Content and Teaching Methods in Elementary School Mathematics

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Abstract

We look at the historical roots of the content of the mathematics taught in elementary grades, and at the prevailing spiral method of teaching it. We conclude that the content is badly outdated and that the method is not in agreement with modern educational goals as outlined by the National Council of Teachers of Mathematics (1989). We suggest some changes, both in the content and in the method of teaching it.

Content and Teaching Methods in Elementary School Mathematics

This article is in two parts, dealing with the content and methods of teaching mathematics in elementary grades. Part I is about the spiral method of teaching, and Part II is about three systems of numbers (whole numbers, fractions, and decimals).

I. The spiral method.

Mathematics methods courses taught in Schools and Colleges of Education are supposed to provide teachers with techniques used in teaching mathematics in classrooms. Unfortunately, the most strongly established techniques found in these courses were designed during the nineteenth century, and when they are used in a late twentieth century environment, they are one of the main sources of children's systematic errors and misunderstandings. Here we discuss critically several recommended teaching techniques and suggest (we will only suggest, not recommend) some alternatives.

Let's start first with a truism. Teachers must know well the material that they are going to cover in their classrooms. This has obvious consequences for methods courses, because how you teach depends primarily on what you teach.

Today's basic mathematical literacy is rather easy to describe, and we think that it should form the core knowledge that elementary school teachers must have and young children should attempt to acquire. It consists of:

- Arithmetic of real numbers, with a stress on approximations.
- Algorithms, with a stress on design, correctness, and complexity.

This forms a foundation for the basic skills, which are:

- Mental arithmetic.
- Efficient use of calculators and other computing devices.

This knowledge and these skills should be related to the following applications:

- Money.
- Measurements.
- Geometry.
- Statistics.

One hundred years ago, basic mathematical literacy consisted of:

- Arithmetic of three types of numbers: whole numbers, common fractions, and decimal fractions (see Part II).

This literacy was the foundation for the following skills:

- Fast and accurate execution of paper and pencil algorithms for the four basic operations (+, -, *, /) on the three basic types of numbers mentioned above.

The main applications were:

- Word problems of several standard types.
- Accounting.

The dominant teaching strategy: The spiral method.

This method is often described as returning over and over again to the same topic, each time adding some new information after reviewing the previously learned material. Here is an example from a text for a course in math methods for future teachers:
The teaching of addition and subtraction of whole numbers distributed between grades one to six should be done in the following order:
One digit numbers, 2-digit numbers without regrouping, 2-digit numbers with regrouping, 3-digit numbers without regrouping, three digit numbers with regrouping, three 2-digit numbers, ...
Every other topic, such as multiplication of whole numbers or addition of decimals, has a similar schedule, which starts with simple special cases and slowly spirals toward complete coverage.

Origins of the method.

It seems that the spiral method was never really invented. In the United States it was developed during the nineteenth century, when arithmetic became a part of general education. Before that, when arithmetic was mainly a part of vocational training for shopkeepers, artisans, merchants, and other people who were expected to use it professionally, teaching followed the pattern:
(1) Memorize the facts.
(2) Learn the algorithms by doing a few very complex multi-digit examples.
(3) Practice for accuracy and speed.
Arithmetic skills were called "ciphering", and the prerequisite for learning them was the ability to read and write. Specific goals were rather clear for each profession, and every student needed to master only a very limited set of techniques important to his

profession. (Using "his" reflects the reality of those times. We found only one reference in books on arithmetic to women using it professionally. It was in Daboll's Schoolmaster's Assistant (1823), in which he recommends that either a son or a *daughter* of a farmer should learn single entry bookkeeping to administer the finances of the farm.)

The change started with Warren Colburn (1826, 1849), who in the 1820's brought the "inductive" or gradual method, based on the teaching of Johann Pestalozzi (1746-1827). It was suitable for young children, and began with counting small collections of objects, slowly progressing toward bigger numbers and more complex operations. By 1860, series of three or four books on different levels become standard, for example, Primary Arithmetic, Intellectual Arithmetic, and Practical Arithmetic (Ray, 1887). But the next book in a series was not a continuation of the previous one. It contained all the material of the former one(s) in slightly compressed form. Graded books of the twentieth century have preserved (even today) the overlap of material between grades (Flanders, 1987). In the past, there was one practical reason for this. Speed of computation was considered to be very important, and constant practice was the way to maintain it. So during each year, a considerable amount of time was spent practicing all algorithms that had been learned before. At the same time, this way of teaching became recognized as a method.

The relation of the spiral method to psychological theories.

Faculty psychology, which was popular in the nineteenth century, treated mental capacity in the same way as physical ones: you train your brain as you train your muscle. So repetitive practice of arithmetic was considered to be good intellectual training. The metaphor of the child's mind as an empty vessel which schooling must slowly fill (John Locke's (1690; 1975) tabula rasa) suggested a 'drop by drop' method, which was in good agreement with the spiral method.

In contrast, modern cognitive science (e.g., Gardner, 1985; Klahr & Kotovsky, 1989) stresses the process of concept formation which is a rather discrete, yes or no process, and not a process of slow organic growth. Exemplars for formation of a concept need to be selected broadly, so that the concept formed is of sufficient generality (e.g., Rosch & Lloyd, 1978). Modern situated learning theories (e.g., Brown et al., 1989; Lave et al., 1991; 1993; Nunes et al., 1993; Cognition and Technology Group, 1993) also suggest learning in real contexts, not those that are unnaturally simplified. (See also Dewey, 1938.)

So the spiral method has lost its psychological support.

Observations.

Assessments which we have given to young children to check their concept of numbers, and to check the scope of the algorithms they have learned, are in agreement with modern cognitive science. Most children in early grades form a complete, although often incorrect, concept of a number, based on the exemplars they are given. They do not form a partial concept, to be extended later, which is the basic tenet of the spiral method. We have observed three distinct versions of what a number is, among second and fourth graders (Baggett & Ehrenfeucht, 1995):

1. An adult's version of a decimal number.

- A number may be positive or negative (it can have a sign), and it may have a decimal point.

2. A positive decimal number.

- All numbers are positive, but they may have a decimal point.

3. A whole number.

• Numbers are just whole numbers. (We did not find any children who had a concept of a negative number but not of a decimal.)

(In our sample of 83 children from three second grades and one fourth grade, there were 46 children or 55% in Group 1; 26 or 31 % in Group 2; and 11 or 13% in Group 3.) When presented with arithmetic problems which included both decimal points and negative numbers, almost all children "solved" all of them in a way that was consistent with their concept of a number. Children in groups 2 and 3 above showed systematic, and predictable, errors, due to their inadequate understanding of numbers. The incorrect strategies they used in subtraction problems were direct extensions of the methods they had been taught in school.

Conclusion.

The existing spiral method of teaching arithmetic leads to the formation of a concept of number that is too limited. It leads to incorrect generalizations of arithmetic algorithms, which almost immediately leads to systematic errors that are hard to correct.

Suggestions.

(1) From the very beginning, children should learn that a number can also be negative, and have a fractional part. They should be exposed to full decimal notation, which can be done easily with the help of four-operation calculators. Care should be taken that each algorithm children learn generalizes properly to an algorithm using decimals.

(2) The whole concept of the spiral method of teaching mathematics should be revised and probably abandoned, because it creates unnecessary difficulties and is contrary to the goals of modern education, which give high priority to understanding and low priority to mechanical skills (e.g., NCTM, 1989).

II. Three systems of numbers.

The system of real numbers was not created until the nineteenth century, and has been clarified only by the work of algebraists and logicians of the twentieth century. This is the system that combines continuous aspects of geometry with discrete aspects of integers and rational numbers. In addition, by including negative numbers and "formal" aspects of algebraic notation, it removes the boundary between arithmetic and algebra.

All of these changes, which have happened during the last two hundred years, somehow did not trickle down to elementary education. Textbooks in arithmetic of the early nineteenth century were based on texts written for vocational purposes, often a hundred years before, and they reflected the old view of arithmetic. Somehow this has never changed; and besides a more "child friendly" approach and lavish illustrations, so-called modern textbooks for early grades teach the same arithmetic that was taught around the year 1750. But the view of arithmetic which was the state of art at that time is hopelessly outdated now and at odds with the theory and practice of modern mathematics. Methods of explanation, examples and clarifications, and more generally, methods of teaching arithmetic based on this outdated view form an obstacle to the understanding of modern mathematics.

The three number systems of the eighteenth century.

The three systems of numbers of arithmetic were whole numbers, fractions and decimals.

A whole number was considered to be either a unit or collection of units, but zero was not considered to be a number. A fraction was a part of a unit, also called a whole, or a collection of such parts. (We will talk about decimals later). Negative numbers and other "quantities" such as surds and other irrational numbers were not a part of arithmetic but of algebra. Depending on the unit, numbers were abstract or concrete, and also similar or dissimilar. For example, a foot and an inch were similar units and a foot and a pound were dissimilar. Only numbers based on similar units could be added. Numbers that were sums of similar but not identical units were called compound numbers. A sum of a whole number and a fraction based on the same unit was a mixed number. You could multiply only an abstract number by a concrete one, or two abstract numbers. That led to two kinds of division, of two concrete numbers based on the same unit, and a concrete number by an abstract one. Decimals could be viewed either as a special case of mixed numbers: 2.23 = 2 23/100, or as composite numbers: 2 ones and 23 hundredths. The second was attractive after federal money was introduced. Just compare the English 2 pounds, 3 shillings, 3 pence, to the American 2 eagles, 1 dollar, 3 dimes, 5 cents, written as 21.35.

Computations were done in several ways; business computations were done in "modern" written forms, often helped by extensive tables ("ready reckoners", e.g. The World's Ready Reckoner and Rapid Calculator, 1890). The slide rule was a tool for craftsmen and artisans. Scientific and other more advanced computations were supported by logarithmic and trigonometric tables. The Oriental abacus was never used in Western Europe or the Americas, and the Roman abacus was not used any longer in Europe.

School arithmetic.

The three number systems, with small modifications such as the inclusion of zero and some use of negative numbers, still dominate arithmetic in early grades today. The description of a whole number as a collection of units, and a fraction as a part of a whole, are still basic ways of explaining the concept of a number, in spite of the fact that these explanations disappeared from theoretical mathematics two hundred years ago and never became a part of college level mathematics. There are serious problems created by this way of teaching mathematics. They are:

1. Contradictions which make mathematics appear to be arbitrary and illogical.
2. Inadequate and misleading explanations based on inadequate definitions.
3. Separations of the three systems, which makes applications spotty and difficult.

1. Some examples of contradictions.

You cannot divide 3 by 2 because 3 is odd (Whole numbers, (W)), but $3/2 = 1\ 1/2$ (Fractions, (F)) and $3/2 = 1.5$ (Decimals, (D)).

You cannot subtract 3 from 2 because $3 > 2$ (W), but $2 - 3 = -1$ (Algebra, (A)).

The number 2 is not a square (W), but the square root of 2 is irrational (A).

After 3 the next number is 4 (W); but $3 < 3.5 < 4$ (D).

One can avoid these problems in a classroom by never mixing such topics together. Children learn fast that in math class you cannot divide 3 by 2, but in science class you can. But without such external cues children make the most unusual errors, and they form the opinion that "math rules" are arbitrary. (Davidson (1987) has made similar observations.)

One high school graduate explained division of 3 by 2 as follows: "To divide 3 by 2 you MAKE 3 even and then divide it and get 1.5." In modern mathematics, consistency of a system is of primary importance. When we give children contradictory information and ask them to "reason mathematically", we present them with an impossible task.

2. Some examples of inadequate definitions and explanations.
Multiplication is repeated addition (W). Does this explain $1.2*3.7 = 4.44$ (D) or $-1*-2 = 2$ (A), or $\sqrt{2}*\sqrt{3} = \sqrt{6}$ (A)?

Exponentiation is repeated multiplication (W). What about 2^-1 = .5 (D)?

To see which number is bigger, look at the FIRST different digit: $123 < 141$.
Unfortunately this rule also yields $12.3 < 1.41$, which is a typical error.

These are examples of explanations which may be valid in a very limited context, but outside that context, they only create errors and confusion. Mathematical definitions should be viewed from the perspective of children learning not just one isolated topic, but mathematics as a whole.

3. The three systems are applied in different situations.
Whole numbers deal mainly with sets of objects, and also with common measurements with non standard units.
Fractions deal mainly with partitions of geometric figures. (They are often taught in this way, being treated more as a part of geometry than arithmetic.)
Decimals deal with money and the metric system.

This increases the impression that the three systems are disconnected. And again children learn from secondary cues which mathematics to use when. We saw an example of children who already were skillful with both common and decimal fractions but who, after measuring a stick and finding it was 5 inches long, said that half of it was 2 (or 3) inches, because they had learned measurement earlier within the framework of whole numbers. Most children in higher grades still have no idea that they can compare common fractions on a simple calculator (convert to decimals and then subtract), because common fractions and decimals are for them different systems.

Skills.

Mental computations were never sufficient, so additional techniques were always used. In Europe the Roman abacus (abacists) and paper and pencil algorithms (algorists) coexisted for centuries. Later, paper and pencil algorithms coexisted with the slide rule. The Oriental abacus is still used in many places together with other techniques. Computers and calculators have completely automated arithmetic computation in industry and commerce. In the present day in the United States, the *only* environment in which paper and pencil calculations are done on a daily basis is in classrooms. Does this activity still have enough value, after it has lost all its practical importance? (See also NCTM, 1989.)

Suggestions.

- School mathematics should be based on the arithmetic of real numbers, and skills which are useful in the adult world should be taught, namely mental arithmetic and the use of computing devices.
- This requires a radical change in both content and methods courses, in the preparation of future math teachers.

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