GEOMETRIC REALIZATION OF STRATA IN THE BOUNDARY OF THE INTERMEDIATE JACOBIAN LOCUS

by

KRISZTIÁN HAVASI

M.Sc., Technical University of Budapest, 1996

MMath, ELTE Univeristy of Budapest, 2007

M.A., University of Colorado, 2013

A thesis submitted to the

Faculty of the Graduate School of the

University of Colorado in partial fulfillment

of the requirement for the degree of

Doctor of Philosophy

Department of Mathematics

2016

This thesis entitled:
Geometric Realization of Strata in the Boundary of the Intermediate Jacobian Locus
written by Krisztián Havasi
has been approved for the Department of Mathematics

Sebastian Casalaina-Martin

Jonathan Wise

Date ________________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

ABSTRACT

Havasi, Krisztián (Ph.D., Mathematics, Department of Mathematics)

Geometric Realization of Strata in the Boundary of the Intermediate Jacobian Locus

Thesis directed by Associate Professor Sebastian Casalaina-Martin

In this thesis we describe intermediate Jacobians of threefolds obtained from singular cubic threefolds. By this we mean two things. First, we describe the intermediate Jacobian of a desingularization of a cubic threefold with isolated singularities. Second, we describe limits of intermediate Jacobians of smooth cubic threefolds, as the family of cubic threefolds acquires isolated singularities. In regards to the first question, generalizing a result of Clemens–Griffiths we show specifically that the intermediate Jacobian of a distinguished desingularization of a cubic threefold with a single singularity of type A_3 is the Jacobian of the normalization of an associated complete intersection curve in $\mathbb{P}^3$, the so called $(2,3)$-curve. In regards to degenerations, we describe how the limit intermediate Jacobian, under certain conditions, can be described as a semi-abelian variety as the extension of a torus by the finite quotient of the product of Jacobians of curves, where one of the curves is the normalization of the $(2,3)$-curve associated to the cubic threefold and a choice of singularity, and the other curves are so-called tails arising from stable reduction of plane curve singularities.

ACKNOWLEDGEMENTS

I give many thanks to my advisor Professor Sebastian Casalaina-Martin for all his help throughout these years. I would also like to thank the Dance Department at CU Boulder, its wonderful faculty and dancers without whom this thesis would never have seen the light of day.

Contents

Chapter 1

Introduction

Cubic threefolds (smooth cubic hypersurfaces in $\mathbb{P}^4$) have played an important role in the development of algebraic geometry. By the global Torelli theorem for cubic threefolds, a well known result of Clemens and Griffiths [CG72], smooth cubic threefolds can be recovered from their principally polarized intermediate Jacobians, just like smooth curves can be recovered from their principally polarized Jacobians. In their celebrated paper [CG72], using the intermediate Jacobian as a tool of study, Clemens and Griffiths also showed that smooth cubic threefolds are not rational. Since these results the intermediate Jacobian has played a central role in the study of smooth cubic threefolds. Much attention has also been paid to the degeneration of the intermediate Jacobian as the cubic threefold becomes singular. It turns out that intermediate Jacobians of smooth cubic threefolds are in fact Prym varieties [Mum74], therefore one way to study degenerations of intermediate Jacobians is through degenerations of Prym varieties. Our goal is to compute the degenerate intermediate Jacobian as a degenerate Prym variety for several cubic threefolds with various singularities. Another interesting question is that if we have a singular cubic threefold, how can we describe the intermediate Jacobian of a desingularization? We answer this question in the cases of a cubic threefold with a single singular point of type either A_1, A_2 or A_3 (the A_1 case is due to Clemens and Griffiths, and the A_2 case essentially follows immediately from that argument).

If X is a cubic threefold with isolated singularities only and with a distinguished double point P, we can project from P the lines of X passing through P. The projection is then a $(2,3)$-complete intersection curve C in $\mathbb{P}^3$ with the property that the blow-up of $\mathbb{P}^3$ along C is isomorphic to the blow-up of X at P, and the singularities of C are the same as the singularities of the blow-up of

X at P. Thus C is nonsingular if X has an A_1 or A_2 singularity at P and no other singularities. To find the intermediate Jacobian of the desingularization of X in this case (i.e. the intermediate Jacobian of the blow-up of X at P), using that $\mathrm{Bl}_P X \cong \mathrm{Bl}_C \mathbb{P}^3$ we only need to know how the cohomology changes when we blow up a manifold along a smooth submanifold. This is well-known (we review the details in this thesis for completeness), and so we find that the intermediate Jacobian of the desingularization of a cubic threefold with a single A_1 or A_2 singularity is the Jacobian of the curve C. In the case of a single A_3 singularity the situation is more complicated, because C is now singular with one node. Using the theory of toric varieties we prove

Theorem 1.0.1. If X is a cubic threefold with a unique singularity, which is of type A_3, the intermediate Jacobian of the desingularization of X obtained by two successive blow-ups of the singular point, is the Jacobian of the normalization of the associated $(2, 3)$-curve.

Let $L \subset X$ be a sufficiently general line. The planes of $\mathbb{P}^4$ containing L can be parameterized by $\mathbb{P}^2$. Let $D \subset \mathbb{P}^2$ be the discriminant curve belonging to L, i.e. the curve whose points represent planes intersecting X in L and two residual lines. Then D has a double cover $\widetilde{D} \to D$, where points of $\widetilde{D}$ represent the residual lines. The singularities of D are in type-preserving bijection with the singularities of X. According to a theorem of Mumford's, in the case of a smooth X the Prym variety of this double cover is isomorphic to the intermediate Jacobian of X. Thus in the case of a singular cubic threefold describing the degenerate intermediate Jacobian is the same as describing the degenerate Prym variety of the double cover $\widetilde{D} \to D$.

A degeneration of a Prym variety is an extension of some quotient of the Prym variety of the desingularization of the double cover $\widetilde{D} \to D$ (the compact part) with a non-compact torus $(\mathbb{C}^*)^n$:

$$1 \to (\mathbb{C}^*)^n \to P_{\widetilde{D}/D} \to P_{N\widetilde{D}/ND}/G \to 0, \tag{1.0.2}$$

where $N\widetilde{D}$ and ND are desingularizations of $\widetilde{D}$ and D, respectively, and G is some finite group, often trivial. We prove that the Prym variety $P_{N\widetilde{D}/ND}$ is isomorphic to the Jacobian of the desingularization of C if X has an A_n singularity and ND is non-hyperelliptic, or X has a D_4 singularity. In fact we have:

Theorem 1.0.3. Let $\mathcal{X} \to \Delta$ be a general family of cubic threefolds over the unit disk with smooth general fiber and central fiber a cubic threefold X with isolated AD singularities $P_1, \dots, P_n$ of types $S_1, \dots, S_n$ respectively, with $S_1 = A_k$ for some $1 \le k \le 6$, or $S_1 = D_4$. Assume that there is a general line $L \subset X$ so that projection from L yields an irreducible discriminant curve $D \subset \mathbb{P}^2$ (which will be the case for instance if $n = 1$). We further assume that if P_1 is of type A_k, the normalization ND is non-hyperelliptic. Then the limit intermediate Jacobian $IJ(X)$ will be a semi-abelian variety

$$1 \to (\mathbb{C}^*)^r \to IJ(X) \to \big(J(NC) \times JT_1 \times \cdots \times JT_n\big)/G \to 0 \tag{1.0.4}$$

where NC is the normalization of the $(2,3)$-curve C obtained from projection from P_1, T_i is a so-called tail curve associated to stable reduction of a plane curve singularity of type S_i, G is some finite group, and $r = 5 - g(NC) - \sum_{i=1}^{n} g(T_i)$.

For this theorem, the main contribution of this thesis is to identify the abelian variety $J(NC)$ in the compact part (the remainder of the theorem can be found in [CMGHL15]). This is proven in theorems 3.2.5 and 3.2.14. The special cases where X has a unique singularity, which is of type A_1 or A_2 is due to [CM78] and [CML09], respectively. We also note, that we are able to describe the extension data for the semi-abelian variety explicitly, although for brevity, we direct the reader to the body of the thesis.

One of the main goals of this investigation was to identify boundary strata in the closure of the intermediate Jacobian locus in the second Voronoi compactification of $\mathcal{A}_5$. In chapter 4 we give explicit descriptions of degenerations of intermediate Jacobians in a number of examples; we describe the semi-abelian varieties. While this is not the complete degeneration data to give a point of the second Voronoi compactification for torus rank 2 or more, this does describe a good amount about how various loci in the boundary can arise geometrically. Nevertheless, we point out that one question we had hoped to answer was what geometric locus of cubic threefolds gives rise to the locus $\mathbf{B}_{22}$ described in [CMGHL15]. Unfortunately, at this point, we have not yet identified this locus.

Chapter 2

Intermediate Jacobians of Desingularizations of Cubic Threefolds

2.1 Cubic Threefold with Isolated Singularities

Let X be a cubic hypersurface in $\mathbb{P}^4$ having a double point at $P = (1 : 0 : 0 : 0 : 0)$. Let π denote the projection of $\mathbb{P}^4$ through the point P onto the hyperplane $\mathbb{P}^3$ given by $\{x_0 = 0\}$. We denote by C the image through π of all lines that lie in X and pass through P. We start with the following well-known result (see e.g., the references in [CMJL12, §1.2, p. 6]). For completeness, we include a proof here.

Theorem 2.1.1. If X has isolated singularities only, then C is a curve and a complete intersection of type $(2, 3)$, and

$$\mathrm{Bl}_C\, \mathbb{P}^3 \cong \mathrm{Bl}_P\, X, \tag{2.1.2}$$

i.e. the blow-up of $\mathbb{P}^3$ along the curve C is isomorphic to the blow-up of X at the singular point P.

Proof. As X is a cubic hypersurface in $\mathbb{P}^4$, it is given as the zero set of a single homogeneous polynomial $f(x_0, x_1, x_2, x_3, x_4)$ of degree three. The coefficient of x_0^3 in f must be zero, otherwise X would not contain the point P. Similarly, the coefficients of the terms $x_0^2 x_i$, $(1 \le i \le 4)$ must be zero, otherwise P would be a smooth point of X. Therefore the equation of X can be written in the form

$$x_0Q(x_1, x_2, x_3, x_4) + F(x_1, x_2, x_3, x_4) = 0, \tag{2.1.3}$$

where Q is a homogeneous polynomial of degree two and F is a homogeneous polynomial of degree three. Here Q must be nonzero, because otherwise the singularity at P would have multiplicity three. Similarly, F is nonzero, since otherwise X would be the union of a hyperplane and a quadric, and it would have nonisolated singularities. For the same reason Q and F cannot have common factors.

Next, we want to find the image C of all lines lying in X and passing through P along the projection π. Let $(0 : x_1 : x_2 : x_3 : x_4)$ be an arbitrary point of the hyperplane $\{x_0 = 0\}$. Then the line $\mathbb{P}^1$ connecting $(0 : x_1 : x_2 : x_3 : x_4)$ and $P = (1 : 0 : 0 : 0 : 0)$ is given by $(\mu : \lambda x_1 : \lambda x_2 : \lambda x_3 : \lambda x_4)$, where $(\mu : \lambda)$ are homogeneous parameters. Plugging this into equation (2.1.3) we get

$$\mu Q(\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4) + F(\lambda x_1, \lambda x_2, \lambda x_3, \lambda x_4) = 0, \tag{2.1.4}$$

then using that $\deg Q = 2,\ \deg F = 3$:

$$\mu\lambda^2 Q(x_1, x_2, x_3, x_4) + \lambda^3 F(x_1, x_2, x_3, x_4) = 0. \tag{2.1.5}$$

When an entire line lies in X, then this equation must hold for any $(\mu : \lambda)$. The only way this can happen is when the coefficients of $\mu\lambda^2$ and λ^3 are zero. Thus the projection C into $\mathbb{P}^3$ of the lines lying inside X and containing the singular point P is given by the equations

$$\begin{cases} Q(x_1, x_2, x_3, x_4) &= 0 \\ F(x_1, x_2, x_3, x_4) &= 0. \end{cases} \tag{2.1.6}$$

Since Q and F have no common factors, C has pure dimension one, and it is a complete intersection of type $(2, 3)$.

Next, we want to find $\mathrm{Bl}_C\,\mathbb{P}^3$, the blow-up of $\mathbb{P}^3$ along the curve C. The result below (the equation of $\mathrm{Bl}_C\,\mathbb{P}^3$) also follows from standard results on blowing up regular sequences, see e.g. [EH00, Exercise IV-26, p. 173]. Let's consider the open set $U_1 \cong \mathbb{C}^3 \subset \mathbb{P}^3$ given by $x_1 \neq 0$. In U_1 the curve C is given by the equations

$$\begin{cases} Q(1, x_2, x_3, x_4) &= 0 \\ F(1, x_2, x_3, x_4) &= 0. \end{cases} \tag{2.1.7}$$

Then $\operatorname{Bl}_{C \cap U_1} U_1 \subset \mathbb{C}^3 \times \mathbb{P}^1$ is given by the equation

$$a_0 Q(1, x_2, x_3, x_4) = a_1 F(1, x_2, x_3, x_4), \tag{2.1.8}$$

where $(a_0 : a_1)$ are homogeneous coordinates of $\mathbb{P}^1$. We get similar equations for the open sets $U_i,\ 1 \leq i \leq 4$. The blow-up $\operatorname{Bl}_C \mathbb{P}^3$ is then a variety that we receive by gluing the blow-ups $\operatorname{Bl}_{C \cap U_i} U_i$ together. That we can do this simply follows from the fact that the blow-up of a manifold along a subvariety exists.

Next, we compute $\operatorname{Bl}_P X$, the blow-up of the variety X at the singular point P which will be a subvariety of $\mathbb{P}^4 \times \mathbb{P}^3$. If the homogeneous coordinates of $\mathbb{P}^3$ are given by $(y_1 : y_2 : y_3 : y_4)$, we consider the open set $\mathbb{P}^4 \times \mathbb{C}^3 \subset \mathbb{P}^4 \times \mathbb{P}^3$ given by $y_1 \neq 0$. Then we have the following equations for the blow-up in $\mathbb{P}^4 \times \mathbb{C}^3$:

$$x_0 Q(x_1, x_2, x_3, x_4) + F(x_1, x_2, x_3, x_4) = 0 \tag{2.1.9}$$

$$x_2 = x_1 y_2 \tag{2.1.10}$$

$$x_3 = x_1 y_3 \tag{2.1.11}$$

$$x_4 = x_1 y_4. \tag{2.1.12}$$

After substitutions, the first equation becomes

$$x_0 Q(x_1, x_1 y_2, x_1 y_3, x_1 y_4) + F(x_1, x_1 y_2, x_1 y_3, x_1 y_4) = 0. \tag{2.1.13}$$

Then pulling out x_1 gives:

$$x_1^2 \big(x_0 Q(1, y_2, y_3, y_4) + x_1 F(1, y_2, y_3, y_4)\big) = 0. \tag{2.1.14}$$

Here the factor x_1^2 represents the exceptional set ($x_1 = 0$ gives $x_2 = x_3 = x_4 = 0$), so we must disregard it. Thus the blow-up in $\mathbb{P}^4 \times \mathbb{C}^3$ is given by:

$$x_0 Q(1, y_2, y_3, y_4) + x_1 F(1, y_2, y_3, y_4) = 0, \tag{2.1.15}$$

$$x_2 = x_1 y_2, \quad x_3 = x_1 y_3, \quad x_4 = x_1 y_4. \tag{2.1.16}$$

We can in fact write this as a subvariety of $\mathbb{P}^1 \times \mathbb{C}^3$ by simply taking the projection from $\mathbb{P}^4 \times \mathbb{C}^3$ by forgetting the coordinates x_2, x_3, x_4 and the last three equations. Then the blow-up becomes a subvariety of $\mathbb{P}^1 \times \mathbb{C}^3$ given by the single equation

$$x_0 Q(1, y_2, y_3, y_4) + x_1 F(1, y_2, y_3, y_4) = 0. \tag{2.1.17}$$

We get similar equations for the other open patches inside the sets $\{y_i \neq 0\}$. Comparing equations (2.1.8) and (2.1.17) we see that the open patches of $\mathrm{Bl}_C \mathbb{P}^3$ and $\mathrm{Bl}_P X$ inside $\mathbb{P}^1 \times \mathbb{C}^3$ are isomorphic. The transition functions between the different open patches must agree for the two blow-ups, because in both cases they are derived from the usual transition functions between the standard open affine subsets of the projective space $\mathbb{P}^3$. Thus $\mathrm{Bl}_C \mathbb{P}^3$ is isomorphic to $\mathrm{Bl}_P X$. □

We now show the following well-known result (see e.g., the references in [CMJL12, §1.2, p. 6]). For completeness, we include a proof here.

Theorem 2.1.18 (e.g., [CML09, §3], [CMJL12, Prop. 1.3])**.** If X has isolated singularities only, the singularities of the blow-up $\mathrm{Bl}_P X$ of X at P are in bijection with the singularities of the $(2,3)$-curve C, and along this bijection the corresponding singularities have the same singularity types.

Proof. By Theorem 2.1.1 we have $\mathrm{Bl}_C \mathbb{P}^3 \cong \mathrm{Bl}_P X$, therefore it is enough to see that the singularities of $\mathrm{Bl}_C \mathbb{P}^3$ are in bijection with the singularities of C, and this bijection respects the singularity type. All singularities of $\mathrm{Bl}_C \mathbb{P}^3$ must be in the exceptional set, therefore it is enough to prove that each point above a non-singular point of C is non-singular, while above any singular point p of C there is exactly one singular point in the blow-up with the same singularity type as p.

So let's fix a point $p = (y_1 : y_2 : y_3 : y_4) \in C$ and let's examine the points of the blow-up above p. From the proof of Theorem 2.1.1 we know that C is given by the equations

$$\begin{cases} Q(x_1, x_2, x_3, x_4) &= 0 \\ F(x_1, x_2, x_3, x_4) &= 0, \end{cases} \tag{2.1.19}$$

where Q and F are nonzero homogeneous quadratic and cubic polynomials. By our assumptions it is not possible that both Q and F are singular at p, because then X would be singular along the entire line connecting P and the point $p \in \mathbb{P}^3 \cong \{x_0 = 0\}$ and this would contradict the assumption that X has isolated singularities only. To see this we take the derivative of the defining polynomial of X (see equation (2.1.3)) to get

$$\begin{aligned} \Big[Q(x_1, \ldots, x_4) \qquad x_0 \tfrac{\partial}{\partial x_1} Q(x_1, \ldots, x_4) + \tfrac{\partial}{\partial x_1} F(x_1, \ldots, x_4) \quad \ldots \\ \ldots \quad x_0 \tfrac{\partial}{\partial x_4} Q(x_1, \ldots, x_4) + \tfrac{\partial}{\partial x_4} F(x_1, \ldots, x_4)\Big]. \end{aligned} \tag{2.1.20}$$

If we plug in the general point $(\mu : \lambda y_1 : \lambda y_2 : \lambda y_3 : \lambda y_4)$ of the line connecting P and p we get

$$\begin{aligned} \Big[Q(\lambda y_1, \ldots, \lambda y_4) \qquad \mu\, \partial_1 Q(\lambda y_1, \ldots, \lambda y_4) + \partial_1 F(\lambda y_1, \ldots, \lambda y_4) \quad \ldots \\ \ldots \quad \mu\, \partial_4 Q(\lambda y_1, \ldots, \lambda y_4) + \partial_4 F(\lambda y_1, \ldots, \lambda y_4)\Big] = \\ \Big[\lambda^2 Q(y_1, \ldots, y_4) \qquad \mu\lambda \tfrac{\partial}{\partial y_1} Q(y_1, \ldots, y_4) + \lambda^2 \tfrac{\partial}{\partial y_1} F(y_1, \ldots, y_4) \quad \ldots \\ \ldots \quad \mu\lambda \tfrac{\partial}{\partial y_4} Q(y_1, \ldots, y_4) + \lambda^2 \tfrac{\partial}{\partial y_4} F(y_1, \ldots, y_4)\Big]. \end{aligned} \tag{2.1.21}$$

If p is a singular point of both Q and F, this Jacobian is zero for any $(\mu : \lambda)$, therefore at p at least one of Q and F is non-singular.

Let's apply a local analytic transformation to a neighborhood of p in $\mathbb{P}^3$ such that p is taken to the origin of $\mathbb{C}^3$, while $z = 0$ gives locally either the surface Q or F (we pick one that is non-singular at p). Then the other surface of Q and F will be given by some equation $h(x, y, z) = 0$, with h complex analytic. The curve C is then given locally by the equations $z = 0,\ h(x, y, 0) = 0$ (thus C becomes a planar curve) and the singularity type of C at p will be given by the function $h(x, y, 0)$. By another analytic transformation (in the x, y coordinates only) we can write $h(x, y, 0)$ in the standard polynomial form $q(x,y)$ according to the singularity type of C at the point p. Then the blow-up of $\mathbb{C}^3$ along the curve given by $z = q(x,y) = 0$ is given in $\mathbb{C}^3 \times \mathbb{P}^1$ by the equation

$$a_0 z - a_1 q(x,y) = 0, \tag{2.1.22}$$

where $(a_0 : a_1)$ are homogeneous coordinates of $\mathbb{P}^1$. The Jacobian of this is given by

$$\begin{bmatrix} -a_1\partial_x q(x,y) & -a_1\partial_y q(x,y) & a_0 & z & -q(x,y) \end{bmatrix}. \tag{2.1.23}$$

To get a singular point we must have $a_0 = 0$, therefore above the origin there can be at most one point w where the blow-up is singular, and that is given by $(a_0 : a_1) = (0 : 1),\ (x,y,z) = (0,0,0)$. From the first two entries of the Jacobian we see that if C is non-singular at p, the blow-up is non-singular at w, therefore above a non-singular point of C all points of the blow-up are non-singular. It is also clear that if p is a singular point of C (i.e. $q(x,y)$ is singular at $(0,0)$), then w is also a singular point of the blow-up. To see that the two singularity types are the same, let's write equation (2.1.22) in the open set $\mathbb{C}^3 \times \mathbb{C}$ given by $a_1 \neq 0$. We can pick $a_1 = 1$ and write

$$q(x,y) - a_0 z = 0 \tag{2.1.24}$$

as an equation in $\mathbb{C}^4$. Applying the linear transformation

$$a_0 = u + iv, \quad z = -u + iv \tag{2.1.25}$$

the equation transforms into

$$q(x,y) + u^2 + v^2 = 0. \tag{2.1.26}$$

Thus the type of the singularity is determined by $q(x,y)$ and therefore C has the same singularity type at p as the blow-up has at w. □

2.2 Cohomology of a Blow-up

In this section we review some basic facts about the cohomology of a blow-up. This is standard (see e.g., [GH78, p. 605]). For completeness, we include detailed proofs here. Let M be a complex manifold and $Y \subset M$ a submanifold. Let $\widetilde{M}$ denote $\mathrm{Bl}_Y M$, the blow-up of M along Y, $\pi : \widetilde{M} \to M$ the map of the blow-up, and $E = \pi^{-1}(Y) \subset \widetilde{M}$ the exceptional set. For computing the cohomology of $\widetilde{M}$ we have the following

Theorem 2.2.1 ([GH78, p. 605])**.** For any $n \geq 0$ the cohomology of the blow-up is given by

$$H^n(\widetilde{M}, \mathbb{C}) = \pi^*\big(H^n(M, \mathbb{C})\big) \oplus \Big(H^n(E, \mathbb{C}) \Big/ \pi^*_{|_E}\big(H^n(Y, \mathbb{C})\big)\Big), \tag{2.2.2}$$

and this formula respects the Hodge-decomposition, i.e. for any $p, q \geq 0$ we have

$$H^{p,q}(\widetilde{M}) = \pi^*\big(H^{p,q}(M)\big) \oplus \Big(H^{p,q}(E) \Big/ \pi^*_{|_E}\big(H^{p,q}(Y)\big)\Big). \tag{2.2.3}$$

Formula (2.2.2) is also true with the integer cohomologies $H^n(\cdot, \mathbb{Z}) \big/ \{\text{torsion cocycles}\}$.

Proof. We want to use Mayer-Vietoris sequences for M and $\widetilde{M}$. We will decompose M as the union of a tubular neighborhood of Y and the open set $M - Y$, and similarly, we will write $\widetilde{M}$ as the union of a neighborhood of E and the open set $\widetilde{M} - E$. The two decompositions will be compatible along π. First let's choose U to be a tubular neighborhood of Y, and $\widetilde{U} = \pi^{-1}(U) \subset \widetilde{M}$ its preimage along π, which is a neighborhood of E. We also introduce the following open sets:

$$M^\circ = M - Y \tag{2.2.4}$$

$$\widetilde{M}^\circ = \widetilde{M} - E = \pi^{-1}(M^\circ) \tag{2.2.5}$$

$$U^\circ = U - Y \tag{2.2.6}$$

$$\widetilde{U}^\circ = \widetilde{U} - E = \pi^{-1}(U^\circ). \tag{2.2.7}$$

With this notation we have

$$M = M^\circ \cup U, \qquad U^\circ = M^\circ \cap U, \tag{2.2.8}$$

which gives the Mayer-Vietoris sequence

$$\cdots \to H^{n-1}(U^\circ, \mathbb{C}) \to H^n(M, \mathbb{C}) \to H^n(M^\circ, \mathbb{C}) \oplus H^n(U, \mathbb{C}) \to H^n(U^\circ, \mathbb{C}) \to \cdots, \tag{2.2.9}$$

and similarly

$$\widetilde{M} = \widetilde{M}^\circ \cup \widetilde{U}, \qquad \widetilde{U}^\circ = \widetilde{M}^\circ \cap \widetilde{U}, \tag{2.2.10}$$

gives the sequence

$$\cdots \to H^{n-1}(\widetilde{U}^\circ, \mathbb{C}) \to H^n(\widetilde{M}, \mathbb{C}) \to H^n(\widetilde{M}^\circ, \mathbb{C}) \oplus H^n(\widetilde{U}, \mathbb{C}) \to H^n(\widetilde{U}^\circ, \mathbb{C}) \to \cdots. \tag{2.2.11}$$

The map π is an isomorphism outside of E and Y, therefore we have the isomorphisms

$$\pi|_{\widetilde{M}^\circ} : \widetilde{M}^\circ \to M^\circ \tag{2.2.12}$$

$$\pi|_{\widetilde{U}^\circ} : \widetilde{U}^\circ \to U^\circ, \tag{2.2.13}$$

which in turn induce cohomology isomorphisms

$$\pi^*_{|_{\widetilde{M}^\circ}} : H^n(M^\circ, \mathbb{C}) \to H^n(\widetilde{M}^\circ, \mathbb{C}) \tag{2.2.14}$$

$$\pi^*_{|_{\widetilde{U}^\circ}} : H^n(U^\circ, \mathbb{C}) \to H^n(\widetilde{U}^\circ, \mathbb{C}). \tag{2.2.15}$$

Now U being a tubular neighborhood of Y means that there is a contraction

$$f : U \times [0,1] \to U \tag{2.2.16}$$

$$f(z,0) = z, \quad \forall z \in U \tag{2.2.17}$$

$$f(y,t) = y, \quad \forall y \in Y,\ \forall t \in [0,1] \tag{2.2.18}$$

$$f(z,1) \in Y, \quad \forall z \in U, \tag{2.2.19}$$

and Y is a deformation retract of U. This means that they are homotopy equivalent and the inclusion $i : Y \hookrightarrow U$ induces an isomorphism $i^* : H^n(U,\mathbb{C}) \to H^n(Y,\mathbb{C})$. Using the universal property of blow-up ([Har77, Prop. II.7.14., Cor. II.7.15.]) f induces a contraction $\tilde{f}$ of $\widetilde{U}$ to E and therefore the inclusion $\tilde{\imath} : E \hookrightarrow \widetilde{U}$ induces an isomorphism $\tilde{\imath}^* : H^n(\widetilde{U},\mathbb{C}) \to H^n(E,\mathbb{C})$.

Using the above isomorphisms and the functorial property of the Mayer-Vietoris sequence, which we apply to π, we get the following commutative diagram:

$$\begin{array}{ccccccc}
H^{n-1}(U^\circ,\mathbb{C}) & \longrightarrow & H^n(\widetilde{M},\mathbb{C}) & \longrightarrow & H^n(M^\circ,\mathbb{C}) \oplus H^n(E,\mathbb{C}) & \longrightarrow & H^n(U^\circ,\mathbb{C}) \\
\Big\| \pi^*|_{\widetilde{U}^\circ} & & \Big\| & & \Big\| \pi^*|_{\widetilde{M}^\circ} \quad \Big\| \tilde{\imath}^* & & \Big\| \pi^*|_{\widetilde{U}^\circ} \\
H^{n-1}(\widetilde{U}^\circ,\mathbb{C}) & \longrightarrow & H^n(\widetilde{M},\mathbb{C}) & \longrightarrow & H^n(\widetilde{M}^\circ,\mathbb{C}) \oplus H^n(\widetilde{U},\mathbb{C}) & \longrightarrow & H^n(\widetilde{U}^\circ,\mathbb{C}) \\
\uparrow \pi^* & & \uparrow \pi^* & & \uparrow \pi^* & & \uparrow \pi^* \\
H^{n-1}(U^\circ,\mathbb{C}) & \longrightarrow & H^n(M,\mathbb{C}) & \longrightarrow & H^n(M^\circ,\mathbb{C}) \oplus H^n(U,\mathbb{C}) & \longrightarrow & H^n(U^\circ,\mathbb{C}) \\
\Big\| & & \Big\| & & \Big\| \quad \Big\| i^* & & \Big\| \\
H^{n-1}(U^\circ,\mathbb{C}) & \longrightarrow & H^n(M,\mathbb{C}) & \longrightarrow & H^n(M^\circ,\mathbb{C}) \oplus H^n(Y,\mathbb{C}) & \longrightarrow & H^n(U^\circ,\mathbb{C})
\end{array} \tag{2.2.20}$$

The second row is the same as (2.2.11), the third row as (2.2.9) and the arrows between them come from the functorial property of the Mayer-Vietoris sequence. Copying the first and fourth rows we get:

$$\begin{array}{ccccccc} H^{n-1}(U^\circ,\mathbb{C}) & \longrightarrow & H^n(\widetilde{M},\mathbb{C}) & \longrightarrow & H^n(M^\circ,\mathbb{C})\oplus H^n(E,\mathbb{C}) & \longrightarrow & H^n(U^\circ,\mathbb{C}) \\ \| & & \uparrow{\scriptstyle \pi^*} & & \| \quad \uparrow{\scriptstyle \pi^*_{|_E}} & & \| \\ H^{n-1}(U^\circ,\mathbb{C}) & \longrightarrow & H^n(M,\mathbb{C}) & \longrightarrow & H^n(M^\circ,\mathbb{C})\oplus H^n(Y,\mathbb{C}) & \longrightarrow & H^n(U^\circ,\mathbb{C}) \end{array} \tag{2.2.21}$$

The map $\pi^* : H^n(M,\mathbb{C}) \to H^n(\widetilde{M},\mathbb{C})$ is injective. To see this, we consider de-Rham cohomology and the fact that π is birational, i.e. an isomorphism away from closed sets. If φ is a C^∞ n-form on M that represents a non-zero cocycle, then by the de Rham theorem there is a piecewise smooth n-chain σ, such that $\int_\sigma \varphi \neq 0$. We can choose σ such that it is transversal to Y. Then if $\tilde{\sigma}$ is the proper transform of σ (which is an n-chain in $\widetilde{M}$), we have $\int_{\tilde{\sigma}} \pi^*\varphi = \int_\sigma \varphi \neq 0$, because π is an isomorphism away from Y and E. This proves that π^* is injective. By theorem 2.2.24 below $\pi^*_{|_E} : H^n(Y,\mathbb{C}) \to H^n(E,\mathbb{C})$ is also injective. We also need to prove that the map $H^n(\widetilde{M},\mathbb{C}) \to H^n(E,\mathbb{C})$ is surjective. Again, to see this, we consider de Rham cohomology: if φ is a C^∞ n-form on E, it can be extended into an C^∞ n-form on $\widetilde{M}$ using a tubular neighborhood of E.

Using the above we can transform diagram (2.2.21) into the following:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \uparrow & & \uparrow & & \\ & & \operatorname{Coker}\beta & \xrightarrow{\delta} & \operatorname{Coker}\gamma & & \\ & & {\scriptstyle \beta'}\uparrow & & {\scriptstyle \gamma'}\uparrow & & \\ A & \xrightarrow{\mu'} & B' & \xrightarrow{\alpha'} & C_1 \oplus C_2' & \xrightarrow{\lambda'} & D \\ {\scriptstyle \cong}\uparrow & & {\scriptstyle \beta}\uparrow & & {\scriptstyle \gamma=\mathrm{id}\oplus\gamma_2}\uparrow & & {\scriptstyle \cong}\uparrow \\ A & \xrightarrow{\mu} & B & \xrightarrow{\alpha} & C_1 \oplus C_2 & \xrightarrow{\lambda} & D \\ & & \uparrow & & \uparrow & & \\ & & 0 & & 0 & & \end{array} \tag{2.2.22}$$

The columns are exact, the last two rows are the rows of diagram (2.2.21), therefore exact, and we want to prove that δ, which is induced from α', is an isomorphism. To prove injectivity, let $x \in \operatorname{Coker}\beta$ with $\delta(x) = 0$. There is a $y \in B'$, such that $\beta'(y) = x$. Since $\gamma'(\alpha'(y)) = 0$, there exists $z \in C_1 \oplus C_2$, such that $\gamma(z) = \alpha'(y)$. Now $\lambda'(\alpha'(y)) = 0$, therefore $\lambda'(\gamma(z)) = 0$ and $\lambda(z) = 0$. Thus there exists some $b \in B$, such that $z = \alpha(b)$. Next, we have $\alpha'(y - \beta(b)) = \alpha'(y) - \alpha'(\beta(b)) = \alpha'(y) - \gamma(\alpha(b)) = \alpha'(y) - \gamma(z) = 0$, which gives that there exists $u \in A$, such that $\mu'(u) = y - \beta(b)$. But since $\mu'(u) = \beta(\mu(u))$, we have $y = \beta(b) + \beta(\mu(u))$. This gives that $x = \beta'(y) = 0$ and injectivity is proven. To prove surjectivity we use the fact that $\mathrm{pr}_2 \circ \alpha'$ is surjective (this is the map $H^n(\widetilde{M}, \mathbb{C}) \to H^n(E, \mathbb{C})$ above). Let $x \in \operatorname{Coker}\gamma$ be arbitrary element. Since γ' is surjective, there exists $(u, v) \in C_1 \oplus C_2'$ with $\gamma'(u, v) = x$. Since $\mathrm{pr}_2 \circ \alpha'$ is surjective, there exists $b \in B'$, such that $\alpha'(b) = (u_2, v)$, for some $u_2 \in C_1$. Then $\gamma'(\alpha'(b)) = \gamma'(u_2, v) = \gamma'(u, v) + \gamma'(u_2 - u, 0) = \gamma'(u, v) + \gamma'(\gamma(u_2 - u, 0)) = x$. Therefore $x = \delta(\beta'(b))$, and δ is surjective.

Returning to our original notation and to diagram (2.2.21), the above gives that

$$H^n(\widetilde{M}, \mathbb{C})/\pi^*\big(H^n(M, \mathbb{C})\big) \cong H^n(E, \mathbb{C})/\pi^*_{|E}\big(H^n(Y, \mathbb{C})\big), \tag{2.2.23}$$

which proves the first part of the theorem, since these cohomologies are vector spaces and thus subspaces are direct summands.

To prove the theorem for the $H^{p,q}$ cohomology, we notice that the above proof works for that case as well. We can use the sheaf cohomology version of the Mayer-Vietoris theorem, since $H^{p,q}(M) \cong H^q(M, \Omega^p)$. In the diagram (2.2.20) all vertical maps are pullbacks, therefore they respect the Hodge decomposition. The maps π^* and $\pi^*_{|E}$ in diagram (2.2.21) are injective on the $H^{p,q}$ direct summands, since they are pullbacks and respect the Hodge decomposition, and similarly the map $H^n(\widetilde{M}, \mathbb{C}) \to H^n(E, \mathbb{C})$ is surjective on the $H^{p,q}$ components. □

Theorem 2.2.24 ([BT82, §20])**.** If k is the codimension of Y in M, we have

$$H^n(E, \mathbb{C}) \cong \bigoplus_{i=0}^{k-1} H^{n-2i}(Y, \mathbb{C}) = H^n(Y, \mathbb{C}) \oplus H^{n-2}(Y, \mathbb{C}) \oplus \cdots \oplus H^{n-2k+2}(Y, \mathbb{C}), \tag{2.2.25}$$

and

$$H^{p,q}(E) \cong \bigoplus_{i=0}^{k-1} H^{p-i,q-i}(Y) = H^{p,q}(Y) \oplus H^{p-1,q-1}(Y) \oplus \cdots \oplus H^{p-k+1,q-k+1}(Y). \tag{2.2.26}$$

Proof. The exceptional set E is the projectivization $\mathbb{P}(N_{Y/M})$ of the normal bundle $N_{Y/M}$ of Y in M. The fibers on $N_{Y/M}$ have dimension k, and the fibers of $E \to Y$ are projective spaces $\mathbb{P}^{k-1}$. From the theory of the cohomology of the projectivization of a vector bundle ([BT82, §20]) we know that additively

$$H^*(E, \mathbb{C}) = H^*(Y, \mathbb{C}) \otimes H^*(\mathbb{P}^{k-1}, \mathbb{C}). \tag{2.2.27}$$

We can rewrite this as

$$H^n(E, \mathbb{C}) = \bigoplus_{r+s=n} H^r(Y, \mathbb{C}) \otimes H^s(\mathbb{P}^{k-1}, \mathbb{C}) \tag{2.2.28}$$

for any $n \geq 0$. Since the cohomology of the projective space is

$$H^s(\mathbb{P}^{k-1}, \mathbb{C}) = \begin{cases} \mathbb{C} & \text{if} \quad 2 \mid s, \quad 0 \leq s \leq 2k-2 \\ 0 & \text{else,} \end{cases} \tag{2.2.29}$$

we obtain the first statement of the theorem. To prove (2.2.26), we note that the tensor product in (2.2.28) is realized as the wedge product of differential forms when we consider de Rham cohomology. In fact, let η be a 2-form on E, such that the restriction to any fiber gives a cocycle $[\eta_{|_{\mathbb{P}^{k-1}}}]$ that generates $H^2(\mathbb{P}^{k-1}, \mathbb{C}) = \mathbb{C}$, i.e. it represents a hyperplane cocycle. Such η exists by [BT82, p. 270]. Then (2.2.28) becomes

$$H^n(E, \mathbb{C}) = \bigoplus_{i=0}^{k-1} H^{n-2i}(Y, \mathbb{C}) \wedge [\eta^i], \tag{2.2.30}$$

where $H^r(Y, \mathbb{C}) \subset H^r(E, \mathbb{C})$ through the inclusion $\pi^*_{|_E}$. Since $[\eta^i] \in H^{i,i}(E)$, we obtain the second statement of the theorem. □

2.3 Cubic Threefold with a Single A_1 or A_2 Singularity

In this section we describe the intermediate Jacobian of the desingularization of a cubic threefold with a single A_1 or A_2 singularity. We show in theorem 2.3.15 that it is the Jacobian of the $(2,3)$-curve described above. These results are due to Clemens and Griffiths [CG72], and we include the proof here for completeness. Let $X \subset \mathbb{P}^4$ be a cubic threefold with a single A_1 or A_2 singularity at $P \in X$. By theorem 2.1.1 $\mathrm{Bl}_P X = \mathrm{Bl}_C \mathbb{P}^3$, where C is the $(2,3)$-curve introduced earlier, and by theorem 2.1.18 the singularities of C are the same as that of $\mathrm{Bl}_P X$. Blowing up X at P will desingularize X at P, since it is either an A_1 or an A_2 point, therefore both $\mathrm{Bl}_P X$ and the curve C are non-singular. Our goal is to find the intermediate Jacobian of the desingularization $\widetilde{X} = \mathrm{Bl}_P X = \mathrm{Bl}_C \mathbb{P}^3$. For this we find the cohomology and the Hodge diamond of $\widetilde{X}$. Let $\pi : \widetilde{X} \to \mathbb{P}^3$ be the map of the blow-up along C, and let $E = \pi^{-1}(C)$ be the exceptional divisor. Then $\dim E = 2$ and the fibers of $\pi_{|_E}$ are $\mathbb{P}^1$. By theorem 2.2.24 we have

$$H^n(E,\mathbb{C}) \cong H^n(C,\mathbb{C}) \oplus H^{n-2}(C,\mathbb{C}) \tag{2.3.1}$$

$$H^{p,q}(E) \cong H^{p,q}(C) \oplus H^{p-1,q-1}(C). \tag{2.3.2}$$

More precisely, formula (2.2.30) gives

$$H^n(E,\mathbb{C}) = H^n(C,\mathbb{C}) \oplus H^{n-2}(C,\mathbb{C}) \wedge [\eta], \tag{2.3.3}$$

where η is a $(1,1)$-form on E, such that its restrictions on the fibers generate $H^2(\mathbb{P}^1,\mathbb{C})$. We then have the following cohomologies:

$$H^4(E,\mathbb{C}) = \mathbb{C} \tag{2.3.4}$$

$$H^3(E,\mathbb{C}) = H^1(C,\mathbb{C}) \wedge [\eta] \tag{2.3.5}$$

$$H^2(E,\mathbb{C}) = H^2(C,\mathbb{C}) \oplus H^0(C,\mathbb{C}) \wedge [\eta] \cong \mathbb{C} \oplus \mathbb{C} \tag{2.3.6}$$

$$H^1(E,\mathbb{C}) = H^1(C,\mathbb{C}) \tag{2.3.7}$$

$$H^0(E,\mathbb{C}) = \mathbb{C}. \tag{2.3.8}$$

The Hodge diamond of E:

$$\begin{array}{ccccc} & & \mathbb{C} & & \\ & H^{1,0}(C) & & H^{0,1}(C) & \\ 0 & & \mathbb{C}^2 & & 0 \\ & H^{1,0}(C) & & H^{0,1}(C) & \\ & & \mathbb{C} & & \end{array} \tag{2.3.9}$$

Next we want to use theorem 2.2.1, so we compute

$$H^2(E,\mathbb{C})\Big/\pi^*_{|E}\big(H^2(C,\mathbb{C})\big) = H^0(C,\mathbb{C}) \wedge [\eta] \cong \mathbb{C} \tag{2.3.10}$$

$$H^1(E,\mathbb{C})\Big/\pi^*_{|E}\big(H^1(C,\mathbb{C})\big) = 0 \tag{2.3.11}$$

$$H^0(E,\mathbb{C})\Big/\pi^*_{|E}\big(H^0(C,\mathbb{C})\big) = 0. \tag{2.3.12}$$

After "factoring out" the Hodge diamond of C from that of E we get:

$$\begin{array}{ccccc} & & \mathbb{C} & & \\ & H^{1,0}(C) & & H^{0,1}(C) & \\ 0 & & \mathbb{C} & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \tag{2.3.13}$$

To get the Hodge diamond of $\widetilde{X}$, we have to add the Hodge diamond of $\mathbb{P}^3$ to the above, which gives:

$$\begin{array}{ccccccc} & & & \mathbb{C} & & & \\ & & 0 & & 0 & & \\ & 0 & & \mathbb{C}^2 & & 0 & \\ 0 & & H^{1,0}(C) & & H^{0,1}(C) & & 0 \\ & 0 & & \mathbb{C}^2 & & 0 & \\ & & 0 & & 0 & & \\ & & & \mathbb{C} & & & \end{array} \tag{2.3.14}$$

This means that $H^{1,2}(\widetilde{X}) \oplus H^{0,3}(\widetilde{X}) \cong H^{0,1}(C)$, and $H^3(\widetilde{X},\mathbb{C}) \cong H^1(C,\mathbb{C})$. This proves the following

Theorem 2.3.15 ([CG72]). The intermediate Jacobian of the desingularization $\widetilde{X} = \mathrm{Bl}_P X$ of a cubic threefold with a single A_1 or A_2 singularity is isomorphic to the Jacobian JC of the $(2,3)$-curve

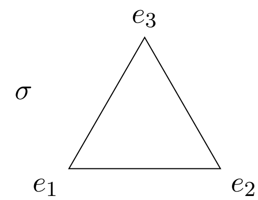
C.

2.4 Cubic Threefold with a Single A_3 Singularity

Let $X \subset \mathbb{P}^4$ be a cubic threefold with a single A_3 singularity at $P \in X$. Blowing up X once at P results in a singular threefold with a node, so we have to blow it up a second time to get the desingularization $\widetilde{X}$ of X. Since the singularities of $\operatorname{Bl}_P X \cong \operatorname{Bl}_C \mathbb{P}^3$ are the same as the singularities of C, the curve C is singular with a node. Thus we do not want to consider the Jacobian of C, but rather the Jacobian of the normalization of C, i.e. the blow-up of C at the node. To find the intermediate Jacobian of $\widetilde{X}$, we want to see how it relates to the Jacobian of the normalization of C. For this we want to find some relationship between $\widetilde{X}$ and $\operatorname{Bl}_{NC} \operatorname{Bl}_Q \mathbb{P}^3$, where Q is the node of C and $NC = \operatorname{Bl}_Q C$ is the normalization of C. In other words to get $Y = \operatorname{Bl}_{NC} \operatorname{Bl}_Q \mathbb{P}^3$ we first blow up $\mathbb{P}^3$ at the node of C, then blow up the result along the proper transform of C. Notice that an alternative way to get $\widetilde{X}$ is first blowing up $\mathbb{P}^3$ along C, then blowing up the result at the node (which is a point above Q, the node of C), thus to get Y we essentially switch the order of these two blow-ups.

First we consider the picture locally around Q. Since Q is a node of C, there exists some neighborhood U of Q and an analytic isomorphism $\xi : U \to \mathbb{C}^3$, that gives an analytic isomorphism between $C \cap U$ and $D = \{(x, y, z) \in \mathbb{C}^3 \mid xy = z = 0\}$, i.e. the union of the x and y axes. We need to compare two series of blow-ups: one, the blow-up of $\mathbb{C}^3$ along D followed by the blow-up of the result at the node of the result (there has to be one node above the origin); two, the blow-up of $\mathbb{C}^3$ at the origin followed by the blow-up of the result along the proper transform of D. We hope to find further blow-ups of these spaces that will result in the same threefold. To do this we use the theory of toric varieties. We follow the treatment of the theory as in [CLS11], and we rely on results of Chapters 1-3, 7, 11.

Let $N = \mathbb{N}^3$, $N_{\mathbb{R}} = N \otimes \mathbb{R} = \mathbb{R}^3$ and (e_1, e_2, e_3) the standard basis of $N_{\mathbb{R}}$. Let $\sigma = \operatorname{Cone}(e_1, e_2, e_3) \subset N_{\mathbb{R}}$ be the cone of the first octant, which is strongly convex and rational. The following picture depicts the intersection of σ with the plane lying on the endpoints of the vectors e_1, e_2, e_3.



Below we use the same plane to get two dimensional representations of cones and fans. The affine toric variety of σ is $U_\sigma = \mathbb{C}^3$. We note the closures of the orbits of the faces of σ:

$$V(\text{Cone}(e_1)) = \text{the } yz\text{-plane} \tag{2.4.1}$$

$$V(\text{Cone}(e_2)) = \text{the } xz\text{-plane} \tag{2.4.2}$$

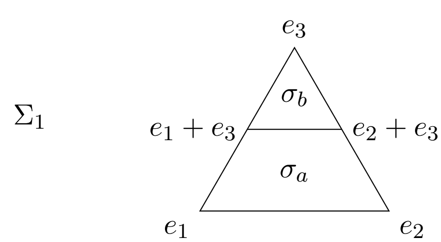
$$V(\text{Cone}(e_3)) = \text{the } xy\text{-plane} \tag{2.4.3}$$

$$V(\text{Cone}(e_1, e_2)) = \text{the } z\text{-axis} \tag{2.4.4}$$

$$V(\text{Cone}(e_1, e_3)) = \text{the } y\text{-axis} \tag{2.4.5}$$

$$V(\text{Cone}(e_2, e_3)) = \text{the } x\text{-axis.} \tag{2.4.6}$$

To construct $\text{Bl}_D\,\mathbb{C}^3$, let Σ_1 be the fan containing the cones $\sigma_a = \text{Cone}(e_1, e_2, e_2 + e_3, e_1 + e_3)$ and $\sigma_b = \text{Cone}(e_1 + e_3, e_2 + e_3, e_3)$ and all of their faces. Thus we get Σ_1 if we subdivide σ along the new face $\text{Cone}(e_1 + e_3, e_2 + e_3)$.



Proposition 2.4.7. The toric variety T_{Σ_1} is the blow-up of $\mathbb{C}^3$ along the union of the x and y axes, i.e. $T_{\Sigma_1} \cong \text{Bl}_D\,\mathbb{C}^3$.

Proof. We need to construct U_{σ_a} and U_{σ_b} and see how they glue together. To get the dual cones we need to find the facet normals of σ_a and σ_b. A facet normal can be found by taking the cross product in $\mathbb{R}^3$ of the two generating vectors of the facet, e.g. $e_1 \times e_2 = e_3$ gives the normal to the facet that is represented by the bottom edge of the triangle in the diagram of Σ_1. This computation gives:

$$\sigma_a^\vee = \text{Cone}(e_3, e_1, e_1 + e_2 - e_3, e_2) \subset M_\mathbb{R} \tag{2.4.8}$$

$$\sigma_b^\vee = \text{Cone}(e_3 - e_2 - e_1, e_1, e_2) \subset M_\mathbb{R}, \tag{2.4.9}$$

where $M = N^\vee$ and $M_\mathbb{R} = M \otimes \mathbb{R} = N_\mathbb{R}^\vee$. We have to find generators of the semigroups $\sigma_a^\vee \cap M$ and $\sigma_b^\vee \cap M$. In fact in both cases the generators of the cones in $M_\mathbb{R}$ are also generators of the semigroups in M. To see this let

$$x = t_1 e_3 + t_2 e_1 + t_3(e_1 + e_2 - e_3) + t_4 e_2 \in \sigma_a^\vee \cap M, \quad t_i \in \mathbb{R}_{\geq 0}. \tag{2.4.10}$$

This gives

$$x = (t_2 + t_3)e_1 + (t_4 + t_3)e_2 + (t_1 - t_3)e_3, \tag{2.4.11}$$

where the coefficients must be integers. If $t_1 \geq t_3$, we are done. Otherwise we write

$$x = (t_1 + t_2)e_1 + (t_1 + t_4)e_2 + (t_3 - t_1)(e_1 + e_2 - e_3). \tag{2.4.12}$$

Since now $t_3 - t_1$ is non-negative and integer, $t_1 + t_2$ and $t_1 + t_4$ are also integers, this proves that $\{e_1, e_2, e_3, e_1 + e_2 - e_3\}$ are generators of the semigroup $\sigma_a^\vee \cap M$. Similarly let

$$x = t_1(e_3 - e_2 - e_1) + t_2 e_1 + t_3 e_2 \in \sigma_b^\vee \cap M, \quad t_i \in \mathbb{R}_{\geq 0}. \tag{2.4.13}$$

We can rewrite this as

$$x = (t_2 - t_1)e_1 + (t_3 - t_1)e_2 + t_1 e_3, \tag{2.4.14}$$

where the coefficients are integers, but this is only possible if the original coefficients in (2.4.13) are integers, which proves that $\{e_3 - e_2 - e_1, e_1, e_2\}$ are generators of $\sigma_b^\vee \cap M$.

The affine toric variety U_{σ_a} is then defined as the Zariski closure of the image of $(\mathbb{C}^*)^3 \ni$

$(q_1, q_2, q_3) \mapsto (q_3, q_1, q_1 q_2 q_3^{-1}, q_2) \in \mathbb{C}^4$. Thus

$$U_{\sigma_a} = \mathbf{V}(\langle u_1 u_3 - u_2 u_4 \rangle), \qquad \langle u_1 u_3 - u_2 u_4 \rangle \lhd \mathbb{C}[u_1, u_2, u_3, u_4]. \tag{2.4.15}$$

Similarly U_{σ_b} is the Zariski closure of the image of $(\mathbb{C}^*)^3 \ni (q_1, q_2, q_3) \mapsto (q_3 q_1^{-1} q_2^{-1}, q_1, q_2) \in \mathbb{C}^3$. Thus $U_{\sigma_b} \cong \mathbb{C}^3$, and the coordinate ring is $\mathbb{C}[U_{\sigma_b}] = \mathbb{C}[v_1, v_2, v_3]$ (this also follows from the fact that σ_b is generated by a $\mathbb{Z}$-basis of N). We can glue $U_{\sigma_a} \setminus \mathbf{V}(\langle u_3 \rangle)$ and $U_{\sigma_b} \setminus \mathbf{V}(\langle v_1 \rangle)$ with the isomorphism given by

$$u_1 = v_1 v_2 v_3, \qquad u_2 = v_2, \qquad u_3 = \frac{1}{v_1}, \qquad u_4 = v_3. \tag{2.4.16}$$

Since the union D of the x and y axes is given by the ideal $\mathfrak{a} = \langle xy, z \rangle \lhd \mathbb{C}[x, y, z]$, the blow-up of $\mathbb{C}^3$ along D is given in $\mathbb{P}^1 \times \mathbb{C}^3$ by the equation $a_0 xy = a_1 z$, where $(a_0 : a_1) \in \mathbb{P}^1$. We get an open affine cover by the sets $\{xy = a_1 z\} \subset \mathbb{C}^4$ and $\{a_0 xy = z\} \subset \mathbb{C}^4$ by setting $a_0 = 1$ first, then $a_1 = 1$. The identifications

$$a_0 = v_1, \qquad a_1 = u_3, \qquad x = u_2 = v_2, \qquad y = u_4 = v_3, \qquad z = u_1 \tag{2.4.17}$$

then show that the toric variety T_{Σ_1} is indeed the blow-up $\mathrm{Bl}_D \mathbb{C}^3$.

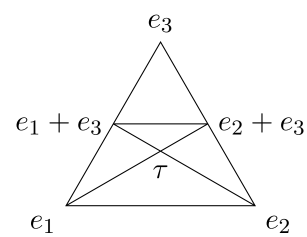
For an alternative proof, we can construct the lattice polyhedron P belonging to $\mathfrak{a}$ (as in [CLS11, §11.3]). The ideal $\mathfrak{a}$ is radical because it is generated by square-free monomials, therefore it is integrally closed (see [CLS11, Example 11.3.7.]). The polyhedron belonging to $\mathfrak{a}$ is then $P = \mathrm{Conv}(e_1 + e_2, e_3) + \sigma^\vee$, where $\mathrm{Conv}(e_1 + e_2, e_3)$ is the line segment between the two points. Since the facets of the recession cone of P are coordinate planes, P is normal and therefore $X_P = \mathrm{Bl}_{\mathfrak{a}} \mathbb{C}^3$. To find the fan in $N_{\mathbb{R}}$ that gives X_P we note that P has two vertices, $e_1 + e_2$ and e_3. The facets joining $e_1 + e_2$ have normal vectors that are the generators of σ_b and the facets joining e_3 have normal vectors that are the generators of σ_a. This means that the fan corresponding to P is Σ_1 and therefore $X_{\Sigma_1} = X_P = \mathrm{Bl}_D \mathbb{C}^3$. □

The blow-up $\mathrm{Bl}_D \mathbb{C}^3$ has a node, and in the next step we want to find the blow-up of $\mathrm{Bl}_D \mathbb{C}^3$ at the node. From Examples 11.1.12 and 11.2.12 of [CLS11] we know that this blow-up is given by the star subdivision $\Sigma_2 = \Sigma_1^*(\tau)$ of Σ_1, where

$$\tau = \mathrm{Cone}(e_1 + e_2 + e_3) = \mathrm{Cone}(e_1, e_2 + e_3) \cap \mathrm{Cone}(e_2, e_1 + e_3) \tag{2.4.18}$$

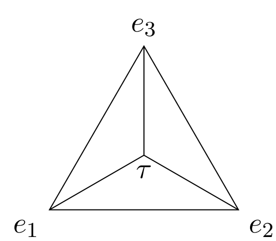
is the center line of σ_a. The toric variety X_{Σ_2} is the one that we get by first blowing up $\mathbb{C}^3$ along the union of the x and y axes, then blowing up the singular point.

Σ_2



In the second series of blow-ups we first blow up $\mathbb{C}^3$ at the origin, then blow up the result along the proper transform of the union of the x and y axes. Again, we start from the cone $\sigma \subset N_{\mathbb{R}}$ which corresponds to $\mathbb{C}^3$. The origin is the orbit belonging to σ, i.e. $V(\sigma) = \{0\}$. If we let Σ denote the fan whose only maximal cone is σ, as σ is smooth the blow-up $\mathrm{Bl}_0\,\mathbb{C}^3$ is given by the star subdivision $\Sigma_3 = \Sigma^*(\tau)$, where $\tau = \mathrm{Cone}(e_1 + e_2 + e_3)$, as defined earlier.

Σ_3



In Σ_3 we have the following orbit closures:

$$V(\text{Cone}(e_1, e_3)) = \text{proper transform of the } y \text{ axis} \tag{2.4.19}$$

$$V(\text{Cone}(e_2, e_3)) = \text{proper transform of the } x \text{ axis} \tag{2.4.20}$$

$$V(\text{Cone}(e_3)) = \text{proper transform of the } xy \text{ plane} \tag{2.4.21}$$

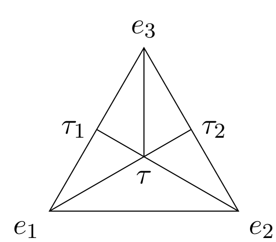
$$V(\tau) = \text{the exceptional set} \tag{2.4.22}$$

$$V(\text{Cone}(\tau, e_3)) = V(\text{Cone}(e_3)) \cap V(\tau). \tag{2.4.23}$$

The orbit closure $V(\text{Cone}(e_3))$ can also be described as the xy plane blown up at the origin and $V(\text{Cone}(\tau, e_3))$ as the exceptional set of the blow-up of the xy plane at the origin.

In $\text{Bl}_0 \mathbb{C}^3$ the proper transforms of the x and y axes are disjoint, therefore blowing up $\text{Bl}_0 \mathbb{C}^3$ along the proper transform of the union of the two axes is the same as blowing it up twice consecutively along the proper transform of one axis then the other. To get the two consecutive blow-ups we take two consecutive star subdivisions of Σ_3. This gives $\Sigma_4 = (\Sigma_3^*(\tau_1))^* (\tau_2)$, where $\tau_1 = \text{Cone}(e_1 + e_3)$ and $\tau_2 = \text{Cone}(e_2 + e_3)$. This gives the blow-up that we want, because all cones of Σ_3 and $\Sigma_3^*(\tau_1)$ are smooth. Thus X_{Σ_4} is the blow-up of $\mathbb{C}^3$ at the origin followed by the blow-up of the proper transform of the union of the x and y axes.

Σ_4



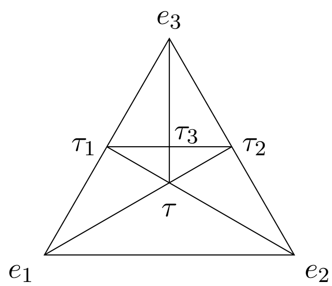
In the next step we construct a common refinement of the fans Σ_2 and Σ_4. It is given by adding the new ray

$$\tau_3 = \text{Cone}(e_1 + e_2 + 2e_3) = \text{Cone}(\tau_1, \tau_2) \cap \text{Cone}(\tau, e_3), \tag{2.4.24}$$

and taking the star subdivision

$$\Sigma_5 = \Sigma_2^*(\tau_3) = \Sigma_4^*(\tau_3). \tag{2.4.25}$$

Σ_5



Since Σ_2 and Σ_4 are smooth, these refinements represent blow-ups. We obtain Σ_5 from Σ_2 by adding τ_3 to $\text{Cone}(\tau_1, \tau_2)$ and subdividing, therefore $X_{\Sigma_5} \to X_{\Sigma_2}$ is the blow-up $\text{Bl}_{V(\text{Cone}(\tau_1,\tau_2))} X_{\Sigma_2}$. By [CLS11, Prop. 11.1.10] the orbit closure $V(\text{Cone}(\tau_1, \tau_2))$ in Σ_1 is the total transform (the preimage) of $0 \in \mathbb{C}^3$ in $\text{Bl}_D \mathbb{C}^3$, which is some curve F isomorphic to $\mathbb{P}^1$. The orbit closure $V(\text{Cone}(\tau_1, \tau_2))$ in Σ_2 is then the proper transform of F along the blow-up $X_{\Sigma_2} \to X_{\Sigma_1}$. This latter blow-up was at a node of X_{Σ_1} which is actually on F. The proper transform of F is still isomorphic to $\mathbb{P}^1$, therefore we get X_{Σ_5} from X_{Σ_2} by blowing it up along some curve isomorphic to $\mathbb{P}^1$. Similarly, we obtain Σ_5 from Σ_4 by adding τ_3 to $\text{Cone}(\tau, e_3)$ and subdividing, therefore $X_{\Sigma_5} \to X_{\Sigma_4}$ is the blow-up $\text{Bl}_{V(\text{Cone}(\tau,e_3))} X_{\Sigma_4}$. The orbit closure $V(\text{Cone}(\tau, e_3))$ in Σ_3 is the exceptional set of the blow-up of the xy plane at the origin, therefore it is isomorphic to $\mathbb{P}^1$. Its proper transform in X_{Σ_4} must also be isomorphic to $\mathbb{P}^1$, therefore we get X_{Σ_5} from X_{Σ_4} by blowing it up along some curve isomorphic to $\mathbb{P}^1$.

Getting back to the global picture, let's consider the following two series of blow-ups, both starting from $\mathbb{P}^3$:

$$
\begin{array}{ccc}
\mathrm{Bl}_\eta \mathrm{Bl}_R \mathrm{Bl}_C \mathbb{P}^3 = X^{\#} & \qquad & \mathrm{Bl}_\nu \mathrm{Bl}_{\widetilde{C}} \mathrm{Bl}_Q \mathbb{P}^3 = Z^{\#} \\
\downarrow \pi_3 & & \downarrow \rho_3 \\
\mathrm{Bl}_R \mathrm{Bl}_C \mathbb{P}^3 = \widetilde{X} & & \mathrm{Bl}_{\widetilde{C}} \mathrm{Bl}_Q \mathbb{P}^3 = \widetilde{Z} \\
\downarrow \pi_2 & & \downarrow \rho_2 \\
\mathrm{Bl}_C \mathbb{P}^3 & & \mathrm{Bl}_Q \mathbb{P}^3 \\
\downarrow \pi_1 & & \downarrow \rho_1 \\
\mathbb{P}^3 & & \mathbb{P}^3,
\end{array}
\qquad \pi = \pi_1 \circ \pi_2 \circ \pi_3, \quad \rho = \rho_1 \circ \rho_2 \circ \rho_3
\tag{2.4.26}
$$

where R is the node of $\mathrm{Bl}_C \mathbb{P}^3$, η is the proper transform along π_2 of the preimage along π_1 of $Q \in C$, $\widetilde{C} = NC = \mathrm{Bl}_Q C$ is the proper transform of C along ρ_1 and ν is the proper transform along ρ_2 of the curve that consists of all the directions from Q that are in the tangent plane of C at Q (since C has a node at Q, it has a tangent plane and the directions in that tangent plane at Q give a curve in the exceptional set of $\mathrm{Bl}_Q \mathbb{P}^3$). The curves η and ν are isomorphic to $\mathbb{P}^1$.

There is an isomorphism between $X^{\#} \setminus \pi^{-1}(C)$ and $Z^{\#} \setminus \rho^{-1}(C)$, since both are isomorphic to $\mathbb{P}^3 \setminus C$. These are isomorphisms between open dense sets. In fact more is true:

$$X^{\#} \setminus \pi^{-1}(Q) \cong Z^{\#} \setminus \rho^{-1}(Q) \cong \mathrm{Bl}_{C \setminus \{Q\}} \left(\mathbb{P}^3 \setminus \{Q\} \right), \tag{2.4.27}$$

since the blow-ups of (2.4.26) only differ at Q and its preimages. To see that this isomorphism extends to an isomorphism $X^{\#} \cong Z^{\#}$ we use our result about toric varieties. This gives that

$$\pi^{-1}(U) \cong X_{\Sigma_5} \cong \rho^{-1}(U), \tag{2.4.28}$$

where $U \subset \mathbb{P}^3$ was defined above as some small open neighborhood of Q. The isomorphism $\pi^{-1}(U) \cong \rho^{-1}(U)$ is compatible with $X^{\#} \setminus \pi^{-1}(Q) \cong Z^{\#} \setminus \rho^{-1}(Q)$, therefore $X^{\#} \cong Z^{\#}$. This proves the following

Theorem 2.4.29. Let $X \subset \mathbb{P}^4$ be a cubic threefold with a single A_3 singularity, and let $\widetilde{X}$ be its desingularization which we get by blowing up the singular point twice. Then $\widetilde{X}$ can be obtained from a series of blow-ups and blow-downs as follows: blow up $\mathbb{P}^3$ at the node of the $(2,3)$-curve C, blow up the result along the normalization of C, blow up the result along the curve $\nu \cong \mathbb{P}^1$ and finally blow down the result along the curve $\eta \cong \mathbb{P}^1$.

Corollary 2.4.30. The intermediate Jacobian of the desingularization of X is the Jacobian of the normalization of the $(2,3)$-curve C, i.e.

$$IJ(\widetilde{X}) \cong J(NC). \tag{2.4.31}$$

Proof. Using the theorems of §2.2 and the process in §2.3 we see that blowing up at a point or along a curve isomorphic to $\mathbb{P}^1$ does not change the third cohomology. On the other hand blowing up along NC adds $H^1(NC,\mathbb{C})$ to $H^3(\cdot,\mathbb{C})$ and adds $H^{0,1}(NC)$ to $H^{1,2}(\cdot)$. □

Chapter 3

Degenerations of Intermediate Jacobians of Cubic Threefolds

3.1 Cubic Surfaces

In this section X is a surface in $\mathbb{P}^3$ given by a single homogeneous degree three polynomial. It will frequently be the case that we will want to cut a cubic threefold with a hyperplane in $\mathbb{P}^4$. The result will be a cubic surface, and we will be interested in lines on these possibly singular cubic surfaces. We give a short summary of the description of lines on cubic surfaces with some basic isolated singularities. The source of all these classical results is Cayley's famous Memoir on Cubic Surfaces ([Cy869]). In this section we use Cayley's original notation.

3.1.1 Non-singular cubic surfaces

If X has no singularities, it contains exactly 27 lines, each with multiplicity one. There are 45 planes in $\mathbb{P}^3$ that intersect X in a union of lines. The intersection in all 45 cases is a union of three distinct lines. Each of the 27 lines appears in 5 different planes as part of the intersection. The incidence relations between the lines and the planes are the following. There are six disjoint lines labelled from 1 to 6. Another set of six disjoint lines are labelled from $1'$ to $6'$. Line i intersects line j' if and only if $i \neq j$. There are 15 lines labelled ij with $1 \leq i < j \leq 6$, i.e. lines $12, 13, \ldots, 16, 23, \ldots, 56$. Line ij intersects line k if and only if $k = i$ or $k = j$ and similarly line ij intersects line l' if and only if $l = i$ or $l = j$. Finally, line ij intersects line kl if and only if i, j, k, l are distinct indexes. The lines i and j' determine the plane ij', where the third line of intersection is ij if $i < j$ and ji

otherwise. There are $6 \times 5 = 30$ such planes. The lines ij, kl, mn also determine a plane if all these indexes are distinct, that gives the remaining 15 planes.

3.1.2 A_1 cubic surfaces

In this case there are six lines labelled from 1 to 6 that go through the singular point and lie on a quadric cone which is the tangent cone to X at the node. These lines have multiplicity two. There are 15 lines with multiplicity one, not containing the node. These are labelled ij, with $1 \leq i < j \leq 6$. For $i < j$ the lines i, j and ij make up the plane ij. These planes have multiplicity two. The lines ij, kl, mn lie in a plane if and only if these indexes are distinct. These 15 planes have multiplicity one.

3.1.3 A_2 cubic surfaces

Now the tangent cone at the singular point is a union of two planes which are labelled 123 and 456. Each of these planes has multiplicity six. In plane 123 the three lines labelled 1, 2 and 3 intersect at the singular point, and similarly in plane 456 the lines 4, 5 and 6 intersect there. Thus the lines going through the singular point are 1 through 6, and each of these lines has multiplicity three. The remaining lines have multiplicity one and they are labelled ij, with $i \in \{1,2,3\},\ j \in \{4,5,6\}$. The planes ij with $i \in \{1,2,3\},\ j \in \{4,5,6\}$ contain the lines i, j and ij, each having multiplicity three. There are six more planes, each with multiplicity one, made up of the lines ij, kl, mn with distinct indexes.

3.1.4 A_3 cubic surfaces

The tangent cone at the singularity is the union of two planes, and the intersection line of these planes is a line of X which is labelled line 3 and called the edge. It has multiplicity six. One of the biplanes is labelled 12 and it contains the additional lines 1 and 2, also called rays, the other plane is labelled $1'2'$ and contains the additional lines $1'$ and $2'$. The rays $1, 2, 1', 2'$ each have multiplicity four, while the biplanes 12 and $1'2'$ have multiplicity 12. The above are the five lines that contain the singular point. Next, a transversal line labelled 4 intersects the edge 3. It has multiplicity one. The plane 0 contains the lines 3 and 4, line 3 having multiplicity two in this plane. The plane 0 has multiplicity three. There are four more line, each with multiplicity one, labelled ij' with $i, j \in \{1, 2\}$. The plane that is spanned by the lines i and j' is labelled ij' and the third line of intersection in this plane is naturally ij'. These four planes each have multiplicity four. There are two more planes remaining: $\overline{11'.22'}$ containing the lines $11'$, $22'$ and 4, and the plane $\overline{12'.21'}$ containing the lines $12'$, $21'$ and 4. These planes have multiplicity one.

3.1.5 $4A_1$ cubic surfaces

The four nodes of X are labelled from 1 to 4. The six lines 12, 13, 14, 23, 24, 34 called axes, each connect two of the nodes. The axes have multiplicity four. The planes 1, 2, 3, 4 each contain three nodes and the three axes among them. For example plane 1 contains the nodes opposite to node 1, i.e. the nodes 2, 3, 4 and the axes 23, 24, 34. The multiplicity of these planes is eight. There are three transversal lines 12.34, 13.24, 14.23, each intersecting two of the axes and not touching the nodes. They have multiplicity one. The planes 12, 13, 14, 23, 24, 34 each contain the corresponding axis with multiplicity two and one of the transversals with multiplicity one. For example the plane 12 contains the lines 12 and 12.34. The multiplicity of these planes is two. There is one more plane labelled 1234 having multiplicity one and containing the three transversals.

3.1.6 $2A_2$ cubic surfaces

There is a line called the axis joining the two singular points. It is labelled 0, and has multiplicity nine. Three additional lines, labelled 1, 2, 3, go through the first singular point only, and similarly the lines labelled 4, 5, 6 go through the second singular point only. These six lines are called rays and they have multiplicity three. These are all the lines in X. The plane 0 is the common biplane of the two singular points, and it contains the line 0 only with multiplicity three. The plane 0 has multiplicity six. The plane labelled 7 is the other biplane of the first singular point, and it contains the lines 1, 2 and 3. Similarly, plane 8 is the other biplane of the second singular point and it contains the lines 4, 5 and 6. The planes 7 and 8 have multiplicity six. Finally, the plane 14 contains the lines 1, 4 and 0, the plane 25 contains the lines 2, 5 and 0 and the plane 36 contains the lines 3, 6 and 0. These planes have multiplicity nine.

3.2 Plane Quintic and the $(2,3)$-curve

Let X be a cubic threefold in $\mathbb{P}^4$ which has at least one singularity of type A_n or D_4 and contains isolated singularities only. Let the point Q be a distinguished singularity of type A_n or D_4 on X. Then the projection ρ to $\mathbb{P}^3$ through the point Q defines a birational map from the blow-up of X at Q onto $\mathbb{P}^3$. This birational map is in fact the blow-up of $\mathbb{P}^3$ along the curve C, introduced in §2.1. The curve C is a complete intersection of type $(2,3)$ and is the image along the projection ρ of all the lines that go through Q and are contained in X. By theorem 2.1.18 the singularities of C are in bijection with the singularities of $\operatorname{Bl}_Q X$ and this bijection preserves singularity type.

Next, let's fix some line L contained in X that does not go through Q or any other singularity and is not contained in any plane that may be contained in X. To be more precise, we need L to be non-special in the sense of [CML09, §3.2]. The two dimensional subspaces $\mathbb{P}^2$ of $\mathbb{P}^4$ containing the line L can be parameterized by the projective plane $\mathbb{P}^2$. Let's denote this parameter space by Π. Then a point $V \in \Pi$ represents a 2-space in $\mathbb{P}^4$ containing L. The intersection $X \cap V$ has to be a degree three curve, since X is a cubic hypersurface. This intersection already contains the line L, therefore other than L it has to be a degree two curve, i.e. either a conic curve or two lines. Let's denote by D the set of points $V \in \Pi$ that represent 2-spaces in $\mathbb{P}^4$ such that the intersection $X \cap V$ consists of L and two other lines. We know that D is a quintic curve in $\Pi = \mathbb{P}^2$, and the singularities of D are in bijection with the singularities of X and this bijection respects singularity type (see [CML09, Prop. 3.6 and its proof]). Let's denote by R the point of Π that corresponds to the 2-space that is the span of the line L and the singular point Q. Then $R \in D$ and R has the same singularity type in D as Q in X, i.e. it is either a double point of D of type A_n or a triple point of type D_4.

We can construct a double cover of the curve D in the following way. Let $\widetilde{D}$ be the curve in the Fano scheme of the lines of X whose points represent lines of X intersecting L (but $L \notin \widetilde{D}$). Then a point $L_1 \in \widetilde{D}$ is a line of X that spans a 2-space V together with L, since L and L_1 intersect. As both L and L_1 are in the intersection $X \cap V$, this intersection must be three lines (L, L_1 and some

L_2), therefore $V \in D$. Thus we can define a map $\pi : \widetilde{D} \to D$ sending the line L_1 to the 2-space V. This is a double cover, since the line L_2 is also mapped to V and L_1 and L_2 are the only lines that are mapped to V. By [CML09, Prop. 3.6] π is étale, i.e. L_1 and L_2 are always different. We can normalize both curves D and $\widetilde{D}$ and obtain the commutative diagram

$$\begin{array}{ccc} N\widetilde{D} & \xrightarrow{\hat{\pi}} & ND \\ \downarrow & & \downarrow \\ \widetilde{D} & \xrightarrow{\pi} & D, \end{array} \tag{3.2.1}$$

where $\hat{\pi}$ is an étale double cover of smooth, possibly disconnected curves and the vertical arrows are the desingularizations. In the remaining part of this section we assume that D is irreducible, which also implies that ND is connected.

3.2.1 Trigonal construction. The A_n case.

In this subsection we assume that Q is an A_n singularity of the cubic threefold X.

Proposition 3.2.2. For a cubic threefold X with an A_n singularity (and possibly other isolated singularities), and an irreducible discriminant curve D, the curve ND is trigonal, i.e. there exists a degree three map $ND \to \mathbb{P}^1$.

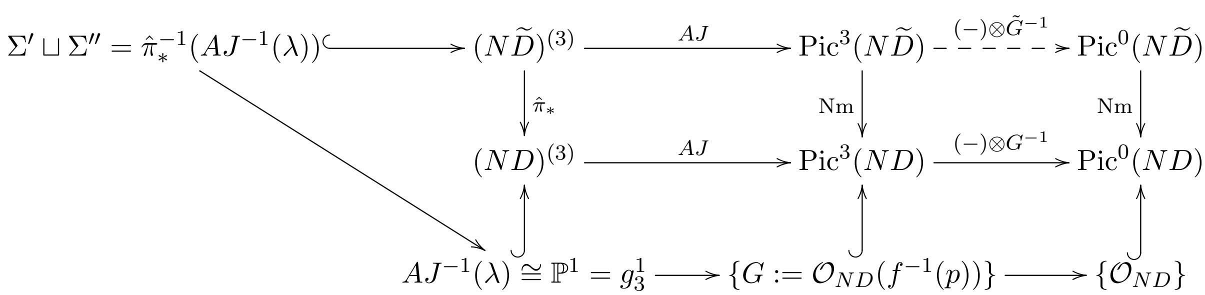
Proof. Recall that R is the point in $\Pi = \mathbb{P}^2$ that represents the plane spanned by the line L and the point Q. Thus R is a double point of D of type A_n. Projection from R gives a rational map $D \to \mathbb{P}^1$. Since R is a double point, the map is generically $3 : 1$. The normalization map is birational, so the induced map $ND \to \mathbb{P}^1$ is degree three. □

Recillas' theorem ([Rec74], [BL04, §12.7]) says that the Prym variety belonging to an étale double cover of a trigonal, non-hyperelliptic curve is the Jacobian of a tetragonal curve. Tetragonal means that the curve has a g^1_4, or equivalently, that it has a degree four map onto $\mathbb{P}^1$. We can construct this tetragonal curve the following way. Using the above notation consider the composition

$$N\widetilde{D} \xrightarrow{2:1} ND \xrightarrow{3:1} \mathbb{P}^1. \tag{3.2.3}$$

The preimage of a general point in $\mathbb{P}^1$ is then six points in $N\widetilde{D}$ consisting of three pairs of points where each point in a pair is mapped to the same point in ND. Say, if $p \in \mathbb{P}^1$, above p we have the points $p_1, p_2, p_3 \in ND$ and above p_i we have the points $p_i^0, p_i^1 \in N\widetilde{D}$, $1 \leq i \leq 3$. A point of the tetragonal curve will be then a triple of points of $N\widetilde{D}$ (thus a point of the symmetric product $(N\widetilde{D})^{(3)}$), such that the three points of the triple are mapped to different points of ND, but are mapped to the same point of $\mathbb{P}^1$ (except above branch points of $ND \to \mathbb{P}^1$). In other words, from each of the three pairs of points in $N\widetilde{D}$ above a single point of $\mathbb{P}^1$ we pick one point to get a triple of points of $N\widetilde{D}$. In the above example a triple $(p_1^{i_1}, p_2^{i_2}, p_3^{i_3})$ would be such a point. There are $2^3 = 8$ ways to do this, thus we get a degree eight cover of $\mathbb{P}^1$. It turns out that the curve that we get will have two identical components. Picking one of the components gives us the desired tetragonal curve corresponding to the double cover $\widetilde{D} \to ND$. Alternatively, we can identify complementary triples, i.e. identify $(p_1^{i_1}, p_2^{i_2}, p_3^{i_3})$ with $(p_1^{1-i_1}, p_2^{1-i_2}, p_3^{1-i_3})$. According to Recillas, the Jacobian of this tetragonal curve is the same as the Prym-variety of the double cover of the trigonal curve.

Remark 3.2.4. Beauville [Bea82] gives the following diagram to describe the above situation:



Here $\Sigma' = \Sigma''$ is the tetragonal curve, AJ is the Abel-Jacobi map, $f : ND \xrightarrow{3:1} \mathbb{P}^1$, Nm is the norm map induced by $\hat{\pi} : N\widetilde{D} \to ND$, $p \in \mathbb{P}^1$ is arbitrary and $\lambda \subset \mathrm{Pic}^3(ND)$ represents the g_3^1 of ND.

The following theorem generalizes a result of Collino–Murre ([CM78, Thm. 4.22]) in the case of A_1 singularities, and [CML09, Thm. 4.1] in the case of A_2 singularities. The argument we give here is inspired by that of [CMGHL15, Prop. 6.7], which gives another proof of the Collino–Murre result for cubics with an A_1 singularity. The theorem below answers affirmatively the question posed in [CML09, Rem. 4.2].

Theorem 3.2.5. For a cubic threefold X having an A_n singularity (and possibly other isolated singularities), an irreducible discriminant D, and a non-hyperelliptic ND, the tetragonal curve constructed above is isomorphic to the normalization NC of the $(2,3)$-curve C, where C is obtained from projection from an A_n singularity.

Remark 3.2.6. By [CML09, Cor. 3.7 and its proof] for an X with only one singular point, which is of type $A_1, \dots, A_6$, the discriminant D is irreducible and thus ND is connected.

Proof of 3.2.5. We only need to define a birational map from C to the tetragonal curve. That will also define a birational map from the normalization NC to the tetragonal curve, but then a birational map between two compact smooth curves can be extended to an isomorphism.

Let's take therefore a general point $\ell \in C$ which represents a line in X through the distinguished A_n singularity Q. Since ℓ is a general point, we may assume that it is non-singular in C and that it does not intersect L in X. Because of this, L and ℓ span some 3-space W in $\mathbb{P}^4$. The set of 2-spaces contained in W and containing L defines a line in Π, which we denote by w. Since ℓ is a general point of C and all singularities of X are isolated, we may assume that W does not contain any singularity of X other than Q.

Let's consider the intersection $Y := X \cap W$. Then Y is a cubic surface and since ℓ is a general point we can assume that the only singularity of Y is at Q (i.e. X and W are not tangent to each other). We now show that for a general W (therefore a general $\ell \in C$) the singularity Q of Y is of type A_1. Let

$$x^{n+1} + y^2 + z^2 + v^2 = 0 \tag{3.2.7}$$

be the equation of a threefold in $\mathbb{C}^4$ with an A_n singularity in the origin. Let a general hyperplane through the origin be given by the equation

$$v = ax + by + cz. \tag{3.2.8}$$

Then the equation of the intersection is given by

$$x^{n+1} + y^2 + z^2 + a^2x^2 + b^2y^2 + c^2z^2 + 2abxy + 2acxz + 2bcyz = 0. \tag{3.2.9}$$

We use the method described in [All03, §2] (in particular Theorem 2.1 of [All03], attributed to Bruce and Wall) to find the singularity type at the origin. Applying the weights $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ to the variables (x, y, z), we see that for general (a, b, c) there are no terms with weight less than 1 and the terms of weight 1 (all terms except the first) determine an isolated singularity. Thus we have an A_1 singularity in the origin. This means that if we intersect a threefold at an A_n singularity with a general hyperplane, we get an A_1 singularity. Of course, a general W is not necessarily a general hyperplane through Q, since W must contain the fixed line L. However, it is possible to choose L in a way that a general W containing L and Q intersects X in an A_1 cubic surface. Indeed it is enough to find only one pair of (W, L), such that L is non-special, $\langle L, Q\rangle \subset W$ and $X \cap W$ is an A_1 cubic surface, because then fixing L and moving W in some neighborhood will not change the fact that $X \cap W$ is an A_1 cubic surface, since an open set of hyperplanes containing Q intersect X in an A_1 cubic surface, as was shown above. It is for dimension reasons why there must be a pair (W, L) as given above: by [CML09, Lemma 3.9] a general L is non-special, therefore there can only be a one dimensional variety of special lines in X (lines that cannot be picked as L). There is however a three dimensional family of hyperplanes W through Q intersecting X in an A_1 cubic surface, and in each W there are finitely many lines of X not touching Q. These lines make up a two dimensional family, because each line belongs to a one dimensional family of hyperplanes containing the line and Q. Therefore among these lines there has to be a non-special line that we can choose to be L. The above gives that we can fix L such that for a general W (and thus a general $\ell \in C$) $Y = X \cap Q$ is a cubic surface with one A_1 singularity.

Recall from §3.1.2 that a cubic surface with an ordinary double point contains 21 lines with the following incidence relations. The lines $l_1, l_2, \ldots, l_6$ go through the singular point. The plane determined by the lines l_i and l_j cuts out a third line l_{ij}, giving 15 more lines for $1 \leq i < j \leq 6$. Then l_i meets l_{rs} if and only if $i = r$ or $i = s$, and l_{ij} meets l_{rs} if and only if i, j, r, s are all different.

In the cubic surface Y we already have two lines, one is L that does not meet Q, the other is ℓ that does. We can therefore identify L as the line l_{12} of the cubic surface, while ℓ will be identified with l_6. Now the 3-space W determines the line w in the plane Π. The line w intersects the double point R (because $Q \in W$) and in general it intersects the quintic D in three additional points. The double point R and the three additional points represent the planes of W containing L and meeting Y in three lines (and not a line and a conic curve). The point R represents the plane that is the span of Q and $L = l_{12}$. Thus this plane cuts out the lines l_{12}, l_1, l_2 from Y. The three additional intersection points of w and D give three planes in W that cut out three line triples in Y, where in each triple one of the lines is l_{12}. To see which are the other lines in the triples we enumerate the lines in Y that intersect l_{12} other than l_1, l_2. These lines are $l_{34}, l_{35}, l_{36}, l_{45}, l_{46}, l_{56}$. The three pairs that determine three planes are

$$(l_{34}, l_{56}), (l_{35}, l_{46}), (l_{36}, l_{45}), \tag{3.2.10}$$

since l_{ij} intersects l_{rs} if and only if the indexes are distinct. Thus in the double cover $\widetilde{D} \to D$, the line $l_{34} \in \widetilde{D}$ for example is mapped to the point of D that represents the plane spanned by l_{34} and l_{56}, and so is the line l_{56}.

Our original goal in the proof was to find a point of the tetragonal curve that we map the point $\ell \in C$ to. Now to give a point of the tetragonal curve means that we have to pick one line l_{ij} from each of the pairs in (3.2.10). Our choice cannot be arbitrary, because it has to move continuously with the point $\ell \in C$ and we also want to stay on one of the components of the curve obtained in Recillas' construction. Thus from each pair of lines we pick the one that intersects l_6, i.e. l_{56} from the first pair, l_{46} from the second and l_{36} from the third. Thus we have defined a map from an open set of C to the tetragonal curve. It is clear that this map is not constant.

Notice that the lines l_3, l_4 and l_5 of Y are also points of C and they would define the same 3-space W and thus the same line $w \subset \Pi$. They then also give the same three additional intersection points on $w \cap D$ and thus the same three pairs of lines as in (3.2.10). However, they will give different points on the tetragonal curve, because of our convetion of how we pick the three lines from the three pairs of lines. This is the reason why the map defined above is injective and therefore birational. □

3.2.2 Hyperelliptic construction. The D_4 case.

In this subsection we assume that the distinguished singularity Q is a D_4 singularity of the cubic threefold X, and that the discriminant D is irreducible, which also implies that ND is connected.

Proposition 3.2.11. For a cubic surface X with a D_4 singularity (and possibly other isolated singularities), and an irreducible discriminant D, the curve ND is hyperelliptic, i.e. there exists a degree two map $ND \to \mathbb{P}^1$.

Proof. The proof is similar to that of Proposition 3.2.2. The difference is that R is a triple point of D, therefore a general line passing through it has two residual intersections with D. Thus the map $ND \to \mathbb{P}^1$ has degree two. □

Mumford's theorem ([Mum74, §7]) says that the Prym variety of an étale double cover of a hyperelliptic smooth curve is the Jacobian of a hyperelliptic curve or the product of two such. The hyperelliptic construction gives the curve whose Jacobian is the same as the Prym of the double cover, and it goes as follows. Similarly to the trigonal construction, we consider the composition

$$N\widetilde{D} \xrightarrow{2:1} ND \xrightarrow{2:1} \mathbb{P}^1. \tag{3.2.12}$$

Above a general point of $\mathbb{P}^1$ there are two points of ND and four points, or two pairs of points, of $N\widetilde{D}$. From each pair of points we pick one to get a point of the newly created curve. There are four ways to do this, so the new curve has four points above a general point of $\mathbb{P}^1$. The new curve is a subvariety of the symmetric product $(N\widetilde{D})^{(2)}$. The point $(p,q) \in (N\widetilde{D})^{(2)}$ is a point of the new curve if and only if p and q are mapped to different points of ND and the same point of $\mathbb{P}^1$ (except over branch points of $ND \to \mathbb{P}^1$). According to Mumford the newly constructed curve is either hyperelliptic or the disjoint union of two hyperelliptic curves.

Remark 3.2.13. As in the case of the trigonal construction (see remark 3.2.4) we have the following diagram to describe the above situation [Bea82]:

$$\begin{array}{ccccccc}
\Sigma = \hat{\pi}_*^{-1}(AJ^{-1}(\lambda)) & \hookrightarrow & (N\widetilde{D})^{(2)} & \xrightarrow{AJ} & \mathrm{Pic}^2(N\widetilde{D}) & \overset{(-)\otimes\tilde{G}^{-1}}{\dashrightarrow} & \mathrm{Pic}^0(N\widetilde{D}) \\
 & \searrow & \downarrow \hat{\pi}_* & & \downarrow \mathrm{Nm} & & \downarrow \mathrm{Nm} \\
 & & (ND)^{(2)} & \xrightarrow{AJ} & \mathrm{Pic}^2(ND) & \xrightarrow{(-)\otimes G^{-1}} & \mathrm{Pic}^0(ND) \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & AJ^{-1}(\lambda) \cong \mathbb{P}^1 = g^1_2 & \longrightarrow & \{G := \mathcal{O}_{ND}(f^{-1}(p))\} & \longrightarrow & \{\mathcal{O}_{ND}\}
\end{array}$$

Here Σ is the newly constructed curve, AJ is the Abel-Jacobi map, $f : ND \xrightarrow{2:1} \mathbb{P}^1$, Nm is the norm map induced by $\hat{\pi} : N\widetilde{D} \to ND$, $p \in \mathbb{P}^1$ is arbitrary and $\lambda \subset \mathrm{Pic}^2(ND)$ represents the g^1_2 of ND.

The following is the D_4-version of theorem 3.2.5:

Theorem 3.2.14. For a cubic threefold X with a D_4 singularity (and possibly other isolated singularities), and an irreducible discriminant D, the curve constructed above is isomorphic to the normalization NC of the $(2,3)$-curve C, where C is obtained from projection from a D_4 singularity.

Remark 3.2.15. By [CML09, Cor. 3.7] for an X with only one singular point which is of type D_4, the discriminant D is irreducible and thus ND is connected.

Proof of 3.2.14. The proof follows the ideas of the proof of Theorem 3.2.5. Again, it is enough to construct a birational map between the curve C and the curve arising from the hyperelliptic construction. If we pick a general point of C, that gives us a line ℓ in X passing through Q, a three-space W (the span of L and ℓ), and a line w in Π passing through R. If we intersect a threefold at a D_4 singularity with a general hyperplane, we get an A_2 singularity in the intersection. To see this let's take the standard form of a D_4 singularity at the origin of $\mathbb{C}^4$:

$$x^3 + xy^2 + z^2 + v^2 = 0. \tag{3.2.16}$$

The equation of a general hyperplane is given by

$$y = ax + bz + cv. \tag{3.2.17}$$

Then the intersection becomes

$$x^3 + a^2x^3 + b^2xz^2 + c^2xv^2 + 2abx^2z + 2acx^2v + 2bcxzv + z^2 + v^2 = 0. \tag{3.2.18}$$

Applying the weights $\left(\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right)$, we see that all terms have weight ≥ 1 and for general a the terms having weight 1 (i.e. $x^3 + a^2x^3 + z^2 + v^2$) define an isolated singularity. Therefore the singularity has type A_2. As in the proof of Theorem 3.2.5, we can find a non special line L, such that for a general W (and thus a general $\ell \in C$) $Y := X \cap W$ is a cubic surface with a single A_2 singularity.

Recall from §3.1.3 that a cubic surface with an A_2 singularity has 15 lines, six of which pass through the singular point. These are labelled as $l_1, l_2, \ldots, l_6$. There are nine lines labelled l_{ij}, with $i \in \{1, 2, 3\}$, $j \in \{4, 5, 6\}$. The lines l_1, l_2, l_3 are coplanar meeting at the singular point, and similarly the lines l_4, l_5, l_6 are coplanar meeting at the singular point. The lines l_i, l_j, l_{ij} are coplanar for $i \in \{1, 2, 3\}$, $j \in \{4, 5, 6\}$. Finally the lines l_{ij}, l_{kl}, l_{mn} are coplanar if and only if i, j, k, l, m, n are distinct.

We can identify the line ℓ with l_6 of Y and the non-special line L with l_{14}. Then the plane spanned by L and the point Q intersects X in the lines $L = l_{14}, l_1, l_4$. The line $w \subset \Pi$ intersects D in two additional points besides R. These intersection points correspond to two planes in W containing L but not Q and intersecting X in three lines. These two planes intersect X in the lines $L = l_{14}, l_{25}, l_{36}$ and $L = l_{14}, l_{26}, l_{35}$, respectively. In the map $\widetilde{D} \to D \to \mathbb{P}^1$ the line w represents a point ξ in $\mathbb{P}^1$, the two intersection points (besides R) η_1 and η_2 of w with D are the points above ξ, while the lines l_{25}, l_{36} represent points of $\widetilde{D}$ above η_1 and the lines l_{26}, l_{35} represent points of $\widetilde{D}$ above η_2. To pick a point of the hyperelliptic construction we pick l_{36} from the pair of points above η_1 and we pick l_{26} from the pair of points above η_2, that is we pick the lines that intersect $\ell = l_6$. Notice that besides l_6 there are three other lines in Y that come from different points of C, namely the lines l_2, l_3, l_5. These lines determine the same line $w \subset \Pi$, and the same intersection points η_1, η_2, however they pick different lines from the two pairs of lines l_{25}, l_{36} and l_{26}, l_{35}. That is why the rational map we created from C to the hyperelliptic construction is injective. Also, it is clearly non-constant, therefore it is birational. □

3.3 Generalized Intermediate Jacobians of Singular Cubic Threefolds

In §2.3 and §2.4 we desingularized a cubic threefold by successive blow-ups at singular points, before we could talk about its intermediate Jacobian. In this section we consider degenerate intermediate Jacobians of cubic threefolds with certain types of isolated singularities. Degenerate intermediate Jacobians are limit intermediate Jacobians of 1-parameter degenerations $\mathcal{X} \to \Delta$ of cubic threefolds, where Δ is the unit disk in $\mathbb{C}$, and the fibers are smooth cubic threefolds on $\Delta \setminus \{0\}$. Any such degeneration can be filled in with a semi-stable cubic threefold above $0 \in \Delta$, which means that we only have to consider cubic threefolds with at worst A_k, $k \leq 5$ or D_4 singularities or the chordal cubic. The latter is not treated in this paper, although we recall from [Co82] that degenerations to the chordal cubic give rise to genus five hyperelliptic Jacobians. For a computation on the stability of the cubic threefold we refer to [All03], for a brief summary see [CMGHL15, Thm. 1.1]. By a result due to Mumford, the intermediate Jacobian for a smooth cubic threefold is isomorphic to the Prym variety of the connected étale double cover of the plane quintic introduced above. Thus taking limits of intermediate Jacobians of 1-parameter degenerations is equivalent to taking degenerations of Prym varieties along 1-parameter families of unramified double covers of plane quintics ([CML09, §5.1]).

If the double cover $\widetilde{D} \to D$ at the center of a 1-parameter degeneration of covers has worse than node singularities, we can perform a stable reduction to replace the fiber at the center with a double cover that only has nodes. In this case the new 1-parameter family above $\Delta \setminus \{0\}$ only differs from the previous one by a finite base change, so the limit Prym variety is the same. To perform a stable reduction on an A_{2k+1}, $k \geq 1$ singularity $p \in D$, we blow up D at p k-times to normalize it and attach a hyperelliptic, genus-k tail T at the two points of $\mathrm{Bl}_p^{(k)} D$ that are above p. Thus we replace an A_{2k+1} singularity with two nodes, the attachment points of T and $\mathrm{Bl}_p^{(k)} D$. In case of an A_{2k}, $k \geq 1$ singularity, similarly we have to blow D up k-times and add a hyperelliptic, genus-k tail, but since there is only one point in $\mathrm{Bl}_p^{(k)} D$ above p, we attach the blow-up to T at only one point, which then becomes a node of the stable reduction. In case of a D_4 singularity, since in two dimensions such a point is a triple point, the elliptic tail T will be attached at the three points of $\mathrm{Bl}_p D$ that are above p. After dealing with the singularities, if there are genus zero irreducible components with less than three marked points (connections with other components), we have to collapse such components to a point. As we perform the stable reduction on D we have to do the same on the double cover $\widetilde{D}$ by adding the same hyperelliptic tails twice, since each singularity $p \in D$ has two preimages along $\widetilde{D} \to D$. The properties of the marked points used to attach the tail to the rest of the curve are summarized in [CMGHL15, p. 25], and for more details see [Has00].

The Jacobian of a curve C with nodes can be defined as the group of line bundles on C with multidegree 0, meaning the degree must be 0 on all irreducible components. Equivalently, it can be defined as the limit of Jacobians of 1-parameter degenerations of smooth curves. Such a Jacobian is a semiabelian variety which can be given by a short exact sequence

$$1 \to H^1(\Gamma, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C}^* \to JC \to J(NC) \to 0, \tag{3.3.1}$$

where Γ is the dual graph of C, $\mathbb{C}^*$ is the multiplicative group of $\mathbb{C}$, and NC is the normalization (or desingularization) of C. Thus JC is the extension of the principally polarized abelian variety $J(NC)$ (the compact part) by some torus $(\mathbb{C}^*)^n$ (the non-compact part). We can thus give a Jacobian by specifying the compact and non-compact parts, as well as an element of the extension group $\mathrm{Ext}\big(J(NC), H^1(\Gamma,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{C}^*\big)$. The latter is equivalent to giving a homomorphism $H_1(\Gamma, \mathbb{Z}) \to {}^tJ(NC)$, where ${}^tJ(NC)$ is the dual of the compact part, and $H_1(\Gamma, \mathbb{Z})$ is the character group or dual of the non-compact part. This map is defined as follows: we identify ${}^tJ(NC)$ with $J(NC)$ through the principal polarization, then any edge e in a cycle is mapped to $t(e) - s(e)$, where $t(e)$ and $s(e)$ are points that are the results of the desingularization of the node represented by e, $t(e)$ is located on the target component of e and $s(e)$ is located on the source component of e (see [ABH02, §1.1]).

Now let $\widetilde{D} \to D$ be an étale double cover of nodal curves. We can define its Prym variety by taking limits of Prym varieties of 1-parameter degenerations of étale double covers of smooth curves, or equivalently by setting

$$P_{\widetilde{D}/D} = \mathrm{Ker}(1+\iota)_0 = \mathrm{Im}(1-\iota), \tag{3.3.2}$$

where $\iota : J\widetilde{D} \to J\widetilde{D}$ is induced by the fixed point free involution $\iota : \widetilde{D} \to \widetilde{D}$, which is induced by the étale double cover $\widetilde{D} \to D$. Such a Prym variety is a semiabelian variety which can be given by a short exact sequence

$$1 \to H^1(\widetilde{\Gamma}, \mathbb{Z})^- \otimes_{\mathbb{Z}} \mathbb{C}^* \to P_{\widetilde{D}/D} \to P_{N\widetilde{D}/ND}/G \to 0, \tag{3.3.3}$$

where $\widetilde{\Gamma}$ is the dual graph of $\widetilde{D}$, $H^1(\widetilde{\Gamma}, \mathbb{Z})^-$ is the -1-eigenspace of $H^1(\widetilde{\Gamma}, \mathbb{Z})$ with respect to $\iota : \widetilde{\Gamma} \to \widetilde{\Gamma}$, the involution induced by $\iota : \widetilde{D} \to \widetilde{D}$. The curves $N\widetilde{D}$ and ND are normalizations and $G \subset P_{N\widetilde{D}/ND}$ is a finite group that will be described later. The dual or the character group of the non-compact part is

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^{[-]} = H_1(\widetilde{\Gamma}, \mathbb{Z})/H_1(\widetilde{\Gamma}, \mathbb{Z})^+, \tag{3.3.4}$$

where $H_1(\widetilde{\Gamma}, \mathbb{Z})^+$ is the $+1$-eigenspace of $H_1(\widetilde{\Gamma}, \mathbb{Z})$. Equivalently $H_1(\widetilde{\Gamma}, \mathbb{Z})^{[-]}$ can also be defined as the image of the map

$$\pi^- : H_1(\widetilde{\Gamma}, \mathbb{Z}) \longrightarrow H_1(\widetilde{\Gamma}, \tfrac{1}{2}\mathbb{Z}) \tag{3.3.5}$$

$$h \longmapsto \tfrac{1}{2}(h - \iota(h)), \tag{3.3.6}$$

see [ABH02, p. 10]. It is possible to choose a basis $\langle u_1, \dots, u_k, v_1, \dots, v_l, w_1, \dots, w_{2m} \rangle$ of $H_1(\widetilde{\Gamma}, \mathbb{Z})$ such that $\iota : H_1(\widetilde{\Gamma}, \mathbb{Z}) \to H_1(\widetilde{\Gamma}, \mathbb{Z})$ is the identity on $\langle u_i \rangle_{i=1}^k$, is multiplication by -1 on $\langle v_i \rangle_{i=1}^l$ and interchanges w_{2i-1} and w_{2i} for $1 \leq i \leq m$. Then we have

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^+ = \langle u_1, \ldots, u_k, w_1 + w_2, \ldots, w_{2m-1} + w_{2m} \rangle \tag{3.3.7}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^- = \langle v_1, \ldots, v_l, w_1 - w_2, \ldots, w_{2m-1} - w_{2m} \rangle \tag{3.3.8}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^{[-]} = \operatorname{Im}(\pi^-) = \langle v_1, \ldots, v_l, \tfrac{1}{2}(w_1 - w_2), \ldots, \tfrac{1}{2}(w_{2m-1} - w_{2m}) \rangle \tag{3.3.9}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^{[-]} / H_1(\widetilde{\Gamma}, \mathbb{Z})^- \cong (\mathbb{Z}/2\mathbb{Z})^m \tag{3.3.10}$$

$$(1+\iota) H_1(\widetilde{\Gamma}, \mathbb{Z}) = \langle 2u_1, \ldots, 2u_k, w_1 + w_2, \ldots, w_{2m-1} + w_{2m} \rangle \tag{3.3.11}$$

$$\begin{aligned} &H_1(\widetilde{\Gamma}, \mathbb{Z})^{[-]} = H_1(\widetilde{\Gamma}, \mathbb{Z}) / H_1(\widetilde{\Gamma}, \mathbb{Z})^+ = \\ &\quad \Big(H_1(\widetilde{\Gamma}, \mathbb{Z}) / (1+\iota) H_1(\widetilde{\Gamma}, \mathbb{Z})\Big) \Big/ \Big(H_1(\widetilde{\Gamma}, \mathbb{Z})^+ / (1+\iota) H_1(\widetilde{\Gamma}, \mathbb{Z})\Big) = \\ &\quad \Big(H_1(\widetilde{\Gamma}, \mathbb{Z}) / (1+\iota) H_1(\widetilde{\Gamma}, \mathbb{Z})\Big) \Big/ \big(\text{Torsion}\big). \end{aligned} \tag{3.3.12}$$

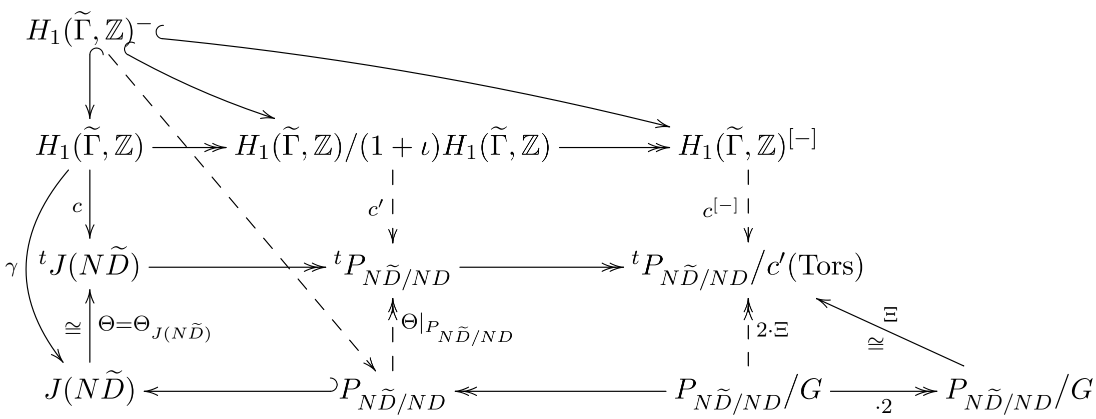
By part 1 of [ABH02, Lemma 1.4] the extension homomorphism of (3.3.3) can be defined by the last column of the diagram:

$$\begin{array}{ccccccc} H_1(\widetilde{\Gamma}, \mathbb{Z}) & \xrightarrow{1+\iota} & H_1(\widetilde{\Gamma}, \mathbb{Z}) & \longrightarrow & H_1(\widetilde{\Gamma}, \mathbb{Z}) / (1+\iota) H_1(\widetilde{\Gamma}, \mathbb{Z}) & \longrightarrow & H_1(\widetilde{\Gamma}, \mathbb{Z})^{[-]} \\ \downarrow c & & \downarrow c & & \downarrow c' & & \downarrow c^{[-]} \\ {}^tJ(N\widetilde{D}) & \xrightarrow{1+\iota} & {}^tJ(N\widetilde{D}) & \longrightarrow & {}^tJ(N\widetilde{D}) / (1+\iota)\, {}^tJ(N\widetilde{D}) & \longrightarrow & K \end{array} \tag{3.3.13}$$

where

$$K = \frac{{}^tJ(N\widetilde{D}) / (1+\iota)\, {}^tJ(N\widetilde{D})}{c'\Big(\text{Torsion}\big(H_1(\widetilde{\Gamma}, \mathbb{Z}) / (1+\iota) H_1(\widetilde{\Gamma}, \mathbb{Z})\big)\Big)}, \tag{3.3.14}$$

c is the extension homomorphism of $J\widetilde{D}$, c' and $c^{[-]}$ are induced from c and the rightmost horizontal map of the first row is factoring by the torsion group. To get the element in the upper right corner we used (3.3.12). If we dualize the second row and add the dualization as an extra row, delete the first column, then using [ABH02, Prop. 1.5] we get the diagram



(3.3.15)

Here γ is the extension map of $J\widetilde{D}$ giving $\gamma(e) = t(e) - s(e)$ for a cycle e, as was defined earlier. The maps between the last two rows come from the polarizations, Θ being the principal polarization of $J(N\widetilde{D})$, and Ξ being the principal polarization of $P_{N\widetilde{D}/ND}$. By a well known property of Prym varieties

$$\Theta|_{P_{N\widetilde{D}/ND}} \cong 2 \cdot \Xi. \tag{3.3.16}$$

The composition $H_1(\widetilde{\Gamma},\mathbb{Z}) \to H_1(\widetilde{\Gamma},\mathbb{Z})^{[-]}$ of the second row is in fact π^- of (3.3.5). For the finite group G we have $G \cong (\mathbb{Z}/2\mathbb{Z})^d$, where $0 \le d \le k$, k being the dimension of the subspace of $H_1(\widetilde{\Gamma},\mathbb{Z})$ on which ι is the identity. Then some diagram chasing gives:

Proposition 3.3.17. If $H_1(\widetilde{\Gamma},\mathbb{Z})/(1+\iota)H_1(\widetilde{\Gamma},\mathbb{Z}) = H_1(\widetilde{\Gamma},\mathbb{Z})^{[-]}$, (i.e. $(1+\iota)H_1(\widetilde{\Gamma},\mathbb{Z}) = H_1(\widetilde{\Gamma},\mathbb{Z})^+$), then the Prym variety is an extension

$$1 \to H^1(\widetilde{\Gamma},\mathbb{Z})^- \otimes_{\mathbb{Z}} \mathbb{C}^* \to P_{\widetilde{D}/D} \to P_{N\widetilde{D}/ND} \to 0 \tag{3.3.18}$$

with extension data given by the map

$$\gamma^{[-]} : H_1(\widetilde{\Gamma},\mathbb{Z})^{[-]} \to P_{N\widetilde{D}/ND}, \tag{3.3.19}$$

which is defined as follows. For any $z \in H_1(\widetilde{\Gamma},\mathbb{Z})^{[-]}$, one has $2z \in H_1(\widetilde{\Gamma},\mathbb{Z})^- \subseteq H_1(\widetilde{\Gamma},\mathbb{Z})$, and we set $\gamma^{[-]}(z) := \gamma(2z)$; i.e.,

$$\sum n_i e_i \mapsto \mathcal{O}_{ND}\left(\sum 2n_i(t(e_i) - s(e_i))\right). \tag{3.3.20}$$

Note that here, we may have $n_i \in \frac{1}{2}\mathbb{Z}$, under the natural identification of $H_1(\widetilde{\Gamma}, \mathbb{Z})^{[-]}$ with the image of the map $H_1(\widetilde{\Gamma}, \mathbb{Z}) \to H_1(\widetilde{\Gamma}, \frac{1}{2}\mathbb{Z})$, given by $z \mapsto \frac{1}{2}(z - \iota z)$.

Chapter 4

Geometric Realization of Strata in the Boundary of the Intermediate Jacobian Locus

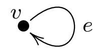
4.1 Irreducible Quintic

In this section X is a cubic threefold with isolated singularities, $\ell \subset X$ is a non-special line (as defined in [CML09, §3.2]), and the discriminant curve D is a plane quintic that we assume to be irreducible. We also assume that the double cover $\widetilde{D}$ is irreducible. Note that for cubic threefolds with a unique singularity of type A_k, $k \leq 6$ or D_4, the discriminant is automatically irreducible [CML09, Cor. 3.7 and its proof].

4.1.1 A_1 cubic threefolds

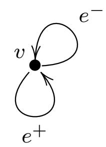
For completeness, here we review the case of cubic threefolds with an A_1 singularity. The result is essentially due to [CM78] and the argument we give here follows [CMGHL15, Thm. 6.4]. We include the proof since it helps to motivate the approach we take in other, more complicated cases.

Let X be a cubic threefold with a unique singular point, which is an A_1 singularity. Let $\widetilde{D} \to D$ be the étale double cover of the plane quintic. Then D has one A_1 singularity p, and recall that D is automatically irreducible. Above p the two nodes of $\widetilde{D}$ are p^+ and p^-. Both D and $\widetilde{D}$ are irreducible. The dual graph Γ of D is therefore



, (4.1.1)

where v represents the only irreducible component of D and e represents the node p. The dual graph $\widetilde{\Gamma}$ of $\widetilde{D}$ is



, (4.1.2)

where v represents the only irreducible component of $\widetilde{D}$ and e^+ and e^- represent the nodes p^+ and p^- respectively. Using results from page 42 we have

$$H_1(\widetilde{\Gamma}, \mathbb{Z}) = \mathbb{Z}\langle e^+, e^-\rangle \tag{4.1.3}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^+ = \mathbb{Z}\langle e^+ + e^-\rangle \tag{4.1.4}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^- = \mathbb{Z}\langle e^+ - e^-\rangle \tag{4.1.5}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^{[-]} = \mathbb{Z}\langle \tfrac{1}{2}(e^+ - e^-)\rangle. \tag{4.1.6}$$

Since $(1+\iota)(e^+) = (1+\iota)(e^-) = e^+ + e^-$, we have $(1+\iota)H_1(\widetilde{\Gamma}, \mathbb{Z}) = H_1(\widetilde{\Gamma}, \mathbb{Z})^+$. Therefore the Prym-variety of $\widetilde{D} \to D$ is given by

$$1 \to H^1(\widetilde{\Gamma}, \mathbb{Z})^- \otimes_{\mathbb{Z}} \mathbb{C}^* \to P_{\widetilde{D}/D} \to P_{N\widetilde{D}/ND} \to 0, \tag{4.1.7}$$

and the intermediate Jacobian is given by

$$1 \to \mathbb{C}^* \to IJ(X) \to P_{N\widetilde{D}/ND} \to 0, \tag{4.1.8}$$

where the extension data is

$$g \mapsto \mathcal{O}_{N\widetilde{D}}(p_1^+ - p_2^+ - p_1^- + p_2^-), \tag{4.1.9}$$

obtained from proposition 3.3.17. Here p_1^+ and p_2^+ are the points on $N\widetilde{D}$ that we receive from blowing up the point p^+ and similarly for the points p_1^-, p_2^-, p^-.

We want to show that ND is non-hyperelliptic. Since D is planar degree five with a node, ND is genus five. Let's consider all conic curves in the plane of D that go through the node of D. Since five points determine a conic and one is fixed at the node, we can choose such a conic by picking four points on the plane. This gives us a choice of dimension eight, however, once we pick the four points and thus the conic, if we move the points around on the conic, the conic will not change. Therefore there is a four dimensional family of conics passing through the node of D. The conic and D must have 10 intersection points, but two of these are at the node. This gives us a base point free g_8^4 on D and ND, which means ND can be embedded into $\mathbb{P}^4$ as a degree 8 curve. Thus it cannot be hyperelliptic by standard algebraic curve theory. Alternatively, since $g(ND) = 5$ and ND has a base point free g_3^1 (which must be complete by Clifford's Theorem), ND is non-hyperelliptic by [ACGH85, p. 13]. This in fact works for all genus at least three.

Since the curve ND is genus five, non-hyperelliptic and trigonal, by Theorem 3.2.5 the Prym-variety $P_{N\widetilde{D}/ND}$ is the Jacobian of the $(2,3)$-curve C, which is smooth (Theorem 2.1.18) and genus four. From the proof of Theorem 2.1.1, C is the intersection of the quadric $Q = 0$ and the cubic $F = 0$, where Q gives the tangent cone to the node of X if considered as a polynomial in the open affine $\mathbb{C}^4$, and gives the projectivization of the tangent cone if considered as a homogeneous polynomial in $\mathbb{P}^3$. For an A_1 singularity the tangent cone is a quadric cone whose projectivization is a smooth quadric. Under the indentification of the Prym of $N\widetilde{D} \to ND$ with the Jacobian of C the extension data is given by the difference of the two g_3^1's on the $(2,3)$-curve (see [CMGHL15, §6.1 and Theorem 6.4]). The two g_3^1's come from the two rulings of the smooth quadric that contains C.

To summarize, the intermediate Jacobian is

$$1 \to \mathbb{C}^* \to IJ(X) \to JC \to 0, \tag{4.1.10}$$

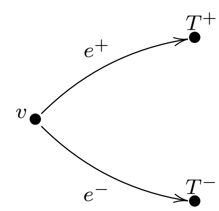
with extension data

$$g \mapsto N \otimes \hat{N} \in JC, \tag{4.1.11}$$

where N and $\hat{N}$ are the two g^1_3's on C. (In Theorem 6.4 of [CMGHL15] we see $(N\otimes\hat{N})^{\otimes\pm1}$, but that is because in fact the extension data should be in $JC/\operatorname{Aut}(JC)$. Now $\operatorname{Aut}(JC)\cong\{\pm1\}$, explaining the ambiguity.)

4.1.2 A_2 cubic threefolds

This case is also covered in [CML09]. Let X be a cubic threefold with a unique singular point, which is an A_2 singularity and let $\widetilde{D}\to D$ be the étale double cover of the plane quintic. Recall that D is automatically irreducible. Then the $(2,3)$-curve is smooth and genus four, while D has an A_2 singularity p. This means we need to perform a stable reduction. If the result is the cover $\widetilde{F}\to F$, then F consists of two irreducible components, one being the normalization ND, the other an elliptic tail T. The two components are attached at a node which on ND rests over the point received from the blow-up of D at p. Then the curve $\widetilde{F}$ consists of three components, two of which are identical to T (T^+ and T^-) and the third is the normalization $N\widetilde{D}$. This gives the following dual graph $\widetilde{\Gamma}$:



(4.1.12)

We can see that $H_1(\widetilde{\Gamma},\mathbb{Z})=H_1(\widetilde{\Gamma},\mathbb{Z})^+=H_1(\widetilde{\Gamma},\mathbb{Z})^-=0$, and thus $IJ(X)$ does not have a non-compact part. Therefore

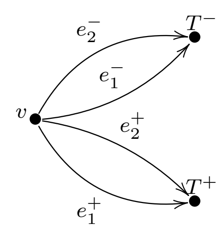
$$IJ(X)=JT\times P_{N\widetilde{D}/ND}=JT\times JC, \tag{4.1.13}$$

where JT is the Jacobian of the elliptic curve T, and JC is the Jacobian of the $(2,3)$-curve C. To see that $JC=P_{N\widetilde{D}/ND}$, we have to show that ND is non-hyperelliptic. This can be done similar to the A_1 case, since ND is genus five, just like it was genus five in the A_1 case.

4.1.3 A_3 cubic threefolds

We now address the case of cubic threefolds with an A_3 singularity. This case is dealt with in [CMGHL15, Thm. 6.3, §6.2]; we follow the proof there, and also provide one strengthening, that the extension data for the degeneration of the intermediate Jacobian, lying in the Jacobian of the normalization of the $(2,3)$-curve, is in fact due to the difference of points lying over the node in the $(2,3)$-curve; this is explained in more detail below.

With the usual notation, D (automatically irreducible) now has an A_3 singularity p and the $(2,3)$-curve C has an A_1 singularity. The stable reduction F consists of an elliptic tail T and the normalization ND which is genus four. They are attached to each other at the two nodes $p_1 = t_1$ and $p_2 = t_2$, where $p_i \in ND$ come from the blow-up of p and $t_i \in T$. Then $\widetilde{F}$ consists of two elliptic curves T^+ and T^- which are isomorphic to T and the normalization $N\widetilde{D}$. The tail T^+ is attached to $N\widetilde{D}$ by the two nodes $p_1^+ = t_1^+$ and $p_2^+ = t_2^+$ and similarly for T^-, p_i^- and t_i^-. The double cover $\widetilde{F} \to F$ then consists of a trivial cover $T^+ \sqcup T^- \to T$ and the cover $N\widetilde{D} \to ND$ which is induced from $\widetilde{D} \to D$. The points $p_i^\pm \in N\widetilde{D}$ are above $p_i \in ND$ and $t_i^\pm \in T^\pm$ are above $t_i \in T$, for $i = 1, 2$. This arrangement gives the dual graph $\widetilde{\Gamma}$:



(4.1.14)

For the cohomology we have

$$H_1(\widetilde{\Gamma}, \mathbb{Z}) = \mathbb{Z}\langle e_1^+ - e_2^+, e_1^- - e_2^- \rangle \tag{4.1.15}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^+ = \mathbb{Z}\langle e_1^+ - e_2^+ + e_1^- - e_2^- \rangle \tag{4.1.16}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^- = \mathbb{Z}\langle e_1^+ - e_2^+ - e_1^- + e_2^- \rangle \tag{4.1.17}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^{[-]} = \mathbb{Z}\langle \tfrac{1}{2}(e_1^+ - e_2^+ - e_1^- + e_2^-) \rangle. \tag{4.1.18}$$

It is straightforward to check that $(1+\iota)H_1(\widetilde{\Gamma}, \mathbb{Z}) = H_1(\widetilde{\Gamma}, \mathbb{Z})^+$. For the intermediate Jacobian we have

$$1 \to \mathbb{C}^* \to IJ(X) \to JT \times P_{N\widetilde{D}/ND} \to 0, \tag{4.1.19}$$

with extension data $g \mapsto \big(\mathcal{O}_T(t_1 - t_2), \mathcal{O}_{N\widetilde{D}}(p_2^+ - p_1^+ - p_2^- + p_1^-)\big)$.

The curve ND is genus four and trigonal. To prove that it is non-hyperelliptic, we proceed like in the A_1 case with some modifications. Let's take the conics in the plane of D that go through the singularity of D and have the same tangent there as the tangent given by the A_3 singularity. That means we can use three points to determine the conic and this gives us $6 - 3 = 3$ degrees of freedom. The intersection multiplicity of the conic and D is four, therefore there has to be six other intersection points. This gives us a base point free g_6^3 and an embedding of ND to $\mathbb{P}^3$ as a degree six curve. Therefore ND is non-hyperelliptic by standard algebraic curve theory. Alternatively, we can use [ACGH85, p. 13]. As ND is non-hyperelliptic the Prym-variety of $N\widetilde{D} \to ND$ is the Jacobian of the normalization of the $(2,3)$-curve, or $P_{N\widetilde{D}/ND} = J(NC)$.

Next we want to find the extension data on $J(NC)$. The curve C is the intersection of a singular quadric cone and a cubic surface and it has an A_1 singularity at the node of the cone. The normalization NC is genus three and it is hyperelliptic, because the ruling of the cone gives a g_2^1 on C (each line of the ruling intersects C at three points, but one of these is always the node). Lines in the plane of D passing through $p \in D$ give a g_3^1 on ND. One of these lines, the tangent line of the A_3 singularity $p \in D$ gives the divisor $p_1 + p_2 + r$ on ND which belongs to the g_3^1. The point $r \in D$ is the fifth intersection point of the tangent line with D, as the intersection multiplicity at p is four. Let $\tilde{r} \in N\widetilde{D}$ be one of the points above r. If we pick the points $\xi_1 = p_1^+ + p_2^- + \tilde{r}$ and $\xi_2 = p_1^- + p_2^+ + \tilde{r}$ on NC using the trigonal construction, then the extension data becomes $p_2^+ - p_1^+ - p_2^- + p_1^- = \xi_2 - \xi_1$ in $J(NC)$. Using that NC is genus three and hyperelliptic, this is a point on the particular symmetric theta divisor $NC - NC$ (see also [CMGHL15, §6.2]).

Let's denote the node of C by ξ. We want to prove that the points $\xi_1, \xi_2 \in NC$ found above come from ξ after desingularization. To see this we need to prove that the line of X represented by ξ is mapped to the tangent line of $p \in D$ through the projection from the non-special line ℓ. The point ξ represents the null-line of the A_3 singularity of X. This line together with ℓ spans a three-space W that intersects X. Now $W \cap X$ is a cubic surface with an A_3 singularity in general. If we cut X with a three-space in general position, the singularity becomes A_1, but now we are in the special case where the three-space W contains the null-line of A_3. In general the singularity remains A_3 in this case (it can get worse for a special three-space). According to §3.1.4 line 4 in $W \cap X$ cannot be picked as ℓ, because in plane 0 besides line 4, there is line 3 with multiplicity two, therefore the cover $\widetilde{D} \to D$ would not be étale. Line $11'$ can be a good choice for ℓ. There are only two planes containing line $11'$, plane $11'$ having multiplicity four and containing the singular point, and plane $11'.22'$ having multiplicity one and not containing the singular point. This means that the line in the plane of D that corresponds to the three-space W must intersect D at two points and therefore it has to be the tangent line of the singular point p. Even the multiplicities of the planes coincide with the intersection multiplicities of the tangent line and D.

To summarize, the intermediate Jacobian of X is given by the sequence

$$1 \to \mathbb{C}^* \to IJ(X) \to JT \times J(NC) \to 0, \tag{4.1.20}$$

with extension data $g \mapsto \big(\mathcal{O}_T(t_1 - t_2), \mathcal{O}_{NC}(\xi_2 - \xi_1)\big)$, where $\xi_1, \xi_2 \in NC$ come from blowing up the point $\xi \in C$.

4.1.4 A_4 cubic threefolds

This case is also covered in [CML09], see table on page 52. The A_4 case is very similar to the A_2 case. Now the $(2,3)$-curve has an A_2 singularity which lies at the node of the cone. Thus NC is genus three and hyperelliptic, the g^1_2 coming from the ruling of the cone. The stable reduction of $\widetilde{D} \to D$ (D being automatically irreducible) gives a curve F with two components, one being a genus two, hyperelliptic tail T, the other being the normalization ND, which is genus four, non-hyperelliptic and trigonal. We can see that it is non-hyperelliptic similar to the A_3 case, noting that the tangent line of a planar A_4 singularity has intersection multiplicity four with the curve at the A_4 singularity in general. The Prym-variety of the double cover of ND is thus the Jacobian of NC. The dual graph $\widetilde{\Gamma}$ is exactly as in the A_2 case. For the intermediate Jacobian we then have

$$IJ(X) = JT \times P_{N\widetilde{D}/ND} = JT \times J(NC), \tag{4.1.21}$$

where T is genus two and hyperelliptic, ND is genus four, non-hyperelliptic and trigonal and NC is genus three and hyperelliptic.

4.1.5 A_5 cubic threefolds

This is very similar to the A_3 case. Now D has an A_5 singularity at p and C has an A_3 singularity at the node of the cone. Thus ND is genus three and NC is genus two and hyperelliptic. In the stable reduction the tail T is genus two. We have the same configuration for $T, T^{\pm}, p_i, p_i^{\pm}, t_i, t_i^{\pm}$ as in the A_3 case, and the dual graph $\widetilde{\Gamma}$ is exactly the same. We also know that $t_1 + t_2$ is a divisor on T belonging to the g_2^1. The extension data also has the same format as in the A_3 case.

We need to prove that ND is non-hyperelliptic. The proof is similar to the A_1 and A_3 cases. Now we consider conics in the plane that go through the A_5 singularity at p, and has the same tangent to the second degree as that given by the A_5 singularity. In other words conics that approximate the two branches of A_5 locally to degree two. This means that we are fixing three of the five points that determine a conic, so we have two points left that we can freely choose on the plane. That gives us $4 - 2 = 2$ degrees of freedom in picking a conic. Since the conic approximates both branches locally to the second degree, the intersection multiplicity of the conic and the quintic at p is six. We therefore have four other intersection points that vary with the conics. This gives us a base point free g_4^2 and an embedding to $\mathbb{P}^2$ as a degree four curve. Therefore ND cannot be hyperelliptic (alternatively, use [ACGH85, p. 13]). Thus we can use the trigonal construction and the Prym-variety of the double cover of ND is the Jacobian of NC.

We want to locate the extension data on $J(NC)$. We want to prove that this is just $\xi_2 - \xi_1$, where ξ_1, ξ_2 come from blowinp up the singularity of C, exactly as in the A_3 case. To see this we need to prove that the singular point ξ on C is mapped to the tangent line of the singularity p along the projection from ℓ. But this proof is the exact same as in the A_3 case, because a general three-space that contains the null-line of an A_5 singularity gives an A_3 singularity on the intersection surface, while for a general planar A_5 singularity the tangent line of the singularity has intersection multiplicity four with the singular point. Thus the situation is the same as in the A_3 case, and the same proof works.

As a result the intermediate Jacobian is given by

$$1 \to \mathbb{C}^* \to IJ(X) \to JT \times J(NC) \to 0, \tag{4.1.22}$$

with extension data $g \mapsto \big(\mathcal{O}_T(t_1 - t_2), \mathcal{O}_{NC}(\xi_2 - \xi_1)\big)$, where T is genus two, hyperelliptic, $t_1 + t_2 \in g^1_2$, NC is genus two, hyperelliptic and $\xi_1, \xi_2 \in NC$ come from blowing up the singular point $\xi \in C$.

4.1.6 A_6 cubic threefolds

This case is similar to the A_4 case. The discriminant D is automatically irreducible. The intermediate Jacobian is given by

$$IJ(X) = JT \times P_{N\widetilde{D}/ND} = JT \times J(NC), \tag{4.1.23}$$

where the tail T is genus three and hyperelliptic, ND is genus three, non-hyperelliptic and trigonal, and finally NC is genus two and hyperelliptic. To prove that ND is non-hyperelliptic we can use [ACGH85, p. 13].

Remark 4.1.24. According to Allcock (see e.g., [CMJL12, Thm. 2.1]), X with an A_n, $n \geq 6$ singularity is semi-stable if and only if it does not contain any of the planes containing its null line. In this case X degenerates (in the sense of [CMJL12, §2.1]) to the chordal cubic, and this means that $IJ(X)$ should be a degeneration of a hyperelliptic Jacobian, as is the case above.

4.1.7 A_7 cubic threefolds with irreducible discriminants

Similar to the A_5 case. Remark 4.1.24 also applies here. The curve D has a unique singularity, which is A_7, therefore ND is genus two and necessarily hyperelliptic. The tail T is now genus three and hyperelliptic. Using the hyperelliptic construction the Prym-variety $P_{N\widetilde{D}/ND}$ is the Jacobian of an elliptic curve G. Thus the intermediate Jacobian is

$$1 \to \mathbb{C}^* \to IJ(X) \to JT \times JG \to 0, \tag{4.1.25}$$

where T is genus three, hyperelliptic, and G is elliptic.

Suspicion: C is an irreducible $(2,3)$-curve with an A_5 singularity, thus NC is an elliptic curve and $G = NC$ via the hyperelliptic construction. The extension data is $g \mapsto \big(\mathcal{O}_T(t_1 - t_2), \mathcal{O}_{NC}(\xi_2 - \xi_1)\big)$, where ξ_1, ξ_2 come from the normalization of the singularity of C.

4.1.8 A_8 cubic threefolds with irreducible discriminants

Similar to the A_6 case. Remark 4.1.24 also applies here. The intermediate Jacobian is given by

$$IJ(X) = JT \times P_{N\widetilde{D}/ND} = JT \times JG, \tag{4.1.26}$$

where the tail T is genus four and hyperelliptic, ND is genus two and hyperelliptic and G is elliptic.

Suspicion: C is an irreducible $(2,3)$-curve with an A_6 singularity, thus NC is an elliptic curve and $G = NC$ via the hyperelliptic construction.

4.1.9 A_9 cubic threefolds with irreducible discriminants

Similar to the A_5, A_7 cases. Remark 4.1.24 also applies here. Now ND is elliptic, therefore the Prym of the double cover is trivial. The intermediate Jacobian:

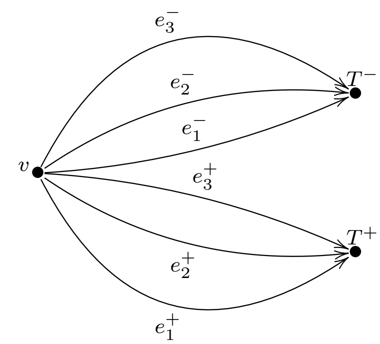
$$1 \to \mathbb{C}^* \to IJ(X) \to JT \to 0, \tag{4.1.27}$$

where the tail T is genus four and hyperelliptic and the extension data is $g \mapsto \mathcal{O}_T(t_1 - t_2)$.

4.1.10 D_4 cubic threefolds

If X has one D_4 singularity, the plane quintic D (automatically irreducible) has one D_4 singularity at p. The curve $\widetilde{D}$ is irreducible with two D_4 singularities p^+ and p^-, which are the preimage of p along the double cover $\widetilde{D} \to D$. The stable reduction then gives the double cover $\widetilde{F} \to F$, where F has two irreducible components, one being an elliptic tail T, the other being the normalization ND. They are attached at three nodes $p_1 = t_1, p_2 = t_2, p_3 = t_3$, where $p_i \in ND$ come from the blow-up of $p \in D$. The curve $\widetilde{F}$ has three components, two of which being identical to T (T^+ and

T^-), one of which being the normalization $N\widetilde{D}$. We have the points $p_i^\pm, t_i^\pm$ on $\widetilde{F}$ above the points $p_i, t_i \in F$. Then the dual graph $\widetilde{\Gamma}$ of $\widetilde{F}$ is the following:



(4.1.28)

For the cohomology we have

$$H_1(\widetilde{\Gamma}, \mathbb{Z}) = \mathbb{Z}\langle e_3^+ - e_2^+, e_3^+ - e_1^+, e_3^- - e_2^-, e_3^- - e_1^- \rangle \tag{4.1.29}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^+ = \mathbb{Z}\langle e_3^+ - e_2^+ + e_3^- - e_2^-, e_3^+ - e_1^+ + e_3^- - e_1^- \rangle \tag{4.1.30}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^- = \mathbb{Z}\langle e_3^+ - e_2^+ - e_3^- + e_2^-, e_3^+ - e_1^+ - e_3^- + e_1^- \rangle \tag{4.1.31}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^{[-]} = \mathbb{Z}\langle \tfrac{1}{2}(e_3^+ - e_2^+ - e_3^- + e_2^-), \tfrac{1}{2}(e_3^+ - e_1^+ - e_3^- + e_1^-)\rangle. \tag{4.1.32}$$

It is easy to check that $(1+\iota)H_1(\widetilde{\Gamma}, \mathbb{Z}) = H_1(\widetilde{\Gamma}, \mathbb{Z})^+$. The intermediate Jacobian is an extension

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT \times P_{N\widetilde{D}/ND} \to 0, \tag{4.1.33}$$

with extension data

$$g_1 \mapsto \left(\mathcal{O}_T(t_3 - t_2), \mathcal{O}_{N\widetilde{D}}(p_3^+ - p_2^+ - p_3^- + p_2^-)\right) \tag{4.1.34}$$

$$g_2 \mapsto \left(\mathcal{O}_T(t_3 - t_1), \mathcal{O}_{N\widetilde{D}}(p_3^+ - p_1^+ - p_3^- + p_1^-)\right). \tag{4.1.35}$$

By Theorem 3.2.14 the Prym of $N\widetilde{D} \to ND$ is the Jacobian of the normalization NC. We also know that this is eiher a hyperelliptic Jacobian or the product of two such. Let's consider the singular point of X. It is a D_4 singularity, whose tangent cone is the union of two hyperplanes. The intersection of the two hyperplanes give a plane that contains three lines of X. These lines represent the three directions that would give the three A_1 singularities in the blow up at p and they give the three A_1 singularities of C. In fact the tangent cone when projectivized is the union of two planes and the three nodes of C lie on their intersection line. Let's denote these nodes by q_1, q_2, q_3. The cubic surface in $\mathbb{P}^3$ then intersects the projectivized tangent cone in two elliptic curves in the two planes of the projectivized tangent cone. These two elliptic curves make up the curve C and they are attached at the three nodes q_1, q_2, q_3. The normalization NC is then the disjoint union of two elliptic curves with three marked points on each.

Now let's pick the point $q_1 \in C$. It represents one of the three lines of X in the intersection of the two hyperplanes that make up the tangent cone of the singular point. An easy computation shows that in general a three-space that contains such a line intersects X in a cubic surface where the D_4 singularity becomes A_3. Let W be the three-space spanned by the non-special line ℓ and the line represented by q_1. For similar considerations that we made in the case of the A_3 singularity, the image of W in the plane of the quintic D is a line that intersects D at two points, one with multiplicity four, the other with multiplicity one. Therefore this image can only be one of the three tangent lines of the singular point p.

The above suggest the following: the intermediate Jacobian is

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT \times JC_a \times JC_b \to 0, \tag{4.1.36}$$

with extension data

$$g_1 \mapsto \left(\mathcal{O}_T(t_3 - t_2), \mathcal{O}_{C_a}(q_3^a - q_2^a), \mathcal{O}_{C_b}(q_3^b - q_2^b)\right) \tag{4.1.37}$$

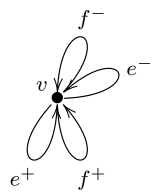
$$g_2 \mapsto \left(\mathcal{O}_T(t_3 - t_1), \mathcal{O}_{C_a}(q_3^a - q_1^a), \mathcal{O}_{C_b}(q_3^b - q_1^b)\right), \tag{4.1.38}$$

where T, C_a, C_b are elliptic, C_a, C_b are the two components of C and q_i^a, q_i^b are the marked points of C_a and C_b.

Remark 4.1.39. According to Allcock (see e.g., [CMJL12, Thm. 2.1]), X with a unique singularity, which is D_4 is strictly semi-stable, and X degenerates (in the sense of [CMJL12, §2.1]) to a cubic threefold with three D_4 singularities. In particular, we should get the same type of data in the two cases for $IJ(X)$, which in fact we do, as we will see later.

4.1.11 $2A_1$ cubic threefolds

If X has two nodes, D also has two nodes, say p and q. The double cover $\widetilde{D}$ is then irreducible with four nodes p^+, p^-, q^+, q^-, with $p^\pm$ lying above p and $q^\pm$ lying above q. The dual graph $\widetilde{\Gamma}$ of $\widetilde{D} \to D$ is



, (4.1.40)

For the cohomology we have

$$H_1(\widetilde{\Gamma}, \mathbb{Z}) = \mathbb{Z}\langle e^+, e^-, f^+, f^- \rangle \tag{4.1.41}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^+ = \mathbb{Z}\langle e^+ + e^-, f^+ + f^- \rangle \tag{4.1.42}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^- = \mathbb{Z}\langle e^+ - e^-, f^+ - f^- \rangle \tag{4.1.43}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^{[-]} = \mathbb{Z}\langle \tfrac{1}{2}(e^+ - e^-), \tfrac{1}{2}(f^+ - f^-) \rangle. \tag{4.1.44}$$

Obviously $(1 + \iota)H_1(\widetilde{\Gamma}, \mathbb{Z}) = H_1(\widetilde{\Gamma}, \mathbb{Z})^+$. Therefore the intermediate Jacobian is

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to P_{N\widetilde{D}/ND} \to 0, \tag{4.1.45}$$

with extension data

$$g_1 \mapsto \mathcal{O}_{N\widetilde{D}}(p_2^+ - p_1^+ - p_2^- + p_1^-) \tag{4.1.46}$$

$$g_2 \mapsto \mathcal{O}_{N\widetilde{D}}(q_2^+ - q_1^+ - q_2^- + q_1^-), \tag{4.1.47}$$

where p_1^+ and p_2^+ are obtained by desingularizing the node p^+, and similarly for the other points.

The normalization ND is a genus four, trigonal curve. To see that it is non-hyperelliptic, we consider the conics of the plane of D that go through the points p and q. Since five points determine a conic and two are now fixed, we can move three points freely on the plane. This gives us $6-3=3$ degrees of freedom. The intersection multiplicity of the conic in general is two at each the nodes, therefore there are six intersection points on D other than the nodes. This gives us a base point free three dimensional family of degree six effective divisors (a g^3_6) on ND, and thus an embedding of ND into $\mathbb{P}^3$ as a degree six curve. Therefore it cannot be hyperelliptic (for an alternative proof use [ACGH85, p. 13]) and the Prym of ND is the Jacobian of NC. This also means that NC has to be genus three and connected.

The curve C has one singularity, a node at say ξ. The point ξ represents the line l' of X that connects the two singular points. The lines ℓ and l' span a three-space whose image along the projection from ℓ is the line that connects the points p and q of D. Now let's assume that C was obtained from the projection from the node that corresponds to $p \in D$. Consider the g^1_3 of ND that we get if we intersect D with lines going through p. Then the line connecting p and q gives the divisor q_1+q_2+r of the g^1_3, where q_1, q_2 are obtained from desingularizing q, and r is the fifth intersection point between D and the line connecting p and q. Thus the extension data $(q_2^+ - q_1^+ - q_2^- + q_1^-)$ above can be given as the difference of two points of NC, namely

$$q_2^+ - q_1^+ - q_2^- + q_1^- = (q_1^- + q_2^+ + \tilde{r}) - (q_1^+ + q_2^- + \tilde{r}) = \xi_1 - \xi_2 \in J(NC), \tag{4.1.48}$$

where $\tilde{r} \in N\widetilde{D}$ is one of the points above $r \in ND$. Since the point $\xi \in C$ is mapped to the line connecting p and q, this suggests that ξ_1 and ξ_2 come from the desingularization of ξ. We can of course do the projection of X from the other node, the one that correspondes to q and take the g^1_3 using lines going through q and we get a similar result for the other extension data. Thus we have for the intermediate Jacobian:

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to J(NC) \to 0, \tag{4.1.49}$$

with extension data

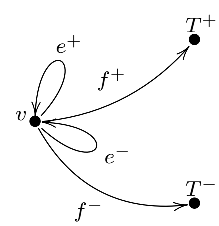
$$g_1 \mapsto \mathcal{O}_{NC}(\eta_1 - \eta_2) \tag{4.1.50}$$

$$g_2 \mapsto \mathcal{O}_{NC}(\xi_1 - \xi_2) \tag{4.1.51}$$

where η_i and ξ_i come from the desingularization of the node of C when we use one of the nodes of X as projection point, then the other one. When we use different nodes of X for the projection to get C, we get different curves C, but their normalization NC does not depend on the point of projection, and that is why we can talk about the points η_i and ξ_i on NC.

4.1.12 $A_1 + A_2$ cubic threefolds

Let's denote the A_1 singularity of D by p and the A_2 singularity by q. Then $\widetilde{D}$ has the singularities p^+, p^-, q^+, q^-. The stable reduction gives the curve F with components ND and an elliptic tail T, which are attached at the node $q = t, q \in ND, t \in T$. The curve $\widetilde{F}$ has components $N\widetilde{D}, T^+, T^-$ and nodes $p^+, p^-, q^+ = t^+, q^- = t^-$. The dual graph of $\widetilde{F} \to F$ is



(4.1.52)

The cohomology is given by

$$H_1(\widetilde{\Gamma}, \mathbb{Z}) = \mathbb{Z}\langle e^+, e^- \rangle \tag{4.1.53}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^+ = \mathbb{Z}\langle e^+ + e^- \rangle \tag{4.1.54}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^- = \mathbb{Z}\langle e^+ - e^- \rangle \tag{4.1.55}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^{[}-] = \mathbb{Z}\langle \tfrac{1}{2}(e^+ - e^-) \rangle. \tag{4.1.56}$$

Obviously $(1+\iota)H_1(\widetilde{\Gamma},\mathbb{Z}) = H_1(\widetilde{\Gamma},\mathbb{Z})^+$. Therefore the intermediate Jacobian is

$$1 \to \mathbb{C}^* \to IJ(X) \to JT \times P_{N\widetilde{D}/ND} \to 0, \tag{4.1.57}$$

with extension data

$$g \mapsto \left(\mathcal{O}_T, \mathcal{O}_{N\widetilde{D}}(p_2^+ - p_1^+ - p_2^- + p_1^-)\right). \tag{4.1.58}$$

The curve ND is genus four and trigonal. It is non-hyperelliptic for the same reason given in the $2A_1$ case. Therefore its Prym is the Jacobian of NC, which is necessarily genus three. If we consider the projection from the A_2 singularity of X and the g_3^1 received from lines going through $q \in D$, for reasons explained in previous cases (e.g. $2A_1$) the extension data is $\xi_2 - \xi_1$, where the ξ_i come from the desingularization of the A_1 singularity of X. Thus for the intermediate Jacobian we have

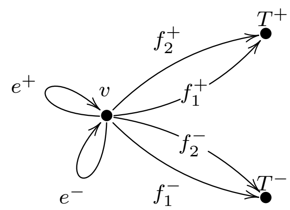
$$1 \to \mathbb{C}^* \to IJ(X) \to JT \times J(NC) \to 0, \tag{4.1.59}$$

with extension data

$$g \mapsto (\mathcal{O}_T, \mathcal{O}_{NC}(\xi_2 - \xi_1)). \tag{4.1.60}$$

4.1.13 $A_1 + A_3$ cubic threefolds with irreducible discriminants

Let the A_1 singularity of D be the point p, and the A_3 singularity of D the point q. Then $\widetilde{D}$ has the singularities p^+, p^-, q^+, q^-. To get the stable reduction F, we desingularize q and attach an elliptic tail T to the points q_1 and q_2. These points come from the desingularization of q and are attached to t_1 and t_2 of T. In $\widetilde{F}$, the stable reduction of $\widetilde{D}$, we then have the tails T^+ and T^- and the points $q_i^\pm$, $t_i^\pm$. The dual graph $\widetilde{\Gamma}$ then looks like



(4.1.61)

The cohomology is given by

$$H_1(\widetilde{\Gamma},\mathbb{Z}) = \mathbb{Z}\langle e^+, e^-, f_2^+ - f_1^+, f_2^- - f_1^-\rangle \tag{4.1.62}$$

$$H_1(\widetilde{\Gamma},\mathbb{Z})^+ = \mathbb{Z}\langle e^+ + e^-, f_2^+ - f_1^+ + f_2^- - f_1^-\rangle \tag{4.1.63}$$

$$H_1(\widetilde{\Gamma},\mathbb{Z})^- = \mathbb{Z}\langle e^+ - e^-, f_2^+ - f_1^+ - f_2^- + f_1^-\rangle \tag{4.1.64}$$

$$H_1(\widetilde{\Gamma},\mathbb{Z})^{[-]} = \mathbb{Z}\langle \tfrac{1}{2}(e^+ - e^-), \tfrac{1}{2}(f_2^+ - f_1^+ - f_2^- + f_1^-)\rangle. \tag{4.1.65}$$

Again we have $(1+\iota)H_1(\widetilde{\Gamma},\mathbb{Z}) = H_1(\widetilde{\Gamma},\mathbb{Z})^+$. Therefore the intermediate Jacobian is

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT \times P_{N\widetilde{D}/ND} \to 0, \tag{4.1.66}$$

with extension data

$$g_1 \mapsto \left(\mathcal{O}_T, \mathcal{O}_{N\widetilde{D}}(p_2^+ - p_1^+ - p_2^- + p_1^-)\right) \tag{4.1.67}$$

$$g_2 \mapsto \left(\mathcal{O}_T(t_2 - t_1), \mathcal{O}_{N\widetilde{D}}(q_2^+ - q_1^+ - q_2^- + q_1^-)\right), \tag{4.1.68}$$

where $p_i^\pm$ come from the blow-ups of $p^\pm$. The curve ND is genus three, trigonal and non-hyperelliptic for the usual reason, therefore the Prym is the Jacobian of NC, where NC is connected, genus two and hyperelliptic. Thus for the intermediate Jacobian we have

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT \times J(NC) \to 0. \tag{4.1.69}$$

For the usual reason the extension data is given by

$$g_1 \mapsto (\mathcal{O}_T, \mathcal{O}_{NC}(\xi_2 - \xi_1)) \tag{4.1.70}$$

$$g_2 \mapsto (\mathcal{O}_T(t_2 - t_1), \mathcal{O}_{NC}(\eta_2 - \eta_1)), \tag{4.1.71}$$

where the ξ_i and the η_i come from the desingularization of the singularities of the $(2,3)$-curve C, the ξ_i belonging to the original A_1 singularity and the η_i belonging to the original A_3 singularity of X.

4.1.14 $A_1 + A_4$ cubic threefolds with irreducible discriminants

This is very similar to the $A_1 + A_2$ case. The only difference is that the tail T is hyperelliptic genus two and ND is genus three. The intermediate Jacobian is

$$1 \to \mathbb{C}^* \to IJ(X) \to JT \times P_{N\widetilde{D}/ND} \to 0, \tag{4.1.72}$$

with extension data

$$g \mapsto \left(\mathcal{O}_T, \mathcal{O}_{N\widetilde{D}}(p_2^+ - p_1^+ - p_2^- + p_1^-)\right), \tag{4.1.73}$$

where the $p_i^\pm$ come from the desingularization of the nodes of $\widetilde{D}$ above the A_1 singularity of D. The normalization NC of the $(2,3)$-curve is connected and genus two. The intermediate Jacobian using NC is

$$1 \to \mathbb{C}^* \to IJ(X) \to JT \times J(NC) \to 0, \tag{4.1.74}$$

with extension data

$$g \mapsto (\mathcal{O}_T, \mathcal{O}_{NC}(\xi_2 - \xi_1)), \tag{4.1.75}$$

where the ξ_i come from the blow-up of the node of C if the A_4 singularity of X was used as center of projection to create C.

4.1.15 $A_1 + A_5$ cubic threefolds with irreducible discriminants

This is similar to the $A_1 + A_3$ case. The point $q \in D$ is now an A_5 singularity, the tail T is now a hyperelliptic, genus two curve. The dual graph and the cohomology groups are the same, therefore the intermediate Jacobian is given as

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT \times P_{N\widetilde{D}/ND} \to 0, \tag{4.1.76}$$

with extension data

$$g_1 \mapsto \left(\mathcal{O}_T, \mathcal{O}_{N\widetilde{D}}(p_2^+ - p_1^+ - p_2^- + p_1^-)\right) \tag{4.1.77}$$

$$g_2 \mapsto \left(\mathcal{O}_T(t_2 - t_1), \mathcal{O}_{N\widetilde{D}}(q_2^+ - q_1^+ - q_2^- + q_1^-)\right). \tag{4.1.78}$$

The normalization ND is hyperelliptic and genus two, therefore we cannot use the trigonal construction. By the hyperelliptic construction the Prym of the double cover is the Jacobian of some elliptic curve G, therefore we have

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT \times JG \to 0. \tag{4.1.79}$$

Suspicion: C is an irreducible $(2,3)$-curve with an A_5 singularity (or an A_1 and an A_3 singularity), thus NC is an elliptic curve and $G = NC$ via the hyperelliptic construction. The extension data is

$$g_1 \mapsto (\mathcal{O}_T, \mathcal{O}_{NC}(\xi_2 - \xi_1)) \tag{4.1.80}$$

$$g_2 \mapsto (\mathcal{O}_T(t_2 - t_1), \mathcal{O}_{NC}(\eta_2 - \eta_1)), \tag{4.1.81}$$

where the ξ_i come from the desingularization of the node and the η_i come from the desingularization of the A_5 singular point.

4.1.16 $A_1 + A_7$ cubic threefolds with irreducible discriminants

This is similar to the $A_1 + A_3$ and the $A_1 + A_5$ cases. Remark 4.1.24 also applies here. The dual graph and the cohomology groups are the same. The difference is that the tail T is now hyperelliptic and genus three, while ND is elliptic and thus its Prym variety is trivial. Therefore the intermediate Jacobian is given by

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT \to 0, \tag{4.1.82}$$

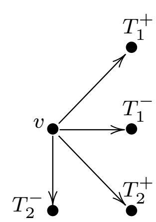
with extension data

$$g_1 \mapsto \mathcal{O}_T \tag{4.1.83}$$

$$g_2 \mapsto \mathcal{O}_T(t_2 - t_1). \tag{4.1.84}$$

4.1.17 $2A_2$ cubic threefolds with irreducible discriminants

In the stable reduction we have the elliptic tails T_1, T_2, and $T_1^{\pm}, T_2^{\pm}$. The dual graph $\widetilde{\Gamma}$:



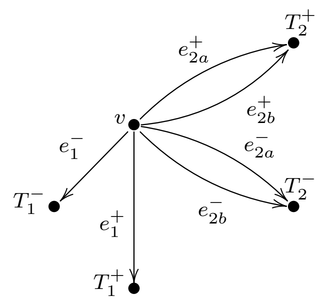
(4.1.85)

The cohomology groups are trivial and the intermediate Jacobian is compact. The normalization ND is a trigonal, non-hyperelliptic, genus four curve. It is non-hyperelliptic for the usual reasons, see e.g. the $2A_1$ case. The Prym of ND is the Jacobian of the normalization NC of the $(2, 3)$-curve C. The curve NC is genus three. Thus for the intermediate Jacobian we have:

$$IJ(X) = JT_1 \times JT_2 \times J(NC). \tag{4.1.86}$$

4.1.18 $A_2 + A_3$ cubic threefolds with irreducible discriminants

Let p denote the A_2 singularity, and q denote the A_3 singularity of D. In the stable reduction T_1 denotes the elliptic tail belonging to p, and T_2 denotes the hyperelliptic genus two tail belonging to q. In F the tail T_2 is connected to ND at the nodes $q_a = t_{2a}$ and $q_b = t_{2b}$. We have the tails $T_1^\pm$ and $T_2^\pm$ as usual and the nodes $q_a^\pm, q_b^\pm, t_{2a}^\pm, t_{2b}^\pm$. The dual graph $\widetilde{\Gamma}$:



(4.1.87)

The cohomology is then given by

$$H_1(\widetilde{\Gamma}, \mathbb{Z}) = \mathbb{Z}\langle e_{2a}^+ - e_{2b}^+, e_{2a}^- - e_{2b}^- \rangle \tag{4.1.88}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^+ = \mathbb{Z}\langle e_{2a}^+ - e_{2b}^+ + e_{2a}^- - e_{2b}^- \rangle \tag{4.1.89}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^- = \mathbb{Z}\langle e_{2a}^+ - e_{2b}^+ - e_{2a}^- + e_{2b}^- \rangle \tag{4.1.90}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^{[-]} = \mathbb{Z}\langle \tfrac{1}{2}(e_{2a}^+ - e_{2b}^+ - e_{2a}^- + e_{2b}^-) \rangle. \tag{4.1.91}$$

Clearly $(1+\iota)H_1(\widetilde{\Gamma}, \mathbb{Z}) = H_1(\widetilde{\Gamma}, \mathbb{Z})^+$. The curve ND is trigonal and genus three. It is also non-hyperelliptic for the usual reason. Its Prym is therefore the Jacobian of NC, which is a genus two curve. If $\xi_1, \xi_2 \in NC$ are obtained by desingularizing the point of C representing the A_3 singularity of X, the intermediate Jacobian is given by

$$1 \to \mathbb{C}^* \to IJ(X) \to JT_1 \times JT_2 \times J(NC) \to 0, \tag{4.1.92}$$

with extension data

$$g \mapsto (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}(t_{2b} - t_{2a}), \mathcal{O}_{NC}(\xi_2 - \xi_1)) \,. \tag{4.1.93}$$

4.1.19 $A_2 + A_4$ cubic threefolds with irreducible discriminants

Similar to the $2A_2$ case. The intermediate Jacobian is

$$IJ(X) = JT_1 \times JT_2 \times J(NC), \tag{4.1.94}$$

where T_1 is elliptic, T_2 is hyperelliptic, genus two and NC is genus two.

4.1.20 $A_2 + A_5$ cubic threefolds with irreducible discriminants

The intermediate Jacobian is given by

$$1 \to \mathbb{C}^* \to IJ(X) \to JT_1 \times JT_2 \times JG \to 0, \tag{4.1.95}$$

where T_1 is elliptic, T_2 is hyperelliptic genus two and G is elliptic.

Suspicion: NC is connected and elliptic and $G = NC$. The extension data is given by

$$g \mapsto (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}(t_2 - t_1), \mathcal{O}_{NC}(\xi_2 - \xi_1)), \tag{4.1.96}$$

where the points $\xi_1, \xi_2 \in NC$ come from the desingularization of the singular point that represents the A_5 singularity of X.

4.1.21 $A_2 + A_7$ cubic threefolds with irreducible discriminants

Remark 4.1.24 also applies here. The intermediate Jacobian is given by

$$1 \to \mathbb{C}^* \to IJ(X) \to JT_1 \times JT_2 \to 0, \tag{4.1.97}$$

where T_1 is elliptic and T_2 is hyperelliptic genus three. The extension data is given by

$$g \mapsto (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}(t_2 - t_1)). \tag{4.1.98}$$

4.1.22 $2A_3$ cubic threefolds with irreducible discriminants

The intermediate Jacobian is given by

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT_1 \times JT_2 \times JG \to 0, \tag{4.1.99}$$

where T_1, T_2, G are elliptic.

Suspicion: NC is connected and elliptic and $G = NC$. The extension data is given by

$$g_1 \mapsto (\mathcal{O}_{T_1}(t_{1b} - t_{1a}), \mathcal{O}_{T_2}, \mathcal{O}_{NC}(\xi_{1b} - \xi_{1a})) \tag{4.1.100}$$

$$g_2 \mapsto (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}(t_{2b} - t_{2a}), \mathcal{O}_{NC}(\xi_{2b} - \xi_{2a}))\,, \tag{4.1.101}$$

where the points $\xi_{1a}, \xi_{1b}, \xi_{2a}, \xi_{2b} \in NC$ come from the desingularization of the singular points that represent the A_3 singularities of X.

4.1.23 $A_3 + A_4$ cubic threefolds with irreducible discriminants

The intermediate Jacobian is given by

$$1 \to \mathbb{C}^* \to IJ(X) \to JT_1 \times JT_2 \times JG \to 0, \tag{4.1.102}$$

where T_1, G are elliptic and T_2 is hyperelliptic genus two.

Suspicion: NC is connected and elliptic and $G = NC$. The extension data is given by

$$g \mapsto (\mathcal{O}_{T_1}(t_2 - t_1), \mathcal{O}_{T_2}, \mathcal{O}_{NC}(\xi_2 - \xi_1))\,, \tag{4.1.103}$$

where the points $\xi_1, \xi_2, \in NC$ come from the desingularization of the singular point that represents the A_3 singularity of X.

4.1.24 $A_3 + A_5$ cubic threefolds with irreducible discriminants

The intermediate Jacobian is given by

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT_1 \times JT_2 \to 0, \tag{4.1.104}$$

where T_1 is elliptic and T_2 is hyperelliptic, genus two. The extension data is given by

$$g_1 \mapsto (\mathcal{O}_{T_1}(t_{1b} - t_{1a}), \mathcal{O}_{T_2}) \tag{4.1.105}$$

$$g_2 \mapsto (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}(t_{2b} - t_{2a}))\,. \tag{4.1.106}$$

4.1.25 $2A_4$ cubic threefolds with irreducible discriminants

The intermediate Jacobian is

$$IJ(X) = JT_1 \times JT_2 \times JG, \tag{4.1.107}$$

where T_1, T_2 are hyperelliptic, genus two and G is elliptic.

Suspicion: NC is connected and elliptic and $G = NC$.

4.1.26 $A_4 + A_5$ cubic threefolds with irreducible discriminants

The intermediate Jacobian is given by

$$1 \to \mathbb{C}^* \to IJ(X) \to JT_1 \times JT_2 \to 0, \tag{4.1.108}$$

where T_1, T_2 are hyperelliptic, genus two. The extension data is given by

$$g \mapsto (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}(t_2 - t_1))\,. \tag{4.1.109}$$

4.1.27 $3A_1$ cubic threefolds with irreducible discriminants

Similar to the $2A_1$ case. The intermediate Jacobian is given by

$$1 \to (\mathbb{C}^*)^3 \to IJ(X) \to J(NC) \to 0, \tag{4.1.110}$$

with extension data

$$g_1 \mapsto \mathcal{O}_{NC}(\xi_2 - \xi_1) \tag{4.1.111}$$

$$g_2 \mapsto \mathcal{O}_{NC}(\eta_2 - \eta_1) \tag{4.1.112}$$

$$g_3 \mapsto \mathcal{O}_{NC}(\mu_2 - \mu_1), \tag{4.1.113}$$

where NC is connected, hyperellitpic and genus two and the points $\xi_i, \eta_i, \mu_i \in NC$ come from the desingularization of nodes of C corresponding to the nodes of X.

4.1.28 $2A_1 + A_2$ cubic threefolds with irreducible discriminants

Similar to the $A_1 + A_2$ case. The intermediate Jacobian is given by

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT \times J(NC) \to 0, \tag{4.1.114}$$

with extension data

$$g_1 \mapsto (\mathcal{O}_T, \mathcal{O}_{NC}(\xi_2 - \xi_1)) \tag{4.1.115}$$

$$g_2 \mapsto (\mathcal{O}_T, \mathcal{O}_{NC}(\eta_2 - \eta_1)), \tag{4.1.116}$$

where T is an elliptic tail, NC is connected, hyperellitpic and genus two and the points $\xi_i, \eta_i \in NC$ come from the desingularization of nodes of C corresponding to the nodes of X.

4.1.29 $2A_1 + A_3$ cubic threefolds with irreducible discriminants

Similar to the $A_1 + A_3$ case. The intermediate Jacobian is given by

$$1 \to (\mathbb{C}^*)^3 \to IJ(X) \to JT \times JG \to 0, \tag{4.1.117}$$

where T is an elliptic tail and G is also elliptic.

Suspicion: NC is connected and elliptic and $G = NC$. The extension data is given by

$$g_1 \mapsto (\mathcal{O}_T, \mathcal{O}_{NC}(\xi_2 - \xi_1)) \tag{4.1.118}$$

$$g_2 \mapsto (\mathcal{O}_T, \mathcal{O}_{NC}(\eta_2 - \eta_1)) \tag{4.1.119}$$

$$g_3 \mapsto (\mathcal{O}_T(t_2 - t_1), \mathcal{O}_{NC}(\mu_2 - \mu_1)), \tag{4.1.120}$$

where the points $\xi_i, \eta_i \in NC$ come from the desingularization of nodes of C corresponding to the A_1 singularities of X, and the points μ_i are desingularization of the singularity corresponding to the A_3 point of X.

4.1.30 $2A_1 + A_4$ cubic threefolds with irreducible discriminants

Similar to the $2A_1 + A_2$ case. The intermediate Jacobian is given by

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT \times JG \to 0, \tag{4.1.121}$$

where T is a hyperelliptic, genus two tail and G is elliptic.

Suspicion: NC is connected and elliptic and $G = NC$. The extension data is given by

$$g_1 \mapsto (\mathcal{O}_T, \mathcal{O}_{NC}(\xi_2 - \xi_1)) \tag{4.1.122}$$

$$g_2 \mapsto (\mathcal{O}_T, \mathcal{O}_{NC}(\eta_2 - \eta_1)), \tag{4.1.123}$$

where the points $\xi_i, \eta_i \in NC$ come from the desingularization of nodes of C corresponding to the node of X.

4.1.31 $2A_1 + A_5$ cubic threefolds with irreducible discriminants

Similar to the $2A_1 + A_3$ case. The intermediate Jacobian is given by

$$1 \to (\mathbb{C}^*)^3 \to IJ(X) \to JT \to 0, \tag{4.1.124}$$

with extension data

$$g_1 \mapsto \mathcal{O}_T \tag{4.1.125}$$

$$g_2 \mapsto \mathcal{O}_T \tag{4.1.126}$$

$$g_3 \mapsto \mathcal{O}_T(t_2 - t_1), \tag{4.1.127}$$

where T is a hyperelliptic, genus two tail.

4.1.32 $A_1 + 2A_2$ cubic threefolds with irreducible discriminants

The intermediate Jacobian is given by

$$1 \to \mathbb{C}^* \to IJ(X) \to JT_1 \times JT_2 \times J(NC) \to 0, \tag{4.1.128}$$

with extension data

$$g \mapsto \left(\mathcal{O}_{T_1}, \mathcal{O}_{T_2}, \mathcal{O}_{NC}(\xi_2 - \xi_1)\right), \tag{4.1.129}$$

where T_1, T_2 are elliptic tails, NC is connected, genus two, hyperelliptic.

4.1.33 $A_1 + A_2 + A_3$ cubic threefolds with irreducible discriminants

The intermediate Jacobian is given by

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT_1 \times JT_2 \times JG \to 0, \tag{4.1.130}$$

where T_1, T_2 are elliptic tails and G is also elliptic.

Suspicion: NC is connected, elliptic and $G = NC$. The extension data is

$$g_1 \mapsto (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}, \mathcal{O}_{NC}(\xi_2 - \xi_1)) \tag{4.1.131}$$

$$g_2 \mapsto (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}(t_2 - t_1), \mathcal{O}_{NC}(\eta_2 - \eta_1)), \tag{4.1.132}$$

where the ξ_i belong to the A_1 singularity and the η_i belong to the A_3 singularity of X.

4.1.34 $A_1 + A_2 + A_4$ cubic threefolds with irreducible discriminants

The intermediate Jacobian is given by

$$1 \to \mathbb{C}^* \to IJ(X) \to JT_1 \times JT_2 \times JG \to 0, \tag{4.1.133}$$

where T_1 is an elliptic tail, T_2 is a hyperelliptic, genus two tail and G is elliptic.

Suspicion: NC is connected, elliptic and $G = NC$. The extension data is

$$g \mapsto (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}, \mathcal{O}_{NC}(\xi_2 - \xi_1)), \tag{4.1.134}$$

where the ξ_i belong to the A_1 singularity of X.

4.1.35 $A_1 + A_2 + A_5$ cubic threefolds with irreducible discriminants

The intermediate Jacobian is given by

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to JT_1 \times JT_2 \to 0, \tag{4.1.135}$$

with extension data

$$g_1 \mapsto (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}) \tag{4.1.136}$$

$$g_2 \mapsto (\mathcal{O}_{T_1}, \mathcal{O}_{T_2}(t_2 - t_1)), \tag{4.1.137}$$

where T_1 is an elliptic tail and T_2 is a hyperelliptic, genus two tail.

4.2 Quintic = 2 Conics + 1 Line

4.2.1 $8A_1$ cubic threefolds

Two conics and a line in the plane in general position have eight nodal intersections. Since there is no need for a stable reduction, and ND and $N\widetilde{D}$ are disjoint unions of genus zero curves, the intermediate Jacobian has no compact part, therefore

$$IJ(X) \cong (\mathbb{C}^*)^5. \tag{4.2.1}$$

4.3 Quintic = Conic + 3 Lines

4.3.1 $9A_1$ cubic threefolds

A conic and three lines in the plane in general position have nine nodal intersections. Since there is no need for a stable reduction, and ND and $N\widetilde{D}$ are disjoint unions of genus zero curves, the intermediate Jacobian has no compact part, therefore

$$IJ(X) \cong (\mathbb{C}^*)^5. \tag{4.3.1}$$

4.3.2 $3D_4$ cubic threefolds

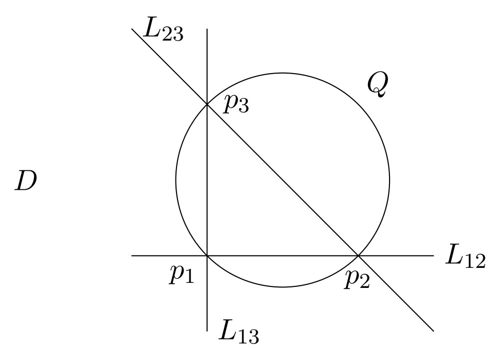
There is a unique cubic threefold X with three D_4 singularities with equation

$$x_0x_1x_2 + x_3^3 + x_4^3 = 0, \tag{4.3.2}$$

in the sense that any other cubic threefold with three D_4 singularities can be obtained under projective transformation of this equation. The three singularities occur at the points $p_1 = (1:0:0:0:0)$, $p_2 = (0:1:0:0:0)$, $p_3 = (0:0:1:0:0)$. Since permuting the coordinates x_0, x_1, x_2 does not change the equation of X, we can see that X is symmetric with respect to the three singularities. The equation of X also tells us about the $(2,3)$-curve C, namely it is the intersection of the quadric $\{x_1x_2 = 0\}$ and the cubic $\{x_3^3 + x_4^3 = 0\}$ in $\mathbb{P}^3$. The former is the union of the planes

$\{x_1 = 0\}$ and $\{x_2 = 0\}$, while the latter is the union of three planes intersecting one another in the common line $\{x_3 = x_4 = 0\}$. Therefore C consists of two sets of three lines, each set of three lines being on the same plane and intersecting in a common point giving a D_4 singularity. Lines from different sets meet each other in three pairs, giving three A_1 singularities on the intersection of the planes $\{x_1 = 0\}$ and $\{x_2 = 0\}$. The three A_1 singularities come from the blow-up of X at p_1, while the two D_4 singularities (located at $(1 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0)$) come from p_2 and p_3. Since C consists of six lines, there are six planes contained in X and containing p_1. Three of them contains the line through p_1 and p_2 and the other three contains the line through p_1 and p_3. Because of symmetry the same statement about the six planes is true for p_2 and p_3, therefore for any two singular points there are three planes in X containing the two singular points.

The plane quintic discriminant curve D has three D_4 singularities which we also denote by p_1, p_2, p_3. They are formed the following way: the three lines L_{12}, L_{13}, L_{23} of the quintic intersect each other in the points p_1, p_2, p_3, and the conic Q goes through these intersections. Thus D consists of four irreducible components, each non-singular and genus zero.



By Hurwitz's theorem a connected double cover of a genus zero curve must have ramification points, therefore in $\widetilde{D}$ there must be two identical copies above each component of D. Therefore $\widetilde{D}$ consists of eight genus zero curves labelled $L_{ij}^{\pm}, Q^{\pm}$, connected through six D_4 singularities $p_i^{\pm}$. These eight curves must be connected in a way that the resulting $\widetilde{D}$ is connected. We want to figure out the configuration of these connections.

Let p_i^+ be three points on Q^+ and p_i^- be three points on Q^-. These are isomorphic copies of Q with the three points p_i. The curves Q^+ and Q^- are obviously disjoint, because the double cover of Q is their disjoint union. We want to find how the six curves $L_{ij}^{\pm}$ are connected to the curves $Q^{\pm}$. The curve L_{ij}^{+} must connect to one of p_i^+, p_i^- and also one of p_j^+, p_j^-. Then L_{ij}^{-} will connect to the other points of each of these pairs.

Let's consider the straight line L_{12} of D. This represents a three-space W_{12} in $\mathbb{P}^4$. We want to describe the cubic surface $X \cap W_{12}$. The three-space W_{12} can be obtained as the span of the non-special line ℓ and the line through p_1 and p_2 of X. Since the entire line L_{12} is part of the discriminant D, any plane of W_{12} conaining ℓ intersects X in three distinct lines. This means that the cubic surface $X \cap W_{12}$ has infinitely many straight lines. Thus the possibilities for $X \cap W_{12}$ is that it is either a reducible cubic surface, an irreducible cone, or an irreducible ruled surface (a scroll). It is easy to see that $X \cap W_{12}$ cannot be an irreducible cone, since all lines go through one point in an irreducible cone, while in $X \cap W_{12}$ the lines ℓ and the one through p_1 and p_2 are skew. To find out about $X \cap W_{12}$, we have to go back to the equation of X and see the possible intersections that we get with three-spaces. The general equation of a three-space in $\mathbb{P}^4$ is

$$ax_0 + bx_1 + cx_2 + dx_3 + ex_4 = 0. \tag{4.3.3}$$

Since the three-space belonging to L_{12} contains p_1 and p_2, but not p_3, we have $a = b = 0$ and $c \neq 0$. We can choose $c = -1$ and get $x_2 = dx_3 + ex_4$, where d and e are non-homogeneous coordinates now. In the case $d = e = 0$, W_{12} is given by $x_2 = 0$ and the equation of $X \cap W_{12}$ is just $x_3^3 + x_4^3 = 0$, which is the union of three planes intersecting one another in a common line, the line through p_1 and p_2. This cannot be the case, because no matter how we would pick the non-special line $\ell \subset X$, the image of the entire line through p_1 and p_2 would be a point in the plane of the discriminant curve. Therefore $d = e = 0$ cannot be, and because of symmetricity we can assume that $d \neq 0$. For $X \cap W_{12}$ we have

$$dx_0x_1x_3 + ex_0x_1x_4 + x_3^3 + x_4^3 = 0. \tag{4.3.4}$$

To find the singularities we consider the Jacobian

$$\begin{bmatrix} x_1(dx_3 + ex_4) & x_0(dx_3 + ex_4) & dx_0x_1 + 3x_3^2 & ex_0x_1 + 3x_4^2 \end{bmatrix}. \tag{4.3.5}$$

It is easy to see that if we assume that $dx_3 + ex_4 \neq 0$, we eventually have to conclude that $x_0 = x_1 = x_3 = x_4 = 0$, which is not possible, so we assume that $dx_3 + ex_4 = 0$. Since $d \neq 0$, we have $x_3 = -\frac{e}{d}x_4$ and the equation of $X \cap W_{12}$ becomes

$$x_0x_1(dx_3 + ex_4) - \frac{e^3}{d^3}x_4^3 + x_4^3 = \left(1 - \frac{e^3}{d^3}\right)x_4^3 = 0. \tag{4.3.6}$$

Case I: $e^3 \neq d^3$. Then necessarily $x_4 = 0$ and therefore $x_3 = 0$. From the third and fourth entries of the Jacobian we see that either x_0 or x_1 must be zero, therefore in this case the only singularities are at $(1 : 0 : 0 : 0)$ and $(0 : 1 : 0 : 0)$ and they come from p_1 and p_2. Since we only have isolated singularities, the cubic surface can only have finitely many lines, so this case cannot give the cubic surface we are looking for.

Case II: $e^3 = d^3$. Under the assumption $dx_3 + ex_4 = 0$ and $e^3 = d^3$, it is easy to see that the third and fourth entries of the Jacobian give the same equation. The locus of the singularities is given by the equations

$$dx_3 + ex_4 = 0 \tag{4.3.7}$$

$$dx_0x_1 + 3x_3^2 = 0, \tag{4.3.8}$$

which is a plane quadric. Therefore $X \cap W_{12}$ cannot be an irreducible ruled cubic surface, because that has a double line singularity. Thus $X \cap W_{12}$ is reducible, and is the union of a plane and a quadric surface. Indeed, $(dx_3 + ex_4)$ can be factored out from the equation (4.3.4) if $e^3 = d^3$, and what remains is an irreducible quadric. The singular points p_1 and p_2 must be singular in $X \cap W_{12}$ as well, therefore they belong to both the plane and the quadric surface.

Remark: we see that case I is the more general case, so we might wonder if we could pick the non-special line ℓ as one of the lines that occur in $X \cap W_{12}$, when W_{12} belongs to case I, i.e. when $e^3 \neq d^3$. In fact it is not possible, because in case I $X \cap W_{12}$ has at best two A_2 singularities (a D_4 in a threefold of $\mathbb{P}^4$ becomes an A_2 when intersected with a general three-space), and we know from §3.1.6, that all lines in a cubic surface with two A_2 singularities pass through at least one of the singularities. Therefore we could not find a non-special line there.

We have shown above that if W_{12} is the three-space belonging to L_{12} of D, then $X \cap W_{12}$ is the union of a plane Π_{12} and a quadric surface R_{12}, where the singularities p_1 and p_2 lie on the intersection $\Pi_{12} \cap R_{12}$ and Π_{12} is one of the three planes of X mentioned above containing these two singular points. As the non-special line cannot intersect the line through p_1 and p_2, it must lie on the quadric R_{12}.

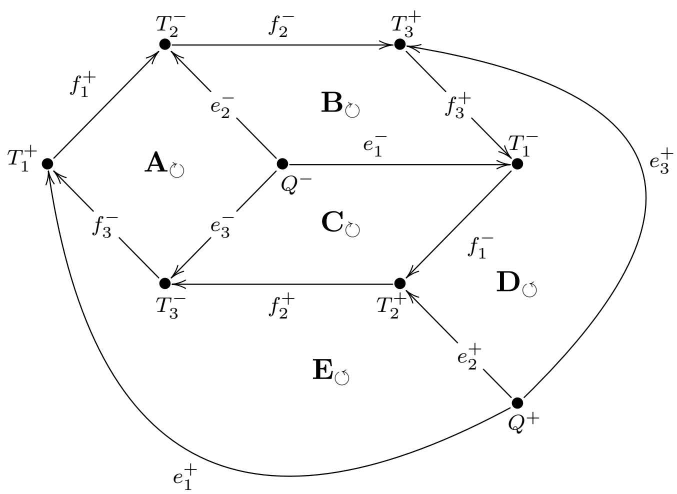
Next we want to identify the curves L_{12}^+, L_{12}^- and see how they connect to the other curves $L_{ij}^\pm$. If we pick a point of L_{12}, that corresponds to a plane Y of W_{12} containing ℓ. Then Y necessarily has two additional lines when intersected with X, one in the plane Π_{12}, the other in the quadric R_{12}. One of these lines represent a point of L_{12}^+, the other a point of L_{12}^-. Let's say that the lines on Π_{12} correspond to the points of L_{12}^+, while the lines on R_{12} correspond to the points of L_{12}^-, and similarly for the other $L_{ij}^\pm$. The question is does L_{12}^+ connect to L_{13}^+ or to L_{13}^-? This connection occurs above the point p_1, so we have to consider the plane spanned by ℓ and p_1 and the two residual lines of intersection with X, say l^+ and l^-, where l^+ is the line on the plane Π_{12} and l^- is the one on R_{12}. The question is does the line l^+ lie on the plane Π_{13} or on the quadric R_{13} of the cubic surface $X \cap W_{13}$ belonging to L_{13}? The plane Π_{13} of $X \cap W_{13}$ is one of the three planes in X containing both p_1 and p_3. Two of these three planes intersect Π_{12} in only a point (which is p_1), and only the third intersect it in a line (the curve C explains why). This line corresponds to one of the A_1 singularities of the $(2,3)$-curve C and it is one of the three distinguished lines in the tangent cone of the D_4 singular point p_1. If Π_{12} and Π_{13} intersect only in a point, l^+ must lie on R_{13}, and thus L_{12}^+ connects with L_{13}^-. Let's assume now that Π_{13} is the plane that intersects Π_{12} in a line. The question is therefore is it possible that l^+ is the same as the intersection of Π_{12} and Π_{13}? If it was, then the non-special line ℓ would intersect $\Pi_{12} \cap \Pi_{13}$, because ℓ and l^+ intersect. But that is not possible, because $\Pi_{12} \cap \Pi_{13}$ represents the tangent direction at $p_1 \in X$ that gives one of the A_1 singularities when p_1 is blown up, thus the span of ℓ and $\Pi_{12} \cap \Pi_{13}$ must be a three-space that projects to one of the tangent lines at $p_1 \in D$ in the plane of the discriminant D. Thus ℓ cannot intersect $\Pi_{12} \cap \Pi_{13}$, and l^+ does not lie in Π_{13}, but in R_{13}. Therefore L_{12}^+ connects to L_{13}^- in this case, as well, and thus L_{12}^- must connect to L_{13}^+. Because of symmetricity, similar statements are true above the other singular points, as well. Therefore the $L_{ij}^\pm$ lines are connected to each other in a single cycle of six elements (as opposed to two cycles of three, for example).

The next question is how do the curves Q^+ and Q^- fit into this arrangement? As the configuration must be symmetric with respect to the singularities, the only possible way to do this (up to some relabelling) is if the lines $L_{ij}^{\pm}$ keep alternating between Q^+ and Q^-, that is each $L_{ij}^{\pm}$ is connected to Q^+ at one point and Q^- at the other.

Now that we have a clear understanding of the double cover $\widetilde{D} \to D$, we want to describe the stable reduction $\widetilde{F} \to F$. To resolve the D_4 singularities of $\widetilde{D}$ and D, we add an elliptic tail for each singularity. The three irreducible components that are connected at a singularity of $\widetilde{D}$ or D will connect to three distinct points of the corresponding tail curve. We have the tails T_i in F for the singularities p_i and the tails $T_i^{\pm}$ in $\widetilde{F}$ for the singularities $p_i^{\pm}$. On each $T_i^{\pm}$ there are connection points where the other components connect through a node, these nodes are labelled as ${}^{x}t_i^{\pm}$, where x can be one of $(q, 12, 13, 23)$ depending on what component is connected to $T_i^{\pm}$ at that node. To get the stable reduction of $\widetilde{D}$ it is not enough to blow up the singularities and add the tails described above. The problem is that the genus zero curves $L_{ij}^{\pm}$ have only two marked points, so their automorphism groups are infinite. This means that they have to be contracted to a point and the two tails that used to be connected to a curve $L_{ij}^{\pm}$ will now be connected to each other through a node that comes from the contraction of these curves. The components and the connections are described by the following table:

	Q^+	Q^-	T_1^+	T_2^+	T_3^+	T_1^-	T_2^-	T_3^-
Q^+	-	-	${}^q t_1^+$	${}^q t_2^+$	${}^q t_3^+$	-	-	-
Q^-	-	-	-	-	-	${}^q t_1^-$	${}^q t_2^-$	${}^q t_3^-$
T_1^+	${}^q t_1^+$	-	-	-	-	-	${}^{12}t_1^+ = {}^{12}t_2^-$	${}^{13}t_1^+ = {}^{13}t_3^-$
T_2^+	${}^q t_2^+$	-	-	-	-	${}^{12}t_2^+ = {}^{12}t_1^-$	-	${}^{23}t_2^+ = {}^{23}t_3^-$
T_3^+	${}^q t_3^+$	-	-	-	-	${}^{13}t_3^+ = {}^{13}t_1^-$	${}^{23}t_3^+ = {}^{23}t_2^-$	-
T_1^-	-	${}^q t_1^-$	-	${}^{12}t_1^- = {}^{12}t_2^+$	${}^{13}t_1^- = {}^{13}t_3^+$	-	-	-
T_2^-	-	${}^q t_2^-$	${}^{12}t_2^- = {}^{12}t_1^+$	-	${}^{23}t_2^- = {}^{23}t_3^+$	-	-	-
T_3^-	-	${}^q t_3^-$	${}^{13}t_3^- = {}^{13}t_1^+$	${}^{23}t_3^- = {}^{23}t_2^+$	-	-	-	-

This gives the dual graph $\widetilde{\Gamma}$:



(4.3.9)

With 8 vertices and 12 edges the Euler number of the graph is 4, therefore $H_1(\widetilde{\Gamma}, \mathbb{Z}) = \mathbb{Z}^5$. In fact we have:

$$\begin{aligned}H_1(\widetilde{\Gamma},\mathbb{Z}) = \mathbb{Z}\langle & f_1^+ - e_2^- + e_3^- + f_3^-, \\ & f_2^- + f_3^+ - e_1^- + e_2^-, \\ & f_1^- + f_2^+ - e_3^- + e_1^-, \\ & e_3^+ + f_3^+ + f_1^+ - e_2^+, \\ & - e_1^+ + f_3^- + f_2^+ + e_2^+\rangle\end{aligned} \tag{4.3.10}$$

$$\begin{aligned}H_1(\widetilde{\Gamma},\mathbb{Z})^+ = \mathbb{Z}\langle & f_1^+ - e_2^- + e_3^- + f_3^- + e_3^+ + f_3^+ + f_1^+ - e_2^+, \\ & f_2^- + f_3^+ - e_1^- + e_2^- - e_1^+ + f_3^- + f_2^+ + e_2^+, \\ & f_1^+ + f_2^- + f_3^+ + f_1^- + f_2^+ + f_3^-\rangle\end{aligned} \tag{4.3.11}$$

$$\begin{aligned}H_1(\widetilde{\Gamma},\mathbb{Z})^- = \mathbb{Z}\langle & f_1^+ - e_2^- + e_3^- + f_3^- - e_3^+ - f_3^+ - f_1^+ + e_2^+, \\ & f_2^- + f_3^+ - e_1^- + e_2^- + e_1^+ - f_3^- - f_2^+ - e_2^+\rangle,\end{aligned} \tag{4.3.12}$$

$$\begin{aligned}H_1(\widetilde{\Gamma},\mathbb{Z})^{[-]} = \mathbb{Z}\langle & \tfrac{1}{2}(f_1^+ - e_2^- + e_3^- + f_3^- - e_3^+ - f_3^+ - f_1^+ + e_2^+), \\ & \tfrac{1}{2}(f_2^- + f_3^+ - e_1^- + e_2^- + e_1^+ - f_3^- - f_2^+ - e_2^+)\rangle,\end{aligned} \tag{4.3.13}$$

or using the cycles indicated on the graph:

$$H_1(\widetilde{\Gamma},\mathbb{Z}) = \mathbb{Z}\langle \mathbf{A}_\circlearrowleft, \mathbf{B}_\circlearrowleft, \mathbf{C}_\circlearrowleft, \mathbf{D}_\circlearrowleft, \mathbf{E}_\circlearrowleft\rangle \tag{4.3.14}$$

$$H_1(\widetilde{\Gamma},\mathbb{Z})^+ = \mathbb{Z}\langle \mathbf{A}_\circlearrowleft + \mathbf{D}_\circlearrowleft, \mathbf{B}_\circlearrowleft + \mathbf{E}_\circlearrowleft, \mathbf{A}_\circlearrowleft + \mathbf{B}_\circlearrowleft + \mathbf{C}_\circlearrowleft\rangle \tag{4.3.15}$$

$$H_1(\widetilde{\Gamma},\mathbb{Z})^- = \mathbb{Z}\langle \mathbf{A}_\circlearrowleft - \mathbf{D}_\circlearrowleft, \mathbf{B}_\circlearrowleft - \mathbf{E}_\circlearrowleft\rangle \tag{4.3.16}$$

$$H_1(\widetilde{\Gamma},\mathbb{Z})^{[-]} = \mathbb{Z}\langle \tfrac{1}{2}(\mathbf{A}_\circlearrowleft - \mathbf{D}_\circlearrowleft), \tfrac{1}{2}(\mathbf{B}_\circlearrowleft - \mathbf{E}_\circlearrowleft)\rangle. \tag{4.3.17}$$

The picture may become clearer if we introduce

$$\mathbf{F}_\circlearrowleft := \mathbf{A}_\circlearrowleft + \mathbf{B}_\circlearrowleft + \mathbf{C}_\circlearrowleft = f_1^+ + f_2^- + f_3^+ + f_1^- + f_2^+ + f_3^-, \tag{4.3.18}$$

a cycle around the hexagone of the $T_i^{\pm}$ in a clockwise direction. Then choosing a new basis we have

$$H_1(\widetilde{\Gamma}, \mathbb{Z}) = \mathbb{Z}\langle \mathbf{A}_{\circlearrowleft}, \mathbf{D}_{\circlearrowleft}, \mathbf{B}_{\circlearrowleft}, \mathbf{E}_{\circlearrowleft}, \mathbf{F}_{\circlearrowleft} \rangle, \tag{4.3.19}$$

and using this basis the matrix of ι is

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4.3.20}$$

and we have

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^+ = \mathbb{Z}\langle \mathbf{A}_{\circlearrowleft} + \mathbf{D}_{\circlearrowleft}, \mathbf{B}_{\circlearrowleft} + \mathbf{E}_{\circlearrowleft}, \mathbf{F}_{\circlearrowleft} \rangle \tag{4.3.21}$$

$$H_1(\widetilde{\Gamma}, \mathbb{Z})^- = \mathbb{Z}\langle \mathbf{A}_{\circlearrowleft} - \mathbf{D}_{\circlearrowleft}, \mathbf{B}_{\circlearrowleft} - \mathbf{E}_{\circlearrowleft} \rangle. \tag{4.3.22}$$

It is not true that $(1+\iota)H_1(\widetilde{\Gamma}, \mathbb{Z}) = H_1(\widetilde{\Gamma}, \mathbb{Z})^+$, because $\iota(\mathbf{F}_{\circlearrowleft}) = \mathbf{F}_{\circlearrowleft}$ for the basis element $\mathbf{F}_{\circlearrowleft}$. Therefore $k = 1$ in (3.3.7), and $G = (\mathbb{Z}/2\mathbb{Z})^d$, where d is either 0 or 1. Thus the intermediate Jacobian is given by the sequence

$$1 \to (\mathbb{C}^*)^2 \to IJ(X) \to (JT_1 \times JT_2 \times JT_3)\big/G \to 0. \tag{4.3.23}$$

The extension data is given by

$$g_1 \mapsto \left[\mathcal{O}_{T_1}({}^{13}t_1 - {}^{12}t_1), \mathcal{O}_{T_2}({}^{q}t_2 - {}^{12}t_2), \mathcal{O}_{T_3}({}^{13}t_3 - {}^{q}t_3)\right] \tag{4.3.24}$$

$$g_2 \mapsto \left[\mathcal{O}_{T_1}({}^{q}t_1 - {}^{13}t_1), \mathcal{O}_{T_2}({}^{23}t_2 - {}^{q}t_2), \mathcal{O}_{T_3}({}^{23}t_3 - {}^{13}t_3)\right], \tag{4.3.25}$$

where ${}^{jk}t_i$ is the node of T_i and L_{jk} on T_i and ${}^{q}t_i$ is the node of T_i and Q on T_i.

4.4 Quintic = 5 Lines

4.4.1 $10A_1$ cubic threefolds

For reference, see [Gw05]. The cubic threefold X with ten nodes is the Segre Cubic, and it is unique up to projective transformation. It can be given in $\mathbb{P}^5$ as the intersection of the cubic

$$\sum_{i=0}^{5} x_i^3 = 0 \tag{4.4.1}$$

and the hypersurface

$$\sum_{i=0}^{5} x_i = 0. \tag{4.4.2}$$

In $\mathbb{P}^4$ it is defined by the equation

$$\sum_{\substack{i<j<k \\ i,j,k\in\{0,\ldots,4\}}} 2x_ix_jx_k \quad + \sum_{\substack{i\neq j \\ i,j\in\{0,\ldots,4\}}} x_i^2x_j = 0. \tag{4.4.3}$$

X contains 15 planes, in each plane there are four nodes, and each node is contained in six planes. The $(2,3)$-curve C must have 9 nodes. It has to contain six lines which correspond to the six planes that contain the node which is the center of projection. Since C is the intersection of a smooth quadric and a cubic, it must consist of the six lines, and the cubic is in fact the union of three planes, each tangent to the quadric. Thus C is the union of three pairs of lines, where each pair is the intersection of the quadric and a plane. Two lines on a plane have one intersection and any pair of lines intersects any other pair of lines in two points, thus we have nine nodes altogether, as we wanted.

The plane quintic D consists of five lines that intersect one another in nodes, this gives the $\binom{5}{2} = 10$ nodes. We do not need to do a stable reduction, so the normalization ND is the disjoint union of five curves of genus zero, and the double cover $N\widetilde{D}$ is the disjoint union of ten curves of genus zero. (For the dual graph $\widetilde{\Gamma}$ of $\widetilde{D}$ see [Gw05, Figure 1].) Thus the intermediate Jacobian does not have a compact part, and for dimension reasons it is

$$IJ(X) = (\mathbb{C}^*)^5 \, . \tag{4.4.4}$$

There is more degeneration data provided in [Gw05]; here we only wanted to show that the compact part is trivial.

Bibliography

[ACGH85] E. Arbarello, M. Cornalba, P. A. Griffiths, J. Harris: Geometry of Algebraic Curves. Springer Science+Business Media, LLC, 1985

[All03] D. Allcock: The Moduli Space of Cubic Threefolds. J. Algebraic Geometry 12 (2003) 201-223, S 1056-3911(02)00313-2

[ABH02] V. Alexeev, Ch. Birkenhake, K. Hulek: Degenerations of Prym varieties. J. Reine Angew. Math., 553:73-116 (2002)

[Bea82] A. Beauville: Sous-variétés spéciales des variétés de Prym, Compositio Math. 45 (1982), no. 3, 357-383. MR MR656611 (83f:14025)

[BL04] Ch. Birkenhake, H. Lange: Complex Abelian Varieties. Second, Augmented Edition. Springer-Verlag (2004)

[BT82] R. Bott, L. W. Tu: Differential Forms in Algebraic Topology. Springer-Verlag (1982)

[CG72] C. H. Clemens, P. A. Griffiths: The intermediate Jacobian of the cubic threefold, Ann. Math. (2) 95 (1972), 281-356

[CLS11] D. Cox, J. Little, H. Schenck: Toric Varieties. Graduate Studies in Mathematics Vol. 124, American Mathematical Society (2011)

[CM78] A. Collino, J. P. Murre: The intermediate Jacobian of a cubic threefold with one ordinary double point; an algebraic-geometric approach, I and II. Indag. Math. 40 (1978), no. 1, 43-55 and 56-71

[CMGHL15] S. Casalaina-Martin, S. Grushevsky, K. Hulek, R. Laza: Complete Moduli of Cubic Threefolds and their Intermediate Jacobians. (2015)

[CMJL12] S. Casalaina-Martin, D. Jensen, R. Laza: The geometry of the ball quotient model of the moduli space of genus four curves. In Compact moduli spaces and vector bundles, volume 564 of Contemp. Math., pages 107-136. Amer. Math. Soc., Providence, RI, 2012

[CML09] S. Casalaina-Martin, R. Laza: The moduli space of cubic threefolds via degenerations of the intermediate Jacobian. J. Reine Angew. Math. 633 (2009), 29-65. MR 2561195 (2011a:14071)

[Co82] A. Collino: The fundamental group of the Fano surface. I, II. In Algebraic threefolds (Varenna, 1981), volume 947 of Lecture Notes in Math., pages 209-218, 219-220. Springer, Berlin, 1982

[Cy869] Prof. Cayley: A Memoir on Cubic Surfaces. Philosophical Transactions of the Royal Society of London, Vol. 159 (1869), pp. 231-326

[EH00] D. Eisenbud, J. Harris: The Geometry of Schemes. Graduate Texts in Mathematics 197, Springer-Verlag (2000)

[GH78] P. Griffiths, J. Harris: Principles of Algebraic Geometry. John Wiley & Sons (1978)

[Gw05] T. Gwena: Degenerations of cubic threefolds and matroids. Proc. Amer. Math. Soc., 133(5):1317-1323 (electronic) (2005)

[Har77] R. Hartshorne: Algebraic Geometry. Graduate Texts Math 52, Springer (1977)

[Has00] B. Hassett: Local stable reduction of plane curve singularities. J. Reine Angew. Math., 520:169-194, 2000

[Mum74] D. Mumford: Prym varieties. I. In Contributions to analysis (a collection of papers dedicated to Lipman Bers), pages 325-350. Academic Press, New York (1974)

[Rec74] S. Recillas: Jacobians of curves with g_4^1's are the Prym's of trigonal curves. Bol. Soc. Mat. Mexicana (2), 19(1):9-13 (1974)