Gluings and Stratifications of Bundles and Infinite-Dimensional Spaces

by

Harold Lee "Howy" Jordan

B.A., University of South Carolina Upstate, 2017

M.A., University of Colorado Boulder, 2021

A thesis submitted to the

Faculty of the Graduate School of the

University of Colorado in partial fulfillment

of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

2025

Committee Members:

Markus Pflaum, Chair

Sheagan John

Agnés Beaudry

Robin Deeley

Michael Hermele

Jordan, Harold Lee "Howy" (Ph.D., Mathematics)

Gluings and Stratifications of Bundles and Infinite-Dimensional Spaces

Thesis directed by Prof. Markus Pflaum

The study of stratified spaces is concerned with topological spaces that can be decomposed nicely into more familiar spaces, typically manifolds. The classical theory is concerned soley with spaces whose strata are finite-dimensional manifolds. Infinite-dimensional spaces arise naturally even in the study of finite-dimensional spaces and are fundamental to modern physics. However, the notion of stratification of a space into such infinite-dimensional spaces has only been studied incidentally. Here we will extend much of the theory of finite-dimensional stratifications to allow for this. In the finite-dimensional case, the space can be seen as being glued together along the strata and this data can be used to glue together sheaves on the strata into sheaves on the original space. We will extend this notion to consider gluing together bundles more general than the étalé bundles that correspond to sheaves. This allows for a reformulation of the Whitney A-regularity condition that guarantees the existence of a stratified tangent bundle on classical stratified spaces. In the infinite-dimensional case, the space is not necessarily given by gluing together the strata in the same way. When it is, we can extend Artin gluing of bundles to this context as well and use the reformulated A-regularity condition to construct stratified tangent bundles for these infinite-dimensional manifolds. Finally, we will apply these results to problems in mathematical physics. In particular, we will stratify and construct stratified tangent bundles on the state spaces of a C^*-algebra, corresponding to the states of a quantum system.

Dedication

To my daughter Alora and to my fiancé Khalil.

Acknowledgements

Thanks to my advisor Markus Pflaum for his direction and advice, and for letting me choose my own thesis problems that align with my interests, much as I choose my own partner.

Thanks to Sheagan John for catching numerous grammatical and typesetting errors, and for suggestions for future research.

Thanks to the committee for their careful consideration, insightful questions, and being present with me while I cross this bridge.

Thanks to the members of my family: those connected by birth and the chosen family connected by life. Everytime I wanted to give up it was the memories of you all that kept me going. Thanks to my friends in the mathematics department at the University of Colorado for many inspiring discussions over the years. Thanks to my daughter Alora for being a consistent inspiration and helping me realize that everything will be okay. Thanks to my fiancè Khalil for being my rock, reminding me of what I can accomplish, and seeing me through the highs and lows.

This material is based on work supported by the National Science Foundation's Division of Mathematical Sciences via grant No. 2055501.

Contents

Figures

Figure

Chapter 1

Introduction

In the study of topology, we often restrict focus to spaces that have some desirable properties amenable to the analysis at hand. The study of stratified spaces allows for an extension of these contexts by considering spaces that, while not having the desired properties, can nevertheless be decomposed into spaces which do have the desired properties, where the decomposition itself is required to be "nice" in some sense. Classically, a decomposition is considered nice when the partition is a locally finite partition of locally closed subsets, each a smooth manifold, with the partition satisfying the frontier condition and the smooth manifolds compatible in some sense expressed by conditions such as those due to Whitney. This basic structure allows one to generalize theorems of algebraic topology of manifolds, in particular Poincaré duality, as well as theorems of differential topology and geometry.

In an effort to both understand the general theory of finite-dimensional stratified spaces and generalize results further to infinite-dimensional manifolds, we begin by organizing the basic language to single out a topologically useful notion of decomposed space as the fundamental object of study. There are a couple desirable properties. We will begin in Chapter 2 by considering the ability to reconstruct an open set from data defined in terms of the pieces of the decomposition. The theory of **Artin gluing**, which originated in the work of Artin, Grothendieck, and Verdier [2, exposè IV §9.5], is integral here. Artin gluing was generalized by Wraith in [26] to elementary toposes, and we will follow his exposition. Already, this theory specifies how to recover the topology of a space from a finite decomposition. We introduce a new notion of **conjunctive decompositions** to characterize those decompositions which satisfy the Artin gluing conditions in general. In Theorem 2.3.4, we show that locally finite decompositions are conjunctive. The theory also extends to general toposes, allowing sheaves over the pieces of a finite decomposition to be glued together with additional data provided. We show in Theorem 2.4.7 that, provided the decomposition is conjunctive, then this theory can be extended to glue together sheaves over the decomposition. This provides an original generalization of the theory in Wraith [26] when applied to topological spaces equipped with decompositions into infinitely many pieces. The proof we have given is also much simpler than that of Wraith, again thanks to our restriction to topological spaces, hence spatial locales and toposes.

In Chapter 3, we consider another way in which a decomposition can relate to the topology of the underlying space. We consider decompositions of a space that are given by a continuous open map from a topological space to an ordered set with the **specialization** or **Alexandroff topology**, following Jacob Lurie's notion of a **poset-stratified space** as in [13]. Such a map might carry additional unnecessary information. In particular, the codiscrete topology on the set of pieces of the decomposition is given by the chaotic order. Thus we will also ask that the decomposition map be a quotient map. In fact, we will ask for something stronger, namely that it be an open map. This corresponds precisely to the notion of a frontier decomposition of a space. The frontier condition also implies that the partition can be equipped with a partial order, even if a priori no order is given. We show in Lemma 3.2.4 that the frontier condition implies a simplification of the data typically needed to reconstruct a space and its sheaves from a decomposition as in Chapter 1. We will also consider when two poset-stratifications of a given space give decompositions which carry essentially the same information. In the classical theory of stratified spaces, this redundancy is handled by defining a stratification to be an equivalence class of decompositions, two decompositions being declared equivalent when they give the same set-germs around each point. Here we show in Theorem 3.3.7 that this notion can be captured via a map of the posets, i.e. a monotone map, satisfying an injectivity condition on chains of elements. We then formally define a stratification (with no other qualifications) to be an equivalence class of poset-stratified spaces satisfying the frontier condition.

In Chapter 4, we consider how to extend the construction of stratified tangent bundles from the classical theory of stratified spaces to our more general context. In fact, we will show how to glue general bundles together using data on related étalé bundles, provided that the decomposition is conjunctive. This provides another original extension to the theory of Artin gluing. We then extend this theory further to fiber bundles by requiring the gluing data to be equivariant with respect to the structure group of the fiber bundles. This will enable us to rephrase the Whitney A-regularity condition in this context, thus providing groundwork towards stratified vector fields in the infinite-dimensional case. Our new theory of gluing fiber bundles is also useful for providing alternative viewpoints on nonstratified fiber bundles such as the Möbius strip, as illustrated in Example 4.1.8.

In Chapter 5, we extend the control theory of Mather from [15] to the case of infinite dimensional Riemannian manifolds. Control theory in the infinite dimensional case as presented in this work appears to be new. This requires some additional conditions on the poset underlying the decomposition. In these cases, we can construct generalized tubular neighborhoods, required to be compatible with a given map. This generalized tubular neighborhood theorem is fundamental in the construction of **control data**, systems of tubular neighborhoods around the strata which are compatible with each other. We show in Theorem 5.1.10 that mild additional conditions on the stratification allow the construction of control data, generalizing Pflaum's proof in [16]. We will also apply the theory of generalized slices of Diez and Rudolph in [6] to the particular case of a Banach Lie group action on a Banach submanifold of Banach space.

Finally, in Chapter 6 we briefly consider applications of the previous chapters. We show that stratifications can be given of $\mathbb{R}^\infty$, of infinite jet bundles over finite-dimensional vector bundles, and of the state space of an infinite-dimensional C^*-algebra. Moreover, these stratifications are Whitney A-regular in our generalized sense and thus give rise to stratified tangent bundles and hence stratified vector fields. While a notion of stratification applicable to these cases is already present in, for instance, Diez and Rudolph [6], this is the first instance of a notion of stratified tangent bundle being available for such spaces.

Chapter 2

Artin Gluing of Spaces and Sheaves

While it is well-known that a space can be reconstructed from an open cover, in certain circumstances spaces can be reconstructed from a decomposition of the space into disjoint subsets. The general theory of such reconstructions is commonly called **Artin gluing** (sometimes spelled glueing), in honor of Artin's original work on this topic in [2]. There is also a succint exposition of the general method of gluings in the context of toposes in [26]. Artin gluing makes use of so-called **fringe functors** between the lattices of open subsets of the pieces of the decomposition. These functors specify for a given open set of one piece of the decomposition what the largest open set of another piece is whose union with the first open set is open in the original space. Taken together, these give conditions on tuples of open sets which specify which of these have union open in the original space.

The classical theory is concerned primarily with finite decompositions. We begin by reviewing this theory and setting notation. Then we will demonstrate how to extend this procedure to more general decompositions. For locally-finite decompositions, the generalization is immediate. In other cases, we demonstrate counterexamples where the **Artin gluing topology** on the space is finer than the original topology given. That is, we construct a decomposition of a space and give a subset that satisfies the hypothesis of Artin gluing but which is nevertheless not open in the original space.

2.1 Gluing Pairs of Subsets

We begin by considering a certain map between subspaces of a topological space. Let X be a topological space with open sets $\mathcal{O}(X)$. For any $A \subset X$ with injection $\iota_A : A \to X$ there is an induced pair of adjoint functors

$$\mathcal{O}(A) \underset{\iota_*^A}{\overset{\iota_A^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathcal{O}(X).$$

These functors act respectively by

$$U \cap A \hookleftarrow U$$
$$V \mapsto \mathrm{int}_X(V \cup X \setminus A).$$

That is, the first map is simply given by the usual intersection defining the subspace topology, whereas the second is the largest open subset of X contained in $V \cup X \setminus A$.

Lemma 2.1.1. *For all A, $\iota_*^A \iota_A^* \geq \mathrm{id}_{\mathcal{O}(X)}$, where the inequality is pointwise, i.e. $\iota_*^A \iota_A^*(U) \supseteq U$ for all $U \in \mathcal{O}(X)$.*

Proof. Consider a $U \in \mathcal{O}(X)$. Then

$$\begin{aligned}\iota_*^A \iota_A^*(U) &= \mathrm{int}_X((U \cap A) \cup X \setminus A)\\ &= \mathrm{int}_X(U \cup X \setminus A)\end{aligned}$$

by distribution inside the interior operator. Since int_X is monotone, $\mathrm{int}_X(U) \subseteq \mathrm{int}_X(U \cup X \setminus A)$. Since U is open in X, we then have $U \subseteq \mathrm{int}_X(U \cup X \setminus A)$. □

Since $\iota_A^* \dashv \iota_*^A$, the first preserves all unions and the second preserves all intersections. But moreover, the left adjoint ι_A^* is left exact, that is, it preserves all finite intersections as well. Thus, the composition of two such maps will preserve all finite intersections.

In particular, consider two subsets A, B with the ι_A and ι_B their respective inclusions. Then we have a map

$$\mathcal{O}(A) \xrightarrow{\iota_B{}^* \iota_{A*}} \mathcal{O}(B)$$

$$V \longmapsto B \cap \operatorname{int}_X(V \cup X \setminus A)$$

which also preserves all finite intersections. This map is the key ingredient in the gluing of disjoint subsets.

Definition 2.1.2. The **fringe functor** from $\mathcal{O}(A) \to \mathcal{O}(B)$ is given by

$$\mathrm{f}_{BA} := \iota_B{}^* \iota_{A*}.$$

Likewise, there is an $\mathrm{f}_{AB} : \mathcal{O}(B) \to \mathcal{O}(A)$. Figure 2.1 shows an example of the fringe of an open subset of the right-half plane of $\mathbb{R}^2$ in the subspace given by the y-axis. Observe that the fact that f_{BA} preserves finite intersections corresponds to it being a meet-homomorphism of the meet-semilattice underlying the opens of A to that of B. Equivalently, regarding the posets as thin categories, f_{AB} is a left exact functor between left exact categories. Though this follows immediately from general abstract properties, there is also an easy direct verification of this result.

Lemma 2.1.3. *The fringe functor preserves intersections, that is*

$$\mathrm{f}(V \cap V') = \mathrm{f}(V) \cap \mathrm{f}(V')$$

for all $V, V' \in \mathcal{O}(A)$*, hence it is a meet-homomorphism, i.e. a morphism of meet-semilattices.*

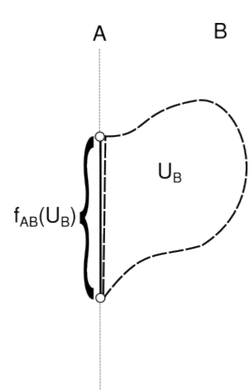


Figure 2.1: The fringe of an open set U_B of the subspace B in the subspace A of $\mathbb{R}^2$.

Proof. Observe that if $W \in \mathcal{O}(B)$ is such that $W \cup (V \cap V')$ is open in X, then $W \cup (V \cap V') \cup V = W \cup V$ is open in X. Likewise $W \cup (V \cap V') \cup V' = W \cup V'$ is open in X, hence if $W \subset \mathrm{f}(V \cap V')$ then $W \subset \mathrm{f}(V) \cap \mathrm{f}(V')$. It remains to show the reverse inclusion. Suppose that $W \subset \mathrm{f}(V) \cap \mathrm{f}(V')$. Then $W \subset \mathrm{f}(V)$ and $W \subset \mathrm{f}(V')$. So we have $W \cup V, W \cup V' \in \mathcal{O}(X)$. Thus, $(W \cup V) \cap (W \cup V') = W \cup (V \cap V')$ is open in X so $W \subset \mathrm{f}(V \cap V')$. □

Notice also that

$$\mathrm{f}_{BA}(A) = B$$

which follows immediately from the left-exactness. We also have that

$$\mathrm{f}_{BA}(\varnothing) = B \cap \mathrm{int}_X(X \setminus A).$$

In particular then we have the following trivial but important observations, which we collect into a lemma.

Lemma 2.1.4. *Let A, B be subsets of a topological space X. If $\mathrm{int}_X(X \setminus A) = \varnothing$, then $\mathrm{f}_{BA}(\varnothing) = \varnothing$. If instead $B \subset \mathrm{int}_X(X \setminus A)$ (or equivalently, if $B \cap \overline{A} = \varnothing$), then $\mathrm{f}_{BA}(\varnothing) = B$ and f_{BA} is the trivial meet homomorphism with constant value B. Conversely, if f_{BA} is the constant function with value B then $B \subset \mathrm{int}_X(X \setminus A)$.*

Example 2.1.5. Let A be a closed subset of a topological space and B its complement. Then $B = \mathrm{int}_X(X \setminus A)$, so the fringe functor f_{BA} is constant. However, f_{AB} is not. In fact here we see that $\mathrm{f}_{AB}(\varnothing) = \mathrm{int}_X(A)$. This is the motivating example for the name "fringe". For in this case, $\mathrm{f}_{AB}(W)$ for $W \subset B$ open is the interior of A union those points of A which are limits of points of B. But then $\mathrm{f}_{AB}(W)$ is exactly the union of all V such that $V \cup W$ is open in X. That is to say, it is the largest open set of A for which the union with W is open in X, and an arbitrary open set V of A is such that $V \cup W$ is open in X if and only if $V \subset \mathrm{f}_{AB}(W)$.

If A and B partition X, that is, their union is X and their intersection is empty, we can try to reconstruct the topology on X from those of A and B. Note that if A and B

cover X but their interiors have non-empty intersection, then X is already recoverable as the pushout of A and B along that intersection.

This technical issue will not concern us, but note that even a partition of X might be such that the topology is already recoverable without any additional data. For instance, consider the decomposition of $\mathbb{R}$ into $\mathbb{Q}$ and its complement. This happens because although the subsets have empty intersection, the corresponding sublocales of open sets do not. They intersect in the so-called **regular open sets**, i.e. those open sets U for which $U = \mathrm{int}_X \overline{U}$. The regular opens form a sublocale but the join of regular opens is not the usual union of open subsets, and furthermore the regular opens do not admit any representation as the topology of open subsets for some space. Thus we can recover the topology on $\mathbb{R}$ by taking the pushout in the category of locales, and no additional data is required.

We will restrict to partitioning X into pieces which are complemented as sublocales. The open and closed subsets are of course complemented, but so are the locally closed subsets and even the constructible subsets (i.e. those finite unions of locally closed subsets) [9].

Suppose now that we are given two arbitrary spaces A and B and an arbitrary meet homomorphism $f : \mathcal{O}(A) \to \mathcal{O}(B)$. It is natural to ask if there is a topology on the disjoint union of A and B for which f is the induced fringe. In fact, we can construct such a topology.

Definition 2.1.6. Let $f : \mathcal{O}(A) \to \mathcal{O}(B)$ be some meet-homomorphism. We construct a topology on $X \coloneqq A \coprod B$ by declaring $U \subset X$ open whenever $(U \cap B) \subset f(U \cap A)$. We call X with this topology the **disjoint gluing** or **Artin gluing of** A **and** B **by** f, and denote it $A \cup_f B$.

The condition which we imposed on pairs of subsets will arise often in our considerations of decomposed spaces, so will give it a precise name.

Definition 2.1.7. Let A and B be spaces (for instance, disjoint subspaces of a given space). Let $f : \mathcal{O}(A) \to \mathcal{O}(B)$ be a meet homomorphism. Let U_A, U_B be open sets in A, B respectively. Then we will say that the pair (U_A, U_B) satisfies the **fringe conditions** or the

disjoint gluing conditions given by f if and only if

$$U_B \subseteq f(U_A).$$

It is not hard to see that disjoint gluing topology is indeed a topology on X.

Lemma 2.1.8. *The disjoint gluing topology is a topology on X.*

Proof. Since f is a meet-homomorphism, $f(A) = B$ hence $A \cup B = X$ is open in X. We also have that $\varnothing \subset f(\varnothing)$ regardless of the value of $f(\varnothing)$, so that $\varnothing$ is open in X.

For an arbitrary collection $\{U_\alpha := V_\alpha \cup W_\alpha\}$ of opens of X, we have $W_\alpha \subset f(V_\alpha)$ for any α. We have $f(V_\alpha) \subset f(\cup_{\alpha'} V_{\alpha'})$ by monotonicity of f, hence $\cup_\alpha W_\alpha \subset f(\cup_\alpha V_\alpha)$. Thus

$$\bigcup_\alpha U_\alpha = \bigcup_\alpha V_\alpha \cup \bigcup_\alpha V_\alpha$$

is again open in X.

Lastly, consider two sets $V_1 \cup W_1$ and $V_2 \cup W_2$. Then $(V_1 \cup W_1) \cap (V_2 \cup W_2) = (V_1 \cap V_2) \cup (W_1 \cap W_2)$ since the V_i are disjoint from the W_i and $W_1 \cap W_2 \subset f(V_1) \cap f(V_2) = f(V_1 \cap V_2)$ since f is a meet-homomorphism, so that the intersection is open as well. □

The Artin gluing induces a bijection between $\mathcal{O}(X)$ and a certain subset of $\mathcal{O}(A) \times \mathcal{O}(B)$. Namely, that subset of $\mathcal{O}(A) \times \mathcal{O}(B)$ consisting of those pairs satisfying the fringe condition with respect to f. Observe that if $\mathcal{O}(A) \to \mathcal{O}(B)$ is the constant function to B, then this is onto $\mathcal{O}(A) \times \mathcal{O}(B)$ and we obtain the disjoint union topology $A \coprod B$. In this case we obtain the same by using the constant function $\mathcal{O}(B) \to \mathcal{O}(A)$ to A. Note that the coproduct on spaces corresponds to the product on the frames of opens, as explained in [9].

Observe that the Artin gluing construction for an arbitrary meet-homomorphism $f : \mathcal{O}(A) \to \mathcal{O}(B)$ always produces a space in which A is open and B is its closed complement. That is, since $A \cap B = \varnothing$ as subsets of the disjoint union, $\varnothing \subset f(A)$ makes A open trivially. We have thus shown that there is suitable topology on the disjoint union of a pair of sets making a given one open and another closed. Of course, the disjoint union topology also satisfies this, but there both possible fringe functors are trivial. But when the fringe information given is taken into account, the Artin topology is unique.

Theorem 2.1.9. *Let A and B be two topological spaces with open sets $\mathcal{O}(A)$ and $\mathcal{O}(B)$. Let $f : \mathcal{O}(A) \to \mathcal{O}(B)$ be a meet-homomorphism. Then the Artin gluing topology $A \cup_f B$ is the coarsest topology on the disjoint union $A \coprod B$ such that if U is such that $U \cap A$ and $U \cap B$ are open respectively in A and B and with $U \cap B \subset f(U \cap A)$ then U is open. It is thus the unique topology such that a set U is open if and only if $U \cap A$ is open in A, $U \cap B$ is open in B, and $U \cap B \subset f(U \cap A)$.*

Proof. It is clear that the Artin gluing topology $A \cup_f B$ gives always a coarser topology than the usual disjoint union topology, i.e. it picks out a certain subset of the pairs of opens of A and B. Any other topology on the disjoint union such that A is open and B closed with the fringe condition holding for pairs of subsets of A and B must be finer than $A \cup_f B$. That is, if U is open in $A \cup_f B$, then it must also be open for any finer topology, by definition of the Artin topology and the fringe condition. Hence, $A \cup_f B$ is the coarsest such topology. □

Remark 2.1.10. Anders Kock and Till Plewe first commented that Artin gluing could be generalized to a situation beyond an open set and its complement in [11]. In the case of topological spaces, this specializes to an analogous construction to the one given above except that we also require $U \cap A \subset \mathrm{f}_{AB}(U \cap B)$. When this fringe is constant, this reduces to the above. Though we will soon generalize to larger partitions of spaces with sets which may be locally closed, we will see that a so-called "frontier condition", a natural and familiar condition from the common theory of stratified spaces, is tantamount to imposing triviality of half of the fringe data, so this will not concern us here.

2.2 Fringe Data

Definition 2.2.1. Let X be a topological space. Let $\{X_i\}_{i \in I}$ be a partition of X into locally closed subsets. We will say X is a **decomposed space with decomposition** $\{X_i\}_{i \in I}$. We will often abuse notation and not explicitly give the indexing set for the decomposition.

If the collection $\{X_i\}$ is finite then we say the pair $(X, \{X_i\})$ is a **finitely decomposed space**. If the collection is locally finite in the sense that for all x there is a neighborhood U of x such that the set $\{U \,|U \cap X_i \neq \varnothing\}$ is finite, then we say $(X, \{X_i\})$ is a **locally finite decomposed space**. We will also denote a decomposed space by giving the associated set map $X \to \{X_i\}$ mapping $x \in X$ to the unique X_i for which $x \in X_i$, equipped with the quotient topology.

Any decomposition induces families of fringe functors between the open sets of each pair X_i, X_j. We will introduce some notation for this now.

Definition 2.2.2. Let $X \to \{X_i\}_{i \in I}$ be a decomposed space. Denote the induced morphisms on open sets by $\iota_i := \iota_{X_i}$. Then the **fringe data on** $X \to \{X_i\}$ is the family of fringe functors

$$\left\{ \mathrm{f}_{ij} := \iota_i^* \iota_*^j \right\}.$$

We can show that this family of data is compatible, in the following sense.

Lemma 2.2.3. *Let $X \to \{X_i\}$ be a decomposed space with fringe data $\{\mathrm{f}_{ij}\}$. Then for all i, j, k we have*

$$\mathrm{f}_{ik} \leq \mathrm{f}_{ij}\, \mathrm{f}_{jk}$$

where the inequality is taken pointwise, i.e.

$$\mathrm{f}_{ik}(U_k) \subseteq \mathrm{f}_{ij}\, \mathrm{f}_{jk}(U_k)$$

for all $U_k \in \mathcal{O}(X_k)$.

Proof. Let $U_k \in \mathcal{O}(X_k)$. Then we have that

$$\begin{aligned}
\mathrm{f}_{ij}\,\mathrm{f}_{jk}(U_k) &= X_i \cap \mathrm{int}_X(\mathrm{f}_{jk}(U_k) \cup X \setminus X_j) \\
&= X_i \cap \mathrm{int}_X((X_j \cap \mathrm{int}_X(U_k \cup X \setminus X_k) \cup X \setminus X_j) \\
&= X_i \cap \mathrm{int}_X((X_j \cup X \setminus X_j) \cap (\mathrm{int}_X(U_k \cup X \setminus X_k) \cup X \setminus X_j)) \\
&= X_i \cap \mathrm{int}_X(\mathrm{int}_X(U_k \cup X \setminus X_k) \cup X \setminus X_j) \\
&\supset X_i \cap \mathrm{int}_X(U_k \cup X \setminus X_k) \\
&= \mathrm{f}_{ik}(U_k).
\end{aligned}$$

□

If we equip the indexing set I with the codiscrete preordering and the associated topology, then $X \to I$ (or equivalently, $X \to \{X_i\}$ with the codiscrete topology on $\{X_i\}$) is what we will call later a **preset stratified space**. Regarding this preordered I as a category then, we have that fringe data constitutes a lax contravariant 2-functor from I to the category of locales and left exact maps between them. Even when the X_i are locally closed, we can not always refine the ordering on I to a partial order such that we obtain a poset stratified space. If we can always refine a decomposition by locally closed subsets into a decomposition satisfying the frontier condition however, then we can always refine the order on I into one with less relations imposed.

One can also give a more abstract proof of the above result.

Proof. Since $\iota^j_* \iota^*_j(U) \supseteq U$ for all U and j by Lemma 2.1.1, and everything is a monotone function, we immediately have $\iota^*_i \iota^k_*(U_k) \subseteq \iota^*_i \iota^j_* \iota^*_j \iota^k_*(U_k)$. □

If a set U is open in X, then it necessarily satisfies fringe conditions as the following lemma shows. As we will soon see, the converse does not always hold, at least not when the decomposition is infinite.

Lemma 2.2.4. *Let $(X, \{X_i\})$ be a decomposed space and let $f = \{\mathrm{f}_{ij}\}$ be the associated fringe data. Then for all open sets U of X we have that*

(1) for all i, $U_i := U \cap X_i$ is open in X_i, and

(2) for all i, j, $U_i \subseteq \mathrm{f}_{ij}(U_j)$.

Proof. The first condition follows simply from the definition of the subspace topology on each X_i. For the second condition, note that

$$\begin{aligned}
\mathrm{f}_{ij}(U_j) &= X_i \cap \mathrm{int}_X(U_j \cup X \setminus X_j) \\
&= X_i \cap \mathrm{int}_X((U \cap X_j) \cup X \setminus X_j) \\
&= X_i \cap \mathrm{int}_X((U \cup X \setminus X_j) \cap (X_j \cup X \setminus X_j)) \\
&= X_i \cap \mathrm{int}_X(U \cup X \setminus X_j) \\
&\supseteq X_i \cap U \\
&= U_i.
\end{aligned}$$

□

We will now extend the definitions to consider the case of a collection of topological spaces, not given as subsets of a fixed space.

Definition 2.2.5. Let $\{X_i\}$ be an arbitrary collection of topological spaces with topologies $\mathcal{O}(X_i)$. Then **fringe data f for $\{X_i\}$** is given by $f = \{\mathrm{f}_{ij}\}$ a family of meet-homomorphisms

$$\{\mathrm{f}_{ij} : \mathcal{O}(X_j) \to \mathcal{O}(X_i)\}$$

such that for each $i \le j \le k$ we have

$$\mathrm{f}_{ik} \le \mathrm{f}_{ij}\,\mathrm{f}_{jk}$$

where again the inequality is taken pointwise.

This fringe data is exactly a contravariant functor from $(I, \leq)$ to the category of locales and left exact maps.

Definition 2.2.6. Let $\mathrm{f} = \{\mathrm{f}_{ij}\}$ be gluing data for topological spaces $\{X_i\}$. Let $X \coloneqq \coprod_i X_i$ be the disjoint union of sets. The **disjoint gluing** or **Artin gluing topology on** X **induced by** f is given by

$$\mathcal{O}(X) \coloneqq \left\{ \cup_i U_i \,\middle|\, U_i \subseteq \mathrm{f}_{ij}(U_j) \forall i \leq j \right\}.$$

In other words, a set $U \subset X$ is open in the Artin gluing topology if and only if $U \cap X_i$ is open in X_i for all i, and for all i, j we have

$$U \cap X_i \subseteq \mathrm{f}_{ij}(U \cap X_j).$$

In a generalization of the earlier theorem for pairs of spaces, we obtain a universal topology on X induced by the gluing data.

Theorem 2.2.7. *Let $f = \{\mathrm{f}_{ij}\}$ be fringe data for $\{X_i\}$. Then $\mathcal{O}(X)$ is a topology for the disjoint union X.*

Proof. This is a simple generalization of Lemma 2.1.8 □

Note that the definitions given above are nothing more than the those given by [26], specialized to the case of localic topoi gluing to localic topoi.

2.3 Conjunctive Decompositions

By the earlier lemma, the Artin gluing topology is at least as fine as the original topology. Given $(X, \{X_i\})$ a decomposed space, we would like to know under what conditions we can recover precisely the topology on X from the topologies on the X_i much as we did for the case of a partitioning into two subsets. That is to say, when is the Artin gluing

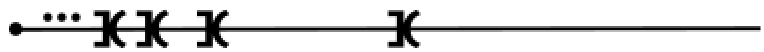


Figure 2.2: A non-conjunctive decomposition of $\mathbb{R}_{\geq 0}$.

topology on X induced by the fringe data associated to $(X, \{X_i\})$ the same as the original topology. By the general theory of Artin gluing as outlined in [26], this is guaranteed when the decomposition is finite and the pieces X_i are locally closed. When the decomposition is infinite though, this is not necessarily true, as the following example shows.

Example 2.3.1. Let $X = \mathbb{R}_{\geq 0}$ with the usual topology. Let $X_0 = \{0\}$. Let $X_1 = (1, \infty)$ and for $i \in \mathbb{N}$ with $i > 1$ let

$$X_i = \left(\frac{1}{i}, \frac{1}{i-1}\right].$$

Consider the set $U = X_0 \cup X_1$. For all i, let $U_i = X_i \cap U$. Notice that U_i is open in X_i for all i. Furthermore, for all i, j we have

$$U_i \subseteq \mathrm{f}_{ij}(U_j)$$

where $f = \{\mathrm{f}_{ij}\}$ is the fringe data associated to the decomposition $(X, \{X_i\})$. Thus U is open in the Artin gluing topology on X associated to the fringe data f. However, U is obviously not open in the usual topology on X. Figure 2.2 illustrates the situation given by this decomposition. In particular, note that this decomposition is not locally finite around 0.

Thus, in general, the Artin gluing topology is finer than the original topology. As we are interested in cases where the two topologies coincide, we give a special name to decompositions for which this holds.

Definition 2.3.2. Let $(X, \{X_i\})$ be a decomposition of a space X. We say the decomposition is **conjunctive** if given an arbitrary set $U \subseteq X$,

(1) for all i $U_i \coloneqq U \cap X_i$ and

(2) for all i, j $U_i \subseteq \mathrm{f}_{ij}\, U_j$

imply that U is open in X.

As already observed, every finite decomposition into locally closed subsets is conjunctive. We can generalize this immediately to locally finite decompositions. First we state the following observation as a lemma.

Lemma 2.3.3. *Let $(X, \{X_i\}_{i \in I})$ be a decomposed space where each X_i is locally closed in X. For a finite subset $I' \subseteq I$, let $X_{I'}$ be given by*

$$X_{I'} := \bigcup_{i \in I'} X_i$$

as a subspace of X. Then a decomposition is conjunctive if and only if the topology on X is the final topology given by the inclusions

$$X_{I'} \hookrightarrow X$$

over all finite subsets $I' \subset I$.

Proof. This follows by considering maps into the Sierpinski space Ω which classifies open subsets. By Artin gluing for finite decompositions, a set is open in $X_{I'}$ if it is open in each X_i for $i \in I'$ and satisfies the fringe conditions for $i, j \in I'$. If X is conjunctive, then if the set is open in each X_i then for any $i, j \in I$ it satisfies the fringe conditions for $I' = \{i, j\}$ and hence is open in U. But this holds if and only if U is open in the colimit topology on X given by $X_{I'} \to X$ by definition. □

Theorem 2.3.4. *Let $(X, \{X_i\})$ be a locally finite decomposition. Then the decomposition is conjunctive.*

Proof. Since the decomposition is locally finite, there is a cover $\{V_\lambda\}$ of X such that on each V_λ the restricted decomposition of V_λ into $\{V\lambda \cap X_i\}$ is finite. Then a set U is open if and only if all $U \cap V_\lambda$ are open. Since each V_λ carries a finite decomposition, each $U \cap V_\lambda$ is open if and only if it satisfies the conjunctive conditions. □

One would like to know other sufficient conditions which ensure that a given decomposition is conjunctive. We will explore this question in the next chapter when considering stratified spaces. In particular, we will see that there are conditions on the poset underlying the stratification which give conjuncitivity of the decomposition associated to the stratification.

2.4 Gluing Sheaves

Let $\mathrm{Sh}(X)$ be the category of sheaves on X. Then the inclusion map ι_A for $A \subset X$ induces a pair of adjoint functors $\iota_A^* \dashv \iota_*^A$ with left adjoint $\iota_A^* : \mathrm{Sh}(X) \to \mathrm{Sh}(A)$ also left exact, just as in the case of the categories of open sets. Such a pair of adjoints is also called a **geometric morphism**, for instance in Mac Lane and Moerdijk [14], and is often denoted simply by $\iota_A : \mathrm{Sh}(A) \to \mathrm{Sh}(X)$ using the direction of the right adjoint, in line with the original continuous map inducing it. Note that this is not in fact a drastic abuse of notation. In-particular, $\mathcal{O}(X)$ exists in $\mathrm{Sh}(X)$ isomorphically in the form of the locale of subterminal objects, or equivalently, elements of the suboject classifier sheaf. Then the adjoint functors restrict to the definitions given above for $\mathcal{O}(A) \to \mathcal{O}(X)$. This is also outlined in [14] as well as in Johnstone's famous topos theory treatise [8, 9].

Given a sheaf $\mathcal{F} \in \mathrm{Sh}(X)$, $\iota_A^*(\mathcal{F})$ is the sheaf whose stalk $\iota_A^*(\mathcal{F})_x$ is given by the stalk $\mathcal{F}_{\iota_A(x)} = \mathcal{F}_x$. Thus it is also often denoted $\mathcal{F}|_A$, the sheaf $\mathcal{F}$ restricted to A. Note however that unless A is open, $\iota_A^*(\mathcal{F})(V) \neq \mathcal{F}(V)$ as $V \in \mathcal{O}(A)$ is not necessarily open in X so the right hand side is not even defined in general! Instead, we have

$$\iota_A^*(\mathcal{F}) = \mathrm{Sh}\left(V \mapsto \operatorname*{colim}_{V \subset U \in \mathcal{O}(X)} \mathcal{F}(U)\right)$$

so that the sheaf $\mathcal{F}$ restricted to A is the sheaf of germs of sections of $\mathcal{F}$ over the open subsets of A.

Given now $\mathcal{G} \in \mathrm{Sh}(A)$ we likewise obtain $\iota_*^A(\mathcal{G}) \in \mathrm{Sh}(X)$, where

$$\iota_*^A(\mathcal{G})(U) = \mathcal{G}(U \cap A).$$

This is the trivial extension of the sheaf $\mathcal{G}$ to the opens of X, which assigns identical sections to sets U, U' which have the same set germ over A, i.e. for which $U \cap A = U' \cap A$.

For a pair of subspaces A, B we thus obtain a left exact functor which we call the **fringe functor**. Note that this is also a direct generalization of the case from the previous sections.

Definition 2.4.1. The **fringe functor** $\mathrm{Sh}(A) \to \mathrm{Sh}(B)$ is the functor

$$\mathrm{f}_{BA}^* := \iota_B^* \iota_*^A : \mathrm{Sh}(A) \to \mathrm{Sh}(B)$$

given by

$$\mathrm{f}_{BA}^*(\mathcal{G}) = \mathrm{Sh}\left(W \mapsto \operatorname*{colim}_{V \in \mathcal{O}(A), W \subset \mathrm{f}_{BA}(V)} \mathcal{G}(V) \right).$$

By lemma 2.1.4, f_{BA} is constant with value B if and only if $B \cap \overline{A} = \varnothing$. In these cases we have immediately the following.

Lemma 2.4.2. *Suppose that* f_{BA} *is the trivial meet-homomorphism. Then*

$$\mathrm{f}_{BA}^*(\mathcal{G})(V) = \mathcal{G}(\varnothing) \cong \{*\}$$

is a singleton set for all $V \in \mathcal{O}(B)$*. Thus,* f_{BA}^* *is the constant functor to the terminal sheaf, i.e. the trivial left exact map.*

Proof. If f_{BA} is trivial, then $V \subset \mathrm{f}_{BA}(W)$ for all $W \in \mathcal{O}(A)$, hence the colimit is taken over $\mathcal{O}(A)$. But, $\varnothing$ is initial in $\mathcal{O}(A)$ hence the colimit is given by the value at $\varnothing$, which is a singleton for all sheaves. □

Thus if the fringe functor f_{BA} is trivial so is the induced functor f_{BA}^*. When considering frontier decompositions (or equivalently, those fringe data over posets with all morphisms nontrivial), the same simplifications from the gluing to open sets lift to sheaves.

When $\mathcal{F} \in \mathrm{Sh}(X)$, we have two sheaves $\mathcal{F}|_A \in \mathrm{Sh}(A)$ and $\mathcal{F}|_B \in \mathrm{Sh}(B)$, as well as $\mathrm{f}_{BA}^* \mathcal{F}|_A$ and $\mathrm{f}_{AB}^* \mathcal{F}|_B$.

Lemma 2.4.3. *For any $\mathcal{F} \in \mathrm{Sh}(X)$, there is a natural map*

$$\mathcal{F}|_B \to \mathrm{f}^*_{BA}\,\mathcal{F}|_A.$$

This map has components at $W \in \mathcal{O}(B)$

$$\mathcal{F}|_B(W) \to \operatorname*{colim}_{W \subset \mathrm{f}_{BA}(V)} \mathcal{F}|_A(V)$$

given by

$$\langle s \rangle_B \mapsto \langle s|_A \rangle_B$$

where $s \in \mathcal{F}(U)$ for some $U \supseteq W$ and $s|_A \coloneqq s|_{U \cap A}$.

Proof. Recall that the adjoint pair $\iota^*_A \dashv \iota^A_*$, like all adjoint pairs, induces a unit map

$$\mathcal{F} \xrightarrow{\eta^A} \iota^A_* \iota^*_A.$$

Observe that the component η^A_U of this map acts by taking a section t at U to the germ of t around A, denoted $\langle t \rangle_A$. Applying the functor ι^*_B to η^A yields the given map. □

There is, of course, likewise a map $\mathcal{F}|_A \to \mathrm{f}^*_{AB}\,\mathcal{F}|_B$. When we later impose a recurring simplification that one of the fringe functors, here f_{AB}, is trivial we will have that $\mathrm{f}^*_{AB}\,\mathcal{F}|_B$ is the terminal sheaf. Thus the map $\mathcal{F}|_A \to \mathrm{f}^*_{AB}\,\mathcal{F}|_B$ is the terminal map in those cases.

The map given in the previous lemma can be interpreted as specifying for a section of $\mathcal{F}|_A$ the germ relative to the fringe of those sections of $\mathcal{F}|_B$ which can be consistently glued to the given section. We make this generalization of the situation for gluing spaces precise in the following definition and accompanying lemma.

This definition simply names in general the data obtained in Lemma 2.4.3 for the case of a general fringe functor between arbitrary topological spaces.

Definition 2.4.4. Let A and B be two spaces with $\mathrm{f} : \mathrm{Sh}(A) \to \mathrm{Sh}(B)$ a left exact functor. We call f **the fringe functor**. Let $\mathcal{F} \in \mathrm{Sh}(A)$ and $\mathcal{G} \in \mathrm{Sh}(B)$ be two sheaves over A and

B respectively. Then **gluing data for $\mathcal{F}$ and $\mathcal{G}$ with respect to the fringe data** f is given by a morphism

$$g : \mathcal{G} \to \mathrm{f}\,\mathcal{F}$$

of sheaves on B.

Note that the fringe functor f induces also a fringe functor $\mathrm{f}\,|_{\mathcal{O}(A)} : \mathcal{O}(A) \to \mathcal{O}(B)$ by restricting to the lattice of subterminal objects in $\mathrm{Sh}(A)$, which is canonically isomorphic to $\mathcal{O}(A)$. We can now give necessary and sufficient conditions for the gluing of sections of arbitrary sheaves, analogous to those given in Definition 2.2.6.

Theorem 2.4.5. *Let $f : \mathrm{Sh}(A) \to \mathrm{Sh}(B)$ be a fringe functor (i.e., a left exact functor). Let $X := A \cup_f B$ be the gluing of A with B by f. Let $\mathcal{F} \in \mathrm{Sh}(A)$ and $\mathcal{G} \in \mathrm{Sh}(B)$ be two sheaves, with gluing data $g : \mathcal{G} \to f\mathcal{F}$ with respect to f. Then there is an essentially unique sheaf $\mathcal{F} \cup_g \mathcal{G} \in \mathrm{Sh}(X)$ such that $(\mathcal{F} \cup_g \mathcal{G})|_A = \mathcal{F}$, $\mathcal{F} \cup_g \mathcal{G})|_B = \mathcal{G}$, and with g equal to the gluing data induced by lemma 4.2. This sheaf, say $\mathcal{H} := \mathcal{F} \cup_g \mathcal{G}$, has the property that sections r of $\mathcal{H}(U)$ are in bijection with pairs of sections s, t of $\mathcal{F}(U \cap A)$ and $\mathcal{G}(U \cap B)$ respectively for which $g_{U \cap B}(t) = \langle s \rangle_B$ where*

$$\langle s \rangle_B$$

is the image of s under the inclusion into the colimit $f^*_{BA}(\mathcal{G})(U \cap B)$. Note that this image is well defined, since U is open in X if and only if $U \cap B \subset \mathrm{f}_{BA}(U \cap A)$.

Proof. Consider the associated étalé spaces to $\mathcal{F}$ and $\mathcal{G}$, denoted respectively by $\acute{\mathcal{F}}$ and $\acute{\mathcal{G}}$. We topologize this by taking as a basis the images of all pairs (s, t), where s is a section of $\mathcal{F}$ over $U \cap A$, t is a section of $\mathcal{G}$ over $U \cap B$, and the set U is open in X with the Artin gluing topology, provided that $g_{U \cap B}(t) = \langle s \rangle_B$. That is, given such a pair (s, t) we define

$$s \cup_g t : U \to \acute{\mathcal{F}} \coprod \acute{\mathcal{G}}$$

by the rule

$$x \mapsto \begin{cases} s(x) & x \in A \\ t(x) & x \in B. \end{cases}$$

The basis is then given by the image $(s \cup_g t)(U)$. We denote $\acute{\mathcal{F}} \coprod \acute{\mathcal{G}}$ with the resulting topology by

$$\acute{\mathcal{F}} \cup_g \acute{\mathcal{G}}.$$

By construction, this makes the projection $\acute{\mathcal{F}} \cup_g \acute{\mathcal{G}} \to X$ a local-homeomorphism, hence the domain is étalé over X and the sections of this bundle give a sheaf on X. Denoting the corresponding sheaf of sections by $\mathcal{F} \cup_g \mathcal{G}$, we have the desired sheaf. This sheaf is essentially unique as the sheaf must have the same germs as the desired sheaf, and sheaves are determined by their germs. □

The generalization to a family $\{X_i\}_I$ of spaces indexed by a poset $(I, \leq)$ with fringe data $\{\mathrm{f}_{ij} : \mathrm{Sh}(X_j) \to \mathrm{Sh}(X_i)\}$ is straightforward.

Definition 2.4.6. Let $\{X_i\}$ be a family of spaces with fringe data $f = \{\mathrm{f}_{ij}\}$. Let

$$\{\mathcal{F}_i \in \mathrm{Sh}(X_i)\}$$

be a family of sheaves over the respective X_i. Then **gluing data for $\{\mathcal{F}_i\}$ relative to f** is given by a family of sheaf morphisms $\{g_{ij} : \mathcal{F}_i \to \mathrm{f}_{ij}\,\mathcal{F}_j\}$ such that for all $i < j < k$ we have

$$g_{ik} = \mathrm{f}_{ij}(g_{jk})g_{ij}$$

We will now give one of our main results. The following theorem gives a construction of the disjoint gluing (a.k.a. Artin gluing) of a family of sheaves over a family of respective spaces into a sheaf on the disjoint gluing of those spaces. Note that there is already a proof of this result for the case of finite decompositions in for instance Johnstone [9] and in Wraith [26]. Our result is somewhat more general in that it applies to any conjunctive decomposition, for instance locally finite decompositions. It is also more simple in that, by restricting to the case of topological spaces, we can construct the sheaf directly via the étalé spaces in a manner similar to how the sheafification of a presheaf is constructed.

Theorem 2.4.7. *Let X be the disjoint gluing of the spaces $\{X_i\}$ with respect to the fringe data $\{\mathrm{f}_{ij}\}$. (For instance, if the $\{X_i\}$ form a conjunctive decomposition of a space X.) Let $\{\mathcal{F}_i\}$ be a family of sheaves over the respective X_i and let $\{g_{ij}\}$ be gluing data for the $\{\mathcal{F}_i\}$. Then there is an essentially unique sheaf $\mathcal{F}$ on X such that $\mathcal{F}|_{X_i} = \mathcal{F}_i$ and such that g_{ij} is the induced gluing data.*

Proof. Define $\mathcal{F}$ on open sets U of X by

$$\mathcal{F}(U) = \left\{(s_i)_{i\in I} \,\middle|\, s_i \in \mathcal{F}_i(U \cap X_i), g_{ij}(s_i) = \langle s_j \rangle_{X_i}\right\}$$

We then topologize $\coprod \acute{\mathcal{F}}_i$ by taking images of open sets U on X under the appropriate families of sections. That is, for each $(s_i) \in \mathcal{F}(U)$ as defined above, we have an associated map $\cup_g s_i : U \to \coprod \acute{\mathcal{F}}_i$ defined by

$$x \mapsto \begin{cases} s_i(x) & x \in X_i \end{cases}$$

and we take as a basis the images of all such maps.

Denoting the resulting topological space $\acute{\mathcal{F}}$, we have that the projection

$$\acute{\mathcal{F}} \to X$$

is a local homeomorphism by construction. □

Regarding the sheaves as étalé bundles, the $\mathcal{F}$ constructed from gluing $\mathcal{F}_i$ corresponds to $\acute{\mathcal{F}} \to X$ glued from $\acute{\mathcal{F}}_i \to X_i$. In particular, we have induced fringe morphisms $\mathcal{O}(\acute{E}_j) \to \mathcal{O}(\acute{E}_i)$ from the topology on $\acute{E}$. Note however that it is not enough to simply specify these fringe morphisms, as one has to ensure that the projection is a local homeomorphism. That is, the gluing data is necessary along with the fringe data to ensure that we obtain a space that is an étalé bundle over X.

Chapter 3

Stratified Spaces

Our fundamental concern is partitions of topological spaces. When lifting a surjective set function witnessing the partition to the category of topological spaces, there are two basic topologies that can be placed on the codomain. The coarsest, called the codiscrete or chaotic topology. The finest is the quotient topology, and any other lies between these two. While the quotient topology is useful for many purposes, it will also prove useful to have an additional structure on the codomain, namely an order relation that is compatible with the topology. We will primarily consider a special case which in particular will be the quotient topology, but at first we will consider potentially coarser topologies and study the general behavior of continuous maps into ordered sets.

3.1 Continuous maps to posets

A decomposition of a space X into $\{X_i\}_{i \in I}$ can be usefully equipped with topological information by way of asking that the set function $\pi : X \to I$ (or equivalently, $X \to \{X_i\}$) sending x to the (index of) its piece be a continuous function. This function is the surjection of sets witnessing the underlying set partition. It is interesting to consider topologies given by preorders on I. In particular, every continuous surjection onto a finite set is of this type.

Definition 3.1.1. Given a preorder $(P, \leq_P)$, the **specialization** or **Alexandroff** topology on P is the topology given by taking a set $U \subset P$ to be open if and only if U is **upwards**

closed, that is iff

$$p \in U \implies \forall q \geq p, q \in U.$$

Remark 3.1.2. The specialization topology defines a fully faithful embedding of the preorders into the topological spaces. We use the term **specialization topology** for these spaces as their specialization order is exactly the preordering defining the topology. Alexandroff spaces are often defined as spaces for which every intersection of open sets is open (equivalently, every union of closed sets is closed) and this definition is another characterization of spaces whose opens sets are exactly the sets upwards closed in the specialization order. They can also be characterized as those spaces which have the finest topology for which the inclusions of finite subspaces are continuous.

The closure of a singleton $\{p\}$ in the specialization topology is exactly the closed set given by the principle downset $\{q \,|q \leq p\}$. Thus p is a limit point of the set $\{p_\lambda\}$ if and only if p is in the set of all elements below p_λ for some choice of λ. Each point also has a **smallest open neighborhood** given by the principal upset $\{q \,|p \leq q\}$. Note that if the order on P is a partial order then each singleton is in fact locally closed as in that case $\{p\}$ is the intersection of the principal open upset and principal closed downset generated by it.

It will be helpful to understand specialization topologies in terms of their converging nets, in particular when we consider continuous maps valued in these spaces. The following proposition is an easy consequence of the definitions.

Proposition 3.1.3. *Let $(P, \leq_P)$ be equipped with the specialization topology. Let $\Lambda \to P$ given by $\lambda \mapsto p_\lambda$ be a net in P. Then p_λ converges to p_∞ if and only if there is an α such that $p_\infty \leq p_\lambda$ for all $\lambda \geq \alpha$.*

Proof. If p_∞ is the limit of the net p_λ, then in particular the net is eventually in the smallest neighborhood of p_∞, given by the principal upset. Hence the net is eventually above p_∞ as claimed.

Conversely, if it is eventually above p_∞ then it is eventually in the principal upset and hence eventually in any neighborhood of p_∞. □

Notation 3.1.4. It will be helpful to introduce a notational and linguistic convention to indicate the preorder generated by a set of relations on a set. For instance, we would like to quickly indicate the partial order $\{1 \leq 1, 1 \leq 2, 2 \leq 2\}$ by simply giving $\{1 \leq 2\}$ and inducing the rest by closing under reflexivity and transitivity. Thus, we will say **the preorder** R or the **preorder given by** R for the preorder generated by R.

Now we introduce a definition originally due to Jacob Lurie in [13]. Actually we will start with a slight generalization of his definition, allowing for preorders. Some of the basic results are true even at this level of generality, and we can use these notions to rule out the existence of certain decompositions. The idea is that this fundamental structure arises in all the familiar cases of a locally finite decomposition of a space into locally closed subsets. We will see that this structure provides the fundamental data needed to work with stratified spaces in arbitrary contexts.

Definition 3.1.5. Let X be a topological space. A **preorder stratified space** or **prestratified space** is a continuous map $\pi : X \to P$ where P is a preorder with the specialization order. If $(P, \leq)$ is in fact a partial order, then $\pi : X \to P$ is called a **poset stratified space**, **partially stratified space**, or for short **parstratified space**. We call the nonempty sets $X_p = \pi^{-1}(p)$ for $p \in P$ the **strata** of the prestratification. When we wish to emphasize the decomposition, we will specify, for instance, $X \to \{X_i\}$ and leave the preorder on the index set I implicit.

While we will mainly be interested in special parstratified spaces, there are useful observations to be made even of prestratified spaces so we will phrase some early results in terms of these.

Example 3.1.6. (1) Any surjective function $X \to P$ with the codomain equipped with the codiscrete topology, with no open sets beyond the empty set and the whole codomain itself, corresponds to a prestratification by the codiscrete poset where $p \leq q$ for all $p, q \in P$.

(2) More interestingly, consider any partition of a space X into an open set U and its closed complement $K = X \setminus U$. This can be naturally equipped with the preorder given by $K \leq U$, which is in fact a partial order. Then the quotient map $X \to \{U, K\}$ is a parstratified space.

(3) Consider a manifold M with boundary ∂M. Then ∂M is a closed subset, so this is a special case of the above.

(4) Consider a space partitioned into a locally closed set L and the open and closed sets U and K forming its complement (note that the complement of any locally closed set is a union of an open and a closed set). Then the order $K \leq L \leq U$ gives a parstratified space structure on the evident map.

(5) Let $X^0 \subset X^1 \subset \dots X^n = X$ be a filtration of a space X by closed subsets. We allow $n = \infty$, in which case X is the inductive limit of the x_i for $0 \leq i < \infty$. Then the $X_i = X^i \setminus X^{i-1}$ are locally closed and the order $X_1 \leq X_2 \leq \dots X$ gives the partition function a prestratified space structure. Classically, stratified space theory is the study of refinements of this approach typically requiring a notion of compatible smooth structures on the strata. We will consider sheaf theoretic approaches to these structures and generalizations later.

(6) As a special case of the previous example, consider $\mathbb{R}^\infty$ the inductive limit over the diagram generated by the inclusion maps $\mathbb{R}^n \to \mathbb{R}^{n+1}$ mapping $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 0)$. The filtration is given by the images of $\mathbb{R}^n \to \mathbb{R}^\infty : x \mapsto (x, 0)$.

(7) Consider the decomposition of $\mathbb{R}$ into the rationals $\mathbb{Q}$ and the irrationals $\mathbb{R} \setminus \mathbb{Q}$. This can be equipped with the codiscrete topology to obtain a prestratified space.

However, we will see soon that there can be no parstratified space structure on this partition.

(8) Consider the decomposition of $X \coloneqq \mathbb{R}_{\geq 0}$ given in the previous chapter as Example 2.3.1, with $X_0 = \{0\}$, $X_1 = (1, \infty)$, and for $i \in \mathbb{N}$ with i at least 1, $X_i = \left(\frac{1}{i}, \frac{1}{i+1}\right]$. Then the map $X \to \mathbb{N}$ with the atypical order $0 < \cdots i+1 < i < \cdots 2 < 1$ is a parstratification, and in fact is a quotient map. Note the odd ordering that is required to ensure continuity. In particular, since X_1 is open it must be maximal in the order and as X_0 is closed it must be minimal As we observed in the previous chapter, the associated decomposition is pathological in the sense that it is not conjunctive.

(9) Now consider a slight modification of the above example. We still take $X \coloneqq \mathbb{R}_{\geq 0}$ with $X_0 = \{0\}$. However, now we will let $X_1 = [1, \infty)$ and $X_i = \left[\frac{1}{i}, \frac{1}{i+1}\right)$. Now in order to ensure that $X \to \mathbb{N}$ is continuous we will have to use a different ordering on $\mathbb{N}$, and in fact there is no obvious best choice. Since X_1 is now closed it will need to be minimal. By continuity, we will also need to contain the order $1 < 2 < \cdots i < i+1 < \cdots$ since for each $i > 1$ there is a sequence with limit contained in X_{i-1}. There is no open set in the decomposition, so no index need be maximal. However, X_0 is still closed and thus 0 should be minimal in addition to 1. One option is to make $0 \leq 1 \leq 0$ so that both are minimal and the order on $\mathbb{N}$ is a preorder but not a partial ordering. With this we obtain a prestratification $X \to \mathbb{N}$. However, there is a pathology here in that, even though each X_i is locally closed we have not given $\mathbb{N}$ a partial ordering so that each index is locally closed in the specialization topology. We can rectify this by choosing, say, 2 to be the first index that 0 is less than, i.e. imposing $0 < 2$, so that 0 and 1 are both minimal but are themselves incomparable. Of course there is nothing special about 2 here, and we can choose instead any other natural number, say 3 or 4 or so on, to be the successor to 0 in our strange ordering. Each of these

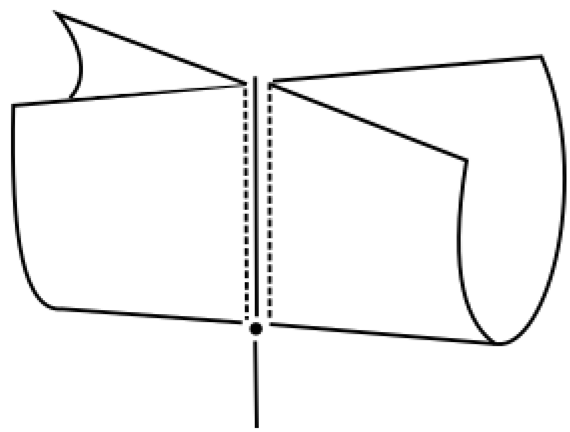


Figure 3.1: The Whitney umbrella, given by the real locus $V(x^2 = y^2 z)$ as a subspace of $\mathbb{R}^3$.

is in turn a refinement of the prior one, and there is no finest choice. In particular, these maps, though continuous, are not quotient maps.

(10) Let $X = V(x^2 = y^2 z)$ be the (real) vanishing locus of the polynomial equation $x^2 = y^2 z$ in $\mathbb{R}^3$. This space is also called the **Whitney umbrella**, as in [16]. Decompose this is via $X_0 = \{(0,0,0)\}$ the origin, $X_1 = \{(x,y,z) \mid x = y = 0, z \neq 0\}$ the z-axis minus the origin, and X_2 the remaining part of X given by the "leaves" of the umbrella. The Whitney umbrella will serve as part of the inspiration for our generalization of the Whitney (A)-regularity conditions in the sequel. This space is illustrated in Figure 3.1.

A basic motivation for prestratified spaces is that they capture a way in which (the image of) a prestratified space gives a partition of the space with a special topological relation between the subsets. Before we show this precisely, we introduce a piece of notation. Let $x_\cdot : \Lambda \to X$ be a net in a space X. We equip Λ with the discrete topology so that $x_\cdot$ is trivially a continuous map. Now let $\bar{\Lambda}$ be the set $\Lambda \coprod \{\infty\}$. We topologize $\bar{\Lambda}$ by declaring the neighborhoods of ∞ to be those subsets $U \subset \bar{\Lambda}$ with $\infty \in U$ such that $U \cap \Lambda$ contains an upwards closed subset of Λ, and so that Λ as a subspace carries the discrete topology. Note then that the net $\Lambda \to X$ extends to a continuous function $\bar{\Lambda} \to X$ given by assigning

$\infty \mapsto x_\infty$ if and only if the net converges to x_∞.

Lemma 3.1.7. *If a net $\Lambda \to X$ in a prestratified space $\pi : X \to P$, given by x_λ, converges to some x_∞ then $\pi(x_\infty) \leq \pi(x_\lambda)$ for all λ in some upwards closed set D of the directed set Λ.*

Proof. As noted above, a net converges if and only if extends to a continuous function $\bar{\Lambda} \to X$. This immediately induces a prestratified structure on $\bar{\Lambda}$, in particular, a continuous map, and thus a converging net in P. But a net converges to p_∞ in P with the specialization order if and only if the inverse image of any principle upset $\{q \,|p \leq q\}$ is open in $\bar{\Lambda}$. That is, if and only if there exists a $\lambda \in \Lambda$ such that $p_{\lambda'} \geq p_\lambda$ for all $\lambda' \geq \lambda$. Then $D = \left\{\lambda' \,\middle|\, \lambda' \geq \lambda\right\}$ is the desired upwards closed set. □

Corollary 3.1.8. *Given a converging net as above, x_∞ is in the closure of $\cup_{\lambda \in D} X_\lambda$. In particular, if $x_\infty \in X_p$ and if the net is eventually in X_q, then $X_p \cap \overline{X_q}$ is nonempty.*

This can usefully be phrased in terms of a requirement for the order $\leq$ on I for the codomain of any prestratified space.

Corollary 3.1.9. *For any prestratified space $\pi : X \to P$, we have $p \leq q$ whenever $X_p \cap \overline{X_q} \neq \varnothing$. Thus given any set partitioning $\{X_i\}_{i \in I}$ of a topological space X, any prestratified space structure on the indices I must contain the preorder generated by the relations $i \leq j$ whenever $X_i \cap \overline{X_j} \neq \varnothing$.*

By the above lemma, a prestratified space decomposes a space in a way that allows the preorder to retain some information about how points of one strata are limit points of other strata. In particular, the preorder necessarily includes the relation $p \leq q$ whenever X_p contains a limit point of X_q. This implies that, for instance, the decomposition of $\mathbb{R}$ into the rationals and irrationals thus cannot be given any refinement from a prestratified space into a partially stratified space.

Corollary 3.1.10. *There does not exist a parstratified space structure for the decomposition of* $\mathbb{R}$ *with its usual topology into the rationals and irrationals.*

We have seen that for any prestratified space $\pi : X \to P$ we have must have $p \leq q$ whenever $X_p \cap \overline{X_q}$ is nonempty. It is not hard to see that we can have $p \leq q$ also when $X_p \cap \overline{X_q} = \varnothing$. One can see this by considering simply the codiscrete topology on the codomain. More interestingly though, there are decompositions for which such relations are forced by transitivity and continuity.

Example 3.1.11. Consider $\mathbb{R}$ partitioned into $\{(-\infty, 0], (0, 1], (1, \infty)\}$. Then in order to have a prestratified structure on this partition we are forced to include the relations $(-\infty, 0] \leq (0, 1]$ and $(0, 1] \leq (1, \infty)$. When we close under transitivity, we thus also have $(-\infty, 0] \leq (1, \infty)$ even though $(-\infty, 0]$ contains no limit points of $(1, \infty)$.

A natural question is to ask whether for any partition $\{X_i\}_{i \in I}$ of a space X there is a finest prestratification with underlying set function the $\pi : X \to I$ witnessing the partition. We can show that this is not the case.

Example 3.1.12. Consider $X = \mathbb{R}$ decomposed into $X_0 = (-\infty, -1] \cup [1, \infty)$ and $(-1, 1)$. We will refine $(-1, 1)$ by taking the set difference with the decreasing sets $U_2 = (-\frac{1}{n}, \frac{1}{n})$ for $n \geq 2$. That is, we set $X_1 = (-1, 1) \setminus U_2$ and $X_n = U_n \setminus U_{n+1}$ for $n \geq 2$. The intersection of the U_n is $\{0\}$, so we take $X_\infty = \{0\}$. Then the $\{X_i\}$ indexed by $I = \{0, 1, 2, \ldots\} \cup \{\infty\}$ partition X.

Any preorder on I for which $X \to I$ is a prestratified space must have $0 < 1 < 2 < \ldots$. But since X_∞ is closed we must have ∞ minimal. It is further not open, so it cannot be also maximal, hence we must have $\infty < k$ for some $k \neq 0$. Each choice of k suffices for a prestratified space structure, and each $k' > k$ provides a refinement. There is hence no finest prestratified space structure.

This example can be generalized using any closed set X_0 of an arbitrary space X, and a closed set X_∞ that is an intersection of open sets contained in $X \setminus X_0$. In particular, this works for a choice of X_∞ given by a closed G_δ subset of X (such as arbitrary G_δ subsets of G_δ spaces such as $\mathbb{R}$).

Of course, the quotient topology associated to $X \to I$ is the finest continuous map factoring through $X \to I$. But it is not always a specialization topology on I for some preorder $\leq$, as the above example shows. Another good example of this phenomena is to take a more familiar quotient, say $\mathbb{R} \to \mathbb{S}^1$. The pieces of the partition that the underlying set map represents are all discrete as subspaces of $\mathbb{R}$.

Any prestratified space structure refining the codiscrete case demonstrates some non-trivial topological information contained in the decomposition, but even when the strata X_i are locally closed we cannot guarantee a **poset** structure on I. Thus we are led to consider an extra condition which is sufficient for the existence of a finest prestratification.

In fact, the extra condition we will consider will have a few equivalent forms. It will in particular entail for prestratified spaces that the map is open. This follows from [22, Lemma 2.3] and the observation by [27, Remark 3.23] that the result easily generalizes to preorders. Thus in these cases the given preorder results in a quotient map and hence the finest possible.

3.2 The Frontier Condition

Given a prestratified space, one would like to remove redundant information from the preorder as much as possible. We have already seen that if the pieces of the partition are not "nice" enough then it might not be possible to remove any of this information, as in the case of the reals decomposed into the rationals and irrationals. However, even asking that the partition be equippable with a partial order is not enough. In particular, transitivity sometimes forces us to maintain some unnecessary information relating strata whose closures are disjoint. For instance, consider $\mathbb{R}$ decomposed into $(-\infty, 0] \leq (0,1] \leq (1,\infty)$. This is an example of the parstratified space induced by the locally closed set $(0,1]$. But by transitivity we have $(-\infty, 0] \leq (1, \infty)$. Notice also that the image of the open set $(-1, 1)$ under the decomposition map $X \to \{(-\infty, 0], (0, 1], (1, \infty)\}$ is the non-open set $\{(-\infty, 0], (0, 1])\}$. The **frontier condition** on a partition $\{X_i\}$ will prevent such redundant information by asking that the boundaries of strata should be decomposed as part of the partitioning.

Definition 3.2.1. Let $\{X_i\}_{i \in I}$ be a partition of a space X. We say that it is a **frontier decomposition** or that it satisfies the **frontier condition** if for any i, j we have

$$X_i \cap \overline{X_j} \neq \varnothing \iff X_i \subset \overline{X_j}.$$

Note that even the prestratification of $\mathbb{R}$ into the rationals and irrationals gives a frontier decomposition of $\mathbb{R}$. Yet the prestratification is necessarily codiscrete and so still does not carry additional topological information.

However, when the strata are locally closed the frontier condition guarantees an interesting partial order refining the codiscrete order. If this order also gives rise to a parstratified space, then it is an open map and hence a quotient map. Thus we are in the most interesting case of a finest possible order that carries exactly the relevant topological information and no more than that.

If a decomposition satisfies the frontier condition, then we can impose a canonical partial ordering on the X_i (equivalently, on the indexing set I) by the following.

Definition 3.2.2. Let $\{X_i\}$ be a partition of a space X. The **frontier ordering** is the relation $\leq_\partial$ on the partition given by

$$X_i \leq_\partial X_j \iff X_i \cap \overline{X_j} \neq \varnothing.$$

When the frontier ordering is clear, we will simply write $\leq$.

Of course, if the partition doesn't satisfy the frontier condition then the partition function $X \to \{X_i\}$ can easily fail to be continuous when $\{X_i\}$ has the topology induced by the frontier ordering. In particular, the frontier ordering might be discrete! For a locally finite frontier partition, it is automatically true that we have as finite spaces are necessarily specialization topologies. However, for an infinite frontier partition the map still might fail to be continuous.

In Yokura's [27, Proposition 4.2], a sufficient condition for this map to be continuous (due originally to Hiro Lee Tanaka) is noted by the author. Namely, there is the following.

Proposition 3.2.3 ([27]). *Let $X \to \{X_i\}_{i \in I}$ be a frontier partition of a space X and let $\leq$ be the frontier ordering on I. If $X \to (\{X_i\}, \leq)$ is continuous if and only if for all downwards closed subsets $C \subset (I, \leq)$ we have*

$$\bigcup_{i \in C} \overline{X_i}$$

is closed in X.

We even obtain a particularly nice poset stratified space. That is, the map is an open map and hence a quotient map.

Using the earlier lemmas, we can show that for a frontier decomposition the fringe data is particularly simple. In fact, we can give an equivalent condition on the fringe data guaranteeing it arises from a frontier decomposition.

Lemma 3.2.4. *Let X be a topological space. Let $\{X_i\}$ be a frontier decomposition. Then for all $i < j$, the fringe morphism f_{ji} is constant to X_j and f_{ij} is non-constant.*

Furthermore, for an arbitrary poset stratified space structure on I, if there exists an $i < j$ for which f_{ij} is constant then the frontier condition fails.

Proof. By Lemma 2.1.4, f_{ji} is trivial with constant value X_j if and only if $X_j \cap \overline{X_i} = \varnothing$. By the frontier condition, $i < j$ if and only if $X_i \cap \overline{X_j} \neq \varnothing$ and $i \neq j$. Hence, we have the first claims immediately.

Likewise, if the poset structure on I is such that for all $i < j$ the above conditions hold, then $i < j$ is exactly the frontier ordering. □

Note that for an arbitrary poset stratified space, whenever $X_i \cap \overline{X_j} \neq \varnothing$, by continuity we then have $i \leq j$. Hence, when we fail to have $j \leq i$ we must have $X_j \cap \overline{X_i} = \varnothing$ so that f_{ji} is constant. The important point is that the f_{ij} are all non-constant if and only if the frontier condition holds. In particular, one can construct poset stratified spaces such as $(-\infty, 0] \cup (0, 1] \cup (1, \infty) = \mathbb{R}$ with ordering $(-\infty, 0] \leq (0, 1] \leq (1, \infty)$, for which $\mathcal{O}((1, \infty)) \to \mathcal{O}((-\infty, 0])$ is constant where each interval is given the subspace topology relative to the standard topology on $\mathbb{R}$.

Corollary 3.2.5. *If U is open and $A \subset X \setminus U$, then the fringe $\mathcal{O}(A) \to \mathcal{O}(U)$ is constant.*

3.3 Equivalence of Prestratified Spaces

Consider the parstratification of $\mathbb{R}$ given earlier by the decomposition

$$P_0 = \{(-\infty, 0), \{0\}, (0, \infty)\}.$$

The decomposition $P_1 = \{(-\infty, 0) \cup (0, \infty), \{0\}\}$ is a coarser one. Equipping it with the obvious order $\{0\} \leq (-\infty, 0) \cup (0, \infty)$, the map $\mathbb{R} \to P_1$ is also a parstratified space. Around each point $x \in \mathbb{R}$, these decompositions are essentially the same. In the classical setting of stratified spaces, one can define a stratified space a la Mather in [15] as an equivalence class of decompositions which induce the same set germs at each point to make this notion precise and eliminate this redundancy. Here we will show that this notion can also be captured by a notion of equivalence generated from certain monotone maps.

The map $P_0 \to P_1$ mapping $(-\infty, 0), (0, \infty) \mapsto (-\infty, 0) \cup (0, \infty)$ and $\{0\} \mapsto \{0\}$ is monotone, and so is continuous when the posets are equipped with the specialization topology. It also commutes with the stratifications. To capture that this map also induces equivalent set germs, it will suffice to observe that this map also satisfies the property that it is **bijective on chains**. We will show that such an equivalence induces an equivalence of decompositions in the sense of Mather.

First we recall the definition of (maximal) chains in ordered sets.

Definition 3.3.1. Let $(P, \leq)$ be a preordered set with associated strict order $<$. A **chain** is a subset C of P that is linearly ordered by $<$. That is, for any $p_1, p_2 \in C$ either $p_1 < p_2$, $p_2 < p_1$, or $p_1 = p_2$. A chain is called **maximal** if $C \cup \{p\}$ is not a chain for any $p \in P \setminus C$. That is, C is maximal if any $p \in P \setminus C$ is incomparable with some element of C.

Recall that a functor $F : C \to D$ between two categories is called essentially surjective if for each $d \in D$ there is a $c \in C$ such that $F(c)$ is isomorphic to d in D. We will only use the particular case where C, D are posetal categories and F is a monotone map.

Definition 3.3.2. Let $f : P \to Q$ be a monotone map of posets. We call f a **weak equivalence of specialization spaces** if it is essentially surjective and bijective on chains. By bijective on chains, we mean that given any chain C, $f|_C : C \to f(C)$ is a bijection. Equivalently, f is surjective and bijective on maximal chains.

We call this a weak equivalence, as for instance it is not reflexive. Though we have phrased the definition in terms of chains, it suffices to show that the map does not collapse any strict inequalities $p < q$, as the following lemma shows.

Lemma 3.3.3. *A monotone map $f : P \to Q$ of posets is a weak equivalence if and only if f is surjective and for all $p, q \in P$, $p < q$ implies that $f(p) < f(q)$.*

Proof. The forward direction is immediate, since $p < q$ is a chain in P. For the reverse direction, the easiest proof is by contraposition. So, suppose that f is a surjection that is not a weak equivalence, i.e. that there is some chain C in P such that $f|_C$ is not injective. Then there is at least one pair $p \neq q \in C$ such that $p < q$ but $f(p) = f(q)$. □

Now we consider the equivalence relation induced by this notion of weak equivalence.

Definition 3.3.4. Two posets P and Q are said to be **weak equivalent** if there is a zigzag of weak equivalences $P \to P_1 \leftarrow P_2 \to \ldots Q$ or $P \leftarrow P_1 \to P_2 \leftarrow \ldots Q$. That is, weak equivalence is the equivalence relation generated by the relations $P \sim Q$ whenever there is a weak equivalence of specialization spaces $P \to Q$.

To see that such a map induces equivalent set germs, we will use the following lemmas. The first shows that the (nontrivial) images of a weak equivalence are disconnected subsets.

Lemma 3.3.5. *Let $X \to P$ and $X \to Q$ be two prestratifications of a topological space X. Let $f : P \to Q$ be a weak equivalence commuting with the stratifications. Let $f^{-1}(q)$ be the preimage of some $q \in Q$. Suppose that $f^{-1}(q)$ contains at least two points. Then*

$$X_q = \bigcup_{p \in f^{-1}(q)} X_p$$

is disconnected a subspace of X.

Proof. Observe that the fact that f is a weak-equivalence implies that $f^{-1}(q)$ is a cochain in P. Thus for all $p \neq p' \in f^{-1}(q)$ (of which at least one such pair exists) we have that

$$X_p \cap \overline{X_{p'}} = \varnothing.$$

Hence X_q is disconnected. □

This next lemma shows that the set germ determined by a disconnected set is determined by the connected component of the set containing the point around which the germ is considered.

Lemma 3.3.6. *Suppose $Y \subset X$ is a disconnected subspace of the topological space X. Let $x \in Y$, and let Y' be the connected component of Y containing x. Denote the set germ of a set A around a point x by $[A]_x$. Then*

$$[Y]_x = [Y']_x.$$

Proof. Since Y is disconnected, $Y = Y' \cup Y''$ and there exists open sets U, V of X such that $U \cap V = \varnothing$, $Y' \subset U$, and $Y'' \subset V$. Then

$$[Y]_x = [Y \cap U]_x = [Y']_x.$$

□

Now we can show that the notion of weak equivalence gives rise to the usual notion of equivalence of decomposition due to Mather, namely that the set germs around each point are equal.

Theorem 3.3.7. *Let $X \to P$ and $X \to Q$ be two prestratifications of a topological space X. Let $x \in X_p \cap X_q$. Let $f : P \to Q$ be a weak equivalence commuting with the prestratifications. Then*

$$[X_p]_x = [X_q]_x.$$

Proof. Immediate from the previous two lemmas. □

The upshot is that we can define the equivalence of decompositions categorically, i.e. in terms of the existence of certain maps. This is almost enough for us to define a useful notion of stratification that applies in the case of decompositions which are not necessarily locally finite. The last piece we will need is an understanding of the so-called frontier condition, which is typically taken as part of the definition of decomposition in the finite dimensional contexts.

When considering stratified spaces, we often will not care much about the specific choice of decomposition up to the equivalence given above. Thus as a matter of notation we will introduce the following.

Definition 3.3.8 (Stratified Spaces)**.** Let X be a topological space. Then a **stratified space** structure on X is an equivalence class of poset-stratified space structures $X \to P$ satisfying the frontier condition. We will denote a stratified space by $X \to \mathcal{S}$.

3.4 Stratifications and Conjunctive Decompositions

Later we will need to know if the decomposition induced by a stratification is conjunctive in order to perform gluing operations on bundles over the strata. We have already seen an example of a decomposition which is not conjunctive. Here we begin by elaborating on this example and showing that the decomposition is associated to a parstratification.

Example 3.4.1. Recall the decomposition given earlier of $X := \mathbb{R}$ into $X_0 = \{0\}$, $X_1 = (1,\infty)$ and $X_n = \left(\frac{1}{n}, \frac{1}{n-1}\right]$ for $n \geq 2$. Then the map $X \to \mathbb{N}$ taking $x \mapsto n$ where $x \in X_n$ is continuous if we equip $\mathbb{N}$ with the specialization topology associated to the order

$$0 < \ldots 3 < 2 < 1.$$

In particular, notice that since X_0 is closed we must have 0 be minimal in the ordering even though $X_0 \cap \overline{X_n}$ is empty for $n > 0$. This relation is obviously a partial order, hence we have a parstratification of $\mathbb{R}$. However, as we have already seen, the decomposition $\{X_n\}_{n \in \mathbb{N}}$ is not conjunctive. Thus having a prestratification or even a parstratification is insufficient in general to ensure that the associated decomposition is conjunctive.

Observe that the decomposition given above fails to satisfy the frontier condition. We can also show that a decomposition satisfying the frontier condition is likewise insufficient, using a modification of the above decomposition.

Example 3.4.2. Again let $X := \mathbb{R}$. Take $X_0 = \{0\}$ and $X_1 = (1,\infty)$ as before. However, now let $X_{1'} = \{1\}$, $X_n := \left(\frac{1}{n}, \frac{1}{n-1}\right)$, and $X_{n'} := \left\{\frac{1}{n}\right\}$ for $n \geq 2$. Then this decomposition satisfies the frontier condition. The same set from the previous example, $U = X_0 \cup X_1$, is a witness to the failure of conjunctivity.

Notice that in the previous example, even though each stratum is locally closed we cannot equip the indexing set $P := \mathbb{N} \cup \{1', 2', \ldots\}$ with a partial ordering making the map continuous. In particular, as $\{0\}$ is closed, 0 must be minimal in the ordering, so there must be some other elements which are larger. However, in order to ensure continuity, the set of elements above 0 must give an open set in X. With the given decomposition, there must be a chain of elements $\{n, n'\}$ with $n \geq N$ for some $N \in \mathbb{N}$ large which lie above 0 in the ordering. However, the sets $X_{n'} = \left\{\frac{1}{n}\right\}$ are also closed hence n' should also be minimal.

Hence, we must have a preorder for which 0 and some infinite chain of n' are equivalent, i.e there must be some $N \in \mathbb{N}$ such that for $n \geq N$ we have $0 \leq n'$ and $n' \leq 0$. This shows that even a frontier decomposition in locally closed subsets can fail to be conjunctive.

Finally, one might conjecture that it suffices to have a frontier decomposition of the space X into locally closed subsets X_i such that the map $X \to \{X_i\}$ is a parstratification for some ordering on the indices. Note that in this case, we furthermore have a quotient map onto the set of strata which are equipped with the frontier ordering. Unfortunately, we have neither a proof nor a counterexample for this claim. Thus we simply state the following conjecture.

Conjecture 3.4.3. *Let $X \to \mathcal{S}$ be a stratified space. That is, $\mathcal{S}$ carries a partial order given by the frontier ordering and each X_i is locally closed in X. Then $\{X_i\}_{i\in\mathcal{S}}$ is a conjunctive decomposition.*

However, we can demonstrate conjunctivity in a case sufficiently general for our purposes.

Theorem 3.4.4. *Let $\pi : X \to \mathcal{S}$ be a stratified space. Suppose that for each $i \in \mathcal{S}$ that the subset $\uparrow i := \{j \mid i \leq j\}$ satisfies the descending chain condition. Then the induced decomposition $X \to \{X_i\}$ is conjunctive.*

Proof. Let U be a subset of X such that for all $i \in \mathcal{S}$ we have $U_i := U \cap X_i$ is open in X_i and for which $U_i \subseteq \mathrm{f}_{ij}(U_j)$ for all $i, j \in \mathcal{S}$. We will show that U is open in X by taking a net $(x_\lambda)_{\lambda\in\Lambda}$ with limit point $x_\infty \in U$ and showing that this net is eventually contained in U.

By continuity of the stratification $X \to \mathcal{S}$, the net $i_\lambda := \pi(x_\lambda)$ converges in $\mathcal{S}$ to some $i_\infty := \pi(x_\infty)$. By the definition of the topology on $\mathcal{S}$, we have that i_λ is eventually in $\uparrow i_\infty$. So without loss of generality we can assume that x_λ is contained in $X_{\uparrow i_\infty}$.

Now, since the set $\uparrow i_\infty$ satisfies the descending chain condition, for any index i_λ there are only finitely many indices $I \subset \Lambda$ such that $x_{\lambda'} \in X_I$ for all $\lambda' > \lambda$, where $X_I = \bigcup_{i\in I} X_i$.

Again, without loss of generality, we assume that the net x_λ is contained in X_I. Take $I' = I \cup \{i_\infty\}$ which is still a finite set. By the classical theory of Artin gluing, we have that $U_{I'} \coloneqq \bigcup_{i \in I'} U_i$ is open in $X_{I'} = X_I \cup X_{i_\infty}$. Since $X_{I'}$ contains the net x_λ and the limit point x_∞ the net also converges in $X_{I'}$. Hence the net x_λ is eventually contained in $U_{I'}$. As $U_{I'} \subseteq U$ and the net was arbitrary we have that the net is eventually contained in U. Hence U is open. □

Chapter 4

Gluing Bundles

Sheaves and hence étalé bundles can be glued provided fringe and gluing data are provided. This is outlined in the appendix. However, sheaves often arise not from étalé bundles (to which they are in fact equivalent) but from more general bundles. It is natural to ask if, given a family of bundles $E_i \to X_i$, we have a notion of gluing that applies to this case.

In particular, consider an arbitrary bundle $E \to X$. Suppose that there is a decomposition of X into an open set U and its complement K such that the pullback bundles U^*E and K^*E are fiber bundles with respective fibers F_U and F_K. Note that this does not mean that the sheaf of sections of $E \to X$ restricts to locally trivial bundles. In particular, the germs of $E \to X$ over $x \in K$, if x is a limit of some net in U, carry information about the topology over U as well. This is most obvious if K is a single point. In this case, the pullback bundle is just the fiber of E over the point.

Thus we must consider not just the bundles but some additional data specifying how to glue the bundles together over X. We will show that gluing data on a family of sheaves can be used to glue any family of bundles provided that there is a map of bundles from the étalé bundles to the given bundles.

4.1 Gluing Data for Bundles

A bundle over a space X is, in its simplest form, a surjective map $\pi : E \to X$. The continuity of the map creates a sense in which the topology of the space E must vary over the space X. If the space X is pre- or parstratified, say $X \to P$, then $E \to P$ is likewise. However if $X \to S$ is stratified, i.e. is an open map, then $E \to S$ is not necessarily so. It will also be convenient technically to allow for a refined stratification of the total space E of a bundle. Therefore we will give the following definitions.

Definition 4.1.1. A **prestratified bundle** over a prestratified space $X \to P$ is a prestratified space $E \to Q$ together with a map of stratified spaces $E \to X$ and $Q \to P$ such that $E \to X$ is surjective. A **parstratified bundle** over a parstratified space is a prestratified bundle such that Q is partially ordered. A stratified bundle is an equivalence class of parstratified bundles $E \to R$ over a stratified space $X \to \mathcal{S}$. In the cases we will consider here, we will typically have $Q = P$ so that E and X are have the same poset underlying the stratification.

Note that in each case, the stratified notion is merely a bundle in the appropriate category. For each (pre)stratified bundle $\pi : E \to X$ and each strata X_i of X, there is a bundle $E_i \to X_i$ where $E_i = \pi^*(X_i)$ is the pullback.

Let $X \to \{X_i\}$ be a decomposed space (e.g., through a stratification). Suppose that $\{\pi_i : E_i \to X_i\}$ is a family of bundles. We mean this in the most general sense of a surjective map $\pi_i : E_i \to X_i$. We have already seen how to glue together sheaves, and hence étalé bundles, together over conjunctive decompositions. In particular, we can apply this to locally trivial sheaves corresponding to fiber bundles.

In the case of classical stratified spaces, a particular instance of this occurs when the sections of the tangent bundles on the strata (each stratum being a smooth manifold) are glued together to give sections of the stratified tangent bundle. However, the stratified tangent bundle is not itself an étalé bundle but instead is a stratified vector bundle. That is, it is a stratified fiber bundle in the above sense where the fiber bundles over each stratum are vector bundles with general linear structure groups. We would like to extend the notion of Artin gluing given for étalé bundles to understand how vector bundles can be glued together. It is clear that we must specify in some way how the desired sections of this bundle should glue together, and so utilizing a choice of étalé bundles is necessary.

In particular, consider a simple vector bundle such as the Möbius strip $M \to S^1$. Decompose $X \coloneqq S^1$ into a single point $X_0 \coloneqq \{0\}$ and its complement $X_1 \coloneqq X \setminus X_0$. The restricted bundle over X_0 is simply $\mathbb{R} \to \{0\}$, and the restricted bundle over X_1 is likewise trivial, $X_1 \times \mathbb{R} \to X_1$. This example shows that it is not enough to consider the associated étalé bundles to the specified vector bundles.

In particular, the associated étalé bundle to $\mathbb{R} \to X_0$ is the sheaf of functions $X_0 \to \mathbb{R}$, equivalently the set of points of $\mathbb{R}$ with the discrete topology. Hence, as part of the data for Artin gluing fiber bundles we expect to require in general an auxiliary choice of étalé bundle whose germs carry the additional topological information required. So we begin by defining gluing data for bundles as gluing data for a choice of étalé bundles along with quotient bundle maps to the given bundles.

Definition 4.1.2. **Gluing data** for the bundles $\{E_i \to X_i\}$ is given by a collection of étalé bundles $\left\{\acute{E}_i \to X_i\right\}$ together with bundle morphisms $\left\{\epsilon_i : \acute{E}_i \to E_i\right\}$, along with gluing data for the sheaves $g_{ij} : \acute{E}_i \to \mathrm{f}_{ij}\, \acute{E}_j$.

Our goal is to construct a bundle $\pi : E \to X$ whose fibers over $x \in X_i$ are given by the fibers of $E_i \to X$ over x, and whose topology is determined by the gluing data.

Definition 4.1.3. Let $\acute{E}$ be the Artin gluing of the $\acute{E}_i$ by the gluing data g_{ij}. Then the **Artin gluing bundle** of the E_i by the gluing data is given by taking the quotient topology on

$$E \coloneqq \coprod E_i$$

by the map $\epsilon : \acute{E} \to E$ where

$$\epsilon(e_i) = \epsilon_i(e_i).$$

Typically a more structured bundle is needed, i.e. a fiber bundle. However, a stratified fiber bundle carries more structure than simply being a stratified bundle such that over each strata the pullback bundle $E_i \to X_i$ is a fiber bundle. In particular, the fibers and structure groups over adjacent strata are compatible in a natural way. We can capture this by asking that the bundle map not only be made of fiber bundles strata-wise, but that the map also be a quotient map, a condition which comes for free as part of the definition of a fiber bundle.

Definition 4.1.4. Suppose X is stratified by $X \to \{X_i\}$. A bundle $\pi : E \to X$ is a **stratified fiber bundle** if

(1) For each i, $\pi_i := \pi|_{\pi^{-1}X_i}$ is a fiber bundle, and

(2) π is a quotient map.

If $X \to \mathcal{S}$ is a stratified space, and $E \to X$ a stratified fiber bundle with F_i the respective fibers over E_i and G_i the respective group actions, we will sometimes denote the stratified fiber bundle by

$$G \circlearrowright F \to E \to X \to \mathcal{S}.$$

When the original bundles are fiber bundles, we would like the result to be a stratified fiber bundle in the sense we defined here. In particular, we would like to confirm that the resulting bundle is a quotient map.

Lemma 4.1.5. *Let $X \to \{X_i\}$ be a conjunctive decomposition. Let $\{\pi_i : E_i \to X_i\}$ be a family of bundles which are quotient maps. Let $\epsilon = \{\epsilon_i : \acute{E}_i \to E_i\}$ with $g = \{g_{ij} : \acute{E}_i \to \mathrm{f}_{ij}\, \acute{E}_j\}$ be gluing data for the family of bundles relative to the fringe data $\mathrm{f} = \{\mathrm{f}_i\, j\}$, such that each ϵ_i is a quotient map. Let $\pi : E \to X$ be the Artin gluing of the $\{E_i\}$ via this gluing data. Then π is a quotient map.*

Proof. Suppose that U is an open subset of E that is saturated with respect to $E \to X$. Then each $U_i := U \cap E_i$ is a saturated open set of E_i. Hence each $\pi_i(U_i)$ is open in X_i as each π_i is a quotient map. Since $\epsilon^{-1}(U)$ is open in $\acute{E}$, we have that for all i, j

$$\pi_i^{-1}(U_i) \subset \mathrm{f}_{ij}(\pi_j^{-1}(U_j))$$

where we have taken the fringe in $\acute{E}$. Then by continuity and commutativity we have that for all i, j that

$$\acute{\pi}_i(\pi^{-1}(U_i)) \subset \mathrm{f}_{ij}(\acute{\pi}_j(\pi_j^{-1}(U_j)).$$

Since $X \to \{X_i\}$ is a conjunctive decomposition, we then have that

$$\pi(U) = \cup_i \acute{\pi}_i(\pi^{-1}(U_i)) = \cup_i \pi_i(U_i)$$

is open in X. Since U was an arbitrary open saturated set, we have that $\pi : E \to X$ is a quotient map.. □

We must also consider how the group actions are respected in the gluing data. Suppose that $E \to X \to \mathcal{S}$ is a stratified fiber bundle. Let $i < j$ in $\mathcal{S}$, i.e. X_i is contained in the frontier of X_j. Let U be an open set of X such that $U_i := U \cap X_i$ and $U_j := U \cap X_j$ each carry two trivializations over the respective fiber bundles E_i and E_j. We will denote the change of coordinates from one trivialization to the other by ϕ_i and ϕ_j respectively. Then for all $x \in U_i$, $\phi_i(x, -)$ is an element of the structure group G_i of the fiber F_i and likewise for $x \in U_j$, $\phi_j(x, -)$ is an element of G_j. Now suppose that x_λ is a net in U_j with limit $x_\infty \in U_i$, and suppose further that this net has a lift to the bundle e_λ with limit e_∞. In one trivialization, denote $e_\lambda \mapsto (x_\lambda, f_\lambda^1)$ and in the other $e_\lambda \mapsto (x_\lambda, f_\lambda^2)$ with $f_\lambda^1, f_\lambda^2 \in F_j$. Likewise denote $e_\infty \mapsto (x_\infty, f_\infty^1)$ and $e_\infty \mapsto (x_\infty, f_\infty^2)$ in the respective local trivializations. Then for each λ, let $g_\lambda = \phi_j(x_\lambda, -)$ and let $g_\infty = \phi_i(x_\infty, -)$. So, $g_\lambda \cdot f_\lambda^1 = f_\lambda^2$ and $g_\infty \cdot f_\infty^1 = f_\infty^2$. By continuity, we expect that g_λ has limit g_∞ so that G_i is indeed a subgroup of G_j. Thus we expect that a stratified fiber bundle have a stratified structure group in a sense we make precise below. We also expect that the induced gluing data be equivariant. So we give the following definitions.

Definition 4.1.6. A **stratified group** is a topological group G with a stratification $G \to \mathcal{S}$.

Before we give the next definition, we need to note some relevant facts from categorical logic. A fiber bundle with structure group G is a model of a certain algebraic theory, the theory of G-objects. Algebraic theories (also called Lawvere theories) are in particular examples of essentially algebraic theories which are exactly the theories whose models are preserved by left exact functors. Since each fringe morphism $f : \mathrm{Sh}(X_j) \to Sh(X_i)$ is a left exact functor, if $\mathcal{E} \in \mathrm{Sh}(X_j)$ is the sheaf of sections of a fiber bundle with structure group G then $f\mathcal{E} \in \mathrm{Sh}(X_i)$ likewise carries a G-action. For details, see a textbook on categorical logic such as [9]. Thus the following definition is well-formed.

Definition 4.1.7. Let $X \to \mathcal{S}$ be a stratified space. Let $\{G_i \circlearrowleft F_i \to E_i \to X_i\}_{i \in \mathcal{S}}$ be a family of fiber bundles. Then gluing data $\{g_{ij}\}$ for the bundles is called **fiber bundle gluing data** if each map g_{ij} is equivariant with respect to actions of G_i and G_j.

Now we give a few examples of these constructions for a few choices of bundles.

Example 4.1.8. (1) Given any conjunctive decomposition $X \to \{X_i\}$ and any fiber bundle $E \to X$ we obtain immediately a stratified fiber bundle. We can view this bundle as being glued together by the maps

$$\epsilon_i : \acute{E}_i \to E_i$$

where each $\acute{E}_i \to X_i$ is given by $\acute{E}|_{X_i}$, i.e. the restriction of the associated sheaf to the subset X_i. Each map ϵ_i acts by taking a germ $\langle s \rangle_x$ of a section s to x to its evaluation $s(x)$, i.e.

$$\epsilon_i(\langle s \rangle_x) = s(x).$$

(2) Now we will elaborate on the construction of the Möbius band bundle $M \to S^1$. First, we will illustrate the importance of the choice of étalé bundles by considering a pathological bundle.

Let $X := S^1 \subset \mathbb{R}^2$ be decomposed as $X_0 := \{(1,0)\}$ and $X_1 := X \setminus X_0$. Let $E_0 := \mathbb{R}$ with $\pi_0 : E_0 \to X_0$ the terminal map. Let $E_1 := X_1 \times \mathbb{R}$ with $\pi_1 : E_1 \to X_1$ the projection onto the first component. For $\acute{E}_0$ and $\acute{E}_1$, we will simply take the associated étalé bundles to E_0 and E_1 respectively with the canonical maps $\epsilon_0 : \acute{E}_0 \to E_0$ and $\epsilon_1 : \acute{E}_1 \to E_1$. As noted earlier, $\acute{E}_0$ is simply $\mathbb{R}$ with the discrete topology. For the gluing data $g_{01} : \acute{E}_0 \to \mathrm{f}_{01}\, \acute{E}_1$ we can take each $s \in \acute{E}_0$ to a fringe germ defined on a small enough fringe neighborhood U of x in X_1. Since for small enough U, U is disconnected, say as $U = V \cup W$, we can take s to the fringe germ associated to the mapping

$$x \mapsto \begin{cases} s & x \in V \\ -s & x \in W \end{cases}$$

to try to imitate the familiar twist on the Möbius strip. However, since $\acute{E}_0$ is discrete our choices for input germs of $\acute{E}_0$ were limited to discrete real numbers. The resulting Artin gluing bundle $\pi : E \to X$ is such that germs around 0 are **locally constant**, though they do in fact exhibit a twist. Thus we have failed to reproduce the familiar Möbius strip.

(3) Now we will consider the previous construction, but with some modifications to the étalé bundles and gluing data. Let $\acute{E}_1$ be as before, but let $\acute{E}_0$ be the étalé bundle of germs of real valued functions on X at the point $\{(1,1)\}$. For the gluing data $g_{01} : \acute{E}_0 \to \mathrm{f}_{01}\, \acute{E}_1$, we take a germ $\langle s \rangle_x$ of a section $s : U \to \mathbb{R}$ defined around $(1,0)$

to the fringe germ given by

$$x \mapsto \begin{cases} s(x) & x \in V \\ -s(x) & x \in W \end{cases}$$

where $U \setminus \{(1,0)\} = V \cup W$ as before. Note that formally the gluing data is much the same as before, but now germs of the resulting bundle $\pi : E \to X$ around the point $(1,0)$ are more diverse. We have successfully recreated the usual Möbius bundle by way of accessing these germs.

The last example illustrates a common theme among fiber bundles. Namely, that we will wish to have data about how the topology of a fiber over a lower stratum sits in the topology of a fiber over a higher stratum. This could be specified in different ways, but for our purposes it will suffice to consider stratified fibers over the same ordered set. In this way, each fiber over a stratum is given directly as a subspace of a fiber over a higher stratum.

4.2 The Whitney (A) Condition

Throughout this section we will consider spaces that are stratified by manifolds. That is, by a stratified space $X \to \{X_i\}$ we mean that in addition to the definition given before that each X_i is

Definition 4.2.1. Let $X \to \{X_i\}$ be a stratified space. Let $\mathcal{C} \subset \mathrm{Top}$ be a subcategory of the category of topological spaces. Then we say that $X \to \{X_i\}$ is **$\mathcal{C}$-stratified** if each $X_i \subset X$ is, in the subspace topology, an object of $\mathcal{C}$.

More generally, suppose that $\mathcal{C}$ is a category with a faithful functor to Top. Then we say that $X \to \{X_i\}$ is **$\mathcal{C}$-stratified** if each X_i is given as the image of an object of $\mathcal{C}$. In particular, each X_i must carry a given structure making it an object of $\mathcal{C}$.

Example 4.2.2. The most important examples of categories of interest are categories of manifolds and smooth maps. For ease of notation, we only consider spaces which are C^∞ in each case. We fix notation for these now.

- $\mathrm{Mfld}_{<\omega}$ is the category of finite dimensional smooth manifolds and smooth maps.
- $\mathrm{Mfld}_{\mathrm{Hilb}}$ is the category of manifolds and smooth maps modeled on Hilbert spaces.
- $\mathrm{Mfld}_{\mathrm{Ban}}$ is the category of manifolds and smooth maps modeled on Banach spaces.
- $\mathrm{Mfld}_{\mathrm{Fre}}$ is the category of manifolds and smooth maps modeled on Frechet spaces.

For each of the categories listed above, we can consider spaces stratified by the respective manifolds. In each case, the objects of the category come naturally equipped with a tangent bundle structure. It is useful to combine these tangent bundles together into a stratified tangent bundle. In the case of $\mathrm{Mfld}_{<\omega}$-stratified spaces, there is a natural condition due to Whitney which ensures that this can be done. The Whitney (A)-regularity condition is a necessary and sufficient condition for the tangent bundles $\{TX_i\}$ to glue to a stratified fiber bundle TX.

Definition 4.2.3. Let X_i, X_j be two strata. We say that (X_i, X_j) is **Whitney (A) regular** if for any sequence $\{x_\lambda\}$ in X_j converging to $x \in X_i$, if the tangent spaces $T_{x_\lambda}X_j$ converge to τ (in the Grassmanian bundle) then $\tau \supset T_x X_i$.

It is easy to see that the Whitney condition is equivalent to the existence of equivariant injective gluing data between the tangent bundles.

Lemma 4.2.4. *A locally closed subset $X \subset \mathbb{R}^n$ with stratification $\{X_i\}$ is (A) regular if and only if the induced maps*

$$\Gamma T X_i \to \mathrm{f}_{ij}\, \Gamma T X_j$$

are inclusions, where Γ is the functor $\mathrm{Top}/X \to \mathrm{Sh}(X)$ *taking bundles to their sheaves of sections.*

As an example of a stratification which fails to satisfy the Whitney (A) regularity condition, see Figure 4.1.

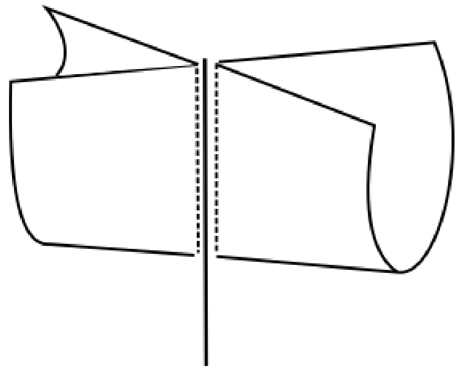


Figure 4.1: A stratification of the Whitney umbrella which fails to satisfy the Whitney (A) regularity condition at the origin. In particular, note that there are sequences which approach the origin but whose tangent plans do not contain the tangent space to origin as a point in the z-axis.

This motivates the use of conditions on the gluing data to generalize the Whitney (A) regularity condition to infinite dimensional manifolds, in particular those that fail to be Banach.

Definition 4.2.5. Let $X \to \{X_i\}$ be a $\mathrm{Mfld}_{\mathrm{Fre}}$-stratified space with tangent bundles $TX_i \to X_i$. We will say the stratified space is **(Whitney) (A)-regular** when it is equipped with injective gluing data.

This ensures a version the usual Whitney (A) criterion in the sense that if a sequence of points of a given stratum have limit in a lower stratum and a lift to the tangent bundle likewise has a limit then this limit is in some sense a subspace of the limiting space. In particular, this can be true even if there is not a well-behaved notion of Grassmanian bundle available to discuss the topology of subspaces with directly.

Chapter 5

Control Theory, Lie Group Actions, and Slices

In this chapter we explore some other aspects from the theory of stratfied spaces. We begin with control data. Control data gives a family of generalized tubes (typically arising from tubular neighborhoods) around the strata that are compatible in a certain sense. This is generalized from the presentation of Pflaum in [16]. Smooth proper Lie group actions often induce a singular foliation into orbits. When slices exist, one can introduce an equivalence relation on M that is coarser than the identification by orbits, namely the equivalence by orbit types. The associated decomposition turns out to refine the foliation into a stratification. These ideas share the common thread of being concerned essentially with neighborhoods of points which are normal to certain submanifolds, in the first case to strata of a stratified subspace and contained in higher strata, and in the second case normal to orbit spaces.

5.1 Control Theory

Control data in the context of finite dimensional stratified spaces was introduced by John Mather. Control data is closely related to tubular neighborhoods. Control data essentially specifies a way in which in which locally around a given stratum the higher strata are submersed onto it. In particular, this provides a useful way to characterize integrable stratified vector fields on the stratified space. Here we outline a generalization to certain infinite dimensional spaces.

5.1.1 Tubular Neighborhoods

The existence of control data hinges on a generalized tubular neighborhood theorem. We recall the definition of tubular neighborhoods and the classical existence theorem and then state and prove a generalization requiring compatibility with a prescribed map.

Definition 5.1.1. Let M be a manifold and S a submanifold of M, of class C^p. A **tubular neighborhood** of S is a vector bundle $\pi : E \to S$ along with a map $\phi : E' \to M$ that is a diffeomorphism of a neighborhood $E' \subset E$ of the zero section with a neighborhood of S in M.

There is a famous existence result for tubular neighborhoods that applies in at least the generality of paracompact Banach manifolds. We quote here the result of Lang's.

Theorem 5.1.2. *Let M be a manifold of class C^p for $p \geq 3$ that admits partitions of unity. Let S be a closed submanifold. Then there exists a tubular neighborhood of S in M of class C^{p-2}.*

To prove this, we use the split exact sequence defining the normal bundle to S to view the normal bundle as a subbundle of the tangent bundle of M restricted to S. Then we restrict a spray on $T(M)$ (whose existence is guaranteed) to $N(S)$, show that this is a local diffeomorphism, and then show that the domain can be shrunk so that it is in fact a diffeomorphism.

We will need an generalization of this theorem, in the style of Mather as in [15]. Namely, we will need a notion of compatibility of a tubular neighborhood with a given submersive map. Intuitively, a tubular neighborhood of a submanifold X in a manifold M is compatible with a smooth map f when the projection from the neighborhood onto the space X is done along level curves of f. More precisely, we give the following definition.

Definition 5.1.3. Let (T, ϵ, ρ) be a tubular neighborhood of a submanifold X of a manifold M. Let $f : M \to N$ be a smooth map. Then we say the tubular neighborhood is **compatible**

with f if $(f \circ \rho)|_T = f|_T$.

We will prove a simple extension of Mather's original existence result to Hilbert manifolds. We will use this theorem later to construct systems of tubular neighborhoods which are compatible with each other in a precise sense.

Theorem 5.1.4. *Let X be a C^{m+2} (locally closed) submanifold of a C^{m+2} manifold M, where $m \geq 1$ and with both manifolds modeled on Hilbert spaces and admitting partitions of unity. Let $f : M \to N$ be a smooth map that is submersive over X. Let $U \subset X$ be open in X and $A \subset U$ closed in X, and T_0 a tubular neighborhood of U in M compatible with f. Then there exists a tubular neighborhood T of of X compatible with f and equivalent to T_0 over A*

Proof. We will proceed by taking an arbitrary Riemannian metric θ of class C^{m+1} over M and adjusting the metric to one for which the associated flows move along the level curves of f. Let T be a neighborhood of X for which X is closed in T.

Let $P^f : \mathrm{T} \to \mathrm{End}(TM)$ be the projection valued map onto the kernel bundle of Tf. We choose P^f so that is self-adjoint with respect to θ, i.e. such that the projection is moreover θ-orthogonal onto the kernel bundle.

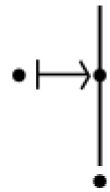
Let T^θ be the tubular neighborhood associated to θ of X. Let $P^\theta : \mathrm{T} \to \mathrm{End}(TM)$ be the θ-orthogonal projection onto the horizontal space of T^θ. Let $P^{f,\theta}$ be the θ-orthogonal projection onto the kernel bundle of $P^f - P^\theta$. Let $Q^{f,\theta} = P^f - P^{f,\theta}$. Note then that at each x the image of $Q^{f,\theta}_x$ is complementary to the image of P^θ_x in T_xM. Thus, there is a positive definite map $A_x \in \mathrm{End}(TM)$ such that the restriction of A_x to the image of P^θ_x is the identity and such that the image of $Q^{f,\theta}_x$ is the same as the image of Q^θ_x, where $Q^\theta_x = \mathrm{id} - P^\theta_x$.

Now we define a new Riemannian metric μ as follows:

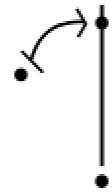
$$\mu_x(v_x, w_x) := \theta(A_x v_x, A_x w_x)$$

for all x, $v_x, w_x \in T_xM$. Thus, the images of P^θ_x and $Q^{f,\theta}_x$ are μ-orthogonal.

Now let $\mathcal{N} \to X$ be the μ-orthogonal bundle of TX in TM. Let exp be the exponential map of the connection associated to μ, and let $T^{\max}$ be the maximal tubular neighborhood



(a) A typical tubular neighborhood of a subspace in $\mathbb{R}^2$. Note that the projection to the subspace is orthogonal in the usual Riemannian metric on $\mathbb{R}^2$.



(b) The projection of a tube as part of a collection of control data. Note that now the projection is required to be radial with respect to the origin, to maintain compatibility with the projection to the origin.

Figure 5.1: A comparison of typical tubular neighborhoods with those arising in control theory.

of the zero section of $\mathcal{N}$. Then by construction we have for any vector x and $v \in \mathcal{N}_x \cap T^{\max}$ that

$$f(\exp(tv)) = f(x)$$

so that $T^{\max}$ is the desired tubular neighborhood. □

5.1.2 Control Data

Let X be stratified space with stratification $\mathcal{S}$. Control data will consist intuitively of a system of tubular neighborhoods around each of the strata along with projections and fiber-wise distance maps. This data is asked to be compatible in the sense that everything commutes wherever the composition is defined. We will give definitions that are slightly weaker than requiring actual tubular neighborhoods, though frequently these are available and will be singled out as a special case.

Definition 5.1.5. Let S be a stratum of the stratified space X. A **tube** around S is a neighborhood τ_S of S in X along with continuous maps $\pi_S : \tau_S \to S$ and $\rho_S : \tau_S \to \mathbb{R}_{\geq 0}$ such that the following conditions hold.

(1) Putting $\tau_{S,R} \coloneqq \tau_S \cap R$ for any stratum R, $\tau_{S,R} \neq$ implies $R \geq S$.

(2) π_S is a continuous retraction of S and for all strata $R > S$ the restricted map $\pi_{S,R} \coloneqq \pi_S|_{\tau_{S,R}}$ is smooth.

(3) $\rho_S^{-1}(0) = S$ and for all strata $R > S$, $\rho_{S,R} \coloneqq \rho_S|_{\tau_{S,R}}$ is smooth.

(4) The map $(\pi_{S,R}, \rho_{S,R}) : \tau_{S,R} \to S \times \mathbb{R}_{>0}$ is a submersion for every $R > S$.

We will abuse notation and refer to the triple (τ_S, π_S, ρ_S) by the short name τ_S. Note that tubular neighborhoods in a Riemannian manifold induce tubes in the above sense. When this is the case, we say that the induced tube is a **normal tube**. Observe that a smooth manifold M with its trivial stratification has a tube with $\tau_M = M$, $\pi_M = id_M$, and $\rho_M = 0$. We now give compatibility conditions on families of tubes.

Definition 5.1.6. Let $X \to \mathcal{S}$ be a stratified space. **Control data** for X consists of a family $(\tau_S)_{S \in \mathcal{S}}$ of tubes, each τ_S respectively a tube for S, such that for every pair of strata $R > S$ we have

(1) $\pi_S \circ \pi_R(x) = \pi_S(x)$

(2) $\rho_S \circ \pi_R(x) = \rho_S(x)$

wherever this composition is defined (that is, when $x \in \tau_S \cap \tau_R \cap \pi_R^{-1}(\tau_S)$).

When each tube is normal, we say the control data is **normal control data**. Control data which differs only by enlarging the neighborhoods should be regarded as equivalent. We define this notion now.

Definition 5.1.7. Let τ_S, τ_S' be two tubes for a strata S of a stratified space X. We say they are **equivalent** if there exists a neighborhood $U \subset \tau_S \cap \tau_S'$ of S such that $\pi_S|_U = \pi_S'|_U$ and $\rho_S|_U = \rho_S'|_U$.

Given pairs of control data (τ_S) and (τ'_S), we say they are **equivalent** if for each $S \in \mathcal{S}$ the tubes τ_S and τ'_S are equivalent.

A **controlled space** is a space X together with an equivalence class of control data.

Figures 5.1a and 5.1b illustrates the distinction between a tubular neighborhood in the traditional sense and a tube with the required compatibility conditions. We also define an appropriate notion of morphism for controlled spaces, and will in particular be concerned with a special case.

Definition 5.1.8. Let X and Y be controlled spaces. A stratified continuous map $f : X \to Y$ is called a **weakly controlled map** if control data $(\tau_S)_{S\in\mathcal{S}}$ and $(\tau_R)_{R\in\mathcal{R}}$ exists respectively for X and Y such that the following holds for every stratum $S \in \mathcal{S}$ with $f(S) \subset R$ and $x \in \tau_S$.

(1) $f(\tau_S) \subset \tau_R$.

(2) $f \circ \pi_S(x) = \pi_R \circ f(x)$

If in addition we have $\rho_S(x) = \rho_R \circ f(x)$, we say f is **controlled**.

Note that any manifold with its trivial stratification and control data give a controlled space. In the special case that $Y = M$ is such a controlled space and $f : X \to M$ is a weakly controlled map, we say also that the (equivalence class of) the control data (τ_S) for X is **compatible** with f.

Now we will establish a generalization of a theorem of Mather, namely the existence of control data compatible with a stratified submersion on a Whitney stratified space. This proof is largely the same as that given by Mather, but rephrased so that the original proof by induction on the dimensions of the strata is replaced by induction over the poset structure on the strata. We introduce some notation first to support this, replacing the usual k-skeleton of the space X with a notion of P-skeleton for a stratum P. As the k-skeleton is the union of all strata of dimension at most k, and strata of the same dimension can be incomparable when considering, for instance, the coarsest decomposition inducing a particular stratification, we define the P-skeleton so that it includes those strata which are incomparable to P. This allows us to abstract away unnecessary details of the decomposition $X \to \mathcal{P}$.

Definition 5.1.9. For a strata $p \in \mathcal{P}$ the p**-skeleton** of X, denoted X^p, is the union of those strata p' such that $p' \leq p$ or p' is incomparable to p. That is,

$$X^p = \left(\bigcup \{p' | p' \leq p\}\right) \cup \left(\bigcup \{p' | p' \nleq p, p \nleq p'\}\right).$$

Note of course that if p' is incomparable with p then $X^p = X^{p'}$.

Theorem 5.1.10. *Let X be a Whitney space stratified by Hilbert manifolds and $f : X \to M$ a smooth stratified submersion to a Hilbert manifold. Suppose that there is a minimal (necessarily closed) strata X_0 and that for each $p \in P$ the downset $\downarrow p$ satisfies the ascending chain condition. Then there exists normal control data on X compatible with F.*

Proof. We adapt the proof of [15] to this infinite dimensional context. See also [16].

Let $X_{\downarrow p} := \cup_{q \leq p} X_q$ be the p-skeleton of X We also set $X_{-\infty} = \emptyset$. We will proceed by induction on p.

Suppose that for some $k \in P$ we have a system of normal tubes $(T_s, \pi_s, \rho_s)_{s \in \downarrow k}$ that are compatible with f and such that for all $r, s < k$ the mapping

$$(\pi_r, \rho_r) : X_s \cap T_r \to X_r \times \mathbb{R}$$

is a submersion.

To construct the compatible tube for k, we will work via reverse induction on $l \leq k$. Note that since there is a minimal element $0 \in P$ that this reverse induction necessarily terminates. Note also that by the ascending chain condition on $\downarrow k$, there are at each step $l \leq k$ only finitely many q such that $l \leq q \leq p$.

For $l = k$ the empty data suffices. Suppose that the desired tube T_{l+1} has been constructed compatible with all q such that $l < q \leq k$. (Here the notation $l + 1$ refers to all q that are successors of l in P, which exist by the Noetherian condition on $\downarrow k$). By shrinking T_{l+1} if necessary, we assume without loss of generality that for each $x \in T_{l+1}$ there exists a $l < q < k$ for which $x \in T_q$ and $\pi_{l+1}(x) \in T_q$. Note in particular that the ascending chain condition on $\downarrow k$ implies that we need only take finitely many intersections to shrink T_{l+1}. Thus, after shrinking T_{l+1} it is still a nonempty open set.

By the primary inductive hypothesis, there is already a tube T_l that is compatible for the inductively constructed tubes T_q for $q < k$. We must possibly shrink T_l so that the desired commutation relations are satisfied for the newly modified tubes, in particular for $q = k$. To do this, we replace T_l by replacing it with

$$T_l' \coloneqq T_l \setminus \bigcup_{\{q|l<q<k\}} \pi_q^{-1}(T_q \setminus T_l).$$

We will denote the resulting tube again by T_l from here out.

Since for each x there is a $l < q < k$ with $x \in T_q$ and $\pi_{l+1}(x) \in T_q$, we have $\pi_{l+1}(x) \in T_l \cap T_q$. By the shrinking on T_l, $\pi_q(x) \in T_l$, hence the control conditions hold. By the commutativity conditions, the tube $T_l \cap T_k$ is compatible with the submersive map (π_l, ρ_l). Thus, by the existence theorem for generalized tubular neighborhoods, there exists a tubular neighborhood on X_l compatible the submersive map, hence satisfying the control conditions. Thus we have constructed control data for T_l. By induction on l, we have constructed control data on T_k, and then by induction on k we have control data for all of X as desired.

□

5.2 Lie Group Actions and Slices

5.2.1 Lie Group Actions

Definition 5.2.1. A **Lie group** is a group G with the structure of a smooth manifold such that the multiplication and inversion maps are smooth. A **Lie group action** on a smooth manifold M is a group action $G \circlearrowright M$, written $g \cdot x$ for $g \in G$, $x \in M$, such that G is a Lie group and the map

$$G \times M \to M$$

given by

$$(g, x) \mapsto g \cdot x$$

is a smooth map.

In order to distinguish between the different notions of infinite dimensional manifolds, we will specify that G is a **Hilbert Lie group**, **Banach Lie group**, or **Frechet Lie group** if the underlying smooth manifold structure is a Hilbert, Banach, or Frechet group structure and the multiplication and inversion are smooth in the appropriate sense. For our applications, we will only need to consider a Banach Lie group G acting on a Banach manifold M. We will need the notion of a principal Lie subgroup.

Definition 5.2.2. Let G be a Lie group and H a Lie subgroup of G. H is called a **principal** subgroup if the map $G \to G/H$ is a locally trivial principal bundle.

We will also need to recall the definition of proper group actions.

Definition 5.2.3. A Lie group action $G \circlearrowright M$ is **proper** when the shear map

$$G \times M \to M \times M$$

acting by

$$(g, x) \mapsto (g \cdot x, x)$$

is a proper map.

The orbits of such a proper Lie group action, denoted by $Gx := \{g \cdot x \,|\, g \in G\}$ for $x \in M$, do not typically form a stratification. However, we can take unions of certain families of orbits to achieve a stratification. We will now define the necessary equivalence on orbits.

Definition 5.2.4. Let Gx and Gy be two orbits of a smooth group action of a Frechet Lie group G on a Frechet manifold M. We say that x and y have the same **orbit type** if the stabilizer subgroups G_x and G_y are conjugate to each other, that is, if there exists a $g \in G$ such that

$$gG_xg^{-1} = G_y.$$

We will denote the orbit type associated to G_x by (G_x). We will denote the subspace of points of M having orbit type (G_x) by for some $x \in M$ by $M_{(G_x)}$.

Note in particular that points in the same orbit are automatically in the same orbit type. Thus this relation does refine the foliation into orbits.

5.2.2 Slices for Banach Submanifolds of a Banach Space

The decomposition of a manifold into subspaces by orbit type does not automatically form a stratification. To demonstrate that we achieve a stratification into manifolds, we will need a so-called slice theorem. In the finite-dimensional case, the slice theorem follows for any proper group action, as shown for instance in Pflaum's [16, Theorem 4.2.6]. In the infinite-dimensional case, more is needed. A very general version has been presented by Diez and Rudolph in [6, Theorem 3.5], which applies in the generality of Lie groups and manifolds modelled on locally convex topological vector spaces. They then go on to demonstrate a version of the theorem which applies more specifically to Frechet manifolds. We will only need the theorem in the case of a Banach submanifold of a Banach space. In this section we will outline the necessary components and establish that slices exist in such cases, following an argument due to Tobias Diez. We will begin by formally defining slices, following [6, Definition 2.2].

Definition 5.2.5. Let $G \circlearrowright M$ be a Lie group action on a smooth manifold. A **slice** at $x \in M$ is a submanifold $S \subset M$ with $x \in S$ satisfying the following:

(1) S is invariant under the induced action of G_x, i.e. $G_x \cdot S \subset S$.

(2) If $g \in G$ is such that $(g \cdot S) \cap S \neq \emptyset$ then $g \in G_x$.

(3) G_x is a principal Lie subgroup of G.

(4) The principal bundle $G \to G/G_x$ admits a local section $\chi : U \to G$ on an open neighborhood of the identity coset such that the map $U \times S \to M$ given by

$$(\langle g \rangle, s) \mapsto \chi(\langle g \rangle) \cdot s$$

is diffeomorphic onto a **slice neighborhood** V of x in M.

(5) The partial slice $S_{(G_x)} := \{s \in S \,|\, (G_s) = (G_x)\}$ is a closed submanifold of S.

Intuitively, a slice at x is a submanifold normal to the orbit of x such that, modulo the action of the stabilizer of x, is mapped by a neighborhood of the identity onto an open neiborhood of x. Showing that a Lie group action admits slices is the content of a **slice theorem**. We will now, following Diez, demonstrate that the conditions of the general slice theorem in [6, Theorem 3.5] are satisfied in the particular case of a locally exponential Banach Lie group acting properly on a Banach submanifold of a Banach space.

Lemma 5.2.6. *Let $G \circlearrowright M$ be a proper Banach Lie group action on a submanifold M of a Banach space B, such that G is locally exponential. Then for each $x \in M$ the action admits a slice at x.*

Proof. There are four conditions we must check to satisfy the general slice theorem. First, we need that G_x is a principal Lie subgroup of G. This is exactly [6, Lemma 2.11]. Second, we need that the orbit Gx is locally closed. Jotz and Neeb show in [10] that the infinitesimal orbit is closed in T_xM. Since the group is locally exponential, the image of the infinitesimal orbit, i.e. the orbit, is closed in M. The final two conditions are the existence of a local addition and G-invariant topological metric. However, since we are restricting to the case of Banach submanifold of a Banach space, then we automaticallly have a global addition and metric as required. □

With the slice theorem, we can invoke in our special cases the theorem establishing the orbit type stratification. Namely, in Diez and Rudolph [6, Theorem 4.2], it shown that provided $M_{\geq(H)} \subset \overline{M_{(H)}}$ holds for every stabilizer subgroup H then the orbit type decomposition is a stratification.

Chapter 6

Applications

We will now demonstrate applications of the theory outlined in previous chapters.

6.1 Stratification of $\mathbb{R}^\infty$

We will be following Saunders as in [18]. We start by considering the Frechet space $X := R^\infty$ given by the limit of the maps

$$\mathbb{R}^{n+1} \to \mathbb{R}^n : (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n).$$

This space can be given a stratification by setting

$$X_0 := \{0\},$$

$$X_n := \left\{(x_1, x_2, \ldots) \middle| x_{n-1} \neq 0, x_i = 0 \forall i \geq n\right\},$$

and

$$X_\omega := \left\{(x_1, x_2, \ldots) \middle| \forall i \exists j > i x_j \neq 0\right\}.$$

Note that with the natural ordering

$$0 < 1 < \ldots n < n+1 < \ldots \omega$$

that the map $X \to \{X_i\}$ is a stratification. For each i, X_i is a manifold. When $n < \omega$, X_n is a finite dimensional manifold, diffeomorphic to $\mathbb{R}^n \setminus \mathbb{R}^{n-1}$, whereas X_ω is an infinite dimensional Frechet manifold, given by an open subset of the Frechet space.

We will now give gluing data for the tangent bundles to achieve a stratified tangent bundle. Note that since X is itself a Frechet space, there is already the usual tangent bundle $TX \to X$ as in [20]. For concreteness, we take the tangent bundle to consist of differentiable germs of smooth paths. Furthermore, each X_n, being a finite dimensional manifold, already has a well-defined tangent bundle $T_{X_n} \to X_n$. There is an obvious inclusion map

$$\iota_n : T_{X_n} \to TX$$

taking each path jet as an element of T_{X_n} to the same jet thought of as an element of TX. For the choice of sheaves, we will choose sections of the bundle TX for which the image of each stratum lands inside the image of the inclusion of the tangent bundle over that stratum. That is, we take the sheaves $\mathcal{E}_n$ given by, for $x \in X_n$,

$$(\mathcal{E}_n)_x = \left\{ \langle s \rangle_x \,\middle|\, s : U \to TX_n, x \in U, s(U \cap X_n) \subseteq \iota_n(T_{X_n}) \right\}.$$

In this way, the germs carry the appropriate information about the surrounding topology to facilitate the gluing process. In particular, $\mathcal{E}_\omega$ is simply the sheaf of sections of TX over X_ω.

For the gluing maps, we take the map

$$(g_{n,i})_x : (\mathcal{E}_n)_x \to (\mathrm{f}_{n,i}\, \mathcal{E}_i)_x$$

given by

$$\langle s \rangle_x \mapsto \langle s|_{X_i} \rangle_x.$$

Note that this is well-defined, since s is given by a germ of a map defined on a neighborhood U of x in X. Observe that this gluing data is also equivariant with respect to the action of smooth maps to the groups $\mathrm{Gl}(X_i)$, and it is injective by continuity of the sections. Thus we have a stratified fiber bundle, which we denote here $T_{\mathcal{S}}X \to X \to \mathcal{S}$ to distinguish it from the aforementioned tangent bundle $TX \to X$ when thinking of X as a single Frechet space. Moreover, this gluing data satisfies our generalized Whitney (A)-regularity condition. In particular, for each $X_{\downarrow n}$ for $n < \omega$, the pullback bundle is isomorphic to the usual stratified fiber bundle as defined in the finite dimensional case.

It is worth noting that we could also have constructed the bundle as a subspace of the (unstratified) tangent bundle $TX \to X$. That is, we could have restricted to the subspace of the jets of those paths which remain in the appropriate stratum. However, then it would not be clear how to characterize Whitney (A)-regularity. In particular, there is no independent developed theory of the Grassmanian bundle in a Frechet space, so it is unclear how to characterize the limit of tangent spaces in this context without essentially using the same data as we have outlined.

6.2 Stratification of Jet Bundles as Frechet Manifolds

Let $E \to M$ be a finite dimensional smooth vector bundle over a finite dimensional manifold M. Recall that two sections ϕ, ψ defined at some $x \in M$ are said to give the same jet of order k at x if for all multi-indices α with $|\alpha| = k$ we have that

$$\frac{\partial^{|\alpha|}\phi}{\partial x^I} = \frac{\partial^{|\alpha|}\psi}{\partial x^I}$$

in any well-adapted coordinate system. That is, the partial derivatives for ϕ and ψ agree up to order k. We denote the equivalence class of such sections by $j_x^k\phi$. If this is true for all k, we say that ϕ and ψ give the same infinite jet, denoted $j_x^\infty\phi$. We denote the sets of all such jets at all $x \in M$ by J^kE and $J^\infty E$.

The infinite jet bundle $J^\infty E$ is then a Frechet manifold modeled on R^∞. Given adapted coordinates (x^i, u^i) for the bundle $E \to M$ we construct coordinates for $J^\infty E$ by specifying in addition the partial derivatives, i.e.

$$(x^i, u^i, u_I^i)$$

give coordinates for $J^\infty E$ where $u_I^i = \frac{\partial^{|I|}u^i}{\partial x^I}$ for any multi-index I.

We will stratify the manifold $X = J^\infty E$ as follows. We start with

$$X_0 = \left\{ j^\infty\phi \,\middle|\, u_I^i(j^\infty\phi) = 0, |I| > 0 \right\},$$

i.e. the set of all stationary jets. In general we set

$$X_n = \left\{ j^\infty \phi \,\middle|\, u^i_I(j^\infty \phi) = 0, |I| > n, \exists I\, |I| = n u^i_I(j^\infty \phi) \neq 0 \right\}.$$

Finally we set

$$X_\omega = \left\{ j^\infty \phi \,\middle|\, \forall n \exists n' > n, \exists I\, |I| = n', u^i_I(j^\infty \phi) \neq 0 \right\}.$$

We follow the same process as for R^∞ to construct a stratified tangent bundle. Namely, we take $\mathcal{E}_n$ to be the sheaf of sections for which, on X_n, the image lands inside the tangent bundle TX_n as included in TX. Again, this data is injective and equivariant so we have a Whitney (A)-regular stratification of the infinite jet bundle.

6.3 State Spaces of Infinite Dimensional C^*-algebras

Let $\mathcal{A}$ be a unital C^*-algebra, possibly infinite dimensional. We will denote by $\mathcal{S}(\mathcal{A})$ the state space of $\mathcal{A}$, i.e. the space of positive normalized linear functionals. For more details on this notion, we refer to [1] and [4]. We endow $\mathcal{S}(\mathcal{A})$ with the norm topology given by the norm

$$|\phi| = \sup_{A, |A|_{\mathcal{A}} = 1} |\phi(A)|_{\mathbb{C}}.$$

Note that $\mathcal{S}(\mathcal{A})$ can carry other topologies, notably the weak-* topology, i.e. the weak topology with respect to the maps $\phi \mapsto \phi(a)$ for all $a \in \mathcal{A}$. However, in this topology the (pure) state space is compact, hence is a finite dimensional manifold. This is elaborated on in, for instance, [21].

Note that $\mathcal{A}$ and in particular $U(\mathcal{A})$ is a Banach Lie group under multiplication and that $\mathcal{S}(A)$ is a Banach submanifold (with boundary) of the Banach space $\mathcal{A}^*$. We will use the stratification of $\mathcal{S}(\mathcal{A})$ by orbit types of the induced action from the conjugate action of the unitary group $U(\mathcal{A})$ of $\mathcal{A}$, invoking in particular Lemma 5.2.6. That is, for all $u \in U(\mathcal{A})$ let u act on $a \in \mathcal{A}$ by

$$u \cdot a := uau^*.$$

This in turn induces an action on $\mathcal{A}^*$ by

$$(u \cdot \phi)(a) := \phi(u \cdot a).$$

In particular, note that as u is a unitary it has norm 1 so that when $\phi \in \mathcal{S}(\mathcal{A})$ is a state so is $u \cdot \phi$. We denote orbit types by indexing via the conjugacy classes of the stabilizer subgroups. Writing (H) for such a conjugacy class, then we denote an orbit type of $X = \mathcal{S}(\mathcal{A})$ by $X_{(H)}$. This stratification is already explored in the finite-dimensional case in [7], but the work of Diez and Rudolph in [6] allow for the extension of this to arbitrary dimensions.

The conjugacy classes are partially ordered by sub-conjugacy, i.e. $(H_1) \leq (H_2)$ if H_1 is conjugate to a subgroup of H_2. This partial order carries over the the orbit type decomposition. Furthermore, the frontier ordering on such a decomposition is antisymmetric so that we indeed have a stratification in the sense given here, i.e. the decomposition map on $X = \mathcal{S}(\mathcal{A})$ given by $X \to \{X_i\}$ is continuous when $\{X_i\}$ is given the specialization topology associated with the frontier ordering. What has not yet been explored is the possibility of constructing an infinite-dimensional stratified tangent bundle over this stratification. We do this now.

Consider $\mathcal{S}(\mathcal{A})$ as a subspace of $\mathcal{A}^*$, the space of continuous linear functionals with the norm topology. Then $\mathcal{A}^*$ is also a Banach space, hence a Banach manifold. Thus there is a well defined tangent bundle $T\mathcal{A}^* \to \mathcal{A}^*$. Since $\mathcal{S}(\mathcal{A})$ is closed in $\mathcal{A}^*$, and each orbit type is locally closed in $\mathcal{S}(\mathcal{A})$, each orbit type is also locally closed in $\mathcal{A}^*$. We take each tangent space $TX_{(G_x)}$, where G_x is the stabilizer subgroup of x, and embed it into TX. For each, we take the sheaf $\mathcal{E}_{(G_x)}$ to be those sections of $TX \to X$ for which the image of $X_{(G_x)}$ lands inside $TX_{(G_x)}$. As before, the gluing data is injective and equivariant with respect to the action of $\mathrm{Gl}(\mathcal{A}^*)$. Hence we have a Whitney A regular stratification of the state space, enabling us to make sense of stratified vector fields.

Bibliography

[1] Erik Alfsen and Frederic Shultz. Geometry of State Spaces of Operator Algebras. Birkhauser Boston, MA, 2003.

[2] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. Théorie des Topos et Cohomologie Etale des Schémas Tome 1: Théorie des topos, Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA4). Number 269 in Lecture Notes in Mathematics. Springer Heidelberg, 1972.

[3] Hans-Joachim Baues and Davide L. Ferrario. Stratified fibre bundles. Forum Mathematicum, 16, 1 2004.

[4] Ingemar Bengtsson and Karol Życzkowski. Geometry of Quantum States. Cambridge University Press, 8 2017.

[5] Chruscinski Dariusz, Ciaglia Florio M., Ibort Alberto, Marmo Giuseppe, and Ventriglia Franco. Stratified manifold of quantum states, actions of the complex special linear group. Nov 2018. Annals of Physics, Volume 400, January 2019.

[6] Tobias Diez and Gerd Rudolph. Slice theorem and orbit type stratification in infinite dimensions. Differential Geometry and its Applications, 65:176–211, 8 2019.

[7] Francesco D' Andrea and Davide Franco. On the pseudo-manifold of quantum states. Nov 2020. Diff. Geo. App. 78 (2021) 101800.

[8] Peter T. Johnstone. Sketches of an Elephant - A Topos Theory Compendium, vol. 2 (Oxford Logic Guides, 44). Oxford University Press, USA, 2002.

[9] Peter T. Johnstone. Sketches of an Elephant - A Topos Theory Compendiumm vol. 1 (Oxford Logic Guides, 43). Oxford University Press, USA, 2002.

[10] Madeleine Jotz and Karl-Hermann Neeb. Closedness of the tangent spaces to the orbits of proper actions. April.

[11] Anders Kock and Till Plewe. Glueing analysis for complemented subtoposes. Theory and Applications of Categories, 2(9):100–112, 1996.

[12] Serge Lang. Fundamentals of differential geometry. Number 191 in Graduate texts in mathematics. Springer, New York, 1999.

[13] Jacob Lurie. Higher algebra. 2017.

[14] Saunders Mac Lane and Ieke Moerdijk. Sheaves in Geometry and Logic: A First Introduction to Topos Theory. Universitext. Springer New York, New York, NY, 1994.

[15] John Mather. Notes on topological stability. Bulletin of the American Mathematical Society, 49:475–506, 2012.

[16] Markus Pflaum. Analytic and geometric study of stratified spaces: contributions to analytic and geometric aspects. Number 1768 in Lecture Notes in Mathematics. Springer Science & Business Media, 2001.

[17] Ethan Ross. Stratified vector bundles: Examples and constructions. Journal of Geometry and Physics, 198:105114, 4 2024.

[18] D. J. Saunders. The Geometry of Jet Bundles. Cambridge University Press, 3 1989.

[19] Varun Kher Scarlett. Smooth stratified vector bundles and obstructions to their orthonormal frame bundles. Master's thesis, Virginia Tech, 2023.

[20] Alexander Schmeding. An Introduction to Infinite-Dimensional Differential Geometry. Cambridge University Press, 12 2022.

[21] Daniel Spiegel, Juan Moreno, Marvin Qi, Michael Hermele, Agnès Beaudry, and Markus J. Pflaum. Continuous dependence on the initial data in the kadison transitivity theorem and gns construction. Dec 2021. Reviews in Mathematical Physics Vol. 33 (2022) 2250031 (84 pages).

[22] Dai Tamaki. Cellular Stratified Spaces, pages 305–435. World Scientific, 12 2017.

[23] R. Thom. Ensembles et morphismes stratifiés. Bulletin of the American Mathematical Society, 75:240–284, 1969.

[24] A. Uhlmann and B. Crell. Geometry of state spaces. In Andreas Buchleitner, Carlos Viviescas, and Markus Tiersch, editors, Entanglement and Decoherence, volume 768, pages 1–60. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009. ISSN: 0075-8450, 1616-6361 Series Title: Lecture Notes in Physics.

[25] Lukas Waas, Jon Woolf, and Shoji Yokura. On stratifications and poset-stratified spaces. 7 2024.

[26] Gavin Wraith. Artin glueing. Journal of Pure and Applied Algebra, 4(3):345–348, 1974.

[27] Shoji Yokura. Decomposition spaces and poset-stratified spaces. Tbilisi Mathematical Journal, 13(2):101–127, 2020.