Fast Numerical Computation of the Input Impedance and Electric Field Distribution for a Printed Strip Dipole in a Layered Medium

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Chapter 1

INTRODUCTION

The problem of finding the electromagnetic radiation from an electric current source in the presence of a stratified medium has been of interest for some time. In 1909, Sommerfeld solved the problem of a vertical electric Hertzian dipole over a homogeneous half-space [1]. Later, Hörschelman treated the case of a horizontal electric dipole in air [2], and Elias analyzed the vertical magnetic dipole in air [3]. Then in 1926, Sommerfeld treated all four cases of elementary Hertzian dipole sources in air [4].

There has been a substantial amount of work done on the Sommerfeld problem since 1926. For a good historical overview see [5]. Many of the authors who worked on the Sommerfeld problem obtained asymptotic expansions which hold for large observation distances from the source, or for large values of other parameters such as the relative permittivity of the earth (see [5] – [9]). These asymptotic expansions can be used to efficiently compute the far-fields and quasi-static fields. When the index of refraction is large, it is even possible to obtain asymptotic expansions for the near-fields.

While asymptotic expansions provide valuable physical insight into the Sommerfeld problem, they unfortunately can’t be used to compute the fields at all points in space for any given set of material parameters. However, with the introduction of high speed digital computers, it became possible to compute the Sommerfeld integrals at any point in space by using numerical techniques. Now, asymptotic techniques can be used in conjunction with numerical techniques to obtain an efficient algorithm for the computation of Sommerfeld integrals (see [10]).

Green’s functions for more complicated multilayer problems can also be
calculated with the use of a computer (see [11]-[14]). Once the Green's
function for a specific problem has been obtained, the fields can then be
computed for any given distribution of current by convolving the Green's
function with the source distribution. This operation can be written as

\[ \mathbf{E}(x) = \int_S \mathbf{G}_e(x - x') \cdot \mathbf{J}(x') \, ds', \] (1.1)

where \( x = \hat{a}_x x + \hat{a}_y y + \hat{a}_z z \). When the current distribution is unknown,
as is the case for a perfect conducting scatterer or antenna in a stratified
medium, an electric field integral equation (EFIE) can be obtained by forcing
the tangential component of the electric field to vanish at the surface of the
perfect conducting body (see [15]-[20]). The method of moments (MOM) [21]
can then be used to reduce the EFIE to a set of linear equations which can
be solved using standard matrix techniques; thereby yielding an approximate
distribution for the current on the body.

Before the matrix equation can be solved, the elements of the impedance
matrix must be computed. A typical matrix element can be written in the
general form

\[ Z_{mn} = \int_S \int_S w_m(x) \cdot \mathbf{G}_e(x - x') \cdot f_n(x') \, ds \, ds', \] (1.2)

where \( w_m / f_n \) is one of the chosen weighting/basis functions, respectively.
Since the evaluation of the Green's function involves a two-dimensional (2-d)
inverse Fourier transform, the computation of a typical matrix element will
involve a six-fold integration. It is important to develop efficient techniques
for the computation of these matrix elements since the calculation of the
elements in the impedance matrix usually requires a large percentage of the
total computation time in a MOM problem. There are two different methods
that are commonly used for this purpose.

In the first method, a polar transformation is applied to the Green's func-
tion, and then the angular integration is carried out in closed form — yielding
Bessel functions of the first kind. The remaining semi-infinite integral can
be classified as a one-dimensional (1-d) Sommerfeld integral. Asymptotic ex-
traction techniques, singularity extraction techniques, and other numerical
techniques can then be applied to the 1-d Sommerfeld integral (see [14], [17],
[19], and [22] – [27]). In [17] and [23], the authors point out that since the
Green's function only depends on the distance between the source and field
points, an interpolation scheme can be used in problems where the Green's function must be computed a large number of times. Once the Green's function can be efficiently computed, then the matrix elements are obtained by using a four-dimensional numerical integration routine for the remaining finite integrals.

In the second method, the convolution theorem is used to rewrite the expressions for the electric field and the matrix element as (see (1.1), (1.2), and (2.21))

\[
E(x) = (2\pi)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{G}_e(\alpha_1, \alpha_2, z - z') \cdot \tilde{J}(\alpha_1, \alpha_2, z') e^{-i(\alpha_1 x + \alpha_2 y)} \, d\alpha_1 \, d\alpha_2
\]

\[
Z_{mn} = (2\pi)^4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{w}_m(\alpha_1, \alpha_2, z) \cdot \tilde{G}_e(\alpha_1, \alpha_2, z - z') \cdot \tilde{f}_n(\alpha_1, \alpha_2, z') \, d\alpha_1 \, d\alpha_2
\]

where the weighting functions are assumed to be real valued functions and the * denotes the complex conjugate operation. In order to obtain these equations, we applied the Fourier transform pair in (2.21). Now, if we choose basis functions and weighting functions whose Fourier transforms can be represented by combinations of algebraic functions and special functions, then the computation of a typical matrix element will only involve carrying out a 2-d inverse Fourier transform (see (1.3)). A polar transformation once again yields a Sommerfeld integral, but this time the angular integral can't be carried out in closed form. Therefore, this method involves the computation of 2-d Sommerfeld integrals. In [15] and [28], the 2-d Sommerfeld integrals are computed using a 2-d numerical integration routine. Once again, asymptotic extraction techniques can be used to enhance the efficiency for the computation of the semi-infinite integral (see [29] and [30]).

Both of these methods have their advantages and disadvantages. If we assume that an interpolation table has already been constructed for the Green's function in the first method, then for each matrix element, a four-dimensional, definite integral must be computed using numerical techniques. On the other hand, a definite integral and a semi-infinite integral must be computed in the second method. The fact that the Green's function becomes singular as the distance between the source point and field point goes to zero presents some difficulties in the first method. In the second method, this problem shows up in the form of a slowly converging integral. Luckily, there
are techniques which can be used to circumvent these difficulties (see [14], [24]–[27], [29], and [30]). For a given problem, one method may be better suited than the other; but in general, both of these methods are important and need to be investigated further.

In [31], an efficient numerical method was presented for evaluating the 2-d Sommerfeld integrals that are encountered when computing the variational input impedance for printed strip dipole antennas. In that paper, the inner angular integral, in the polar representation of the Sommerfeld integral, was rewritten in terms of a finite number of incomplete Lipschitz-Hankel integrals (ILHI's). Then, numerical techniques were developed which allowed for the efficient computation of the resulting ILHI's. It was found that expressing the inner angular integral in terms of ILHI's provides a more efficient method for computing the 2-d Sommerfeld integrals than using a 2-d adaptive quadrature routine.

In this report, we will apply techniques that are similar to the ones discussed in [31] to the evaluation of the 2-d Sommerfeld integrals that are encountered when taking the inverse Fourier transform of the spectral domain fields, or when evaluating the elements in the impedance matrix using the form given in (1.3). The techniques which are presented in this report have been developed for the special case of piecewise-sinusoidal (PWS) basis and weighting functions, however, similar techniques can also be applied for other choices of basis and weighting functions.

First, in Chapter 2 we will use spectral domain techniques to find the electromagnetic radiation due to a distribution of electric surface currents in the presence of a stratified medium. The fields will be expressed as 2-d Sommerfeld integrals. Then, in Chapter 3 we will use the Sommerfeld integral representation for the electric field in order to construct an EFIE for a printed strip dipole antenna in a stratified medium. Also in that Chapter, we will demonstrate how Galerkin's method can be used to obtain an approximate distribution for the currents on the antenna. Once the current distribution is known, the near-zone fields and the driving point impedance for the antenna can be computed.

Finally, in Chapter 4 and Appendix A, we will show how the angular integral, in the 2-d Sommerfeld integrals, can be rewritten in terms of ILHI's. Then, we will show how to efficiently compute these ILHI's in Appendix B. Also, in Appendix C we will show how writing the angular integral in terms of ILHI's allows us to develop a new asymptotic extraction technique for the
elements in the impedance matrix. In order to demonstrate the usefulness of these techniques, we will apply them in Chapter 5 to the problem of computing the driving-point input impedance and the near-zone electric field distribution for a printed strip dipole antenna in a layered medium.
Chapter 2

SPECTRAL DOMAIN TECHNIQUES APPLIED TO STRATIFIED MEDIA

2.1 Introduction

In this Chapter, we will use spectral domain techniques to solve Maxwell’s equations in a stratified medium which has an arbitrary number of layers (see Figure 2.1). Spectral domain techniques have been used by a number of authors to study this problem (see [6], [7], [11]–[14], [20], and [32]). In [12]–[14] and [32], the fields are expressed as a superposition of transverse electric (TE) and transverse magnetic (TM) (to z) fields. Since decomposing the field into TE and TM fields simplifies the problem, we will use this technique in our analysis.

2.2 Maxwell’s Equations

We will assume that the only sources in this problem are the electric surface currents which are located at the interface at $z = 0$. We will further assume that these sources vary sinusoidally with time ($e^{j\omega t}$ time convention);
Figure 2.1: General stratified media
therefore, we can use the time-harmonic form of Maxwell’s equations

\[
\begin{align*}
\nabla \cdot (\varepsilon \mathbf{E}) &= \rho \\
\nabla \cdot (\mu \mathbf{H}) &= \rho_m \\
\n
\nabla \times \mathbf{H} &= \mathbf{J} + j\omega \varepsilon \mathbf{E} \\
\n\nabla \times \mathbf{E} &= -\mathbf{M} - j\omega \mu \mathbf{H}
\end{align*}
\]

(2.1)

with the constitutive relations

\[
\begin{align*}
\mathbf{D} &= \varepsilon \mathbf{E} \\
\mathbf{B} &= \mu \mathbf{H}
\end{align*}
\]

(2.2)

The inhomogeneous nature of this problem can be characterized by specifying a complex permittivity,

\[
\varepsilon_{pq} = \varepsilon'_{pq} - j\varepsilon''_{pq} = \varepsilon'_{pq} - \frac{j\sigma_{pq}}{\omega},
\]

(2.3)

and a permeability, \(\mu_{pq}\), for each layer. The subscript \(pq\) denotes the \(q\)th layer in the positive \(z\) half-space when \(p = 1\), and the \(q\)th layer in the negative \(z\) half-space when \(p = -1\) (see Figure 2.1). Defining \(\varepsilon_{pq}\) as a complex number provides a simple way to account for any losses that may exist in the layers.

### 2.3 Hertz Potentials

The solution procedure can be simplified by introducing the Hertz vector potentials. First we will consider the case of a homogeneous isotropic medium that contains no magnetic current sources. Reference to (2.1) shows that for this case the divergence of the magnetic field will be zero. This enables us to define

\[
\mathbf{H} := j\omega \varepsilon \nabla \times \mathbf{\Pi}_e,
\]

(2.4)

where \(\mathbf{\Pi}_e\) is the electric Hertz potential. After substituting (2.4) into (2.1), we find that

\[
\mathbf{E} = \nabla \times \nabla \times \mathbf{\Pi}_e - \frac{\mathbf{J}}{j\omega \varepsilon},
\]

(2.5)

and

\[
\nabla \times (\mathbf{E} - k^2 \mathbf{\Pi}_e) = 0,
\]

(2.6)
where $k := \omega \sqrt{\mu \varepsilon}$. Now, the vector identity,

$$\nabla \times \nabla \times \Pi_e = \nabla (\nabla \cdot \Pi_e) - \nabla^2 \Pi_e,$$

(2.7)

can be applied to (2.5), yielding

$$E = \nabla (\nabla \cdot \Pi_e) - \nabla^2 \Pi_e - \frac{J}{j \omega \varepsilon}.$$  

(2.8)

Also, since the curl of the gradient of a scalar is equal to zero, we can use (2.6) to show that

$$E = \nabla \phi + k^2 \Pi_e,$$

(2.9)

where $\phi$ is an arbitrary scalar function. If we choose $\phi := \nabla \cdot \Pi_e$ (the Lorentz gauge condition), then we find that

$$E = \nabla (\nabla \cdot \Pi_e) + k^2 \Pi_e,$$

(2.10)

where $\Pi_e$ is the solution to the non-homogeneous vector Helmholtz equation

$$(\nabla^2 + k^2) \Pi_e = -\frac{J}{j \omega \varepsilon}.$$  

(2.11)

Likewise, for the case of a homogeneous isotropic medium that contains no electric current sources, defining

$$E := -j \omega \mu \nabla \times \Pi_m,$$

(2.12)

where $\Pi_m$ is the magnetic Hertz potential, leads to the following two equations:

$$H = \nabla (\nabla \cdot \Pi_m) + k^2 \Pi_m,$$

(2.13)

$$(\nabla^2 + k^2) \Pi_m = -\frac{M}{j \omega \mu}.$$  

(2.14)

Finally, in a general medium that contains both electric and magnetic current sources, the total field will be a superposition of the partial fields that are given in (2.4), (2.10), (2.12), and (2.13).

$$E = \nabla (\nabla \cdot \Pi_e) + k^2 \Pi_e - j \omega \mu \nabla \times \Pi_m$$

$$H = \nabla (\nabla \cdot \Pi_m) + k^2 \Pi_m + j \omega \varepsilon \nabla \times \Pi_e$$

(2.15)
2.4 Spatial Domain Fields

In a homogeneous layer which is infinite in two directions (say \(x\) and \(y\)) and bounded in the third (\(z\)), any source-free field can be expressed in terms of two scalar potentials called Whittaker potentials. Therefore, in the \(pq\) layer (see Figure 2.1), we will define

\[
\begin{align*}
\Pi_x^{pq} & := a_x V_{pq} \\
\Pi_y^{pq} & := a_y U_{pq}
\end{align*}
\]

(2.16)

After substituting (2.16) into (2.11), (2.14), and (2.15), we find that the fields in the \(pq\) layer can be represented in terms of the Whittaker potentials in the following manner:

\[
\begin{align*}
E_x^{pq}(x, y, z) &= -j\omega \mu_{pq} \frac{\partial U_{pq}(x, y, z)}{\partial y} + \frac{\partial^2 V_{pq}(x, y, z)}{\partial z^2} \\
E_y^{pq}(x, y, z) &= j\omega \mu_{pq} \frac{\partial U_{pq}(x, y, z)}{\partial z} + \frac{\partial^2 V_{pq}(x, y, z)}{\partial y^2} \\
E_z^{pq}(x, y, z) &= \left( \frac{\partial^2}{\partial y^2} + k_{pq}^2 \right) V_{pq}(x, y, z) \\
H_x^{pq}(x, y, z) &= j\omega \epsilon_{pq} \frac{\partial V_{pq}(x, y, z)}{\partial y} + \frac{\partial^2 U_{pq}(x, y, z)}{\partial z^2} \\
H_y^{pq}(x, y, z) &= -j\omega \epsilon_{pq} \frac{\partial V_{pq}(x, y, z)}{\partial z} + \frac{\partial^2 U_{pq}(x, y, z)}{\partial y^2} \\
H_z^{pq}(x, y, z) &= \left( \frac{\partial^2}{\partial z^2} + k_{pq}^2 \right) U_{pq}(x, y, z)
\end{align*}
\]

(2.17)

where \(U_{pq}(x, y, z)\) and \(V_{pq}(x, y, z)\) are solutions to the following homogeneous scalar Helmholtz equations:

\[
\left( \nabla^2 + k_{pq}^2 \right) \left\{ \begin{array}{c} U_{pq}(x, y, z) \\ V_{pq}(x, y, z) \end{array} \right\} = 0.
\]

(2.18)

These equations show that the Whittaker potentials allow us to express the fields as a superposition of TE and TM fields, where \(U_{pq}\) corresponds to the TE field and \(V_{pq}\) corresponds to the TM field. From now on, we will be able to treat the TE and TM fields separately, thereby greatly simplifying the problem.

It should be pointed out at this time that we could have also formulated the problem in terms of the Lorentz potentials and their duals. The Lorentz potentials are defined by

\[
\begin{align*}
\mathbf{B} & := \nabla \times \mathbf{A} \\
-c\mathbf{E} & := \nabla \times \mathbf{F}
\end{align*}
\]

(2.19)
and are related to the Hertz potentials in the following manner:

\[ \Pi_e = \frac{A}{j\omega \mu} \]
\[ \Pi_m = \frac{F}{j\omega \epsilon} \]

Therefore, choosing \( A = a_x A_x \) and \( F = a_z F_z \), as was done in [13], will yield the same results as those given in (2.17). (Note: In [13], \( H := \nabla \times A \) and \( E := -\nabla \times F \))

### 2.5 Spectral Domain Fields

In order to obtain a representation for the Whittaker potentials in a given layer, we must solve the partial differential equations in (2.18), and enforce the appropriate boundary conditions at the interfaces on the two sides of the layer. The process of solving the partial differential equations can be greatly simplified by Fourier transforming the \( x \) and \( y \) dependence out of the problem, thereby reducing the partial differential equations to ordinary differential equations. The use of Fourier transform methods in the solution of electromagnetic field problems is commonly referred to as the spectral domain technique.

First we define the Fourier transform pair

\[
\begin{align*}
\{ \tilde{U}(\alpha_1, \alpha_2, z) \} &:= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ U(x, y, z) \right\} e^{j(\alpha_1 x + \alpha_2 y)} \, dx \, dy \\
\{ \tilde{V}(\alpha_1, \alpha_2, z) \} &:= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ V(x, y, z) \right\} e^{j(\alpha_1 x + \alpha_2 y)} \, d\alpha_1 \, d\alpha_2
\end{align*}
\]

Next, expressions for the spectral domain TE and TM fields (see Table 2.1) are obtained by taking the Fourier transform of the equations in (2.17). Likewise, the spectral domain Helmholtz equations are obtained by taking the Fourier transform of (2.18), yielding

\[
\left( \frac{\partial^2}{\partial x^2} + \tau^2_{pq}(\alpha_1, \alpha_2) \right) \left\{ \tilde{U}_{pq}(\alpha_1, \alpha_2, z) \right\} = 0,
\]

\( (2.22) \)
Table 2.1: Spectral domain fields

<table>
<thead>
<tr>
<th>Field Component</th>
<th>TE Fields</th>
<th>TM Fields</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{E}_x^{(pq)} (\alpha_1, \alpha_2, z)$</td>
<td>$-\omega \mu_{pq} \alpha_2 \hat{U}_{pq}(\alpha_1, \alpha_2, z)$</td>
<td>$-j\alpha_1 \frac{\partial \hat{V}_{pq}(\alpha_1, \alpha_2, z)}{\partial z}$</td>
</tr>
<tr>
<td>$\tilde{E}_y^{(pq)} (\alpha_1, \alpha_2, z)$</td>
<td>$\omega \mu_{pq} \alpha_1 \hat{U}_{pq}(\alpha_1, \alpha_2, z)$</td>
<td>$-j\alpha_2 \frac{\partial \hat{V}_{pq}(\alpha_1, \alpha_2, z)}{\partial z}$</td>
</tr>
<tr>
<td>$\tilde{E}_z^{(pq)} (\alpha_1, \alpha_2, z)$</td>
<td>0</td>
<td>$(\frac{\partial^2}{\partial z^2} + k_{pq}^2) \hat{V}_{pq}(\alpha_1, \alpha_2, z)$</td>
</tr>
<tr>
<td>$\tilde{H}_x^{(pq)} (\alpha_1, \alpha_2, z)$</td>
<td>$-j\alpha_1 \frac{\partial \hat{V}_{pq}(\alpha_1, \alpha_2, z)}{\partial z}$</td>
<td>$\omega \epsilon_{pq} \alpha_2 \hat{V}_{pq}(\alpha_1, \alpha_2, z)$</td>
</tr>
<tr>
<td>$\tilde{H}_y^{(pq)} (\alpha_1, \alpha_2, z)$</td>
<td>$-j\alpha_2 \frac{\partial \hat{V}_{pq}(\alpha_1, \alpha_2, z)}{\partial z}$</td>
<td>$-\omega \epsilon_{pq} \alpha_1 \hat{V}_{pq}(\alpha_1, \alpha_2, z)$</td>
</tr>
<tr>
<td>$\tilde{H}_z^{(pq)} (\alpha_1, \alpha_2, z)$</td>
<td>$(\frac{\partial^2}{\partial z^2} + k_{pq}^2) \hat{U}_{pq}(\alpha_1, \alpha_2, z)$</td>
<td>0</td>
</tr>
</tbody>
</table>
where
\[ r_{pq}^2(\alpha_1, \alpha_2) := k_{pq}^2 - (\alpha_1^2 + \alpha_2^2). \] (2.23)

The general solution for a one-dimensional, homogeneous Helmholtz equation, such as (2.22), can be expressed as a superposition of plane waves; one traveling in the positive z-direction and one in the negative z-direction. In the two semi-infinite regions (i.e. \( pq = 1n \) and \( pq = -1m \)), there will only be outgoing waves present, but in all the other regions, both waves will exist. Therefore, the spectral domain Whittaker potentials can be written as

\[
\begin{aligned}
\begin{bmatrix}
\tilde{U}_{pq}(\alpha_1, \alpha_2, z) \\
\tilde{V}_{pq}(\alpha_1, \alpha_2, z)
\end{bmatrix} &= \begin{bmatrix}
A_{\tilde{U}}^{(pq)} \\
A_{\tilde{V}}^{(pq)}
\end{bmatrix} \left[ e^{-j\rho_{pq}z} + \begin{bmatrix}
\Gamma_U^{(pq)} \\
\Gamma_V^{(pq)}
\end{bmatrix} e^{j\rho_{pq}z} \right],
\end{aligned}
\] (2.24)

where
\[
\Gamma_U^{(1n)} = \Gamma_V^{(1n)} = \Gamma_U^{(-1m)} = \Gamma_V^{(-1m)} = 0,
\] (2.25)

and
\[
p = \begin{cases}
+1; & z > 0 \\
-1; & z < 0
\end{cases}.
\] (2.26)

Since the fields must remain finite valued as \( z \to \pm \infty \), we must restrict

\[
\Im(r_{pq}) = \Im(\sqrt{k_{pq}^2 - (\alpha_1^2 + \alpha_2^2)}) \leq 0,
\] (2.27)

for \( pq = 1n \) and \( pq = -1m \). For other values of \( pq \), the above restriction is unnecessary; however, for the sake of consistency, we will use (2.27) for all values of \( pq \).

### 2.6 Boundary Conditions

The reflection coefficients and the amplitude constants in (2.24) can be obtained by enforcing the proper boundary conditions at each interface. A single interface is shown in Figure 2.2. The boundary conditions for the tangential electric and magnetic fields at such an interface are given by:

\[
a_n \times (E^{[n,q+1]} - E^{[pq]}) = -M
\] (2.28)

\[
a_n \times (H^{[n,q+1]} - H^{[pq]}) = J.
\] (2.29)
At the source-free interfaces (ie. \( z \neq 0 \)), it is possible to obtain decoupled boundary conditions for the TE and TM fields of the spectral domain Whittaker potentials. This can be accomplished by substituting the relevant equations from Table 2.1 into the Fourier transformed versions of the boundary conditions (see (2.28) and (2.29)), where the sources are set equal to zero. For the TE field, we obtain

\[
\begin{align*}
\mu_{pq} \tilde{U}_{pq}(\alpha_1, \alpha_2, z_{pq}) &= \mu_{p,q+1} \tilde{U}_{p,q+1}(\alpha_1, \alpha_2, z_{pq}) \\
\left. \frac{\partial \tilde{U}_{pq}(\alpha_1, \alpha_2, z)}{\partial z} \right|_{z=z_{pq}} &= \left. \frac{\partial \tilde{U}_{p,q+1}(\alpha_1, \alpha_2, z)}{\partial z} \right|_{z=z_{pq}} \end{align*}
\tag{2.30}
\]

and for TM field

\[
\begin{align*}
\varepsilon_{pq} \tilde{V}_{pq}(\alpha_1, \alpha_2, z_{pq}) &= \varepsilon_{p,q+1} \tilde{V}_{p,q+1}(\alpha_1, \alpha_2, z_{pq}) \\
\left. \frac{\partial \tilde{V}_{pq}(\alpha_1, \alpha_2, z)}{\partial z} \right|_{z=z_{pq}} &= \left. \frac{\partial \tilde{V}_{p,q+1}(\alpha_1, \alpha_2, z)}{\partial z} \right|_{z=z_{pq}} \end{align*}
\tag{2.31}
\]

Now we can find relationships between the amplitude constants and the reflection coefficients in the \( pq \) layer, and those in the \( p, q+1 \) layer by enforcing the boundary conditions (2.30) and (2.31) at \( z = z_{pq} \). Substituting the top equation in (2.24) into (2.30) gives two equations with four unknowns:

\[
\begin{align*}
A_U(pq) \mu_{pq} e^{j \rho \rho_{pq} z_{pq}} [e^{-2j \rho \rho_{pq} z_{pq}} + \Gamma_U(pq)] &= \\
A_U(p+1,q) \mu_{p,q+1} e^{j \rho \rho_{p+1,q} z_{pq}} [e^{-2j \rho \rho_{p+1,q} z_{pq}} + \Gamma_U(p,q+1)] 
\end{align*}
\tag{2.32}
\]
An expression for $\Gamma_u^{(pq)}$ is then obtained by dividing (2.32) by (2.33), and then solving for $\Gamma_u^{(pq)}$:

$$
\Gamma_u^{(pq)} = e^{-2j p r_{pq} z_p} \\
\frac{|(e^{-2j p r_{pq} z_p + \Gamma_v^{(pq+1)}}) - \frac{\mu_{pq} T_{pq+1}}{\mu_{pq+1} z_p} (e^{-2j p r_{pq} z_p + \Gamma_v^{(pq+1)}})|}{|(e^{-2j p r_{pq} z_p + \Gamma_v^{(pq+1)}}) + \frac{\mu_{pq} T_{pq+1}}{\mu_{pq+1} z_p} (e^{-2j p r_{pq} z_p + \Gamma_v^{(pq+1)}})|} \tag{2.34}
$$

Using the same procedure for the TM field, we obtain the following set of equations:

$$
A_v^{(pq)} e_{pq} e^{2j p r_{pq} z_p} [e^{-2j p r_{pq} z_p + \Gamma_v^{(pq)}}] = \\
A_v^{(pq+1)} e_{pq+1} e^{2j p r_{pq+1} z_p} [e^{-2j p r_{pq+1} z_p + \Gamma_v^{(pq+1)}}] \tag{2.35}
$$

$$
A_v^{(pq)} e_{pq} e^{2j p r_{pq} z_p} [e^{-2j p r_{pq} z_p - \Gamma_v^{(pq)}}] = \\
A_v^{(pq+1)} e_{pq+1} e^{2j p r_{pq+1} z_p} [e^{-2j p r_{pq+1} z_p - \Gamma_v^{(pq+1)}}] \tag{2.36}
$$

$$
\Gamma_v^{(pq)} = e^{-2j p r_{pq} z_p} \\
\frac{|(e^{-2j p r_{pq} z_p + \Gamma_v^{(pq+1)}}) - \frac{\mu_{pq} T_{pq+1}}{\mu_{pq+1} z_p} (e^{-2j p r_{pq} z_p + \Gamma_v^{(pq+1)}})|}{|(e^{-2j p r_{pq} z_p + \Gamma_v^{(pq+1)}}) + \frac{\mu_{pq} T_{pq+1}}{\mu_{pq+1} z_p} (e^{-2j p r_{pq} z_p + \Gamma_v^{(pq+1)}})|} \tag{2.37}
$$

Since the reflection coefficients in the two semi-infinite regions are equal to zero (see (2.35)), we can recursively compute the values of the reflection coefficients in the other layers by using (2.34) and (2.37). Also, reference to (2.32) and (2.35) shows that all of the amplitude coefficients in the upper/lower layers can be expressed in terms of the amplitude coefficients in the layers which are next to the sources (i.e. $A_{U,V}^{(11)} / A_{U,V}^{(-11)}$, respectively). Therefore, if we can find expressions for these amplitude coefficients, then the spectral domain fields will be uniquely determined.
2.7 Sources

The four unknown amplitude coefficients can be obtained by enforcing the boundary conditions on the tangential fields at the position of the sources (i.e. $z = 0$). At all of the other interfaces, the boundary conditions were applied to the TE and TM fields separately; however, the TE and TM fields are coupled in regions where sources are present, so the boundary conditions must be applied to the total tangential fields. Substituting the relevant equations from Table 2.1 and (2.24) into the Fourier transformed versions of (2.28) and (2.29), we find that (Note: We will assume that $M = 0$):

\[ \omega \mu_{11} \alpha_2 A_u^{(11)}(1 + \Gamma_u^{(11)}) + \alpha_1 \tau_{11} A_v^{(11)}(1 - \Gamma_v^{(11)}) = \omega \mu_{-11} \alpha_2 A_u^{(-11)}(1 + \Gamma_u^{(-11)}) - \alpha_1 \tau_{-11} A_v^{(-11)}(1 - \Gamma_v^{(-11)}) \]  

(2.38)

\[ \omega \mu_{11} \alpha_1 A_u^{(11)}(1 + \Gamma_u^{(11)}) - \alpha_2 \tau_{11} A_v^{(11)}(1 - \Gamma_v^{(11)}) = \omega \mu_{-11} \alpha_1 A_u^{(-11)}(1 + \Gamma_u^{(-11)}) + \alpha_2 \tau_{-11} A_v^{(-11)}(1 - \Gamma_v^{(-11)}) \]  

(2.39)

\[ \omega \epsilon_{11} \alpha_2 A_v^{(11)}(1 + \Gamma_v^{(11)}) - \alpha_1 \tau_{11} A_u^{(11)}(1 - \Gamma_u^{(11)}) = \omega \epsilon_{-11} \alpha_2 A_v^{(-11)}(1 + \Gamma_v^{(-11)}) - \alpha_1 \tau_{-11} A_u^{(-11)}(1 - \Gamma_u^{(-11)}) = \tilde{J}_y \]  

(2.40)

\[ \omega \epsilon_{11} \alpha_1 A_v^{(11)}(1 + \Gamma_v^{(11)}) + \alpha_2 \tau_{11} A_u^{(11)}(1 - \Gamma_u^{(11)}) = \omega \epsilon_{-11} \alpha_1 A_v^{(-11)}(1 + \Gamma_v^{(-11)}) + \alpha_2 \tau_{-11} A_u^{(-11)}(1 - \Gamma_u^{(-11)}) = \tilde{J}_z \]  

(2.41)

We need to solve these four equations in order to obtain the four amplitude coefficients. If we multiply (2.38) by $\frac{\alpha_1}{\alpha_2}$ and add it to (2.39), then we can eliminate $A_v^{(11)}$ and $A_v^{(-11)}$ from the equations.

\[ \mu_{11} A_u^{(11)}(1 + \Gamma_u^{(11)}) - \mu_{-11} A_u^{(-11)}(1 + \Gamma_u^{(-11)}) = 0 \]  

(2.42)

Likewise, if we multiply (2.38) by $\frac{-\alpha_1}{\alpha_2}$ and add it to (2.39), then we obtain:

\[ \tau_{11} A_v^{(11)}(1 - \Gamma_v^{(11)}) + \tau_{-11} A_v^{(-11)}(1 - \Gamma_v^{(-11)}) = 0. \]  

(2.43)
Applying the same techniques to (2.40) and (2.41), we obtain the following two equations:

\[
\tau_{11} A_U^{(11)} (1 - \Gamma_U^{(11)}) + \tau_{-11} A_U^{(-11)} (1 - \Gamma_U^{(-11)}) = \frac{\alpha_2 \tilde{J}_x - \alpha_1 \tilde{J}_y}{\alpha_1^2 + \alpha_2^2} \quad (2.44)
\]

\[
\epsilon_{11} A_V^{(11)} (1 + \Gamma_V^{(11)}) - \epsilon_{-11} A_V^{(-11)} (1 + \Gamma_V^{(-11)}) = \frac{\alpha_1 \tilde{J}_x + \alpha_2 \tilde{J}_y}{\omega(\alpha_1^2 + \alpha_2^2)} \quad (2.45)
\]

Finally, combining (2.42) with (2.44), and (2.43) with (2.45), we obtain the following expressions for the amplitude coefficients:

\[
A_U^{(11)} = \left\{ \frac{\tau_{11}(1+\Gamma_U^{(11)})}{\mu_{11}(1+\Gamma_U^{(11)})} + \frac{\tau_{-11}(1+\Gamma_U^{(-11)})}{\mu_{-11}(1+\Gamma_U^{(-11)})} \right\}^{-1} \frac{\alpha_2 \tilde{J}_x - \alpha_1 \tilde{J}_y}{\mu_{11}(1+\Gamma_U^{(11)})(\alpha_1^2 + \alpha_2^2)}
\]

\[
A_V^{(11)} = \left\{ \frac{\tau_{11}(1+\Gamma_V^{(11)})}{\tau_{11}(1+\Gamma_V^{(-11)})} + \frac{\tau_{-11}(1+\Gamma_V^{(-11)})}{\tau_{-11}(1+\Gamma_V^{(-11)})} \right\}^{-1} \frac{\alpha_1 \tilde{J}_x + \alpha_2 \tilde{J}_y}{\omega(1+\Gamma_V^{(11)})(\alpha_1^2 + \alpha_2^2)}
\]

We can use the results that have been derived in this Chapter to compute the spectral domain fields which are due to electric surface currents placed at \( z = 0 \) in a stratified medium. The procedure that is used to obtain the amplitude constants and the reflection coefficients is similar to the method that is employed in the transmission line analysis (see [13] and [33]).

Next, we can obtain the desired spatial domain fields by taking the inverse Fourier transform (see (2.21)) of the spectral domain fields which are listed in Table 2.1. The resulting expressions for the spatial domain fields are listed in Table 2.2.
Table 2.2: Spatial domain fields

**TE Fields:**

\[
E_U^{(pq)}(x, y, z) = -\omega \mu_{pq} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\alpha_2 a_x - \alpha_1 a_y] A_U^{(pq)} [e^{-j \rho r_{pq}^2}] e^{-j(\alpha_1 z + \alpha_2 y)} \, d\alpha_1 \, d\alpha_2
\]

\[
H_U^{(pq)}(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{ -\rho r_{pq} [\alpha_1 a_x + \alpha_2 a_y] A_U^{(pq)} [e^{-j \rho r_{pq}^2} - \\
\Gamma_U^{(pq)} e^{j \rho r_{pq}^2}] + a_z (\alpha_1^2 + \alpha_2^2) A_U^{(pq)} [e^{-j \rho r_{pq}^2} + \\
\Gamma_U^{(pq)} e^{j \rho r_{pq}^2}] \} e^{-j(\alpha_1 z + \alpha_2 y)} \, d\alpha_1 \, d\alpha_2
\]

**TM Fields:**

\[
E_V^{(pq)}(x, y, z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\rho r_{pq} [\alpha_1 a_x + \alpha_2 a_y] A_V^{(pq)} [e^{-j \rho r_{pq}^2} - \\
\Gamma_V^{(pq)} e^{j \rho r_{pq}^2}] + a_z (\alpha_1^2 + \alpha_2^2) A_V^{(pq)} [e^{-j \rho r_{pq}^2} + \\
\Gamma_V^{(pq)} e^{j \rho r_{pq}^2}] \} e^{-j(\alpha_1 z + \alpha_2 y)} \, d\alpha_1 \, d\alpha_2
\]

\[
H_V^{(pq)}(x, y, z) = \omega \epsilon_{pq} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\alpha_2 a_x - \alpha_1 a_y] A_V^{(pq)} [e^{-j \rho r_{pq}^2} + \\
\Gamma_V^{(pq)} e^{j \rho r_{pq}^2}] e^{-j(\alpha_1 z + \alpha_2 y)} \, d\alpha_1 \, d\alpha_2
\]
Chapter 3

PRINTED STRIP DIPOLE ANTENNAS IN STRATIFIED MEDIA

3.1 Introduction

In this Chapter, we will derive an electric field integral equation (EFIE) for a printed strip dipole antenna in a stratified medium. Galerkin's method will then be used to reduce the EFIE to a matrix equation which can be solved using standard matrix techniques. We will expand the current on the dipole antenna in terms of piecewise-sinusoidal (PWS) basis functions.

The printed strip dipole antenna, whose geometry is shown in Figure 3.1, is assumed to be a perfectly conducting, infinitely thin piece of metal which is located at \( z = 0 \). The surface of the antenna is designated by \( \mathcal{S} \). The antenna is assumed to be fed in the center by an idealized delta-function gap voltage source. We will further assume that the antenna is narrow enough so that the transverse component of the current can be neglected.

3.2 Formulation of the EFIE

If we insert a printed strip dipole antenna, like the one shown in Figure 3.1, into the stratified medium shown in Figure 2.1, then we can use the results which were obtained in Chapter 2 to formulate an EFIE for this problem.
Figure 3.1: Printed strip dipole antenna geometry

We will use the scattering techniques which are presented in Chapter 5 of [21] to analyze this problem. Scattering techniques can be used since the source for the antenna can be modeled as an impressed electric field, \( \mathbf{E}^i \). Since the antenna is a perfect conductor, surface currents, \( \mathbf{J} \), will be induced on \( S \), and these currents will in turn produce a scattered electric field, \( \mathbf{E}^s \).

Let us first define a linear operator,

\[
L(\mathbf{J}) := (-\mathbf{E}^s)_{\text{tan}},
\]

(3.1)

where \( (\quad)_{\text{tan}} \) means the tangential component on \( S \). For problems involving antennas in a stratified media, we can use the results which are given in Table 2.2 to show that

\[
L(\mathbf{J}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \omega \mu_{11} [\alpha_2 \mathbf{a}_z - \alpha_1 \mathbf{a}_y] A_{ij}^{(11)} [1 + \Gamma_{ij}^{(11)}] + \
\tau_{11} [\alpha_1 \mathbf{a}_z + \alpha_2 \mathbf{a}_y] A_{ij}^{(11)} [1 - \Gamma_{ij}^{(11)}] \right\} e^{-j(\alpha_1 z + \alpha_2 y)} \, d\alpha_1 \, d\alpha_2.
\]

(3.2)

Now, the EFIE can be obtained by forcing the tangential component of the electric field to vanish on \( S \) (ie. \( \mathbf{a}_z \times (\mathbf{E}^i + \mathbf{E}^s) = 0 \)):

\[
L(\mathbf{J}) = (\mathbf{E}^i)_{\text{tan}}.
\]

(3.3)
3.3 Application of Galerkin's Method

Once we have the desired EFIE, then we can apply the method of moments (see [21]) in order to obtain an approximation for the current distribution on the antenna. A suitable inner product for this problem is a quantity which is called a reaction:

$$\langle J, E \rangle = \int_S J \cdot E \, ds.$$  \hfill (3.4)

Since the antenna is assumed to be driven by a delta-function gap voltage source which is distributed uniformly across the width of the antenna, the impressed field, $\mathbf{E}$, will only have an $x$-directed component. This, together with the assumption that the width of the antenna is much less than a wavelength means that the transverse component of the current, $J_y$, will be much smaller in magnitude than the longitudinal component, $J_z$. Therefore, we can neglect $J_y$ and expand $J_z$ in a series of basis functions, $\{J_z^{(n)}\}$, which are independent of each other and defined on $S$.

$$J \approx a_x \sum_{n=1}^{N} I_n J_z^{(n)}.$$  \hfill (3.5)

Once we find the complex coefficients, $\{I_n\}$, then (3.5) can be used to obtain the approximate current distribution on the antenna. If we substitute (3.5) into (3.3), and make use of the linearity of $L$, then the EFIE can be written as

$$\sum_{n=1}^{N} I_n L(J_z^{(n)}) \approx a_x E_z^i; \quad \text{on } S.$$  \hfill (3.6)

Next, we define a set of weighting functions, $\{w_n\}$, which are also independent of each other. Since we will be using Galerkin's method, we will choose

$$w_n = a_x J_z^{(n)}; \quad n = 1, 2, \ldots, N.$$  \hfill (3.7)

Now, the matrix equation is formed by forcing the weighted average of the left-hand-side of (3.6) to be equal to the weighted average of the right-hand-side of (3.6), for each weighting function. This can be written in terms of reactions (see (3.4)):

$$\sum_{n=1}^{N} I_n \langle a_x J_z^{(m)}, L(J_z^{(n)}) \rangle = \langle a_x J_z^{(m)}, a_x E_z^i \rangle; \quad m = 1, 2, \ldots, N.$$  \hfill (3.8)
If we define the following matrices:

\[
[I_n] := \begin{bmatrix}
I_1 \\
\vdots \\
I_N
\end{bmatrix} \quad [V_m] := \begin{bmatrix}
(a_x J_x^{(1)}, a_x E_x^{(1)}) \\
\vdots \\
(a_x J_x^{(N)}, a_x E_x^{(N)})
\end{bmatrix},
\]

\[
[Z_{mn}] := \begin{bmatrix}
(a_x J_x^{(1)}, L(J_x^{(1)})) & \cdots & (a_x J_x^{(1)}, L(J_x^{(N)})) \\
\vdots & \ddots & \vdots \\
(a_x J_x^{(N)}, L(J_x^{(1)})) & \cdots & (a_x J_x^{(N)}, L(J_x^{(N)}))
\end{bmatrix},
\]

then (3.8) can be written in matrix form as

\[
[Z_{mn}][I_n] = [V_m],
\]  
(3.10)

where \( [Z_{mn}] \) is the generalized impedance matrix, \( [I_n] \) is the generalized current matrix, and \( [V_m] \) is the generalized voltage matrix. Finally, approximations for the expansion coefficients of \( J \) are obtained by solving the system of linear equations. This can be written symbolically in matrix form as

\[
[I_n] = [Y_{mn}][V_m],
\]  
(3.11)

where the generalized admittance matrix, \( [Y_{mn}] \), is defined as

\[
[Y_{mn}] := [Z_{mn}]^{-1}.
\]  
(3.12)

Before the matrix equation (3.10) can be solved, the elements in the voltage and impedance matrices must be evaluated. If we assume that the weighting functions are real valued functions, then we can use (2.21), (3.2), (3.4), and (3.9) to show that the elements in these matrices have the general form:

\[
V_m = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J_x^{(m)}(x, y) E_x^{(i)}(x, y) \, dx \, dy
\]  
(3.13)

\[
Z_{mn} = (2\pi)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\omega \mu_{11} \alpha_2 A_U^{(11)}[1 + \Gamma_U^{(11)}] + \tau_{11} \alpha_1 A_V^{(11)}[1 - \Gamma_V^{(11)}] \} J_x^{(m)}(\alpha_1, \alpha_2)^* \, d\alpha_1 \, d\alpha_2,
\]  
(3.14)

where the reflection coefficients can be found using (2.34) and (2.37), and expressions for the amplitude constants are given in (2.46).
We will assume that the dipole antenna is fed in the center by an idealized delta-function gap voltage source. We can model this voltage source by defining an incident electric field (see [21])

$$E^i := a_x V_s \delta(x).$$  \hspace{1cm} (3.15)

Substituting (3.15) into (3.13), we find that

$$V_m = V_s \int_{-v}^{v} J_z^{(m)}(0, y) \, dy.$$  \hspace{1cm} (3.16)

The evaluation of the impedance elements, on the other hand, is much more difficult, and will be dealt with in Chapter 4.

### 3.4 Basis Functions

The next step in Galerkin's method is to choose a set of basis functions. As was previously mentioned, we will assume that the transverse component of the current can be neglected; therefore, it is desirable to choose a set of basis functions which will accurately model the longitudinal component of the current on the dipole antenna with only a small number of terms (see (3.5)). Also, since the expression for $Z_{mn}$ involves Fourier transforms of the current (see (2.46) and (3.14)), it is important to choose basis functions which can be expressed as simple functions in the spectral domain.

One possible choice is the PWS basis function which is defined as

$$J_z^{(m)}(x, y) := \begin{cases} \frac{\sin[k_A(|x| - n d + |l|)]}{\sin(k_A d)}; & |y| \leq v, |x - n d + l| \leq d \\ 0 & \text{otherwise} \end{cases},$$  \hspace{1cm} (3.17)

where $d := \frac{2l}{N+1}$, $N$ is the number of basis functions, and $k_A = \omega \sqrt{\mu_0 \epsilon_A}$. The value of $k_A$ can be chosen arbitrarily, however, in some cases a judiciously chosen value for $k_A$ can significantly improve the convergence of the solution. This basis function satisfies the boundary condition that requires the normal component of the current to vanish at the ends of the antenna and guarantees that the overall trial function will be continuous. It also has the advantage that the desired accuracy can often be obtained by using only one basis function when the antenna is near a resonance. This property is very useful
— especially in array problems (see [34] and [35]). The usefulness of this PWS basis function is demonstrated by the large number of authors that have chosen to use it in their work (see [15], [24], [26], [29], [30], and [34] – [37]).

Another useful basis function is the PWS basis function which incorporates the static distribution for the current across the width of the antenna.

\[
J_x^{(n)}(x, y) := \begin{cases} 
\frac{\sin[k_A(d - |x - nd + l|)]}{\pi \sqrt{y^2 - y^2 \sin(k_A d)}} ; & |y| \leq v, |x - nd + l| \leq d \\
0 ; & \text{otherwise}
\end{cases} \tag{3.18}
\]

This basis function has all of the advantages of (3.17), and it has the added advantage that it models the singular behavior of the component of the current that is tangential to the edge of the antenna. This basis function, and other basis functions that try to model the singular behavior of the current, also frequently appear in the literature (see [38] – [43]).

In this report, we will use the PWS basis function which is given in (3.17). We will also need to find the Fourier transform of this basis function. Therefore, applying the Fourier transform (see (2.21)) to (3.17), yields

\[
J_x^{(n)}(\alpha_1, \alpha_2) = \frac{k_A \sin(\alpha_2 v)[\cos(\alpha_1 d) - \cos(k_A d)]}{2v \pi^2 \alpha_2 \sin(k_A d)[k_A^2 - \alpha_1^2]} e^{j\alpha_1(nd-l)}. \tag{3.19}
\]

### 3.5 Variational Driving Point Impedance

Once we have an approximation for the current distribution on the antenna, we can use the equations listed in Table 2.2 to find the electric and magnetic fields, or we can find the driving point impedance for the antenna by using (see [32], pp. 8–9, or [44], pp. 348–349):

\[
Z = \frac{-1}{|I(0)|^2} \int_S \mathbf{J} \cdot \mathbf{E} \, ds. \tag{3.20}
\]

It is well known that this impedance expression is stationary in the sense that a small change, \(\delta \mathbf{J}\), about the correct surface current density, \(\mathbf{J}\), gives a zero first order change, \(\delta Z\), in the impedance \(Z\).

We can use (3.1) and (3.4) to rewrite (3.20) as

\[
Z = \frac{\langle \mathbf{J}, L(\mathbf{J}) \rangle}{\langle \mathbf{J}, a_\mathbf{x} \delta(\mathbf{x}) \rangle^2}. \tag{3.21}
\]
Now, if we substitute (3.5) into (3.21), we find that \( Z \) can be expressed in terms of the generalized currents and voltages:

\[
Z = \frac{[I_m]^T[V_m]}{\sum_{n=1}^{N} I_n \langle a_x J_x^{(n)}, a_x \delta(x) \rangle^2}.
\] (3.22)
Chapter 4

EVALUATION OF THE 2-D SOMMERFELD INTEGRALS

4.1 Introduction

In this report, we are interested in finding an efficient way to compute both the elements in an impedance matrix which results from a MOM formulation; and the electric field which is due to a given current distribution on a printed dipole antenna in a layered medium. As we have shown in Chapter 2 and Chapter 3, the computation of these quantities requires the evaluation of 2-d Sommerfeld integrals. A 2-d numerical integration routine can be used to compute these integrals, but it will require a large amount of CPU time. Therefore, it is desirable to find a more efficient method for the computation of these integrals.

In this Chapter, we will develop a new technique for the computation of the 2-d Sommerfeld integrals that are encountered when using PWS basis functions with Galerkin’s method. First, the 2-d Sommerfeld integrals are expressed in polar form. Then, the inner angular integral is decomposed into a finite number of incomplete Lipschitz-Hankel integrals (ILHI’s), Bessel functions, and other elementary functions. The outer semi-infinite integration is then handled using a numerical integration routine.
The ILHI,

\[ J_n(a, z) := \int_0^\infty e^{-at}t^n J_n(t) \, dt; \quad z \in \mathbb{R}, \; a \in \mathbb{C}, \]  

(4.1)
is an important special function which arises in a number of problems in mathematical physics (see [45] and [46]). There have been several papers written on the computation of this integral (see [45]–[49]). In [46], an algorithm was presented which efficiently computes \( J_n(a, z) \) to a user defined number of significant digits (SD). This algorithm uses one of three different series expansions to compute \( J_n(a, z) \). The choice of which expansion to use depends on the parameters \( a, z \), and SD. The ILHI, \( J_n(a, z) \), which appears in the decomposition of the 2-d Sommerfeld integrals, can be efficiently computed by using the techniques that were developed in [46].

For the purposes of this report, we are interested in computing the 2-d Sommerfeld integrals which are encountered when investigating the use of printed strip dipole antennas as applicators for hyperthermia. We will assume that the tissue, which is to be heated, can be modeled as a layered medium; thus enabling us to use the model of a printed strip dipole in a stratified medium (see Chapter 5). Since the muscle tissue is very lossy at high frequencies, we will be able to use a real-axis integration for the outer integral since the poles and branch-points will be located off the real-axis. The results which are presented in this Section can also be applied to problems in which the dielectrics are assumed to be lossless provided that the pole extraction technique (see [24]) is employed.

For the application mentioned above, we will be interested in finding the input impedance for the antenna, and the heating distribution (i.e. magnitude of the electric field squared) in the muscle tissue. Since we will only be interested in computing the electric field at points which are located at a non-zero vertical distance from the antenna, the outer integral in the 2-d Sommerfeld integral will converge fairly rapidly due to the exponential decay factor which is present in the integrand. On the other hand, when computing the elements in the impedance matrix, the source and field points are both located on the surface of the antenna; therefore, the exponential decay factor will be absent. This means that the upper limit of integration for the semi-infinite integral will have to be taken to be very large in order to obtain the desired accuracy when using a numerical integration routine. In order to avoid this problem, we will apply asymptotic extraction techniques to
the Sommerfeld integrals which are associated with the computation of the impedance elements.

4.2 Computation of the Angular Integral in the 2-D Sommerfeld Integrals

The 2-d Sommerfeld integrals that we are interested in computing are listed in Table 2.2 and equation (3.14). Since we are assuming that the $y$-directed component of the current is negligible, and that the $z$-directed component of the current can be expanded in terms of a set of basis functions (see 3.5), these integrals can be rewritten as

$$
\mathbf{E}^{(pq)}(x,y,z) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ -a_x \alpha_2 f_1^{(pq)}(\lambda, z) + \alpha_2^2 f_2^{(pq)}(\lambda, z) + a_y \alpha_1 \alpha_2 \left[ f_1^{(pq)}(\lambda, z) - f_2^{(pq)}(\lambda, z) \right] + \alpha_1 \alpha_2 f_3^{(pq)}(\lambda, z) \right\} 
\tilde{J}_z^{(n)}(\alpha_1, \alpha_2) e^{-j(\alpha_1 x + \alpha_2 y)} \, d\alpha_1 \, d\alpha_2
$$

(4.2)

and

$$
Z_{mn} = (2\pi)^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ \alpha_2^2 f_1^{(11)}(\lambda, 0) + \alpha_1^2 f_2^{(11)}(\lambda, 0) \right\} 
\tilde{J}_z^{(m)}(\alpha_1, \alpha_2) \tilde{J}_z^{(n)}(\alpha_1, \alpha_2) \, d\alpha_1 \, d\alpha_2,
$$

(4.3)

where

$$
\begin{align*}
 f_1^{(pq)}(\lambda, z) &:= \frac{\omega \mu_0 \delta_{U}^{(pq)} e^{-j \lambda z} + \Gamma^{(pq)} e^{j \lambda z}}{\alpha_2 \tilde{J}_z^{(n)}(\alpha_1, \alpha_2)} \\
 f_2^{(pq)}(\lambda, z) &:= \frac{\omega \sigma_0 \delta_{U}^{(pq)} e^{-j \lambda z} - \Gamma^{(pq)} e^{j \lambda z}}{\alpha_1 \tilde{J}_z^{(n)}(\alpha_1, \alpha_2)} \\
 f_3^{(pq)}(\lambda, z) &:= \frac{\lambda^2 \sigma_0 \delta_{V}^{(pq)} e^{-j \lambda z} + \Gamma^{(pq)} e^{j \lambda z}}{\alpha_1 \tilde{J}_z^{(n)}(\alpha_1, \alpha_2)}
\end{align*}
$$

(4.4)

and

$$
\lambda := \sqrt{\alpha_1^2 + \alpha_2^2}.
$$

(4.5)
Now, if we apply the polar transformation,

\[
\lambda := \sqrt{\alpha_1^2 + \alpha_2^2}
\]
\[
\theta := \tan^{-1}\left(\frac{\alpha_2}{\alpha_1}\right)
\]

(4.6)

to (4.2) and (4.3), then we will obtain the following polar representations for these Sommerfeld integrals:

\[
E^{(pq)}(x, y, z) = \sum_{n=1}^{N} I_n \int_{0}^{\infty} \int_{-\pi}^{\pi} \{ -a_x \lambda \sin^2 \theta f_1^{(pq)}(\lambda, z) + \cos^2 \theta f_2^{(pq)}(\lambda, z) \} + \left(\frac{\omega}{\alpha_1}\right) f_3^{(pq)}(\lambda, z) \}
\]
\[
\cdot \tilde{J}_{\theta}^{(m)}(\lambda \cos \theta, \lambda \sin \theta) e^{-j\lambda(x \cos \theta + y \sin \theta)} \lambda^2 \, d\theta \, d\lambda,
\]

(4.7)

\[
Z_{mn} = (2\pi)^2 \int_{0}^{\infty} \int_{-\pi}^{\pi} \{ \sin^2 \theta f_1^{(11)}(\lambda, 0) + \cos^2 \theta f_2^{(11)}(\lambda, 0) \}
\]
\[
\cdot [J_{\theta}^{(m)}(\lambda \cos \theta, \lambda \sin \theta)]^* \tilde{J}_{\theta}^{(n)}(\lambda \cos \theta, \lambda \sin \theta) \lambda^2 \, d\theta \, d\lambda.
\]

(4.8)

In this report, we will use the PWS basis functions which are defined in (3.17). If we substitute the spectral domain representation for the basis functions (see (3.19)) into (4.7) and (4.8), then we find that

\[
E^{(pq)}(x, y, z) = \frac{k_A}{2\nu \pi^2 \sin(k_A d)} \sum_{n=1}^{N} I_n \int_{0}^{\infty} \{ -a_x \lambda \{ f_1^{(pq)}(\lambda, z) - f_2^{(pq)}(\lambda, z) \} \}
\]
\[
\cdot \mathcal{I}_1(k_A, \lambda, x + l - nd, y, (1, 0, 0, 0)) + \left(\frac{\omega}{\alpha_1}\right) f_3^{(pq)}(\lambda, z) \}
\]
\[
\cdot \mathcal{I}_1(k_A, \lambda, x + l - nd, y, (1, 0, 1, 0)) + a_y \lambda \{ f_1^{(pq)}(\lambda, z) - f_2^{(pq)}(\lambda, z) \} \}
\]
\[
\cdot \mathcal{I}_1(k_A, \lambda, x + l - nd, y, (1, 1, 0, 0)) + a_z \mathcal{I}_1(k_A, \lambda, x + l - nd, y, (1, 1, 1, 0)) \}
\]
\[
\cdot f_3^{(pq)}(\lambda, z) \lambda \, d\lambda
\]

(4.9)

and

\[
Z_{mn} = \left[ \frac{k_A}{\nu \pi \sin(k_A d)} \right]^2 \int_{0}^{\infty} \{ f_1^{(11)}(\lambda, 0) - f_2^{(11)}(\lambda, 0) \}
\]
\[
\cdot \mathcal{I}_1(k_A, \lambda, d(m - n), 0, (0, 0, 0, 1)) + \left(\frac{\omega}{\alpha_1}\right) f_3^{(11)}(\lambda, 0) \}
\]
\[
\cdot \mathcal{I}_1(k_A, \lambda, d(m - n), 0, (0, 0, 1, 1)) \lambda \, d\lambda.
\]

(4.10)
where

\[
I_1(k_A, \lambda, x, y, S) := \int_{-\pi}^{\pi} \left\{ \frac{[\cos(d\lambda \cos \theta) - \cos(k_A d)] \sin(v \lambda \sin \theta)}{k_A^2 - \lambda^2 \cos^2 \theta} \right\}^{S_4+1} \\
\frac{\sin S_1 \theta \cos S_2 \theta}{1 - S_3^2 \cos^2 \theta} e^{-\lambda(x \cos \theta + y \sin \theta)} d\theta 
\]  

(4.11)

and \( S := (S_1, S_2, S_3, S_4) \). Therefore, in order to evaluate the inner angular integral in the Sommerfeld integrals, we need to be able to compute integrals which have the general form of (4.11).

We will use two different methods to compute \( I_1(k_A, \lambda, x, y, S) \). In the first method, an adaptive quadrature routine is used to compute this integral (see Appendix D for details). In the second method, \( I_1(k_A, \lambda, x, y, S) \) is expressed in terms of ILH1's, Bessel functions, and other elementary functions. We will look at the details of the second method in the remainder of this Chapter.

We have found that it is convenient to express (4.11) in terms of a new integral,

\[
I^*_2(k_A, \lambda, x, y, v, S) := \int_{-\pi}^{\pi} \sin S_1 \theta \cos S_2 \theta \\
[1 - S_3^2 \cos^2 \theta] \\
\frac{[e^{-j\lambda r(x, y, v)} \cos[\theta - \theta_0(x, y, v)] + e^{-j\lambda r(x, y, -v)} \cos[\theta - \theta_0(x, y, -v)]]}{(k_A^2 - \lambda^2 \cos^2 \theta)^{S_4+1}} d\theta, 
\]

(4.12)

where \( r(x, y) := \sqrt{x^2 + y^2} \) and \( \theta_0(x, y) := \tan^{-1}\left(\frac{y}{x}\right) \). If we represent the trigonometric functions in the integrand of (4.11), which have trigonometric functions in their arguments, in terms of exponentials, and apply the identity,

\[
x \cos \theta + y \sin \theta = r \cos(\theta - \theta_0),
\]

(4.13)

then it can be shown that (4.11) can be rewritten as:

\[
\text{For } S_4 = 0 \\
I_1(k_A, \lambda, x, y, S) = \frac{-1}{4j} \{ I^*_2(k_A, \lambda, x + d, y, v, S) + I^*_2(k_A, \lambda, x - d, y, v, S) - 2 \cos(k_A d)I^*_2(k_A, \lambda, x, y, v, S) \} 
\]

(4.14)
For $S_4 = 1$

\[
I_1(k_A, \lambda, x, y, S) = \frac{1}{16} \left\{ (I_2^+(k_A, \lambda, x, y, 0, S) - I_2^+(k_A, \lambda, x, y, 2v, S))(2 + 4 \cos^2(k_Ad)) - [I_2^+(k_A, \lambda, x + d, y, 0, S) + I_2^+(k_A, \lambda, x - d, y, 0, S)] - I_2^+(k_A, \lambda, x + d, y, 2v, S) - I_2^+(k_A, \lambda, x - d, y, 2v, S)]4 \cos(k_Ad) + I_2^+(k_A, \lambda, x + 2d, y, 0, S) + I_2^+(k_A, \lambda, x - 2d, y, 0, S)
- I_2^+(k_A, \lambda, x + 2d, y, 2v, S) - I_2^+(k_A, \lambda, x - 2d, y, 2v, S) \right\}
\] (4.15)

Finally, if we define another integral,

\[
I_3(k_A, \lambda, x, y, S) := \int_{-\pi}^{\pi} \sin S_1 \theta \cos S_2 \theta \ e^{-j \lambda r(x,y) \cos \theta \ - k_h(x,y)} \frac{1}{1 - S_3^2 \cos^2 \theta} \frac{1}{(k_A^2 - \lambda^2 \cos^2 \theta)^{S_4 + 1}} \ d\theta,
\] (4.16)

then we find that

\[
I_2^+(k_A, \lambda, x, y, v, S) = I_2(k_A, \lambda, x, y + v, S) \pm I_3(k_A, \lambda, x, y - v, S).
\] (4.17)

Therefore, if we can efficiently compute integrals which have the general form of $I_3(k_A, \lambda, x, y, S)$, then we will be able to calculate the angular integrals in (4.9) and (4.10) by using (4.14), (4.15), and (4.17). In Appendix A, $I_3(k_A, \lambda, x, y, S)$ is decomposed into a finite number of ILHI's, Bessel functions, and other elementary functions. It is also shown in Appendix A that we can use the expression for $I_3(k_A, \lambda, x, y, S)$, which is given in Table A.1, in place of $I_3(k_A, \lambda, x, y, S)$ in (4.17) and still obtain the correct result for $I_1(k_A, \lambda, x, y, S)$ when $S$ takes on one of the values in (A.2) or (A.3).

We have shown that the expression for $I_3(k_A, \lambda, x, y, S)$ in Table A.1 holds for $0 \leq \lambda < k_A$, as well as $\lambda > k_A$ (see §A.4 and §A.5, respectively), however, there will be numerical instabilities when $\lambda$ is close to $k_A$. In order to avoid this problem, we have chosen to use the numerical integration routine (see Appendix D) to evaluate $I_1(k_A, \lambda, x, y, S)$ when $|\lambda - k_A|$ is small. We could have also expanded (4.11) in a Taylor series expansion about the point $\lambda = k_A$, and then decomposed each resulting integral in terms of ILHI's; however, this method would require a lot of extra algebra.

Now that we know how to express $I_3(k_A, \lambda, x, y, S)$ in terms of ILHI's, we must find an efficient way to compute these ILHI's. In Appendix B, we show how the results in [46] can be used to construct an efficient algorithm for the computation of $I_3(k_A, \lambda, x, y, S)$.
In the next three Sections, we will demonstrate how an efficient algorithm can be constructed for the computation of the elements in the impedance matrix, and the near zone electric field distribution.

4.3 Computation of the Elements in the Impedance Matrix

In this Section, we will discuss some of the special techniques which can be applied in order to improve the computational efficiency for the elements in the impedance matrix.

For the set of basis functions that we have chosen (see (3.17)), we will only need to compute $I_3(k_A, \lambda, x, y, S)$ for the special case of $y = 0$ (see (4.10)). If we had also chosen to segment the antenna in the $y$-direction, then $y$ would have taken on non-zero values just as $x$ does. At this point it is beneficial to look at some special cases, including the one which is discussed above. One special case of interest occurs when $-y$ is substituted for $y$ in (4.16). Using the change of variables, $\theta := -\theta$, it is easy to show that

$$I_3(k_A, \lambda, x, -y, S) = (-1)^{S_7} I_3(k_A, \lambda, x, y, S).$$  \hspace{1cm} (4.18)

For the special case of $y = 0$, we can then use (4.18) to show that (4.17) simplifies to

$$I_2^\pm(k_A, \lambda, x, 0, v, S) = [1 + (-1)^{S_7}] I_3(k_A, \lambda, x, v, S).$$ \hspace{1cm} (4.19)

Now we can use (4.19) and (4.17) to show that (4.15) can be rewritten as

$$I_3(k_A, \lambda, x, 0, S) = \frac{-1}{8} \{(2 + 4 \cos^2(k_A d))I_2^-(k_A, \lambda, x, v, v, S) -$$
$$4 \cos(k_A d)[I_2^-(k_A, \lambda, x + d, v, v, S) + I_2^-(k_A, \lambda, x - d, v, v, S)] +$$
$$I_2^-(k_A, \lambda, x + 2d, v, v, S) + I_2^-(k_A, \lambda, x - 2d, v, v, S)}\},$$ \hspace{1cm} (4.20)

for the values of $S$ in (A.2).

Similar simplifications arise for some special cases of $x$. This time, if we substitute $-x$ for $x$ in (4.16), then we can use the change of variables, $\theta := \pi - \theta$, to show that

$$I_3(k_A, \lambda, -x, y, S) = (-1)^{S_7} I_3(k_A, \lambda, x, y, S).$$ \hspace{1cm} (4.21)
Now, if we apply (4.21) to (4.20), then we find that

$$I_1(k_A, \lambda, 0, 0, S) = \frac{-1}{4} \{(1 + 2 \cos^2(k_A d))I_2^1(k_A, \lambda, 0, v, v, S) - 4 \cos(k_A d)I_2^2(k_A, \lambda, d, v, v, S) + I_2^3(k_A, \lambda, 2d, v, v, S)\}, \quad (4.22)$$

for the values of $S$ in (A.2). If we refer to (A.60) and (A.62), we find that (4.18) and (4.21) also hold if $I_3^j(k_A, \lambda, x, y, S)$ is replaced by $I_3^j(k_A, \lambda, x, y, S)$. Therefore, the results in (4.20) and (4.22) will also hold if $I_3^j(k_A, \lambda, x, y, S)$ is replaced by $I_3^j(k_A, \lambda, x, y, S)$ in (4.19).

We now turn our attention to the form of the impedance matrix (see (3.9)). Referring to (4.10), (4.17), and (4.21), we find that the impedance matrix is a symmetric Toeplitz matrix (see [50], pp. 43-52). Therefore, we only need to compute $Z_{m1}$ for $m = 1, 2, \ldots, N$, where $N$ is the number of basis functions (see (3.5)). Once the impedances have been computed, we can use a version of the subroutine TOEPLZ [see [50]], which has been modified to handle complex valued matrix elements, to solve the system of equations in (3.10).

Next we will look at some ways to improve the computational efficiency for the elements in the impedance matrix. We will use versions of the adaptive quadrature routines D01AJF and D01AKF (see [51]), which have been modified to handle complex valued integrands, to compute the outer integral in $Z_{m1}$ (see (4.10)). As was previously mentioned, the inner angular integral, $I_1(k_A, \lambda, d(m - 1), 0, S)$, will be computed by using either an adaptive quadrature routine (see Appendix D), or the decomposition in terms of ILHJ’s (see §4.2, Appendix A, and Appendix B). In both of these methods, we need to compute $I_1(k_A, \lambda, d(m - 1), 0, S)$ at a number of values of $\lambda$ for each different value of $m$. Since $f_1^{(11)}(\lambda, 0)$ and $f_3^{(11)}(\lambda, 0)$ are independent of $m$ (see (4.4)), we can compute these functions at the values of $\lambda$ that are required for the computation of $Z_{N1}$, and then store them away so that they can be reused when computing $Z_{m1}$ for $m = N - 1, N - 2, \ldots, 1$.

When we are using the expansion in terms of ILHJ’s, we can also reuse the computed values of $I_2^j(k_A, \lambda, x, v, v, S)$. When we compute the first integral, $Z_{N1}$, we will need to compute $I_2^j(k_A, \lambda, x, v, v, S)$, for $x = (N - 1)d$, $(N - 1 \pm 1)d$, and $(N - 1 \pm 2)d$, at a number of values of $\lambda$ (see (4.10) and (4.20)). If we store these results, then we will only need to calculate $I_2^j(k_A, \lambda, (N - 4)d, v, v, S)$ when we compute $Z_{N-1,1}$ since the other four pieces of (4.20) have already been computed. Actually, this is only true
if the integrand is sampled at the same values of $\lambda$ when computing both $Z_{N_1}$ and $Z_{N-1,3}$. Therefore, we must keep track of the values of $\lambda$ at which the integrand has been evaluated. We can use this storage trick to compute $I_3(k_A, \lambda, d(m - 1), 0, S)$ for $m = N - 1, N - 2, \ldots, 3$. When $m = 1$ or 2, however, we no longer even need to compute $I_2(k_A, \lambda, (m - 3)d, v, v, S)$ since we can obtain its value through the use of (4.17) and (4.21). The special case for $m = 1$ is shown in (4.22).

In order to apply one of the numerical integration routines, we will truncate the outer semi-infinite integral at a finite upper limit. Since the integral in (4.10) only has algebraic decay for large values of $\lambda$, this upper limit will have to be chosen very large in order to obtain an accurate approximation for this integral. It will be shown in the next Section that an asymptotic extraction technique provides one way to lower the value of the upper integration limit, thereby greatly improving the efficiency for the outer integration.

### 4.4 Application of the Asymptotic Extraction Technique to the Impedance Elements

The asymptotic extraction technique (AET) provides a way to significantly improve the computational efficiency when a numerical integration routine is used to compute a semi-infinite integral. This technique involves finding an asymptotic extraction term which adequately approximates the asymptotic behavior of the integrand, and which when integrated can be expressed in terms of algebraic functions or special functions. After subtracting this asymptotic extraction term from the integrand, a smaller upper limit can be used, thereby decreasing the number of sample points in the numerical integration.

The AET is used in [24] and [27] to improve the efficiency for computing 1-d Sommerfeld integrals (i.e., semi-infinite integrals associated with the evaluation of the electric field due to an electric Hertzian dipole in a layered medium). The authors of [27] demonstrate that the AET can be used to lower the value of the upper integration limit; however, no results are given for the amount of CPU time saved by using this method.

The AET can also be applied to 2-d Sommerfeld integrals, as is shown
in [29] and [30]. Pozar shows in [29] that a homogeneous-space term can be extracted from the integrand, thereby giving an integral that converges more rapidly. We will call this method the homogeneous-space term extraction technique (HSTET).

The ILHI expansions, which we have developed for $\hat{J}_3(k_A, \lambda, x, y, S)$, can also be used to formulate a novel AET for 2-d Sommerfeld integrals. The details of this new AET are given in Appendix C. In this Section, we will compare this AET with the HSTET which is used in [29] and [30].

In Appendix C, we show how the integral in (C.4) can be rewritten in terms of a series of special functions which are of the form in (C.39). Appendix C also shows how these special functions can be efficiently computed.

In order to apply the results in Appendix C, the lower limit of integration in (C.4), $L$, must satisfy the two inequalities in (C.77) and (C.78). We would like to choose $L$ as small as possible since it corresponds to the upper limit of integration in the numerical integration routine that is used to compute (4.10). The inequality in (C.78) is required for the approximations in (C.7) to hold. A similar approximation is required in the HSTET; therefore, the inequality in (C.78) applies to both methods.

If we carry out all of the expansions in Appendix C to the same order of $\lambda$ as in (C.5), then we will obtain $SD$ significant digits in the approximation for the integrand of (C.4) provided that the inequality in (C.77) is satisfied. In [29], the asymptotic form for $Q$ (see [29], (8)) can be obtained from ([29], (2)) by using what is equivalent to the first term in the expansion (C.5) (Pozar’s $k_1$ and $k_2$ correspond to $\tau_{-11}$ and $\tau_{11}$, respectively). Therefore (see [29], (8) and (13)),

$$\frac{Q - Q^h}{Q} < \frac{1}{2} \times 10^{-SD},$$

for all $\beta > L$ provided that

$$\left[ \frac{k_0 \sqrt{\epsilon}}{L} \right]^2 < \frac{1}{2} \times 10^{-SD}. \quad (4.24)$$

The two inequalities in (4.24) and (C.77) can be used to find the approximate upper limit of integration, $L$, for the numerical integration routine. The inequality in (C.77) corresponds to a relative error bound for $Z^a_{mn}$:

$$\frac{Z^a_{mn} - \bar{Z}_{mn}^a}{Z^a_{mn}} < \frac{1}{2} \times 10^{-SD},$$

(4.25)
Table 4.1: $\frac{L}{k_m}$ required for a given accuracy

<table>
<thead>
<tr>
<th>No. of Significant digits</th>
<th>SD 2</th>
<th>SD 3</th>
<th>SD 4</th>
<th>SD 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\frac{k_m}{L}]^6$</td>
<td>2.42</td>
<td>3.55</td>
<td>5.21</td>
<td>7.65</td>
</tr>
<tr>
<td>$[\frac{k_m}{L}]^2$</td>
<td>14.1</td>
<td>44.8</td>
<td>141.</td>
<td>447.</td>
</tr>
</tbody>
</table>

where $\hat{Z}_{mn}^a$ is the approximation which is obtained for $Z_{mn}^a$ by using the results in Appendix C. A more appropriate error bound would be

$$\frac{Z_{mn}^a - \hat{Z}_{mn}^a}{Z_{mn}^a} < \frac{1}{2} \times 10^{-SD},$$

(4.26)

however, this error bound is more difficult to use and doesn’t provide insight into the problem. Therefore, we will use (4.24) and (C.77) to compare the two methods. Table 4.1 shows that carrying out more terms in the expansions, as is done in the AET, significantly reduces the ratio $[L/k_m]$ (where $k_m = \max(k_{11}, k_{-11}, k_A)$ and $k_m = k_0 \sqrt{\epsilon_r} = \max(k_{11}, k_{-11})$ for the AET and the HSTET, respectively) which is required to satisfy (4.24) and (C.77). Therefore, a smaller upper limit of integration, and a correspondingly lower number of sample points will be required if the numerical integration routine is used in conjunction with the AET, instead of the HSTET, to compute the semi-infinite integral in (4.10). In order to insure that the desired accuracy is achieved, we will use a value of $L$ that is one and one-half times greater than the minimum value of $L$ which is found using (C.77) and (C.78).

The HSTET also requires that the expansion mode wavenumber, $k_A$ (see (3.17)), be chosen as

$$k_A = k_0 := \omega \sqrt{\frac{\mu_0}{2} (\epsilon_{11} + \epsilon_{-11})}.$$

(4.27)
As Pozar points out in [29], "The use of $k_e$ as the wavenumber for the PWS modes, however, is sometimes a disadvantage in terms of overall convergence of the moment method solution for microstrip patches. As discussed in [15] and [18], the number of expansion modes needed for convergence of the input impedance of patches can be significantly reduced if an expansion mode wavenumber is chosen to correspond to the effective dielectric constant of the microstrip medium. This wavenumber is basically the same as $k_e$ for thin substrates, but may differ for thick substrates, in which case the moment method solution using $k_e$ will require a larger number of expansion modes for good results. In other words, the number of expansion modes needed for a given accuracy depends on the choice of the expansion mode wavenumber, and $k_e$ is not always the optimum choice." In comparison, we are free to choose the optimal value for $k_A$ in the AET.

When the HSTET is applied to a thin dipole antenna, $Z_{mn}^h$ can be represented solely in terms of exponential integrals and other elementary functions (see [30], (18)). However, an integration of the filamentary PWS modes across the width of the dipole is required for wide dipole antennas. On the other hand, the expansions which are given in Appendix C can be used for dipoles of any width; therefore, no numerical integration is required with the AET to obtain an approximation for $Z_{mn}^a$.

As we have shown in this Section, there are a number of advantages associated with using the AET as compared with the ESTET. The major disadvantage of the AET lies in the complexity of the expansions (see Appendix C).

In Chapter 5, we present some numerical results which show the increase in the computational efficiency which can be obtained by using the AET, instead of a direct numerical integration, to compute the 2-d Sommerfeld integrals.

### 4.5 Computation of the Near-Zone Electric Field

In §4.3, we discussed some special techniques that can be used to improve the computational efficiency for the elements in the impedance matrix. Now, in this Section we will apply some of the same ideas to the computation of the near-zone electric field.
In Chapter 5, we will apply the techniques which have been developed in this report to the problem of using a printed strip dipole antenna as a hyperthermia applicator. One of the goals is to find the distribution of heat in the muscle tissue (see §5.3). This involves computing the near-zone electric field distribution. Therefore, we will develop an algorithm which computes the three components of the electric field at a grid of points. We will assume that the points are separated by the distances $\Delta x$ and $\Delta y$ in the $x$ and $y$ directions, respectively.

Due to the symmetry in the problem (see Figure 2.1 and Figure 3.1), we only need to compute the electric field in the first quadrant (i.e. $x, y \in \{x \geq 0\} \cap \{y \geq 0\}$). Actually, for the case of a symmetric dipole antenna, like the one shown in Figure 3.1, we can use (4.9) and (D.9)–(D.11) to show that:

$$
E_z^{(\text{po})}(-x, y, z) = E_z^{(\text{po})}(x, y, z)
$$

$$
E_y^{(\text{po})}(-x, y, z) = -E_y^{(\text{po})}(x, y, z)
$$

$$
E_z^{(\text{po})}(-x, y, z) = -E_z^{(\text{po})}(x, y, z)
$$

and

$$
E_x^{(\text{po})}(x, -y, z) = E_x^{(\text{po})}(x, y, z)
$$

$$
E_y^{(\text{po})}(x, -y, z) = -E_y^{(\text{po})}(x, y, z)
$$

$$
E_z^{(\text{po})}(x, -y, z) = E_z^{(\text{po})}(x, y, z)
$$

Therefore, once we have computed the electric field at a grid of points in the first quadrant, then we can use the above equations to obtain the electric field in the other three quadrants.

We will once again use versions of D01AJF and D01AKF (see [51]) which have been modified to handle complex valued integrands to compute the semi-infinite integral in (4.9); however, this time we will also modify these routines so that they will enable us to compute the three components of the electric field in parallel. Since $T_1(k_A, \lambda, z, y, S)$ decomposes into the same set of ILH1's for all of the cases of $S$ in (A.3) (see (4.14), (4.17), and Table A.1), parallel computation of $T_1(k_A, \lambda, z, y, S)$, for the four values of $S$ which are needed in (4.9), eliminates having to store each ILH1 in the decomposition.
for each value of \( \lambda \). We will use this technique since storing all of these ILHI's would require large amounts of memory, and recomputing the ILHI's for each component of the electric field would be very inefficient.

Once again, we can also save computation time by storing some of the results which are obtained during the computation of the electric field at the first point in the grid. For a fixed value of \( z \), we can store \( f_1^{(pq)}(\lambda, z) \), \( f_2^{(pq)}(\lambda, z) \), and \( f_3^{(pq)}(\lambda, z) \) the first time they are computed, and then reuse them in the computation of the electric field at the other points in the grid. However, each time \( z \) is changed, these functions will have to be recomputed and stored.

There are a couple of other ways to save computation time. Assuming that the points in the grid are separated by a distance \( \Delta x = d/m \), where \( m \) is an integer value, we can first compute the electric field that is due to the \( n = 1 \) basis function (see (4.9)) at all of the points in the grid, and then we can reconstruct the total electric field by shifting the grid, multiplying by the proper current coefficient, and then summing the results. Also, we can store the values of \( I_2^*(k_A, \lambda, x, y, v, S) \) which are required to compute \( E^{(pq)}(x_0, y, z) \) (ie. \( x = x_0 \) and \( x = x_0 \perp d \), see (4.14)), and later reuse them in the computation of \( E^{(pq)}(x_0 - d, y, z) \). Whenever \( z \) is negative, we can use (4.21) to compute \( I_3(k_A, \lambda, x, y, S) \).

We also used an AET to improve the convergence for the elements in the impedance matrix. However, since we are only interested in computing the electric field at points where \( z \neq 0 \), the integrand of (4.9) will have an exponential decay factor associated with it (see (2.34), (2.37), and (4.4)). Therefore, application of the AET is unnecessary.
Chapter 5

NUMERICAL EXAMPLES AND OBSERVATIONS

5.1 Introduction

To demonstrate the power of the techniques which have been developed in this report, we will apply them in this Chapter to the analysis of a printed strip dipole antenna in a layered medium. Specifically, we will investigate the use of printed strip dipole antennas as applicators for hyperthermia.

We will use a simplistic model in which the body tissue is represented as a layered medium. Therefore, if we place the antenna in Figure 3.1 into the general stratified media of Figure 2.1, then we can use all of the previously obtained results. The specific geometry that we are interested in is shown in Figure 5.1 (refer to Figure 3.1 for the antenna dimensions). The presence of the perfectly conducting ground plane at $z = z_{-11}$ can be handled by finding the limiting cases for (2.34) and (2.37) when $\epsilon_{-12} \rightarrow 1 - j\infty$. This procedure yields the following expressions for the reflection coefficients:

$$\begin{align*}
\Gamma_U^{(-11)} &= -e^{2j\tau_{-11}z_{-11}} \\
\Gamma_V^{(-11)} &= e^{2j\tau_{-11}z_{-11}}
\end{align*}$$

(5.1)

Before we proceed with the analysis of the hyperthermia applicator, we will check our algorithm by comparing our results with the results in Figure 4 of [30]. In this example, we are interested in computing the input impedance
of a printed strip dipole antenna on a grounded lossy substrate for different lengths of the antenna ($\lambda_0$ is the free space wavelength and shouldn't be confused with the spectral variable $\lambda$). Since there are only two layers in this problem, we will set $z_{11} = z_{12} = 0$. The current on the dipole is expanded in terms of five PWS basis functions, and then D01AJF is used to compute the angular integral. The outer semi-infinite integral is computed by using the AET in conjunction with a modified version of D01AJF (see Chapter 4). The results are shown in Figure 5.2. A visual comparison between Figure 5.2 and Figure 4 in [30] confirms that our program is working properly.

This example also shows that a real-axis integration can be used to compute the outer semi-infinite integral even when the losses in the problem are very small. If there are no losses present in the problem, then a real-axis integration can still be used provided that the pole extraction technique is employed (see [24]).

In the next two Sections, we will look at the efficiency of the techniques that have been developed in this report. The numerical examples were carried out on a Hewlett Packard 9000 series 300 computer. In the first Section, we will focus on the computation of the impedance elements. Then, in the second Section we will look at examples in which the near-zone electric field distribution is obtained.
\[ 2v = 0.01\lambda_0 \]
\[ \varepsilon_{11} = \varepsilon_{12} = 0. \]
\[ \varepsilon_{-11} = -0.1\lambda_0 \]
\[ \varepsilon_A = 1.765\varepsilon_0 \]
\[ \varepsilon_{13} = \varepsilon_0 \]
\[ \varepsilon_{-11} = (2.53 - j0.00167)\varepsilon_0 \]
\[ \varepsilon_{-12} = (1. - j\infty)\varepsilon_0 \]

Figure 5.2: Input impedance versus dipole antenna length
We will use these techniques to analyze a hyperthermia applicator which has the same material and geometry parameters as those used in the example in [53]. However, we will use a printed strip dipole antenna, instead of a microstrip patch antenna, to excite the fields. An operating frequency of 915 MHz is used and the antenna is assumed to be driven in the center by an idealized delta function gap voltage source. The antenna lies on a 0.5 cm thick grounded Rexolit substrate \((z_{-11} = -0.5 \text{ cm} \text{ and } \varepsilon_{-11} = 2.53\varepsilon_0)\). A water bolus \((\varepsilon_{11} = 80\varepsilon_0)\) is placed between the antenna and the muscle tissue in order to prevent overheating of the tissue which lies close to the antenna. Muscle tissue \((\varepsilon_{13} = (58 - j12)\varepsilon_0)\) lies in the upper half-space which starts at \(z_{11} = z_{12} = 0.5 \text{ cm}\). All of the materials are nonmagnetic.

The only other parameters that need to be specified are the wavenumber for the basis functions, \(k_A\), and the antenna dimensions (see Figure 3.1). If we use (4.27), we find that \(\varepsilon_A = 41.26\varepsilon_0\). We will arbitrarily choose the width of the antenna to be \(2u = 0.4 \text{ cm}\), and we will use antenna lengths that correspond to the first and third resonances in this example. In Figure 5.3, the input impedance is plotted versus the antenna length for the parameters given above. The results in this plot were obtained by expanding the current in terms of five PWS basis functions. From this plot we find that the first and third resonances occur at \(2l = 2.34 \text{ cm}\) and \(2l = 7.76 \text{ cm}\), respectively.

5.2 Examples Illustrating the Computation of the Impedance Matrix

In this Section, we will apply the techniques in this report to the computation of the impedance elements for the problem of the hyperthermia applicator. If we represent the current distribution on the antenna in terms of PWS basis functions (see (3.17)), then we need to compute 2-d Sommerfeld integrals that are of the form in (4.10). The impedance matrix for this problem is a symmetrical Toeplitz matrix. See §4.3 for details on the computation of the impedance elements. If we assume that the antenna is driven by a \(V_s = 1V\) delta-function gap voltage source (see Figure 3.1) and use an odd number of PWS basis functions, then we find that (see (3.16) and (3.17))

\[
V_m = \begin{cases} 1; & m = \frac{N+1}{2} \\ 0; & \text{otherwise} \end{cases}.
\]  

(5.2)
Figure 5.3: Input impedance versus antenna length for the hyperthermia applicator
Once the impedance and voltage matrices have been computed (see (3.10)), then we can use standard matrix techniques to solve for the complex valued current amplitudes. The driving point input impedance is found by using (3.22).

In Table 5.1, we have listed the amount of CPU time that is required to compute the elements in the impedance matrix to four significant digits of accuracy when using the different methods that have been developed in this report for a variable number of basis functions. The matrix elements must be computed accurately since accuracy is lost when the matrix equation (see (3.10)) is solved to obtain the current amplitudes and the input impedance.

The current distribution that is obtained for the shorter dipole antenna when five PWS basis functions are used is shown in Figure 5.4a. As expected, the antenna is found to have a sinusoidal type of current distribution. The individual basis functions are also shown in this Figure. The amplitudes of the basis functions are listed in Table 5.2. We can use these results along with (3.22) and (5.2) to show that \( Z = 8.708 \pm j0.024\Omega \).

A slightly modified current distribution is obtained when the number of basis functions is increased to nine (see Figure 5.4b). The calculated input impedance also changes slightly — \( Z = 8.983 + j0.073\Omega \). A comparison between Figure 5.4a and Figure 5.4b shows that the current is adequately modeled by five PWS basis functions.

We computed the impedance matrix using four different methods. In the columns headed by "ILHI's", the angular integrals, in the 2-d Sommerfeld integrals, were computed using the expansion in terms of incomplete Lipschitz-Hankel integrals. Numerical tests showed that requesting seven significant digits of accuracy from these expansions assured the convergence of the outer numerical integration routine. On the other hand, the headings "D01AJF" and "D01AKF" imply that the angular integrals were computed using the adaptive quadrature routines from NAG (see [51]). For these routines, it was found that requesting five significant digits was adequate. Numerical tests showed that D01AJF worked better when \( L \) was small and D01AKF worked better when \( L \) was larger, therefore we used both of these routines. For more details on the computation of the angular integral see §4.3 and Appendix D.

Two different upper limits of integration were used for the outer semi-infinite integral in the 2-d Sommerfeld integrals. In the columns headed by "\( L = 1485.52 \)"; the AET (see §4.4) was used in conjunction with a version
Table 5.1: Typical CPU times for the computation of the elements in the impedance matrix

a. $2l = 2.34cm$

<table>
<thead>
<tr>
<th>No. of Basis Functions</th>
<th>Computation time for $Z_{mn}^e(L)$</th>
<th>Upper limit of integration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$L = 1485.52$</td>
</tr>
<tr>
<td></td>
<td>ILHI's</td>
<td>D01AJF</td>
</tr>
<tr>
<td>1</td>
<td>0.54 sec.</td>
<td>8.38 sec.</td>
</tr>
<tr>
<td>3</td>
<td>1.68</td>
<td>16.02</td>
</tr>
<tr>
<td>5</td>
<td>2.62</td>
<td>26.64</td>
</tr>
<tr>
<td>7</td>
<td>3.78</td>
<td>36.92</td>
</tr>
<tr>
<td>9</td>
<td>4.96</td>
<td>43.98</td>
</tr>
</tbody>
</table>

b. $2l = 7.76cm$

<table>
<thead>
<tr>
<th>No. of Basis Functions</th>
<th>Computation time for $Z_{mn}^e(L)$</th>
<th>Upper limit of integration</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$L = 1485.52$</td>
</tr>
<tr>
<td></td>
<td>ILHI's</td>
<td>D01AJF</td>
</tr>
<tr>
<td>1</td>
<td>0.26 sec.</td>
<td>14.82 sec.</td>
</tr>
<tr>
<td>3</td>
<td>0.86</td>
<td>30.74</td>
</tr>
<tr>
<td>5</td>
<td>1.10</td>
<td>45.10</td>
</tr>
<tr>
<td>7</td>
<td>1.78</td>
<td>60.44</td>
</tr>
<tr>
<td>9</td>
<td>2.06</td>
<td>72.16</td>
</tr>
</tbody>
</table>
Figure 5.4: Surface current distribution on the hyperthermia applicator
Table 5.2: Current amplitudes for the hyperthermia applicator (2l = 2.34cm)

<table>
<thead>
<tr>
<th>n</th>
<th>$I_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0688 - j0.0164</td>
</tr>
<tr>
<td>2</td>
<td>0.1022 - j0.0193</td>
</tr>
<tr>
<td>3</td>
<td>0.1148 + j0.0003</td>
</tr>
<tr>
<td>4</td>
<td>0.1022 - j0.0193</td>
</tr>
<tr>
<td>5</td>
<td>0.0688 - j0.0164</td>
</tr>
</tbody>
</table>

of D01AJF which was modified to compute complex valued integrals. This upper limit of integration was automatically chosen by the algorithm to be equal to one and one-half times the smallest value of $L$ that satisfied the inequalities in (C.77) and (C.78). In the numerical tests that we performed, this larger value for $L$ assured that the results satisfied the desired relative error criterion. The column labeled “Computation time for $Z_{mn}^a(L)$” gives the amount of CPU time required for the computation of the asymptotic portion of the integral. Subtracting this column from the columns headed by “$L = 1485.52$” gives the amount of CPU time that was required for the numerical integration of the remainder of the semi-infinite integrals. For the cases that we looked at, the AET always took less than one-eighth of the total computation time. A comparison between Table 5.1a and Table 5.1b shows that the AET is more efficient for longer antennas. The reason for this is that the inequality in (C.26) is satisfied more often for the longer antenna since the values of $r$ are larger for the longer antenna then they are for the shorter antenna. When (C.26) is satisfied, the expansions in (C.29) and (C.31) can be used to compute the integrals in (C.94). On the other hand, when (C.26) isn't satisfied, the integrals must be split into two pieces (see (C.95)) and a more time consuming computational procedure must be applied (see §C.4).

The AET was not applied in the columns headed by “$L = 40000$.” For these cases, the numerical integration of the semi-infinite integral had to be carried out to large values of the spectral variable in order to obtain the desired accuracy. This time we found that it was beneficial to use a modified version of D01AKF to compute the semi-infinite integral. The † indicates that the upper limit of integration, $L$, had to be adjusted by trial
and error as the number of basis functions increased. For example, we had to use \( L = 50000 \) to obtain the desired accuracy when either five or seven basis functions were used, and we had to increase it to \( L = 60000 \) for the case of nine basis functions. It was very inconvenient and time consuming to manually adjust the upper limit of integration in order to obtain the desired accuracy. When \( \lambda \) appears in a space, it means that the numerical integration routine couldn't produce the desired accuracy for the given set of inputs. These problems didn't occur when the AET was employed because \( L \) was chosen automatically by the algorithm, and a more accurate result could be obtained since a numerical integration was only required over the region \( 0 \leq \lambda \leq 1485.52 \) instead of \( 0 \leq \lambda \leq 40000 \).

Now we will compare the computational efficiency of the different techniques. Comparing the third and fifth columns, or the fourth and sixth columns, we find that the AET provided a significant increase in the computational efficiency when it was used with either one of the methods that were employed to compute the angular integrals. A closer look at these columns shows that there were more dramatic improvements in the efficiency when a larger number of basis functions were used. A comparison between columns five and six shows that it was much more efficient to expand the angular integral in terms of ILHI's, than to compute it using a quadrature routine when the AET wasn't used. A comparison between columns three and four shows a similar behavior when the AET was employed, however, the improvement was not as dramatic. Comparing Table 5.1a with Table 5.1b we find that a larger amount of computation time was required for the longer antenna than the shorter antenna. We also find that the ILHI expansion is much more efficient than the numerical integration routine when it is used in conjunction with the AET for the analysis of the longer antenna.

In conclusion, we found that the AET provides a very accurate and efficient method for the computation of the elements in the impedance matrix. The AET is also very convenient to use since the user only has to specify the desired number of significant digits in the result, and then the AET automatically chooses the upper limit for the numerical integration.

Also, an additional improvement in the computational efficiency can be obtained by expressing the angular integral in terms of ILHI's instead of using a numerical integration routine. When the basis functions are relatively close together, the improvement is modest and the pay-off may not be worth the added complexity. However, more spectacular improvements
are obtained when the basis functions are widely separated. The reason for this is that the integrand of the angular integral will oscillate faster as the separation distance increases (i.e. as \(d|m - n|\) increases in (4.10)). Therefore, the amount of computation time will also increase as \(d|m - n|\) increases if a numerical integration routine is used to compute the angular integral. On the other hand, if the angular integral is written in terms of ILHI’s, then an increase in \(d|m - n|\) won’t necessarily lead to a dramatic change in the computation time since the program automatically chooses to use the most efficient expansion in Table B.1 to compute the ILHI’s. As \(z = \lambda r\) becomes larger, the efficient asymptotic factorial-Neumann series expansion will be used more often, thereby partially offsetting the increases in computation time that occur when one of the other two expansions are used. It is our belief that expanding the angular integral in terms of ILHI’s will prove to be a very useful technique for large array problems.

5.3 Heating Distribution for a Dipole Hyperthermia Applicator

In the last Section, we used Galerkin’s method to obtain the current distribution on a printed strip dipole antenna. It was shown that the computational efficiency can be improved by expressing the angular integral, in the 2-d Sommerfeld integrals, in terms of ILHI’s. Now we will show that this technique can also be used when computing the near-zone electric field distribution.

We also used the AET in the last Section to improve the computational efficiency. In this Section, however, the AET won’t be required since the integrand of the semi-infinite integral will decay exponentially with increasing values of the spectral variable for field points which are located off of the interface at \(z = 0\).

Once the current amplitudes have been obtained, then the electric field distribution is found by computing the 2-d Sommerfeld integrals in (4.9) (for more details see §4.5). When analyzing hyperthermia applicators, it is important to find the heating distribution in the muscle tissue (i.e. \(z \geq z_{11}\) in this example). The amount of heating that will occur in this simplified tissue model is proportional to \(\sigma_{13}|E^{(13)}(x, y, z)|^2\).

In order to demonstrate the efficiency of the techniques in this report, we will obtain heating patterns for the previously described hyperthermia...
Table 5.3: Relative values for the heating distributions in Figure 5.5

| $z$ (cm) | $\frac{|E(x,y,z)|^2_{max}}{|E(x,y,0.5cm)|^2_{max}}$ | $\frac{|E(0,0,r)|^2}{|E(x,y,0.5cm)|^2_{max}}$ |
|---------|--------------------------------|---------------------------------|
| 0.5     | 1.000                           | 0.250                           |
| 0.8     | 0.312                           | 0.147                           |
| 1.1     | 0.139                           | 0.100                           |
| 1.5     | 0.067                           | 0.064                           |
| 2.0     | 0.038                           | 0.038                           |
| 2.5     | 0.024                           | 0.024                           |
| 3.0     | 0.015                           | 0.015                           |
| 3.5     | 0.010                           | 0.010                           |

appplicator at the same values of $z$ as those in Figure 5 of [53]. We will use a dipole antenna whose length is $2l = 2.34\text{cm}$. In the previous Section we showed that a good approximation to the current distribution can be obtained by using five PWS basis functions (see Figure 5.4). Therefore, we will use five PWS basis functions for the computations in this Section. The current amplitudes for the basis functions are listed in Table 5.2.

Using this current distribution, we will compute the electric field at a number of points in a grid. We will use a square grid that extends out to $|x_{max}| = |y_{max}| = 3l$, where $2l$ is the length of the dipole antenna. Due to symmetry properties (see (4.28) and (4.29)), we will only have to compute the electric field in the first quadrant. If we compute 37 points in both the $x$ and $y$ directions, then the spacing between grid points will be (see (3.17))

$$\Delta x = \Delta y = \frac{3l}{36} = \frac{d}{4}. \quad (5.3)$$

This number was chosen so that we can apply the computational techniques that are discussed in §4.5. The computed relative heating patterns are shown in Figure 5.5. The peak values for these heating patterns are listed in the second column of Table 5.3. The third column shows the relative values of the heating pattern on the $z$-axis.

The heating distribution that is obtained when using a dipole antenna
Figure 5.5: Relative heating patterns for the hyperthermia applicator
differs significantly from the heating distribution for a microstrip antenna (see [53], Figure 4). The dipole heating distribution has two large peaks which are caused by the fringing fields at the ends of the dipole antenna. The fringing fields are static fields and therefore are only present in the near-field. As the distance from the antenna increases, the contribution from the fringing fields becomes negligible.

The heating distribution for the microstrip applicator in [53] doesn’t exhibit the fringing field effects that were observed with the dipole applicator. If we compare the results in Table 5.3 with Figure 4 of [53], we find that deeper penetration can be obtained by using the dipole applicator than the microstrip applicator. At a distance of 3cm from the microstrip applicator, the peak power absorption has fallen to $\frac{1}{256}$ of the power that was absorbed at a distance of 1.1cm from the applicator. In the case of the dipole applicator, the peak power absorbed at $z = 3.0cm$ is nine times less than the power absorbed at $z = 1.1cm$. Referring to Figure 5.5 and Figure 6 of [53] we find that the heating distribution defocuses faster in the case of the microstrip applicator.

Actually, neither one of these antennas would make a very good applicator if it was used by itself. Referring to Table 5.3 we find that the magnitude of the peak electric field has decayed to $e^{-1}$ of the maximum value that exists at the surface of the muscle tissue by the time it has penetrated only 0.6cm into the muscle tissue. In comparison, a normally incident plane wave has a skin depth of 6.6cm. Therefore, in order to obtain a useful applicator, an array of dipoles or microstrip antennas would have to be used.

The observations that we have made are only valid for the specific antennas that were analyzed. It may be possible to obtain a better applicator by operating one of the antennas in a different mode (ie. change the antenna dimensions) or changing the applicator geometry.

We used two different methods (see §4.5 for details) to compute the heating distribution at $z = 0.5cm$ (see Figure 5.5). In both methods the outer semi-infinite integral was computed using a version of D01AJF which was modified to handle complex valued integrands. In order to obtain four significant digits of accuracy in the results, the numerical integration was carried out to $L = 1000.0$.

In the first method, the inner angular integral was expressed in terms of ILH’s. This method took 17122.3 CPU seconds to compute a $37 \times 37$ grid of points. This corresponds to an average of 12.51 CPU seconds per grid point.
In the second method, the inner angular integral was computed using the adaptive quadrature routine D01AJF. This method required 35096.4 CPU seconds, or 25.64 CPU seconds per point.

This example shows that expanding the angular integral in terms of ILH1's greatly the efficiency for computing the near-zone electric field distribution. An even greater improvement in the efficiency would be obtained if the three components of the magnetic field were computed in addition to the three components of the electric field.

As we have shown, it is important to have an efficient algorithm for computing the near-zone electric field distribution when analyzing hyperthermia applicators. There are also other applications that require the computation of the near-zone fields. In array problems, the near-zone electric field distribution is often measured and then used to reconstruct the far-field pattern. An algorithm which can efficiently compute a theoretical near-field distribution for an array of printed strip dipole antennas would be very useful for comparison purposes. Also, probing techniques will probably be used in the future to test complex printed microstrip circuits. Therefore, there will be a need to efficiently compute the theoretical electric field distribution for comparison purposes. The techniques in this report can be used for these purposes.

5.4 Observations

We have demonstrated in this Chapter that expanding the angular integral in terms of ILH1's improves the computational efficiency for 2-d Sommerfeld integrals (for PWS basis functions). A new AET has also been developed for the evaluation of the outer semi-infinite integral in the expression for the impedance elements. This AET provides a very accurate and efficient way to compute the elements in the impedance matrix. It was also shown in §4.4 that the AET has a number of advantages when compared with the HSTET.

We have only applied these techniques to the analysis of a single printed strip dipole antenna in a layered media; however, they can be used to analyze any planar structure whose current distribution can be adequately modeled by piecewise-sinusoidal basis functions. Arrays of dipole antennas (see [34]), microstrip antennas (see [15] and [36]), and arrays of microstrip antennas (see [18] and [35]) are a few examples. In this report, we have assumed that the
transverse component of the current on the dipole antenna was negligible, and therefore it was neglected. It would be a straightforward procedure to modify the techniques in this report to incorporate the second component of the current. The details of this procedure will not be carried out in this report, however.

Other basis functions have also been used in the analysis of planar circuits. In [15], the entire domain (EB) sinusoidal basis function,

\[ J_x^{(m)}(x, y) := \sin \frac{m \pi}{2l} (x + l); \quad |x| \leq l, \quad (5.4) \]

was used in the analysis of microstrip antennas. It is easy to show that this basis function has the following spectral domain representation:

\[ J_x^{(m)}(\alpha_1, \alpha_2) = \frac{(m \pi/2l) \sin(\alpha_2 v)}{2v \pi^2 \alpha_2 \left( \frac{m \pi}{2l} \right)^2 - \alpha_2^2} \times \begin{cases} \cos(\alpha_1 l); & m \text{ odd} \\ -j \sin(\alpha_1 l); & m \text{ even} \end{cases}, \quad (5.5) \]

If we set

\[ k_A = \frac{m \pi}{2l} \begin{cases} n = 1 \\ d = l \end{cases}, \quad (5.6) \]

in (3.19), then we obtain an expression which is equivalent to the product of \((-1)^{m-1}/m\) and (5.5) for odd values of \(m\). Therefore, we can use (5.6) to modify the expressions in this report so that they hold for the odd EB basis functions. As it turns out, only the odd basis functions are required for the analysis of a center fed dipole antenna. Next we will take a closer look at the required modifications for the odd EB basis functions.

A different computational procedure will have to be used when we use the EB basis functions. If we apply (5.6) to (4.14) and (4.15), we find that some of the terms drop out since \(\cos(k_A d) = 0\) for odd values of \(m\). At first sight it looks like fewer ILHIs will have to be computed when using the EB, instead of PWS, basis functions. A second advantage of the EB basis functions is that once \(I_I(\lambda, x, y, S_1, 1)\) (see Table A.1) has been computed for each value of \(\lambda, x,\) and \(y\) during the computation of the first impedance element, \(Z_{11}\), then these values can be stored, and later reused for the computation of the other impedance elements.
There are also disadvantages associated with using EB basis functions. The partial fraction expansion in (A.6) is still valid for the self term (ie. \(Z_{mn}, m = n\)), however, for the mutual coupling terms (ie. \(Z_{mn}, m \neq n\)), we will have to expand

\[
\frac{1}{(1 - S_3^2 \cos^2 \theta)[(\frac{m}{2\pi})^2 - \lambda^2 \cos^2 \theta][[(\frac{m}{2\pi})^2 - \lambda^2 \cos^2 \theta]].
\]

(5.7)

Therefore, in order to compute \(\tilde{J}_3(k_A, \lambda, x, y, S)\) for the mutual coupling terms, we will have to compute the ILHI's \(J_{e0}(-jF_\pm, \lambda r)\) for both \(k_A = \frac{m\pi}{2\ell}\) and \(k_A = \frac{n\pi}{2\ell}\) (see Table A.1). Also, a symmetric Toeplitz impedance matrix was obtained when PWS basis functions were used to analyze the dipole antenna. This meant that we only had to compute one row in the impedance matrix. On the other hand, the impedance matrix for the EB basis functions is still symmetric, however, this time all of the elements on the diagonal, and in the upper triangular portion of the matrix need to be computed. Without actually writing a computer program, it is hard to determine which set of basis functions, EB or PWS, would provide the most efficient computational scheme. If we were to actually perform this test, we would probably find that the choice of which basis function to use depends on the problem that is being analyzed. If we applied the techniques in this paper to the even EB basis functions, we would obtain expressions that are similar to those obtained for the odd case.

Triangular basis functions are also useful in the analysis of planar structures (see [16]). When a large number of PWS basis functions are used, the basis functions start to look like triangular basis functions (see Figure 5.4). Actually, an expression for the triangular basis functions can be obtained by applying the limit \(k_A \rightarrow 0\) in (3.17). There are indications that a more efficient computational algorithm can be constructed for triangular basis functions. Therefore, it may be advantageous to use triangular basis functions when a large number of basis functions are required in the analysis of a structure. We intend to look at this problem in more detail in the next phase of our research.

Traveling wave basis functions have been used by a number of authors in the analysis of problems which include microstrip lines (see [54]–[58]). The technique of expanding the angular integral, of the 2-d Sommerfeld integrals, in terms of ILHI's, and the AET can be modified for traveling wave basis functions.
An important point that is demonstrated in [54]–[58] is that all of the previously mentioned types of basis functions are useful for different problems. One should choose to use a type of basis function, or a combination of different basis functions, that models the physics in the problem with the smallest number of unknowns. The techniques which have been developed in this report can be modified to handle any of these basis functions. Therefore, they should be investigated further.

As a final remark, there are indications that it may also be possible to develop a new AET for the PWS basis functions which incorporate the static distribution for the current across the width of the antenna (see (3.18)). However, if this turns out to be possible, the expansions will probably be much more complicated than the ones that were developed in Appendix C.
Appendix A

DECOMPOSITION OF
$I_3(k_A, \lambda, x, y, S)$ INTO A FINITE
NUMBER OF ILHI'S

A.1 Introduction

In this Appendix, we will decompose $I_3(k_A, \lambda, x, y, S)$, where

$$I_3(k_A, \lambda, x, y, S) := \int_{-\pi}^{\pi} \sin^{S_1} \theta \cos^{S_2} \theta \ e^{-j\lambda r(x,y)\cos[\theta - \theta_0(x,y)]} \ \frac{d\theta}{(1 - S_3 \cos^2 \theta)(k_A^2 - \lambda^2 \cos^2 \theta)^{S_4+1}}$$  \hspace{1cm} (A.1)

into a sequence of ILHI's. This integral will exhibit a different behavior for each different value of $S$. If we are interested in computing an impedance element, then we will need to compute $I_3(k_A, \lambda, x, y, S)$ for (see (4.10))

$$S = \left\{ \begin{array}{c}
(0,0,0,1) \\
(0,0,1,1)
\end{array} \right\}.$$  \hspace{1cm} (A.2)

On the other hand, if we are interested in computing the electric field, then (see (4.9))

$$S = \left\{ \begin{array}{c}
(1,0,0,0) \\
(1,0,1,0) \\
(0,1,0,0) \\
(1,1,0,0)
\end{array} \right\}.$$  \hspace{1cm} (A.3)
A.2 Partial Fraction Decomposition

The first step in the decomposition involves performing a partial fraction expansion on the integrand of \( T_3(\kappa_A, \lambda, z, y, S) \). For the special case \( S_4 = 0 \), it can be shown that

\[
\frac{\cos S \theta}{(1 - S_3^2 \cos^2 \theta)(k^2_A - \lambda^2 \cos^2 \theta)} = \frac{D_1}{\cos \theta + \frac{1}{S_3}} + \frac{D_2}{\cos \theta - \frac{1}{S_3}} + \frac{D_3}{\cos \theta + \kappa_A^2} + \frac{D_4}{\cos \theta - \kappa_A^2},
\]

(A.4)

where

\[
D_1 = (-1)^{1-S_1}, D_2 = \frac{(-S_2)^{1+S_2}}{2(\lambda^2-k^2_A S_3^2)}
\]

\[
D_3 = (-1)^{1-S_2}, D_4 = \frac{(-\kappa_A^2)^{1-S_2}}{2(\lambda^2-k^2_A S_3^2)}
\]

(A.5)

for the values of \( S \) in (A.3). We can also perform a partial fraction expansion for the case \( S_4 = 1 \), where for this case \( S_1 = S_2 = 0 \) (see (A.2)):

\[
\frac{1}{(1 - S_3^2 \cos^2 \theta)(k^2_A - \lambda^2 \cos^2 \theta)^2} = \frac{D_1}{\cos \theta + \frac{1}{S_3}} + \frac{D_2}{\cos \theta - \frac{1}{S_3}} + \frac{D_3}{\cos \theta + \kappa_A^2} + \frac{D_4}{\cos \theta - \kappa_A^2} + \frac{D_5}{(\cos \theta + \kappa_A^2)^2} + \frac{D_6}{(\cos \theta - \kappa_A^2)^2},
\]

(A.6)

where

\[
D_1 = -D_2 = \frac{S_3^2}{2(\lambda^2-k^2_A S_3^2)^2}
\]

\[
D_3 = -D_4 = \frac{\lambda(\lambda^2-3k^2_A S_3^2)}{4k^2_A(\lambda^2-k^2_A S_3^2)^3}
\]

\[
D_5 = D_6 = \frac{1}{4k^2_A(\lambda^2-k^2_A S_3^2)^3}
\]

(A.7)

Now, if we define two new integrals,

\[
I_4(\lambda, z, y, S_1, P) := \int_{-\pi}^{\pi} \frac{\sin S_1 \theta e^{-j \lambda r(z,y) \cos \theta - S_2(z,y)}}{\cos \theta + P} \, d\theta
\]

(A.8)
\[ I_5(\lambda, x, y, P) := \int_{-\pi}^{\pi} \frac{e^{-j\lambda r(x,y)\cos(\theta-x_0)}\cos(\theta-x_0)}{(\cos \theta + P)^2} \, d\theta, \quad (A.9) \]

then we can use the results in (A.4)–(A.7) to show that

\[ I_3(k_\lambda, \lambda, x, y, S) = D_1 I_4(\lambda, x, y, S_1, 1) + (-1)^{1-S_2} I_4(\lambda, x, y, S_1, -1) + \]

\[ D_3 I_4(\lambda, x, y, S_1, \frac{k_\lambda}{\lambda}) + (-1)^{1-S_2} I_4(\lambda, x, y, S_1, -\frac{k_\lambda}{\lambda}) + \]

\[ D_5 I_5(\lambda, x, y, \frac{k_\lambda}{\lambda}) + I_5(\lambda, x, y, -\frac{k_\lambda}{\lambda}), \quad (A.10) \]

where

\[ D_1 = \frac{-(-S_2)^{1-S_2}}{2(k_\lambda^2 S_2^2 - \lambda^2)^{1-S_2}} \]

\[ D_3 = \frac{(\frac{k_\lambda}{\lambda})^{1-S_2}}{2(k_\lambda^2 S_2^2 - \lambda^2)} \left[ \frac{3k_\lambda^2 S_2^2 - \lambda^2}{2k_\lambda^2 (k_\lambda^2 S_2^2 - \lambda^2)} S_4 \right] \]

\[ D_5 = \frac{1}{4k_\lambda^2 (k_\lambda^2 S_2^2 - \lambda^2)} \]

This equation holds for all the values of \( S \) that are shown in (A.2) and (A.3).

By making the change of variables, \( \hat{\theta} = \theta - \pi \), in (A.8) and (A.9), it is easy to show that

\[ I_4(\lambda, x, y, S_1, -P) = -(-1)^{S_2} I_4(\lambda, x, y, S_1, P)^* \]

\[ I_5(\lambda, x, y, -P) = [I_5(\lambda, x, y, P)]^* \]

Application of these two identities to (A.10), gives

\[ I_3(k_\lambda, \lambda, x, y, S) = D_1 I_4(\lambda, x, y, S_1, 1) + (-1)^{S_2} I_4(\lambda, x, y, S_1, 1)^* + \]

\[ D_3 I_4(\lambda, x, y, S_1, \frac{k_\lambda}{\lambda}) + (-1)^{S_2} I_4(\lambda, x, y, S_1, \frac{k_\lambda}{\lambda})^* + \]

\[ D_5 I_5(\lambda, x, y, \frac{k_\lambda}{\lambda}) + I_5(\lambda, x, y, \frac{k_\lambda}{\lambda})^*. \quad (A.13) \]

Now we need to evaluate integrals which are of the form given in (A.8) and (A.9). First, we will concentrate on evaluating \( I_4(\lambda, x, y, S_1, P) \). In order to simplify the exponential factor in (A.8), we make the change of variables, \( \hat{\theta} = \theta - \theta_0 \), and apply the addition formulas to the resulting trigonometric functions, thereby obtaining:

\[ I_4(\lambda, x, y, S_1, P) = \int_{-\pi}^{\pi} \frac{[\sin \hat{\theta} \cos \theta_0 + \cos \hat{\theta} \sin \theta_0]^{S_1}}{[\cos \hat{\theta} \cos \theta_0 - \sin \hat{\theta} \sin \theta_0 + P]} e^{-j\lambda r \cos \hat{\theta}} \, d\hat{\theta}. \quad (A.14) \]

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Since the variable of integration in (A.8) ranges over one full period of the integrand, the only restriction that was placed on choosing the limits of integration in (A.14) was that they also allow for integration over one full period.

At this point, it is beneficial to rearrange the integrand of (A.14) in such a way that \( \hat{\theta} \) only appears in the form of \( \cos \hat{\theta} \) in the denominator. This can be accomplished by multiplying the numerator and denominator by \((\cos \hat{\theta} \cos \theta_0 + P + \sin \hat{\theta} \sin \theta_0)\):

\[
I_4(\lambda, x, y, S_1, P) = \int_{-\pi}^{\pi} [\sin \hat{\theta} \cos \theta_0 + \cos \hat{\theta} \sin \theta_0]^{S_1} \frac{[\cos \hat{\theta} \cos \theta_0 + P + \sin \hat{\theta} \sin \theta_0]}{[P^2 - \sin^2 \theta_0 + 2P \cos \hat{\theta} \cos \theta_0 + \cos^2 \hat{\theta}]} e^{-j\lambda r \cos \theta} d\hat{\theta}. \quad (A.15)
\]

Now, the numerator can be multiplied out, and the integral can be simplified by neglecting any odd functions of \( \hat{\theta} \) which appear in the numerator since they will integrate to zero. Two cases are possible since \( S_1 = 0 \) or \( S_1 = 1 \):

\[
\begin{align*}
I_4(\lambda, x, y, 0, P) &= \int_{-\pi}^{\pi} \frac{[P + \cos \theta_0 \cos \hat{\theta}]}{[P^2 - \sin^2 \theta_0 + 2P \cos \hat{\theta} \cos \theta_0 + \cos^2 \hat{\theta}]} e^{-j\lambda r \cos \theta} d\hat{\theta} \\
I_4(\lambda, x, y, 1, P) &= \sin \theta_0 \int_{-\pi}^{\pi} \frac{[P \cos \hat{\theta} + \cos \theta_0]}{[P^2 - \sin^2 \theta_0 + 2P \cos \hat{\theta} \cos \theta_0 + \cos^2 \hat{\theta}]} e^{-j\lambda r \cos \theta} d\hat{\theta}
\end{align*}
\]

(A.16)

By factorizing the denominator, it can be rewritten as

\[
P^2 - \sin^2 \theta_0 + 2P \cos \theta_0 \cos \hat{\theta} + \cos^2 \hat{\theta} =
\]

\[
[\cos \hat{\theta} - F_+(\theta_0, P)][\cos \hat{\theta} - F_-(\theta_0, P)], \quad (A.17)
\]

where

\[
F_\pm(\theta_0, P) := \begin{cases} 
- P \cos \theta_0 \pm j \sin \theta_0 \sqrt{P^2 - 1} &; \ P \geq 1 \\
- P \cos \theta_0 \pm \sin \theta_0 \sqrt{1 - P^2} &; \ 0 \leq P < 1 
\end{cases} \quad (A.18)
\]

After defining a new integral,

\[
I_6(\lambda, x, y, F) := \int_{-\pi}^{\pi} \frac{e^{-j\lambda r \cos \theta}}{\cos \theta - F(\theta_0, P)} d\theta, \quad (A.19)
\]

we can once again use a partial fractions expansion to decompose (A.16) into

\[
I_4(\lambda, x, y, S_1, P) = D_7I_6(\lambda, x, y, F_+) + D_8I_6(\lambda, x, y, F_-), \quad (A.20)
\]
where
\[
D_7 := \left\{ \frac{\cos \theta \sqrt{P^2 - 1} - j P \sin \theta}{2 \sqrt{P^2 - 1}} \right\} j \sqrt{P^2 - 1} \}
\]
\[
D_8 := \left\{ \frac{\cos \theta \sqrt{P^2 - 1} + j P \sin \theta}{2 \sqrt{P^2 - 1}} \right\} - j \sqrt{P^2 - 1} \}
\]
\[
; \quad P \geq 1. \tag{A.21}
\]

### A.3 Representation in Terms of ILHI’s

We are finally at a point where we have an integral, \(I_6(\lambda, x, y, F_\pm)\), which can be rewritten in terms of special functions. If we recall that
\[
J_0(z) = \frac{1}{\pi} \int_0^\pi e^{-jz \cos \theta} d\theta, \tag{A.22}
\]
then it is easy to show that \(I_6(\lambda, x, y, F_\pm)\) satisfies the following first-order, non-homogeneous differential equation:
\[
\frac{dI_6}{dr} + j \lambda F_\pm I_6 = -2\pi j \lambda J_0(\lambda r). \tag{A.23}
\]

Using the well known theory for ordinary differential equations, we find that the solution of (A.23) is given by
\[
I_6(\lambda, x, y, F_\pm) = -2\pi j e^{-j\lambda F_\pm r} \int_\epsilon^{\lambda r} e^{iF_\pm t} J_0(t) \, dt, \tag{A.24}
\]
where \(\epsilon\) is a constant which is yet to be determined. If we substitute (A.22) into (A.24), interchange the order of integration, and carry out the resulting integral, then we find that
\[
I_6(\lambda, x, y, F_\pm) = e^{-j\lambda F_\pm r} \int_{-\pi}^{\pi} \left. \frac{e^{i(tF_\pm - \cos \theta)}}{\cos \theta - F_\pm} \right|_\epsilon^{\lambda r} d\theta. \tag{A.25}
\]

By comparing (A.19) with (A.25), we find that \(\delta\) can be defined as
\[
\delta := \begin{cases} 
\infty & ; \quad \Re(jF_\pm) \leq 0 \\
-\infty & ; \quad \Re(jF_\pm) > 0
\end{cases} \tag{A.26}
\]

In addition, reference to (4.1) shows that (A.24) can be rewritten in terms of ILHI’s.
\[
I_6(\lambda, x, y, F_\pm) = -2\pi j e^{-j\lambda F_\pm r} \{ J_0(-jF_\pm, \lambda r) - J_0(-jF_\pm, \delta) \} \tag{A.27}
\]
ILHI’s were studied in great detail in [46]. We will use many of the results that are contained in [46] in this Appendix.
A.4 \( \hat{I}_3(k_A, \lambda, x, y, S) \) for \( 0 \leq \lambda < k_A \)

If we refer back to the definitions of \( I_4(\lambda, x, y, S_1, P) \) and \( I_5(\lambda, x, y, P) \) in (A.8) and (A.9), we find that there are apparent singularities in the denominators of the two integrands when \( 0 \leq P \leq 1 \). On the other hand, if we refer to the definition of \( I_1(k_A, \lambda, x, y, S) \) (see (4.11)), it can be shown that the apparent singularities in the denominator are actually removable singularities for the values of \( S \) given in (A.2) and (A.3). Since \( I_1(k_A, \lambda, x, y, S) \) can be decomposed into a finite number of integrals which have the form of \( I_4(\lambda, x, y, S_1, P) \) and \( I_5(\lambda, x, y, P) \), it must be possible to remove the apparent singularities when the integrals are added together.

When \( P > 1 \), there are no singularities, so no problems exist. Therefore, we will first deal with this case (i.e. \( P > 1 \)), and later we will use analytic continuation arguments to obtain a solution which is valid when \( 0 \leq P < 1 \). Using the results that are given in an earlier report (see [46], (2.34), (2.35), and (2.37)), we find that

\[
J e_0(-j F_\pm, \delta) = \frac{1}{\sqrt{1 - F_\pm^2}}, \quad (A.28)
\]

where the branch cut for the square root is defined by

\[
\left\{ \begin{array}{c}
\Re(\sqrt{1 - F_\pm^2}) \geq 0; \quad \Re(j F_\pm) \leq 0 \\
\Re(\sqrt{1 - F_\pm^2}) < 0; \quad \Re(j F_\pm) > 0
\end{array} \right. \quad (A.29)
\]

Using the definition in (A.18), we find that

\[
\Re(j F_\pm) = \mp \sin \theta_0 \sqrt{P^2 - 1} \quad 1 - F_\pm^2 = (\pm P \sin \theta_0 + j \cos \theta_0 \sqrt{P^2 - 1})^2 \quad (A.30)
\]

In order to satisfy the condition in (A.29), we must define the branch cut as

\[
\sqrt{1 - F_\pm^2} = \pm P \sin \theta_0 + j \cos \theta_0 \sqrt{P^2 - 1}; \quad P \geq 1. \quad (A.31)
\]

Referring to (A.21), we find that the expressions for \( D_7 \) and \( D_9 \) can be
simplified by applying (A.31):
\[
D_7 = \frac{\sqrt{1 - P_2^2}}{2i \sqrt{P_2^2 - 1}} [j \sqrt{P_2^2 - 1}]^{S_1} \\
D_8 = \frac{\sqrt{1 - P_2^2}}{2i \sqrt{P_2^2 - 1}} [-j \sqrt{P_2^2 - 1}]^{S_1}
\]

Finally, combining the results in (A.20), (A.27), (A.28), and (A.32), yields
\[
I_4(\lambda, x, y, S_1, P) = -\pi j (j \sqrt{P_2^2 - 1})^{S_1 - 1} e^{-j\lambda F_+} \frac{\sqrt{F_0^2 - 1}}{\sqrt{1 - F'_2}} J_{\nu}(j F_+, \lambda r) - 1 + (-1)^{S_1} e^{-j\lambda F_-} \frac{\sqrt{F_0^2 - 1}}{\sqrt{1 - F'_2}} J_{\nu}(j F_-, \lambda r) - 1 \quad ; \quad P > 1. \tag{A.33}
\]

Now, if we define
\[
I_7(\lambda, x, y, S_1, P) := -\pi j (j \sqrt{P_2^2 - 1})^{S_1 - 1} \left\{ \sqrt{1 - F_2^2} e^{-j\lambda F_+} J_{\nu}(j F_+, \lambda r) + (-1)^{S_1} \sqrt{1 - F_2^2} e^{-j\lambda F_-} J_{\nu}(j F_-, \lambda r) \right\} \tag{A.34}
\]
and
\[
I_8(\lambda, x, y, S_1, P) := \pi j (j \sqrt{P_2^2 - 1})^{S_1 - 1} \{ e^{-j\lambda F_+} r + (-1)^{S_1} e^{-j\lambda F_-} \}, \tag{A.35}
\]
then we can rewrite (A.33) as
\[
I_4(\lambda, x, y, S_1, P) = I_7(\lambda, x, y, S_1, P) + I_8(\lambda, x, y, S_1, P). \tag{A.36}
\]

Writing $I_4(\lambda, x, y, S_1, P)$ in this way simplifies the analysis since, as is shown below, the term $I_8(\lambda, x, y, S_1, P)$ will cancel with other terms when the pieces are recombined to yield $I_5(k_A, \lambda, x, y, S)$. First, it is easy to show that
\[
I_8(\lambda, x + d, y, S_1, P) + I_8(\lambda, x - d, y, S_1, P) - 2 \cos(k_d d) I_8(\lambda, x, y, S_1, P) = 0, \tag{A.37}
\]
when $S_4 = 0$ and $P = \frac{k_A}{\lambda}$. Since $I_8(\lambda, x, y, S_1, P)$ only appears in linear combinations which have the form given in (A.37) (see (4.14), (4.17), (A.13), and (A.36)), we can cancel $I_8(\lambda, x, y, S_1, P)$ from (A.36) and still obtain the
correct result for $I_1(k_A, \lambda, x, y, S)$ when $S_4 = 0$ and $P > 1$. Likewise, it can also be shown that

$$(2 + 4 \cos^2(k_A d))I_6(\lambda, x, y, S_1, P) - 4 \cos(k_A d)$$

$$[I_6(\lambda, x + d, y, S_1, P) + I_6(\lambda, x - d, y, S_1, P)] +$$

$$I_6(\lambda, x + 2d, y, S_1, P) + I_6(\lambda, x - 2d, y, S_1, P)] = 0,$$  \hspace{1cm} (A.38)

when $S_4 = 1$ and $P = \frac{k_A}{\lambda}$. Therefore, we can once again cancel $I_6(\lambda, x, y, S_1, P)$ from the expression for $I_4(\lambda, x, y, S_1, P)$ and still obtain the correct result for $I_4(k_A, \lambda, x, y, S)$ when all of the pieces are added together (see (4.15), (4.17), (A.13), and (A.36)). This enables us to replace $I_4(\lambda, x, y, S_1, P)$, in the expression for $I_3(k_A, \lambda, x, y, S)$ (see (A.13)), by $I_7(\lambda, x, y, S_1, P)$, yielding

$$\hat{I}_3(k_A, \lambda, x, y, S) = D_1[I_7(\lambda, x, y, S_1, 1) + \frac{(-1)^{S_1}}{S_2}I_7(\lambda, x, y, S_1, 1)^*] +$$

$$D_3[I_7(\lambda, x, y, S_1, \frac{k_A}{\lambda}) + \frac{(-1)^{S_1}}{S_2}I_7(\lambda, x, y, S_1, \frac{k_A}{\lambda})^*] +$$

$$D_5[I_5(\lambda, x, y, \frac{k_A}{\lambda}) + |I_5(\lambda, x, y, \frac{k_A}{\lambda})|^*].$$  \hspace{1cm} (A.39)

The hat on $\hat{I}_3(k_A, \lambda, x, y, S)$ serves as a reminder that this expression will give an incorrect value for $I_3(k_A, \lambda, x, y, S)$ (see (4.16)), but it will yield the correct results for the integral of interest, $I_4(k_A, \lambda, x, y, S)$ (see (4.11)), when all of the pieces are added together.

In order to evaluate $\hat{I}_3(k_A, \lambda, x, y, S)$ for $0 \leq \lambda < k_A$ (ie. $P > 1$), we need to find an expression for $I_7(\lambda, x, y, S_1, 1)$ (see (A.39)). When $S_1 = 1$, it is easy to show that

$$I_7(\lambda, x, y, 1, 1) = -2\pi j \sin \theta_0 e^{j\lambda x} J_5(e_j \cos \theta_0, \lambda r),$$  \hspace{1cm} (A.40)

where we have made use of (A.18), (A.31), and (A.34). This case was easy to handle since the singularity in $I_4(\lambda, x, y, 1, 1)$ is a removable singularity (see (A.8)). On the other hand, when $S_1 = 0$, it is more difficult to find $I_7(\lambda, x, y, 0, 1)$ since $I_4(\lambda, x, y, 0, P)$ has a non-removable singularity at $P = 1$. As was previously mentioned, the singularity at $P = 1$ is a removable singularity in the original integral $I_4(k_A, x, y, S)$ (see (4.11)); therefore, it must also be possible to remove this singularity by adding the pieces, which $I_4(k_A, x, y, S)$ was decomposed into, back together.

It turns out that $I_6(\lambda, x, y, 0, 1)$ contains the singular part of $I_4(\lambda, x, y, 0, 1)$. We have previously shown that the term $I_6(\lambda, x, y, S_1, P)$, which is given in
(A.36), will cancel with other terms when all of the pieces are added together to obtain \( \mathcal{I}_1(k_A, \lambda, x, y, S) \); therefore, we will not have to deal with this singular piece. The other piece, \( \mathcal{I}_T(\lambda, x, y, 0, 1) \), only contains a removable singularity. Therefore, we can obtain an expression for \( \mathcal{I}_T(\lambda, x, y, 0, 1) \) by removing this singularity. If we substitute (A.18) and (A.31) into (A.34), then we can use (4.1) in order to rewrite \( \mathcal{I}_T(\lambda, x, y, 0, 1) \) as

\[
\mathcal{I}_T(\lambda, x, y, 0, 1) = \lim_{P \to 1} \frac{2\pi}{\sqrt{P^2 - 1}} \int_0^{\lambda r} e^{-ip \cos \theta_0 (t - \lambda r)}
\{ P \sin \theta_0 \sinh[\sin \theta_0 \sqrt{P^2 - 1}(t - \lambda r)] -
\sin \theta_0 \sqrt{P^2 - 1} \cosh[\sin \theta_0 \sqrt{P^2 - 1}(t - \lambda r)] \} J_0(t) \, dt. \tag{A.41}
\]

Expanding the hyperbolic functions in power series expansions, yields

\[
\mathcal{I}_T(\lambda, x, y, 0, 1) = 2\pi \{ -e^{i\lambda x} I_0(j \cos \theta_0, \lambda r)[\lambda r \sin^2 \theta_0 + j \cos \theta_0] +
\sin^2 \theta_0 \int_0^{\lambda r} e^{-j \cos \theta_0 (t - \lambda r)} t J_0(t) \, dt \}. \tag{A.42}
\]

In order to simplify this equation, we need to find an expression for integrals which have the form

\[
\mathcal{I}_9(a, z) := e^{az} \int_0^z e^{-at} t J_0(t) \, dt. \tag{A.43}
\]

After integrating by parts, with \( u := e^{-at} \) and \( dv := t J_0(t) \, dt \), we find that

\[
\mathcal{I}_9(a, z) = z J_1(z) + a e^{az} J_1(a, z). \tag{A.44}
\]

In [46], it was shown that (see [46], (2.8))

\[
J_\nu(a, z) = \left[ \frac{a^2 + 1}{2n + 1} \right] J_{\nu + 1}(a, z) = \frac{e^{-az}}{2n + 1} [J_n(z) + a J_{n+1}(z)]; \quad n \geq 0. \tag{A.45}
\]

Therefore, (A.44) can be rewritten as

\[
\mathcal{I}_9(a, z) = \frac{1}{a^2 + 1} \{ a e^{az} J_0(a, z) + z J_1(z) - az J_0(z) \}. \tag{A.46}
\]
Now we can find an expression for $I_7(\lambda, x, y, 0, 1)$ by applying (A.46) to (A.42):

$$I_7(\lambda, x, y, 0, 1) = -2\pi \lambda \{ y \sin \theta_0 e^{i\lambda z} J_0(j \cos \theta_0, \lambda r) + jxJ_0(\lambda r) - rJ_1(\lambda r) \}. \quad (A.47)$$

Reference to (A.40) and (A.47) shows that the general form of $I_7(\lambda, x, y, S_1, 1)$ is given by

$$I_7(\lambda, x, y, S_1, 1) = -2\pi j \{ (-j\lambda y)^{1-S_1} \sin \theta_0 e^{i\lambda z} J_0(j \cos \theta_0, \lambda r) + (1 - S_1)\lambda |xJ_0(\lambda r) + jrJ_1(\lambda r)| \}. \quad (A.48)$$

Before we can evaluate $I_3(k_A, \lambda, x, y, S)$ for $\lambda < k_A$, we still need to find an expression for $I_5(\lambda, x, y, P)$ when $P > 1$ (see (A.39)). Using the definitions in (A.8) and (A.9), it is easy to show that

$$I_5(\lambda, x, y, P) = -\frac{dI_4(\lambda, x, y, 0, P)}{dP}. \quad (A.49)$$

Therefore, we can obtain an expression for $I_5(\lambda, x, y, P)$ by differentiating (A.33) with respect to $P$.

Before actually carrying out the differentiation, we will derive some intermediate results. Using the results in (A.18) and (A.31), it can be shown that

$$\frac{dF_\pm}{dP} = \frac{j \sqrt{1 - P_\pm^2}}{\sqrt{P^2 - 1}}; \quad P > 1. \quad (A.50)$$

It can also be shown that (see (4.1))

$$\frac{d}{da} e^{az}J_0(a, z) = e^{az} \left\{ zJ_0(a, z) - \int_0^z e^{-at}tJ_0(t) \, dt \right\}. \quad (A.51)$$

We have previously shown that the second integral on the right-hand-side of this equation can be rewritten in terms of $J_0(a, z)$ (see (A.43) and (A.46)); therefore, (A.51) can be written as

$$\frac{d}{da} e^{az}J_0(a, z) = \left[ z - \frac{a}{(a^2 + 1)} \right] e^{az}J_0(a, z) + \frac{z}{(a^2 + 1)}[aJ_0(z) - J_1(z)]. \quad (A.52)$$
Now, using the results in (A.18), (A.33), (A.49), (A.50), and (A.52), we find that

\[ I_5(\lambda, z, y, P) = \left( \frac{P}{P_2 - 1} \right) I_4(\lambda, z, y, 0, P) + \frac{\pi \lambda r}{(P_2^2 - 1)} (1 - F_+^2) e^{-\lambda F_+ r} \]

\[ J_0(-jF_+, \lambda r) - (1 - F_-^2) e^{-\lambda F_- r} J_0(-jF_-, \lambda r) - 2|J_1(\lambda r) - jP \cos \theta_0| J_0(\lambda r) - \sqrt{1 - F_+^2} e^{-\lambda F_+ r} - \sqrt{1 - F_-^2} e^{-\lambda F_- r} \quad ; \quad P > 1. \]  

(A.53)

As was the case with \( I_4(\lambda, x, y, S_1, P) \), it will be easier to handle \( I_5(\lambda, x, y, P) \) if we break this expression into two pieces. If we let

\[ I_{10}(\lambda, x, y, P) := \frac{\pi \lambda r}{(P_2^2 - 1)} ((1 - F_+^2) e^{-\lambda F_+ r} J_0(-jF_+, \lambda r) + (1 - F_-^2) e^{-\lambda F_- r} J_0(-jF_-, \lambda r) - 2|J_1(\lambda r) - jP \cos \theta_0 J_0(\lambda r)|) \]  

(A.54)

and

\[ I_{11}(\lambda, x, y, P) := -\frac{\pi \lambda r}{(P_2^2 - 1)} \left\{ \sqrt{1 - F_+^2} e^{-\lambda F_+ r} + \sqrt{1 - F_-^2} e^{-\lambda F_- r} \right\}, \]  

(A.55)

then we can use the results in (A.36) and (A.53) to show that

\[ I_5(\lambda, x, y, P) = \left( \frac{P}{P_2^2 - 1} \right) \left[ I_7(\lambda, x, y, 0, P) + I_8(\lambda, x, y, 0, P) \right] - I_{10}(\lambda, x, y, P) + I_{11}(\lambda, x, y, P); \quad P > 1. \]  

(A.56)

Using the same argument as before, we can neglect \( I_8(\lambda, x, y, 0, P) \) when we derive the expression for \( I_6(k_A, \lambda, x, y, S) \). Likewise, since

\[ (2 + 4 \cos^2(k_A d)) I_{11}(\lambda, x, y, P) - 4 \cos(k_A d) I_{11}(\lambda, x + d, y, P) + I_{11}(\lambda, x - d, y, P) \]

\[ I_{11}(\lambda, x + 2d, y, P) + I_{11}(\lambda, x - 2d, y, P) = 0, \]  

(A.57)

when \( S_1 = 1 \) and \( P = \frac{k_A}{4} \), we can also neglect \( I_{11}(\lambda, x, y, P) \) in the expression for \( I_6(k_A, \lambda, z, y, S) \). Therefore, using (A.39) and (A.56), we obtain the expression for \( I_6(k_A, \lambda, x, y, S) \) which is given in Table A.1. Now, if we use (4.14), (4.15), and (4.17), where \( I_6(k_A, \lambda, x, y, S) \) is used in place of \( I_3(k_A, \lambda, x, y, S) \), then we will obtain the correct result for \( I_1(k_A, \lambda, x, y, S) \) for the values of \( S \) in (A.2) and (A.3), when \( 0 \leq \lambda < k_A \). We have also listed the other important expressions which have been derived in this Appendix in Table A.1.
Table A.1: \( \hat{J}_0(k_A, \lambda, x, y, S) \) decomposed into a finite number of ILH1's

\[
\hat{J}_0(k_A, \lambda, x, y, S) = D_1 \{ I_7(\lambda, x, y, S_1, 1) + (-1)^{S_1-S_2} |I_7(\lambda, x, y, S_1, 1)|^* \} + \\
D_9 \{ I_7(\lambda, x, y, S_1, \frac{k_A}{\lambda}) + (-1)^{S_1-S_2} |I_7(\lambda, x, y, S_1, \frac{k_A}{\lambda})|^* \} + \\
D_5 \{ I_{10}(\lambda, x, y, \frac{k_A}{\lambda}) + |I_{10}(\lambda, x, y, \frac{k_A}{\lambda})|^* \}
\]

\[
D_1 = \frac{-(-S_2)^{1-S_2}}{2(k_A^2 S_2^2 - \lambda^2)^{1-S_2}}
\]

\[
D_5 = \frac{S_2}{4k_A^2 (k_A^2 S_2^2 - \lambda^2)}
\]

\[
D_9 = \frac{(\frac{k_A}{\lambda})^{1-S_2}}{2(e^{2S_2} - \lambda^2)} \left[ \frac{3k_A^2 S_2^2 - \lambda^2}{2k_A^2 (k_A^2 S_2^2 - \lambda^2)} \right]^{S_2} + \frac{\lambda S_2}{k_A^2 - \lambda^2} D_5
\]

\[
F_\pm(\theta_0, P) = \left\{ \begin{array}{ll}
-P \cos \theta_0 \pm j \sin \theta_0 \sqrt{P^2 - 1} & ; \ P \geq 1 \\
-P \cos \theta_0 \pm \sin \theta_0 \sqrt{1 - P^2} & ; \ 0 \leq P < 1
\end{array} \right.
\]

\[
\sqrt{1 - F_\pm^2} = \left\{ \begin{array}{ll}
\pm P \sin \theta_0 + j \cos \theta_0 \sqrt{P^2 - 1} & ; \ P \geq 1 \\
\pm P \sin \theta_0 + \cos \theta_0 \sqrt{1 - P^2} & ; \ 0 \leq P < 1
\end{array} \right.
\]

\[
r(x, y) = \sqrt{x^2 + y^2} \quad \theta_0(x, y) = \tan^{-1}(y)
\]

\[
i_7(\lambda, x, y, S_1, 1) = -2\pi j \{ (-j \lambda y)^{1-S_1} \sin \theta_0 e^{j\lambda x} J_0(j \cos \theta_0, \lambda r) + (1 - S_1) \lambda [x J_0(\lambda r) + j r J_1(\lambda r)] \}
\]

\[
i_7(\lambda, x, y, S_1, P) = -\pi j (j \sqrt{P^2 - 1})^{S_1-1} \{ e^{-j\lambda F_+ r} \sqrt{1 - F_+^2} J_0(-j F_+, \lambda r) + (-1)^{S_1} e^{j\lambda F_- r} \sqrt{1 - F_-^2} J_0(-j F_-, \lambda r) \}
\]

\[
i_{10}(\lambda, x, y, P) = \frac{\pi j}{(P^2 - 1)} \{ e^{-j\lambda F_+ r} (1 - F_+^2) J_0(-j F_+, \lambda r) + e^{-j\lambda F_- r} (1 - F_-^2) J_0(-j F_-, \lambda r) - 2 [J_1(\lambda r) - j P \cos \theta_0 J_0(\lambda r)] \}
\]
A.5 \( \hat{I}_3(k_A, \lambda, x, y, S) \) for \( \lambda > k_A \)

The next step is to find an expression for \( \hat{I}_3(k_A, \lambda, x, y, S) \) that holds when \( \lambda > k_A \) (i.e. \( 0 \leq P < 1 \)). As we have already mentioned, the integrand of \( I_3(k_A, \lambda, x, y, S) \) has a non-removable singularity when \( \lambda > k_A \) (see (A.1)). On the other hand, the integral of interest, \( I_3(k_A, \lambda, x, y, S) \) (see (4.11)), only contains removable singularities for this case. Therefore, the contributions from the apparent singularities in \( I_3(k_A, \lambda, x, y, S) \) must cancel out when the desired integral, \( I_1(k_A, \lambda, x, y, S) \), is reconstructed (see (4.14), (4.15), and (4.17)). Since the contributions from the singularities all cancel out in the end, the method which is used to handle the singularities is unimportant, so long as the method is used consistently on each piece in the decomposition.

We found that analytic continuation arguments provided the most efficient method for handling the case when \( \lambda > k_A \). Earlier, we showed how \( I_1(k_A, \lambda, x, y, S) \) can be represented in terms of a finite number of ILHI’s when \( 0 \leq \lambda < k_A \) (see (4.14), (4.15), (4.17), and Table A.1). Now, since \( I_1(k_A, \lambda, x, y, S) \) (see (4.11)) and its expansion in terms of ILHI’s are both analytic functions of \( \lambda \) for the values of \( S \) in (A.2) and (A.3), we can extend the region of validity for the previous results to \( \lambda > k_A \) by letting

\[
j \sqrt{P^2 - 1} = \sqrt{1 - P^2},
\]

(A.58)

where \( P = \frac{k_A}{\lambda} \). It should be pointed out that the complex conjugate operations in the expression for \( \hat{I}_3(k_A, \lambda, x, y, S) \) should be written out explicitly before (A.58) is applied. As it turns out, the expression for \( \hat{I}_3(k_A, \lambda, x, y, S) \), which is given in Table A.1, is still valid when \( \lambda > k_A \).

A.6 Reflection Properties for \( \hat{I}_3(k_A, \lambda, x, y, S) \)

Now that we have an expression for \( \hat{I}_3(k_A, \lambda, x, y, S) \), it is important to look at the reflection properties (i.e., reflections about \( x = 0 \) and \( y = 0 \)) of this expression. Specifically, we would like to determine whether the relationships in (4.18) and (4.21) will still hold when \( I_3(k_A, \lambda, x, y, S) \) is replaced by \( \hat{I}_3(k_A, \lambda, x, y, S) \). First, we will look at \( \hat{I}_3(k_A, \lambda, x, -y, S) \). Referring to Table A.1, we find that

\[
F_{\pm}(-\theta_0, P) = F_{\pm}(\theta_0, P).
\]

(A.59)
Using the above relationship, it is easy to show that

\[ \hat{I}_3(k_A, \lambda, x, -y, S) = (-1)^{S_2} \hat{I}_3(k_A, \lambda, x, y, S). \]  \hspace{1cm} (A.60)

It is more difficult to handle \( \hat{I}_3(k_A, \lambda, -x, y, S) \) since we will need to handle the two cases \( P > 1 \) and \( 0 \leq P < 1 \) separately. First, when \( P > 1 \) we find that

\[
\begin{align*}
    jF_\pm(\pi - \theta_0, P) &= [jF_\pm(\theta_0, P)]^* \\
    \sqrt{1 - F_\pm^2(\pi - \theta_0, P)} &= \left[\sqrt{1 - F_\pm^2(\theta_0, P)}\right]^*.
\end{align*}
\]  \hspace{1cm} (A.61)

Substituting these results into Table A.1, we find that

\[ \hat{I}_3(k_A, \lambda, -x, y, S) = (-1)^{S_2} \hat{I}_3(k_A, \lambda, x, y, S), \]  \hspace{1cm} (A.62)

when \( P > 1 \). On the other hand, when \( 0 \leq P < 1 \),

\[
\begin{align*}
    jF_\pm(\pi - \theta_0, P) &= [jF_\pm(\theta_0, P)]^* \\
    \sqrt{1 - F_\pm^2(\pi - \theta_0, P)} &= -\sqrt{1 - F_\pm^2(\theta_0, P)}.
\end{align*}
\]  \hspace{1cm} (A.63)

Therefore, we can show that (A.62) still holds for this case. The results which are given in (A.60) and (A.62) will be used in Chapter 4.

### A.7 Definition of \( \hat{I}_3(k_A, \lambda, x, y, S) \)

For the purposes of this Appendix, we found that it was advantageous to neglect some of the terms in the expression for \( I_3(k_A, \lambda, x, y, S) \). In doing so, we obtained the expression for \( \hat{I}_3(k_A, \lambda, x, y, S) \) (see (A.33)-(A.38) and (A.53)-(A.57)). The expression for \( \hat{I}_3(k_A, \lambda, x, y, S) \) will be used throughout this report, however, it is not always the most convenient expression to use.

Another expression can be obtained by reintroducing the terms that were neglected in the expression for \( \hat{I}_3(k_A, \lambda, x, y, S) \). These terms can be taken into account by defining the following expression:

\[
\begin{align*}
\hat{I}_3(k_A, \lambda, x, y, S) &= D_1\{I_7(\lambda, x, y, S_1, 1) + (-1)^{S_1-S_2}\{I_7(\lambda, x, y, S_1, 1)\}^*\} + \\
D_2\{I_4(\lambda, x, y, S_1, k_A^x) + (-1)^{S_1-S_2}\{I_4(\lambda, x, y, S_1, k_A^x)\}^*\} + \\
D_5\{I_12(\lambda, x, y, k_A^x) + [I_12(\lambda, x, y, k_A^x)]^*\},
\end{align*}
\]  \hspace{1cm} (A.64)
where an expression for $I_4(\lambda, x, y, S_1, P)$ is given in (A.33), and

\[
I_{12}(\lambda, x, y, P) := \frac{\pi \lambda r}{(P^2 - 1)} \{ \sqrt{1 - F_x^2} e^{-j \lambda F_x r} [\sqrt{1 - F_x^2} J_0(-j F_x, \lambda r) - 1] + \sqrt{1 - F_y^2} e^{-j \lambda F_y r} [\sqrt{1 - F_y^2} J_0(-j F_y, \lambda r) - 1] - 2[J_1(\lambda r) - jP \cos \theta_0 J_0(\lambda r)] \}. \tag{A.65}
\]

Referring to (A.13), (A.33), and (A.53), we find that $\hat{I}_3(k_A, \lambda, x, y, S) = I_3(k_A, \lambda, x, y, S)$ when $P = \frac{k_A}{\lambda} > 1$. However, the above inequality doesn’t hold when $0 < P < 1$. The reason for this is as follows: since $I_3(k_A, \lambda, x, y, S)$ (see (A.1)) isn’t an analytic function for values of $\lambda > k_A$, analytic continuation arguments can only be applied to $I_1(k_A, \lambda, x, y, S)$, which in turn is composed of a number of pieces which have the form of $I_3(k_A, \lambda, x, y, S)$ (see (4.14), (4.15), and (4.17)). Therefore, even though $\hat{I}_3(k_A, \lambda, x, y, S) \neq I_3(k_A, \lambda, x, y, S)$ when $\lambda > k_A$, we can use $\hat{I}_3(k_A, \lambda, x, y, S)$ in place of $I_3(k_A, \lambda, x, y, S)$ and still obtain the correct results for $I_1(k_A, \lambda, x, y, S)$.

### A.8 Reflection Properties for $\hat{I}_3(k_A, \lambda, x, y, S)$

As was the case with $\hat{I}_3(k_A, \lambda, x, y, S)$, it is important to investigate the reflection properties of $\hat{I}_3(k_A, \lambda, x, y, S)$ (see (A.60) and (A.62)). If we apply (A.59) and (A.61) to (A.64), then we find that

\[
\hat{I}_3(k_A, \lambda, x, -y, S) = (-1)^{S_1} \hat{I}_3(k_A, \lambda, x, y, S); \quad \lambda \geq 0 \tag{A.66}
\]

\[
\hat{I}_3(k_A, \lambda, -x, y, S) = (-1)^{S_2} \hat{I}_3(k_A, \lambda, x, y, S); \quad k_A > \lambda > 0. \tag{A.67}
\]

These results are not surprising since we have previously shown that $\hat{I}_3(k_A, \lambda, x, y, S) = I_3(k_A, \lambda, x, y, S)$ when $k_A > \lambda \geq 0$ (see (4.18) and (4.21)). When $\lambda > k_A$, we can use (A.63) to show that

\[
\hat{I}_3(k_A, \lambda, -x, y, S) = (-1)^{S_1} \hat{I}_3(k_A, \lambda, x, y, S) + 2\pi j(\sqrt{1 - P^2})^{S_1 - 1} D_0 \{(-1)^{S_1} e^{j \lambda F_x r} - (-1)^{S_1} e^{-j \lambda F_x r} + e^{j \lambda F_y r} - (-1)^{S_1} e^{-j \lambda F_y r}\} + \frac{2\pi \lambda r}{(P^2 - 1)} D_1 \{\sqrt{1 - F_x^2} [e^{j \lambda F_x r} + e^{-j \lambda F_x r}] + \sqrt{1 - F_y^2} [e^{j \lambda F_y r} + e^{-j \lambda F_y r}]\}. \tag{A.68}
\]
The fact that this result differs from (4.21) should also not be surprising since we have previously shown that there is no reason for $\hat{I}_3(k_A, \lambda, x, y, S)$ to be equal to $I_3(k_A, \lambda, x, y, S)$ when $\lambda > k_A$. 


Appendix B

NUMERICAL COMPUTATION OF \( \hat{I}_3(k_A, \lambda, x, y, S) \)

B.1 Introduction

In Appendix A, we demonstrated how \( \hat{I}_3(k_A, \lambda, x, y, S) \) can be written in terms of ILHI's, Bessel functions, and other elementary functions. Now, in this Appendix, we will show how the ILHI's can be efficiently computed.

As was mentioned earlier, several papers have been written on the computation of the ILHI, \( J_{e_0}(a, z) \) (see [46]–[49]). A very useful Neumann series expansion for \( J_{e_0}(a, z) \) was found by Agrest (see [48]). Also, two new series expansions for \( J_{e_0}(a, z) \), which are classified as factorial-Neumann series expansions, are given in [46].

It is shown in [46] that \( J_{e_0}(a, z) \) can be efficiently computed for all values of \( a \) and \( z \), where \( a \in \mathbb{C} \) and \( z \in \mathbb{R} \), by using either the Neumann series expansion, or one of the two factorial-Neumann series expansions. An algorithm is given in [46] which uses these three expansions to compute \( J_{e_0}(a, z) \) to a user defined number of significant digits (SD). Other expansions that are given in the literature for \( J_{e_0}(a, z) \), were also compared with the three expansions that are used in the algorithm in [46]; however, they were found to offer no significant computational advantages and therefore were not used.

The three expansions that are used in the algorithm in [46] will also
Table B.1: Series expansions for the ILH1, $J_{\ell_0}(a, z)$

<table>
<thead>
<tr>
<th>I. Convergent Factorial-Neumann Series Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_{\ell_0}^N(a, z) = \Gamma\left(\frac{3}{2}\right) z e^{-az} \sum_{k=0}^{N-1} \left[\frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+\frac{1}{2})}\right]^k \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+\frac{1}{2})} J_k(z) \frac{\Gamma(k+\frac{3}{2})}{\Gamma(k+\frac{1}{2})} J_{k+1}(z); \quad z &gt; 0, a \in \mathbb{C}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>II. Asymptotic Factorial-Neumann Series Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_{\ell_0}^N(a, z) \sim \frac{1}{\sqrt{z^2+1}} + \frac{e^{-az}}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}\right)} \sum_{k=0}^{N-1} \left[\frac{2}{z(a^2+1)}\right]^k \Gamma(k+\frac{1}{2}) \Gamma(k+\frac{3}{2}) \Gamma(k+\frac{5}{2}) [J_k(z) - aJ_{k+1}(z)];$</td>
</tr>
<tr>
<td>$z</td>
</tr>
<tr>
<td>where $\Re(\sqrt{a^2 + 1}) &lt; 0; \quad \Re(a) &lt; 0 \land</td>
</tr>
<tr>
<td>$\Re(\sqrt{a^2 + 1}) \geq 0; \quad$ otherwise</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>III. Neumann Series Expansion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$J_{\ell_0}^N(a, z) = \frac{1}{\sqrt{z^2+1}} \left{ 1 + e^{-az} \sum_{k=0}^{N-1} \frac{(-1)^k \epsilon_k J_k(z)}{z(a^2+1)^k} \right}; \quad z &gt; 0, a \in \mathbb{C}, a \neq \pm j,$</td>
</tr>
<tr>
<td>where $\epsilon_k = \begin{cases} 1; &amp; k = 0 \ 2; &amp; k = 1, 2, \ldots \end{cases}$</td>
</tr>
<tr>
<td>and $\Re(\sqrt{a^2 + 1}) \geq 0; \quad \Re(a) \geq 0$</td>
</tr>
<tr>
<td>$\Re(\sqrt{a^2 + 1}) &lt; 0; \quad \Re(a) &lt; 0$</td>
</tr>
</tbody>
</table>

80
be used in this report. These expansions are listed in Table B.1. Along with these expansions, we will also make use of some of the other results in Section 5 of [46].

The three expansions in Table B.1 will behave very differently for different values of the variables $a$ and $z$. In [46], it is shown that the convergent factorial-Neumann series expansion converges most rapidly for small to moderate values of $z|\sqrt{a^2 + 1}|$. On the other hand, the asymptotic factorial-Neumann series expansion can be used when $z|a^2 + 1|$ is large. Finally, the Neumann series expansion fills in the gap left by the other two expansions since it is most useful when $|\sqrt{a^2 + 1} + a| \geq 1$, and $z$ has a small to moderate value.

### B.2 Computation of the Bessel functions

Looking at the three expansions in Table B.1, we see that they all involve series of Bessel functions. Therefore, we will need an efficient method to compute the required sequence of Bessel functions. We will use two different methods to compute the Bessel functions. In both of these methods, we will make use of the following recurrence relation:

$$Z_{n-1}(z) + Z_{n+1}(z) = \frac{2n}{z} Z_n(z). \quad \text{(B.1)}$$

It is a well known fact that $J_n(z)$ is the minimal solution of this recurrence relation, and $Y_n(z)$ is the dominant solution. Therefore, care must be taken when using (B.1) to compute the sequence of Bessel functions $\{J_n(z)\}$.

The first method, which is the most efficient of the two methods and will be used whenever possible, involves the use of (B.1) in the forward direction. Since $J_n(z)$ is the minimal solution of the recurrence relation, forward recurrence will become unstable at some point. For values of $n < z$, $J_n(z)$ and $Y_n(z)$ both behave like damped trigonometric functions of $z$, but when $n > z$, these two functions behave very differently, as is shown below (see [52], (9.3.1)):

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu; \quad \nu \to \infty \quad \text{(B.2)}$$

$$Y_\nu(z) \sim -\sqrt{\frac{2}{\pi\nu}} \left(\frac{ez}{2\nu}\right)^{-\nu}; \quad \nu \to \infty. \quad \text{(B.3)}$$
Using the above equations, it can be shown that forward recurrence will become unstable when \( n \) becomes larger than \( z \) (see [59] and [60]).

Since we can only use forward recurrence to obtain \( \{J_n(z)\} \) for \( n \leq z \), this method will only be useful when \( z \) is large. When this is the case, we can use Hankel's asymptotic expansion to compute the two starting functions, \( J_0(z) \) and \( J_1(z) \), and then we can use (B.1) in the forward direction to compute \( \{J_n(z)\} \) for \( n \leq z \).

When we can't use forward recurrence to compute the sequence of Bessel functions, then we must resort to using a backward recurrence algorithm. We will make use of a backward recurrence algorithm which is based upon a combination of the algorithms due to J. C. P. Miller and F. W. J. Olver (see [61]). We chose this algorithm because it automatically computes the sequence of Bessel functions to a user defined number of significant digits. For more details on the computation of the sequence of Bessel functions using these two methods see Section 5 and Appendix B of [46].

### B.3 Computation of the ILHI's for \( 0 \leq \lambda < k_A \)

Now we can turn our attention to the task of computing the ILHI's which are encountered in the expression for \( \hat{I}_3(k_A, \lambda, z, y, S) \) (see Table A.1). We will first handle the case when \( \lambda < k_A \). As was demonstrated in Section 5 of [46], the values of the parameters, \( a, z, \) and \( S D, \) will determine which expansion to use to compute \( J_{c_0}(a, z). \) For the application given in this report (see Chapter 5), \( r(x, y) \) will have a maximum value of a few wavelengths; therefore, \( z = \lambda r \) will have a small to moderate value when \( \lambda < k_A \). Since \( z \) is relatively small when \( \lambda < k_A \), we will use the backward recurrence algorithm to compute the sequence of Bessel functions.

In order to evaluate \( I_7(\lambda, x, y, S_1, 1) \), we must compute (see Table A.1)

\[
J_{c_0}(j \cos \theta_0, \lambda r).
\]  

(B.4)

For this ILHI, we find that

\[
|a^2 + 1| = \sin^2 \theta_0 \leq 1;
\]  

(B.5)

therefore, the convergent factorial-Neumann series expansion (see Table B.1) is the best expansion to use to compute (B.4). When we are computing the
elements in the impedance matrix, it is possible to save some computation
time because we only need to compute the real part of $I_7(\lambda, x, y, 0, 1)$ (see
(A.2) and Table A.1). Therefore, it can be shown that

$$\mathbb{R}\{I_7(\lambda, x, y, 0, 1)\} = 2\pi \lambda \left\{ \tau J_1(\lambda r) - \right.$$  
$$\left. \frac{\lambda y^2 \Gamma \left( \frac{3}{2} \right)}{2r} \sum_{k=0}^{\infty} \left[ \frac{\lambda y^2}{2r} \right]^k \frac{J_k(\lambda r)}{\Gamma(k + \frac{3}{2})} \right\}; \ S_4 = 1. \quad (B.6)$$

On the other hand, if we are interested in computing the electric field, then
we will have to use

$$I_7(\lambda, z, y, 1, 1) = -2\pi j \lambda y \Gamma \left( \frac{3}{2} \right) \sum_{k=0}^{\infty} \left[ \frac{\lambda y^2}{2r} \right]^k \right.$$  
$$\left[ \frac{J_k(\lambda r) + j \cos \theta_0 J_{k+1}(\lambda r)}{\Gamma(k + \frac{3}{2})} \right]; \ S_4 = 0. \quad (B.7)$$

We only need to compute $I_7(\lambda, x, y, S_1, 1)$ for $S_1 = 1$ since $D_1 = 0$ when
$S = (0, 1, 0, 0)$ (see (A.3) and Table A.1).

The only other ILHII's that need to be computed in order to evaluate
$I_3(k_A, \lambda, x, y, S)$ are

$$\begin{align*}
J_{e_0}(-j F_+, \lambda r) \\
J_{e_0}(-j F_-, \lambda r)
\end{align*} \quad (B.8)$$

where the expression for $F_\pm$ is given in Table A.1. If we refer to Table A.1,
we find that the factor,

$$|a^2 + 1| = |1 - F_\pm^2| = P^2 - \cos^2 \theta_0, \quad (B.9)$$

may be much greater than one when $P = \frac{k_A}{\lambda} > 1$. Therefore, the convergent
factorial-Neumann series expansion (see Table B.1) will not provide the most
efficient method for the computation of the ILHII's in (B.8). On the other
hand, we can show that

$$a + \sqrt{a^2 + 1} = -j F_+ + \sqrt{1 - F_+^2} = \frac{j}{\lambda} e^{\pm j \theta_0} [k_A + \sqrt{k_A^2 - \lambda^2}]. \quad (B.10)$$

Since $|a + \sqrt{a^2 + 1}| \geq 1$ when $\lambda < k_A$, we will use the Neumann series
expansion (see Table B.1) to compute the ILHII's in (B.8).
At this point, it is convenient to define
\[
\mathcal{I}_{13}^\pm(\lambda, x, y, P) := -e^{-\lambda F_x r} \sqrt{1 - F_\mp^2} J_{\mathcal{E}_0}(-jF_\mp, \lambda r) - 1. \tag{B.11}
\]

Now, using (B.10) and the Neumann series expansion which is given in Table B.1, we find that \(\mathcal{I}_{13}^\pm(\lambda, x, y, P)\) can be written as:
\[
\mathcal{I}_{13}^\pm(\lambda, x, y, P) = \sum_{k=0}^{\infty} \frac{(j e^{\pm j\phi})^k e_k J_k(\lambda r)}{[P + \sqrt{P^2 - 1}]^k}. \tag{B.12}
\]

When \(\lambda < k_A\), we find that it is convenient to use the expression for \(\hat{I}_3(k_A, \lambda, x, y, S)\) instead of the expression for \(\hat{I}_3(k_A, \lambda, x, y, S)\). Therefore, using (B.11), we find that \(\hat{I}_3(k_A, \lambda, x, y, S)\) (see (A.64)) can be rewritten as
\[
\hat{\mathcal{I}}_3(k_A, \lambda, x, y, S) = D_1 \{\mathcal{I}_7(\lambda, x, y, S_1, 1) + (-1)^{S_1-S_2}|\mathcal{I}_7(\lambda, x, y, S_1, 1)|^* \} +
D_0 \pi j \left( \frac{j \sqrt{k_A^2 - \lambda^2}}{\lambda} \right)^{S_1-1} \{\mathcal{I}_{13}^+ (\lambda, x, y, \frac{k_A}{\lambda}) + (-1)^{S_2} |\mathcal{I}_{13}^- (\lambda, x, y, \frac{k_A}{\lambda})|^* \} +
(-1)^{S_1} \mathcal{I}_{13}(\lambda, x, y, \frac{k_A}{\lambda}) + (-1)^{S_1-S_2} |\mathcal{I}_{13}(\lambda, x, y, \frac{k_A}{\lambda})|^* \} -
D_0 \frac{2\pi \lambda^3 r}{k_A^2 - \lambda^2} \Re \{\sqrt{1 - F_\mp^2} \mathcal{I}_{13}^+ (\lambda, x, y, \frac{k_A}{\lambda}) + \sqrt{1 - F_\mp^2} \mathcal{I}_{13}^- (\lambda, x, y, \frac{k_A}{\lambda})
+ 2J_1(\lambda r)\}; \quad 0 \leq \lambda < k_A, \tag{B.13}
\]

where (B.6) and (B.7) are used to compute \(\mathcal{I}_7(\lambda, x, y, S_1, P)\), and (B.12) is used to compute \(\mathcal{I}_{13}^\pm(\lambda, x, y, P)\).

As was shown in Appendix A, the expression for \(\hat{I}_3(k_A, \lambda, x, y, S)\) can be used in place of \(\mathcal{I}_3(k_A, \lambda, x, y, S)\) in the expressions for \(\mathcal{I}_1(k_A, \lambda, x, y, S)\) (see (4.14), (4.15), and (4.17)). Also, the reflection equations which are given in (4.18) and (4.21) hold for \(\hat{I}_3(k_A, \lambda, x, y, S)\) (see (A.66) and (A.67)).

### B.4 Computation of the ILHI’s for \(\lambda > k_A\)

Now we must handle the more difficult case when \(\lambda > k_A\). Once again, we need to compute the integrals that are given in (B.4) and (B.8), but this time we need to use the definitions in Table A.1 which are valid when \(0 \leq P < 1\). For the integral in (B.4), the parameter \(|a^2 + 1|\) will still behave as in (B.5).
The expressions which were previously derived (see (B.6) and (B.7)) can still be used when $\frac{\lambda^2}{2r}$ is small, but as the value of $\lambda$ increases, more and more terms will be required in these expansions. Therefore, it is desirable to find a new way to compute (B.4) for large values of $\lambda$. Also, now that $\lambda > k_A$, the parameter $|a^2 + 1|$ will have a very different behavior for the two integrals given in (B.8). In fact, we now find that

$$|a^2 + 1| = |1 - F_1^2| = \cos^2(\theta_0 - \tan^{-1}\left(\frac{P}{\sqrt{1 - P^2}}\right)) \leq 1; \quad \lambda > k_A, \quad (B.14)$$

where we have made use of (4.13). Once again, we will need to use a different method to compute the integrals in (B.8) now that $\lambda > k_A$. This shows that we need a way to determine which of the three expansions in Table B.1 will be best suited for the computation of $J_0(a,z)$ for a given set of parameters, $a$, $z$, and $SD$. In order to accomplish this task, we will make use of some of the results which are given in Section 5 of [46].

For large values of $\lambda$, the argument of the Bessel functions (i.e. $z = \lambda r$), which are present in any of the expansions in Table B.1, may become large. Therefore, it may be possible to use forward recurrence to obtain the desired sequence of Bessel functions. In order to use forward recurrence, we need to first obtain the starting functions $J_0(z)$ and $J_1(z)$. In [46], it was found that Hankel's asymptotic expansion can be used to obtain these starting functions whenever (see [46], (5-4))

$$z > ZJASY = SD + 4. \quad (B.15)$$

It may also be possible to use the asymptotic factorial-Neumann series expansion (see Table B.1) to compute $J_0(a,z)$ when $z = \lambda r$ is large. For the integrals that we are interested in computing (i.e. (B.4) and (B.8)), we have previously shown in (B.5) and (B.14) that $|a^2 + 1| \leq 1$; therefore, we find that (see [46], (5-6))

$$z = \lambda r > k_{\text{max}} := \frac{2|a^2 + 1| - 1}{2}. \quad (B.16)$$

Now, if we use the results in ([46], (5-10) and (5-11)), we find that the asymptotic factorial-Neumann series expansion can be used to compute $J_0(a,z)$ to $SD$ significant digits provided that

$$\frac{1}{2} \times 10^{-SD} |e^{az} J_0(a,z)| > \frac{2}{|a^2 + 1| \sqrt{\pi z}} e^{-\frac{|a^2 + 1|}{2}} \max(1, |a|), \quad (B.17)$$
where the following approximation can be applied:

\[
|e^{az} J_0(a, z)| \sim \frac{e^{az}}{\sqrt{a^2 + 1}} \sqrt{\frac{2}{\pi z}} \frac{|a \cos(z - \frac{\pi}{4}) - \sin(z - \frac{\pi}{4})|}{(a^2 + 1)} \; \text{min}(z, z|a \pm j|) \gg 0. \quad (B.18)
\]

When \( z > ZJASY \), but (B.17) isn't satisfied, we still prefer to use forward recurrence to compute the sequence of Bessel functions; however, this time we would like to use the convergent factorial-Neumann series expansion (see Table B.1) to compute \( J_0(a, z) \). It can be shown that this method can be used if (see [46], (5-13))

\[
\frac{1}{2} \times 10^{-SD} |e^{az} J_0(a, z)| > \sqrt{\frac{z}{\pi}} \frac{e^{\frac{3}{2}}}{2k_{int} + 3} \left[ \frac{ez|a^2 + 1|}{2k_{int} + 3} \right]^{k_{int}} \max(1, |a|),
\]

(B.19)

where

\[
k_{int} := \text{int}(z). \quad (B.20)
\]

When \(|a^2 + 1| \geq 2\), we can use the approximation for \(|e^{az} J_0(a, z)|\) which is given in (B.18). On the other hand, when \(|a^2 + 1| < 2\), we can use (see [46], (5-14))

\[
|e^{az} J_0(a, z)| \approx \sqrt{\frac{2z}{\pi}} |\cos(z - \frac{\pi}{4}) + a \cos(z - \frac{3\pi}{4})| ; \; z|a^2 + 1| < 2. \quad (B.21)
\]

If the inequality in (B.15) is satisfied, and if one out of the two inequalities in (B.17) or (B.19) are satisfied for each one of the integrals in (B.4) and (B.8), then we will use forward recurrence to compute the sequence of Bessel functions. If this is not the case, then a backward recurrence routine will have to be used to compute the Bessel functions.

If neither of the inequalities in (B.17) or (B.19) are satisfied for a given integral (see (B.4) and (B.8)), then we will have to use backward recurrence to compute the Bessel functions, and we will have to find another way to compute that integral. Referring to Table B.1, we find that the convergent factorial-Neumann series expansion will converge faster than the Neumann series expansion when \(|a^2 + 1| \leq 2\). Therefore, we will use the convergent factorial-Neumann series expansion to compute \( J_0(a, z) \) when this is true.

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When $z|a^2 + 1| > 2$ and $|a^2 + 1| \leq 1$, the convergent factorial-Neumann series expansion will still converge faster than the Neumann series expansion, but now we need to worry about round-off errors. In Section 5 of [46], it is shown that in order to use the convergent factorial-Neumann series expansion to compute $J_{e_0}(a, z)$ to $SD$ significant digits when $z|a^2 + 1| > 2$ and $|a^2 + 1| \leq 1$, all operations have to be carried out to $SDN$ significant digits, where (see [46], (5-18))

$$SDN := SD - \log_{10} \left( \frac{|a^2 + 1| \sqrt{\pi z} e^{\pi}[\Re(a) - \frac{|a^2 + 1|}{2}] \max(1, |a|)}{\max(1, |a|)} \right).$$ (B.22)

Therefore, if the computer has at least $SDN$ significant digits of accuracy, then the convergent factorial-Neumann series can be used to compute $J_{e_0}(a, z)$. If the convergent factorial-Neumann series expansion is used for this case, then we will also have to calculate the sequence of Bessel functions (see [46], Appendix B) to $SDN$, instead of $SD$, significant digits.

Finally, if the parameters $a$, $z$, and $SD$ are such that none of the previously mentioned methods can be used, then we will use backward recurrence to compute the sequence of Bessel functions, and the Neumann series expansion (see Table B.1) will be used to calculate $J_{e_0}(a, z)$. An algorithm which is structured as outlined in this Appendix can be used to compute the ILHIs which are encountered in the expression for $\hat{I}_3(k_A, \lambda, z, y, S)$. 

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Appendix C

DETAILS OF THE ASYMPTOTIC EXTRACTION TECHNIQUE

C.1 Introduction

In this Appendix, we will apply the asymptotic extraction technique to the semi-infinite integral in the expression for $Z_{mn}$. The goal is to find an asymptotic expansion of the integrand in (4.10) which holds for large values of the spectral variable $\lambda$, and which can be analytically integrated from some lower limit, $L$, to infinity. If this can be accomplished, then numerical integration will only be required over the range from 0 to $L$, thereby significantly improving the efficiency of the algorithm.

C.2 Asymptotic Expansion of the Integrand

For this part of the analysis, it is beneficial to use the expression for $\hat{I}_3(k_A, \lambda, x, y, S)$ (see (A.64)). As was shown in §A.7, we can use $\hat{I}_3(k_A, \lambda, x, y, S)$ in place of $I_3(k_A, \lambda, x, y, S)$ in (4.19) and still obtain the correct results for the integral of interest, $I_3(k_A, \lambda, x, 0, S)$ (see (4.20)). Before we start the analysis, it is useful to define the following quantities for $S = (0, 0, S_3, 1)$:
\[ I_{14}^{(n)}(\lambda) := 2\lambda \{ f_{1}^{(11)}(\lambda, 0) - f_{2}^{(11)}(\lambda, 0) \} D_{n} |_{\lambda=0} + f_{2}^{(11)}(\lambda, 0) D_{n} |_{\lambda=1}, \]  
\[ (C.1) \]

\[ I_{15}(\lambda, x, y) := I_{14}^{(1)}(\lambda) \Re \{ I_{7}(\lambda, x, y, 0, 1) \} + I_{14}^{(2)}(\lambda) \Re \{ I_{5}(\lambda, x, y, 0, \frac{k_{A}}{\lambda}) \} + I_{14}^{(3)}(\lambda) \Re \{ I_{12}(\lambda, x, y, \frac{k_{A}}{\lambda}) \}, \]  
\[ (C.2) \]

and

\[ I_{16}^{(\pm)}(\lambda, x, y, v) := I_{15}(\lambda, x, y + v) \pm I_{15}(\lambda, x, y - v). \]  
\[ (C.3) \]

Now, the contribution to \( Z_{mm} \), which is due to large values of the spectral variable \( \lambda \), can be separated out from (4.10) by defining a new integral (see (4.10), (4.19), (4.20), (A.64), and (C.1)–(C.3))

\[ Z_{mm}(L) := -\frac{1}{8} \left[ \frac{k_{A}}{\sqrt{\pi}} \sin(k_{A}d) \right]^{2} \int_{L}^{\infty} \{ |2 + 4 \cos^{2}(k_{A}d)| I_{16}(\lambda, d(m - n), v, v) - 4 \cos(k_{A}d) I_{16}(\lambda, d(m - n + 1), v, v) + I_{16}(\lambda, d(m - n - 1), v, v) \} + I_{16}(\lambda, d(m - n + 2), v, v) + I_{16}(\lambda, d(m - n - 2), v, v) \} d\lambda, \]  
\[ (C.4) \]

where \( L \) is chosen large enough so that the asymptotic expansions, which will be derived in this Appendix, will provide the desired accuracy for the integrand.

We will start the analysis by looking at the asymptotic behavior of \( \tau_{pq} \). Referring to (2.23) and (2.27), we find that

\[ \tau_{pq} = -j\lambda \sqrt{1 - \frac{k_{pq}^{2}}{\lambda^{2}}} \sim -j\lambda \left[ 1 - \frac{k_{pq}^{2}}{2\lambda^{2}} - \frac{k_{pq}^{4}}{8\lambda^{4}} + O(\lambda^{-6}) \right]. \]  
\[ (C.5) \]

In order to find the behavior of the reflection coefficients for large values of \( \lambda \), we can substitute (C.5) into (2.34) and (2.37), yielding

\[ |\Gamma_{ip}^{(pq)}| \propto |\Gamma_{ij}^{(pq)}| \propto |e^{-2p\lambda z_{ip}}|, \quad \lambda \gg |k_{pq}|. \]  
\[ (C.6) \]
Now, if we assume that only non-magnetic materials are present in the problem (i.e. \( \mu \rho = \mu_0 \)), then we can use (2.46), (4.4), and (C.6) to show that the asymptotic expansions,

\[
\begin{align*}
 f_1^{(11)}(\lambda, 0) &\sim \frac{\omega \mu_0}{\lambda^2 (\gamma_{11} + \tau_{-11})} \\
 f_2^{(11)}(\lambda, 0) &\sim \frac{\omega \mu_0 (\rho_{11} \tau_{-11})}{\lambda^2 (\gamma_{11} k_{-11}^2 + \tau_{-11} k_{-11}^2)}
\end{align*}
\]

hold when \( \lambda \) is large enough that

\[
|e^{-2\rho \lambda^2 \tau_{-11}}| < \frac{1}{2} \times 10^{-SD}.
\]  

Furthermore, substituting (C.5) into (C.7) enables us to write

\[
\begin{align*}
 f_1^{(11)}(\lambda, 0) &\sim \frac{j \omega \mu_0}{2 \lambda^4} \left\{ 1 + \frac{(k_{11}^4 + \lambda^2)}{4 \lambda^2} + \frac{(k_{11}^4 + k_{-11}^4) k_{-11}^2}{8 \lambda^4} \right\} \\
 f_2^{(11)}(\lambda, 0) &\sim \frac{-j \omega \mu_0}{2 \lambda^4 (k_{11}^2 + k_{-11}^2)} \left\{ 1 - \frac{(k_{11}^4 + k_{-11}^4)}{2 \lambda^4 (k_{11}^2 + k_{-11}^2)} - \frac{(k_{11}^4 + k_{-11}^4) k_{-11}^2}{8 \lambda^4} + \frac{k_{11}^4 k_{-11}^2}{\lambda^4 (k_{11}^2 + k_{-11}^2)^2} \right\}
\end{align*}
\]

In order to obtain an asymptotic expansion for \( I_{14}^{(n)}(\lambda) \), we need to find asymptotic expansions for the coefficients, \( D_n \) (see (C.1)), for \( S = (0, 0, S_1, 1) \). Using the results in Table A.1, it is easy to obtain asymptotic expansions for the coefficients (see Table C.1). Now, combining the results in (C.1), (C.9), and Table C.1, we find that

\[
I_{14}^{(1)}(\lambda) \sim \frac{-j \omega \mu_0}{\lambda^4 (k_{11}^2 + k_{-11}^2)} \left\{ c_0^{(1)} + \frac{c_{1}^{(1)}}{\lambda^2} + \frac{c_{2}^{(1)}}{\lambda^4} \right\},
\]

\[
I_{14}^{(5)}(\lambda) \sim \frac{j \omega \mu_0}{2 \lambda^4 k_{A}^2 (k_{11}^2 + k_{-11}^2)} \left\{ c_0^{(5)} + \frac{c_{1}^{(5)}}{\lambda^2} + \frac{c_{2}^{(5)}}{\lambda^4} \right\},
\]

\[
I_{14}^{(9)}(\lambda) \sim \frac{j \omega \mu_0}{2 \lambda^4 k_{A}^2 (k_{11}^2 + k_{-11}^2)} \left\{ c_0^{(9)} + \frac{c_{1}^{(9)}}{\lambda^2} + \frac{c_{2}^{(9)}}{\lambda^4} \right\},
\]
where

\[
\begin{align*}
  c_0^{(1)} &:= 1 \\
  c_1^{(1)} &:= 2k_A^2 - \frac{(k_A^4 + k_{-11}^4)}{2(k_{11}^2 + k_{11}^2)} \\
  c_2^{(1)} &:= k_A^2 \left[ 3k_A^2 - \frac{(k_{11}^4 + k_{11}^4)}{(k_{11}^2 + k_{-11}^2)} \right] - \frac{(k_{11}^4 + k_{11}^4 + k_{-11}^4)}{8} + \frac{k_{11}^2 k_{-11}^4}{(k_{11}^2 + k_{-11}^2)^2} \\
  c_0^{(5)} &:= \frac{(k_{-11}^2 + k_{-11}^2)}{2} - k_A^2 \\
  c_1^{(5)} &:= \frac{(k_{-11}^2 + k_{-11}^2)^2}{8} + \frac{k_A^2 (k_{11}^4 + k_{11}^4)}{2(k_{11}^2 + k_{-11}^2)} - k_A^4 \\
  c_2^{(5)} &:= \frac{(k_{11}^4 + k_{11}^4 + k_{11}^4)}{8} - \frac{2k_A^2}{(k_{11}^2 + k_{-11}^2)^2} + \frac{k_A^2 k_{11}^4 k_{-11}^4}{(k_{11}^2 + k_{-11}^2)^2} \\
  c_0^{(9)} &:= \frac{(k_{11}^2 + k_{11}^2)}{2} - k_A^2 \\
  c_1^{(9)} &:= \frac{(k_{11}^2 + k_{11}^2)^2}{8} - \frac{k_A^2 (k_{11}^4 + k_{11}^4 + k_{11}^4)}{2(k_{11}^2 + k_{11}^2)} + 4k_A^4 \\
  c_2^{(9)} &:= \frac{(k_{11}^4 + k_{11}^4 + k_{11}^4)}{8} - \frac{2k_A^2}{(k_{11}^2 + k_{-11}^2)^2} + k_A^4 \left[ \frac{k_{11}^4}{(k_{11}^2 + k_{-11}^2)^2} - \frac{(k_{11}^2 + k_{11}^2)^2}{8} \right] \frac{(k_{11}^2 + k_{-11}^2)^2}{(k_{11}^2 + k_{11}^2)^2} + 8k_A^6 \\
\end{align*}
\]

\[
, \quad (C.13)
\]

\[
, \quad (C.14)
\]

\[
, \quad (C.15)
\]

The next step is to obtain an asymptotic expansion for \( \mathcal{I}_{15}(\lambda, x, y) \), which when integrated from \( L \) to \( \infty \), as is required in (C.4), can be represented in terms of special functions. This will involve finding expansions for \( \mathcal{I}_7(\lambda, x, y, 0, 1) \), \( \mathcal{I}_7(\lambda, x, y, 0, 1) \), \( \mathcal{I}_7(\lambda, x, y, 0, 1) \), and \( \mathcal{I}_7(\lambda, x, y, 0, 1) \).

We will first concentrate on finding an expansion for \( \mathcal{I}_7(\lambda, x, y, 0, 1) \). If we substitute the Neumann series expansion, which is given in Table B.1, into the expression for \( \mathcal{I}_7(\lambda, x, y, 0, 1) \), which is given in Table A.1, then we obtain the following convergent series expansion:

\[
\Re\{\mathcal{I}_7(\lambda, x, y, 0, 1)\} = 2\pi \lambda \left\{ r J_1(\lambda r) - y \Re\{e^{i\lambda x} - \sum_{k=0}^{\infty} (j e^{i\theta_0}) e_k J_k(\lambda r) \} \right\}. \quad (C.16)
\]
Table C.1: Asymptotic Expansions of $D_n$ for $S = (0, 0, S_3, 1)$

<table>
<thead>
<tr>
<th></th>
<th>$S_3 = (0, 0, 0, 1)$</th>
<th>$S_3 = (0, 0, 1, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$ $\sim$</td>
<td>0</td>
<td>$\frac{1+2x^4+3x^8}{2\lambda^2}$</td>
</tr>
<tr>
<td>$D_5$ $\sim$</td>
<td>$\frac{1}{4\lambda^2x^2}$</td>
<td>$\frac{1+x^8+x^8}{4x^2\lambda^2}$</td>
</tr>
<tr>
<td>$D_9$ $\sim$</td>
<td>$\frac{1-x^8-x^8-x^8}{4x^2\lambda^2}$</td>
<td>$\frac{1-2x^2-x^8-x^8}{4x^2\lambda^2}$</td>
</tr>
</tbody>
</table>

We can use ([62], (8.511.4)) to rewrite the above equation as

$$\Re\{I_7(\lambda, x, y, 0, 1)\} = 2\pi \lambda \{r J_1(\lambda r) - \sum_{k=0}^{\infty} (-1)^k \sin[(2k + 1)\theta_0] J_{2k+1}(\lambda r)\}.$$  \hspace{1cm} (C.17)

Next, we will obtain expansions for $I_6(\lambda, x, y, 0, \frac{k_4}{\lambda})$ and $I_{12}(\lambda, x, y, \frac{k_4}{\lambda})$. Since we are looking for an expansion which holds for large values of $\lambda$, we will make use of the asymptotic factorial-Neumann series expansion in Table B.1. However, before we proceed with finding the asymptotic expansions, we will derive some useful intermediate results.

Since we will be using the asymptotic factorial-Neumann series expansion, we will encounter terms which have the following form (see (A.33), (A.65), and Table B.1):

$$(\sqrt{1-P^2})^{-c} \left(\sqrt{1-P_+^2}\right)^{-b} = \left(\frac{\sqrt{1-P^2}}{\cos^b \theta_0}\right)^{-b+c} \left[1 \pm \frac{P \tan \theta_0}{\sqrt{1-P^2}}\right]^{-b}.$$  \hspace{1cm} (C.18)

When $P = \frac{k_4}{\lambda} \ll 1$ and $P \tan \theta_0 \ll 1$, we can use ([52], (3.6.9) and (6.1.22))
to obtain the following series expansion:

\[
(\sqrt{1 - P^2})^{-c} (\sqrt{1 - F_\pm^2})^{-b} = \frac{\cos^{-b} \theta_0}{\Gamma(b)} \sum_{m=0}^{\infty} \left(\mp P \tan \theta_0\right)^m \frac{\Gamma(m + b)}{\Gamma(m + 1)} \sum_{n=0}^{\infty} \frac{2^n \Gamma(n + \frac{m+b+1}{2})}{\Gamma(n+1)}; \quad c \geq 0, \quad b \geq 0.
\] (C.19)

Next, we can use the asymptotic factorial-Neumann series expansion along with (C.19) to show that:

\[
\Re\left\{ -\frac{j \pi e^{-j\lambda F_\pm r}}{\sqrt{1 - \frac{(\lambda^2 \pm)^2}{\lambda^2}}} \left[\sqrt{1 - \frac{2r}{F_\pm}} J_0(-j F_\pm, \lambda r) - 1\right] \right\} \sim -\frac{\pi}{\sqrt{\frac{1}{2}}} \sum_{k=0}^{\infty} \frac{2^k \gamma^k}{\lambda^{2k}}
\frac{\Gamma(k + \frac{1}{2})}{\Gamma(2k + 1)} J_k(\lambda r) \sum_{m=0}^{\infty} \left(\frac{y k_A}{x \lambda}\right)^m \frac{\Gamma(m + 2k + 1)}{\Gamma(m + 1)} \sum_{n=0}^{\infty} \frac{(\frac{k_A}{\lambda})^{2n}}{\Gamma(n+1)} \left\{ \frac{k_A}{\lambda} \frac{\Gamma(n + k + 1 + \frac{m}{2})}{\Gamma(\frac{m+1}{2} + k)} + \frac{y \Gamma(n + k + \frac{m+1}{2})}{x \Gamma(\frac{m+1}{2} + k)} \right\}
\] (C.20)

\[
\Re\left\{ \frac{j \pi \lambda r}{\left(\frac{k_A}{\lambda}\right)^2 - 1} \left[\sqrt{1 - \frac{2r}{F_\pm}} e^{-j\lambda F_\pm r} \left[\sqrt{1 - \frac{2r}{F_\pm}} J_0(-j F_\pm, \lambda r) - 1\right] - J_1(\lambda r)\right] \right\} \sim
\frac{\pi \lambda r}{\sqrt{\frac{1}{2}}} \sum_{k=1}^{\infty} \frac{2^k \gamma^k}{\lambda^{2k}} \frac{\Gamma(k + \frac{1}{2})}{\Gamma(2k)} J_{k-1}(\lambda r) \sum_{m=0}^{\infty} \frac{(\frac{y k_A}{x \lambda})^m}{\Gamma(m + 2k + 1)} \frac{\Gamma(m + 2k)}{\Gamma(m + 1)}
\sum_{n=0}^{\infty} \left(\frac{k_A}{\lambda}\right)^{2n} \frac{\Gamma(n + k + 1 + \frac{m}{2})}{\Gamma(n+1)} \right. \left. \frac{\Gamma(\frac{m+1}{2} + k)}{\Gamma(\frac{m+1}{2} + k)} \right\}
\] (C.21)

Using the above two equations, we can now obtain asymptotic expansions for \(I_4(\lambda, x, y, 0, k_A)\) and \(I_{12}(\lambda, x, y, k_A)\). If we substitute (C.20) and (C.21) into (A.33) and (A.65), respectively, and then apply ([52], (6.1.18)), we find that

\[
\Re\{I_4(\lambda, x, y, 0, k_A)\} \sim -2\pi \sum_{k=0}^{\infty} \left[\frac{2r}{k_A^2 z^2}\right]^k \sum_{m=0}^{\infty} \frac{(2y)^m}{x^m} \sum_{n=0}^{\infty} \left[\frac{y}{z}\right]^n \left(\frac{2k}{2m+1} + 1\right) \frac{\Gamma(m + k + \frac{1}{2}) \Gamma(n + m + k + 1)}{\Gamma(\frac{1}{2}) \Gamma(2m+1) \Gamma(k+1) \Gamma(n+1)} J_k(\lambda r) \left(\frac{k_A}{\lambda}\right)^{k+2m+2n+1}
\] (C.22)
\[
\Re \left\{ \frac{J_{12}(\lambda, z, y, \frac{\lambda k_A}{\lambda})}{\lambda} \right\} \sim -2\pi r \sum_{k=1}^{\infty} \left( \frac{2r}{k_A z^2} \right)^k \sum_{m=0}^{\infty} \left( \frac{2y}{x} \right)^m \sum_{n=0}^{\infty} \left( \frac{k}{m + k} \right) \frac{\Gamma(m + k + \frac{1}{2}) \Gamma(n + m + k + 1)}{\Gamma(n + 1) \Gamma(k + 1) \Gamma(2m + 1)} \frac{J_{k+1}(\lambda r)}{\lambda^{k+2m+2n}}. \quad (C.23)
\]

Referring to (C.22) and (C.23), we find that these asymptotic expansions will only be useful when \( \lambda \gg k_A, \lambda z \gg 2yk_A, \) and \( \lambda z^2 \gg 2r \). Actually, we can use (B.17) to show that the asymptotic factorial-Neumann series expansion, which was used to obtain (C.22) and (C.23), will provide the desired accuracy if

\[
\frac{1}{2} \times 10^{-SD} \left| e^{az} \left[ J_0(a, z) - \frac{1}{\sqrt{a^2 + 1}} \right] \right| > \frac{2}{|a^2 + 1| \sqrt{\pi z}} e^{-\frac{a^2+1}{2}} \max(1, |a|). \quad (C.24)
\]

If we modify (B.18), then we find that

\[
\left| e^{az} \left[ J_0(a, z) - \frac{1}{\sqrt{a^2 + 1}} \right] \right| \sim \sqrt{\frac{2}{\pi z}} \frac{\max(1, |a|)}{|a^2 + 1|}; \min(z, z|a \pm j|) \gg 0. \quad (C.25)
\]

Therefore, if we combine (C.24) and (C.25), we find that (C.22) and (C.23) will provide the desired accuracy provided the following inequality holds:

\[
\lambda r |1 - F_+^2| > 2 \ln[2 \sqrt{2} \times 10^{SD}]. \quad (C.26)
\]

The above inequality shows that we need to find other expansions for \( I_4(\lambda, z, y, 0, \frac{\lambda k_A}{\lambda}) \) and \( I_{12}(\lambda, z, y, \frac{\lambda k_A}{\lambda}) \) which will work when \( z \) is small (see Table A.1).

First, we will look at the special case \( z = 0 \). For this case, we can use the convergent factorial-Neumann series expansion, from Table B.1, to show that:

\[
\Re \left\{ -j\pi \lambda e^{-jF_+y} \sqrt{\lambda^2 - k_A^2} J_0(-jF_+, \lambda y) - 1 \right\} = \\
\pi \left\{ \pm \frac{\sin(y \sqrt{\lambda^2 - k_A^2})}{\sqrt{\lambda^2 - k_A^2}} - \frac{3}{2} \frac{k_A}{k_A^2} \sum_{k=0}^{\infty} \frac{y k_A^2}{2\lambda^2} \frac{J_{k+1}(\lambda y)}{\Gamma(k + \frac{3}{2})} \right\}, \quad (C.27)
\]

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\[ \mathbb{R} \left\{ \frac{\pi \lambda y}{(\frac{k_A}{\lambda})^2 - 1} \sqrt{1 - F_{\pm}^2 e^{-j\lambda F_{\pm} y}} \right\} = \]
\[ \frac{\pi k_A y}{(\frac{k_A}{\lambda})^2 - 1} \left\{ \pm \cos(y\sqrt{\lambda^2 - k_A^2}) + \Gamma\left(\frac{3}{2}\right) k_A y \sum_{k=0}^{\infty} \left[ \frac{y k_A^2}{2\lambda} \right]^k \frac{J_k(\lambda y)}{\Gamma(k + \frac{3}{2})} \right\}. \] (C.28)

Now we can use the results in (A.33), (A.65), (C.27), and (C.28) to show that

\[ \mathbb{R}\{I_4(\lambda, 0, y, 0, k_A, \lambda)\} = -2\pi \Gamma\left(\frac{3}{2}\right) k_A y \sum_{k=0}^{\infty} \left[ \frac{y k_A^2}{2\lambda} \right]^k \frac{J_{k+1}(\lambda y)}{\Gamma(k + \frac{3}{2})} \] (C.29)

\[ \mathbb{R}\left\{ \frac{I_{12}(\lambda, 0, y, k_A, \lambda)}{\lambda} \right\} = \frac{2\pi y}{1 - (\frac{k_A}{\lambda})^2} \left\{ J_1(\lambda y) - \right. \]
\[ \left. 2\Gamma\left(\frac{3}{2}\right) \sum_{k=0}^{\infty} \left[ \frac{y k_A^2}{2\lambda} \right]^k \frac{J_{k+1}(\lambda y)}{\Gamma(k + \frac{3}{2})} \right\}. \] (C.30)

Also, when \( \lambda \gg k_A \), we can use ([52], (3.6.9)) to rewrite (C.30) as:

\[ \mathbb{R}\left\{ \frac{I_{12}(\lambda, 0, y, k_A, \lambda)}{\lambda} \right\} \sim 2\pi y \left[ 1 + \frac{k_A^2}{\lambda^2} + \frac{k_A^4}{\lambda^4} \right] \left\{ J_1(\lambda y) - \right. \]
\[ \left. 2\Gamma\left(\frac{3}{2}\right) \sum_{k=0}^{\infty} \left[ \frac{y k_A^2}{2\lambda} \right]^k \frac{J_{k+1}(\lambda y)}{\Gamma(k + \frac{3}{2})} \right\}. \] (C.31)

Next, we will develop an expansion which can be used when the inequality in (C.26) doesn't hold and \( x \neq 0 \). This time we will use the Neumann series expansion in Table B.1. Once again it is beneficial to derive some intermediate results. First, the terms that we will encounter in the Neumann series expansion can be rewritten in the following form by using ([52], (3.6.8) and (6.1.21)):

\[ \left( \sqrt{1 - P^2} \right)^c \left( 1 - \frac{jP}{\sqrt{1 - P^2}} \right)^b = \Gamma(b + 1) \sum_{m=0}^{\infty} \frac{(-j)^m \Gamma\left(\frac{c-m}{2} + 1\right)}{\Gamma(m+1) \Gamma(b-m+1)} \]
\[ \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{k_A}{\lambda}\right)^{2n+m}}{\Gamma(n+1) \Gamma\left(\frac{c-m}{2} - n + 1\right)} ; \quad c \geq b \geq 0. \] (C.32)
Now we can use the Neumann series expansion (see Table B.1) along with (C.32) to show that

\[
\Re\{ -je^{-j\lambda F_\pm^r} \sqrt{1 - F_\pm^r} \, J_0(-jF_\pm, \lambda r) - 1\} = -\Re\left\{ \sum_{k=0}^{\infty} \epsilon_k (-1)^k e^{j\lambda \theta_0} J_k(\lambda r) \right\}
\]

\[
\frac{\Gamma(k + 1)}{\Gamma(m + 1)\Gamma(n + 1)} \sum_{m=0}^{\infty} \frac{(-j)^m \Gamma(k - m + 1)}{\Gamma(k - m + 1)\Gamma(k - m + 1)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma^{2n+m}}{\Gamma(n + 1)} \frac{\Gamma\left(\frac{k-m+3}{2} - n\right)}{\Gamma\left(\frac{k-m+3}{2} - n\right)} \right\}.
\]

Finally, if we substitute the above equations into (A.33) and (A.65), then we obtain the following convergent series expansions:

\[
\Re\{ I_4(\lambda, x, y, 0, \frac{k_A}{\lambda}) \} = \frac{\pi}{\sqrt{1 - \left(\frac{k_A}{\lambda}\right)^2}} \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{k+m+n}}{\Gamma(2m+2)\Gamma(k-2m+1)\Gamma(n+1)\Gamma(k+3/2 - m - n)} \cos k\theta_0(k - 2m)(k + 1 - 2(m + n)) \left(\frac{k_A}{\lambda}\right)^{2m+2n+1} J_k(\lambda r)
\]

\[
\Re\{ I_{12}(\lambda, x, y, \frac{k_A}{\lambda}) \} = \frac{2\pi}{1 - \left(\frac{k_A}{\lambda}\right)^2} \left\{ rJ_4(\lambda r) + \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{k+m+n}\epsilon_k \Gamma(k+1)\Gamma(3/2 - m)}{\Gamma(2m+2)\Gamma(k-2m+1)\Gamma(n+1)\Gamma(3/2 - m - n)} \right\}
\]

\[
\left[ \left(\frac{k_A}{\lambda}\right)^2 y \sin k\theta_0(k - 2m) \left(\frac{k+1}{2} - m - n\right) + x \cos k\theta_0 \left(\frac{k+1}{2} - m\right)(2m+1) \left(\frac{k_A}{\lambda}\right)^{2m+2n} J_k(\lambda r) \right].
\]
Furthermore, the desired expansions can be obtained by applying ([52], (3.6.9)) to the above equations:

\[
\Re\{I_2(\lambda, x, y, kA/\lambda)\} = \pi \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \cos k\theta_0 (k - 2m)(k + 1 - 2(m + n)) \\
\frac{(-1)^{k+m+n}2\Gamma(k+1)\Gamma(k+1/2 - m)}{(k-2m+1)\Gamma(n+1)\Gamma(k+3/2 - m - n)} \\
\left[ 1 + \frac{k_A^2}{2\lambda^2} + \frac{3k_A^4}{8\lambda^4} \right] \left( \frac{k_A}{\lambda} \right)^{2m+2n+1} J_k(\lambda r) \tag{C.37}
\]

\[
\Re \left\{ \frac{I_2(\lambda, x, y, kA/\lambda)}{\lambda} \right\} = 2\pi \left\{ r J_1(\lambda r) + \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left[ \frac{k_A}{\lambda} \right]^2 y \sin k\theta_0 \right. \]

\[
(k - 2m) \left( \frac{k + 1}{2} - m - n \right) + x \cos k\theta_0 \left( \frac{k + 1}{2} - m \right) (2m + 1) \\
\left. \frac{(-1)^{k+m+n}e_k\Gamma(k+1)\Gamma(k+1/2 - m)}{(k-2m+1)\Gamma(n+1)\Gamma(k+3/2 - m - n)} \right[ 1 + \frac{k_A^2}{2\lambda^2} + \frac{k_A^4}{8\lambda^4} \right] \left( \frac{k_A}{\lambda} \right)^{2m+2n} J_k(\lambda r) \right\}. \tag{C.38}
\]

### C.3 Computation of \( j_{m,n}(L, r) \)

Referring to (C.2), (C.3), (C.10)–(C.12), (C.17), (C.22), (C.23), (C.29), (C.31), (C.37), and (C.38), we find that in order to evaluate \( Z_{mn}^{\alpha} \), which is defined in (C.4), we need to evaluate integrals that have the general form

\[
j_{m,n}(L, r) := \int_{L}^{\infty} J_n(\lambda r) \frac{d\lambda}{\lambda^m}. \tag{C.39}
\]

If we carry out all of the expansions in \( \lambda \) to the same order as in (C.5), then we will need to compute \( j_{m,n}(L, r) \) for \( m = 3, 4, \ldots, 10 \). Therefore, all that we need to do is find an efficient way to compute integrals which are of the form given above. Since we will need to compute a sequence of these integrals, it is beneficial to use a recurrence algorithm. We have found that the recurrence relation,

\[
j_{m,n}(L, r) - \frac{(n + 1 + m)}{(n + 1 - m)} j_{m+2,n}(L, r) = - \frac{2(n + 1)}{(n + 1 - m)} \frac{J_{n+1}(Lr)}{rL^m}, \tag{C.40}
\]

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is well suited for this purpose. The above recurrence relation can be obtained by rearranging ([52], (11.3.6)). Examining (C.40), we find that this recurrence relation decouples when \( n + 1 - m = 0 \). At this point, the recurrence relation can be rewritten as

\[
j_{m,m+1}(L,r) = \frac{J_m(Lr)}{r L^m}.
\]

Before we use (C.40), we must determine which direction the recurrence is stable in. We will make use of the results in Appendix A of [46] for the stability analysis. First, when \( n \) is an even integer, (C.40) takes on the general form

\[
j_{m,2n}(L,r) + d^{(1)}_{m,2n} j_{m,2(n+1)}(L,r) = f^{(1)}_{m,2n}; \quad n = 0, 1, \ldots,
\]

where

\[
\begin{aligned}
d^{(1)}_{m,2n} &:= \frac{(2n+1+m)}{(2n+1-m)} \\
f^{(1)}_{m,2n} &:= -\frac{2(2n+1)}{(2n+1-m)} J_{2(n+1)}(Lr)
\end{aligned}
\]

The recurrence relation (C.42) behaves differently for even and odd values of \( m \). When \( m \) is odd, the recurrence relation decouples at the point \( 2n + 1 - m = 0 \). On the other hand, when \( m \) is even the recurrence relation doesn't decouple. Since (C.42) behaves differently for even and odd values of \( m \), we will have to handle the stability analysis for these two cases separately.

We will first handle the case for even values of \( m \). For this case, we can use ([46], (A-4)) and ([52], (6.1.22)) to show that the homogeneous solution for (C.42) is given by

\[
j^{(1h)}_{m,2n} = \prod_{k=0}^{n-1} \left[ -d^{(1)}_{m,2k} \right]^{-1} = (-1)^n \frac{\Gamma(m+1)}{\Gamma\left(\frac{m+1}{2} - n\right)\Gamma\left(\frac{m+1}{2} + n\right)};
\]

\( m \) even, or \( m \) odd and \( 0 \leq 2n \leq m - 3 \).

For odd values of \( m \), it is slightly more difficult to find the homogeneous solution for (C.42). When \( 0 \leq 2n \leq m - 3 \), we can once again directly apply ([46], (A-4)) and ([52], (6.1.22)) since the forward recurrence starts at \( m = 0 \). Doing this, we find that (C.44) still holds for this case. On the other hand, when \( m \) is odd and \( 2n \geq m + 1 \), we will have to modify ([46], (A-4)) since
the forward recurrence will start at $2n = m + 1$ (see (C.41)). This time we find that (see [46], Appendix A)

$$j_{m,2n}^{(1h)} = \prod_{k=m+1}^{n-1} [-d_{m,2k}^{(1)}]^{-1} = \frac{\Gamma(n + \frac{1-m}{2})\Gamma(m + 1)}{\Gamma(n + \frac{m+1}{2})};$$

$m$ odd, $2n \geq m + 1$. \hfill (C.45)

In order to determine the stability of (C.42), we make use of the index of stability (see [46], (A-7) and (A-8)),

$$\alpha^{(1)}(2k, 2n) := \left| \frac{j_{m,2k}(L, r) j_{m,2n}^{(1h)}}{j_{m,2n}(L, r) j_{m,2k}^{(1h)}} \right| = \frac{\rho_{m,2n}^{(1)}}{\rho_{m,2k}^{(1)}},$$

where

$$\rho_{m,2n}^{(1)} := \left| \frac{j_{m,0}(L, r) j_{m,2n}^{(1h)}}{j_{m,2n}(L, r)} \right|. \hfill (C.47)$$

Before we can use (C.46), we need to obtain an approximation for $j_{m,n}(L, r)$. When $n \ll Lr$, we can use the first term in Hankel's asymptotic expansion (see [52], (9.2.1)) to show that

$$j_{m,n}(L, r) \sim \sqrt{\frac{2}{\pi r}} \int_{L}^{\infty} \frac{\cos(r \lambda - \frac{n\pi}{2} - \frac{\pi}{4})}{\lambda^{m+\frac{1}{2}}} d\lambda$$

$$\sim -\sqrt{\frac{2}{\pi r}} \frac{\sin(Lr - \frac{n\pi}{2} - \frac{\pi}{4})}{rL^{m+\frac{1}{2}}}; \quad n \ll Lr. \hfill (C.48)$$

On the other hand, when $n \gg Lr$ and $n + 1 > m > \frac{1}{2}$, we can split the integral into two pieces:

$$j_{m,n}(L, r) = \int_{0}^{\infty} \frac{J_{n}(r \lambda)}{\lambda^{m}} d\lambda - \int_{0}^{L} \frac{J_{n}(r \lambda)}{\lambda^{m}} d\lambda. \hfill (C.49)$$

Now, we can use ([52], (9.3.1)) and ([62], (6.561.14)) to show that

$$j_{m,n}(L, r) \sim r^{m-1} \left[ \left( \frac{1}{2} \right)^{m} \frac{\Gamma(n+1-m)}{\Gamma(n+1)} - \left( \frac{eLr}{2n} \right)^{n} \frac{(Lr)^{1-m}}{\sqrt{2\pi n} (n + 1 - m)} \right];$$

$$n \gg Lr, \quad n + 1 > m > \frac{1}{2}. \hfill (C.50)$$
We are finally at a point where we can calculate the index of stability for (C.42). We will first handle the case where \( m \) is an even integer, or \( m \) is an odd integer and \( 0 \leq 2n \leq m - 3 \). When \( 2n \gg Lr \), we can use (C.44), (C.47), (C.50), and \((52), (6.1.17)) \) to show that

\[
\rho_{m,2n}^{(1)} \sim \left( \frac{2}{Lr} \right)^{m+\frac{1}{2}} \left| \frac{\sin(Lr - \frac{\pi}{4})}{\pi^{\frac{1}{2}}} \right| \left[ \Gamma\left( \frac{m+1}{2} \right) \right]^2; \quad m \text{ even}, \ 2n \gg Lr. \quad (C.51)
\]

On the other hand, when \( 2n \ll Lr \), we can use (C.44), (C.47), and (C.48) to show that

\[
\rho_{m,2n}^{(1)} \sim \frac{|\Gamma\left( \frac{m+1}{2} \right)|^2}{\Gamma\left( \frac{m+1}{2} - n \right)\Gamma\left( \frac{m+1}{2} + n \right)}; \quad m \text{ even or } m \text{ odd and } 0 \leq 2n \leq m - 3, \ 2n \ll Lr. \quad (C.52)
\]

Therefore, if we assume that an initial value, \( j_{m,2k}(L,r) \), is known, then an index a stability for the forward computation of \( j_{m,2n}(L,r) \) from \( j_{m,2k}(L,r) \) can be obtained by substituting (C.51) or (C.52) into (C.46). For the purposes of this report, we will be interested in computing \( j_{m,n}(L,r) \) for \( m = 3, 4, 5, \ldots, 10 \). When \( m \) is even, or when \( m \) is odd and \( 0 \leq 2n \leq m - 3 \), we find that as \( n \) increases, the index of stability will be non-increasing; thus proving that (C.40) can be used stably in the forward direction (for more details, see Appendix A of [46]). Actually, the above results have only been proven for the special case when \( n \) is an even integer; however, we could have used similar techniques to prove that forward recurrence using (C.40) is also stable when \( n \) is an odd integer.

Next, we will handle the case when \( m \) is an odd integer and \( 2n \geq m + 1 \). For this case, the forward recurrence will start at \( 2n = m + 1 \); therefore, it will be easier to directly compute the index of stability (C.46). This time we can use (C.45), (C.46), and (C.50) to show that

\[
\alpha^{(1)}(m + 1,2n) \sim \left( \frac{2}{Lr} \right)^m J_m(Lr) \Gamma(m + 1); \quad m \text{ odd}, \ 2n \geq m + 1, \ 2n \gg Lr. \quad (C.53)
\]

Likewise, when \( 2n \ll Lr \), we can use (C.45), (C.46), and (C.48) to show that

\[
\alpha^{(1)}(m + 1,2n) \sim \sqrt{\frac{\pi Lr}{2}} J_m(Lr) \frac{\Gamma(n + \frac{1-m}{2})\Gamma(m + 1)}{\Gamma(n + \frac{1+m}{2})\sin(Lr - \frac{\pi}{4})}; \quad m \text{ odd}, \ 2n \geq m + 1, \ 2n \ll Lr. \quad (C.54)
\]
Once again we find that the index of stability (see (C.53) or (C.54)) will be non-increasing as \( n \) increases. Therefore, (C.40) can be used stably in the forward direction for all values of \( m \) and \( Lr \).

The stability analysis shows that if we can find a way to obtain the starting functions \( j_{m,0}(L, r) \) and \( j_{m,1}(L, r) \) for \( m = 3, 4, \ldots, 10 \), then we can use (C.40) in the forward direction to obtain \( j_{m,n}(L, r) \) for \( n = 2, 3, \ldots \). When the recurrence relation decouples at the point \( n + 1 - m = 0 \), (C.41) can be used to restart the recurrence. Using ([52], (11.3.4)), we can show that

\[
j_{m,n}(L, r) - \frac{(m + n + 1)}{r} j_{m+1,n+1}(L, r) = -\frac{J_{n+1}(Lr)}{r L^m}.
\]

We are interested in using (C.55) for the two special cases where \( n = -1 \) and \( n = 0 \):

\[
\begin{align*}
    j_{m,1}(L, r) + \frac{m}{r} j_{m+1,0}(L, r) &= \frac{J_0(Lr)}{r L^m} \\
    j_{m,0}(L, r) - \frac{m+1}{r} j_{m+1,1}(L, r) &= -\frac{J_1(Lr)}{r L^m}
\end{align*}
\]

This shows that once the two starting functions, \( j_{m,0}(L, r) \) and \( j_{m,1}(L, r) \), are obtained for one value of \( m \), then the recurrence relations in (C.56) can be used to compute the starting functions for the other values of \( m \). However, before we can use these recurrence relations, we must determine in which direction they’re stable.

Once again we can use the techniques in Appendix A of [46] to handle the stability analysis. In order to simplify the analysis, we will combine the two recurrence relations in (C.56), thereby obtaining:

\[
j_{2m,0}(L, r) + d^{(2)}_{2m,0} j_{2(m+1),0}(L, r) = f^{(2)}_{2m,0}
\]

where

\[
\begin{align*}
    d^{(2)}_{2m,0} &:= \frac{(2m+1)^2}{r^2} \\
    f^{(2)}_{2m,0} &:= \left\{ (2m+1)I_0(Lr) - J_1(Lr) \right\}/(r L^{2m})
\end{align*}
\]

This time we find that the homogeneous solution is given by (see [52], (6.1.12))

\[
j^{(2k)}_{2m,0}(r) := \prod_{k=0}^{m-1} \left[ -d^{(2)}_{2k,0} \right]^{-1} = \left( \frac{-r^2}{4} \right)^m \left[ \frac{\Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})} \right]^2,
\]

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and that
\[ \rho_{2m,0}^{(2)} := \left| \frac{j_{0,0}(L,r) j_{2m,0}^{(2h)}(r)}{j_{m,0}(L,r)} \right| \sim \left( \frac{Lr}{2} \right)^{2m} \left[ \frac{\Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})} \right]^2, \]  
(C.60)

where we have made use of (C.48).

For even values of \( m \), the recurrence relations in (C.56) will have the same kind of behavior as that of (C.57). We could have also performed a similar analysis in order to determine the behavior of (C.56) for odd values of \( m \). Actually, for general values of \( m \), it can be shown that the index of stability for (C.57) can be written as (see [46], (A-7))
\[ \alpha^{(2)}(k,m) := \frac{\rho_{m,0}^{(2)}}{\rho_{k,0}^{(2)}}, \]  
(C.61)

where
\[ \rho_{m,0}^{(2)} := \left| \frac{j_{0,0}(L,r) j_{m,0}^{(2h)}(r)}{j_{m,0}(L,r)} \right| \sim \left( \frac{Lr}{2} \right)^m \left[ \frac{\Gamma(\frac{1}{2})}{\Gamma(m + \frac{1}{2})} \right]^2. \]  
(C.62)

Using ([46], (E-4)), we find that \( \rho_{m,0} \) reaches a maximum value when \( m = m_{\text{max}} \), where
\[ m_{\text{max}} := \text{int}(Lr - 1). \]  
(C.63)

Therefore, if we use \( j_{m_{\text{max}},0}(L,r) \) as a starting function, then reference to (C.61) and (C.62) shows that (C.57) can be used stably in the forward direction to compute \( j_{m,0}(L,r) \) for \( m = m_{\text{max}} + 1, m_{\text{max}} + 2, \ldots, 10 \), and backward recurrence can be used stably for \( m = m_{\text{max}} - 1, m_{\text{max}} - 2, \ldots, 3 \). So, if we can find some way to compute \( j_{m_{\text{max}},0}(L,r) \) and \( j_{m_{\text{max}},1}(L,r) \), then we can use (C.56) to compute the starting functions for the other values of \( m \).

We will use two different techniques to compute these starting functions. When \( Lr > SD + 4 \), where \( SD \) is the number of desired significant digits, we can use Hankel’s asymptotic expansion (see [52], (9.2.5)) to write
\[ j_{m,n}(L,r) \sim r^{m-1} \left\{ \sum_{k=0}^{\infty} \left( \frac{1}{4} \right)^k \right\} \left\{ \begin{array}{l} (n,2k) \left\{ \cos\left[ \frac{\pi}{4}(2n + 1) \right] \int_{Lr}^{\infty} \frac{\cos t}{t^{2k+m+\frac{1}{2}}} dt + \\
\sin\left[ \frac{\pi}{4}(2n + 1) \right] \int_{Lr}^{\infty} \frac{\sin t}{t^{2k+m+\frac{1}{2}}} dt \right\} - \frac{(n,2k+1)}{2} \left\{ \cos\left[ \frac{\pi}{4}(2n + 1) \right] \\
\sin\left[ \frac{\pi}{4}(2n + 1) \right] \int_{Lr}^{\infty} \frac{\cos t}{t^{2k+m+\frac{1}{2}}} dt - \sin\left[ \frac{\pi}{4}(2n + 1) \right] \int_{Lr}^{\infty} \frac{\sin t}{t^{2k+m+\frac{1}{2}}} dt \right\} \right\}. \]  
(C.64)

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Furthermore, the integrals in the above equation can be written in terms of incomplete gamma functions (see [52], (6.5.3))

\[
\begin{align*}
\int_{L_r}^{\infty} \frac{\cos t}{\tau_r} \, dt &= \mathcal{R}[j^{\mu-1}\Gamma(1 - \mu, jL_r)] \\
\int_{L_r}^{\infty} \frac{\sin t}{\tau_r} \, dt &= -\mathcal{I}[j^{\mu-1}\Gamma(1 - \mu, jL_r)]
\end{align*}
\]  

(C.65)

Therefore, if we can find a way to compute the required sequence of incomplete gamma functions, then we can use (C.64) to compute the starting functions, \( j_{m_{max},0}(L, \tau) \) and \( j_{m_{max},1}(L, \tau) \).

Incomplete gamma functions also satisfy a first-order, non-homogeneous recurrence relation (see [52], (6.5.3) and (6.5.22)):

\[
\Gamma(-n + \frac{1}{2}, z) + d_n^{(3)} \Gamma(-(n + 1) + \frac{1}{2}, z) = f_n^{(3)},
\]

(C.66)

where

\[
\begin{align*}
d_n^{(3)} &:= n + \frac{1}{2} \\
f_n^{(3)} &:= z^{-n+\frac{1}{2}} e^{-z}
\end{align*}
\]  

(C.67)

Applying the techniques for stability analysis (see [46], Appendix A) and ([52], (6.1.12)), we find that the homogeneous solution for (C.66) is given by

\[
\Gamma^{(h)}(-n + \frac{1}{2}) := \prod_{k=0}^{n-1} [-d_k^{(3)}]^{-1} = (-1)^n \frac{\Gamma(\frac{1}{2})}{\Gamma(n + \frac{1}{2})}.
\]

(C.68)

Also, when \( z \in S_\pi \), where

\[
S_\pi := \{ z : |\angle z| < \pi \}
\]

(C.69)

it can be shown that

\[
\Gamma(-n + \frac{1}{2}, z) := \int_{z}^{\infty} \frac{e^{-t}}{t^{n+\frac{1}{2}}} \, dt \sim z^{-n+\frac{1}{2}} e^{-z}; \quad |z| \gg 1, \ n \geq 0.
\]

(C.70)

Therefore, we find that the stability index is given by (see [46], (A-7) and (A-8))

\[
\alpha^{(3)}(k, n) := \frac{\rho_n^{(3)}}{\rho_k^{(3)}},
\]

(C.71)

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where
\[
\rho_n^{(3)} := \left| \frac{\Gamma\left(\frac{1}{2}, z\right) \Gamma(\alpha)(-n + \frac{1}{2})}{\Gamma(-n + \frac{1}{2}, z)} \right| \sim \left| \frac{z^n \Gamma\left(\frac{1}{2}\right)}{\Gamma(n + \frac{1}{2})} \right| ; \quad |z| \gg 1.
\] (C.72)

Once again, we can use ([46], (E-4)) to show that \( \rho_n \) reaches a maximum value when \( n = n_{\text{max}} := \text{int}(|z| - \frac{1}{2}) \). Therefore, once we obtain the starting function \( \Gamma(-n_{\text{max}} + \frac{1}{2}, z) \), we can use (C.66) in the backward direction to obtain \( \Gamma(-n + \frac{1}{2}, z) \) for \( n = n_{\text{max}} - 1, \ldots, m_{\text{max}} \); and we can use forward recurrence for \( n = n_{\text{max}} + 1, n_{\text{max}} + 2, \ldots \). The continued fraction expansion (see [63], (12.13-8)),
\[
\Gamma(a, z) = z^a e^{-z} \left( \frac{1}{z + \frac{1 - a}{1 + \frac{2 - a}{z + \frac{2}{1 + \ldots}}} \right) ; \quad z \in S, a \in \mathbb{C}.
\] (C.73)

provides the most efficient method for obtaining the required starting function \( \Gamma(-n_{\text{max}} + \frac{1}{2}, z) \).

In summary, when \( Lr > SD + 4 \), we can use (C.73) to compute the starting function \( \Gamma(-n_{\text{max}} + \frac{1}{2}, jLr) \). Then we can use (C.64) to compute the desired integrals, \( j_{m_{\text{max}},0}(L, r) \) and \( j_{m_{\text{max}},1}(L, r) \), where the recurrence relation (C.66) is used to obtain the incomplete gamma functions in the asymptotic expansion.

When \( Lr < SD + 4 \), a different method is required for computing the starting functions \( j_{m_{\text{max}},0}(L, r) \) and \( j_{m_{\text{max}},1}(L, r) \). For this case, we find that (see (C.63)) \( m_{\text{max}} \leq \text{int}(SD + 3) \). Therefore, if we compute the starting functions \( j_{3,0}(L, r) \) and \( j_{3,1}(L, r) \), then (C.56) can be used relatively stably in the forward direction for \( SD \leq 5 \). It’s true that we may lose some accuracy when we compute \( j_{m,0}(L, r) \) and \( j_{m,1}(L, r) \) for \( m = 4, 5, \ldots, 10 \), but the loss will not be significant.

In order to compute these starting functions, we first split the integral into two pieces:
\[
j_{3,n}(L, r) = \int_L^r \frac{J_3(r\lambda)}{\lambda^3} d\lambda + r^2 j_{3,n}(9,1).
\] (C.74)

The second integral on the right-hand-side of (C.74) can now be computed using (C.64) when \( SD \leq 5 \). This integral only needs to be computed once
since it is independent of \( r \). In fact, we find that
\[
\begin{align*}
  j_{3,0}(9, 1) &= -3.320658 \times 10^{-4} \\
  j_{3,1}(9, 1) &= -1.795596 \times 10^{-5}
\end{align*}
\]  \( \quad \) (C.75)

Now, if we expand the Bessel function, in the other piece, in a power series (see [52], (9.1.10)), and integrate term-by-term, we find that:
\[
\begin{align*}
  j_{3,0}(L, r) &= r^2 \left\{ j_{3,0}(9, 1) + \frac{3}{2} \left[ \frac{1}{(Lr)^2} - \frac{1}{81} \right] + \ln \left( \frac{Lr}{4} \right) + \right. \\
  &\quad \left. \sum_{k=2}^{\infty} \frac{(2k-2)!}{(-4)^k k!^2 (2k-2)!} \right\} \\
  j_{3,1}(L, r) &= r^2 \left\{ j_{3,1}(9, 1) + \sum_{k=0}^{\infty} \frac{(2k-1)!}{(-4)^k k!^2 (k+1)! (4k-2)!} \right\}
\end{align*}
\]  \( \quad \) (C.76)

Once the starting functions have been computed by using (C.64) or (C.76), then we can use (C.56) in the forward and backward directions to compute \( j_{m,0}(L, r) \) and \( j_{m,3}(L, r) \) for \( 3 \leq m \leq 10 \). Finally, (C.40) can then be used in the forward direction to compute \( j_{m,n}(L, r) \) for fixed values of \( m \), where \( n \geq 2 \).

### C.4 Computation of \( f_L^\infty \mathcal{I}_{15}(\lambda, x, y) d\lambda \)

In this Section of the Appendix, we demonstrate how the results from §C.2 and §C.3 can be used to compute \( f_L^\infty \mathcal{I}_{15}(\lambda, x, y) d\lambda \), where \( \mathcal{I}_{15}(\lambda, x, y) \) is defined in (C.2). Once we can compute this integral, then (C.3) and (C.4) can be used to compute \( Z_{mn}^*(L) \).

Up to this point, we haven’t specified a value for \( L \). If we carry out all of the expansions to the same order of accuracy as in (C.5), then we will obtain \( SD \) significant digits in the expansions provided that \( L \) satisfies the following inequality:
\[
\left[ \frac{\max(k_{11}, k_{11}, k_{AA})}{L} \right]^6 < \frac{1}{2} \times 10^{-SD}.
\]  \( \quad \) (C.77)

The inequality in (C.8) gives a second constraint for the choice for \( L \):
\[
e^{-2L \min(z_{11}, z_{-11})} < \frac{1}{2} \times 10^{-SD}.
\]  \( \quad \) (C.78)
We will choose the minimum value of $L$ that satisfies the above two inequalities.

The first integral that we will look at is (see (C.2))
\[
\int_{L}^{\infty} I_{14}^{(1)}(\lambda) \Re \{I_{7}(\lambda, x, y, 0, 1)\} d\lambda.
\] (C.79)

Referring to (C.10), (C.17), and (C.39), we find that
\[
\int_{L}^{\infty} I_{14}^{(1)}(\lambda) \Re \{I_{7}(\lambda, x, y, 0, 1)\} d\lambda = \frac{-j^{2} \pi \omega \mu_{0}}{(k_{11}^{2} + k_{11}^{2})} \sum_{i=0}^{2} c_{i}^{(1)} \{r \}
\]
\[
j_{3+2i,1}(L, r) - 2y \sum_{k=0}^{\infty} (-1)^{k} \sin[(2k + 1)\theta_{0}]j_{3+2i,2k+1}(L, r) \}. (C.80)
\]

This series won't converge until $2k + 1 \gg Lr$ (see (C.48) and (C.50)). In order to investigate the convergence of (C.80), we can use (C.50) to show that
\[
j_{3+2i,2k+1}(L, r) \sim \frac{1}{r} \left(\frac{r}{2}\right)^{3-2i} \frac{\Gamma(k - i - \frac{1}{2})}{\Gamma(k + i + \frac{5}{2})}; \quad 2k + 1 \gg Lr. (C.81)
\]

This shows that when summing over $k$ in (C.80), the series will converge more rapidly when $i = 2$ then when $i = 0$. Actually, we have found that there are numerical problems associated with summing the series over $k$ for $i = 0$ and $i = 1$. Therefore, we will attack the problem from a different angle.

If we define
\[
I_{17}(x, y; n) := \int_{L}^{\infty} \frac{e^{ijx}J_{0}(j \cos \theta_{0}, \lambda r)}{\lambda^{n+1}} d\lambda, (C.82)
\]
then it can be shown that (see Table A.1 and (C.10))
\[
\int_{L}^{\infty} I_{14}^{(1)}(\lambda) \Re \{I_{7}(\lambda, x, y, 0, 1)\} d\lambda = \frac{-j^{2} \pi \omega \mu_{0}}{(k_{11}^{2} + k_{11}^{2})} \sum_{i=0}^{2} c_{i}^{(1)} \{r \}
\]
\[
j_{3+2i,1}(L, r) - y \sin \theta_{0} \Re \{I_{17}(x, y; 2 + 2i)\} \}. (C.83)
\]

Now, comparing (C.80) with (C.83), we find that
\[
\Re \{I_{17}(x, y; 2 + 2i)\} = \frac{2}{\sin \theta_{0}} \sum_{k=0}^{\infty} (-1)^{k} \sin[(2k + 1)\theta_{0}]j_{3+2i,2k+1}(L, r). (C.84)
\]
Using numerical tests, we found that this expansion can be used to compute $I_{17}(x, y; \epsilon)$.  

Next, we need to find an efficient way to compute $I_{17}(x, y; 2 + 2i)$ for $i = 0$ and $i = 1$. We will develop a recurrence relation for this purpose. When $x \neq 0$, we can use integration by parts, where $u := \frac{J_{n+1}(j \cos \theta_0, L \nu)}{\lambda^{n+1}}$ and $dv := e^{i \lambda x} d\lambda$, and (4.1) to show that

$$I_{17}(x, y; \nu) = \frac{j}{x} \left\{ \frac{e^{i \lambda x}}{\lambda^{n+1}} J_{n+1}(j \cos \theta_0, L \nu) - (n + 1) I_{17}(x, y; \nu + 1) + \right.$$  
$$+ \left. \frac{r_{j_{n+1,0}}(L, \nu)}{\lambda^{n+1}} \right\}; \; x \neq 0. \quad (C.85)$$

We only need to compute $\Re\{I_{17}(x, y; \nu)\}$ for even values of $\nu$, therefore, we can use (C.85) to show that

$$\Re\left\{ \frac{2}{\nu} I_{17}(x, y; \nu) + (n + 1)(n + 2) I_{17}(x, y; \nu + 2) \right\} = (n + 1) r_{j_{n+2,0}}(L, \nu) +$$  
$$+ \Re\left\{ \frac{e^{i \lambda x} J_{n+2}(j \cos \theta_0, L \nu)}{\lambda^{n+2}} \right\} - \frac{L x \Re\left\{ e^{i \lambda x} J_{n+2}(j \cos \theta_0, L \nu) \right\}}{L^{n+2}}. \quad (C.86)$$

When $x = 0$, we can rewrite (C.85) as

$$I_{17}(0, y; \nu) = \frac{1}{\nu} \left[ \frac{1}{L \nu} J_{0}(0, L \nu) + y r_{j_{0,0}}(L, \nu) \right]. \quad (C.87)$$

Before we can use (C.85), we need to perform a stability analysis. The homogeneous solution of (C.85) is given by (see [46], Appendix A)

$$I_{17}^{(h)}(x, y; \nu) := \frac{(j x)^n}{\Gamma(n + 1)}, \quad (C.88)$$

and an index of stability for forward recurrence is given by

$$\alpha_{17}(k, n) := \left| \frac{I_{17}(x, y; k) I_{17}^{(h)}(x, y; n)}{I_{17}(x, y; n) I_{17}^{(h)}(x, y; k)} \right| = \frac{\rho_{17}(n)}{\rho_{17}(k)}, \quad (C.89)$$

where

$$\rho_{17}(n) := \left| \frac{I_{17}(x, y; 0) I_{17}^{(h)}(x, y; n)}{I_{17}(x, y; n)} \right|. \quad (C.90)$$

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In order to use (C.89), we need to obtain an approximation for $|T_{17}(x, y; n)|$. This can be accomplished by substituting (B.18) into (C.82):

$$|T_{17}(x, y; n)| \sim \left| \csc \theta_0 \int_{L}^{\infty} \frac{e^{i\lambda z}}{\lambda^{n+1}} d\lambda - \sqrt{\frac{2}{\pi r}} \frac{\csc^2 \theta_0}{\lambda} \int_{L}^{\infty} \lambda^{-n-\frac{3}{2}} |\cos(\lambda r - \frac{\pi}{4})|ight. \left. j \cos \theta_0 - \sin(\lambda r - \frac{\pi}{4}) \right| d\lambda \bigg|_{\min(Lr, Lr| \cos \theta_0 \pm 1|) \gg 0}. \quad (C.91)$$

When $\min(Lr, Lr| \cos \theta_0 \pm 1|) \gg 1$, we can use (C.70) to show that

$$|T_{17}(x, y; n)| \sim \left| \frac{\csc \theta_0}{x L^{n+1}} \right|. \quad (C.92)$$

Now, we can use (C.88), (C.90), and (C.92) to show that

$$\rho_{17}(n) \sim \frac{|Lx|^n}{\Gamma(n + 1)}; \quad |Lx| \gg 0. \quad (C.93)$$

We can also show that (C.93) still holds when $|\cos \theta_0| \approx 1$ by substituting (B.21), instead of (B.18), into (C.82).

Referring to (C.89), (C.93), and (|46|, (E.4)), we find that once $T_{17}(x, y; 6)$ has been computed, we can stably use (C.85) in the backward direction to calculate $T_{17}(x, y; 4)$ and $T_{17}(x, y; 2)$ when $|Lx| > 7.0$. When $|Lx| < 7.0$, we can still use backward recurrence, but accuracy will be lost. This loss of accuracy will not cause problems for the applications which are presented in this report. Therefore, we can use the results in this Section to compute the integral in (C.83).

The next step is to determine how to compute

$$\int_{L}^{\infty} T_{14}^{(6)}(\lambda) \mathcal{R}\{T_{4}(\lambda, x, y, 0, \frac{k_A}{\lambda})\} d\lambda \quad \text{and} \quad \int_{L}^{\infty} T_{14}^{(5)}(\lambda) \mathcal{R}\{T_{12}(\lambda, x, y, \frac{k_A}{\lambda})\} d\lambda \quad \text{.} \quad (C.94)$$

When the inequality in (C.26) holds, we can use the asymptotic expansions in (C.22) and (C.23). On the other hand, when the inequality in (C.26) doesn’t hold, we will have to use another method to compute (C.94). When $x = 0$, we can use the convergent series expansions in (C.29) and (C.31). Finally, when the inequality in (C.26) doesn’t hold and $x \neq 0$, we will make
use of (C.37) and (C.38). In order to do this, we split the integrals in (C.94) into two pieces:

\[ \int_{L}^{\infty} \{ \} d\lambda = \int_{L}^{L'} \{ \} d\lambda + \int_{L'}^{\infty} \{ \} d\lambda, \tag{C.95} \]

where

\[ L' := \frac{2r}{x^2} \ln(2\sqrt{2} \times 10^{5P}). \tag{C.96} \]

For values of \( x \) that are of the same order as \( y \), \( L' \) will be the smallest value of \( \lambda \) that will satisfy the inequality in (C.26) for \( \lambda \gg k_A \). Therefore, the second integral on the right-hand-side of (C.95) can be computed using (C.22) and (C.23). Now, if we substitute the convergent series expansions (C.37) and (C.38) into the first integral on the right-hand-side of (C.95), then the evaluation of this integral involves the computation of

\[ j_{m,n}(L, L', r) := j_{m,n}(L', r) - j_{m,n}(L, r). \tag{C.97} \]

We can use a modified version of (C.40) to compute \( j_{m,n}(L, L', r) \),

\[ j_{m,n}(L, L', r) = \frac{(n + 1 + m)}{(n + 1 - m)} j_{m,n+2}(L, L', r) = \]

\[ \frac{2(n + 1)}{r(n + 1 - m)} \left[ \frac{J_{n+1}(Lr)}{L^m} - \frac{J_{n+1}(L'r)}{(L')^m} \right], \tag{C.98} \]

however, we must first perform a stability analysis. Since we have not changed the homogeneous equation, (C.44) and (C.45) will still hold. When \( n \ll Lr \), we can obtain an approximation for \( j_{m,n}(L, L', r) \) by substituting (C.48) into (C.97). Since \( j_{m,n}(L, L', r) \) has the same kind of behavior as \( j_{m,n}(L, r) \) when \( n \ll Lr \), we can conclude that (C.98) can be used stably in the forward direction. On the other hand, when \( n \gg L'r \), we can use ([52], (9.3.1)) to show that

\[ j_{m,n}(L, L', r) \sim \left( \frac{er}{2n} \right)^n \frac{[(L')^{n+1-m} - L^{n+1-m}]}{\sqrt{2\pi}n(n + 1 - m)}. \tag{C.99} \]

Therefore, we find that (C.98) must be used in the backward direction when \( n \gg L'r \). We have found using numerical tests that forward recurrence can be used to compute \( j_{m,n}(L, L', r) \) for \( n = 0, 1, \ldots, m \), and backward recurrence (see §B.2) can be used for \( n = m + 1, \ldots \).
C.5 Reflection Properties of \( \int_{L}^{\infty} I_{15}(\lambda, x, y) \, d\lambda \)

In this Section, we will look at the reflection properties for \( \int_{L}^{\infty} I_{15}(\lambda, x, y) \, d\lambda \). First, however, we will look at the reflection properties of \( I_{15}(\lambda, x, y) \). Using (A.64), (A.68), (C.1), and (C.2), we find that

\[
I_{15}(\lambda, -x, y) = I_{15}(\lambda, x, y) - \frac{2\pi \lambda}{\sqrt{\lambda^2 - k_{A}^2}} T_{14}^{(9)}(\lambda) \{ \sin(\lambda F_{+} r) + \sin(\lambda F_{-} r) \} - \frac{2\pi \lambda^3 r}{(\lambda^2 - k_{A}^2)} T_{14}^{(5)}(\lambda) \{ \sqrt{1 - F_{+}^2} \cos(\lambda F_{+} r) + \sqrt{1 - F_{-}^2} \cos(\lambda F_{-} r) \}. \tag{C.100}
\]

This expression can be further simplified by applying ([52], (4.3.34)-(4.3.37)):

\[
I_{15}(\lambda, -x, y) = I_{15}(\lambda, x, y) + \frac{4\pi \lambda}{\sqrt{\lambda^2 - k_{A}^2}} T_{14}^{(9)}(\lambda) \sin(k_{A} x) \cos(y \sqrt{\lambda^2 - k_{A}^2}) - \frac{4\pi \lambda^2}{(\lambda^2 - k_{A}^2)} T_{14}^{(5)}(\lambda) \{ k_{A} y \sin(k_{A} x) \cos(y \sqrt{\lambda^2 - k_{A}^2}) + x \sqrt{\lambda^2 - k_{A}^2} \cos(k_{A} x) \cos(y \sqrt{\lambda^2 - k_{A}^2}) \}. \tag{C.101}
\]

When \( y \neq 0 \), we can integrate both sides of (C.101) with respect to \( \lambda \) and apply the change of variables, \( u := y \sqrt{\lambda^2 - k_{A}^2} \), thereby obtaining (see (C.10)-(C.12))

\[
\int_{L}^{\infty} I_{15}(\lambda, -x, y) \, d\lambda = \int_{L}^{\infty} I_{15}(\lambda, x, y) \, d\lambda + \frac{j2\pi \omega_{0}}{k_{A}^2 (k_{11}^2 + k_{11}^2)} \sum_{i=0}^{2} \left\{ \frac{\sin(k_{A} x)}{k_{A}} c_{i}^{(9)} - x \cos(k_{A} x) c_{i}^{(5)} \right\} \frac{1}{y} \int_{u'}^{\infty} \left( \frac{y}{u} \right)^{2i+3} \cos u \, du \, \frac{\sin u \, du}{\left[ 1 + (\frac{k_{A} x}{u})^2 \right]^{i+\frac{3}{2}}} - k_{A} \sin(k_{A} x) c_{i}^{(5)} \int_{u'}^{\infty} \left( \frac{y}{u} \right)^{2i+4} \frac{\sin u \, du}{\left[ 1 + (\frac{k_{A} x}{u})^2 \right]^{i+\frac{3}{2}}}; \quad y \neq 0, \tag{C.102}
\]

where \( u' := y \sqrt{L^2 - k_{A}^2} \). When \( \lambda \gg k_{A} \), we can use ([52], (3.6.9), (6.122), and (6.5.3)) to obtain the following expansion:
\[
\int_L I_{15}(\lambda, -x, y) \, d\lambda = \int_L I_{15}(\lambda, x, y) \, d\lambda + \frac{j2\pi\omega\mu_0}{k_A^2(k_{11}^2 + k_{11}^2)} \sum_{i=0}^{2} \sum_{n=0}^{\infty} (-k_A^2)^n y^{2n+2i+3} \frac{\Gamma(n + i + \frac{3}{2})}{\Gamma(n + 1) \Gamma(i + \frac{3}{2})} \left[ \frac{\sin(k_A x)}{k_A} c_i^{(9)} - x \cos(k_A x) c_i^{(5)} \right] \\
\frac{1}{y} \int_{u'}^{\infty} \cos u \frac{du}{u^{2n+2i+3}} - k_A y \sin(k_A x) c_i^{(5)} \int_{u'}^{\infty} \frac{\sin u \, du}{u^{2n+2i+4}} \right] \quad y \neq 0. \quad (C.103)
\]

Referring to (C.65), we find that the integrals in (C.103) can be represented as incomplete gamma functions. Actually, the results in (C.65)–(C.73) can be modified and used to compute the incomplete gamma functions in (C.103).

We will have to obtain another expansion for the special case \( y = 0 \). We start by integrating both sides of (C.101), where \( y \) is set to equal to zero:

\[
\int_L I_{15}(\lambda, -x, 0) \, d\lambda = \int_L I_{15}(\lambda, x, 0) \, d\lambda + \frac{j2\pi\omega\mu_0}{k_A^2(k_{11}^2 + k_{11}^2)} \sum_{i=0}^{2} \int_L \frac{d\lambda}{\lambda^{2i+3} \sqrt{1 - (k_A^2)^2}}. \quad (C.104)
\]

When \( \lambda \gg k_A \), we can use ((52), (3.6.9) and (6.1.22)) to expand the square root in a power series, and then we can integrate term-by-term, yielding:

\[
\int_L I_{15}(\lambda, -x, 0) \, d\lambda = \int_L I_{15}(\lambda, x, 0) \, d\lambda + \\
\frac{j\pi\omega\mu_0}{k_A^2(k_{11}^2 + k_{11}^2)} \sum_{i=0}^{2} \left[ \frac{\sin(k_A x)}{k_A} c_i^{(9)} - x \cos(k_A x) c_i^{(5)} \right] \\
\sum_{n=0}^{\infty} \frac{(k_A)^{2n} \Gamma(n + \frac{1}{2})}{\Gamma(n + 1) \Gamma(n + i + \frac{1}{2}) \Gamma(n + i + \frac{3}{2})} L^{2n+2i+2}. \quad (C.105)
\]
Appendix D

NUMERICAL INTEGRATION OF
\[ I_1(k_A, \lambda, x, y, S) \]

D.1 Introduction

In this Appendix, we will investigate using a numerical integration routine to compute \( I_1(k_A, \lambda, x, y, S) \) for the values of \( S \) in (A.2) and (A.3). We have chosen to use the two adaptive quadrature routines D01AJF and D01AKF (see [51]) in this report; however, the results which are derived in this Appendix can be applied with any integration routine. We will say more about the choice of the numerical integration routines later.

D.2 Impedance Elements

We will first examine the integrals which are associated with the elements in the impedance matrix. Reference to (4.11) and (A.2) shows that we need to evaluate the following integral:

\[
I_1(k_A, \lambda, x, y, (0, 0, S_3, 1)) = \int_{-\pi}^{\pi} \left\{ \frac{\cos(d\lambda \cos \theta) - \cos(k_A d) \sin(\lambda \sin \theta)}{k_A^2 - \lambda^2 \cos^2 \theta} \right\}^2 \frac{e^{-j\lambda(x \cos \theta + y \sin \theta)}}{(1 - S_3^2 \cos^2 \theta)} d\theta. \tag{D.1}
\]
Due to the symmetry which is present in the problem (see Figure 3.1), it is possible, and desirable, to change the limits of integration in the above integral. If we apply Euler's formula to the exponential function in (D.1), and then cancel the odd functions of $\theta$, we find that

$$I_1(k_A, \lambda, x, y, (0, 0, S_3, 1)) = 2 \int_0^\pi \left\{ \frac{\cos(d\lambda \cos \theta) - \cos(k_A d)}{k_A^2 - \lambda^2 \cos^2 \theta} \right\}^2 \cos(\lambda y \sin \theta) \cos(\lambda x \cos \theta) \sin(\lambda \cos \theta) \cos(\lambda x \sin \theta) \cos(\lambda y \cos \theta) \sin(\lambda \cos \theta) \frac{d\theta}{1 - S^2_3 \cos^2 \theta}. \quad (D.2)$$

Now, if we apply the change of variables, $\hat{\theta} := \theta - \frac{\pi}{2}$, and once again cancel the odd functions of $\hat{\theta}$, then it can be shown that

$$I_1(k_A, \lambda, x, y, (0, 0, S_3, 1)) = 4 \int_\frac{-\pi}{2}^\frac{\pi}{2} \left\{ \frac{\cos(d\lambda \sin \hat{\theta}) - \cos(k_A d)}{k_A^2 - \lambda^2 \sin^2 \hat{\theta}} \right\}^2 \cos(\lambda x \sin \hat{\theta}) \cos(\lambda y \cos \hat{\theta}) \frac{d\hat{\theta}}{1 - S^2_3 \sin^2 \hat{\theta}}. \quad (D.3)$$

Changing back to the variable \( \theta \), we find that

$$I_1(k_A, \lambda, x, y, (0, 0, S_3, 1)) = 4 \int_0^\pi \left\{ \frac{\cos(d\lambda \cos \theta) - \cos(k_A d)}{k_A^2 - \lambda^2 \cos^2 \theta} \right\}^2 \cos(\lambda x \cos \theta) \cos(\lambda y \sin \theta) \frac{d\theta}{1 - S^2_3 \cos^2 \theta}. \quad (D.4)$$

This shows us that \( I_1(k_A, \lambda, x, y, (0, 0, S_3, 1)) \) is a real-valued function. Therefore, if we split the inner angular integral of \( Z_{mn} \) into two pieces, as was done in (4.10), then we can compute this integral by using a numerical integration routine which is designed for real-valued functions, twice. On the other hand, if we try to evaluate the angular integral as one piece, we will have to use a numerical integration routine for complex-valued functions. In order to keep the numerical integration technique consistent with the method where the angular integral is decomposed into a finite number of ILHI's (see §4.2), we will apply the numerical integration to (D.4) and then use (4.10).

Referring to (D.4) we find that the integrand will oscillate very rapidly when \( \lambda \) becomes large and either \( x \) or \( y \) are non-zero. When the AET is applied to the evaluation of the outer integral (see §4.4), the value of the
spectral variable will not become very large, therefore the numerical integration routine D01AJF will be adequate for the evaluation of (D.4). On the other hand, if we don’t use the AET on the outer integral, then the numerical integration routine D01AKF, which is specially designed for oscillatory integrals, is better suited. Numerical tests demonstrated that it is more efficient to use D01AJF to integrate (D.4) when the AET is used, and then apply D01AKF when the AET isn’t used.

In Appendix A, it was mentioned that \( \mathcal{I}_1(k_A, \lambda, x, y, S) \) has removable singularities when \( S \) takes on the values given in (A.2) or (A.3). Therefore, before we can apply a numerical integration algorithm to (D.4), we need to find a way to handle these removable singularities.

When \( S_3 = 1 \), a removable singularity occurs at \( \theta = \frac{\pi}{2} \). This singularity can be handled by expanding the sine function, which is in the numerator, in a Taylor series expansion:

\[
\lim_{\theta \to \frac{\pi}{2}} \frac{\sin(v \lambda \cos \theta)}{\cos \theta} \approx v \lambda \left[ 1 - \frac{(v \lambda \cos \theta)^2}{6} \right]. \tag{D.5}
\]

Another removable singularity also exists when \( \lambda > k_A \). In order to handle this singularity, we use the trigonometric identity (see [52], (4.3.37)),

\[
\cos z_1 - \cos z_2 = -2 \sin \left( \frac{z_1 + z_2}{2} \right) \sin \left( \frac{z_1 - z_2}{2} \right), \tag{D.6}
\]

to write

\[
\frac{\cos(d \lambda \cos \theta) - \cos(k_A d)}{k_A^2 - \lambda^2 \cos^2 \theta} = \frac{2 \sin \left[ \frac{d}{2} (k_A + \lambda \cos \theta) \right] \sin \left[ \frac{d}{2} (k_A - \lambda \cos \theta) \right]}{k_A^2 - \lambda^2 \cos^2 \theta}. \tag{D.7}
\]

Now, when \( |k_A - \lambda \cos \theta| \) is small, we find that

\[
\lim_{k_A - \lambda \cos \theta \to 0} \frac{\cos(d \lambda \cos \theta) - \cos(k_A d)}{k_A^2 - \lambda^2 \cos^2 \theta} \approx \frac{d \sin \left[ \frac{d}{2} (k_A + \lambda \cos \theta) \right]}{k_A + \lambda \cos \theta} \left\{ 1 - \frac{\left[ \frac{d}{2} (k_A - \lambda \cos \theta) \right]^2}{6} \right\}. \tag{D.8}
\]

The approximations in (D.5) and (D.8) can be used to handle the removable singularities when a numerical integration algorithm is used to evaluate (D.4).
D.3 Electric Field

Next, we will look at the integrals which are encountered when computing the near-zone electric field (see (4.9) and (4.11)). Once again, we will use the symmetry in the problem to change the limits of integration. If we apply the techniques which were previously used for the integrals associated with the elements in the impedance matrix, then we find that:

\[
I_1(k_A, \lambda, x, y, (1, 0, S_0, 0)) = 4 \int_0^{\frac{\pi}{2}} \left\{ \frac{\cos(d \lambda \cos \theta) - \cos(k_A d)}{k_A^2 - \lambda^2 \cos^2 \theta} \right\} \sin \theta \cos(\lambda x \cos \theta) \cos(\lambda y \sin \theta) d\theta \quad (D.9)
\]

\[
I_1(k_A, \lambda, x, y, (0, 1, 0, 0)) = -4 \int_0^{\frac{\pi}{2}} \left\{ \frac{\cos(d \lambda \cos \theta) - \cos(k_A d)}{k_A^2 - \lambda^2 \cos^2 \theta} \right\} \cos \theta \sin(\lambda x \cos \theta) \sin(\lambda y \sin \theta) d\theta \quad (D.10)
\]

\[
I_1(k_A, \lambda, x, y, (1, 1, 1, 0)) = -4j \int_0^{\frac{\pi}{2}} \left\{ \frac{\cos(d \lambda \cos \theta) - \cos(k_A d)}{k_A^2 - \lambda^2 \cos^2 \theta} \right\} \cot \theta \sin(\lambda x \cos \theta) \cos(\lambda y \sin \theta) d\theta. \quad (D.11)
\]

As was the case with the elements in the impedance matrix, the integrals that are encountered when computing the near-zone electric field are all either purely real-valued or purely imaginary (see (D.9)–(D.10)). Therefore, we can use a numerical integration routine which is designed for real-valued functions to compute them. Due to the exponential decay factor in the integrand, we will be able to truncate the numerical integration of the outer semi-infinite integral at a relatively small value. Therefore, since the spectral variable, \(\lambda\), won't become very large, we can use the numerical integration routine D01AJF for the inner integrals. Also, the removable singularities that are contained in these integrals can be handled using (D.5) and (D.7).

D.4 Application of D01AJF or D01AKF

As we previously mentioned, we will use the adaptive quadrature routines D01AJF and D01AKF (see [51]) to compute the integrals in (D.4) and (D.9)–(D.11). The removable singularities can be handled using (D.5) and (D.8).
The results in this Appendix can be used to construct a function, \( F(\lambda) \), that returns a value for the integrand of \( I_1(k_A, \lambda, x, y, S) \) for any value of \( S \) in (A.2) or (A.3).

The other inputs for D01AJF and D01AKF are defined as:

\[
\begin{align*}
A &= 0, \\
B &= \frac{\pi}{2}, \\
EPSABS &= 0, \\
EPSREL &= \frac{1}{2} \times 10^{-SD}
\end{align*}
\]  

(D.12)

The subroutines D01AJF and D01AKF will return an approximation for the integral along with some other diagnostic results.
Bibliography


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