a_n) = f(s_{1,1}, s_{2,1}, \ldots, s_{n-1,1}, b_n).$$

So $C^{n+1}_{\tau\mathbf{A}}(\mathbf{p}_i, \mathbf{q}_i)$ is a constant cube. This contradicts the assumption that we chose $C^{n+1}_{\tau\mathbf{A}}(\mathbf{p}_i, \mathbf{q}_i)$ to be a witness of the failure of the $(n+1)$-dimensional term condition. $\square$

The Propositions 5.1, 5.2, and 5.6 of this section immediately entail the following:

Theorem 5.7. *For any natural number $n \geq 2$ there is a simple algebra $\mathbf{A}$ such that*

$$[\underbrace{1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}}_{n \text{ many}}] = 1_{\mathbf{A}} \qquad \text{and} \qquad [\underbrace{1_{\mathbf{A}}, \ldots, 1_{\mathbf{A}}, 1_{\mathbf{A}}}_{(n+1) \text{ many}}] = 0_{\mathbf{A}}.$$

Bibliography

[1] Erhard Aichinger and Nebojša Mudrinski. Some applications of higher commutators in Mal'cev algebras. Algebra Universalis, 63(4):367–403, 2010.

[2] Andrei Bulatov. On the number of finite Mal'cev algebras. In Contributions to general algebra, 13 (Velké Karlovice, 1999/Dresden, 2000), pages 41–54. Heyn, Klagenfurt, 2001.

[3] Ralph Freese and Ralph McKenzie. Commutator theory for congruence modular varieties, volume 125 of London Mathematical Society Lecture Note Series. Cambridge University Press, Cambridge, 1987.

[4] Keith A. Kearnes and Emil W. Kiss. The shape of congruence lattices. Mem. Amer. Math. Soc., 222(1046):viii+169, 2013.

[5] Keith A. Kearnes, William A. Lampe, and Ross Willard. Unpublished notes.

[6] Keith A. Kearnes and Ágnes Szendrei. Does supernilpotence imply nilpotence? Unpublished notes.

[7] Matthew Moore and Andrew Moorhead. Supernilpotence Need Not Imply Nilpotence. arXiv e-prints, page arXiv:1808.04858, August 2018.

[8] Andrew Moorhead. Higher commutator theory for congruence modular varieties. J. Algebra, 513:133–158, 2018.

[9] Alexander Wires. On supernilpotent algebras. Algebra Universalis, 80(1):Art. 1, 37, 2019.