Development of a Protocol for Engineering Applications of Evidence Theory

William Seites-Rundlett[1], Ross B. Corotis[2], Cristina Torres-Machi[3]

Abstract

Recent data trends and analysis have highlighted the need to incorporate more imprecise, ambiguous, and unreliable data into uncertainty analysis traditionally handled by probability theory. Data fraught with potential error and missing information, however, are not well suited for analysis using probability theory due to high epistemic uncertainty. Evidence Theory offers an alternative method of assessing epistemic uncertainty and is well suited for expanded use in engineering applications. Unfortunately, a unified approach to the application of Evidence Theory is lacking. To address this gap, we develop a protocol for engineering applications of Evidence Theory. The protocol proposes a logical procedure for defining the frame of discernment, the initial assignment of belief mass, the selection of combination rule, and sensitivity analysis. A literature review of prevailing methods related to the application of Evidence Theory highlights concepts and considerations to address. The steps of the protocol are then explored and discussed using an example problem including several rule combinations in order to highlight differences in the results and implications of making different analytical decisions. The protocol proposed herein is intended to facilitate engineering applications of Evidence Theory and promote more widespread use of the theory in the field of Civil Engineering.

[1] *Department of Civil, Environmental, and Architectural Engineering, University of Colorado Boulder, 1111 Engineering Drive, Boulder CO 80309-0428, United States of America,* William.seitesrundlett@colorado.edu

[2] *Department of Civil, Environmental, and Architectural Engineering, University of Colorado Boulder, 1111 Engineering Drive, Boulder CO 80309-0428, United States of America,* ross.corotis@colorado.edu, Corresponding author

[3] *Department of Civil, Environmental, and Architectural Engineering, University of Colorado Boulder, 1111 Engineering Drive, Boulder CO 80309-0428, United States of America,* Cristina.torresmachi@colorado.edu

Introduction

The visibility and predominance of uncertainty in our daily lives is a major driver of our thoughts, emotions, and actions. We gather information in order to evaluate our uncertainty and guide our decision-making, and we consider information frivolous unless it contributes to that (Kyburg 1988). The analysis of uncertainties and development of decision-making frameworks has led to the adoption of mathematical formulations to represent uncertainties, resulting in a quantitative analysis of uncertainty. Currently, probability theory is the predominant method employed in uncertainty analysis.

Probability theory-based methods, however, are challenging to apply in situations characterized by ignorance, where lack of information makes estimates of initial (often called prior) probabilities or probability distributions difficult to justify (Shafer 2016). Probability theory, based on the applicability of distributions to model the different states of random variables, is well suited to address aleatory uncertainty of randomness and chance (Oberkampf and Helton 2002). Epistemic uncertainty, the state of imperfect knowledge arising from ignorance, is, however, difficult to analyze accurately using probability theory (Oberkampf et al. 2002). This is because judgments based on probability theory suggest there is precise information not only about the event itself, but also about its contrary, which is often not appropriate in cases of limited quantitative knowledge (Corotis 2015). Furthermore, these subjective judgments pertain not only to the selection of unknown probabilities, but also to the selection of a model and underlying distribution.

Given these circumstances, engineering challenges require novel methods of uncertainty assessment to address this shortcoming of probability theory, and improve both our understanding and our quantification of epistemic uncertainty. An interesting framework of assessing epistemic uncertainty is Evidence Theory, also known as Dempster-Shafer theory or the theory of belief functions. Evidence Theory was originally conceived in the late 1960s and 1970s (Dempster 1968; Shafer 1976), and saw initial applications and concept development within the Artificial Intelligence community in the 1980s. Evidence Theory has recently seen expanded applications to machine learning and practical engineering problems (Attoh-Okine et al. 2009; Behrouz and Alimohammadi 2018; Denoeux 2000, 2013). Notable features of Evidence Theory are that the mathematics are set-based and there is an explicit recognition of ignorance. The recognition of ignorance presents a valuable tool for treating epistemic uncertainty and a methodological alternative to probability theory, in which the probabilities for and against (i.e., its complement) a given event must sum to unity.

Despite the perceived advantages and recent expanded research into Evidence Theory, there is a lack of an agreed upon method of applying Evidence Theory (Smets 2007). Many different approaches have been developed under the umbrella of Evidence Theory; however, it is not clear which methods are appropriate for certain applications and how particular methods influence results. This presents a clear research gap: "There is no single method appropriate for combining all types of evidence in all situations dealing with epistemic uncertainty" (Helton et al. 2004 pp. 10–26). The purpose of this paper, therefore, is to develop a protocol for the application of Evidence Theory. The goal of the protocol is to provide a method of systematically applying Evidence Theory, enabling an understanding of alternative methods of Evidence Theory application. This paper aims to expand the use of Evidence Theory in practical applications through the identification, demonstration, and discussion of the protocol. The proposed protocol will also facilitate and provide guidance for the performance of sensitivity analysis on Evidence Theory applications. A framework for performing sensitivity analysis is a crucial step in enabling practical engineering applications of Evidence Theory (Oberkampf and Helton 2002).

Background

Review of previous engineering applications of Evidence Theory

Evidence Theory has seen use for uncertainty analysis in engineering applications in recent years. The following section provides an example of many of these applications. The list is not comprehensive, but provides an overview of practical applications of Evidence Theory. These applications cover many topics, including system reliability, structural assessment, natural hazard impact assessment, and multicriteria optimization.

Early application of Evidence Theory was primarily to engineering system safety and reliability. Bogler (1987) investigated Evidence Theory for the fusion of data from multiple sensors on an aircraft. Inagaki (1993) looked at the use of Evidence Theory in decision making using the Challenger space shuttle explosion as an example. Hester (2012) analyzed aircraft maintenance times by combining expert opinions of failure sources using Evidence Theory. Alim (1988) explored the use of Evidence Theory in seismic analysis, motivated by the inherent imprecision of seismic parameters and the frequent use of linguistic labels to confer quantitative data. Agarwal (2004) applied Evidence Theory to optimization, using belief functions as constraints in an example sizing an aircraft subject to performance requirements. Chen and Rao (1998) apply Evidence Theory to multi-criteria optimization as well, analyzing a four-bar mechanical linkage for an optimum path of travel. Fetz et al. (2000) analyze queuing times for transport vehicles given constraints on excavator capacity. Hou (2021) proposed a method of sensitivity analysis in order to obtain an overall view of system level reliability.

Evidence Theory has seen limited publication in fields of applied infrastructure research. Attoh-Okine has published research on the use of belief functions in pavement management systems (PMS) decision frameworks, estimating construction costs, infrastructure re-development, and an urban infrastructure resilience index (Attoh-Okine and Martinelli 1994; Attoh-Okine 2002; Attoh-Okine et al. 2009; Attoh-Okine and Gibbons 2001). Seites-Rundlett et al. (2022) uses Evidence Theory in the prediction of pavement condition from remote satellite imagery. Evidence Theory has been applied in hydrological analysis to incorporate uncertainty (Behrouz and Alimohammadi 2018; Zargar et al. 2012). Evidence Theory has seen applications in predicting transportation planning and traffic analysis (Kronprasert and Kikuchi 2011; Soua et al. 2016; Tarko and Rouphail 1997; Truong et al. 2019). Evidence Theory has also seen applications in instances of data fusion to guide decision making (Cai et al. 2018; Zhao et al. 2010; Zhou et al. 2018).

Evidence Theory has also been applied in instances of performance and structural assessment (Ballent et al. 2019; Bao et al. 2012; Talon Aurélie et al. 2014).

Since its formal definition and introduction, Evidence Theory has garnered interest and research from the Expert Systems and Artificial Intelligence Communities (Denoeux 2000). This interest stemmed from the applicability of Evidence Theory to the realm of uncertain judgment, particularly due to the flexibility of the theory and its wide range of uses in decision-making (Murphy 2000). Many recent applications in machine learning take advantage of Dempster's rule of conditioning and Evidence Theory as a tool for fusing and transforming information into useful output (Denoeux 2019). Applications of Evidence Theory to supervised classification include the evidential K-nearest neighbor rule (EK-NN) (Denœux 2008a), binomial logistic regression (Denoeux 2019), and applying Dempster's rule to combine multiple classifiers into ensemble predictions (Bi et al. 2008). Recent research applications for Evidence Theory in unsupervised machine learning include deep learning and neural networks (Denoeux 2000, 2019; Huang et al. 2021) and clustering (Denœux et al. 2015). These have led to the development of machine learning classification models constructed with Evidence Theory at their base (Chen et al. 2014; Denoeux 2019; Liu et al. 2013).

These applications have led to a deeper understanding of the possibilities and potential of Evidence Theory, and the wide potential for the application of evidence theory to civil engineering problems. They have also led to the identification of a research gap in identifying and developing a methodological protocol for engineering applications.

Evidence Theory

Evidence Theory, as initially conceptualized by Dempster (1968), interpreted statistical inference based on the concepts of upper and lower probabilities, as opposed to the confidence intervals developed by Neyman (Lehmann 2011). The theory was then further developed by Shafer (1976) with his introduction of a theory of evidence based on belief functions. Dempster had interpreted upper and lower probabilities as bounds on degrees of knowledge, however Shafer interpreted these upper and lower probabilities as bounds on degrees of belief, and renamed these limits belief functions. Yager and Liu (2008) provide an historical development of the theory, including a collection of published research critical to its development.

Evidence theory is often described as a generalization of the Bayesian subjective degree of belief interpretation. This is because Evidence theory encompasses aspects of probability theory using set-based mathematical approaches to uncertainty analysis. A distinguishing feature of Evidence Theory, however, is that belief functions allow the calculation of three beliefs, each bounded by 0 and 1: the amount of belief favoring an outcome for any given event, the belief against, and the belief of don't know (i.e., ignorance) (Dempster 2008). This explicit recognition of ignorance as belief to quantify is a special feature of Evidence Theory, freeing it from the probability theory restriction that the probabilities for and against (i.e., its complement) for a given event must sum to unity. In addition, the calculation of beliefs on sets allows information to be applied to a set of events without complete distribution of belief to individual events themselves.

Evidence Theory Definitions

The major terms, methods, and mechanics of Evidence Theory will be defined in this section. The first definition is the frame of discernment, which represents the set of all possible events or outcomes. The frame of discernment (often represented as Ω) is analogous to the sample space of probability theory (Yager and Liu 2008). The frame of discernment represents the power set of possible outcomes, meaning that the set is comprised not of just single elements representing mutually exclusive outcomes, but also compound elements representing one or more possible outcomes. For example, in Figure 1 an example set consisting of three mutually exclusive outcomes {A, B, C} is expanded to the power set used for calculation in Evidence theory. The outer ring (black) represents the singleton events A, B, and C. The inner ring represents the compound elements (i.e., elements that represent all multiple event subsets of the power set) {AB}, {BC}, and {AC}. Compound elements allows the explicit representation of non-specificity or ignorance induced by a given piece of evidence. The inner circle (black) represents a unique compound element, the universal set or Ω, which denotes complete ignorance or lack of belief. The presence of compound elements and the universal set is valuable in the task of recognizing non-specificity in highly uncertain data.

The state of belief induced by relevant evidence or data is represented by assigning mass of belief to each element of the frame of discernment. The function for assigning mass of belief is known as the Basic Belief Assignment (BBA), mass function, or Möbius Measure. The term BBA will be used to discuss this function hereafter and its typical representation is m(A) = X, defined as set A has been assigned a mass of belief equal to X. The value of the mass of belief assigned to any given element must be between [0, 1] and all masses of belief assigned across the entire frame of discernment must sum to unity [1.0]. The individual BBAs are said to be normalized when the summation to unity is achieved.

The state of belief induced by relevant evidence can also be represented by functions other than BBAs, such as the belief function (Bel), the plausibility function (Pl), and the commonality function (Q). Each of these functions has advantageous properties in describing the information encompassed in the state of belief in certain situations. However, fundamentally, each of these functions is an equivalent representation of the state of belief, and the transformations between each are accomplished using BBAs. These other functions appear best suited for efficiently performing certain calculations (Reineking 2014). One may consider BBAs, however, as the mathematical foundation of Evidence Theory and as such, all discussions here of assigning belief using Evidence Theory will use them as the basis of discussion. The choice of using BBAs to define belief assignment is motivated by the similarities between BBAs and classical probability measures and the desire to allow the reader to more readily compare the approach applying Evidence Theory to the approach applying probability theory. The choice of defining belief assignment using BBAs or any other belief function would have no impact on the selection of the combination method or the outcome of the analysis.

A number of terms have been defined to describe specific belief structures in Evidence Theory. Any element of the frame of discernment with a BBA greater than 0, e.g., m(X) > 0, is called a ‘focal’ set or element. A

BBA that assigns all belief (1.0) to an element of the frame of discernment other than the universal set is a logical belief and represents certainty. A BBA that assigns all belief (1.0) to the universal set (Ω), and therefore no belief (0.0) to other elements, is a vacuous belief, and represents total ignorance. If the assignment of all belief is to singleton sets, which represents only one unique possible outcome each, then the state of belief is Bayesian, and this represents the situation where the mathematics of Evidence Theory reduce to that of Bayesian Theory. Table 1 summarizes these belief structures, in addition to other names for common specific belief structures, using definitions provided by Denoeux (2006) and Yager and Liu (2008).

Dempster's Rule of Combination

The combination of evidence holds a central role in the application of evidence theory, particularly when combining data from multiple sensors or opinions from multiple experts. The original method for combining the belief induced by two or more pieces of evidence is Dempster's rule of combination. To calculate the combined mass of belief for each element of the frame of discernment, Dempster's rule of combination multiplies the mass of belief assigned to sets whose intersections are not empty, and then sums them, as shown in Equation (1). Dempster's rule therefore represents a Boolean conjunctive rule for combination.

$$m_{1,2}(A) = \frac{\sum_{B\cap C=A} m_1(B)\cdot m_2(C)}{1-c}, \qquad where\ c = \sum_{B\cap C=\emptyset} m_1(B)\cdot m_2(C) \tag{1}$$

Where, *c* is conflict; *A, B, C* are symbolic representations for different sets. BBAs are represented by m(X), with $m_1(X)$ representing the first piece of evidence, $m_2(X)$ representing the second piece of evidence, and $m_{1,2}(X)$ representing the combined result. $B\cap C=\emptyset$ means that sets B and C have no intersections.

The nominator of Equation (1) is the combined belief before normalization. If there is any conflict (c), defined as the mass of belief associated with sets whose intersections are empty, the combined mass of belief for each non-empty set after combination is proportionally normalized so that the sum of the mass of belief for all elements of the frame of discernment (Ω) is 1. Note that Dempster's rule is both commutative and associative. Therefore, for combinations of greater than two independent sources of evidence, one can execute a regression series of combinations, incorporating each unique belief function structure into the combined result.

The sets represented by A, B, and C could be, for example, different states of condition (e.g., good, fair, and poor condition). One of the important features of the rule of combination is the Boolean relationship identified in the summation. $B \cap C = A$ means the common elements that intersect in sets B and C are fully included in the set A. $B \cap C = \emptyset$ means that sets B and C have no intersections (i.e., the conjunction of sets B and C produces the null set), and are thus omitted.

Dempster's rule implies 'subjective independence' among the distinct pieces of evidence combined. Subjective independence requires that the evidence does not share a common source of uncertainty. Therefore, two different outputs from the same sensor cannot be considered subjectively independent, as the uncertainty of the output is dependent on the sensor's functioning. The intent of the independence requirement is that no piece of evidence is counted twice. Therefore, one should apply Dempster's rule only to combine distinct, independent information. It is not appropriate to apply Dempster's rule to synthesize redundant, repetitive, and overlapping information.

Conflict in the Combination of Evidence

One interesting feature of Evidence Theory and Dempster's rule of combination is that it allows the quantification of the conflict between the pieces of evidence being combined. Indeed, one must always identify conflict and use it for normalization in the application of Evidence Theory. One of the primary differences among competing methods of applying Evidence Theory is the treatment of conflict. Conflict, as defined in Evidence Theory, will be present when combining beliefs held in mutually exclusive outcomes. Therefore, holding more belief in singleton outcomes will increase conflict (because singletons cannot share an intersecting set) compared to holding belief in less specific, compound sets that share intersections (e.g., AB, BC, AC, ABC). Conflict can be found when the evidences to be combined are in agreement or disagreement. In cases of agreement, there may be internal conflict. Internal conflict is possible in situations when belief is not held in intersecting sets or when belief mass is assigned to at least two elements of the power set of the frame of discernment besides the universal set (Yager and Liu 2008). Internal conflict results when beliefs for some of the power set events lead to basic belief assignments for mutually exclusive events. Disagreement will produce external and typically larger conflict. These concepts can be illustrated with a simple example in which experts estimate the winner of a race. Only one person can win any given race, and the full belief in a winner will be distributed among the various participants. If two independent experts provide predictions of the outcomes by spreading their belief among the participants, and their predictions are combined with Evidence Theory, there will necessarily be conflict (i.e., internal conflict) (Martin et al. 2008). Using this same example, external conflict derived from a disagreement may exist if one expert places majority belief in Runner A and another expert places majority belief in Runner B.

Reliability and Weights of Evidence

The final core concept of Evidence Theory warranting discussion is the reliability function. The reliability function is a characteristic of the evidence used to define the mass of belief and a belief function structure. The reliability function, with values ranging from 0 to 1, is intended to be combined with the belief function structure to yield an estimate of the total information embodied by the evidence. Reliability could be based on objective specifications (e.g., when prior data are available to mathematically define reliability) or subjective judgment (e.g., when using expert opinions). Reliability, therefore, represents a justification for weighting different pieces of evidence, a concept discussed within the Evidence Theory literature (Shafer 1990; Smets 1992; Yager and Liu 2008). Shafer initially defined the term 'weights of evidence' for the application of a discounting function to Evidence Theory. The concept of weight of evidence, as defined above, is additive when used in conjunction with Dempster's rule, allowing a simple calculation of reliability when multiple pieces of evidence are to be combined. The reliability concept is also equivalent to discounting methods discussed within Evidence Theory. Discounting reduces specificity by moving mass of belief into the universal set to account for unreliable information embodied in evidence (Yang and Xu 2013).

Combination Methods

Previous publications (e.g., Oberkampf and Helton 2002; Reineking 2014; Sentz and Ferson 2002; Smets 1992; Yager and Liu 2008) document well the multitude of combination methods within the field of Evidence Theory. The consequence is that a plethora of combination methods have been developed (Smets 2007), and it is unclear which to apply with Evidence Theory for practical problems involving uncertainty traditionally handled by probabilistic methods. The fundamental consideration here is that different combination methods produce different results, most notably when the number of combinations is increased, and therefore guidance is required about when to use and avoid certain rules.

Rather than trying to determine a priori which combination method is superior, the most important concept to consider is the implication of each, and the relationship of that to the goals of an analysis. One important consideration is that many competing methods are related to each other. For example, many methods incorporate Dempster's rule at their base and primarily differ in the normalization of conflict and the distribution of belief mass to different elements of the frame of discernment (Sentz and Ferson 2002; Smets 2007). Different methods of normalizing conflict or distributing belief masses introduce non-Boolean and case-specific properties to some methods. This makes it clear that the Evidence Theory methods represent a spectrum between precision and explicit recognition of uncertainty. Bayesian updating and Dempster's rule in its original form represent one end of the spectrum, which does not explicitly account for uncertainty but presents precise and repeatable methods of application. Non-Boolean and case specific 'ad hoc' methods of applying Evidence Theory represent the other end of the spectrum, where uncertainty is explicitly incorporated into the analysis, but the result may lack precision, context, or the ability to incorporate further evidence (Sentz and Ferson 2002; Smets 2007).

The evaluation of results must address both ends of the spectrum between precision and explicit recognition of uncertainty in order to avoid making quasi Type-I and quasi Type-II errors in the application of Evidence Theory. Quasi Type-I errors represent instances where a false or uncertain outcome is favored among the results, such as in Zadeh's example (Zadeh 1984). Quasi Type-II errors represent instances where a true or certain outcome is not selected because belief is too widely distributed, reflecting the practicality concerns of Webb and Ayyub (2017). Both of these errors arise from the nature of the initial assignment of belief masses, the combination rule selected, and the applied method of conflict normalization.

The next subsections present a summary of the most common alternative combination methods and a discussion of their relation to each other. Table 2 presents a summary of the attributes of these different methods. Note, that the discussion of alternative combination methods only addresses common rules intended for application to independent and distinct sources of evidence. Alternative methods that handle dependent and non-distinct sources of evidence are discussed, but are not included in the guidance provided, in order to maintain a concise scope addressing the common combination rules in Evidence Theory.

Dempster's Rule

Dempster's rule of combination is appealing due to certain characteristics. Primarily the fact that it is commutative and associative, meaning that the order information is received does not matter. This rule has shortcomings, as new evidence given complete reliability can significantly alter prevailing beliefs.

The mechanics of Dempster's rule concerning normalization of conflict have a significant impact on the results. The most notable effect is that the conflict normalization produces convergence toward the dominant opinion and increases the specificity of the result (Murphy 2000, Ballent et al. 2019). Notably, if any piece of evidence to be combined is represented as a completely Bayesian belief structure (i.e., all belief held in mutually exclusive singleton sets), then this belief structure is repeated in the result, thereby restricting the ability to calculate ignorance or non-specificity in the outcome. Multiple combinations of evidence will converge belief mass towards certainty because this process is repeated over and over again with multiple combinations. The effect of this combination rule is to accumulate belief mass in singletons as opposed to compound elements of the power set of the frame of discernment. Results published in Ballent et al. (2020) show that a belief of 0.15 in an outcome can converge to 0.92 after 20 experts' opinions are combined. Although this convergence behavior has been noted as an advantage of Evidence Theory to converge toward likely outcomes and reject spurious sources of information (Appriou 1997), it is important to consider whether such convergence reflects the desired behavior. A famous critique of Dempster's rule is Zadeh's paradox, where the results converge to an unintuitive result due to significant conflict (Zadeh 1984). Such limitations create the perception that Evidence Theory is best applied summarizing the current state of knowledge, not updating statistical evidence (Oberkampf and Helton 2002). It is important to consider and be sensitive to the fact that any analysis requiring multiple combinations or continual updating will lead toward convergence if Dempster's rule is applied. Furthermore, as with some of the other combination rules, assigning a zero belief to any element of the power set causes a veto effect on beliefs from other sources. This means that assigning zero belief to a given element of the frame of discernment effectively 'vetoes' any potential for that element to hold or be assigned belief after combination. Therefore, the veto effect may produce a quasi-Type I error if the true outcome is incorrectly assigned zero belief by one of the sources to be combined, as one of the other, presumably false, outcomes will necessarily be identified as the favored outcome after combination.

Yager's Rule

The most prominent modification of Dempster's rule of combination is Yager's rule (Sentz and Ferson 2002; Yager 1987). Yager's rule is a modification of Dempster's rule that allocates all conflict to the universal set, Ω (Equation 2). Doing so loses the desirable associative property of Dempster's rule. However, Yager's rule does not require normalization methods and assumes that conflict in reliable pieces of evidence is equivalent to ignorance, therefore moving this belief mass to the universal set. Notably, Yager's rule was developed to address the issue of applying Evidence theory in the role of updating statistical evidence (Sentz and Ferson 2002). Moving conflict to the universal set retains non-specificity in the combination and reduces the potential for future conflict in successive combinations.

$$m_{1,2}(A) = \begin{cases} \sum_{B \cap C = A} m_1(B) \cdot m_2(C), & when\ A \neq \Omega, A \neq \emptyset \\ m_1(\Omega) \cdot m_2(\Omega) + \sum_{B \cap C = \emptyset} m_1(B) \cdot m_2(C), & when\ A = \Omega \\ 0, & when A = \emptyset \end{cases} \quad (2)$$

Conjunctive Rule

The conjunctive rule is central to the transferable belief model (TBM) developed by Smets (1990). This rule is a modification of Dempster's rule that does not require normalization of conflict. Instead of normalizing, the belief mass associated with conflict is allocated to the null set. The TBM framework then provides mechanisms for transferring belief mass held in compound elements to singleton elements for the purpose of decision-making. For a complete discussion of the TBM framework and its associated equations see Smets and Kennes (1994). This method is advantageous because the amount of conflict in each combination is retained and cumulative, whereas Dempster's rule only summarizes conflict in a single combination at a time. The rule, however, is not commutative and leads to convergence of belief in the null set. Therefore, this rule may not be appropriate in applications of significant conflict and repetitive combinations over time (Reineking 2014).

Disjunctive Rule

The disjunctive rule initially proposed by Dubois and Prade (1986) provides an alternative to the conjunctive-based approach of Dempster's Rule. Fundamentally, Dempster's rule and similar conjunctive rules apply 'AND' operations to sets holding belief assignments, while the disjunctive rule applies 'OR' operations to these sets. Equation 3 provides the definition of the disjunctive rule. The disjunctive rule shares a relation to Dubois and Prade's Rule (conjunctive based) as the joint of the basic probability assignments is assigned to the product of the marginals in combination (Sentz and Ferson 2002). Therefore, the disjunctive rule does not calculate conflict and apply normalization as in other combination methods.

$$m_{1,2}(A) = \sum_{B \cup C = A} m_1(B) \cdot m_2(C), \quad when\ A \neq \Omega, A \neq \emptyset \quad (3)$$

The disjunctive rule is intended for an application of mutual discounting of sources, where it is assumed that only one source is reliable. One limitation of this rule is that it is considered the most imprecise of the combination methods (Sentz and Ferson 2002). However, the conjunctive rule does play an important role in the calculation of conditional belief functions (Reinicken 2014). The calculation of conditional belief functions allows the combination of overlapping but non-identical frames of discernment through the assignment of combined belief to the product the focal sets (i.e., the sets holding belief) using the 'OR' operator of the disjunctive rule.

Proportional Combination Rules

Proportional combination rules (PCR) are central to the Dezert-Smarandache Theory (DSmT) for information fusion (Smarandache and Dezert 2005). The PCR rules first apply Dempster's rule, then calculate each partial conflict arising from the combination of any two mutually exclusive focal sets, and finally apply methods to redistribute each partial conflict proportionally. PCR rules represent non-Boolean solutions, because they account for conflict by introducing rules for redistributing belief mass associated with partial conflicts. There have been multiple rules proposed, each intended to maintain certain properties of Dempster's original rules, such as commutativity. PCR Rule 5 is considered the most mathematically exact redistribution of conflict by Smarandache and Dezert (2005) and will be applied in a later example. For brevity the equations associated with PCR Rule 5 are not reproduced here and the reader is referred to Smarandache and Dezert (2005) for further discussion and complete mathematical definitions.

Dubois and Prade's (Conjunctive) Rule

Another prominent modification of Dempster's rule is Dubois and Prade's Rule (Dubois and Prade 1986). Similar to Yager's rule, the mass of belief associated with conflict is not normalized, and instead is moved to coarser elements of the frame of discernment. Equation 4 below describes the method. In Dubois and Prade's rule, conflicting belief mass is moved to the set corresponding to the union of the individual sets producing the conflict. For example, if belief assigned to both element A and element B produces conflict in combination, then the value of conflict is assigned to the joint set, AB.

$$m_{1,2}(A) = \begin{cases} \sum_{B\cap C=A} m_1(B)\cdot m_2(C) + \sum_{B\cap C=\emptyset, B\cup C=A} m_1(B)\cdot m_2(C) & when\ A\subseteq\Omega, A\neq\emptyset \\ 0, \quad when A = \emptyset \end{cases} \tag{4}$$

Additional Methods

There are many additional methods that have been proposed and applied to combine data using Evidence Theory. One primary field of research is into the combination of dependent, non-distinct data that do not meet the subjective independence requirements of the rules identified above. The principal rules for combining dependent information are the conjunctive cautious rule, the normalized cautious rule, and the disjunctive cautious rule (which are parallel alternatives to the conjunctive rule, Dempster's rule, and the disjunctive rule described above).

The cautious rule was created to address an assumption of Dempster's rule, that evidence to be combined must be distinct and subjectively independent. This assumption was intended to prevent any piece of information from being counted twice (Denoeux 2006). Therefore, the cautious rule was designed to be idempotent, that is, the combination of a belief structure with itself will reproduce the original belief structure. The cautious rule is accomplished by calculating weight values in w-space, an alternative representation of BBAs calculated using the commonality function. The method takes the minimum weight of evidence when combining non-distinct pieces of evidence. Therefore, only the minimum support for a given element of the frame of discernment is retained in combination, as opposed to a convergence of belief as observed in Dempster's rule.

Another method of combining data using evidence theory uses the averages of combined beliefs to provide context to Evidence Theory predictions and results. Murphy (2000) studied the tendency of Dempster's rule to either converge to certainty or veto a majority of opinion. Among alternative methods to address these problems, averaging was found to identify unintuitive combinations, showing an alternative distribution of belief.

Although additional combination methods may be useful to any given analysis, these methods will not be applied and discussed further in this paper. This paper will only address the combination of distinct pieces of evidence, which are assumed to meet the subjective independence requirement of Evidence Theory. The identification of impacts on results and guidance for applying these additional rules is considered a topic for future research. For published papers discussing the application of the cautious rules see Denoeux (2006, 2008b).

Combination Methods Summary

Table 2 summarizes the common combination methods discussed. Brief guidance is provided concerning when to use and when to avoid certain rules if conditions are met. The table also identifies particular properties that are implicit in each rule and how the methods differ from the fundamental Dempster's rule.

Methods – Protocol Concepts

This paper has considered a wide assortment of published Evidence Theory literature. We postulate that it is possible to map specific evidence combination methods to specific contexts under which they are applicable. The goal of this paper is to develop a protocol for the application of Evidence Theory to practical problems in the domain of Civil Engineering. In order to develop a protocol, it is necessary to link research gaps discussed above to specific steps in the process of applying Evidence theory. Users of Evidence Theory could utilize such a protocol to guide the proper application of certain analytic methods and improve the applicability and understandability of Evidence Theory.

The unique challenge in the development of the protocol for engineering applications, is to clarify the difference in methods and the interpretation of the results between Evidence Theory and Probability Theory. The primary difference in application between Evidence Theory and Probability Theory is in the pre-processing of data. While Probability Theory must pre-process data in order to fit axioms and constraints of probability theory, Evidence Theory does not require pre-processing of data and can address uncertainty in the data through the development of the frame of discernment and initial assignment of belief masses.

Furthermore, engineering applications must consider the uncertainty embodied in the results and provide a discussion of sensitivity analysis to justify methods applied and decisions made using the model. The proposed protocol addresses this unique challenge by introducing a method to understand the selection of combination method and sensitivity of results.

The key to developing such a protocol is to highlight the commonalities and implications of various existing methods. The definition of terms and discussion of combination methods raised several key concepts to consider when applying evidence theory. These concepts include:

- Definition of the frame of discernment
- Initial assignment of belief mass
- Single or multiple combinations
- Conflict normalization and combination rule selection
- Reliability weighting and sensitivity analysis

Discussion of these five concepts will highlight important implications when evaluating information and making decisions using belief functions. The following sections summarize topics and questions requiring extra attention and clarity relating to these concepts. The flow diagram of Figure 2 provides a visual reference to the methodological steps in the protocol.

Discussion of Concepts

Definition of the Frame of Discernment

An important consideration in the development of the frame of discernment is the granularity of data. Traditional uncertainty analyses based on probability theory assign belief only to singletons, and therefore seek to obtain and process data to the finest granularity possible. An analysis based on probability theory using data with insufficiently coarse granularity must apply assumptions and methods, such as the principle of insufficient reason or interpolation techniques, to process the data to a granularity in agreement with the goals and outcomes upon which decisions and predictions must be made. Evidence Theory, however, is more tolerable to the incorporation of coarser granularity data, owing to the compound sets included in the frame of discernment. The combination methods of Evidence Theory then work to converge belief from less specific coarser sets to more specific finer outcomes and decision points. The definition of the frame of discernment therefore represents a unique difference in approach when applying evidence theory as opposed to probability theory. Additionally, the definition of the frame of discernment enables the incorporation of qualitative and heterogeneous data sources. If these data can be associated with sets defined within the frame of discernment, then initial belief masses can be assigned, and the data can be incorporated into an analysis.

The evidence theory approach is not concerned with processing the data to the granularity needed, but rather evaluating the data available to determine which and how many compound sets to include in the frame of discernment. The combination rules of evidence theory lend themselves to data bearing on these compound sets. Many of the combination rules, however, either generalize to, or do not offer improvement over, prevailing probability theory-based methods when dealing with Bayesian belief structures, where belief is only assigned to singleton elements of the frame of discernment. Another term for such belief structures is dogmatic, i.e., there is no basic belief assigned to the universal set (Ω). However, the application of Evidence Theory in such instances is justified by the argument that belief and evidence are not certain, and all belief should be represented by so-called non-dogmatic belief functions, where some belief is assigned to the universal set (Denoeux 2008b). The user could also ask if they are justified by placing some of this discounted belief in compound sets, and which compound sets are therefore necessary to define. This process forms the core of applying reliability weighting and performing sensitivity analysis to be discussed in below.

Further considerations in the definition of the frame of discernment are whether variables of interest are discrete or continuous and whether they are bounded. In the case of discrete and finite applications, the definition of the frame is less flexible and open to less interpretation. In the case of continuous and infinite applications, definition of the frame is more flexible and subject to additional scrutiny and consideration of the decision consequences. One of the major concerns of Evidence Theory is that adding additional elements to the frame of discernment increases computational complexity (Reineking 2014).

Initial Assignment of Belief Masses

The initial assignment of belief masses follows the definition of the frame of discernment, as there is a need to define belief among the elements of the power set of the frame of discernment. The initial distribution of belief masses can have impacts on the outcome of the analysis. One common method providing guidance for this step is the Least Commitment Principle (Denoeux 2019). According to this principle, when selecting among several equivalent initial assignments of belief, the least informative shall be selected. A mathematical definition may be provided to further specify the application of this rule; however, the general guidance stands as the most widespread approach to selecting initial assignment of belief masses. The general guidance suggests that a belief structure with more belief assigned to less-specific compound sets will be less committed that one with more belief assigned to specific singleton elements. One should also note that assigning all belief from one source to the universal set will not affect the combined beliefs from additional sources (Dezert and Tchamova 2011). Thus, such belief from one source will have negligible effect on the prevailing state of beliefs from all sources.

Single or Multiple Combinations

The applicability of evidence theory to a process of continual updating given new evidence is an open question in the field. The primary concern when performing multiple combinations involving Evidence Theory is the nature of Dempster's Rule to produce convergence towards a favored outcome. Because of this, previous researchers viewed Evidence Theory as inapplicable in domains of continuous updating. For example, "Evidence Theory does not embody the theme of updating probabilities as new evidence becomes available…in Evidence Theory the emphasis is on accurately stating interval valued probabilities given the present state of knowledge." (Oberkampf and Helton 2002 p. 3). However, recent applications of Evidence theory to classification and neural networks (e.g., Denoeux 2019) have demonstrated a role for Evidence Theory in applications of repetitive updating given new information. Given that many Civil Engineering applications require continuous updating, the application of Evidence Theory must include this capability. The identification of whether there will be a single or multiple combinations is a logical consideration for users of Evidence Theory.

Conflict Normalization and Selection of Combination Rule

Conflict normalization is a critical component of decision making with Evidence Theory, as normalization is necessary in order to transform belief masses into probabilities for use in prediction or secondary mathematical analysis. Conflict normalization is also the distinguishing feature among the different conjunctive combination methods (see Table 2). For example, "The issue of conflict and the allocation of the BBA mass associated with it is the critical distinction among all of the Dempster-type rules" (Sentz and Ferson 2002 p. 16). One must define the method of conflict normalization and the impacts of this method on the overall goals of the analysis in any application of evidence theory.

Conflict quantification can be used to redistribute belief mass to specific sets, such as in the case of Yager's rule. Internal or external conflict can be addressed differently, and conflict quantification can be logically linked to threshold values, thereby addressing small conflict and large conflict combinations differently.

One of the primary importance of incorporating the concept of conflict normalization into the protocol is to introduce a common practice of performing sensitivity analysis on conflict normalization rules.

Reliability And Sensitivity Analysis

The reliability of a piece of evidence, often referred to as 'weights of evidence' in the published literature, is often reflected by applying a discounting function. The discounting function moves belief mass to less specific (i.e., compound) sets, e.g., the universal set (Ω), thereby creating a less committed belief function structure. The application of discounting functions is especially important to address conflict between sources. Applying a discounting function and moving belief mass to less specific compound elements such as the universal set will reduce the amount of conflict in a combination using Dempster's rule. Investigations into different methods of applying reliability functions has a clear parallel to training model weighting parameters when applying common machine learning algorithms to data.

The application of a reliability function is well suited to play a major role in follow-up sensitivity analysis. The application of a reliability function provides the user with the ability to manipulate the initial belief mass assignments before combination, thereby offering the opportunity to address any potential complications or unintuitive results that arise from multiple combinations.

Summary

The goal of defining a protocol is motivated to allow a user to be aware of applicable methods of data processing (pre- and post-) and combination given the desire to update existing beliefs or make decisions using Evidence Theory. The protocol is not intended simply to establish a prediction tool, whereby evidence is gathered in order to produce a prediction as output subject to the mathematics of Evidence Theory. The definition of a protocol, rather, facilitates a secondary analysis, which evaluates the information embodied within the results of an uncertainty analysis applying evidence theory. The secondary analysis is of crucial importance and aligns with the research focus of using evidence theory for exposing uncertainty and ignorance embodied within an analysis. The secondary analysis could be a programmed algorithm or expert system that evaluates belief function structures for specific tasks. The point is that the belief functions themselves are not the ultimate step of applying Evidence Theory to uncertainty evaluations, but rather the building blocks.

The development of the protocol is also intended to provide a framework for performing sensitivity analysis. Each concept provides a means to perform sensitivity analysis and determine the implications of decision made in assigning belief mass and combining evidence. The lack of a common approach to sensitivity analysis is a major research gap to be addressed for the widespread adoption of evidence theory to practical engineering applications (Oberkampf and Helton 2002).

Belief functions present information about the nature of the uncertainty considered and evidence available. The development of a protocol, enabling common application methods and sensitivity analysis, allows for an explicit understanding of assumptions and actions made when applying Evidence Theory. The definition of such a protocol also allows the user to consider all possible combination rules, non-Boolean algebras, and calculation methods. Any calculation method, considering that one can define situations when it is and is not applicable, can be incorporated into such a protocol. The advancement of Evidence Theory with such a protocol, therefore, goes beyond the definition of any specific elegant calculation method, because the ultimate product of this protocol development is a more collaborative and mutually understood means of applying Evidence Theory to practical problems. In the next Section, the steps of the protocol will be developed, along with a practical example demonstrating the concepts.

Commentary on Theory Implications

Definition of Example Problem

The guidance embodied in the proposed flow chart will be demonstrated through the discussion and presentation of an example problem. The data chosen for use in the example is from a post-disaster structural damage assessment survey from Ballent et al. (2019). The survey includes 5 different images of Haiti taken shortly after the 2010 earthquake that occurred in the country. The goal is to estimate the amount of destruction (from 0% to 100%) in the area of the image. The participants evaluated each image to assess their belief that the area of the image sustained damage in the range of 0-33%, 34-66%, 67-100%, 0-66%, and 34-100%. The survey asked participants to first assign belief in the smaller ranges (i.e., 0-33%, 34-66%, 67-100%). In the event the participant is not confident assigning all of their belief in these smaller ranges, the remainder of belief was to be assigned to the larger ranges (i.e., 0-66%, and 34-100%). The survey collected 40 valid responses, and combined these into five groups of eight responses each. Ground inspection was also performed at the site of each of the five images used in the survey, so that the actual damage range could be ascertained for each case. These data were chosen because they represent a past application of evidence theory with both expert opinions and the observed real damage amount. The simple example will be used to demonstrate the implications and sensitivities of certain decisions made within the proposed protocol. The results of the survey for a particular image are summarized in Table 3. The results for each group represent a combined belief of eight valid survey responses. The column 'All 40' denotes the combination of all 40 survey responses. Note, that the survey results identify both the belief values (calculated using BBAs and the belief function) and BBAs (*m* values). The juxtaposition of belief and BBA values highlights the difference in data representation when selecting among possible alterative representations of belief. In Table 3, boldface cells are used in the initial assignment of belief mass.

Definition of Frame of Discernment

The frame of discernment is dictated by the survey question. The singleton elements are the smaller ranges.

[0, 33%],[34,66%],[67,100%]

The compound elements represent the possible combinations of the singleton elements.

[0,66%],[34,100%],[0,33%)∪(67,100%],[0,100%]

It is notable that the compound elements [0,33%)∪(67,100%] and [0,100%] (i.e., the universal set) are not included in the survey, but are included in the frame of discernment. The inclusion of these sets in the frame of discernment, however, is necessary in order to apply the methods of evidence theory, including calculating belief and plausibility functions and applying certain combination rules (such as Dubois and Prade's rule or Yager's rule).

Initial Assignment of Belief Masses

The initial assignment of Belief mass is dictated by the survey results. For purposes of this example, two groups of survey results will be combined using Evidence Theory. The two groups chosen to represent the sources of evidence are Group 2 and Group 3 identified in Table 3. Group 3 was chosen, because this group distributes their belief among the possible outcomes most uniformly. Group 5 was not chosen because it assigns the entirety of its belief to one outcome, thereby evoking the veto principle. Group 2 distributes belief in agreement with the other groups, which place most of their belief in one outcome. The choice of Group 2 among the remaining groups was then arbitrary, as either Group 1 or 4 could have also been selected and produced similar results when combined with Group 3.

The compound elements $[0,33\%)\cup(67,100\%]$ and $[0,100\%]$ (i.e., the universal set) are not included in the survey, and therefore no initial belief is explicitly assigned to these sets. The impact of the lack of initial belief assigned to these elements will be discussed in the continued analysis of the example problem. The survey does permit participants to assign less than 100% of their belief, allowing for an indirect initial assignment of belief to the universal set. The presence of only an indirect path for the assignment of belief to the universal set impacts the initial belief assignments, because belief assignment to the compound and universal sets will necessarily be minimal, as observed in the low values (max 0.03) from the survey results above.

Single or Multiple Combinations

The survey data includes 40 valid responses, and the intent is to combine all 40 responses together to evaluate the effect of such a large combination. Therefore, this application represents multiple combinations and we must evaluate the presence of zero belief assignments (step 2a.i) and the potential for conflict (2a.ii).

The evaluation of zero belief assignments reveals that the larger damage ranges (compound sets) all are assigned zero or near zero initial belief. This therefore represent a near Bayesian belief structure (See Table 1). We use the term “near” here because the negligible amount of belief initially assigned to the universal set (0.01 or 0.03). The impact of the Bayesian belief structure is a constraint on the results, thereby restricting the ability of the compound elements (i.e., larger damage ranges) to hold belief after a combination using Dempster’s rule. Therefore, without any modification to the evidence, one would expect the assignment of zero belief to the compound sets to produce high belief assignments to singletons after combination.

Evaluating the potential for conflict, it is necessary to review the initial belief mass assignments held by the singleton elements. Since the majority of belief is held by the singleton elements and distributed among them (i.e., internal conflict,), there is significant potential for conflict. Since belief in the larger ranges is only requested after belief is first assigned to the smaller ranges, nearly all belief sits in the smaller ranges to start. This is a common occurrence, for example when attempting to convert a previous probability theory analysis to evidence theory. The presence of significant potential for conflict will drive convergence behavior, particularly with negligible belief assigned to the compound sets.

Selection of Combination Rule

The example will be continued by the combination of the two initial belief assignments summarized in Table 3. The two initial belief assignments differ slightly. Group 2 assigns belief strongly favoring the lowest damage range, while Group 3 distributes their belief assignments more among the alternative singleton damage ranges, while still favoring the lowest damage range. These two survey responses will be combined using each of the rules identified in Table 2. The results of this combination are summarized in Table 4.

Reviewing the results of Table 4 produces some notable observations. First, the calculated conflict is 0.43, and therefore a good amount of the belief to be assigned after combination must either be normalized or redistributed. As anticipated, the assignment of the majority of belief to the singletons produced results heavily favoring the singletons. The application of Dempster's rule and its normalization method produces convergence behavior, as the combination shows a 0.99 assignment of belief in the lowest damage range after combination, which exceed the belief assigned this range by either piece of evidence (0.55 and 0.97, respectively). Yager's rule mitigates this convergence by assigning conflicting belief into the universal set. The Conjunctive rule prevents this convergence by assigning conflicting belief to the null set, indicating the possibility of unaccounted for outcomes. PCR Rule 5 produces similar convergence behavior to Dempster's rule, but owing to its mechanics for partial redistribution in lieu of normalization, the combined estimate (0.91) does not exceed the highest belief assigned by either of the pieces of evidence (0.97). Dubois and Prade's (conjunctive) rule and the Disjunctive Rule avoid a convergence outcome by assigning conflicting belief to the compound ranges associated with partial conflicts. The belief assignments in these compound sets are informative as to the nature of the partial conflicts, particularly when compared to the results of Dempster's rule. Finally, the Disjunctive rule is nearly equivalent to Dubois and Prade's rule, only differing in a small amount of belief assigned to the universal set after combination.

In order to demonstrate trends in the application of the combination rules, the initial belief assignments of Table 3 were modified in order to perform additional combinations. The process of redistributing belief is fundamental to applying reliability discounting and performing sensitivity analysis, see below for further discussion. The initial belief assignments of Table 3 were adjusted to reassign belief from the singleton element [0,33] to the compound sets that both include this range, namely [0,66] and [0,33)∪(67,100]. This was achieved by reducing 0.5 assigned belief from [0,33] and assigning 0.25 belief to [0,66] and [0,33)∪(67,100], respectively. This is an illustrative example and these values we chosen to demonstrate the effect of holding belief in singleton versus compound sets. The modified initial belief assignments are summarized in Table 5. Similar to Table 4, the modified initial belief assignments are combined with each of the rules identified in Table 2. The results of the combination are summarized in Table 5.

Reviewing the results of conflict in Table 5 reveals interesting trends. First, the calculated conflict is 0.32, which is lower than the amount of conflict in the Table 4 combinations. This reduction of conflict is expected when assigning more belief to compound sets. The nature of convergence moving belief assignment from compound to singleton sets is on display here as well. The total belief assigned to the compound sets before combination is 0.5, but is a maximum total belief of 0.2 (0.1 maximum for belief associated with any individual compound set) for Dempster's rule, Yager's rule, the Conjunctive rule, and PCR rule 5. Only Dubois and Prade's Rule and the Disjunctive rule assign more 0.1 belief to any of the compound sets, due to their assignment of belief associated with partially conflicting belief assignments to the union of the conflicting sets.

In order to further explore trends in the application of the combination rules, the initial belief assignments of Table 4 were modified in order to perform additional combinations. The initial belief assignments of Table 4 were modified to reassign half of the remaining belief from the singleton elements to the universal set. The modified initial belief assignments are summarized in Table 6. Similar to Table 4 and Table 5, the modified initial belief assignments are combined with each of the rules identified in Table 2. The results of the combination are summarized in Table 6.

Reviewing the results of Table 6, conflict is now reduced to 0.11 (compared to 0.43, then 0.32 in the previous combinations). This again reinforces the influence on conflict when assigning more belief to compound sets, including the universal set. One can also notice how the additional belief assigned to the universal set now produces greater belief assignments in the other compound sets. For example, the compound sets [0,66] and [0,33)∪(67,100] now retain most of their initially assigned belief (0.42 out of 0.50) after combination with Dempster's rule. With so much belief assigned to the compound sets, there is now much less convergence towards the singletons. None of the combination rules assign more than 0.40 belief after combination to the set [0,33], although this outcome is still favored. Most interestingly, the results for all of the rules (excluding the Disjunctive Rule) are now more in agreement as compared to Table 4 and Table 5. This highlights the focus of evidence theory on handling coarser granularity data and the applicability of these rules when belief is assigned primarily to the compound sets. The imprecise nature of the disjunctive rule is also on full display in Table 6, as after combination the singletons retain negligible belief and the universal set is the favored outcome.

The results of the combination examples above can also be compared to average survey responses and actual damage in order to evaluate the combination rules (actual damage results were available following ground inspection, see Definition of Example Problem above). Inspection of the survey averages reveals that the estimates of the damage range were far more distributed than the initial belief assignments suggest. Comparison of the survey averages in Table 7 with the original combination results in Table 4 show how the convergence towards certainty in the [0,33] damage range fails to capture this distributed belief and lack of uncertainty among the survey responses, despite the fact that the combinations converge to the actual damage range. This highlights the value of alternative methods, such as Yager's rule retaining belief in the universal set or Dubois and Prade's rule placing belief associated with partial conflicts in compound sets. The more distributed results when applying these rules reveal the lack of certainty in the survey responses.

Inspection of the averaged combined belief (Table 7) and the combined belief for all 40 survey responses (Table 3) also highlights the impact of assigning zero belief and the veto principle. Although the averaged beliefs show there was considerable belief assigned to the damage range [34-66], the combined result produces zero belief in any set including the range [34-66]. One survey response for which zero belief is assigned to this range is sufficient to produce this result and 'veto' any possibility that the truth is in this range. In this example, this result can be justified because the actual damage is in the [0-33] range. However, such circumstances repeated in another scenario could cause an analysis to reject and place zero belief in what could be the actual outcome (i.e., a quasi-Type I error), it is therefore necessary to evaluate and review instances of zero belief assignment before, during, and after performing combinations of data using Evidence Theory.

Reliability and Sensitivity Analysis

The three combination examples above demonstrate the effects of reassigning belief before combination, and therefore the ability to address convergence, zero belief assignments, and unintuitive results. Although belief is reassigned in the above examples in a subjective and ad hoc manner, the combination examples show how reliability could be applied to reassign belief mass to the elements of the frame of discernment. In this case, that meant assigning belief to the compound sets to identify how this belief is redistributed after combination.

For example, consider Figure 3. This shows a simple combination (using Dempster's rule) of two identical belief functions covering a three event (A, B, and C) frame of discernment similar to the one used in the example above. The red box on the left of the figure demonstrates the case of discounting the evidence and moving belief mass to the universal set. Notice that the universal set (m'(ABC) in Figure 3, dark blue triangles) retains the majority of belief after combination and conflict is very low when a reliability function has been applied to move belief to the universal set before combination. Now, in the red box on the right of the figure, the impact of removing the discounting function is presented. Notice that conflict increases and the mass of belief retained in the universal set converges to zero (0.0). Also of note is the fact that the combination converges belief to Event A at the expense of Event B. The discounting of the evidence allows a retention of a higher level of belief in Event B.

Simple sensitivity analysis, as is plotted in Figure 3 could prove useful for a practical application of Evidence Theory. Denoeux (2008b) summarizes a simple method of applying discounting and sensitivity analysis, by transforming a dogmatic belief structure into a non-dogmatic belief structure (by discounting and assigning belief to the universal set), for which many of the rules are intended for application. The amount of belief discounted and reassigned can be modified to observe the impact on results as it approaches 0 and approaches 1, providing a framework to evaluate the sensitivity of the combination to belief assigned to the universal set. The reliability concept of the proposed protocol offers the opportunity to demonstrate the impacts of certain distributions of belief mass and methodological decisions in the analysis. This illuminates the methods upon which Evidence Theory application relies and provides a more detailed application of Evidence Theory to an uncertainty analysis.

Conclusions

This paper has been motivated by the lack of a common method of applying Evidence Theory to engineering applications. Evidence Theory provides a framework to address epistemic uncertainty, and therefore is well positioned to treat data fraught with missing information, imprecise estimates, and metrics of differing granularity. Since uncertainty analysis has been traditionally performed using probability theory, engineering applications using such data must begin with data pre-processing methods and the acceptance of assumptions in order to fit the available data to the constraints of probability theory. The Evidence Theory approach, however, does not place such an emphasis on pre-processing. The Evidence Theory approach instead asks the analyst to evaluate the lack of precision in the data and develop a frame of discernment that can incorporate data of all granularities available. Since the approach can differ so significantly from probability theory, it is necessary to introduce and develop a protocol for engineering applications of evidence theory.

The proposed protocol incorporates two phases. The first phase is necessary to initiate the analysis. The first step of the first phase addresses the development of the frame of discernment. The user should consider things such as data granularity and precision in the frame of discernment and develop a frame that can handle all the data available and meet the goals of the analysis. The protocol reminds the user of the difference in approach between initiating an analysis based on probability theory and one based on Evidence Theory. The second step of the first phase asks the user to review the initial assignment of basic belief masses. Considerations of this step pertain to awareness of convergence of belief in Evidence Theory and the potential to veto majority opinion. The initial assignment of belief mass is a field into itself, with published guidance on assigning belief mass available, for example (Chen et al. 2014; Jiang and Hu 2018). The user, however, is reminded of a few basic concepts to consider. The evaluation of whether their belief structure is as least committed as possible is the first step. The user must also consider whether this will be a single combination or multiple combinations. In instances of multiple combinations, it is necessary that the user review instances of zero initial basic belief assignment and evaluate the potential for conflict.

The second phase of the protocol provides reasons for the selection of a particular combination rule and a framework for performing sensitivity analysis. Guidance pertaining to the selection of a combination rule is linked to the particular properties of the rule, with conditions identified as when to apply or avoid the rule. Primarily, the user should be familiar with the common characteristics associated with Dempster's rule, such as convergence of belief. The application of multiple rules also facilitates discussions concerning the additional context to the combination results that certain rules can reveal. For example, the application of a PCR rule or Dubois and Prade's rule require the calculation of partial conflicting masses and provide context in comparison to Dempster's rule as to how conflict is distributed and what effect normalization is having on converging belief or identifying a most likely outcome.

The application of different combination rules compliments the process of performing sensitivity analysis. A framework for performing sensitivity analysis is seen as a crucial step in enabling practical engineering applications of Evidence Theory (Oberkampf and Helton 2002). A framework for performing sensitivity analysis is discussed using reliability weighting together with comparing the results of applying different rules. The application of reliability weighting permits the reassignment of belief mass before combination, allowing an analysis of sensitivities related to initial assignment of belief and conflict normalization. Such an analysis provides valuable additional context, such as revealing unintuitive convergence or an uncertain outcome assigned majority belief after combination (e.g., Zadeh's Paradox (Zadeh 1984)). The performance of sensitivity analysis is necessary to evaluate pseudo-Type I and pseudo-Type II errors. Pseudo-type I errors are considered by exploring the sensitivity of the result to convergence and information contained in partial conflicts, in order to determine if the result converged on a false or uncertain outcome. Pseudo-type II errors are considered by exploring the sensitivity of the results to holding more belief in imprecise (i.e., compound) sets, thereby exploring whether belief in a true or certain outcome is excessively reduced by imprecise data or slow convergence

The protocol is intended to expand the use of Evidence Theory in practical applications through the identification, demonstration, and discussion of the protocol proposed. It is hoped that the proposed protocol will also facilitate and provide guidance to new users of Evidence Theory and expand its use in engineering applications. The introduction of a logical procedure for evaluating an Evidence Theory analysis is detailed, including how to define the frame of discernment, initial assignment of belief mass, selection of combination rule, and sensitivity analysis. for the performance of sensitivity analysis on Evidence Theory applications. The continued development of guidance and discussion around the application of Evidence Theory, including additional combination rules based on combining dependent data is a topic of future research to improve and expand the protocol. The primary limitation in the acceptance of the proposed protocol is that it lacks an exact method of applying Evidence Theory and obtaining results, but rather focuses on the secondary analysis of results. Continued research and development around the application of Evidence Theory and the framework of the protocol will address this limitation and facilitate more detailed guidance and discussion about conditions for which to use or avoid certain methods. Persistence in the development of a protocol for engineering applications of evidence theory is crucial in expanding Evidence Theory's application and widening tools to address ignorance and epistemic uncertainty.

Data Availability Statement

All data, models, or code that support the finding of this study are available from the corresponding author upon reasonable request.

Acknowledgments

The financial support of the first author through a United States Department of Education Graduate Assistance in Areas of National Need Grant P200A180024 is gratefully acknowledged.

References

Agarwal, H., Renaud, J. E., Preston, E. L., and Padmanabhan, D. (2004). "Uncertainty quantification using evidence theory in multidisciplinary design optimization." *Reliability Engineering & System Safety*, Alternative Representations of Epistemic Uncertainty, 85(1), 281–294.

Alim, S. (1988). "Application of Dempster-Shafer Theory for Interpretation of Seismic Parameters." *Journal of Structural Engineering*, 114(9), 2070–2084.

Appriou, A. (1997). "Multisensor data fusion in situation assessment processes." *Qualitative and Quantitative Practical Reasoning*, Lecture Notes in Computer Science, D. M. Gabbay, R. Kruse, A. Nonnengart, and H. J. Ohlbach, eds., Springer, Berlin, Heidelberg, 1–15.

Attoh-Okine, B. N. O., and Martinelli, D. (1994). "BELIEF-FUNCTION FRAMEWORK FOR HANDLING UNCERTAINTIES IN PAVEMENT MANAGEMENT SYSTEM DECISION MAKING." *Transportation Research Record*, (1455).

Attoh-Okine, N. O. (2002). "Aggregating evidence in pavement management decision-making using belief functions and qualitative Markov tree." *IEEE Transactions on Systems, Man, and Cybernetics, Part C (Applications and Reviews)*, 32(3), 243–251.

Attoh-Okine, N. O., Cooper, A. T., and Mensah, S. A. (2009). "Formulation of Resilience Index of Urban Infrastructure Using Belief Functions." *IEEE Systems Journal*, 3(2), 147–153.

Attoh-Okine, N. O., and Gibbons, J. (2001). "Use of Belief Function in Brownfield Infrastructure Redevelopment Decision Making." *Journal of Urban Planning and Development*, American Society of Civil Engineers, 127(3), 126–143.

Ballent, W., Corotis, R. B., and Torres-Machi, C. (2019). "Dempster–Shafer Theory applications in post-seismic structural damage and social vulnerability assessment." *Sustainable and Resilient Infrastructure*, 0(0), 1–13.

Ballent, W., Corotis, R. B., and Torres-Machi, C. (2020). "Dempster–Shafer Theory applications in post-seismic structural damage and social vulnerability assessment." *Sustainable and Resilient Infrastructure*, 5(4), 1–13.

Bao, Y., Li, H., An, Y., and Ou, J. (2012). "Dempster–Shafer evidence theory approach to structural damage detection." *Structural Health Monitoring*, 11(1), 13–26.

Behrouz, M., and Alimohammadi, S. (2018). "Uncertainty Analysis of Flood Control Measures Including Epistemic and Aleatory Uncertainties: Probability Theory and Evidence Theory." *Journal of Hydrologic Engineering*, 23(8), 04018033.

Bi, Y., Guan, J., and Bell, D. (2008). "The combination of multiple classifiers using an evidential reasoning approach." *Artificial Intelligence*, 172(15), 1731–1751.

Bogler, P. L. (1987). "Shafer-dempster reasoning with applications to multisensor target identification systems." *IEEE Transactions on Systems, Man, and Cybernetics*, 17(6), 968–977.

Cai, J., Li Shuai, and Cai Hubo. (2018). "Accurate Mapping of Underground Utilities: An Information Fusion Approach Based on Dempster-Shafer Theory." *Construction Research Congress 2018*, Proceedings, 712–721.

Chen, L., and Rao, S. S. (1998). "A Modified Dempster-Shafer Theory for Multicriteria Optimization." *Engineering Optimization*, 30(3–4), 177–201.

Chen, Q., Whitbrook, A., Aickelin, U., and Roadknight, C. (2014). "Data classification using the Dempster–Shafer method." *Journal of Experimental & Theoretical Artificial Intelligence*, Taylor & Francis, 26(4), 493–517.

Corotis, R. B. (2015). "An Overview of Uncertainty Concepts Related to Mechanical and Civil Engineering." *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part B: Mechanical Engineering*, 1(4), 040801.

Dempster, A. P. (1968). "A Generalization of Bayesian Inference." *Journal of the Royal Statistical Society. Series B (Methodological)*, 30(2), 205–247.

Dempster, A. P. (2008). “The Dempster–Shafer calculus for statisticians.” *International Journal of Approximate Reasoning*, 48(2), 365–377.

Denoeux, T. (2000). “A neural network classifier based on Dempster-Shafer theory.” *IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans*, 30(2), 131–150.

Denoeux, T. (2006). “The cautious rule of combination for belief functions and some extensions.” *2006 9th International Conference on Information Fusion*, 1–8.

Denœux, T. (2008a). “A k-Nearest Neighbor Classification Rule Based on Dempster-Shafer Theory.” *Classic Works of the Dempster-Shafer Theory of Belief Functions*, Studies in Fuzziness and Soft Computing, R. R. Yager and L. Liu, eds., Springer, Berlin, Heidelberg, 737–760.

Denoeux, T. (2008b). “Conjunctive and disjunctive combination of belief functions induced by nondistinct bodies of evidence.” *Artificial Intelligence*, 172(2), 234–264.

Denoeux, T. (2013). “Maximum Likelihood Estimation from Uncertain Data in the Belief Function Framework.” *IEEE Transactions on Knowledge and Data Engineering*, 25(1), 119–130.

Denoeux, T. (2019). “Logistic regression, neural networks and Dempster–Shafer theory: A new perspective.” *Knowledge-Based Systems*, 176, 54–67.

Denœux, T., Kanjanatarakul, O., and Sriboonchitta, S. (2015). “EK-NNclus: A clustering procedure based on the evidential K-nearest neighbor rule.” *Knowledge-Based Systems*, 88, 57–69.

Dezert, J., and Tchamova, A. (2011). “On the behavior of Dempster’s rule of combination.”

Dubois, D., and Prade, H. (1986). “On the unicity of dempster rule of combination.” *International Journal of Intelligent Systems*, 1(2), 133–142.

Fetz, T., Oberguggenberger, M., and Pittschmann, S. (2000). “Applications of possibility and evidence theory in civil engineering.” *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 08(03), 295–309.

Helton, J. C., Johnson, J. D., and Oberkampf, W. L. (2004). “An exploration of alternative approaches to the representation of uncertainty in model predictions.” *Reliability Engineering & System Safety*, Alternative Representations of Epistemic Uncertainty, 85(1), 39–71.

Hester, P. (2012). “Epistemic Uncertainty Analysis: An Approach Using Expert Judgment and Evidential Credibility.” *Journal of Quality and Reliability Engineering*.

Hou, Y. (2021). “Sensitivity Analysis of Epistemic Uncertainty on Input Parameters and System Structure Using Dempster-Shafer Theory.” *ASCE-ASME J Risk and Uncert in Engrg Sys Part B Mech Engrg*, 7(2).

Huang, L., Ruan, S., and Denoeux, T. (2021). “Covid-19 classification with deep neural network and belief functions.” *arXiv:2101.06958 [cs, eess]*.

Inagaki, T. (1993). “CHAPTER 15 - Dempster-Shafer Theory and its Applications.” *Fundamental Studies in Engineering*, New Trends in System Reliability Evaluation, K. B. Misra, ed., Elsevier, 587–624.

Jiang, W., and Hu, W. (2018). “An improved soft likelihood function for Dempster–Shafer belief structures.” *International Journal of Intelligent Systems*, 33(6), 1264–1282.

Kronprasert, N., and Kikuchi, N. (2011). “Measuring Validity of Reasoning Process for Transportation Planning Using Bayesian Inference and Dempster-Shafer Theory.” *Vulnerability, Uncertainty, and Risk*, Proceedings, 121–128.

Kyburg, H. E. (1988). “Knowledge.” *Machine Intelligence and Pattern Recognition*, Uncertainty in Artificial Intelligence, (J. F. Lemmer and L. N. Kanal, eds.), 5, 263–272.

Lehmann, E. L. (2011). *Fisher, Neyman, and the Creation of Classical Statistics*. Springer Science & Business Media.

Liu, Z.-G., Pan, Q., and Dezert, J. (2013). “A new belief-based K-nearest neighbor classification method.” *Pattern Recognition*, 46(3), 834–844.

Martin, A., Jousselme, A., and Osswald, C. (2008). “Conflict measure for the discounting operation on belief functions.” *2008 11th International Conference on Information Fusion*, 1–8.

Murphy, C. K. (2000). “Combining belief functions when evidence conflicts.” *Decision Support Systems*, 29(1), 1–9.

Oberkampf, W. L., DeLand, S. M., Rutherford, B. M., Diegert, K. V., and Alvin, K. F. (2002). “Error and uncertainty in modeling and simulation.” *Reliability Engineering & System Safety*, 75(3), 333–357.

Oberkampf, W. L., and Helton, J. C. (2002). “Evidence Theory for Engineering Applications.” American Institute of Aeronautics and Astronautics.

Reineking, T. (2014). *Belief Functions: Theory and Algorithms*. Dissertation Report Universitat Bremen.

Seites-Rundlett, W., Bashar, M. Z., Torres-Machi, C., and Corotis, R. B. (2022). “Combined evidence model to enhance pavement condition prediction from highly uncertain sensor data.” *Reliability Engineering & System Safety*, 217, 108031.

Sentz, K., and Ferson, S. (2002). *Combination of evidence in Dempster-Shafer theory*. Sandia National Laboratories.

Shafer, G. (1976). *A Mathematical Theory of Evidence*. Princeton University Press.

Shafer, G. (1990). “Perspectives on the theory and practice of belief functions.” *International Journal of Approximate Reasoning*, 4(5), 323–362.

Shafer, G. (2016). “A Mathematical Theory of Evidence turns 40.” *International Journal of Approximate Reasoning*, 40 years of Research on Dempster-Shafer Theory, 79, 7–25.

Smarandache, F., and Dezert, J. (2005). “Information fusion based on new proportional conflict redistribution rules.” *2005 7th International Conference on Information Fusion*, 8 pp.-.

Smets, P. (1990). “The combination of evidence in the transferable belief model.” *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12(5), 447–458.

Smets, P. (1992). “The transferable belief model for expert judgements and reliability problems.” *Reliability Engineering & System Safety*, 38(1), 59–66.

Smets, P. (2007). “Analyzing the combination of conflicting belief functions.” *Information Fusion*, 8(4), 387–412.

Smets, P., and Kennes, R. (1994). “The transferable belief model.” *Artificial Intelligence*, 66(2), 191–234.

Soua, R., Koesdwiady, A., and Karray, F. (2016). “Big-data-generated traffic flow prediction using deep learning and dempster-shafer theory.” *2016 International Joint Conference on Neural Networks (IJCNN)*, 3195–3202.

Talon Aurélie, Curt Corinne, and Boissier Daniel. (2014). “Performance Assessment Based on Evidence Theory and Fuzzy Logic: Application to Building and Dam Performance.” *Journal of Computing in Civil Engineering*, American Society of Civil Engineers, 28(1), 124–133.

Tarko, A., and Rouphail, R. N. (1997). “Intelligent Traffic Data Processing for ITS Applications.” *Journal of Transportation Engineering*, American Society of Civil Engineers, 123(4), 298–307.

Truong, T. Q., Zhang, J., and Li, Z. (2019). “Dempster-Shafer-Step (DSS) Tradeoff Analysis Method for Multicriteria Transportation Investment Decision Making.”

Webb, D., and Ayyub, B. M. (2017). “Sustainability Quantification and Valuation. II: Probabilistic Framework and Metrics for Sustainable Construction.” *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part A: Civil Engineering*, American Society of Civil Engineers, 3(3), E4016002.

Yager, R. R. (1987). “On the dempster-shafer framework and new combination rules.” *Information Sciences*, 41(2), 93–137.

Yager, R. R., and Liu, L. (Eds.). (2008). *Classic Works of the Dempster-Shafer Theory of Belief Functions*. Studies in Fuzziness and Soft Computing, Springer-Verlag, Berlin Heidelberg.

Zadeh, L. A. (1984). “Review of A Mathematical Theory of Evidence.” *AI Magazine*, 5(3), 81–81.

Zargar, A., Sadiq Rehan, Naser Gholamreza, Khan Faisal I., and Neumann Natasha N. (2012). “Dempster-Shafer Theory for Handling Conflict in Hydrological Data: Case of Snow Water Equivalent.” *Journal of Computing in Civil Engineering*, American Society of Civil Engineers, 26(3), 434–447.

Zhao, X., Gu Haichang, Song Gangbing, Mo Y. L., and Xu Jinwu. (2010). “Structural Health Monitoring with Data Fusion Method.” *Earth and Space*, Proceedings, 2509–2517.

Zhou, D., Wei Tingting, Zhang Huisheng, Ma Shixi, and Wei Fang. (2018). “An Information Fusion Model Based on Dempster–Shafer Evidence Theory for Equipment Diagnosis.” *ASCE-ASME Journal of Risk and Uncertainty in Engineering Systems, Part B: Mechanical Engineering*, American Society of Mechanical Engineers, 4(2), 021005.