On the Consistency and Correctness of School Mathematics

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Consistency and correctness are related but different notions. We say that two statements are inconsistent if they contradict each other; and if they do not, they are consistent. So two incorrect statements may be consistent with each other. On the other hand, we say that a statement is correct if it is true. But truth may depend on the situation. So a statement that is correct in one situation can be incorrect in a different situation.

Teaching school mathematics in a consistent and a correct way is not easy.

Consistency.
One cause of the difficulty is that theoretical mathematics is partitioned into separate theories that are internally consistent when viewed separately, but that are often mutually inconsistent (Kreisel, 1967; Malitz, 1987).

In schools, and also in most applications of mathematics, the results and methods from separate mathematical theories such as Euclidean geometry, graph theory, topology of the plane, and the arithmetic of real numbers, are mixed together and applied at the same time. Such mixtures of results may be inconsistent.

Also, many concepts are clumped together, and they are often simplified. For example, when we look at the four operations, addition, subtraction, multiplication, and division on numbers, we have five main axiomatic systems. But in schools they are taught under at most two headings, arithmetic and algebra. We have these axiomatic systems:

(1) For whole numbers or natural numbers (whole numbers without zero), called Peano's Axioms (Grassman & Tremblay, 1996).
(2) For integers, called axioms for integral domains (Birkhoff & MacLane, 1979).
(3) General axioms for ordered fields (Birkhoff et al., 1979).
(4) Axioms for the field of real numbers (there are several versions) (Birkhoff et al., 1979; Robinson, 1965).
(5) For complex numbers, called axioms for algebraically closed fields of characteristic zero (Birkhoff et al., 1979).

In modern times, the word number most often refers to real numbers, (4) above. Whole numbers, integers, and rational numbers are treated as special types of real numbers. But theories (1), (2), and (3) above are mutually inconsistent. Also, (4) and (5) are mutually inconsistent. Only (4) is a consistent extension of (3). Thus the modern approach does not allow the mixing of different systems without a careful reformulation of definitions. For example, odd numbers can be divided by two within the system of reals.

• We recommend that all statements about numbers that are taught in schools be consistent with the axioms for real numbers, (4), because the system of reals is at present the basis for the arithmetic that is used in everyday life and in business, industry, and science.

Here are some examples of inconsistencies which creep into school mathematics and which are due to a confusion between theories. In each case, statement b should be taught in school, and statement a should not.

Statement:	Axiomatic system:
a. There is no number between 0 and 1.	(1) and (2)
b. $0 < .5 < 1$.	(3) and (4)
a. 0 is the smallest number.	(1)
b. $-1 < 0$.	(2), (3), and (4)
a. You cannot divide 3 by 2.	(1) and (2)
b. $3/2 = 1.5$.	(3) and (4)
a. The number 2 is not a square.	(1) and (2)
b. $(\sqrt{2})*(\sqrt{2}) = 2$.	(4)

Here, (b) is consistent with (3), but it cannot be proved in (3), because the existence of $\sqrt{2}$ cannot be proved.

a. Multiplication can be defined as repeated addition.	(1)
b. Only sometimes can the product of two numbers be computed by repeated addition.	(4)

All the properties of addition, subtraction, multiplication, and division, including operations on fractions, can be derived from (3). When we move to the solution of polynomial equations, we need (4). Properties of exponentiation, logarithms, trigonometric functions, and so on, are consistent with (4), but they may require additional axioms.

The causes of inconsistency in school mathematics are historical. The systems of axioms (1) through (4) were created, and relationships among them were studied, between 1850 and 1950. Earlier arithmetic consisted of two separate systems: whole numbers without 0, whose properties are specified in (1); and positive fractions, whose properties are (mostly) specified in (3) (Pike, 1808; Daboll, 1837; Adams, 1808). So in early arithmetic we had two separate, mutually inconsistent systems of numbers which were used together under the name of mixed numbers. The modern concept of real numbers unified whole numbers, fractions, and also negative and irrational "quantities", which were considered to be part of algebra. A consistent description of the properties of real numbers was provided by (3) and (4). But new school textbooks are written mostly by rewriting older ones, so many outdated statements are often repeated without adequate revision.

There are no inconsistencies in teaching geometry (which does not mean that it is always taught correctly). School geometry evolved from Euclidean geometry (e.g.,

Chauvenet, 1888), from informal, practical geometry which was taught as a part of vocational mathematics (e.g., Hawney, 1813) (metric geometry, in modern terms), and from analytic geometry (e.g., Todhunter, 1874). And these three systems were mutually consistent.

Correctness.
The main cause for making errors is taking a statement that is correct in one situation and using it in another situation in which it is false. Incorrect teaching is rarely due to an explicit error made by the teacher. Children in early grades do not learn by listening; they learn mainly through activities, observation, and imitation. But improperly chosen examples often cause false generalizations and errors.

A classroom study from our research.
In the early grades, kindergarten through second, children measure lengths mostly in non-standard units (paper clips, footsteps), and length is always a whole number. Students in the fourth grade were given the task of measuring a popsicle stick (in inches or centimeters) and putting a mark half way up, and one quarter and three quarters of the way up. The main topic they were studying at that time was common fractions, and rulers and calculators were available. The actual length of the stick was about 4.5 inches, or 11.3 cm.

- Most students rounded the length to a whole number (4 inches or 5 inches, or 11 centimeters).
- Students who recorded the length as 4 inches computed one half of four correctly. Most students who recorded the length as 5 inches, or as 11 cm, said that they could not compute half of 5 (or of 11), because 5 (or 11) is an odd number.
- Many students correctly placed a mark half way up the stick. Some cut a strip of paper the length of the stick, found the midpoint of the paper by folding it in half, and used the fold in the paper to mark the midpoint of the stick.
- Most students marked 1/4 and 3/4 of the length of the stick incorrectly. One student said, “Forget the math. I will fold my strip of paper into four equal parts and use them to mark my stick.” This student marked his stick correctly.

This small study shows two things:
(1) Prolonged experience with length as a whole number had a lasting influence on some students which led to an incorrect generalization, "The length of something is always a whole number."
(2) Children try to reason using mathematics, but they abandon mathematical reasoning when they “know” something is the case, but they cannot get it through mathematics.

The most common error due to misleading and prolonged experience with whole numbers is that children think that a number that has more digits is always bigger, and they use this idea for comparing decimals (so .13 is bigger than .3).

Word problems.

We note that word problems have been a topic of some discussion recently in this journal (Toom, 1999; Boote, 1998; Thomas & Gerofsky, 1997; Gerofsky, 1996; see also Usiskin, 1995). We too have our concerns about them. The main offenders in providing incorrect answers are word problems that wrongly relate mathematics to the physical world. Using mathematics correctly requires that the mathematical process is appropriate for the situation at hand, and such a match is rarely preserved in word problems. Let's look at three examples.

Example 1 (fictitious).
a. If you pour .75 liter of water into a large container that already contains 1.5 liters of water, what is the total volume of water?
Correct answer: 1.5 + .75 = 2.25 liters of water.

b. If you pour .75 l of water into a large container that already contains 1.5 l of dry sand, what is the total volume of the mixture?
Incorrect answer: 1.5 + .75 = 2.25 liters of mixture.
(The operation of addition is not appropriate in this situation.)

Example 2 (found in a textbook).
If you stacked up 1 million pennies, how tall would the stack be?
A suggested solution (metric).
Measure a short stack of pennies (there are 8 pennies in a 1 cm stack).
So 1,000,000/8 is the height of the stack in centimeters, and 1,000,000/8/100 = 10,000/8 = 1,250 is its height in meters.
Answer: The stack would be 1,250 m tall (3/4 of a mile).

Is this a correct answer?
No. Such a stack cannot be built. It would topple over before we reached a height of a few meters. So any conclusions about a stack that cannot exist are unreliable.

Comment.
- In everyday life, and in science, we reason starting with premises that we believe are true.
- Sometimes we start with a hypothesis that we are not sure about and that we suspect to be false, hoping to derive a conclusion that we know is false, and so to reject the hypothesis. (This is the basis for indirect proofs, and for forming statistical inferences.)
- We refuse to reason on the basis of false assumptions, because we know from practice and from logic that any such conclusion may be false (Kleene, 1962; Church, 1956).

Could the penny word problem be redone to be meaningful? One million pennies is worth $10,000, and such amounts of pennies are often handled by mints and banks. So let's ask about the volume and weight of a shipment of 1,000,000 pennies. If we stacked pennies on a square grid of 2 cm by 2 cm (a penny is 1.9 cm wide), arranging them in a 50 by 50 by 400 array, it would have a dimension of 1 cubic meter (37 cubic feet). One penny weighs approximately 2.5 g; thus 10^6 pennies weigh 2,500 kg (5500 LB). So we can package them into 40 boxes, each 25 cm x 25 cm x 10 cm, and each weighing 62.5 kg,

plus the weight of the box (~140 LB). Thus we see that a volume of one million pennies is rather small, but their weight can be a problem. So saying that a million is a big number is meaningless if we do not say a million of what and in what respect. One million pennies are worth $10,000; they don't take up much space, but they are quite heavy.

Unfortunately, such problems, which require that the student abandon logic and common sense and that he or she carry out a prescribed or suggested mathematical procedure, are common. Some are intentional, based on the mistaken belief that writing problems about impossible situations shows imagination and originality. Some are accidental, due to lack of knowledge or oversimplification on the part of the author.

Oversimplification in word problems is very common. For example, most cities and states have sales tax, at least on some goods. But most problems dealing with buying, selling, discounts, and so on, don't take this into account. This often leads to incorrect answers.

Example 3 (fictitious).
You have $1.80. Can you buy 2 items, each costing 89c?
2*89 = 178 < 180; but if there is a sales tax, let's say 3.5%, we should compare 2*89*1.035 = 184.23, rounded to the nearest cent, 184, to 180. But 184 > 180, so $1.80 is not enough.

Using false assumptions in word problems, and in other "practical" problems, has two negative effects. First, it forces students to suspend their world knowledge and their common sense, and to follow blindly some mathematical procedure. This is unfortunate because errors in real life are rarely due to incorrect computation; they are very often the result of choosing an incorrect mathematical procedure. Second, it portrays mathematics as a verbal game that is divorced from reality, and it suggests that mathematics ignores both reason and logic.

The correctness of geometry.
When students learn spatial geometry, some attention needs to be paid to the standard definitions from planar geometry. These definitions may require some changes.
Example.
"A square is a figure with four equal straight sides and at least one right angle." This is an acceptable definition in planar geometry. But in space, a figure with four equal edges and one right angle may be a square bent along one diagonal. So in order to have an acceptable definition of a square in space, we should say: "A square is a figure with four equal edges and at least three right angles."

Statements concerning the meaning of mathematical concepts.
Statements explaining the meaning of mathematical concepts, for example, "Numbers are ratios of quantities," or statements relating mathematics to other domains of life, such as "Mathematics is the language of science," or "Mathematics investigates patterns in nature," are not part of mathematics. But such statements are often made, and we need to pay attention to them. They are opinions, and if they are treated as such, they

can help to paint a broader picture of the role of mathematics. But opinions may differ, and often two opposite opinions can be equally valid, acceptable, and insightful.

The problem arises when an opinion is presented as truth, and any opposite opinion is labeled as false. So, when in doubt, hedge, in order not to present an opinion as something it is not. Let's look at some examples.

(1) There are many opinions about the nature of numbers. One opinion is that numbers are "data", written down, or recorded in some other medium. Another opinion claims that numbers are "abstract entities", and that these "data" are merely symbols representing abstract numbers. Both opinions seem reasonable, and the same person can even hold both of them, depending on the context. But during the New Math period, there was an attempt to teach that data that you write down are not numbers, and to deny legitimacy to the first belief.

(2) When we introduce real numbers (or just irrational numbers) as infinite decimals, we often say categorically, "Real numbers are infinite decimals," instead of hedging a little and saying, "Real numbers are represented by infinite decimals." The problem arises when students start looking at different bases. One third is different in bases 10 and 2,

1/3 = .33333333333333**33333333333333333**... (base 10)
and 1/3 = .0101010101010**1010101010101010101**... (base 2).

By saying that real numbers are infinite decimals, we choose the first representation as the "basic definition", and we deny equal status to the second one (which is an infinite binary, not decimal, fraction). Choosing a representation is just like an opinion; many are possible, and often, as in the example above, there is no reason to claim that one is better that the other.

This problem is aggravated because in the introduction to calculus, Dedekind cuts are often introduced as a definition of real numbers (Royden, 1968). We often say, "The real numbers are Dedekind cuts," rather than that they can be represented by Dedekind cuts.

(3) "A proof of a theorem shows that the theorem is true."
For many centuries, mathematicians and philosophers believed that theorems of geometry and arithmetic represent true knowledge about the world around us which are not only independent of observation, but also superior to any other form of knowledge. The discovery of other geometries and their applications in physics, the development of other number systems, and even the development of other systems of logic, have shown how naive this belief was. According to the modern view, no proof is absolute; every proof relies on some assumptions. And we can only claim that, "If all the assumptions are true in a given situation, then a theorem that is proved based on them is also true."

(4) "A fraction is one of the equal parts of a whole, or a collection of such parts." "A number is a unit or a collection of units."
These statements are relics of the past, when there were two separate systems of numbers: whole numbers and broken numbers (fractured numbers) (Pike, 1808). At that time many mathematicians believed not only that mathematics contains absolute truth, but also that all mathematical concepts can be precisely defined, removing any ambiguity as to their meaning. This again happens to be false. Any chain of definitions has to start with some undefined terms. (In the examples above: What is a part? What is a whole? What is a collection?) And undefined terms, even very simple and very well described, are always open to different interpretations (Ehrenfeucht & Mostowski, 1956). This applies to the most basic concepts, such as number, quantity, and space. Thus each mathematical concept may be viewed in many different ways. And there is no unique, correct way of understanding it (Malitz, 1987; Hintikka, 1995).

So the statements above are better paraphrased as:
"Some time ago, people looked at fractions as one or more parts of 'a whole thing' divided into equal parts. For example, 1/3 was one third of something, maybe an apple, and 2/3 were two such parts." And, "You may think about positive whole numbers as being collections of some units, which can be concrete, such as a penny or a pebble, or abstract, such as one second. You may look at one unit as a special case of a collection that contains just one item. Even zero can by looked at in this way. If someone asks you how many pine cones you have collected, rather than answering none, you may say zero."

Confusion between mathematical facts that must be mutually consistent, and opinions, which can contradict each other, can hamper the learning of mathematics in more than one way.

(1) A discussion of numbers in different contexts makes some students think that the question of whether a whole number is even or odd is a matter of opinion. ("Three is usually odd, but when you write 3/2 = 1.5, you make it even." Here, the fact that 3 is odd is treated as opinion, "We may view 3 as odd, but when we divide it by 2, we change our opinion and think that 3 is even.")

(2) Students who learn only one view, for example, the view that a whole number is the number of elements of a set, have difficulty with operations that do not agree with their point of view. ("What is 3 - 5? I cannot give away five candies, if I have only 3.")

(3) Students who confuse points of view with facts may think that mathematics contains contradictions, and therefore it is pointless to try to understand it. ("A fraction is a part of a whole. And a fraction is the ratio of two quantities. But the ratio of two quantities is not a part of something; it is a relationship. This makes no sense.")

(4) Some students think that having different representations, for example, decimal and binary, indicates that there are different kinds of numbers, "decimal numbers" and "binary numbers". (Some university students in a School of Engineering did not know

whether addition of "binary numbers" is commutative.) Properties of operations, such as commutativity, are facts that do not depend on representations, which are points of view.

Conclusions.
Proposing broad educational reform is always unrealistic, but some things can be done on a local level.

- Courses offered for current and future teachers by mathematics departments should be mutually consistent, so that any definition used in one course can be used without any changes in all other courses.

- Mathematical facts, and opinions about mathematics, should be clearly differentiated.

- Word problems based on incorrect premises should not be used in classrooms. They should be replaced by design and measurement problems which provide opportunities to verify the correctness of conclusions experimentally.

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