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DIFFRACTION BY A DIELECTRIC WEDGE

Final Report

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1. INTRODUCTION AND SUMMARY

1.1 INTRODUCTION

The scattering of an electromagnetic wave by a dielectric wedge is an exceptionally difficult problem which at the present time has no known analytic solution,[1] though Radlow,[2] in the special case of a right-angle wedge, was able to use a double-Laplace transform method that had been found successful in treating the quater-plane problem to analyze the wedge exactly.* All other treatments yield approximations of one kind or another. An approximate numerical solution for the diffraction coefficient has been given,[3] but the shortcoming of this sort of approach is that, although engineering and other applications eventually require numerical calculations, the wedge field is a compound of several different components, such as specularly reflected and refracted rays, surface waves, lateral waves and tip diffraction, and a numerical calculation of the composite effect throws very little light on the real nature of the response. An analytic solution, even though not exact, is highly desirable in order to enable the different components of the solution to be separated, recognized and understood. A study of the functional form of the result can yield considerable insight into the rather complex properties of the wedge, and in particular the behavior near shadow boundaries, as well as the effect of varying such parameters as wedge angle or dielectric constant.

Balling,[4] in his Licentiate thesis, provides an analytic treatment based on tracing the multiply-reflected rays inside the wedge, In two later papers,[5,6] he applies his results to evaluate the role of surface fields

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*Radlow's results are too involved for use, and Kuo and Plonus[11] and Kraut and Lehman[12] have attempted alternative formulations. The latter authors in fact find Radlow's analysis to be erroneous. In any case the method seems incapable of generalization to other angles.

and lateral waves in the radiation from a wedge excited by a line source inside the wedge. All these treatments utilize formulas for successive ray reflections, and Borovikov[7] has also obtained a recurrence formula for the calculation of the terms for an edge wave, provided none of the edges are located near shadow boundaries. A somewhat different approach by Tricoles and Rope[8] based on the penetration of the field in a dielectric slab calculates the approximate effect of a hollow dielectric wedge made by two slabs meeting at a bevelled edge. The application is to radomes, where the hollow structure is used.

Kaminetzky and Keller,[9] in the most recent published results, analyze the wedge in two cases in which the tip diffraction can be calculated approximately as the first term in an expansion of the field in terms of a small expansion parameter. The two cases they treat are i) a small difference in the dielectric constants of the materials inside and outside the wedge, and ii) small wedge angle. The latter complements an earlier treatment[10] where the wedge angle is nearly 180°, and the expansion is relative to the reflection of a wave at a plane dielectric surface.

The wedge problem is an important one in at least two areas. The first concerns radar reflections and EMP pulse response from dielectric objects which may be in free-space or else buried. The second concerns the use of the geometric theory of diffraction (GTD) to calculate the radiation properties of antennas and other reflectors. For example, a dielectric support may be used to locate a subreflector or splash-plate and the effect of the dielectric in modifying the diffracted fields from metallic edges needs to be taken into account. The radiation from most composite reflectors can be represented in terms of specular reflections and edge effects, and the advent of a new class of formulas involving dielectric wedges, either on their own or in contact with a metallic surface, can be expected to extend the range of application of the GTD method. The shadow-boundary terms are an important part of the toal solution and should be shown explicitly.

Although the geometry of the arrangement is quite simple, nevertheless the existence of propagating waves inside and outside the wedge, with different velocities, makes the problem a very formidable one.

1.2 SUMMARY

Three separate investigations were made. The first, based on a spectral decomposition method, was the one on which the contract proposal was originally based. It turned out to have a built-in flaw, which was not discovered, however, until rather late in the analysis. As a consequence the method was abandoned. Just at that time, attention was drawn to two Soviet papers, both of which claimed to have solved the wedge problem rigorously and completely, though by completely different methods. The first paper, by Zavadskii,[13] was examined and found to be faulty. Efforts made to correct the errors were to no avail. However, a semi-trivial result emerged from these studies which, in retrospect, could have been found by elementary methods. The second paper, by Aleksandrova and Khiznyak[14], also turned out to be faulty, though the complex analysis involved a great deal of study to pin-point exactly where the errors occurred. It was not found possible to solve the problem by their method with the errors corrected.

It appears that at the present time there is still no valid and rigorous solution to the wedge problem, though several approximations, particularly Balling's,[4] have appeared in the literature, and should provide useful results in practical applications.

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DIFFRACTION BY A DIELECTRIC WEDGE: SPECTRAL ANALYSIS

ABSTRACT

A spectral analysis of the field inside and outside a dielectric wedge is made, and field matching in the presence of an axial incident wave is made, both on-axis and on the wedge faces. The method involves unknown spectral functions among which a relation (14) is imposed in order to lead to Wiener-Hopf-like equations (21) and (22). However an examination of the pole locations of these equations shows that they self-generate an unending pole-sequence, which makes the method unusable. No modification of the method that would render it valid appears possible.

2.1 INTRODUCTION

Despite its simple geometry, the problem of a dielectric wedge excited by an incident wave is an extremely difficult one, and no exact solutions are known. Various features have been treated by different researchers, including an examination of the field near the tip, specular internal and external reflection, and properties when the dielectric constants of the wedge and the surrounding medium are nearly the same.

In this paper we attempt an exact formulation from which useful results can be extracted. In view of the extreme difficulties encountered, the total wedge angle has been restricted to less than 90° (other features apparently enter when the wedge angle is larger than this), and the analysis is further confined to symmetrical incidence of a plane wave with parallel polarization.

2.2 FORMULATION

Initial attempts to provide a formulation in terms of cylindrical functions were discouraging, and eventually an expansion of the fields as an integral of plane waves was selected as more promising. However, a number of features had to be closely constrained in order to permit field matching in a constructive way and this accounts in part for the form taken by the subsequent analysis.

Figure 2.1 shows the configuration in which a dielectric wedge, of angle 2α and refractive index n is immersed in a medium of refractive index n_o. The latter would normally correspond to free space, but it turns out to be essential to allow both media to possess a small loss term. This will be taken to the zero in the limit, but for much of the analysis it has to be retained, since it determines the all-important feature of whether certain singularities are inside or outside integration contours.

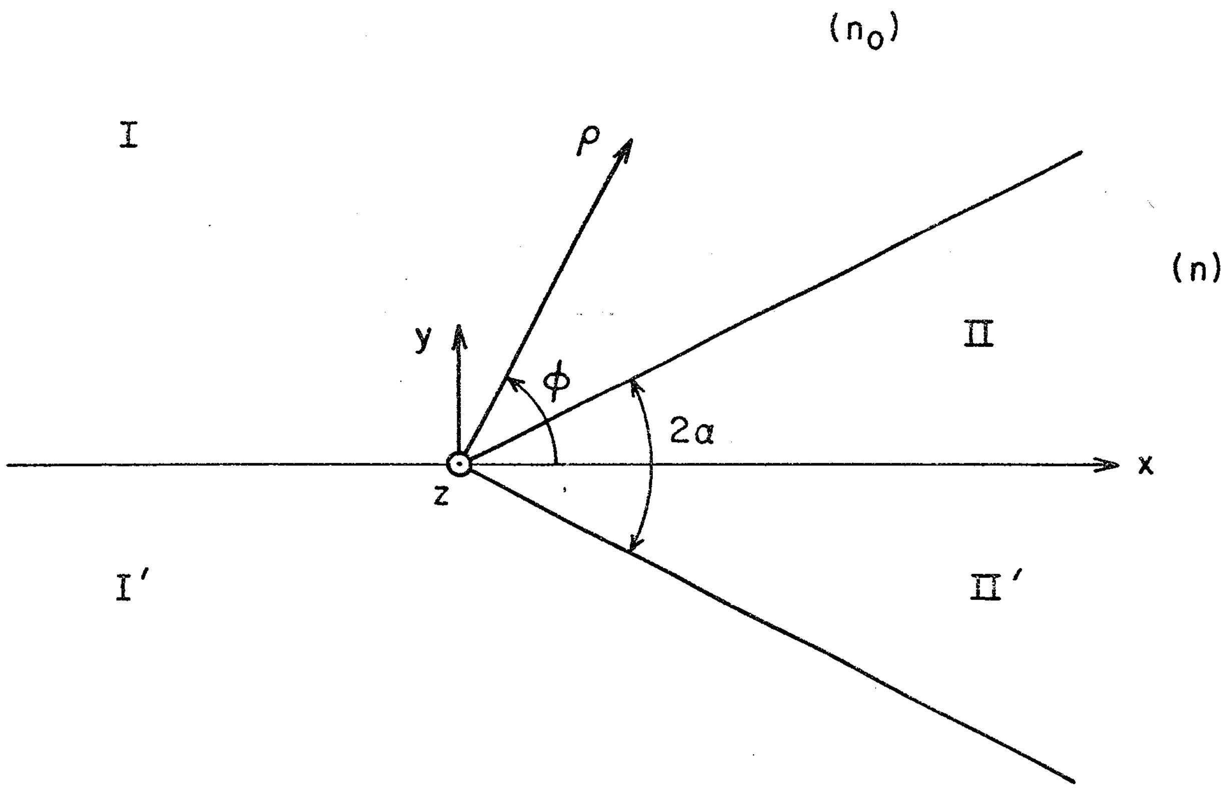

Fig. 2.1 Geometry of Dielectric Wedge

Figure 2.1 shows a rectangular coordinate system with the x-axis bisecting the wedge and the z-axis along its tip. A cylindrical coordinate system ρ, ϕ, z is also erected, with $\phi = \pm\alpha$ at the interfaces.

The total region is divided into four, as follows:

Region I	$\alpha < \phi < \pi$
II	$0 < \phi < \alpha$
I'	$\pi < \phi < 2\pi - \alpha$
II'	$-\alpha < \phi < 0$

In view of the chosen symmetry of the incident wave the fields in Regions I' and II' will be mirror images of those in I and II, and can be allowed for by replacing $(\pi - \phi)$ by $|\pi - \phi|$ and ϕ by $|\phi|$ respectively. However, this does not automatically ensure field matching on the axis, since H_ρ, proportional to $-(1/\rho)\partial E_z/\partial\phi$ will in general be discontinuous on axis due to the presence of the modulus terms in the azimuth angle. However, by requiring $\partial E_z/\partial\phi$ to be zero on axis, corresponding to $H_\rho = 0$ there, as required by symmetry, the necessary continuity is obtained. The reason for this choice, rather than for the more usual one of selecting functions of $\cos\phi$, which are automatically symmetrical on axis, stems from the form taken by the fields at $\phi = \pm\alpha$. To enable matching in a form that is mathematically tractable it seems necessary to work with the above formulation.

2.3 FIELD EXPRESSIONS IN REGION I

Throughout this paper we take w as a spectral variable and build solutions in region I from the basic plane wave

$$E_z = e^{-\rho[(w^2-n_o^2)^{\frac{1}{2}}\sin(\phi-\alpha) - jw\cos(\phi-\alpha)]} \tag{1}$$

In this expression ρ is the radial distance from the tip, normalized with respect to the free-space wave number k_o, i.e. $\rho = k_o\rho_o$ where ρ_o is the actual radial distance. The exponential is put in this form, one of many possible, in anticipation of eventual field matching at $\phi = \alpha$.

As mentioned earlier, n_o contains a small loss term, and we put

$$n_o = |n_o|e^{-j\delta} \simeq |n_o|\ (1-j\delta) \tag{2}$$

Equation (1) has branch cuts at $w = \pm n_o$, and in order to avoid divergent waves at infinity that branch of $(w^2 - n_o^2)^{\frac{1}{2}}$ has to be chosen to give damped waves as $w \to \pm\infty$. This leads as one possibility to the branch cuts shown in Figure 2.2a, but there is some latitude as to where the cuts go.

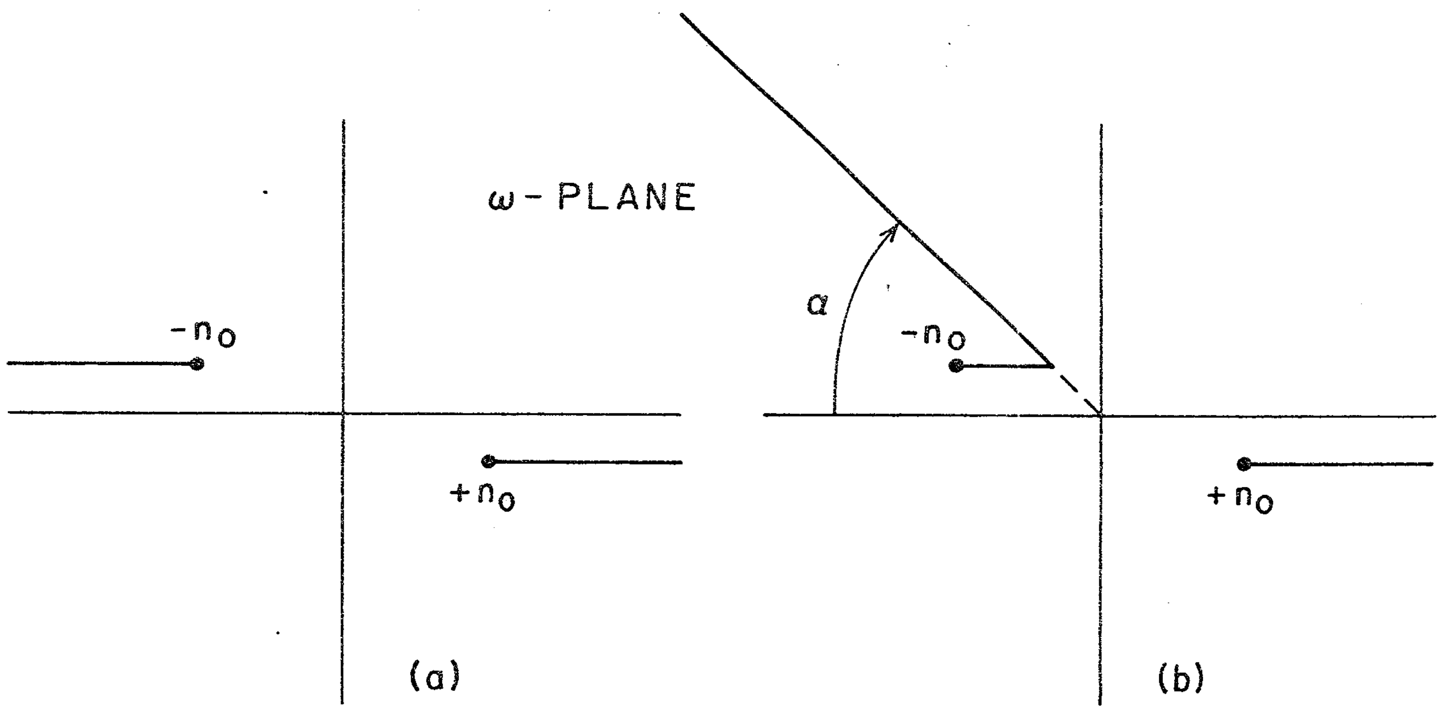

Fig. 2.2 Branch Cuts in w-plane

Since $\alpha < \phi < \pi$ in region I it is found that if $w = Re^{j\beta}$ as $R \to \infty$ that $0 < \beta < \alpha$ is a possible range, whilst if $w = Re^{j\beta}$ as $R \to -\infty$, then $-\alpha < \beta < 0$ is a possible range for β. It turns out from other considerations that a branch cut at $\beta = -\alpha$ is needed at $w \to -\infty$ and this leads to the modified branch cut shown in Figure 2.2b. For the time being we take w exactly on the real axis and get, for the scattered electric field in region I

$$E_z = \int_{-\infty}^{\infty} f(w) e^{-\rho[(w^2-n_o^2)^{\frac{1}{2}}\sin(\phi-\alpha)-jw\cos(\phi-\alpha)]} dw \qquad (3)$$

with f(w) an undetermined spectral function. Before proceeding we need to check both for completeness and performance at infinity. As far as the latter is concerned, this is covered by the convergence of (3) as $\rho \to \infty$ due to the choice of branch cut. However, individual plane wave components need not, in isolation, satisfy the radiation condition, which is a restraint on the total field only. Specifically, when $-|n_o| < w < |n_o|$ we can put $w = |n_o| \cos\theta$ with $0 < \theta < \pi$, and an individual term in (3) becomes like a plane wave* travelling at angle $\phi = \pi + \alpha - \theta$. When $0 < \theta < \alpha$ this gives waves at angles greater than π, which would not, in the absence of attenuation, be acceptable in isolation. The fact that (3) is integrated to $\pm\infty$ ensures in any case the correct performance of the total field, as is eventually confirmed when the fields are found.

As far as completeness is concerned, it might appear that a term similar to (3) but with the sign of $(w^2 - n_o^2)^{\frac{1}{2}}$ reversed would also be needed, since such a term would also satisfy the wave equation. However, for $w^2 > |n_o|^2$ the wave would diverge, whilst for $-|n_o| < w < |n_o|$ the wave components would represent incoming waves at angles from $\pi + \alpha$ to $2\pi + \alpha$; not only are they physically irrelevant when taken in isolation, but they turn out to have inward attenuation when w is real, i.e. they are incoming waves from infinity, and form no part of the solution.

An incident field on axis of unit amplitude is simply $e^{-j\rho n_o \cos\phi}$; whence we get the total fields in $\alpha < \phi < \pi$,

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*The δ term in n_o ensures that these waves are attenuated outward when w is real.

$$E_z = e^{-j\rho n_o \cos\phi} + \int_{-\infty}^{\infty} e^{-\rho[(w^2-n_o^2)^{\frac{1}{2}}\sin(\phi-\alpha)-jw\cos(\phi-\alpha)]} f(w)\,dw \tag{4}$$

$$-\frac{1}{\rho}\frac{\partial E_z}{\partial\phi} = -jn_o\sin\phi\; e^{-j\rho n_o\cos\phi}$$
$$+\int_{-\infty}^{\infty} e^{-\rho[(w^2-n_o^2)^{\frac{1}{2}}\sin(\phi-\alpha)-jw\cos(\phi-\alpha)]} f(w)\,[(w^2-n_o^2)^{\frac{1}{2}}\cos(\phi-\alpha)$$
$$+ jw\sin(\phi-\alpha)]\,dw \tag{5}$$

(The latter expression is proportional to H_ρ).

2.4 FIELD EXPRESSIONS IN REGION II

The analysis follows, in part, the considerations outlined in section 2.3 with $(w^2-n^2)^{\frac{1}{2}}$ replacing $(w^2-n_o^2)^{\frac{1}{2}}$; but there are some important differences. In region II the waves corresponding to reversing the sign of $(w^2-n^2)^{\frac{1}{2}}$ are not only acceptable, from the point of view of convergence at infinity; they are also required, and in fact represent those waves incident in region II from region II', where they are generated by refraction at the boundary at $\phi = -\alpha$ and cross the axis into region II. Accordingly we get, for $0 < \phi < \alpha$

$$E_z = \int_{-\infty}^{\infty} e^{-\rho[(w^2-n^2)^{\frac{1}{2}}\sin(\alpha-\phi)-jw\cos(\alpha-\phi)]} g_1(w)\,dw$$
$$+ \int_C e^{\rho[(w^2-n^2)^{\frac{1}{2}}\sin(\alpha-\phi)+jw\cos(\alpha-\phi)]} g_2(w)\,dw \tag{6}$$

$$-\frac{1}{\rho}\frac{\partial E_z}{\partial\phi} = \int_{-\infty}^{\infty} e^{-\rho[(w^2-n^2)^{\frac{1}{2}}\sin(\alpha-\phi)-jw\cos(\alpha-\phi)]} g_1(w)\,[-(w^2-n^2)^{\frac{1}{2}}\cos(\alpha-\phi)$$
$$-jw\sin(\alpha-\phi)]\,dw \;+$$

$$+ \int_C e^{\rho[(w^2-n^2)^{\frac{1}{2}}\sin(\alpha-\phi)+jw\cos(\alpha-\phi)]} g_2(w)\,[(w^2-n^2)^{\frac{1}{2}}\cos(\alpha-\phi)$$

$$-jw\sin(\alpha-\phi)]\,dw \qquad (7)$$

Herein g_1 and g_2 are two spectral functions, which we may expect to be simply related to each other in view of the fact that the two integrals represent equal downward and upward waves respectively within the wedge. The contour for g_1 is the real-axis, but that for g_2 is not, since its integrand does not converge (for $0 < \phi < \alpha$) at $w = \pm\infty$ on axis. In fact we need $w \sim Re^{j\alpha}$ and $-Re^{-j\alpha}$ as $R \to \infty$ in order to give an integrand which has, for the range $0 < \phi < \alpha$, a non-positive real part to the exponent. We shall later choose this contour to correspond to a purely imaginary exponent at $\phi = \alpha$, but it is sufficient at this stage to note that suitable contours exist, and that the left-hand branch cut must be tilted upwards (at infinity) by an angle α to accommodate this feature.

2.5 FIELD MATCHING ON AXIS AT $\phi = \pi$

As mentioned earlier, the field in I' is obtained from I by replacing $(\pi-\phi)$ by $|\pi - \phi|$. At $\phi = \pi$ this does not in any way affect E_z, which is therefore continuous at the boundary; but $\partial/\partial\phi$ changes sign when operating on $|\pi - \phi|$ as ϕ goes through π. Since $\partial E_z/\partial\phi$ is proportional to H_ρ, which in the symmetrical case vanishes at $\phi = \pi$ anyway, it is clear that matching on the axis requires $\partial E_z/\partial\phi = 0$ at $\phi = \pi$. From (5) with $\phi = \pi$ we thus get

$$0 = \int_{-\infty}^{\infty} e^{-\rho U} f(w)[(w^2-n_o^2)^{\frac{1}{2}}\cos\alpha - jw\sin\alpha]dw \qquad 0 < \rho < \infty \quad (8)$$

where

$$U = (w^2-n_o^2)^{\frac{1}{2}}\sin\alpha + jw\cos\alpha \qquad (9)$$

Now $\partial U/\partial w = j(w^2-n_o^2)^{-\frac{1}{2}}[(w^2-n_o^2)^{\frac{1}{2}}\cos\alpha - jw\sin\alpha]$, and if we define a new function F by

$$f(w)(w^2-n_o^2)^{\frac{1}{2}} = F(U) \qquad (10)$$

then (8) can be written

$$\int_{w=-\infty}^{w=+\infty} e^{-\rho U} F(U)\ dU = 0 \qquad 0 < \rho < \infty \quad (11)$$

It is clear, therefore, that (11) can be satisfied by choosing F(U), qua function of U, to be free of singularities in the region Re $U > 0$. Moreover, convergence of (11) as $\rho \to 0$ requires $F(U) = O(|U|^{-1})$ as U approaches infinity along the contour C_1 in the U-plane corresponding to the real axis of w. This contour is shown in figure 2.3;it starts at $U = \infty e^{-j(\pi/2-\alpha)}$, comes in close to the imaginary U-axis, rises to a maximum at $U = n_o$ and then drops to the right before rising asymptotically to $U = \infty e^{j(\pi/2-\alpha)}$. The closeness of approach to the imaginary axis is proportional to δ. Also shown on the contour is the line

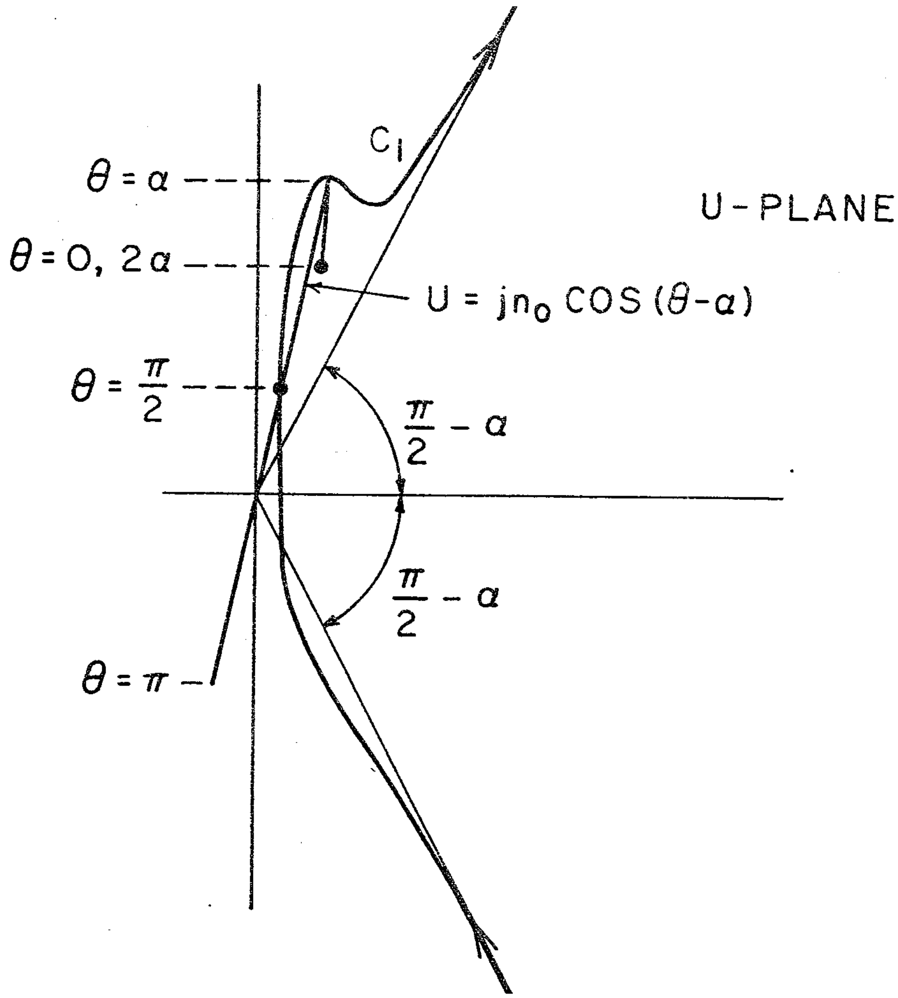

Fig. 2.3 Contour C_1 in the U-plane

$U = jn_o \cos(\theta-\alpha)$ corresponding to $w = n_o \cos\theta$. This intersects the contour C_1 at $\theta = \frac{\pi}{2}$ and (almost) at $\theta = \alpha$; and between these positions the line $w = n_o \cos\theta$ corresponds to values of U to the right of the contour. Since we shall later be concerned with poles at various positions along $w = n_o \cos\theta$ it is clear that poles for $0 < \theta < \pi/2$ cannot be permitted. It is also clear that F(U) free of singularities for Re $U < 0$ is unnecessarily restrictive; it is only necessary for it to be analytic between the contour C_1 and infinity.

It should be noted that F(U) analytic qua U is not the same as f(w) analytic qua w. In fact such a function as $F(U) = U$ is clearly analytic in U but because of (9) it has branch cuts at $w = \pm n_o$.

Around the point $w = n_o\cos(\alpha+\psi)$ we have $U = jn_o\cos\psi$ which is therefore symmetric about $\psi = 0$. Accordingly, if the left hand part of the real-axis contour in the w-plane is swung around the point $w = n_o\cos\alpha$, in fact to be asymptotic to the line $w = \infty e^{-j2\alpha}$, the corresponding contour in the U-plane can be made to collapse to a curve doubling back on itself. Figures 2.4a & 2.4b

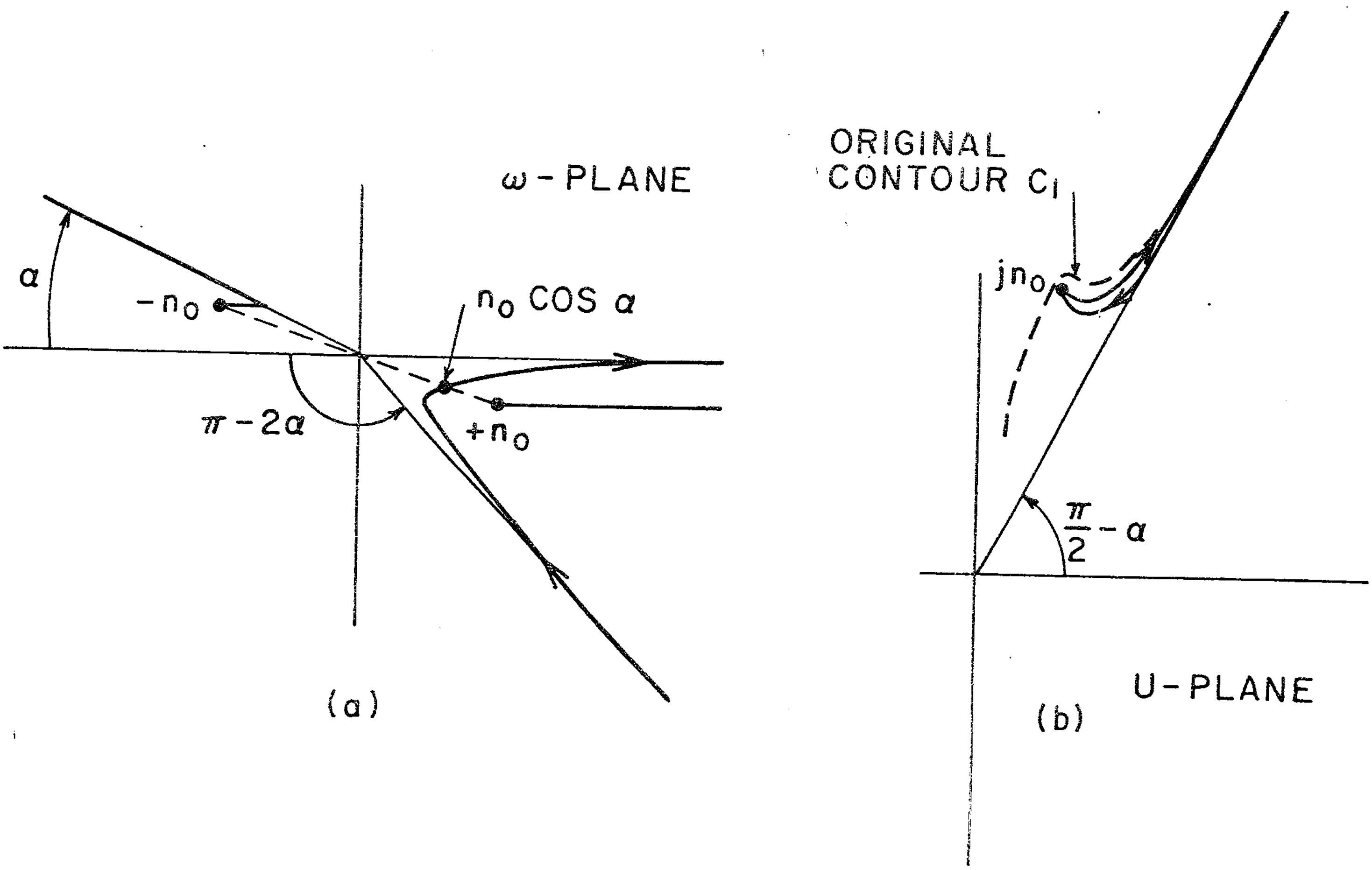

Fig. 2.4 Contours in the w and U-planes

outline the corresponding contours and show that there is a region in Im $w < 0$ which is outside the region which gives the contour C_1. Since the contour of Figure 2.4b lies within the contour C_1 it is apparent that (8) is satisfied for $F(U)$ analytic, qua U, between C_1 and infinity.

2.6 FIELD MATCHING ON AXIS AT $\phi = 0$

In the same way as described in section 2.5 we require $\partial E_z/\partial\phi$ to be zero at $\phi = 0$. If we write

$$W, W^* = \pm (w^2-n^2)^{\frac{1}{2}}\sin\alpha - jw\cos\alpha \qquad (12)$$

then (7) gives

$$0 = \int_{w=-\infty}^{w=+\infty} g_1(w)(w^2-n^2)^{\frac{1}{2}}e^{-\rho W}dW - \int_{w=C} g_2(w)(w^2-n^2)^{\frac{1}{2}}e^{-\rho W^*}dW^* \qquad (13)$$

$$0 < \rho < \infty$$

The contour w real gives the contour C_2 for W, as shown in figure 2.5. The contour C for w in the g_2 integral is somewhat at choice. The larger Im w is on this contour the larger is the real part of W^*. There is a lowest contour, shown in figure 2.6 for which Re $W^* = 0$ and C cannot be taken below this if the second integral in (13) is to be convergent. A little later it will be seen that we wish to make the two contours in the W and W^* plane as close as possible, so we choose $w = C$ to give W^* imaginary on it. If we further choose a new function G such that*

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* This choice is not mandatory, but the method of this section is dependent on it.

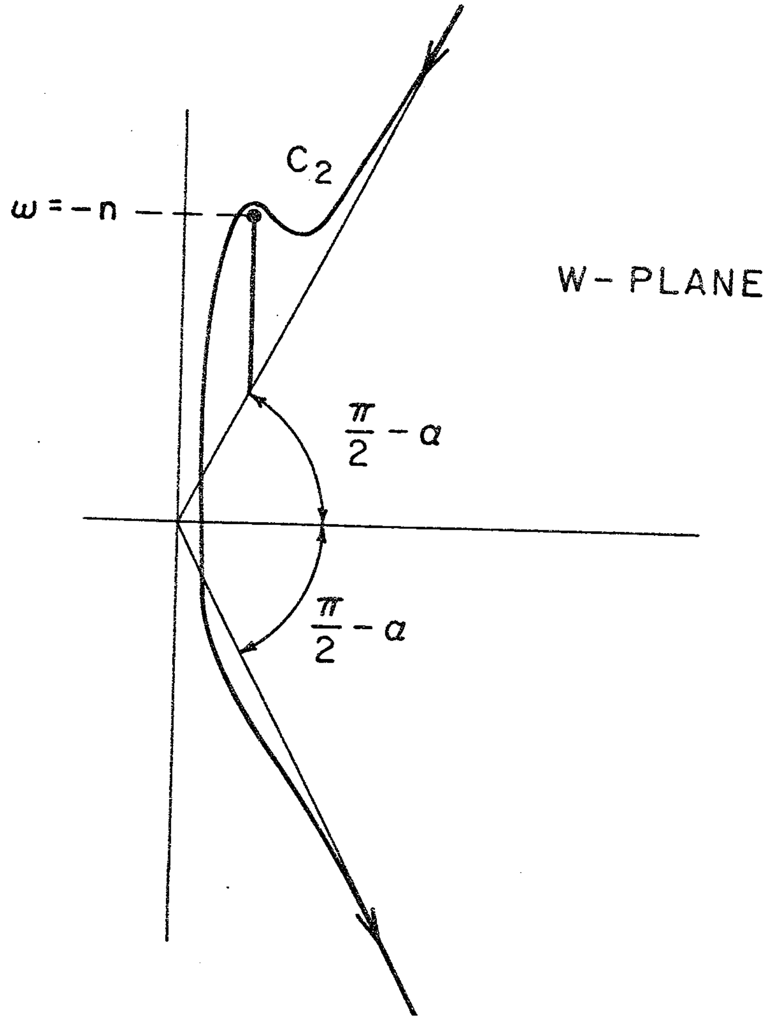


Fig. 2.5 Contour C_2 in the W-plane

$$g_1(w)(w^2-n^2)^{\frac{1}{2}} = G(W), \quad g_2(w)(w^2-n^2)^{\frac{1}{2}} = G(W^*) \tag{14}$$

then (13) becomes

$$\int_{C_2} G(W)e^{-\rho W}dW + \int_{-j\infty}^{j\infty} G(W^*)e^{-\rho W^*}dW^* = 0 \qquad 0 < \rho < \infty \tag{15}$$

Now in (15) both W and W* are functioning merely as dummy variables, so in the second integral we can <u>replace</u> W* by W.

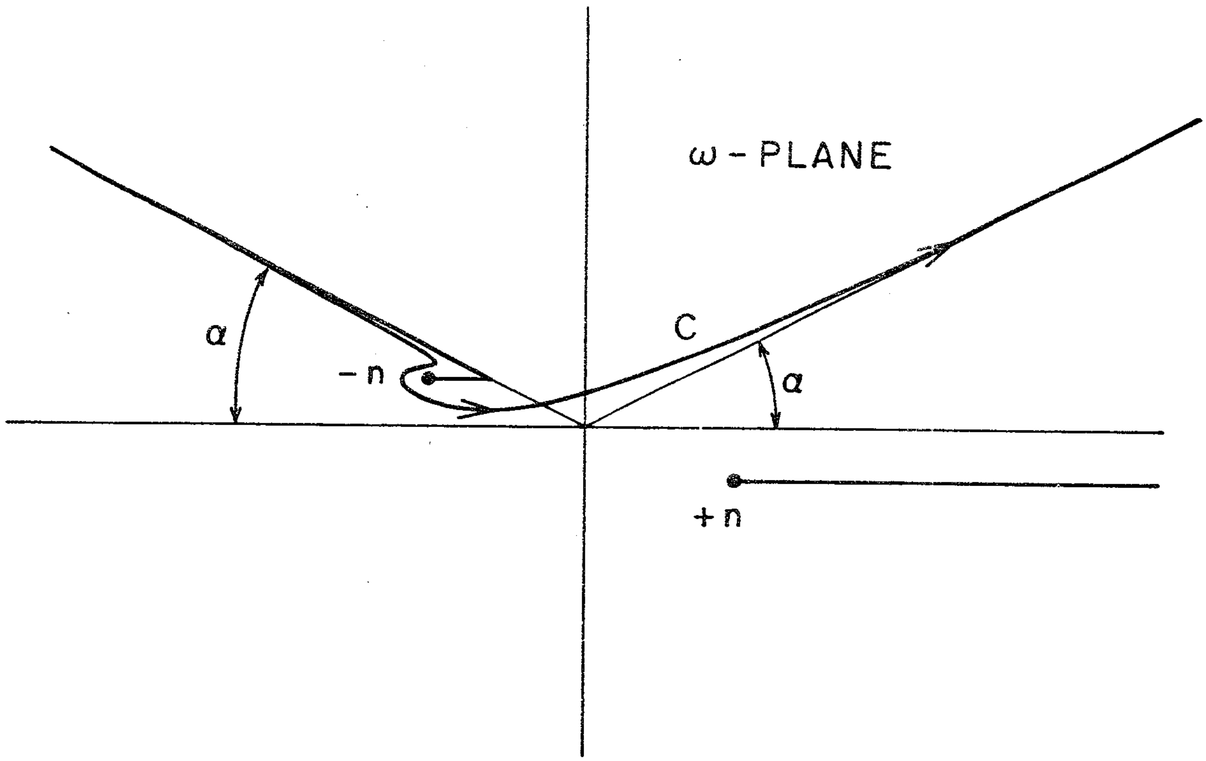


Fig. 2.6 Contour C in the w-plane

Moreover the contour C_2 can be joined to the imaginary axis by arcs at infinity (which give zero on integration), to produce the closed contour C_3 of figure 2.7 consisting of (1) the imaginary axis, (2) arcs at $\pm j\infty$, (3) the contour C_2 of figure 2.5. Equation (15) can now be written simply

$$\int_{C_3} G(W)e^{-\rho W}dW = 0 \qquad 0 < \rho < \infty \qquad (16)$$

and it was in order to achieve this form that g_1 and g_2 were chosen to be related as in (14). Equation (16) is satisfied if G(W) qua W is analytic in C_3. Also shown in figure 2.7 is the line

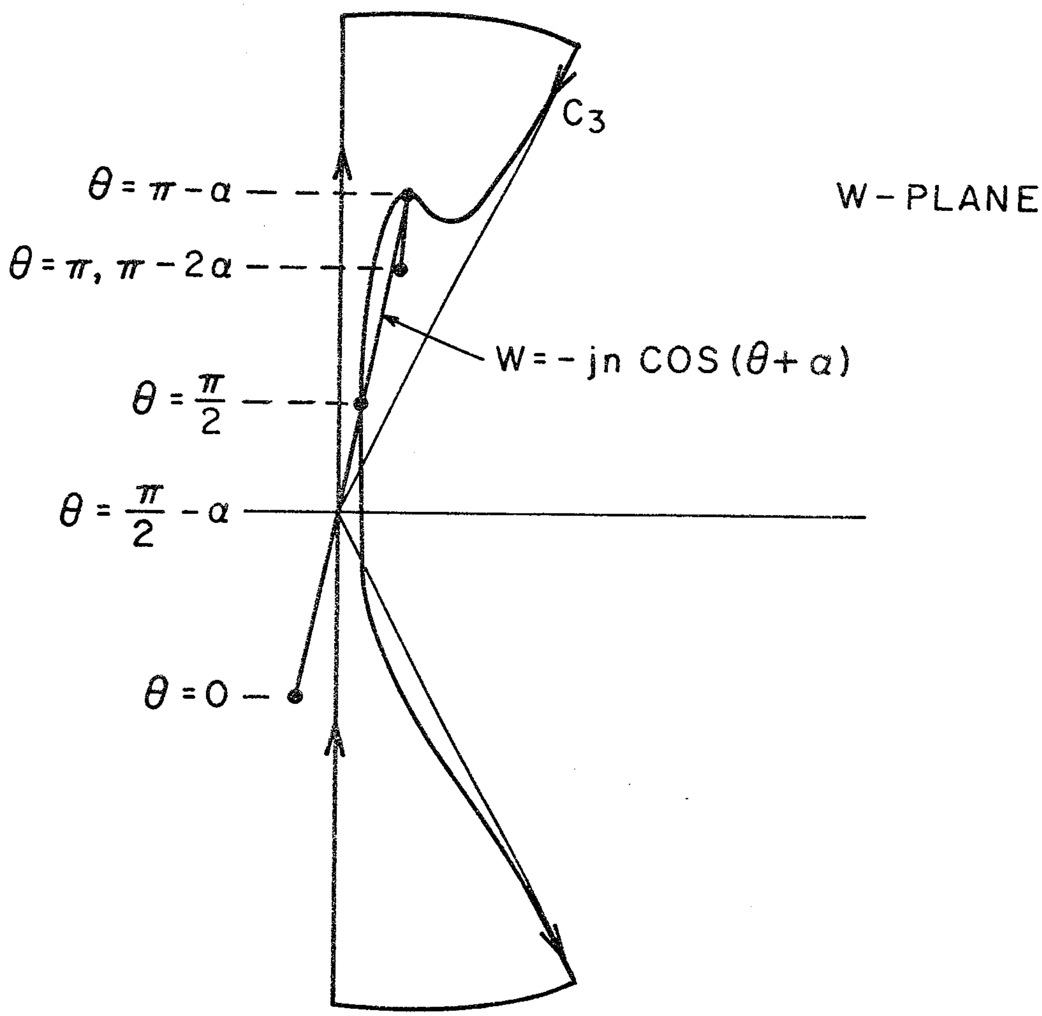


Fig. 2.7 Contour C_3 in the W-plane

$W = -jn\cos(\theta + \alpha)$ corresponding to $w = n\cos\theta$. It lies inside the contour when $\pi/2 - \alpha < \theta < \pi/2$, a range for which $G(W)$ is not permitted to possess singularities. Since we shall later be concerned with poles along the line $w = n\cos\theta$, $0 < \theta < \pi$, the range $\pi/2 - \alpha < \theta < \pi/2$ must be excluded if (16) is to be satisfied.

2.7 FIELD MATCHING AT $\phi = \alpha$

From (4) the incident field at $\phi = \alpha$ varies as $e^{-jn_o\rho\cos\alpha}$ and before we can field match it is necessary to put this in the form of an integral compatible with the others. This is achieved through writing

$$e^{-jn_o\rho\cos\alpha} = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{e^{j\rho w}}{w+n_o\cos\alpha}\, dw \tag{17}$$

a result that is readily verified by deforming the contour into $\mathrm{Im}\, w > 0$, whence the pole at $w = -n_o\cos\alpha$ provides the necessary residue. This is only valid when $\alpha < \pi/2$, but since we are later restricting α to be not greater than $\pi/4$ the constraint is of no consequence. Accordingly, with $\phi = \alpha$, equating (4) to (6) gives

$$\frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{e^{j\rho w}}{w+n_o\cos\alpha}\, dw + \int_{-\infty}^{\infty} f(w)e^{j\rho w}\, dw = \int_{-\infty}^{\infty} g_1(w)e^{j\rho w}\, dw$$

$$+ \int_C g_2(w)e^{j\rho w}\, dw \tag{18}$$

Before proceeding further it is necessary to deform C to the real w-axis. This could not be done before because with the more general exponent for $0 < \phi < \alpha$ the integrand was not convergent; but with $e^{j\rho w}$ in the integrand it can be done, together with non-contributing arcs at $\pm\infty$. Now if $g_2(w)$ has any singularities between the real w-axis and the contour C they will give rise to a contribution on collapsing C to the real axis. To prevent this, write $g_2 = \bar{g}_2 + g_s$ where g_s is that part of g_2 carrying singularities (poles or branch cuts) between C and w real, and is free from singularities above C. Then by deforming C to $+j\infty$ we see that $\int_C g_s(w)e^{j\rho w}dw = 0$ and (18) can be written

$$\int_{-\infty}^{\infty} e^{j\rho w}[f(w)-g_1(w)-\bar{g}_2(w)+(1/2\pi j)(w+n_o\cos\alpha)^{-1}]dw = 0$$

$$0 < \rho < \infty \qquad (19)$$

If we define a plus function $h_+(w)$ to be free of singularities for $\text{Im}\, w > 0$ then (19) can be written in the form

$$f(w) + \frac{1}{2\pi j}\left[\frac{1}{w+n_o\cos\alpha} - h_+(w)\right] = g_1(w) + \bar{g}_2(w) \qquad (20)$$

Since $w = -n$ is above the C-contour the presence of branch cuts at this position doesn't affect $\bar{g}_2(w)$. On putting f and g in terms of F and G through (10) and (14), equation (20) becomes

$$\frac{F(U)}{(w^2-n_o^2)^{\frac{1}{2}}} + \frac{1}{2\pi j}\left[\frac{1}{w+n_o\cos\alpha} - h_+(w)\right] = \frac{G(W)+\bar{G}(W^*)}{(w^2-n^2)^{\frac{1}{2}}} \qquad (21)$$

Here, $\bar{G}(W^*)$ comes from $G(W^*)$ by dropping those singularities, if any, that occur in the w-plane between C and the real axis. If a similar process is used to match H_ρ at $\phi = \alpha$ the following relationship is obtained

$$-F(U) + \frac{n_o\sin\alpha}{2\pi}\left[\frac{1}{w+n_o\cos\alpha} - k_+(w)\right] = G(W) - \bar{G}(W^*) \qquad (22)$$

where k_+ is a similar function to h_+. Equations (21) and (22) are the basic relations from which G and F can, in principle, be found, and which then determine the fields via (4) and (6).

2.8 POLE CONTRIBUTIONS TO THE SOLUTION

Equations (21) and (22) both exhibit a pole at $w = -n_o\cos\alpha$. This is at a position where, if $g_2(w)$ (or $G(W^*)$) possessed a pole it would have to be subtracted out to form $\bar{g}_2(w)$. Hence $\bar{G}(W^*)$ in (21) and (22) cannot contribute such a pole, and only $F(U)$ and $G(W)$ can do so. That both must do so follows at once from the different ways in which F, G and the pole enter into the two equations.

Let U_o and W_o be the values of U and W at $w = -n_o\cos\alpha$. It is easily found that

$$U_o = -jn_o\cos 2\alpha \tag{23}$$

$$W_o = jn\cos(\theta_o - \alpha),\ \text{where } n\cos\theta_o = n_o\cos\alpha \tag{24}$$

Since, from (24), $\alpha < \theta_o < \pi/2$ it can be seen from figures 2.3 and 2.7 that U_o and W_o are in regions where F and G may possess poles, and so we put, in the neighbourhood of these poles

$$F(U) \sim \frac{A}{U-U_o} \quad , \quad G(W) \sim \frac{B}{W-W_o} \tag{25}$$

where A and B are constants to be determined. The residues at $w = -n_o\cos\alpha$ are readily found from (25) and the definitions of U and W in terms of w, and give

$$\frac{A}{U-U_o} \sim \frac{A/2j\cos\alpha}{w+n_o\cos\alpha} \tag{26}$$

$$\frac{B}{U-U_o} \sim \frac{B/2j\cos\alpha}{w+n_o\cos\alpha} \cdot \frac{2n\sin\theta_o}{n_o\sin\alpha - n\sin\theta_o} \tag{27}$$

Hence, to match the poles at $w = -n_o\cos\alpha$ in (21) and (22), since they cannot be provided in that position by h_+ and k_+, we find, after a little simplification, that A and B must satisfy the two equations

$$\frac{A}{n_o\sin\alpha} - \frac{2\cos\alpha}{2\pi j} = \frac{2B}{n_o\sin\alpha - n\sin\theta_o} \tag{28}$$

$$A + \frac{2n_o\sin\alpha\cos\alpha}{2\pi j} = \frac{-2Bn\sin\theta_o}{n_o\sin\alpha - n\sin\theta_o} \tag{29}$$

with solution

$$A = B = \frac{n_o\cos\alpha\sin\alpha}{\pi j} \cdot \frac{n\sin\theta_o - n_o\sin\alpha}{n\sin\theta_o + n_o\sin\alpha} \tag{30}$$

Unfortunately, with G(W) given by (25), it is implied that $G(W^*) \sim B/(W^* - W_o)$ and this introduces an additional pole into (21) and (22) at the value of w for which $W^* = W_o$, i.e. at $w = -n\cos(\theta_o - 2\alpha)$. This pole requires further additional poles in F and G to satisfy (21) and (22), with still further additional sets of poles arising in the same way. Apparently this series does not, as had at first been supposed, terminate in anyway, and so both this method of solution, and also, apparently, equations (21) and (22), must be considered faulty. The reason for this is not known, but may be tied up in the choice implicit in (14). However, without this choice there seems to be no obvious way to proceed beyond the setting up of the initial equations, and the method fails.

2.9 CONCLUSIONS

The spectral analysis has led to the two relations (21) and (22), which bear a superficial resemblance to the more familiar Wiener-Hopf equations. However, an examination of their pole behavior suggests that these relations are faulty and rather reluctantly the method has had to be abandoned.

A CRITIQUE OF
ZAVADSKII'S METHOD OF SOLUTION TO DIFFRACTION
PROBLEMS INVOLVING A RECTANGULAR DIELECTRIC WEDGE

ABSTRACT

A thorough investigation of Zavadskii's method is made in an attempt to obtain solution to electromagnetic diffraction problems involving a rectangular dielectric wedge ($0 \geq \phi \geq \pi/2$) and (i) infinite metal plate along $\phi = \pm\pi/2$, (ii) semi-infinite metal plate along $\phi = -\pi/2$, (iii) perfect magnetic conductor along $\phi = -\pi/2$ and a semi-infinite metal plate along $\phi = \pi/2$. In all the cases it is shown that Zavadskii's method, as it is, gives a solution containing branch cut integrals that grow exponentially in the farfield thus violating the radiation condition. For case (i) involving an infinite metal plate a simple way of modifying Zavadskii's solution is shown so that the resulting solution conforms with the known exact solution. Several modifications to Zavadskii's method are tried to obtain the correct solution to cases (ii) and (iii) but none of them proved to be successful. A method involving a secondary solution, with branch cut integrals alone, is shown to lead to two coupled integral equations. However it is not clear if a solution to these integral equations exists. Finally a unique solution is given to the problem of illuminating a rectangular dielectric wedge, resting on a semi-infinite metal plate, such that there is no net diffracted wave from the edge.

TABLE OF CONTENTS FOR SECTION 3

3.1 INTRODUCTION

The problem of wave diffraction by a perfectly conducting metallic wedge has been solved [1,2]. The analogous problem of diffraction by a wedge with impedance faces has also been solved [3-10]. A generalization of this solution to the case of a dielectric wedge, either free or resting on a semi-infinite metal plate, entails serious mathematical difficulties.

Apart from being a classical boundary value problem the diffraction of electromagnetic waves by a dielectric wedge is of particular interest in the theory of dielectric wave guide matching [11, 12], radio propagation over the earth [13,14], and in radar, for the effect of scattering by dielectric radomes [15]. An analogous problem to that of the dielectric wedge is encountered in the field of acoustics [16] and in seismological situations involving the behavior of Rayleigh waves at the boundary between the ocean and the earth [17, 18]. It is not surprising, then, that considerable research effort has been directed towards the problem of diffraction by a dielectric wedge [19-30].

An understanding of the diffraction by a dielectric wedge resting on a semi-infinite metal plate is of importance in assessing the effects of dielectric supports for a wave guide feed illuminating a reflector antenna. A number of years ago Zavadskii [31] proposed a method, which, he claimed, would give exact analytic solution to a class of two-dimensional wedge diffraction problems. However the solution obtained by his method contains branch cut integrals which he does not evaluate. Upon examining these branch cut integrals we found them to "explode" at infinity in complete violation of the radiation condition.

Several attempts were made to modify his approach so as to remove this drawback but did not meet with success. In this report, after giving a brief summary of Zavadskii's formulation we proceed to discuss various modifications that were tried to circumvent the failure of his approach. We also present the solution to what we call, a "quasi-trivial" problem, that of illuminating, a rectangular dielectric wedge resting on a semi-infinite metal plate, such that there is no diffracted wave from the edge. Though most of our discussion will concentrate on the problem of diffraction by a rectangular dielectric wedge resting on a perfectly conducting metal plate, we will also touch up on other related problems to which Zavadskii's method might be applicable.

It is well known that the problem of diffraction in a wedge with perfectly conducting faces was solved a long time ago by Sommerfeld [2]. Malyuzhinets [3-7] has proposed and developed a method of solving diffraction problems in angular regions, with application to a wedge of arbitrary apex angle with ideal impedance faces, to sectored media representing a system of wedges with a common edge and common faces. This method is based on the representation of the field in a dielectric medium by the Sommerfeld integral and it reduces the diffraction problem to functional equations for the integrands. But solutions to these functional equations could be obtained only in very special cases. To circumvent this difficulty Zavadskii introduces the generalized, two-sided Laplace transform which we denote as the t-transform. Upon application of the t-transform to the functional equations, and after some algebraic manipulation one obtains a functional equation, with periodic coefficients in the transform domain, which is amenable to algebraic solution. In the next section we give a brief summary of Zavadskii's formulation and in the subsequent three sections we proceed to discuss the various approaches that were tried to obtain a correct solution to the diffraction problems involving the following geometries:

i. Rectangular dielectric wedge resting on an infinite metal plate.

ii. Rectangular dielectric wedge resting on a semi-infinite metal plate.

iii. A mixed boundary value problem involving a rectangular dielectric wedge ($0 \geq \phi \geq -\pi/2$) with a metal plate along $\phi = \pi/2$ and a perfect magnetic conductor along $\phi = -\pi/2$.

Conclusions are given in section 3.6.

3.2 ZAVADSKII'S METHOD

3.2.1 Statement of the problem

We begin with the following problem in cylindrical co-ordinates. Consider two homogeneous sectoral media which have a common edge $(\rho = 0)$ and a common face $(\phi = 0)$, each occupying respective wedges with apex angles ϕ_0 and ϕ_1, as shown in Fig. 3.1. The remaining outer faces of the wedge are resting against perfectly conducting metal plates. To

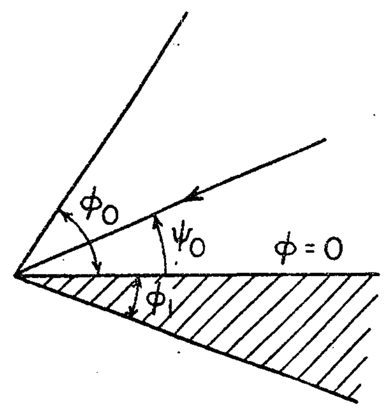


Figure 3.1. Geometry of dielectric wedge diffraction problem.

simplify matters we assume that one of the media is free space and the second medium has a magnetic permeability equal to that of free-space and a refractive index of $n > 1$. Later it will be necessary to restrict ϕ_1 to be a multiple of $\frac{\pi}{2}$ but at present we will assume ϕ_1 to be arbitrary. We assume that a monochromatic, z-polarized plane wave of unit magnitude is incident on the dielectric wedge at an angle ψ_0 with $0<\psi_0<\phi_0$. Then the diffracted electric field will also be polarized in the z-direction and we have a two-dimensional scalar problem.

We use the scalar functions $E(\rho, \phi)$ and $E_1(\rho,\phi)$ where $\rho = kr$ to represent the total electric fields in the free space and the dielectric medium respectively. The fields E and E_1 must satisfy the wave equations in their respective regions, the boundary conditions at $\phi = 0$, ϕ_o, and $-\phi_1$, the radiation condition, and the edge condition. A mathematical description of these requirements is given below.

a) The wave equation:

$$(\Delta + k^2)E(\rho,\phi) = 0; \qquad 0 \leq \phi \leq \phi_0 \tag{1a}$$

$$(\Delta + k^2n^2)E_1(\rho,\phi) = 0; \qquad -\phi_1 \leq \phi \leq 0 \tag{1b}$$

where

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} \quad ; \quad k = \omega/c \tag{1c}$$

ω is the angular frequency and c is the velocity of light in free space.

b) The boundary conditions at $\phi = 0$, ϕ_o, and $-\phi_1$:

$$E(\rho,\phi_o) = 0 \tag{2a}$$

$$E_1(\rho,-\phi_1) = 0 \tag{2b}$$

$$E(\rho,0) = E_1(\rho,0) \tag{2c}$$

$$E'(\rho,0) = E_1'(\rho,0) \tag{2d}$$

where

$$E'(\rho,\phi) = \frac{\partial}{\partial \phi} E(\rho,\phi); \quad \text{and} \quad E_1'(\rho,\phi) = \frac{\partial}{\partial \phi} E_1(\rho,\phi) \tag{2e}$$

The conditions (2a) and (2b) follow from the requirement that the tangential electric field be zero on the metal plate and (2c) and (2d) follow from the requirement that the tangential field and the normal derivative of the field be continuous across the boundary.

c) The radiation condition:

$$\lim_{r\to\infty} r\left(\frac{\partial \tilde{E}}{\partial r} - jk\tilde{E}\right) = 0; \quad \text{for} \quad 0 < \phi < \phi_o \tag{3a}$$

$$\lim_{r\to\infty} r\left(\frac{\partial E_1}{\partial r} - jknE_1\right) = 0, \quad \text{for} \quad -\phi_1 < \phi < 0 \tag{3b}$$

where

$$\tilde{E}(\rho,\phi) = E(\rho,\phi) - E_0(\rho,\phi) \tag{3c}$$

and $E_0(\rho,\phi) = \exp[-j\rho\cos(\phi-\psi_0)]$ is the incident field, and the time reference, following Zavadskii, is taken as $e^{-j\omega t}$.

Physically the conditions (3a) and (3b) mean that all the reflected, refracted, and diffracted fields must be radially outgoing at infinity. A detailed discussion of radiation condition may be found in reference [32].

d) The edge condition [12, 33-35]:

The edge condition requires that the electrical and magnetic energy stored in any finite neighborhood of the edge must be finite; that is,

$$\int_V \{\varepsilon|E|^2 + \mu|H|^2\}dv \to 0 \tag{4}$$

as the volume v contracts to the neighborhood of the edge. For a smooth edge, which may be regarded as locally straight, the differential volume in (4) is $dv = rdrd\phi dz$. Then condition (4) states that in the neighborhood of the edge, none of the field components of (E,H) should grow more rapidly than $e^{-1+\tau}$ with $\tau>0$ as $r\to 0$.

3.2.2 Sommerfeld integrals

To obtain a solution satisfying (1a) - (2d) Zavanskii begins with the following representation for E and E_1.

$$E(\rho,\phi) = \frac{1}{2\pi j}\int_{\gamma+\phi} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha; \quad 0\le\phi\le\phi_0 \tag{5a}$$

$$E_1(\rho,\phi) = \frac{1}{2\pi j}\int_{\Gamma+\phi} e^{-j\rho n\cos(\phi-\zeta)} s_1(\zeta)\, d\zeta \quad -\phi_1\le\phi\le 0 \tag{5b}$$

where γ is the Sommerfeld contour of integration as shown in Fig.3.2. The shaded portions in the α-plane represent the regions where the real part of the exponent of the integrand in (5a) is negative and ensures convergence as α goes to infinity. The path of integration $\gamma+\phi$ shifts as ϕ is varied. $E(\rho,\phi)$ as given by (5a) clearly satisfies the wave equation since it represents an infinite sum of plane waves each of which satisfy the wave equation. Since the end points of the contour $\gamma + \phi$ lie in shaded regions an infinitesimal displacement of the contour does not change the value of the integral, which implies that

$$\frac{\partial^2}{\partial\phi^2} E(\rho,\phi) \equiv \frac{1}{2\pi j} \int_{\gamma+\phi} \left[\frac{\partial^2}{\partial\phi^2} e^{-j\rho\cos(\phi-\alpha)}\right] s(\alpha)\, d\alpha \tag{6}$$

The contour Γ is so chosen that under the transformation

$$\zeta(\alpha) = \cos^{-1} \frac{\cos\alpha}{n} \tag{7a}$$

with the choice of the branch such that

$$\zeta(\alpha) = -\zeta(-\alpha) \tag{7b}$$

and

$$\zeta(\alpha+\pi) = \zeta(\alpha) + \pi \tag{7c}$$

γ transforms to Γ in the ζ-plane.

In the α-plane E_1 is given by

$$E_1(\rho,\phi) = \int_{\gamma} s_1\left(\zeta(\alpha)+\phi\right) e^{-j\rho\cos\alpha}\tau(\alpha)\, d\alpha \tag{8a}$$

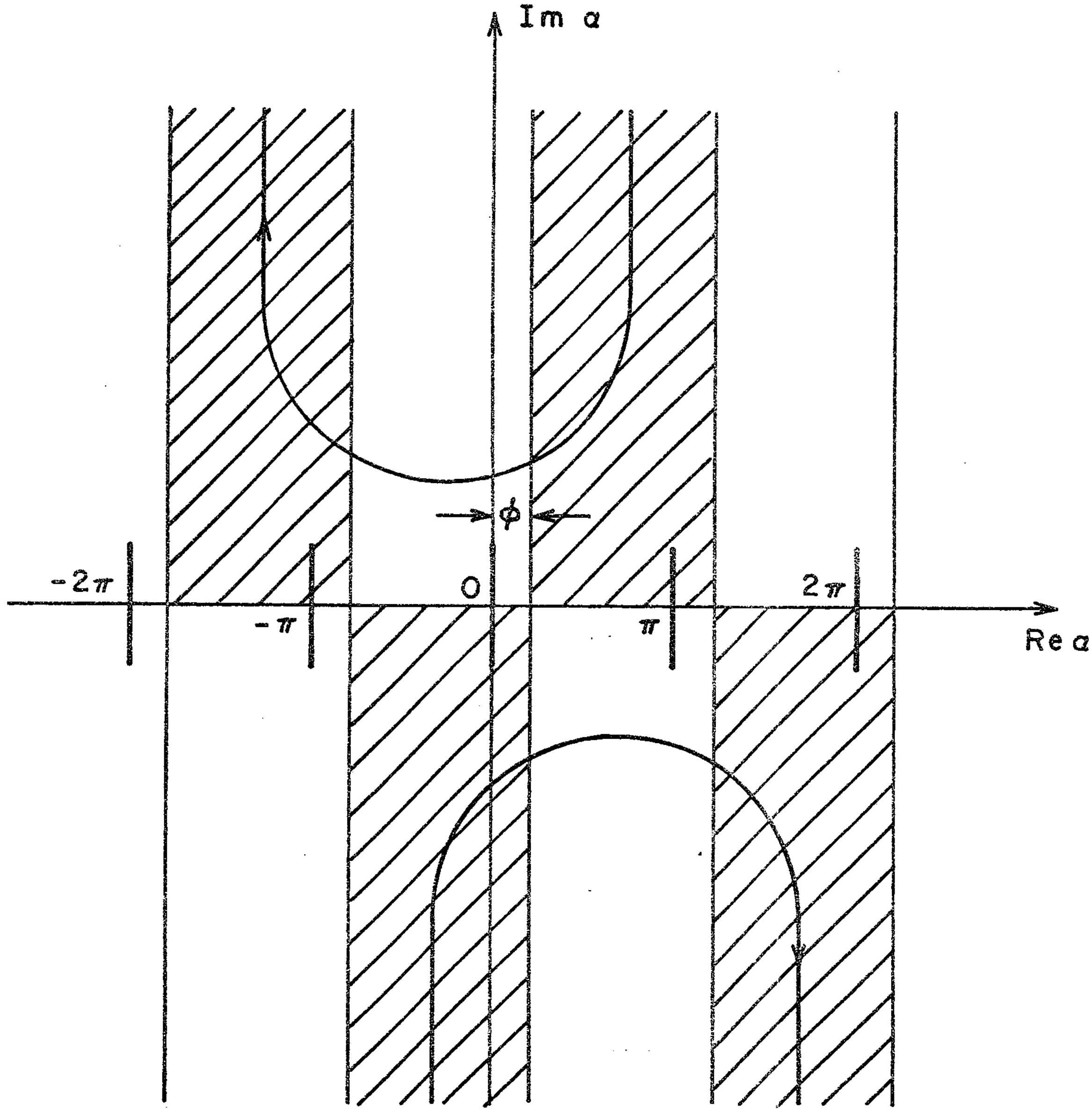


Fig. 3.2 Contour of integration $\gamma + \phi$ in the plane of the complex variable α.

where

$$\tau(\alpha) = \frac{d\zeta(\alpha)}{a\alpha} = \frac{\sin\alpha}{\sqrt{n^2-\cos^2\alpha}} \tag{8b}$$

The branch points at $\alpha = \pm\cos^{-1}n\pm m\pi$, m=1,2,... are joined pair wise as shown in Fig.3.2, and the square root sign is so chosen that the following relation holds.

$$\tau(\alpha) = \tau(-\alpha) \tag{8c}$$

By differentiating (7c) we note that

$$\tau(\alpha+\pi) = \tau(\alpha) \tag{8d}$$

and (8c) is consistent with (7b).

The relations (7a)-(7c) ensure that the mapping from the α-plane to the ζ-plane is one-to-one.

The normal derivatives of E & E_1 are obtained by differentiating (5a) and (5b) with respect to ϕ.

$$2\pi j\,\frac{\partial E(\rho,\phi)}{\partial\phi} = -j\rho\int_{\gamma+\phi}\sin(\alpha-\phi)\,s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha$$

$$= -j\rho\int_{\gamma}\sin\alpha\ s(\alpha+\phi)e^{-j\rho\cos\alpha}d\alpha \tag{9}$$

$$2\pi j\,\frac{\partial E_1}{\partial\phi} = -j\rho\int_{\Gamma+\phi}n\sin(\zeta-\phi)s_1(\zeta)e^{-j\rho n\cos(\zeta-\phi)}d\zeta$$

$$= -j\rho\int_{\Gamma}n\sin\zeta s_1(\zeta+\phi)e^{-j\rho n\cos\zeta}d\zeta$$

$$= -j\rho\int_{\gamma}\sin\alpha s_1(\zeta+\varphi)e^{-j\rho\cos\alpha}d\alpha \tag{10}$$

We have here made use of the relation $\tau(\alpha) = \sin\alpha/n\sin\zeta$.

Using the expressions (5a), (5b), (9) and (10) for E, E_1, $\frac{\partial E}{\partial\phi}$ and $\frac{\partial E_1}{\partial\phi}$ respectively we obtain the following set of functional equations for s and s_1 so as to satisfy (2a)-(2d).

$$s(\alpha+\phi_0) = s(-\alpha+\phi_0) \tag{11a}$$

$$s_1(\zeta(\alpha)-\phi_1) = s_1(-\zeta(\alpha)-\phi_1) \tag{11b}$$

$$s(\alpha)-s(-\alpha) = \tau(\alpha)\ \{s_1[\zeta(\alpha)]-s_1[-\zeta(\alpha)]\} \tag{11c}$$

$$s(\alpha)+s(-\alpha) = s_1[\zeta(\alpha)]+s_1[-\zeta(\alpha)] \tag{11d}$$

3.2.3 Generalized Two-sided Laplace Transform

In order to solve (11a-11d) Zavadskii introduces the following transform relationship.

$$\sigma(\alpha) = \int_{-\infty}^{\infty} t(\alpha,\beta)e^{-\alpha\beta}d\beta \quad \left(\sigma(\alpha) \doteqdot t(\alpha,\beta)\right) \tag{12}$$

which represents a function of one argument in terms of a function of two arguments. If $t(\alpha,\beta)$ is not a function of β then the above transform goes over to ordinary two-sided Laplace transform [36]. It should be noted that given $\sigma(\alpha)$ the function $t(\alpha,\beta)$ is not unique since to any given $t(\alpha,\beta)$ satisfying (12) one could add any other function $t_0(\alpha,\beta)$ with the property,

$$\int_{-\infty}^{\infty} t_0(\alpha,\beta)e^{-\alpha\beta}d\beta = 0 \tag{13}$$

In the case when the function $\sigma(\alpha)$ has a pole, at the point $\alpha=\psi_0$, with unit principle part, in the strip $\mathrm{Re}\,\alpha<\mathrm{Re}\,\psi_0<\mathrm{Re}(\alpha+\phi_0)$, we then have

$$\sigma(\alpha+\phi_0) = \int_{-\infty}^{\infty} [t(\alpha+\phi_0,\beta) + e^{\beta\psi_0}] e^{-\alpha\beta-\phi_0\beta} d\beta \tag{14}$$

The second term in the brackets corresponds to the pole of the function at the point $\alpha = \psi_0$. The convenience of the representation (12) lies in the fact that in certain problems the function $t(\alpha,\beta)$ may be regarded, without contradicting the conditions imposed on the problem, as periodic in the argument α, whereas the function $\sigma(\alpha)$ is not periodic. An example is the case when $\sigma(\alpha)$ satisfies the functional equation

$$\sigma(\alpha) = q(\alpha)\sigma(\alpha+\phi_0) + f(\alpha) \tag{15a}$$

where $q(\alpha) = q(\alpha+\pi)$ and $f(\alpha)$ is representable in the form

$$f(\alpha) = \int_{-\infty}^{\infty} F(\beta)e^{-\alpha\beta}d\beta, \quad \phi_0 = \text{constant} \tag{15b}$$

Taking the transform of (15a) we obtain

$$t(\alpha,\beta) = q(\alpha)e^{-\phi_0\beta}t(\alpha+\phi_0,\beta) + F(\beta) \tag{16}$$

whose solution may be written in the form

$$t(\alpha,\beta) = F(\beta)L_{\phi_0}[q(\alpha)e^{-\phi_0\beta}] \tag{17}$$

where

$$L_{\phi_0}(p(\alpha)) = 1 + \sum_{k=0}^{\infty} \sum_{\nu=0}^{k} p(\alpha+\nu\phi_0) \tag{18a}$$

$$p(\alpha) = q(\alpha)e^{-\phi_0\beta} \tag{18b}$$

Since $q(\alpha) = q(\alpha+\pi)$, it follows that $L_\phi[p(\alpha)]$ is also periodic with period π. In special cases, assuming that the series (18a) converges, we have,

$$L_\pi[p(\alpha)] = 1/[1-p(\alpha)] \tag{19}$$

$$L_{\pi\frac{m}{\ell}}[p(\alpha)] = \frac{1+\sum_{\mu=0}^{\ell-2}\prod_{\nu=0}^{\mu} p(\alpha+\frac{m\nu}{\ell}\pi)}{1-\prod_{\nu=0}^{\ell-1} p(\alpha+\frac{m\nu}{\ell}\pi)}$$

$$m = 1,2,\ldots$$

$$\ell = 1,2,\ldots \tag{20}$$

Now we turn to the functional equations (11a) - (11d) and invoke the following representations for $s(\alpha)$ and $s_1(\alpha)$: $s(\alpha) \doteqdot t(\alpha,\beta)$, $s_1(\zeta(\alpha)) \doteqdot t_1(\alpha,\beta)$, with the restriction that ϕ_1 is an integral multiple of $\pi/2$. Continuing the representation $s(\alpha) \doteqdot t(\alpha,\beta)$ beyond the pole at $\alpha = \psi_0$, we obtain

$$s(\alpha+\phi_0) = \int_{-\infty}^{\infty} [t(\alpha+\phi_0,\beta) + e^{\beta\psi_0}]e^{-\alpha\beta-\phi_0\beta}\,d\beta \tag{21}$$

Since $\tau(\alpha) = \tau(\alpha+\pi)$, the functions $t(\alpha,\beta)$, $t_1(\alpha,\beta)$ have a period π with respect to α. With this fact in mind we obtain the following system of inhomogeneous functional equations for $t(\alpha,\beta)$ and $t_1(\alpha,\beta)$:

$$[t(\alpha+\phi_0,\beta) + e^{\beta\psi_0}]e^{-\phi_0\beta} = [t(-\alpha+\phi_0,-\beta) + e^{-\beta\psi_0}]e^{\phi_0\beta} \tag{22a}$$

$$t_1(\alpha,\beta) = t_1(-\alpha,-\beta)e^{-2\phi_1\beta} \tag{22b}$$

$$t(\alpha,\beta) - t(-\alpha,-\beta) = \tau(\alpha)[t_1(\alpha,\beta) - t_1(-\alpha,-\beta)] \tag{22c}$$

$$t(\alpha,\beta) + t(-\alpha,-\beta) = t_1(\alpha,\beta) + t_1(-\alpha,-\beta) \tag{22d}$$

which can be reduced to

$$t(\alpha,\beta) = [t(\alpha+2\phi_0,\beta)e^{-2\phi_0\beta} + 2e^{-\phi_0\beta}\mathrm{sh}\beta(\psi_0 - \phi_0)]$$

$$\times \frac{1-\tau(\alpha)\mathrm{th}\phi_1\beta}{1+\tau(\alpha)\mathrm{th}\phi_1\beta} \tag{23}$$

The solution to (23) may be written in the form

$$t(\alpha,\beta) = 2e^{\phi_0\beta}\mathrm{sh}(\psi_0-\phi_0)\beta \times \left\{ L_{2\phi_0}\left[e^{-2\phi_0\beta}\frac{1-\tau(\alpha)\mathrm{th}\phi_1\beta}{1+\tau(\alpha)\mathrm{th}\phi_1\beta}\right] -1 \right\}. \tag{24}$$

We may now obtain $s(\alpha)$ by integrating (24).

$$s(\alpha) = \int_{\infty}^{\infty} t(\alpha,\beta)\, e^{-\alpha\beta}d\beta \tag{25}$$

To find the function $s_1(\zeta(\alpha))$, which in turn defines $s_1(\alpha)$ implicitly, we may either find $t_1(\alpha,\beta)$ through (22c), (22d) and (24) and integrate or substitute for $s(\alpha)$ in (11c) and (11d) and eliminate $s_1(-\zeta(\alpha))$.

In principle one could invert (24) for any arbitrary ϕ_0 with ϕ_1 being a multiple of $\pi/2$. However when ϕ_0 is not a multiple of $\pi/2$, it may not be possible to obtain a closed form expression for $s(\alpha)$ through (25).

3.2.4 A note on the solution to functional equations

Even though $s(\alpha)$ and $s_1(\zeta(\alpha))$, as obtained by the above procedure, will satisfy (11a)-(11d) there may be other solutions to this system of equations. There are two reasons for this. The first is that $t(\alpha,\beta)$ as given by (24) may not be the only solution to (22a)-(22d). The second and the more important reason is that as we mentioned earlier, given $s(\alpha)$

the transform $t(\alpha,\beta)$ is not unique. Hence the eqns (22a) - (22d) do not represent a unique transform of the functional equations (11a)-(11d). Hence the solution to $s(\alpha)$ and $s_1(\zeta(\alpha))$ as obtained above does not constitute a unique solution to the problem but it is only a particular solution to the functional equations (11a)-(11d). To check the correctness and the validity of the solution one must evaluate $E(\rho,\phi)$ and $E_1(\rho,\phi)$ and verify if the fields so obtained satisfy the radiation condition and the edge condition. But as we shall show, through specific examples later, the fields obtained through Zavadskii's method diverge at infinity and hence fail to meet the radiation condition. When ϕ_0 is a multiple of $\pi/2$ it is possible to obtain the most general solution to (11a)-(11d) with out taking recourse to t-transform. However <u>any</u> solution to this system of equations is found to result in either an undesired incoming plane wave or a branch cut integral with exponential growth at infinity. Both or either of these conditions constitute a violation of the radiation condition. Since the system of eqns (11a)-(11d) and the representation of the fields as a Sommerfeld integral over the path γ are the key steps in Zavadskii's method we concluded that his method fails to give the correct solution to the diffraction problem. We demonstrate this through a few specific examples.

3.3 RECTANTULAR DIELECTRIC WEDGE RESTING ON AN INFINITE METAL PLATE

3.3.1 Zavadskii's solution

Let us consider the case when $\phi_1 = \phi_0 = \pi/2$, for which the exact solution is well known through geometric optics.

In this case (24) simplifies to

$$t(\alpha,\beta) = \frac{2\text{sh}\beta(\psi_0 - \frac{\pi}{2})\{\text{ch}\frac{\pi}{2}\beta - \tau(\alpha)\,\text{sh}\,\frac{\pi}{2}\beta\}}{\{1+\tau(\alpha)\}\ \text{sh}\pi\beta} \tag{26}$$

which upon integrating results in the following expression for $s(\alpha)$

$$s(\alpha) = \frac{\cos\psi_0}{\sin\alpha - \sin\psi_0} - \frac{1-\tau(\alpha)}{1+\tau(\alpha)}\ \frac{\cos\psi_0}{\sin\alpha + \sin\psi_0} \tag{27a}$$

The function $s_1(\zeta(\alpha))$ may be obtained by substituting for $s(\alpha)$ in (11c) and (11d) and is given by

$$s_1(\zeta(\alpha)) = \frac{2\cos\psi_0}{[1+\tau(\alpha)](\sin\alpha - \sin\psi_0)} \tag{27b}$$

At this point Zavadskii stops with a somewhat bland statement which we quote here. "The poles of the function $s(\alpha)$ situated in the strip $-\pi<\alpha<\pi$ for $\alpha = \pm\,\psi_0$ {with principle parts equal to 1 and $[1-\tau(\psi_0)]/[1+\tau(\psi_0)]$, respectively} and for $\alpha = \pm\,(\pi-\psi_0)$ {with principle parts -1, and $-[1-\tau(\psi_0)]/[1+\tau(\psi_0)]$} make it possible to compute the Sommerfeld integral (5a) and to obtain the field in the wedge $(0,\pi/2)$ in the form of a sum of four plane waves. In this case there is no cylindrical wave radiated by the edge of the wedge."

However Zavadskii overlooks a very important factor. To evaluate the integral (5a) one has to close the contour γ in order to pick up the poles lying on the real axis. One would normally do this by adding two paths of steepest descent D_-, D_+ which are deformed around the branch cuts, due to $\tau(\alpha)$, as shown in Fig.3.3. In this particular case D_-, D_+ are spaced apart by exactly 2π and $s(\alpha)$ being periodic with period 2π the integrals on D_- and D_+ cancel each other. Then we are left with the branch cut integrals (along the paths B_0, B_π as shown in Fig. 4) in addition to the residues from the poles at $\alpha = \pm\psi_0$ and $\alpha = \pm(\pi-\psi_0)$. As we see from Fig.3.4 the upper portion of B_0 and the lower portion of B_π lie in the unshaded region where the exponent of the Sommerfeld integrand has a positive real part. Thus both of these branch cut integrals grow exponentially as $\rho\to\infty$ and fail to meet the radiation condition. In Appendix I we show that the branch cut integrals over the paths B_1 and B_2 do not cancel each other. Thus we conclude that Zavadskii's solution is non-physical and is of no practical use.** We know that for this particular geometry the exact solution is just the sum of four plane waves as obtained by the geometric optics method. As we noted earlier the solution obtained through the t-transform is one particular solution to the system of equations (11a)-(11d), but not necessarily the correct one. Any correct solution to our problem in addition to satisfying these equations must also satisfy the radiation and the edge condition. It turns out that there is in fact a solution to (11a)-(11d) which meets these requirements. We proceed to obtain such a solution as follows.

3.3.2 Correct solution

Let $s(\alpha)$ and $s_1(\zeta(\alpha))$ meet the following requirements.

** This was pointed out to Zavadskii in a personal communication to which there was no response.

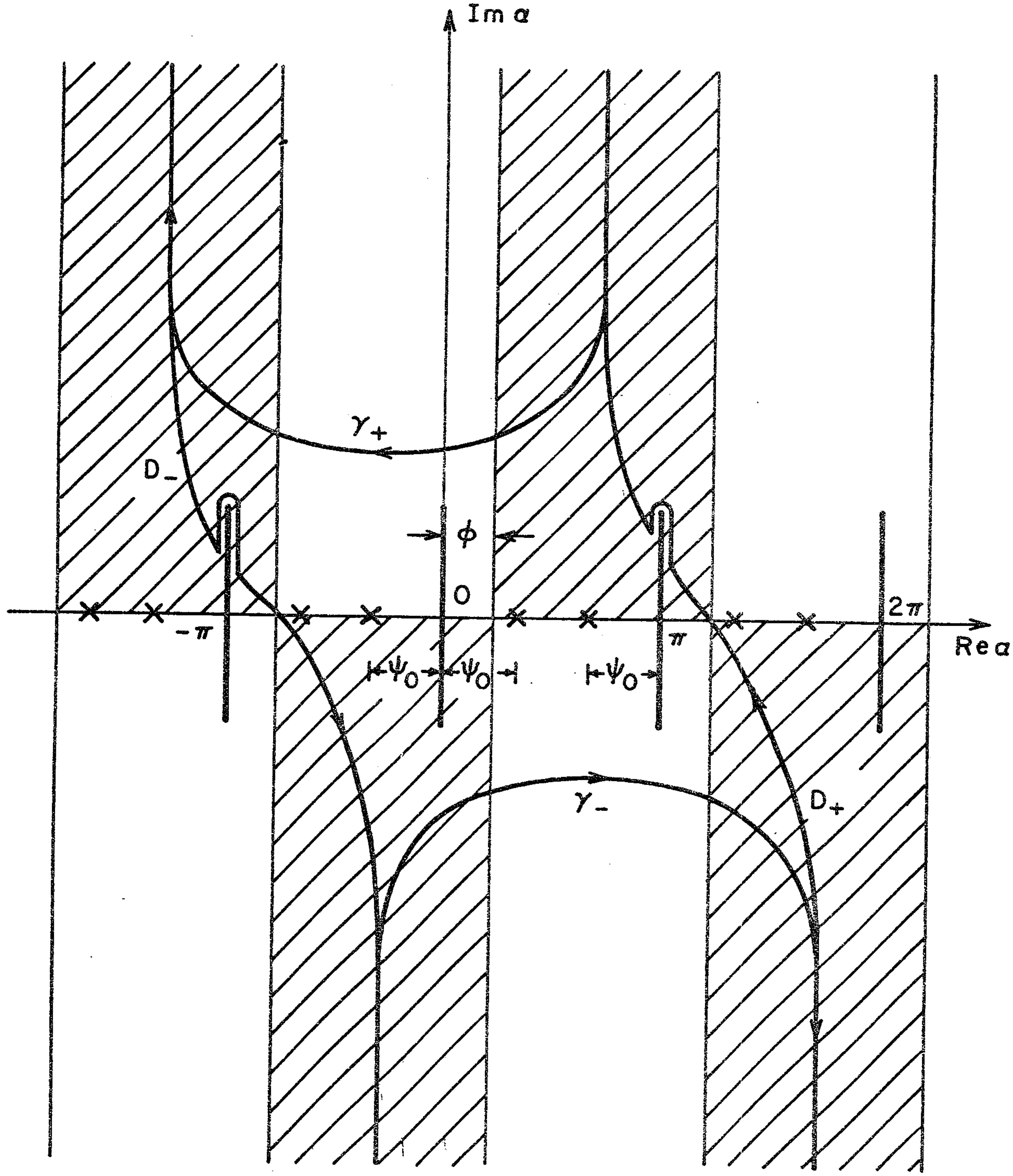


Fig. 3.3 Contours of integration γ_+, γ_-, D_+, and D_- in the plane of the complex variable α.

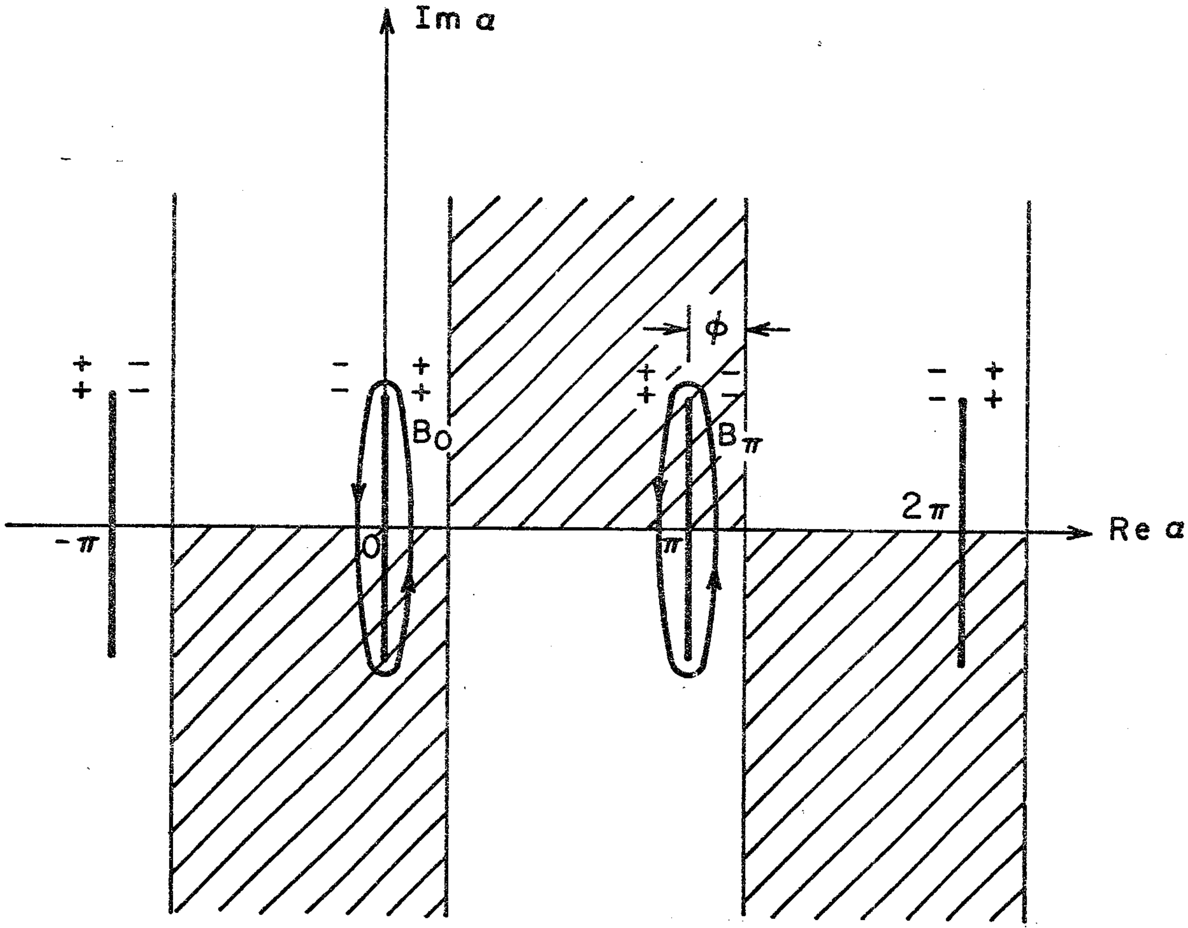


Fig. 3.4 Branch cut contours B_o and B_π in the plane of the complex variable α.

$$\text{i)}\quad s(\alpha+\pi) = s(-\alpha) \tag{28a}$$

$$\text{ii)}\quad s(\alpha) = s(\alpha+2\pi) \tag{28b}$$

$$\text{iii)}\quad s_1(\zeta(\alpha)) = \frac{1}{2}\left\{\frac{s(\alpha)-s(-\alpha)}{\tau(\alpha)} + s(\alpha)+s(-\alpha)\right\} \tag{28c}$$

Then $s(\alpha)$ and $s_1(\zeta(\alpha))$ will also satisfy (11a)-(11d) with $\phi_0 = \phi_1 = \pi/2$. Any odd function with period 2π or any even function with period π will satisfy (28a) and (28b). Thus one has an infinite number of solutions to (11a)-(11d). If we impose the condition that the resulting field must have only one incoming plane wave at $\phi = \psi_0$ we narrow down the solutions to two. One solution is that given by (27). But, as already noted, it fails to meet the radiation condition, and hence is not the desired solution. The second solution which uniquely meets all the requirements is simply obtained by replacing $\tau(\alpha)$ with $\tau(\psi_0)$ in (27). Thus

$$s(\alpha) = \frac{\cos\psi_0}{\sin\alpha - \sin\psi_0} - \frac{1-\tau(\psi_0)}{1+\tau(\psi_0)}\,\frac{\cos\psi_0}{\sin\alpha+\sin\psi_0} \tag{29a}$$

Substituting (29a) in (28c) we obtain the following expression for $s_1(\zeta(\alpha))$.

$$s_1(\zeta(\alpha)) = \frac{2\tau(\psi_0)\cos\psi_0}{1+\tau(\psi_0)}\,\frac{\sin\psi_0/\tau(\psi_0) + \sin\alpha/\tau(\alpha)}{\sin^2\alpha - \sin^2\psi_0}$$

$$= \frac{2\tau(\psi_0)\cos\psi_0}{1+\tau(\psi_0)}\,\frac{\sin\psi_0/\tau(\psi_0) + n\sin\zeta}{\cos^2\psi_0 - n^2\cos^2\zeta} \tag{29b}$$

It immediately follows that $s_1(\alpha)$ is given by

$$s_1(\alpha) = \frac{2\tau(\psi_0)\cos\psi_0}{1+\tau(\psi_0)}\,\frac{\sin\psi_0/\tau(\psi_0) + n\sin\alpha}{\cos^2\psi_0 - n^2\cos^2\alpha}$$

$$= -\frac{2\tau(\psi_0)\cos\theta_0}{1+\tau(\psi_0)}\,\frac{\sin\alpha + \sin\theta_0}{\cos^2\alpha - \cos^2\theta_0} = \frac{2\tau(\psi_0)\cos\theta_0}{1+\tau(\psi_0)}\cdot\frac{1}{\sin\alpha - \sin\theta_0} \tag{29c}$$

where

$$\cos\theta_0 = \cos\psi_0/n \tag{29d}$$

It is now easy to verify that $s(\alpha)$ and $s_1(\zeta(\alpha))$, as given by (29a,b) do indeed satisfy (11a)-(11d). Since the only singularities of $s(\alpha)$ are the poles at $\alpha = \pm\psi_0$ and $\alpha = \pm(\pi-\psi_0)$ we have no branch cut integrals to evaluate and the electric field $E(\rho,\phi)$ is simply given by the following sum of four plane waves.

$$E(\rho,\phi) = e^{-j\rho\cos(\phi-\psi_0)} - e^{j\rho\cos(\phi+\psi_0)} - \frac{1-\tau(\psi_0)}{1+\tau(\psi_0)}\left[e^{-j\rho\cos(\phi+\psi_0)} - e^{j\rho\cos(\phi-\psi_0)}\right] \tag{30}$$

Also since $s_1(\alpha)$ does not have any branch cut singularities, the electric field inside the dielectric, $E_1(\rho,\phi)$ is given by the sum of two plane waves, due to poles at $\alpha = \theta_0$, $\pi - \theta_o$.

After evaluating the residues we find the following expression for $E_1(\rho,\phi)$:

$$E_1(\rho,\phi) = \frac{2\tau(\psi_0)}{1+\tau(\psi_0)}\left\{e^{-j\rho n\cos(\phi-\theta_0)} - e^{j\rho n\cos(\phi+\theta_0)}\right\} \tag{31}$$

The field expressions E & E_1 as given by (30) and (31) correspond exactly to the geometric optical solution to our problem. Thus we have shown a way to overcome the difficulty in finding the solution to this problem through Zavadskii's method. Unfortunately, however, the same kind of approach does not work to other non-trivial geometries as we will demonstrate.

3.4 RECTANGULAR DIELECTRIC WEDGE RESTING ON A SEMI-INFINITE METAL PLATE

3.4.1 Zavadskii's solution:

Consider the case where a semi-infinite metallic plate is resting against a rectangular dielectric wedge as shown in Fig.3.5. This situation corresponds to $\phi_o = \frac{3\pi}{2}$ and $\phi_1 = \pi/2$ in Fig.3.1. Hence all the equations through (24) hold for this problem. Substituting the values of ϕ_0 and ϕ_1 we obtain, after some simplification, the following expression for the function $t(\alpha,\beta)$.

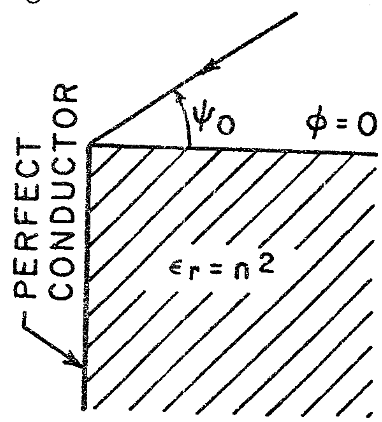


Figure 3.5. Rectangular dielectric wedge resting on a perfectly conducting semi-infinite plate.

$$t(\alpha,\beta) = 2\mathrm{sh}(\psi_0 - \tfrac{3\pi}{2})\beta \, \frac{\left[\mathrm{ch}\,\frac{\pi}{2}\beta - \tau(\alpha)\ \mathrm{sh}\,\frac{\pi}{2}\beta\right]}{[1+\tau(\alpha)]\mathrm{sh}2\pi\beta + [1-\tau(\alpha)]\mathrm{sh}\pi\beta} \tag{32}$$

As shown in Appendix II, eqn. (32) leads to the following expression for the functions $s(\alpha)$ and $s_1(\zeta(\alpha))$:

$$s(\alpha) = \frac{1}{2}\, I(\psi_0+\alpha-\pi) + \frac{1}{2}\, I(\psi_o-\alpha-2\pi)$$

$$+ AI(\psi_0-\alpha-\pi) + AI(\psi_0+\alpha-2\pi) \tag{33a}$$

$$s_1(\zeta(\alpha)) = \frac{1}{1+\tau(\alpha)}\, \{I(\psi_0+\alpha-\pi) + I(\psi_0-\alpha-2\pi)\} \tag{33b}$$

where A and t are defined by

$$A = \frac{1}{2} \frac{1-\tau(\alpha)}{1+\tau(\alpha)} = -\cos t \tag{33c}$$

and [37]

$$I(x) = \int_{-\infty}^{\infty} \frac{\text{sh } xy \, dy}{\text{sh } \pi y(\text{ch}\pi y + A)} = \frac{-1}{(1-A^2)\sin x} \left\{ 2\sin^2 \frac{x(\pi - t)}{2\pi} - (1-A)\sin^2 \frac{x}{2} \right\} \tag{33d}$$

The poles of $s(\alpha)$ are located at $\alpha = \pm\psi_0 \pm m\pi$ where m is an integer. To evaluate the Sommerfeld integral (5a) we close the contour γ by means of two steepest descent paths D_- and D_+ as shown in Fig. 3. Then $E(\rho,\phi)$ is given by the sum of the following three terms.

$$E(\rho,\phi) = E_p(\rho,\phi) + E_b(\rho,\phi) + E_d(\rho,\phi) \tag{34}$$

where

$$E_p(\rho,\phi) = \sum \text{Residues} \tag{35a}$$

$$E_b(\rho,\phi) = \begin{cases} \dfrac{1}{2\pi j} \displaystyle\int_{B_0+B_\pi} s(\alpha)\, e^{-j\rho\cos(\alpha-\phi)} d\alpha & 0<\phi<\pi \\ \dfrac{1}{2\pi j} \displaystyle\int_{B_\pi+B_{2\pi}} s(\alpha) e^{-j\rho\cos(\alpha-\phi)} d\alpha & \pi<\phi<\dfrac{3\pi}{2} \end{cases} \tag{35b}$$

$$E_d(\rho,\phi) = \frac{-1}{2\pi j} \int_{D_-+D_+} s(\alpha) e^{-j\rho\cos(\alpha-\phi)} d\alpha \tag{35c}$$

3.4.1.1 pole contribution

The steepest descent paths D_- and D_+ intersect the real axis of

α at $-\pi+\phi$ and $\pi+\phi$ respectively. Accordingly the combination of poles enclosed by the contour changes as ϕ is varied. After some algebraic manipulation (Appendix III) $s(\alpha)$ may be put into the following form which is more convenient for the evaluation of the pole solution $E_p(\rho,\phi)$:

$$s(\alpha) = \frac{\frac{1}{2}}{1-A^2}\,\frac{(2A^2+A-1)(2\sin\alpha\cos\psi_0) - (1+A-2A^2)\sin 2\psi_0}{\cos 2\alpha - \cos 2\psi_0}$$

$$- \frac{1}{(1-A^2)(\cos 2\alpha - \cos 2\psi_0)}\left\{\sin 2\psi_0 \cos\left(\frac{\psi_0 t}{\pi} - \frac{3t}{2}\right)\cos\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right) + \sin 2\alpha \sin\left(\frac{\psi_0 t}{\pi} - \frac{3t}{2}\right)\sin\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right)\right\}$$

$$- \frac{1}{(1-A^2)}\sin\left(\frac{\psi_0 t}{\pi} - \frac{3t}{2}\right)\cos\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right) \tag{36}$$

The residue $R(\alpha_0)$ at any pole $\alpha = \alpha_0$ is then given by

$$R(\alpha_0) = \frac{\frac{1}{4}}{1-A_0^2}\left\{\left(1+A_0-2A_0^2\right)\frac{\sin 2\psi_0}{\sin 2\alpha_0} - \left(2A_0^2+A_0-1\right)\frac{2\sin\alpha_0\cos\psi_0}{\sin 2\alpha_0}\right\}$$

$$+ \frac{\frac{1}{2}}{(1-A_0^2)}\left\{\frac{\sin 2\psi_0}{\sin 2\alpha_0}\cos\left(\frac{\psi_0 t_0}{\pi} - \frac{3t_0}{2}\right)\cos\left(\frac{\alpha_0 t_0}{\pi} - \frac{3t_0}{2}\right) + \sin\left(\frac{\psi_0 t_0}{\pi} - \frac{3t_0}{2}\right)\sin\left(\frac{\alpha_0 t_0}{\pi} - \frac{3t_0}{2}\right)\right\} \tag{37a}$$

where

$$t_0 = t(\psi_0) \tag{37b}$$

and $A_0 = A(\psi_0)$ (37c)

The residues of poles, that are of interest to us are given by

$$R(-2\pi+\psi_0) = R(-\pi+\psi_0) = R(\pi-\psi_0) = R(\pi+\psi_0) = R(2\pi-\psi_0) = R(2\pi+\psi_0) = 0$$

$$R(-\pi-\psi_0) = 4A_0^2 - 1$$

$$R(-\psi_0) = -2A_0$$

$$R(\psi_0) = 1$$

$$R(3\pi-\psi_0) = -1 \qquad (38)$$

The location of poles, along with the residues, are shown in Fig. 6 for three different ranges of the incident angle. The points of intersection of D_- and D_+ with the real axis of α-plane are also shown. By looking at Fig. 3.6 we can immediately write down the pole solution $E(\rho,\phi)$ for different ranges of ϕ and ψ_0 as given below.

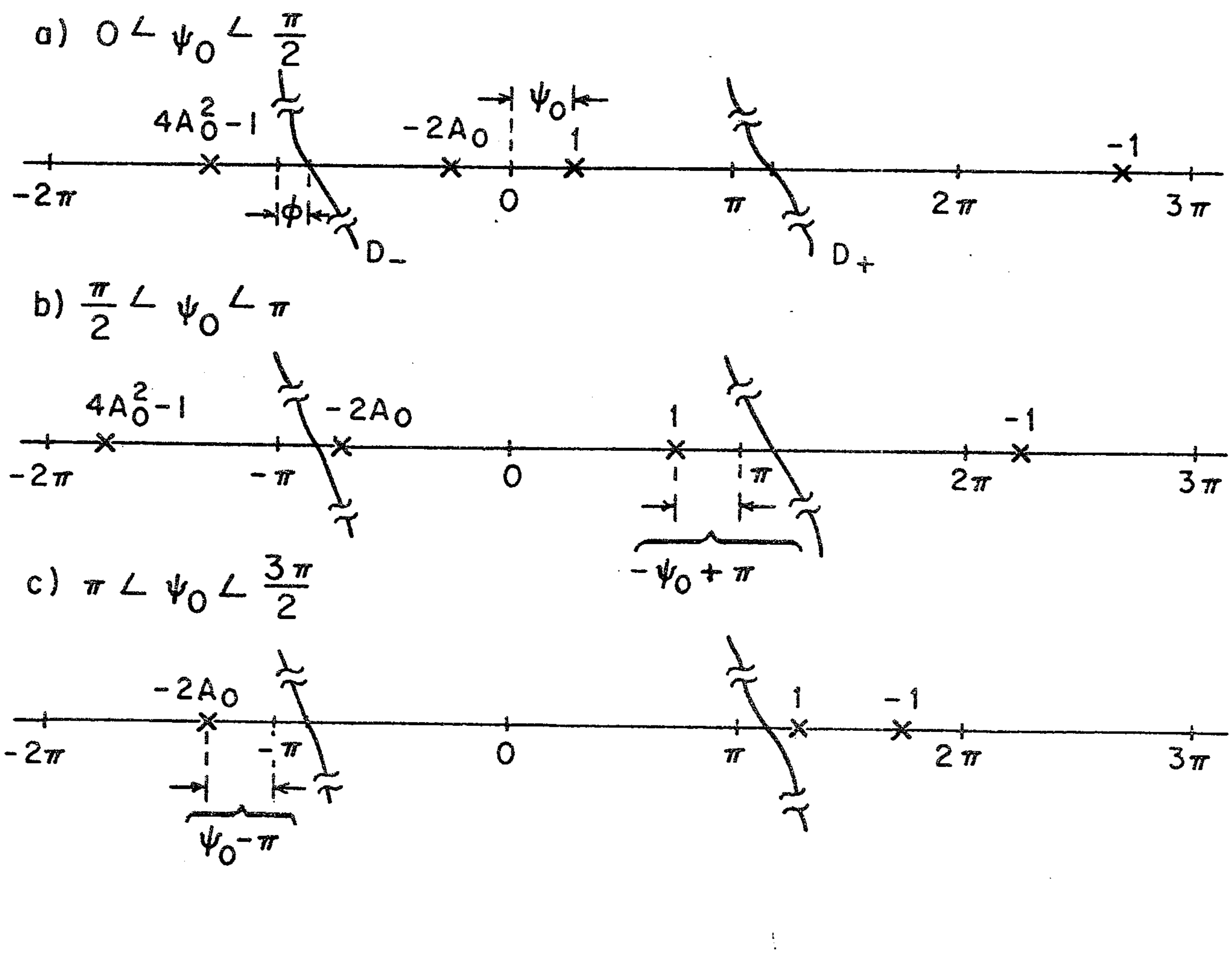


Figure 3.6. Pole and residue structure of $s(\alpha)$ in relation to the contours D_+ and D_-

i) $0<\psi_0<\frac{\pi}{2}$

$$E_p(\rho,\phi) = \begin{cases} P_{\psi_0} - 2A_0\, P_{-\psi_0} & 0<\phi<\pi-\psi_0 \\ P_{\psi_0} & \pi-\psi_0<\phi<\pi+\psi_0 \\ 0 & \pi+\psi_0<\phi<\frac{3\pi}{2} \end{cases} \tag{39a}$$

ii) $\psi_0<\frac{\pi}{2}<\pi$

$$E_p(\rho,\phi) = \begin{cases} P_{\psi_0} - 2A_0 P_{-\psi_0} & 0<\phi<\pi-\psi_0 \\ P_{\psi_0} & \pi-\psi_0<\phi<2\pi-\psi_0 \\ P_{\psi_0} - P_{\pi-\psi_0} & 2\pi-\psi_o<\phi<\frac{3\pi}{2} \end{cases} \tag{39b}$$

iii) $\pi<\psi_0<\frac{3\pi}{2}$

$$E_p(\rho,\phi) \begin{cases} 0 & 0<\phi<\psi_0-\pi \\ P_{\psi_0} & \psi_0-\pi<\phi<2\pi-\psi_0 \\ P_{\psi_0} - P_{\pi-\psi_0} & 2\pi-\psi_0<\phi<\frac{3\pi}{2} \end{cases} \tag{39c}$$

where

$$P_\theta = e^{-j\rho\cos(\phi-\theta)} \tag{39d}$$

When the incident wave is in the first quadrant we observe the incident wave and the reflected wave from the dielectric boundary, until ϕ approaches the value $\pi-\psi_0$ after which the reflected wave disappears. When ϕ exceeds the value $\pi+\psi_0$ the incident wave also disappears and we are in the dark region. When ψ_0 is in the second quadrant the incident wave exists in the entire range $0<\phi<3\pi/2$ and illuminates both the faces of the wedge, thus giving rise to two reflected waves. The reflected wave from the dielectric boundary exists in the range $0<\phi<\pi-\psi_0$ and the reflected wave from the metal plate exists in the range $2\pi-\psi_0<\phi<3\pi/2$. When ψ_0 is in the third quadrant we are in the dark region until ϕ approaches the value $\psi_0-\pi$ after which we just see the incident wave. When ϕ exceeds the value $2\pi-\psi_0$ the reflected wave from the metal plate also comes into the range of observation. Thus the pole solution is consistent with the geometrical optics theory.

3.4.1.2 Diffracted field

The integrals over the contours D_- and D_+ represent the diffracted waves due to the edge. We choose the paths D_- and D_+ such that the following conditions are met for $\alpha = \mu+j\nu \varepsilon D_-,D_+$.

$$\mathrm{Re}\{\cos(\alpha-\phi)\} = \cos(\mu-\phi)\ \mathrm{ch}\nu = -1 \tag{40a}$$

and

$$\mathrm{Im}\{\cos(\alpha-\phi)\} = -\sin(\mu-\phi)\ \mathrm{sh}\nu <0 \tag{40b}$$

Then μ and ν are defined through the relations

$$\mu = -\mathrm{gd}(\nu) + \phi \mp \pi \tag{41a}$$

and

$$\sin(\mu-\phi) = +\ \mathrm{th}\ \nu \tag{41b}$$

where gd(x) is the Gudermann function given by

$$\mathrm{gd}(x) = \cos^{-1}(1/\mathrm{ch}x) \tag{41c}$$

We should note that D_- and D_+ as defined through (41a,b) may but the branch cuts, for certain ranges of ϕ, in which case we deform D_- and D_+ around these branch cuts as shown in Fig.3.3, and the resulting branch cut and end point integrals must be evaluated separately. However for large values of ρ the contribution from these integrals is negligible compared to the dominant contribution from the vicinity of the saddle points at $\alpha = \phi \mp \pi$. Now we proeccd to evaluate these saddle point contributions.

Let $E_{d-}(\rho,\phi)$ and $E_{d+}(\rho,\phi)$ denote the integrals over D_- and D_+ respectively. Then

$$E_{d+}(\rho,\phi) = \frac{1}{2\pi j}\int_{D_+} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha$$

$$= \frac{1}{2\pi j}\int_{-\infty}^{\infty} s(\alpha)e^{-j\rho[\cos(\mu-\phi)ch\nu - j\sin(\mu-\phi)sh\nu]}\left(\frac{d\mu}{d\nu}+j\right)d\nu$$

$$= \frac{1}{2\pi j}\int_{-\infty}^{\infty} s(\alpha)e^{j\rho-\rho th\nu sh\nu}(-1/ch\nu+j)d\nu \qquad (42a)$$

Now we approximate $s(\alpha)(-1/ch\nu+j)$ by its value at the saddle point $\alpha=\pi+\phi+j0$ and $th\nu sh\nu$ by ν^2 to obtain

$$E_{d+}(\rho,\phi) \doteq \frac{1}{2\pi j}\; s(\pi+\phi)(-1+j)\int_{-\infty}^{\infty} e^{j\rho-\rho\nu^2}d\nu$$

$$= s(\pi+\phi)e^{j(\rho+\frac{\pi}{4})}/\sqrt{2\pi\rho} \qquad (42b)$$

Similarly we obtain E_{d-} as

$$E_{d-}(\rho,\phi) \doteq -\, s(-\pi+\phi)e^{j(\rho+\frac{\pi}{4})}/\sqrt{2\pi\rho} \qquad (42c)$$

and taking negative of the sum of (42b) and (42c) we obtain

$$E_d(\pi,\phi) \doteq -\,\{s(\pi+\phi) - s(-\pi+\phi)\}e^{j(\rho+\frac{\pi}{4})}/\sqrt{2\pi\rho}$$

$$= -\left[\frac{2\sin t_1}{(1-A_1^2)(\cos2\phi-\cos2\psi_0)}\left\{\sin2\psi_0\cos\left(\frac{\psi_0 t_1}{\pi}-\frac{3t_1}{2}\right)\sin\left(\frac{\phi t_1}{\pi}-\frac{3t_1}{2}\right)\right.\right.$$
$$\left.-\sin2\phi\sin\left(\frac{\psi_0 t_1}{\pi}-\frac{3t_1}{2}\right)\cos\left(\frac{\phi t_1}{\pi}-\frac{3t_1}{2}\right)\right\}$$
$$\left.+\frac{2\sin t_1}{(1-A_1^2)}\sin\left(\frac{\psi_0 t_1}{\pi}-\frac{3t_1}{2}\right)\sin\left(\frac{\phi t_1}{\pi}-\frac{3t_1}{2}\right)\right]\frac{e^{j(\rho+\frac{\pi}{4})}}{\sqrt{2\pi\rho}} \qquad (43a)$$

where

$$t_1 = t(\phi) \qquad (43b)$$

and $A_1 = A(\phi)$ (43c)

If the refractive index of the dielectric medium is made to approach unity, thus reducing it to free space, then (43a) simplifies to

$$E_d(\rho,\phi)\Big|_{n=1} = E_{d0}(\rho,\phi) = -\frac{e^{j(\rho+\frac{\pi}{4})}}{2\sqrt{2\pi\rho}}\left\{\frac{1}{\sin\frac{\phi+\psi_0}{2}} + \frac{1}{\cos\frac{\phi-\psi_0}{2}}\right\} \qquad (44)$$

which, with proper interpretation of ϕ and ψ_0, corresponds to Sommerfeld's solution [2] to plane wave diffraction by a perfectly conducting semi-infinite plate.

In passing we should note that $E_d(\rho,\phi)$ as given by (43a) is only good when there are no poles, of $s(\alpha)$, in the vicinity of the saddle points $\alpha = \phi\mp\pi$.

3.4.1.3 Branch cut integrals

The contribution due to the branch cuts enclosed within the contour $\gamma+D_-+D_+$ is given by (35b). If we try to evaluate these branch cut integrals we find that the top halves of B_0 and $B_{2\pi}$ and the bottom half of B_π give rise to terms which increase exponentially as ρ increases and thus these terms fail to meet the radiation condition. On these branch cut contours the exponential term in the Sommerfeld integral (5a) has a positive real part which is proportional to $\rho\,\mathrm{sh}(|\nu|)$ where $\nu = \mathrm{Im}\alpha$ It can be easily verified that these exploding branch cut integrals do not anihilate each other and the presence of these terms makes the solution, obtained through Zavadskii's method, physically meaningless. In the next section we describe several approaches to try to correct this problem.

3.4.2 Attempts to correct Zavadskii's solution

3.4.2.1 Is a branch cut free solution possible?

If we replace $\tau(\alpha)$ by $\tau(\psi_0)$ in (36) the resulting function which we call $\tilde{s}(\alpha)$ still satisfies the equation (11a) with $\phi_0 = 3\pi/2$.

$$\tilde{s}(\alpha + \frac{3\pi}{2}) = \tilde{s}(-\alpha + \frac{3\pi}{2}) \tag{45}$$

thus meeting the boundary condition at $\phi = \frac{3\pi}{2}$. However the modified function $\tilde{s}_1(\zeta(\alpha))$ as given by

$$\tilde{s}_1(\zeta(\alpha)) = \frac{1}{2\tau(\alpha)} \left\{ \tilde{s}(\alpha) - \tilde{s}(-\alpha) + \tau(\alpha)[\tilde{s}(\alpha) + \tilde{s}(-\alpha)] \right\} \tag{46}$$

does not satisfy the boundary condition at $\phi = -\pi/2$. That is

$$\tilde{s}_1(\zeta(\alpha)) \neq \tilde{s}_1(\zeta(-\alpha-\pi)) \tag{47}$$

We note that $\tilde{s}(\alpha)$ as obtained above is non-periodic. In Appendix IV we show that any branch cut free solution $\tilde{s}(\alpha)$ to (11a) with $\phi_0 = 3\pi/2$ must be periodic in α, with period 2π, so that the resulting function $\tilde{s}_1(\zeta(\alpha))$ will satisfy (11b) with $\phi_1 = \pi/2$. However, as we shall show later, any such non-trivial periodic solution would give rise to more than one incoming plane wave thus violating the radiation condition. Thus it seems impossible to obtain a solution to the functional equations (11a-11d), with $\phi_0 = 3\pi/2$ and $\phi_1 = -\pi/2$, such that $s(\alpha)$ is free of branch points. Futher, we know that, such a solution would not have any diffracted waves since D_- and D_+ are then spaced by 2π and are in opposite directions. We know that for the problem under consideration there must be diffracted waves from the edge and the associated lateral waves. Normally one would expect to obtain lateral waves from the branch cut integrals; hence seeking a solution which is completely free of branch cuts seems to be a step in the wrong direction. For the trivial problem considered in the earlier section we knew that there would be no diffracted or lateral waves and our attempts to obtain a branch cut free solution then met with success.

3.4.2.2 Is there a solution with zero contribution from the unshaded part of the branch cuts?

To answer this question one must look at the most general solution to (11a-11d). We have obtained such a general solution, in Appendix V, which is given by

$$s(\alpha) = J_{2\pi}(\alpha) + P_{e\pi}(\alpha)\cos\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right) + P_{0\pi}\sin\left(\frac{\alpha t}{\pi}-\frac{3t}{2}\right) \tag{48a}$$

where $P_{e\pi}$ and $P_{0\pi}$ are any even and odd functions of α with period π and $J_{2\pi}(\alpha)$ is any function of α with period 2π and satisfying the condition

$$J_{2\pi}\left(\alpha+\frac{3\pi}{2}\right) = J_{2\pi}\left(-\alpha+\frac{3\pi}{2}\right) \tag{48b}$$

The function $t(\alpha)$ is defined as in (33c). It is easy to see that $s(\alpha)$ as given by (36) is indeed in the form of (48a).

The periodic part $J_{2\pi}(\alpha)$ in (48a) may be chosen such that it is free of any branch points. However the non-periodic part contains branch points due to t. In Appendix VI it is shown that for the branch cut contributions from the unshaded branches to vanish the functions $P_{e\pi}$ and $P_{0\pi}$ must identically vanish. As we shall see a little later any solution of the form $J_{2\pi}(\alpha)$ gives rise to more than one incoming plane wave which is again a violation of the radiation condition which requires that there be no incoming waves other than the incident wave. Thus there is no solution $s(\alpha)$ which, after integration over the contour γ, would give fields satisfying the radiation condition, and we conclude that γ is the wrong contour to start with. This immediately gives rise to the following questions: What other possible contours are there to start with, so that we could write down the boundary conditions as a set of functional equations for $s(\alpha)$? Is it possible to combine the solutions obtained through γ and some other contours in such a manner that the radiation condition is uniquely met? We will make a thorough examination of these possibilities in later sections, but before we do that we would like to make the following comments about the periodic solution $J_{2\pi}(\alpha)$.

If $J_{2\pi}(\alpha)$ is free of singularities then the resultant fields would be identically zero. Any branch cut singularities involving $\tau(\alpha)$ would give rise to exploding waves at infinity. Thus, $J_{2\pi}(\alpha)$ can only have poles. We specifically require a pole at $\alpha = \psi_0$ corresponding to the incident wave. However, because of the periodicity and the requirement (48b), $J_{2\pi}(\alpha)$ cannot have an isolated pole. Any pole with a residue 'a' at $\alpha = \psi_0$ gives rise to a chain of poles with residues as indicated in Fig.3.7a. If we introduce a pole with a residue 'b' at $\alpha = -\psi_0$ this gives rise to a different chain of poles as shown in Fig.3.7b. Thus the most general pole structure of $J_{2\pi}(\alpha)$ is as shown in Fig.3.7c. Since the poles at ψ_0 and $-\psi_0$ correspond to the incident wave and the reflected wave from the dielectric we require that

$$a = 1 \tag{49a}$$

and $$b = -[1-\tau(\psi_0)]/[1+\tau(\psi_0)] \tag{49b}$$

Then $J_{2\pi}(\alpha)$ is given by

$$J_{2\pi}(\alpha) = \frac{\cos\psi_0}{\sin\alpha-\sin\psi_0} - \frac{1-\tau(\psi_0)}{1+\tau(\psi_0)}\,\frac{\cos\psi_0}{\sin\alpha+\sin\psi_0} \tag{50}$$

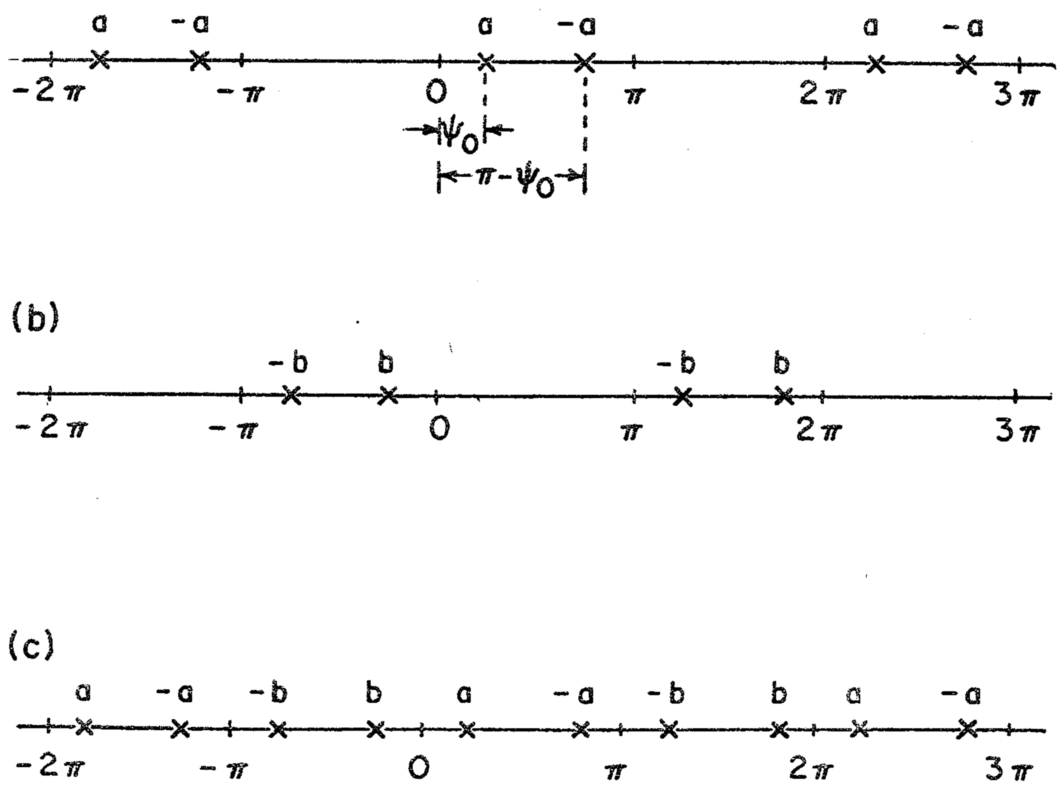


Fig.3.7 Pole and residue structure of $J_{2\pi}(\alpha)$

The resulting solution for the fields is given by

$$\tilde{E}(\rho,\phi) = e^{-j\rho\cos(\phi-\psi_0)} - \frac{1-\tau(\psi_0)}{1+\tau(\psi_0)} e^{-j\rho\cos(\phi+\psi_0)}$$

$$- e^{j\rho\cos(\phi+\psi_0)} + \frac{1-\tau(\psi_0)}{1+\tau(\psi_0)} e^{j\rho(\phi-\psi_0)} \qquad (51)$$

But $\tilde{E}(\rho,\phi)$ has two undesired incoming plane waves at angles $\pi - \psi_0$ and $\pi + \psi_0$, and is not the solution to our problem. However in the process of the above discussion, we have found a unique solution to a 'quasi-trivial' problem which we give next.

3.4.2.3 Solution to a 'quasi-trivial' problem

Can we illuminate a rectangular dielectric wedge, resting on a semi-infinite metal plate, in such a manner that the resulting field will have only plane waves and no diffracted waves? The answer is yes, and the solution is the following combination of incident waves.

$$E_{inc}(\rho,\phi) = P_{\psi_0} - P_{\pi-\psi_0} + \frac{1-\tau(\psi_0)}{1+\tau(\psi_0)} P_{\pi+\psi_0} \qquad (52)$$

where $P_\theta = \exp[-j\rho\cos(\phi-\theta)]$ represents an incident wave of unit amplitude, at an angle θ. The three plane waves P_{ψ_0}, $-P_{\pi-\psi_0}$ and $-RP_{\pi+\psi_0}$ $(R = -[1-\tau(\psi_0)]/[1+\tau(\psi_0)])$ and the corresponding reflected and refracted waves produce continuous fields in the entire region $0\leq\phi\leq 2\pi$ as shown in Fig. 8, where the incident waves are shown with solid lines and the reflected and refracted waves are shown with dotted lines. The discontinuity of I_1 $(=P_{\psi_0})$ at $\phi = \pi+\psi_0$ is cancelled by the reflected wave

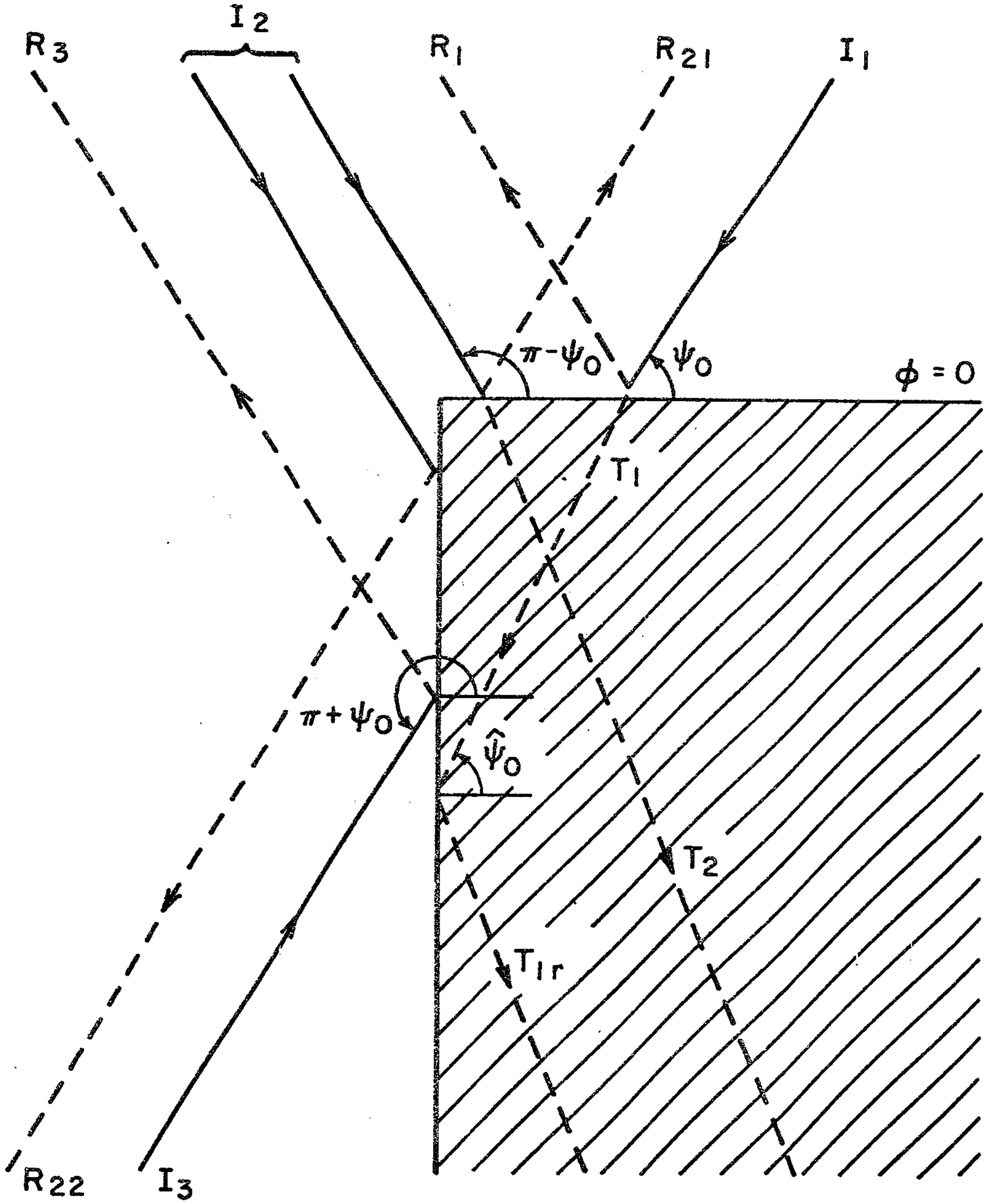


Fig. 3.8 Illumination of a rectangular dielectric wedge, resting on a perfectly conducting semi-infinite plate, such that there is no net diffracted wave from the edge.

___________ incident plane waves
----------- reflected and refracted plane waves

$R_{22}(=P_{\psi_0})$ and the discontinuity of $I_3(=-RP_{\pi+\psi_0})$ is nullified by the reflected wave $R_{21}(=-RP_{\pi+\psi_0})$ which exists only in the range $0\leq\phi\leq\psi_0$. The incident wave I_3 gives rise to a reflected wave R_3 which precisely fills up the shadow region of R_1. The sharow region of T_2 is filled up by T_{1r} which is the reflected wave corresponding to T_1.

If we let $f(\psi)$ to be the diffracted field due to an incident wave P_ψ, then $f(\psi)$ must satisfy the following functional equation.

$$f(\psi) - f(\pi-\psi) - Rf(\pi+\psi) = 0 \tag{53}$$

In Appendix VII we show that the most general solution to (53) is given by

$$f(\psi) = [(n^2-\cos^2\psi)^{\frac{1}{2}} - \sin\psi] \left\{F_{e\pi} \sin \frac{2\psi}{3} + F_{0\pi}\cos \frac{2\psi}{3}\right\} \tag{54}$$

where $F_{e\pi}$ is an arbitrary even periodic function of ψ and $F_{0\pi}$ is an arbitrary odd periodic function of ψ, both of period π. Thus any solution for the diffracted field $f(\psi)$ must satisfy (54).

3.4.2.4 Solution using asymmetric contour $\overline{\gamma}$.

In the previous section we raised the possibility of starting with a contour other than γ and obtaining a solution. Such contours do exist and the asymmetric contour $\overline{\gamma} = \gamma_+ - \gamma_-$, as shown in Fig.3.9, is just one of these. If we write

$$E(\rho,\phi) = \frac{1}{2\pi j} \int_{\overline{\gamma}+\phi} \bar{s}(\alpha) e^{-j\rho\cos(\alpha-\phi)} d\alpha \tag{55a}$$

and

$$E_1(\rho,\phi) = \frac{1}{2\pi j} \int_{\overline{\Gamma}+\phi} \bar{s}_1(\zeta) e^{-j\rho n\cos(\zeta-\phi)} d\zeta \tag{55b}$$

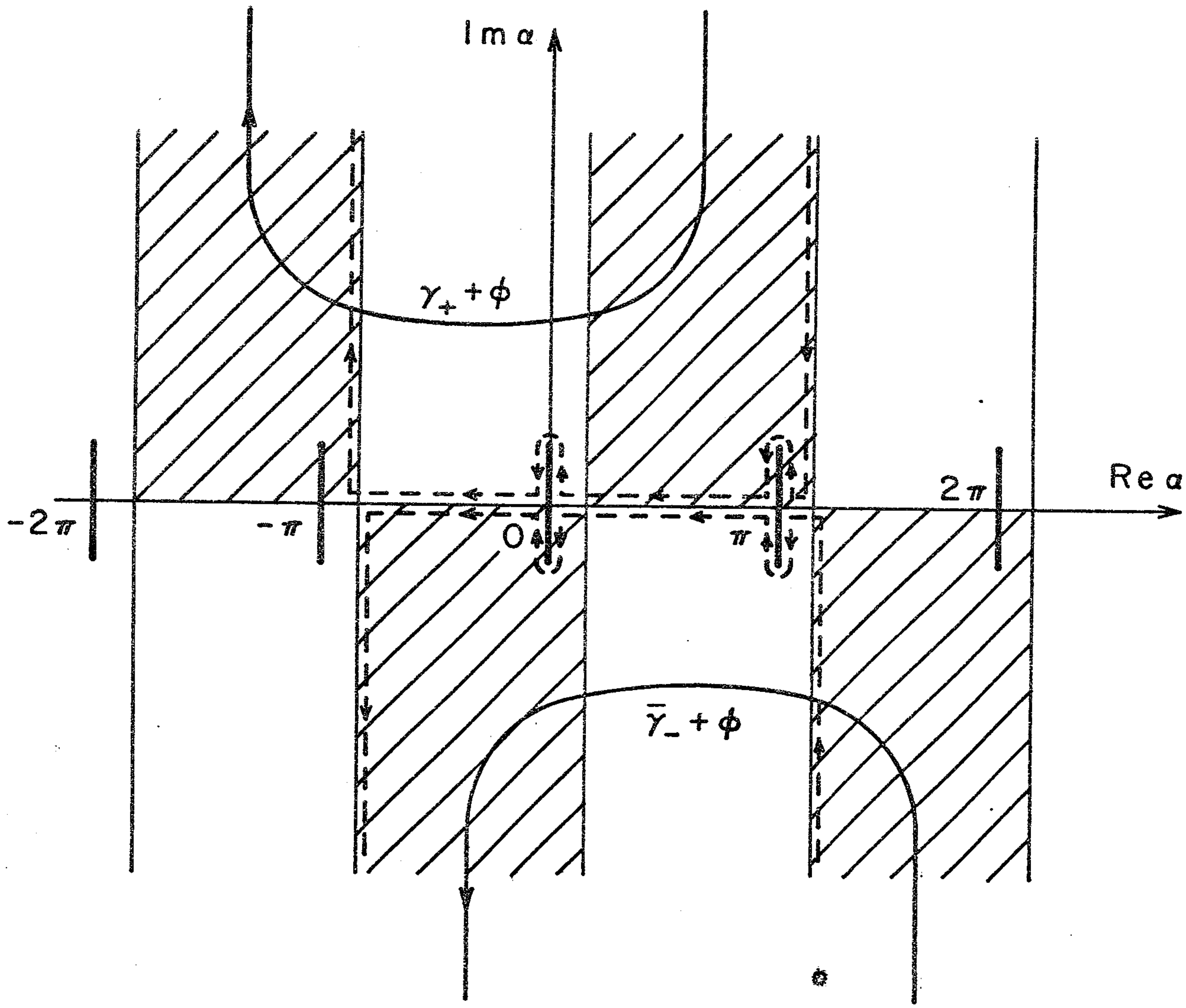


Fig. 3.9 Asymmetric contour $\bar{\gamma}$ (shifted by ϕ) in the plane of the complex variable α.

with $\overline{\Gamma}$ being the mapping of $\overline{\gamma}$ under the transformation (6a,b), we obtain the following set of functional equations for $\bar{s}$ and $\bar{s}_1$:

$$\bar{s}(\alpha+\frac{3\pi}{2}) = -\bar{s}(-\alpha+\frac{3\pi}{2}) \tag{56a}$$

$$\bar{s}_1[\zeta(\alpha)-\frac{\pi}{2}] = -\bar{s}_1[-\zeta(\alpha)-\pi/2] \tag{56b}$$

$$\bar{s}(\alpha) + \bar{s}(-\alpha) = \tau(\alpha)\left\{\bar{s}_1[\zeta(\alpha)] + \bar{s}_1[-\zeta(\alpha)]\right\} \tag{56c}$$

$$\bar{s}(\alpha) - \bar{s}(-\alpha) = \bar{s}_1[\zeta(\alpha)] - \bar{s}_1[-\zeta(\alpha)] \tag{56d}$$

The most general solution to (56a-56d) is given by

$$\bar{s}(\alpha) = \bar{J}_{2\pi}(\alpha) + \bar{P}_{e\pi}\sin(\frac{\alpha t}{\pi}-\frac{3t}{2}) + \bar{P}_{0\pi}\cos(\frac{\alpha t}{\pi}-\frac{3t}{2}) \tag{57a}$$

$$\bar{J}_{2\pi}(\alpha+\frac{3\pi}{2}) = -\bar{J}_{2\pi}(-\alpha+\frac{3\pi}{2}) \tag{57b}$$

where $\bar{J}_{2\pi}(\alpha)$ is any function satisfying (57b) with a period 2π in α and $\bar{P}_{e\pi}(\alpha)$, $\bar{P}_{0\pi}(\alpha)$ are any even and odd functions of α with period π. However, such a solution, apart from having exploding branch cut integrals, gives rise to incoming diffracted waves. To see this we replace $\overline{\gamma}$ by its equivalent sum of the contours D_0, D_-, D_+, B_0; and $-B_\pi$ as shown in Fig3.10. In doing so we pick up the residues of poles situated between the strip $-\pi+\phi<\mathrm{Re}\alpha<\pi+\phi$, and we may write the integral over $\overline{\gamma}$ as the following sum.

$$\int_{\overline{\gamma}+\phi} \bar{s}(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha = 2\pi j\sum \text{Residues} + \int_{B_0-B_\pi} \bar{s}(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha$$

$$+\int_{D_-+D_+} \bar{s}(\alpha)e^{-j\rho\cos(\alpha-\phi)} + 2\int_{D_0} \bar{s}(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha \tag{58}$$

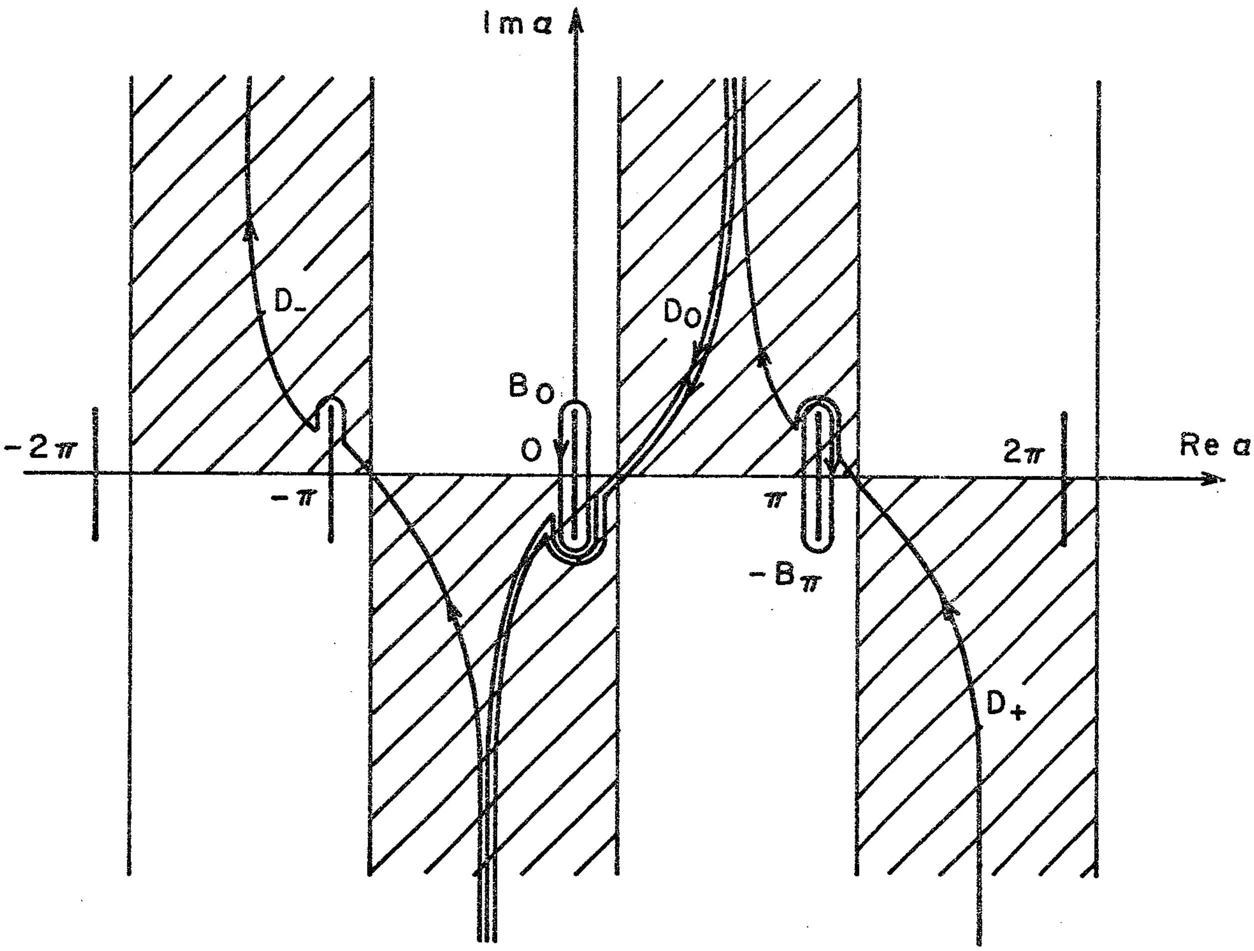


Fig. 3.10 Decomposition of $\bar{\gamma}$ into D_0, D_+, D_-, B_0 and $-B_\pi$.

Equation (57) comtains branch cut integrals over B_0 and B_π. The integrals over the upper half of B_0 and the lower half of B_π diverge as ρ becomes large. Due to the different forms of $s(\alpha)$ and $\bar{s}(\alpha)$ it is not possible to mutually cancel the divergent branch cut integrals. Further the solution from $\bar{s}(\alpha)$ contains an integral over the path D_0 which, when evaluated using the steepest descent method, can be shown to have the form

$$E_{D_0}(\rho,\phi) = F(\phi)e^{-j\rho}/\sqrt{\rho} \tag{59}$$

which represents an incoming diffracted wave, and can form no part of a physically meaningful solution. Thus the use of the asymmetric contour $\bar{\gamma}$ introduces additional difficulties. In the next two sections we examine alternative contours of two other types but discover that any solution obtainable by using these modified contours has identical form to either of the solutions we discussed before.

3.4.2.5 Shifted symmetric contours,

$$\gamma_m = (\gamma_+ + 2m\pi) + (\gamma_- - 2m\pi)$$

We note that any solution obtained by integrating, $s(\alpha)$ of the general form (48a) and the corresponding function $s_1(\zeta(\alpha))$, over any arbitrary contour $\gamma_a+\phi$, where γ_a is symmetric about the origin, will satisfy Maxwell's equations and the boundary conditions. We could choose γ_a to be γ_m where γ_m is shown in Fig 3.11 for m=1. Let us examine the solution $E^{(m)}(\rho,\phi)$ over such a contour.

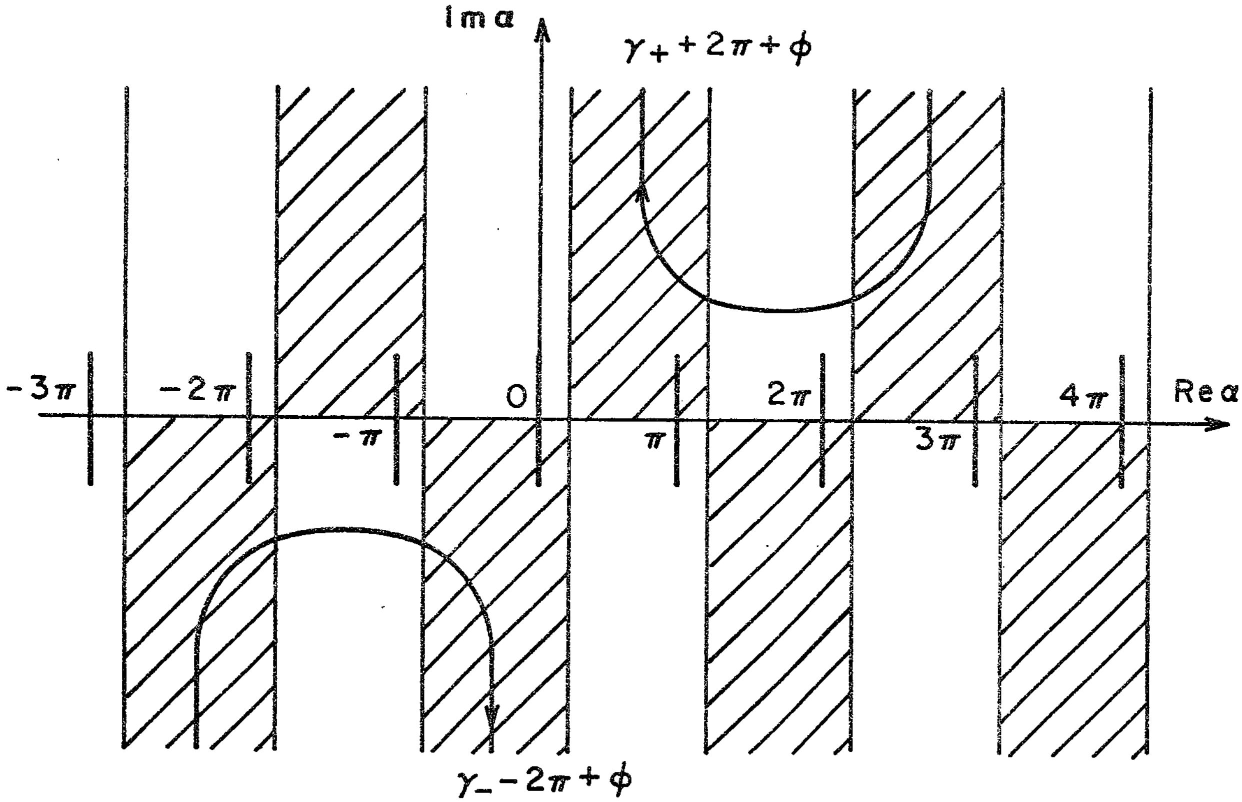


Fig. 3.11 Shifted symmetric contour γ_m for the case $m = 1$.

$$2\pi jE^{(m)}(\rho,\phi) = \int_{\gamma_m'} s^{(m)}(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha$$

$$= \int_{\gamma_+'} s^{(m)}(\alpha+2m\pi)e^{-j\rho\cos(\alpha-\phi)}d\alpha + \int_{\gamma_-'} s^{(m)}(\alpha-2m\pi)e^{-j\rho\cos(\alpha-\phi)}d\alpha \tag{60}$$

where

$$s^{(m)}(\alpha) = J_{2\pi}^{(m)}(\alpha) + P_{e\pi}^{(m)}\cos\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right) + P_{0\pi}^{(m)}\sin\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right) \tag{61}$$

and primes on γ_m, γ_+ and γ_- indicate a shift by ϕ.

By noting the periodicity of the functions $J_{2\pi}^{(m)}$, $P_{e\pi}^{(m)}$, $P_{0\pi}^{(m)}$, t and making use of the decomposition relation.

$$\int_{\Gamma_1} f_1 + \int_{\Gamma_2} f_2 \equiv \frac{1}{2}\int_{\Gamma_1+\Gamma_2}(f_1+f_2) + \frac{1}{2}\int_{\Gamma_1-\Gamma_2}(f_1-f_2) \tag{62}$$

where f_1, f_2 are arbitrary functions integrated over arbitrary contours Γ_1, Γ_2 we can write $E^{(m)}(\rho,\phi)$ in the following form

$$2\pi jE^{(m)}(\rho,\phi) = \int_{\gamma+\phi}\left\{J_{2\pi}^{(m)}(\alpha) + \cos 2mt\left[P_{e\pi}^{(m)}\cos\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right)\right.\right.$$

$$\left.\left. + P_{0\pi}^{(m)}\sin\left(\frac{\alpha t}{\pi} \quad \frac{3t}{2}\right)\right]\right\} \times e^{-j\rho\cos(\alpha-\phi)}d\alpha$$

$$+ \int_{\bar{\gamma}+\phi} \sin 2mt\left[-P_{e\pi}^{(m)}\sin\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right) + P_{0\pi}^{(m)}\cos\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right)\right]$$

$$\times e^{-j\rho\cos(\alpha-\phi)}d\alpha \tag{63}$$

where γ and $\bar{\gamma}$ are symmetric and asymmetric contours that we discussed earlier. We note that $E^{(m)}(\rho,\phi)$ is of the form

$$2\pi j E^{(m)}(\rho,\phi) = \int_{\gamma+\phi} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha + \int_{\bar{\gamma}+\phi} \bar{s}(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha \qquad (64)$$

while s and $\bar{s}$ are of the general form (48a) and (57a) respectively. Thus any solution over γ_m' may be represented as a combination of solutions over γ' and $\bar{\gamma}'$. But we have already noted that any combination of solutions over γ' and $\bar{\gamma}'$ will have diverging branch cut integrals and/or incoming diffracted waves. Hence any combination of solutions over γ_m', γ', $\bar{\gamma}'$ will have the same difficulties and fail to give a physically meaningful solution.

3.4.2.6 Shifted asymmetric contours,

$$\bar{\gamma}_m = (\gamma_+ + 2m\pi) - (\gamma_- - 2m\pi)$$

In a manner similar to the previous section, any solution, over an asymmetric contour of the form $\bar{\gamma}_m' = \bar{\gamma}_m + \phi$, may be represented as a sum of solutions over the contours γ' and $\bar{\gamma}'$. Hence a solution over such a contour does not possess any additional advantage.

3.4.2.7 Contours with the end points separated by $2m\pi$ with $m>1$

All such contours may be decomposed as a sum of the form $\gamma + \gamma_m$ or $\bar{\gamma} + \bar{\gamma}_m$ depending upon their symmetric or asymmetric nature as shown in Fig.3.12 for the case of a symmetric contour with $m = 2$.

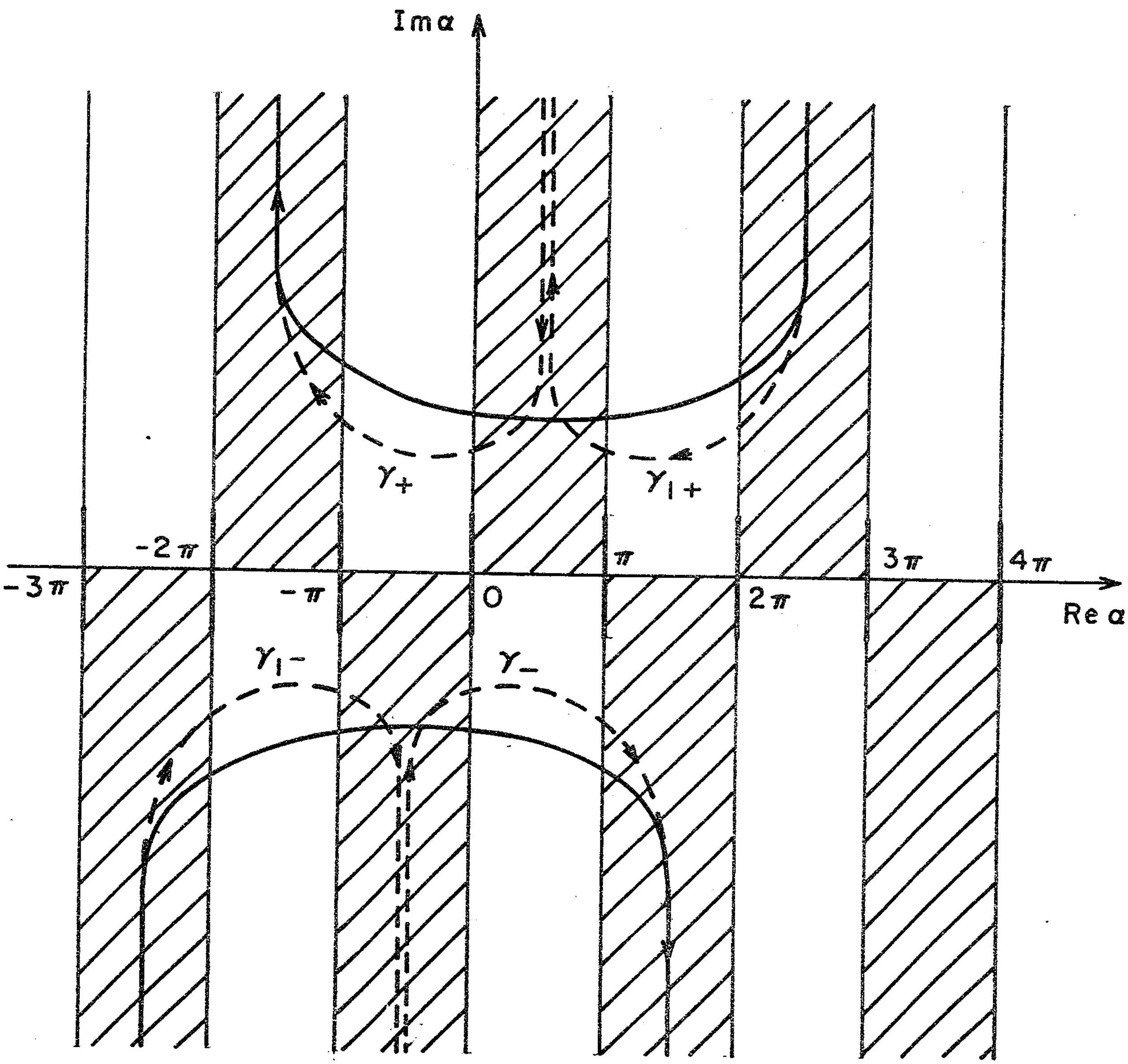


Fig. 3.12 Decomposition of a symmetric contour whose end points are separated by $2\pi m$ for the case $m = 2$.

Thus we have exhausted all possible symmetric and asymmetric contours which do not cross the real axis and whose end points at infinity start in one shaded region and end in another shaded region. The end points of any moving contour must lie in a shaded region at infinity so that the Sommerfeld integral does not alter as the contour is shifted with ϕ. Further any contour whose end points at infinity are in the same shaded region are trivial since any integral over them would identically vanish unless they enclosed singularities by crossing the real axis. Any contour that crosses the real axis is not useful in our method since such a contour would cut a branch cut as it is shifted with ϕ.

We consider two other possibilities. One is to extend the branch cuts up and down wards to infinity rather than joining them pairwise. This is definitely not workable for our present geometry since such a definition of branch cuts would restrict the movement of the contour to less than π whereas we require it to be moveable by $3\pi/2$ without cutting branch cuts. However we explore this possibility in the context of a different problem where a movement of π is sufficient. The second possibility is using finite and fixed contours. We discuss both these approaches in the next section, and conclude that both of the approaches fail to give the desired solution.

3.5 A MIXED BOUNDARY VALUE PROBLEM

Let us consider the problem of diffraction by a rectangular dielectric wedge, resting against an infinite plane whose upper semi-infinite segment ($\phi = \pi/2$) is a perfect electric conductor and the lower segment ($\phi = -\pi/2$) is a perfect magnetic conductor. The incident wave is assumed to be polarized in the z-direction as shown in Fig.3.13. Because of the different boundary conditions on the lower and upper segments of the infinite plane the problem is non-trivial.

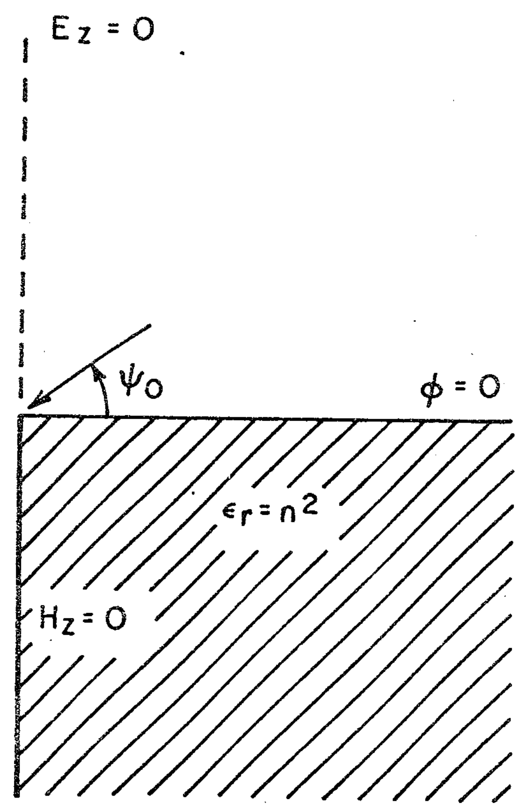


Figure 3.13. Geometry of the mixed boundary value problem.

3.5.1 Zavadskii's solution

Proceeding in the manner described before we seek the fields $E(\rho,\phi)$ and $E_1(\rho,\phi)$ in the following form.

$$E(\rho,\phi) = \frac{1}{2\pi j} \int_{\gamma+\phi} s(\alpha) e^{-j\rho\cos(\alpha-\phi)} d\alpha \qquad 0 \leq \phi \leq \frac{\pi}{2} \tag{65a}$$

$$E_1(\rho,\phi) = \frac{1}{2\pi j} \int_{\Gamma+\phi} s_1(\zeta) e^{-j\rho n\cos(\zeta-\phi)} d\zeta \qquad -\frac{\pi}{2} \leq \phi \leq 0 \tag{65b}$$

where the variables α,ζ and the contours γ and Γ have the same meaning as before.

The boundary conditions are given by

$$E(\rho,\pi/2) = 0 \tag{66a}$$

$$\left.\frac{\partial E_1(\rho,\phi)}{\partial\phi}\right|_{\phi=-\frac{\pi}{2}} = 0 \tag{66b}$$

$$E(\rho,0) = E_1(\rho,0) \tag{66c}$$

$$\left.\frac{\partial E(\rho,\phi)}{\partial\phi}\right|_{\phi=0} = \left.\frac{\partial E_1(\rho,\phi)}{\partial\phi}\right|_{\phi=0} \tag{66d}$$

and (66a)-(66d) lead to the following set of functional equations for $s(\alpha)$ and $s_1(\zeta(\alpha))$:

$$s(\alpha) = s(-\alpha+\pi) \tag{67a}$$

$$s_1(\zeta(\alpha)) = -s_1(\zeta(-\alpha-\pi)) \tag{67b}$$

$$s(\alpha) - s(-\alpha) = \tau(\alpha)[s_1(\zeta(\alpha)) - s_1(\zeta(-\alpha))] \tag{67c}$$

$$s(\alpha) + s(-\alpha) = s_1(\zeta(\alpha)) + s_1(\zeta(-\alpha)) \tag{67d}$$

Proceeding in a manner similar to that shown in Appendix V we obtain the following general solution to $s(\alpha)$.

$$s(\alpha) = P_{e\pi}\cos\left(\frac{\alpha t_2}{\pi} - \frac{t_2}{2}\right) + P_{o\pi}\sin\left(\frac{\alpha t_2}{\pi} - \frac{t_2}{2}\right) \tag{68a}$$

where

$$\cos t_2 = \frac{1 - \tau(\alpha)}{1 + \tau(\alpha)}\ ; \qquad 0 \le t_2 \le \pi \tag{68b}$$

and $P_{e\pi}$ and $P_{o\pi}$ have the same meaning as before.

If we proceed by using the t-transform, as shown in Appendix VIII, we obtain the following particular solutions to $s(\alpha)$ and $s_1(\zeta(\alpha))$.

$$s(\alpha) = \frac{\cos t_2}{\sin t_2}\left[\frac{\sin\{\frac{\psi_0 - \alpha}{\pi}(\pi - t_2)\}}{\sin(\psi_0 - \alpha)} + \frac{\sin\{\frac{\psi_0 + \alpha - \pi}{\pi}(\pi - t_2)\}}{\sin(\psi_0 + \alpha)}\right]$$

$$+ \frac{1}{\sin t_2}\left[\frac{\sin\{\frac{\psi_0 + \alpha}{\pi}(\pi - t_2)\}}{\sin(\psi_0 + \alpha)} + \frac{\sin\{\frac{\psi_0 - \alpha - \pi}{\pi}(\pi - t_2)\}}{\sin(\psi_0 - \alpha)}\right] \tag{69a}$$

$$s_1(\zeta(\alpha)) = \frac{1 + \tau(\alpha)}{2\tau(\alpha)}\left[\frac{\sin\{\frac{\psi_0 + \alpha}{\pi}(\pi - t_2)\}}{\sin(\psi_0 + \alpha)} + \frac{\sin\{\frac{\psi_0 - \alpha - \pi}{\pi}(\pi - t_2)\}}{\sin(\psi_0 - \alpha)}\right]\sin t_2 \tag{69b}$$

$s(\alpha)$ may be put into the following form so as to conform with (68a)

$$s(\alpha) = \left[2\sin 2\psi_0 \cos\left(\frac{\psi_0 t_2}{\pi} - \frac{t_2}{2}\right)\cos\left(\frac{\alpha t_2}{\pi} - \frac{t_2}{2}\right)\right.$$

$$\left. + 2\sin 2\alpha \sin\left(\frac{\psi_0 t_2}{\pi} - \frac{t_2}{2}\right)\sin\left(\frac{\alpha t_2}{\pi} - \frac{t_2}{2}\right)\right]\frac{1}{\cos 2\psi_0 - \cos 2\alpha}$$

$$- 2\sin\left(\frac{\psi_0 t_2}{\pi} - \frac{t_2}{2}\right)\cos\left(\frac{\alpha t_2}{\pi} - \frac{t_2}{2}\right) \tag{70a}$$

Similarly $s_1(\zeta(\alpha))$ may be transformed into the following form.

$$s_1(\zeta(\alpha)) = \frac{1+\tau(\alpha)}{\tau(\alpha)}\left\{\frac{\sin 2\psi_0 \cos(\frac{\psi_0 t_2}{\pi} - \frac{t_2}{2})\sin(\frac{\alpha t_2}{\pi} + \frac{t_2}{2}) - \sin 2\alpha \sin(\frac{\psi_0 t_2}{\pi} - \frac{t_2}{2})\cos(\frac{\alpha t_2}{\pi} + \frac{t_2}{2})}{\cos 2\psi_0 - \cos 2\alpha}\right.$$

$$\left. - \sin(\frac{\psi_0 t_2}{\pi} - \frac{t_2}{2})\sin(\frac{\alpha t_2}{\pi} + \frac{t_2}{2})\right\}\sin t_2 \qquad (70b)$$

From (69b) and (70b) we obtain the following two equivalent expressions for $s_1(\alpha)$:

$$s_1(\alpha) = \left\{\frac{\sin 2\psi_0 \cos(\frac{\psi_0 \hat{t}_2}{\pi} - \frac{\hat{t}_2}{2})\sin(\frac{\hat{\alpha}\hat{t}_2}{\pi} + \frac{\hat{t}_2}{2}) - \sin 2\hat{\alpha}\sin(\frac{\psi_0 \hat{t}_2}{\pi} - \frac{\hat{t}_2}{2})\cos(\frac{\hat{\alpha}\hat{t}_2}{\pi} + \frac{\hat{t}_2}{2})}{(\cos^2\psi_0 - \cos^2\hat{\alpha})}\right.$$

$$\left. - 2\sin(\frac{\psi_o \hat{t}_2}{\pi} - \frac{\hat{t}_2}{2})\sin(\frac{\hat{\alpha}\hat{t}_2}{\pi} + \frac{\hat{t}_2}{2})\right\} \frac{1+\tau(\hat{\alpha})}{2\tau(\hat{\alpha})} \sin\hat{t}_2 \qquad (71a)$$

where

$$\hat{\alpha} = \cos^{-1}(n\cos\alpha) \qquad (71b)$$

$$\hat{t}_2 = t_2(\hat{\alpha}) \qquad (71c)$$

or

$$s_1(\alpha) = \sin\hat{t}_2 \left[\frac{\sin\{\frac{\psi_0 + \hat{\alpha}}{\pi}(\pi - \hat{t}_2)\}}{\sin(\psi_0 + \hat{\alpha})} + \frac{\sin\{\frac{\psi_0 - \hat{\alpha} - \pi}{\pi}(\pi - \hat{t}_2)\}}{\sin(\psi_0 - \hat{\alpha})}\right] \frac{1+\tau(\hat{\alpha})}{2\tau(\hat{\alpha})} \qquad (72)$$

It should be noted that $\alpha = \cos^{-1}\frac{1}{n}$ is not a pole of $s_1(\alpha)$ since $\sin\hat{t}_2$ also vanishes at this point. We write E and E_1 as sums of three terms corresponding to pole contribution, diffraction and branch cut contribution as below.

$$E(\rho,\phi) = E_p(\rho,\phi) + E_d(\rho,\phi) + E_b(\rho,\phi) \tag{73a}$$

$$E_1(\rho,\phi) = E_{p1}(\rho,\phi) + E_{d1}(\rho,\phi) + E_{b1}(\rho,\phi) \tag{73b}$$

3.5.1.1 Pole contribution

To find $E_p(\rho,\phi)$ we consider the following two cases:

i) $0 < \phi < \psi_0 < \pi/2$

In this case the poles at $-\pi + \psi_0, -\psi_0, \psi_0$, and $\pi - \psi_0$ contribute to the solution with the residues $\cos t_{20}, -\cos t_{20}, 1$, and -1 respectively, where

$$t_{20} = t_2(\psi_0) \tag{74}$$

ii) $0 < \psi_0 < \phi < \pi/2$

In this case the poles at $\pi+\psi_0, -\psi_0, \psi_o$, and $\pi-\psi_0$ contribute to the solution with the residues $\cos t_{20}, -\cos t_{20}, 1$, and -1 respectively.

Thus in both cases E_p is given by

$$E_p(\rho,\phi) = e^{-j\rho\cos(\phi-\psi_0)} - e^{+j\rho\cos(\phi+\psi_0)}$$

$$- \frac{1-\tau(\psi_0)}{1+\tau(\psi_0)} \{e^{-j\rho\cos(\phi+\psi_0)} - e^{+j\rho\cos(\phi-\psi_0)}\}; \quad 0 < \phi < \pi/2 \tag{75}$$

Similarly we consider the pole contribution from $s_1(\alpha)$. The only poles of $s_1(\alpha)$ that contribute to the solution are at $\alpha = \hat{\psi}_0$ and $\pi - \hat{\psi}_0$ both of which have a residue of $2\sin\psi_o/[\sqrt{n^2 - \cos^2\psi_0} + \sin\psi_0]$ and E_{p1} is given by

$$E_{p1}(\rho,\phi) = \frac{2\sin\psi_0}{\sqrt{n^2 - \cos^2\psi_0} + \sin\psi_0} \left[e^{-j\rho n\cos(\phi-\hat{\psi}_0)} + e^{+j\rho n\cos(\phi+\hat{\psi}_0)}\right] \tag{76a}$$

where

$$\hat{\psi}_0 = \cos^{-1}\frac{\cos\psi_0}{n}\,; \qquad 0 < \hat{\psi}_0 < \pi/2 \tag{76b}$$

3.5.1.2 Diffracted field

Excluding the shadow boundary regions, where $\phi \simeq \psi_0$ or $\phi \simeq \hat{\psi}_0$, we obtain the following asymptotic expressions for the diffracted fields E_d and E_{d1}:

$$E_d(\rho,\phi) \simeq -\{s(\pi+\phi) - s(-\pi+\phi)\}\, e^{j(\rho+\frac{\pi}{4})}/\sqrt{2\pi\rho}$$

$$= -\sqrt{\frac{2}{\pi\rho}}\, e^{j(\rho+\frac{\pi}{4})} \left\{ \frac{\cos\{\frac{\psi_0 - \phi}{\pi}(\pi - t_\phi)\}\cos t_\phi + \cos\{\frac{\psi_0 - \phi - \pi}{\pi}(\pi - t_\phi)\}}{\sin(\psi_0 - \phi)} \right.$$

$$\left. - \frac{\cos\{\frac{\psi_0 + \phi - \pi}{\pi}(\pi - t_\phi)\}\cos t_\phi + \cos\{\frac{\psi_0 + \phi}{\pi}(\pi - t_\phi)\}}{\sin(\psi_0 + \phi)} \right\} \tag{77a}$$

$$E_{d_1}(\rho,\phi) \simeq -\{s_1(\phi+\pi) - s_1(\phi-\pi)\}e^{j(n\rho+\frac{\pi}{4})}/\sqrt{2\pi n\rho}$$

$$= -\frac{2\sin^2\hat{t}_\phi}{1-\cos\hat{t}_\phi}\left[\frac{\cos\{\frac{\psi_0-\hat{\phi}-\pi}{\pi}(\pi-\hat{t}_\phi)\}}{\sin(\psi_0-\hat{\phi})} - \frac{\cos\{\frac{\psi_0+\hat{\phi}}{\pi}(\pi-\hat{t}_\phi)\}}{\sin(\psi_0+\hat{\phi})}\right]e^{j(n\rho+\frac{\pi}{4})}/\sqrt{2\pi n\rho}$$

$$;\phi \neq -\hat{\psi}_0, \quad \pm\pi-\hat{\psi}_0 \tag{77b}$$

where

$$t_\phi = t_2(\phi) \tag{77c}$$

$$\hat{t}_\phi = \hat{t}_2(\phi) = t_2(\hat{\phi}) \tag{77d}$$

$$\hat{\phi} = \cos^{-1}(n\cos\phi) \tag{77e}$$

If we let the refractive index n to approach unity (77a),(77b) take the identical form given by

$$E_d(\rho,\phi)\Big|_{n=1} = E_{d1}(\rho,\phi)\Big|_{n=1} \simeq -\left\{\frac{1}{\cos(\frac{\psi_0-\phi}{2})} - \frac{1}{\sin(\frac{\psi_0+\phi}{2})}\right\}e^{j(\rho+\frac{\pi}{4})}/\sqrt{2\pi\rho}$$

$$0 < \psi_0 < \pi/2; \ \phi \neq \pm\psi_0 \tag{78}$$

3.5.1.3 Branch cut integrals

The branch cut contribution $E_b(\rho,\phi)$ to the fields in the region $0 < \phi < \pi/2$ is given by (Figs.3.14a,14b),

$$E_b(\rho,\phi) = \int_{B_{0+}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha + \int_{B_{\pi-}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha$$

$$+ \int_{B_{0-}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha + \int_{B_{\pi+}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha \quad (79)$$

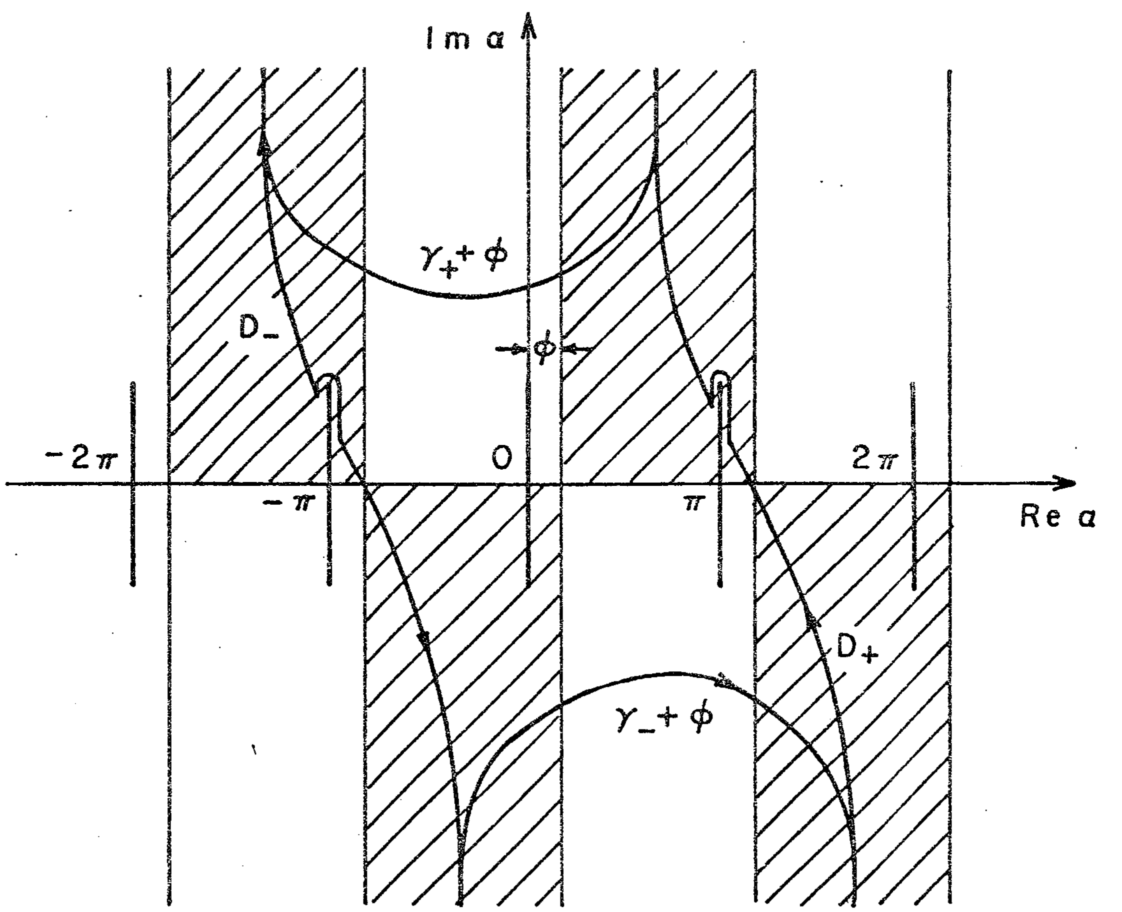


Figure 3.14a. Closing of the contour γ by means of the contours D_+ and D_-.

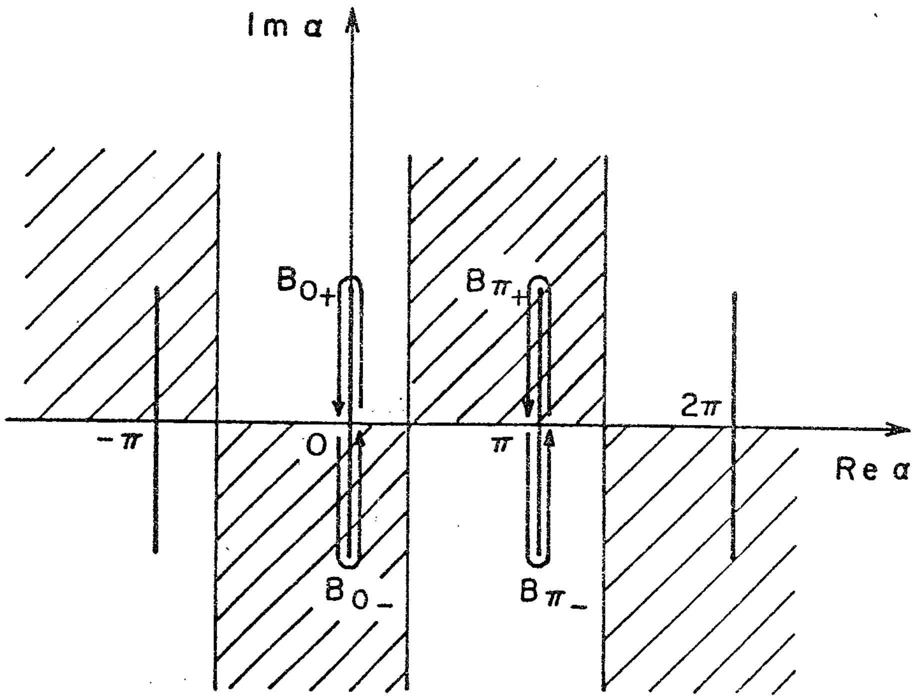


Figure 3.14b. Branch cut contours B_{0+}, B_{0-}, $B_{\pi+}$, and $B_{\pi-}$ in the plane of the complex variable α.

The integrals around B_{0-} and $B_{\pi+}$ are well behaved, but the integrals around B_{0+} and $B_{\pi-}$ are both divergent at infinity as discussed before. In section 3.4.2 we considered all possible solutions over infinite contours and failed to obtain a solution free of diverging waves for the problem of rectangular wedge on a semi-infinite plate. In the next section we explore two other methods of modifying Zavadskii's solution to the present problem.

3.5.2 Attempts to correct Zavadskii's solution

In the first part of this section we aim to find a solution $(\tilde{E},\tilde{E}_1)$ obtained by integrating over fixed finite contours, which when added to Zavadskii's solution would exactly cancel the first two branch cut integrals in (79). In the second part of this section we explore the possibility of obtaining a solution by re-defining the branch cuts vertically to infinity.

3.5.2.1 Secondary solution using fixed finite contours around the branch cuts

We define the fields $\tilde{E}$ and $\tilde{E}_1$ as follows.

$$2\pi j\tilde{E}(\rho,\phi) = -\int_{B_{0+}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha - \int_{B_{\pi-}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha$$

$$+ \int_{B_{0-}} [h(\alpha) + \overline{h}(\alpha)]e^{-j\rho\cos(\alpha-\phi)}d\alpha$$

$$+ \int_{B_{\pi+}} [h(\alpha) - \overline{h}(\alpha)]e^{-j\rho\cos(\alpha-\phi)}d\alpha$$

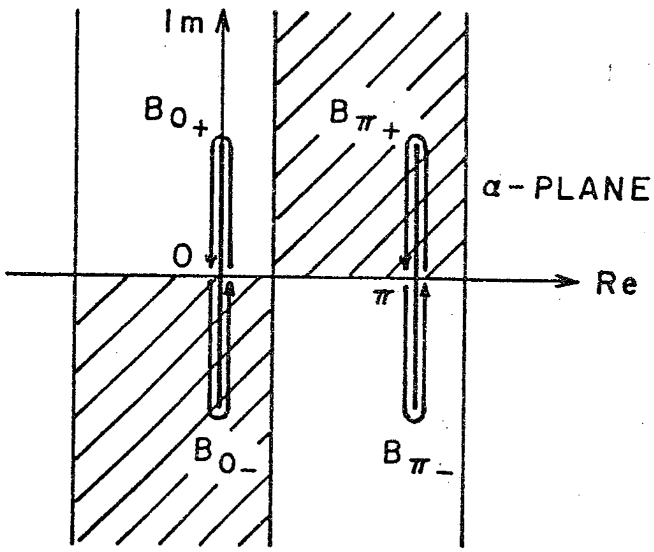
$$0 \leq \phi \leq \pi/2 \qquad (80a)$$

$$2\pi j\tilde{E}_1(\rho,\phi) = \int_{\overline{B}_{0+}} [F(\zeta) + \overline{F}(\zeta)]e^{-j\rho n\cos(\zeta-\phi)}d\zeta + \int_{\overline{B}_{-\pi-}} [F(\zeta) - \overline{F}(\zeta)]e^{-j\rho n\cos(\zeta-\phi)}d\zeta$$

$$+ \int_{\overline{B}_{0-}} [G(\zeta) + \overline{G}(\zeta)]e^{-j\rho n\cos(\zeta-\phi)}d\zeta + \int_{\overline{B}_{-\pi+}} [G(\zeta) - \overline{G}(\zeta)]e^{-j\rho n\cos(\zeta-\phi)}d\zeta$$

$$0 \geq \phi \geq -\pi/2 \qquad (80b)$$

where the contours of integration are shown in Fig 3.14c.



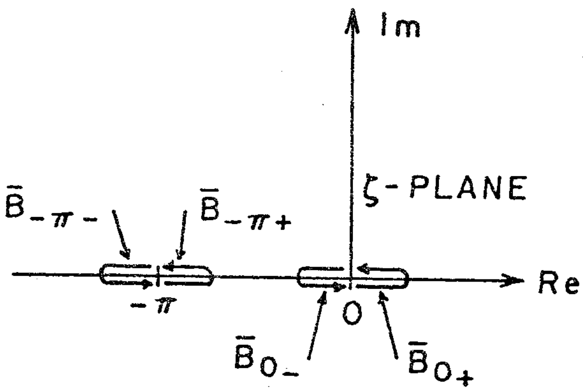


Figure 3.14c. Branch cut contours $\bar{B}_{0+}$, $\bar{B}_{0-}$, $\bar{B}_{-\pi+}$, and $\bar{B}_{-\pi-}$ in the plane of the complex variable ζ.

It should be noted that all the integrals except those over B_{0+} and $B_{\pi-}$ are convergent. The integrals over B_{0+} and $B_{\pi-}$ appear in $\tilde{E}(\rho,\phi)$ in such a manner that the sum $E + \tilde{E}$ will be free of any divergent waves. Since all the branch cuts in ζ-plane are along the real axis E_1 and $\tilde{E}_1$ will be free of any divergent waves. Now we examine the possibility of finding the functions F, $\bar{F}$, G, $\bar{G}$, h and $\bar{h}$ such that $\tilde{E}$ and $\tilde{E}_1$ satisfy the following boundary conditions.

$$\tilde{E}(\rho,\pi/2) = 0 \tag{81a}$$

$$\tilde{E}(\rho,0) = \tilde{E}_1(\rho,0) \tag{81b}$$

$$\tilde{E}'(\rho,0) = \tilde{E}_1'(\rho,0) \tag{81c}$$

$$\tilde{E}_1'(\rho,-\pi/2) = 0 \tag{81d}$$

By a suitable change of variable we can write $\tilde{E}$, $\tilde{E}'$, $\tilde{E}_1$, and $\tilde{E}_1'$ as follows.

$$2\pi j\tilde{E}(\rho,\phi) = -\int_{B_{0+}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha + \int_{B_{0+}} s(-\alpha+\pi)e^{j\rho\cos(\alpha+\phi)}d\alpha$$

$$-\int_{B_{0+}} [h(-\alpha)+\overline{h}(-\alpha)]e^{-j\rho\cos(\alpha+\phi)}d\alpha + \int_{B_{0+}} [h(\alpha+\pi)-\overline{h}(\alpha+\pi)]e^{j\rho\cos(\alpha-\phi)}d\alpha \tag{82a}$$

$$(-\frac{2\pi}{\rho})\tilde{E}'(\rho,\phi) = -\int_{B_{0+}} \sin(\alpha-\phi)s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha + \int_{B_{0+}} \sin(\alpha+\phi)s(-\alpha+\pi)e^{j\rho\cos(\alpha+\phi)}d\alpha$$

$$+\int_{B_{0+}} \sin(\alpha+\phi)[h(-\alpha)+\overline{h}(-\alpha)]e^{-j\rho\cos(\alpha+\phi)}d\alpha - \int_{B_{o+}} \sin(\alpha-\phi)[h(\alpha+\pi)-\overline{h}(\alpha+\pi)]e^{j\rho\cos(\alpha-\phi)}d\alpha \tag{82b}$$

$$2\pi j\tilde{E}_1(\rho,\phi) = \int_{\overline{B}_{0+}} [F(\zeta)+\overline{F}(\zeta)]e^{-j\rho n\cos(\zeta-\phi)}d\zeta - \int_{\overline{B}_{0+}} [F(-\zeta-\pi)-\overline{F}(-\zeta-\pi)]e^{+j\rho n\cos(\zeta+\phi)}d\zeta$$

$$-\int_{\overline{B}_{0+}} [G(-\zeta)+\overline{G}(-\zeta)]e^{-j\rho n\cos(\zeta+\phi)}d\zeta + \int_{\overline{B}_{0+}} [G(\zeta-\pi)-\overline{G}(\zeta-\pi)]e^{+j\rho n\cos(\zeta-\phi)}d\zeta \tag{82c}$$

$$\left(-\frac{2\pi}{\rho}\right)\tilde{E}_1''(\rho,\phi) = \int_{\overline{B}_{0+}} n\sin(\zeta-\phi)\,[F(\zeta)+\overline{F}(\zeta)]\,e^{-j\rho n\cos(\zeta-\phi)}d\zeta$$

$$- \int_{\overline{B}_{0+}} n\sin(\zeta+\phi)\,[F(-\zeta-\pi)-\overline{F}(-\zeta-\pi)]\,e^{j\rho n\cos(\zeta+\phi)}d\zeta$$

$$+ \int_{\overline{B}_{0+}} n\sin(\zeta+\phi)\,[G(-\zeta)+\overline{G}(-\zeta)]\,e^{-j\rho n\cos(\zeta+\phi)}$$

$$- \int_{\overline{B}_{0+}} n\sin(\zeta-\phi)\,[G(\zeta-\pi)-\overline{G}(\zeta-\pi)]\,e^{j\rho n\cos(\zeta-\phi)}d\zeta \qquad (82d)$$

$$2\pi j\tilde{E}_1(\rho,0) = \int_{B_{0+}} (f(\alpha)+\overline{f}(\alpha))e^{-j\rho\cos\alpha}d\alpha - \int_{B_{0+}} [f(-\alpha-\pi)-\overline{f}(-\alpha-\pi)]e^{j\rho\cos\alpha}d\alpha$$

$$- \int_{B_{0+}} [g(-\alpha)+\overline{g}(-\alpha)]e^{-j\rho\cos\alpha}d\alpha + \int_{B_{0+}} [g(\alpha-\pi)-\overline{g}(\alpha-\pi)]e^{j\rho\cos\alpha}d\alpha \qquad (83)$$

where

$$f(\alpha) = F(\zeta)\tau(\alpha) \qquad (84a)$$

$$\overline{f}(\alpha) = \overline{F}(\zeta)\tau(\alpha) \qquad (84b)$$

$$g(\alpha) = G(\zeta)\tau(\alpha) \qquad (84c)$$

$$\overline{g}(\alpha) = \overline{G}(\zeta)\tau(\alpha) \qquad (84d)$$

Similarly we may write $\tilde{E}'_1(\rho,0)$ as

$$(-\frac{2\pi}{\rho})\tilde{E}'_1(\rho,0) = \int_{B_{0+}} \frac{\sin\alpha}{\tau(\alpha)}[f(\alpha) + \overline{f}(\alpha)]e^{-j\rho\cos\alpha}d\alpha$$

$$- \int_{B_{0+}} \frac{\sin\alpha}{\tau(\alpha)}[f(-\alpha-\pi) - \overline{f}(-\alpha-\pi)]e^{j\rho\ \cos\alpha}d\alpha$$

$$+ \int_{B_{0+}} \frac{\sin\alpha}{\tau(\alpha)}[g(-\alpha) + \overline{g}(-\alpha)]e^{-j\rho\cos\alpha}d\alpha$$

$$- \int_{B_{0+}} \frac{\sin\alpha}{\tau(\alpha)}[g(\alpha-\pi) - \overline{g}(\alpha-\pi)]e^{j\rho\ \cos\alpha}d\alpha \qquad (85)$$

The boundary conditions (81a) and 81d) lead to the following functional equations.

$$h(-\alpha) = h(\alpha+\pi) \qquad (86a)$$

$$\overline{h}(-\alpha) = -\overline{h}(\alpha+\pi) \qquad (86b)$$

$$f(\alpha) = -f(-\alpha-\pi) \qquad (86c)$$

$$\overline{f}(\alpha) = +\overline{f}(-\alpha-\pi) \qquad (86d)$$

$$g(\alpha) = -g(-\alpha-\pi) \qquad (86e)$$

$$\overline{g}(\alpha) = \overline{g}(-\alpha-\pi) \qquad (86f)$$

The boundary conditions (81b) and 81c) at $\phi = 0$ lead to the following functional equations.

$$-s(\alpha) - h(-\alpha) - \overline{h}(-\alpha) = f(\alpha) + \overline{f}(\alpha) - g(-\alpha) - \overline{g}(-\alpha) \tag{87a}$$

$$s(\alpha) + h(\alpha+\pi) - \overline{h}(\alpha+\pi) = -f(-\alpha-\pi) + \overline{f}(-\alpha-\pi) + g(\alpha-\pi) - \overline{g}(\alpha-\pi) \tag{87b}$$

$$-s(\alpha) + h(-\alpha) + \overline{h}(-\alpha) = \frac{1}{\tau(\alpha)}[f(\alpha) + \overline{f}(\alpha) + g(-\alpha) + \overline{g}(-\alpha)] \tag{87c}$$

$$s(\alpha) - h(\alpha+\pi) + \overline{h}(\alpha+\pi) = \frac{1}{\tau(\alpha)}[-f(-\alpha-\pi) + \overline{f}(-\alpha-\pi) - g(\alpha-\pi) + \overline{g}(\alpha-\pi)] \tag{87d}$$

Combining (87a)-(87d) with (86a)-(86f) we obtain

$$-s(\alpha) - h(\alpha+\pi) + \overline{h}(\alpha+\pi) = f(\alpha) + \overline{f}(\alpha) + g(\alpha-\pi) - \overline{g}(\alpha-\pi) \tag{88a}$$

$$s(\alpha) + h(\alpha+\pi) - \overline{h}(\alpha+\pi) = f(\alpha) + \overline{f}(\alpha) + g(\alpha-\pi) - \overline{g}(\alpha-\pi) \tag{88b}$$

$$-s(\alpha) + h(\alpha+\pi) - \overline{h}(\alpha+\pi) = [f(\alpha) + \overline{f}(\alpha) - g(\alpha-\pi) + \overline{g}(\alpha-\pi)]/\tau(\alpha) \tag{88c}$$

$$s(\alpha) - h(\alpha+\pi) + \overline{h}(\alpha+\pi) = [f(\alpha) + \overline{f}(\alpha) - g(\alpha-\pi) + \overline{g}(\alpha-\pi)]/\tau(\alpha) \tag{88d}$$

The structure of equations (88a)-(88d) is such that they represent four equations in three unknowns and they are incompatible. Thus we cannot find f, $\overline{f}$, g, $\overline{g}$, h, and $\overline{h}$ such that $\tilde{E}$, $\tilde{E}_1$ would meet the boundary conditions (81a)-(81d). However the boundary conditions (81b) and (81c) may also be satisfied by requiring that

$$-s(\alpha) - h(\alpha+\pi) + \overline{h}(\alpha+\pi) - f(\alpha) - \overline{f}(\alpha) - g(\alpha-\pi) + \overline{g}(\alpha-\pi) = P_1(\alpha) \tag{89a}$$

$$s(\alpha) + h(\alpha+\pi) - \overline{h}(\alpha+\pi) - f(\alpha) - \overline{f}(\alpha) - g(\alpha-\pi) + \overline{g}(\alpha-\pi) = P_2(\alpha) \tag{89b}$$

$$-s(\alpha) + h(\alpha+\pi) - \overline{h}(\alpha+\pi) - [f(\alpha) + \overline{f}(\alpha) - g(\alpha-\pi) + \overline{g}(\alpha-\pi)]/\tau(\alpha) = P_3(\alpha) \tag{89c}$$

$$s(\alpha) - h(\alpha+\pi) + \overline{h}(\alpha+\pi) - [f(\alpha) + \overline{f}(\alpha) - g(\alpha-\pi) + \overline{g}(\alpha-\pi)]/\tau(\alpha) = P_4(\alpha) \tag{89d}$$

where P_1, P_2, P_3, P_4 are arbitrary functions of α without any branch points. By solving (89a)-(89d) we find $s(\alpha)$ given by

$$s(\alpha) = \frac{1}{4}[P_2(\alpha) - P_1(\alpha) + P_4(\alpha) - P_3(\alpha)] \tag{90}$$

which is not possible since $s(\alpha)$ has branch points. There remains just one more way of satisfying the conditions (81b) and (81c), that is by directly equating $\tilde{E}(\rho,0)$ to $\tilde{E}_1(\rho,0)$ and $\tilde{E}'(\rho,0)$ to $\tilde{E}'_1(\rho,0)$ which leads to the following two integral equations.

$$j \int_{B_{0+}} [s(\alpha) + h(\alpha+\pi) - \overline{h}(\alpha+\pi)] \sin(\rho\cos\alpha) d\alpha$$

$$= \int_{B_{0+}} [f(\alpha) + \overline{f}(\alpha) + g(\alpha-\pi) - \overline{g}(\alpha-\pi) \cos(\rho\cos\alpha) d\alpha \tag{91a}$$

$$j \int_{B_{0+}} \sin\alpha[s(\alpha) - h(\alpha+\pi) + \overline{h}(\alpha+\pi)]\sin(\rho\cos\alpha)d\alpha$$

$$= \int_{B_{0+}} [f(\alpha) + \overline{f}(\alpha) - g(\alpha-\pi) + \overline{g}(\alpha-\pi)]\frac{\sin\alpha}{\tau(\alpha)}\cos(\rho\cos\alpha)d\alpha \qquad (91b)$$

Because of the nature of the path of integration it is difficult even to answer the question of existence of a solution to these equations. Thus we seem to have come to a dead end. In the next section we investigate the possibility of obtaining a solution by extending the branch cuts vertically to infinity.

3.5.2.2 Formulation with infinite branch cuts

In this section we formulate the problem by extending the branch cuts vertically to infinity as shown in Fig.3.15. We define E and E_1 as follows

$$2\pi jE(\rho,\phi) = \int_{\gamma+\phi} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha \qquad 0 \leq \phi \leq \pi/2 \qquad (92a)$$

$$2\pi jE_1(\rho,\phi) = \int_{\Gamma+\phi} s_1(\zeta)e^{-j\rho n\cos(\zeta-\phi)}d\zeta \qquad 0 \geq \phi \geq -\pi/2 \qquad (92b)$$

where $\zeta = \cos^{-1}(\frac{\cos\alpha}{n})$ (92c)

and Γ is the mapping of γ into ζ-plane. With such a definition of branch cuts $\zeta(\alpha)$ and $\tau(\alpha)$ satisfy the following relations

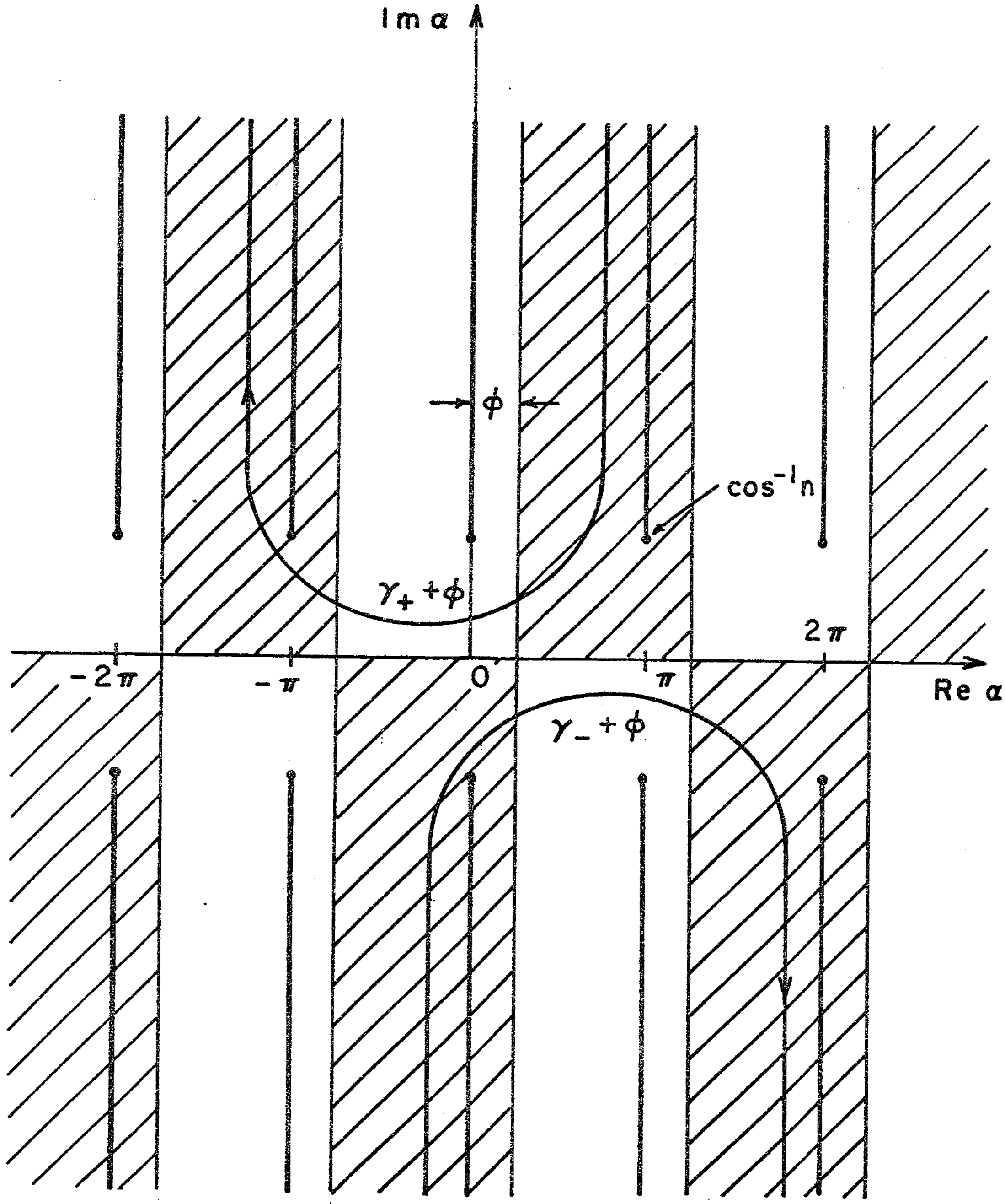


Fig. 3.15 Structure of the infinite branch cuts in the plane of the complex variable α.

$$\zeta(\alpha) = \zeta(-\alpha) \tag{93a}$$

$$\zeta(\alpha+\pi) = -\zeta(\alpha) + \pi \tag{93b}$$

$$\tau(\alpha) = -\tau(-\alpha) \tag{93c}$$

$$\tau(\alpha+\pi) = -\tau(\alpha) \tag{93d}$$

As shown in Appendix IX the boundary conditions of the problem lead to the following functional equations in s and s_1

$$s(-\alpha) = s(\alpha+\pi) \tag{94a}$$

$$s_1(-\zeta) = -s_1(-\zeta(\alpha+\pi)) \tag{94b}$$

$$\frac{1}{\tau(\alpha)}\,[s(\alpha) - s(-\alpha)] = s_1(\zeta) - s_1(-\zeta) \tag{94c}$$

$$s(\alpha) + s(-\alpha) = s_1(\zeta) + s_1(-\zeta) \tag{94d}$$

If we try to solve these equations, as we have done in Appendix X, we obtain expressions for $s_1(\zeta)$ and $s_1(-\zeta)$ which are not compatible with each other.

3.6 CONCLUSIONS

In this report we have made a thorough investigation of Zavadskii's method to try to obtain a solution to electromagnetic diffraction problems involving a rectangular dielectric wedge ($0 \geq \phi \geq -\pi/2$) and

i) Infinite metal plate along $\phi = \pm \pi/2$

ii) Semi-infinite metal plate along $\phi = -\pi/2$

iii) Perfect magnetic conductor along $\phi = -\pi/2$ and a semi-infinite metal plate along $\phi = \pi/2$.

In all the cases that we considered we found that Zavadskii's method, as it is, gives a solution involving branch cut integrals that grow exponentially in the far field thus violating the radiation condition. For the trivial case involving an infinite metal plate we have found a simple way of modifying the solution so as to conform with the known exact solution. We have made several attempts to modify Zavadskii's method to obtain a solution satisfying the radiation condition for the cases ii) and iii) above but none of our attempts proved to be successful. However one of the methods involving a secondary solution with branch cut integrals alone lead to two integral equations whose solution seems to be either very difficult or impossible. In the process of these attempts we discovered the solution to the quasi-trivial problem of illuminating a rectangular dielectric wedge resting on a semi-infinite metal plate with plane waves such that there is no net diffracted wave from the edge.

3.7 ACKNOWLEDGMENTS

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APPENDIX I

In this appendix we obtain an expression I_b for the integrals over the branch cuts $B_0(=B_{0+}+B_{0-})$ and $B_\pi(=B_{\pi+}+B_{\pi-})$ and show that these integrals grow exponentially as $\rho\to\infty$.

$$I_b = \int_{B_{0+}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}\,d\alpha + \int_{B_{0-}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}\,d\alpha$$

$$+ \int_{B_{\pi+}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}\,d\alpha + \int_{B_{\pi-}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}\,d\alpha \qquad \text{(A1.1)}$$

where B_{0+}, B_{0-}, $B_{\pi+}$, and $B_{\pi-}$ are shown in Fig. A1.1.

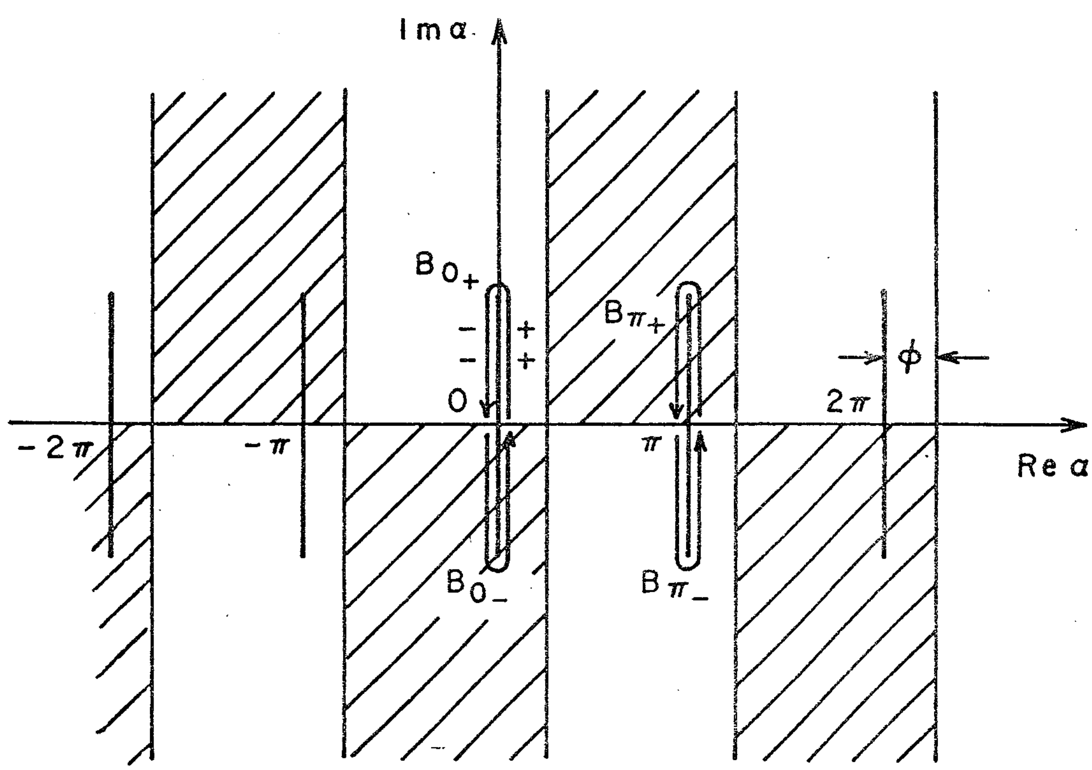

Fig. A1.1 Branch cut contours B_{o+}, B_{o-}, $B_{\pi+}$ and $B_{\pi-}$.

By a suitable change of variable we may write I_b as

$$I_b = \int_{B_{0+}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha - \int_{B_{0+}} s(-\alpha)e^{-j\rho\cos(\alpha+\phi)}$$

$$+ \int_{B_{0+}} s(\alpha+\pi)e^{j\rho\cos(\alpha-\phi)}\,d\alpha - \int_{B_{0+}} s(-\alpha+\pi)e^{j\rho\cos(\alpha+\phi)}\,d\alpha \tag{A1.2}$$

Noting that $s(-\alpha) = s(\alpha+\pi)$ we may write I_b as

$$I_b = \int_{B_{0+}} s(\alpha)\left[e^{-j\rho\cos(\alpha-\phi)} - e^{j\rho\cos(\alpha+\phi)}\right]d\alpha$$

$$- \int_{B_{0+}} s(-\alpha)\left[e^{-j\rho\cos(\alpha+\phi)} - e^{j\rho\cos(\alpha-\phi)}\right]d\alpha$$

$$= -2j\int_{B_{0+}} s(\alpha)e^{-jb\sin\alpha}\sin(a\cos\alpha)d\alpha + 2j\int_{B_{0+}} s(-\alpha)e^{jb\sin\alpha}\sin(a\cos\alpha)d\alpha \tag{A1.3a}$$

where

$$a = \rho\cos\phi \tag{A1.3b}$$

$$b = \rho\sin\phi \tag{A1.3c}$$

$$s(\alpha) = \frac{\cos\psi_0}{\sin\alpha - \sin\psi_0} - \frac{M-\sin\alpha}{M+\sin\alpha}\,\frac{\cos\psi_0}{\sin\alpha+\sin\psi_0} \tag{A1.3d}$$

$$M = (n^2-\cos^2\alpha)^{\frac{1}{2}} \tag{A1.3e}$$

and the sign of M must be chosen as shown in Fig. A1.1. Noting that only the second term in (A1.3d) will contribute to the branch cut integral and choosing the proper sign for M we may write I_b as the following sum of two line integrals.

$$I_b = -8j\cos\psi_0 \int_0^{j\beta_0} \frac{|M|\sin\alpha}{n^2-1} \frac{1}{\sin\alpha + \sin\psi_0} e^{-jb\sin\alpha} \sin(a\cos\alpha)\, d\alpha$$

$$+ 8j\cos\psi_0 \int_0^{j\beta_0} \frac{|M|\sin\alpha}{n^2-1} \frac{1}{\sin\alpha - \sin\psi_0} e^{jb\sin\alpha} \sin(a\cos\alpha)\, d\alpha$$

$$= 8j\cos\psi_0 \int_0^{\beta_0} \frac{|M|\,\mathrm{sh}\beta}{n^2-1} \frac{1}{i\,\mathrm{sh}\beta + \sin\psi_0} e^{b\,\mathrm{sh}\beta} \sin(a\,\mathrm{ch}\beta)\, d\beta$$

$$- 8j\cos\psi_0 \int_0^{\beta_0} \frac{|M|\,\mathrm{sh}\beta}{n^2-1} \frac{1}{i\,\mathrm{sh}\beta - \sin\psi_0} e^{-b\,\mathrm{sh}\beta} \sin(a\,\mathrm{ch}\beta)\, d\beta \qquad \text{(A1.4a)}$$

where $\mathrm{ch}\beta_0 = n$ (A1.4b)

Since a and b are positive when $0 < \phi < \pi/2$, it is immediately clear that the first integral (A1.4a) grows exponentially as $\rho \to \infty$ while the second integral converges for all ρ. We note that the diverging integral corresponds to the unshaded branch cuts B_{0+} and $B_{\pi-}$.

APPENDIX II

In this appendix we obtain $s(\alpha)$ as given by (33a) by inverting $t(\alpha,\beta)$ which is given by (32).

$$t(\alpha,\beta) = 2\mathrm{sh}(\psi_0-3\pi/2)\beta \cdot \frac{\mathrm{ch}(\frac{\pi}{2}\beta)-\tau(\alpha)\,\mathrm{sh}(\frac{\pi}{2}\beta)}{[1+\tau(\alpha)]\,\mathrm{sh}2\pi\beta+[1-\tau(\alpha)]\quad \mathrm{sh}\pi\beta} \tag{A2.1}$$

$$s(\alpha) = \int_{-\infty}^{\infty} t(\alpha,\beta)e^{-\alpha\beta}d\beta \tag{A2.2}$$

$s(\alpha)$ may also be written as the sum of the following integrals.

$$s(\alpha) = \int_0^{\infty} [t(\alpha,\beta)+t(\alpha,-\beta)]\,\mathrm{ch}\alpha\beta d\beta \; - \int_0^{\infty} [t(\alpha,\beta)-t(\alpha,-\beta)]\,\mathrm{sh}\alpha\beta d\beta \tag{A2.3}$$

After substituting for $t(\alpha,\beta)$ and $t(\alpha,-\beta)$ we obtain

$$s(\alpha) = 2 \int_0^{\infty} \frac{\mathrm{sh}[(\psi_0-3\pi/2)\beta]\,[\mathrm{ch}(\frac{\pi}{2}\beta)\,\mathrm{ch}\alpha\beta+\tau(\alpha)\,\mathrm{sh}\frac{\pi}{2}\beta\,\mathrm{sh}\alpha\beta]}{[1+\tau(\alpha)]\;[\mathrm{ch}\pi\beta+A]\,\mathrm{sh}\pi\beta}\,d\beta \tag{A2.4a}$$

where

$$A = \frac{1}{2}\;\frac{1-\tau(\alpha)}{1+\tau(\alpha)} = -\cos t;\; 0 < t < \pi \tag{A2.4b}$$

Noting that

$$2\;[\mathrm{ch}\tfrac{\pi}{2}\beta\,\mathrm{ch}\alpha\beta+\tau(\alpha)\,\mathrm{sh}\tfrac{\pi}{2}\beta\,\mathrm{sh}\alpha\beta] \equiv [1+\tau(\alpha)]\,\mathrm{ch}(\tfrac{\pi}{2}+\alpha)\beta+[1-\tau(\alpha)]\,\mathrm{ch}(\tfrac{\pi}{2}-\alpha)\beta \tag{A2.5}$$

we write $s(\alpha)$ as

$$s(\alpha) = \int_0^\infty \frac{\mathrm{sh}[(\psi_0-3\pi/2)\beta][\mathrm{ch}(\frac{\pi}{2}+\alpha)\beta+2A\mathrm{ch}(\frac{\pi}{2}-\alpha)\beta]}{(\mathrm{ch}\pi\beta+A)\mathrm{sh}\pi\beta}\, d\beta$$

$$= \frac{1}{2}\int_0^\infty \frac{\mathrm{sh}[(\psi_0-\pi+\alpha)\beta]+\mathrm{sh}[(\psi_0-2\pi-\alpha)\beta]+2A\{\mathrm{sh}[(\psi_0-\pi-\alpha)\beta]+\mathrm{sh}[(\psi_0-2\pi+\alpha)\beta]\}}{(\mathrm{ch}\pi\beta+A)\mathrm{sh}\pi\beta}$$

$$.d\beta \qquad \text{(A2.6)}$$

which can be immediately put into the form (33a).

APPENDIX III

In this appendix we transform $s(\alpha)$ as given by (33a) into the form (36).

$$s(\alpha) = \frac{1}{2} I(\psi_0+\alpha-\pi) + \frac{1}{2} I(\psi_0-\alpha-2\pi) + AI(\psi_0-\alpha-\pi) + AI(\psi_0+\alpha-2\pi) \tag{A3.1a}$$

where

$$I(\theta) = \frac{-1}{(1-A^2)\sin\theta}\left\{2\sin^2 \frac{\theta(\pi-t)}{2\pi} - (1-A)\sin^2\frac{\theta}{2}\right\}$$

$$= \frac{-1}{(1-A^2)\sin\theta}\left\{\frac{1+A}{2} + \frac{1-A}{2}\cos\theta - \cos(\theta-\theta t/\pi)\right\} \tag{A3.1b}$$

and

$$A = -\cos t \tag{A3.1c}$$

We note that $I(\theta)$ has a periodic and a nonperiodic component. Denoting $s_{2\pi}$ as the periodic part of $s(\alpha)$ we obtain

$$s_{2\pi}(\alpha) = \frac{-1/2}{(1-A)}\left\{\frac{-1/2}{\sin\psi_s} + \frac{1/2}{\sin\psi_d} - \frac{A}{\sin\psi_d} + \frac{A}{\sin\psi_s}\right\}$$

$$- \frac{1/2}{(1+A)}\left\{\frac{1}{2}\frac{\cos\psi_s}{\sin\psi_s} + \frac{1}{2}\frac{\cos\psi_d}{\sin\psi_d} + A\frac{\cos\psi_d}{\sin\psi_d} + A\frac{\cos\psi_s}{\sin\psi_s}\right\} \tag{A.3.2a}$$

where

$$\psi_s = \psi_0 + \alpha \tag{A3.2b}$$

$$\psi_d = \psi_0 - \alpha \tag{A3.2c}$$

$$s_{2\pi}(\alpha) = \frac{-1/2}{(1-A)} \left\{(A-\tfrac{1}{2})(\sin\psi_d - \sin\psi_s)\right\} \frac{1}{1/2(\cos 2\alpha - \cos 2\psi_0)}$$

$$- \frac{1/2}{(1+A)} \left\{(A+\tfrac{1}{2})(\cos\psi_s \sin\psi_d + \cos\psi_d \sin\psi_s\right\} \frac{1}{\frac{1}{2}(\cos 2\alpha - \cos 2\psi_0)}$$

$$= \frac{1/2}{1-A^2} \left\{\frac{(2A^2+A-1)(2\sin\alpha\cos\psi_0) + (2A^2-A-1)\sin 2\psi_0}{\cos 2\alpha - \cos 2\psi_0}\right\} \tag{A3.3}$$

The non-periodic part of $s(\alpha)$ is given by

$$s(\alpha) - s_{2\pi}(\alpha) = \frac{1/2}{1-A^2} \left\{\frac{\cos[(\psi_s - \pi)(1-t/\pi)]}{-\sin\psi_s} + \frac{\cos[(\psi_d - 2\pi)(1-t/\pi)]}{\sin\psi_d}\right.$$

$$\left. + \frac{2A\cos[(\psi_d - \pi)(1-t/\pi)]}{-\sin\psi_d} + \frac{2A\cos[(\psi_s - 2\pi)(1-t/\pi)]}{\sin\psi_s}\right\}$$

$$= \frac{1/2}{(1-A^2)} \left\{\frac{\cos(\psi_s - \psi_s t/\overline{\pi+t})}{\sin\psi_s} + \frac{\cos(\psi_d - \psi_d t/\overline{\pi+2t})}{\sin\psi_d}\right.$$

$$\left. + \frac{2A\cos(\psi_d - \psi_d t/\overline{\pi+t})}{\sin\psi_d} + \frac{2A\cos(\psi_s - \psi_s t/\overline{\pi+2t})}{\sin\psi_s}\right\}$$

$$= \frac{-1/2}{(1-A^2)} \left\{\frac{\cos(\psi_s - \psi_s t/\overline{\pi+3t})}{\sin\psi_s} + \frac{\cos(\psi_d - \psi_d t/\pi)}{\sin\psi_d}\right\}$$

$$= \frac{-1}{(1-A^2)} \left\{\cos(\psi_s - \psi_s t/\overline{\pi+3t})\sin\psi_d + \cos(\psi_d - \psi_d t/\pi)\sin\psi_s\right\} / (\cos 2\alpha - \cos 2\psi_0) \tag{A3.4}$$

where we have made use of (A3.1c) and decomposed the cosine products. Upon further decomposing (A3.4) we obtain

$$s(\alpha)-s_{2\pi}(\alpha) = \frac{-1/2}{(1-A)^2}\Big\{\sin(2\psi_0-\psi_s t/\overline{\pi+3t}) - \sin(2\alpha-\psi_s t/\overline{\pi+3t})$$

$$+ \sin(2\psi_0-\psi_d t/\pi) + \sin(2\alpha+\psi_d t/\pi)\Big\}(\cos 2\alpha-\cos 2\psi_0)$$

$$= \frac{-1}{(1-A\ )^2}\Big\{\sin(2\psi_0-\psi_0 t/\overline{\pi+3t/2})\ \cos(\alpha t/\overline{\pi-3t/2})$$

$$+ \cos(2\alpha-\alpha t/\overline{\pi+3t/2})\ \sin(\psi_0 t/\overline{\pi-3t/2})\Big\}\ (\cos 2\alpha-\cos 2\psi_0)$$

$$= \frac{-1}{(1-A^2)}\Big\{[\sin 2\psi_0\cos(\psi_0 t/\overline{\pi-3t/2}) - \cos 2\psi_0\sin(\psi_0 t/\overline{\pi-3t/2})]\cos(\alpha t/\overline{\pi-3t/2})$$

$$+ [\cos 2\alpha\cos(\alpha t/\overline{\pi-3t/2}) + \sin 2\alpha\sin(\alpha t/\overline{\pi-3t/2})]\sin(\psi_0 t/\overline{\pi-3t/2})\Big\}/(\cos 2\alpha-\cos 2\psi_0$$

$$= \frac{-1}{(1-A^2)(\cos 2\alpha-\cos 2\psi_0)}\Big\{\sin 2\psi_0\cos(\psi_0 t/\overline{\pi-3t/2})\cos(\alpha t/\overline{\pi-3t/2}) +$$

$$\sin 2\alpha\sin(\psi_0 t/\overline{\pi-3t/2})\sin(\alpha t/\overline{\pi-3t/2})\Big\}$$

$$- \frac{1}{1-A^2}\sin(\psi_0 t/\overline{\pi-3t/2})\cos(\alpha t/\overline{\pi-3t/2}) \qquad \text{(A3.5)}$$

By adding (A3.3) and (A3.5) we immediately obtain (36).

APPENDIX IV

In this appendix we show that any branch cut free solution $s_0(\alpha)$, to the functional equations (11a-11d) with $\phi_0 = 3\pi/2$ and $\phi_1 = \pi/2$, must be periodic in α with period 2π.

The functional equations are as follows:

$$s(\alpha) = s(-\alpha+3\pi) \tag{A4.1a}$$

$$s_1(\zeta(\alpha)) = \frac{1}{2\tau(\alpha)} [s(\alpha)-s(-\alpha)] + \frac{1}{2} [s(\alpha)+s(-\alpha)] \tag{A4.1b}$$

$$s_1(\zeta(\alpha)) = s_1(\zeta(-\alpha-\pi)) \tag{A4.1c}$$

where (A4.1a) and (A4.1c) are modified forms of (11a) and (11b) and (A4.1b) is obtained by combining (11c) and (11d).

Let $s_0(\alpha)$ be any solution of (A4.1a) with no branch points. Then

$$s_0(-\alpha) = s_0(\alpha+3\pi) \tag{A4.2a}$$

$$s_0(-\alpha-\pi) = s_0(\alpha+4\pi) \tag{A4.2b}$$

$$s_1(\zeta(\alpha)) = \frac{1}{2\tau(\alpha)} [s_0(\alpha)-s_0(\alpha+3\pi)] + \frac{1}{2} [s_0(\alpha)+s_0(\alpha+3\pi)] \tag{A4.2c}$$

and

$$s_1(\zeta(-\alpha-\pi) = \frac{1}{2\tau(\alpha)} [s_0(\alpha+4\pi)-s_0(\alpha+\pi)] + \frac{1}{2}[s_0(\alpha+4\pi)+s_0(\alpha+\pi)] \tag{A4.2d}$$

Since $s_1(\zeta(\alpha))$ has two terms one involving $\tau(\alpha)$ and the other independent of $\tau(\alpha)$, we must equate these terms separately to the corresponding terms in $s_1(\zeta(-\alpha-\pi)$ so that (A4.1c) is satisfied. This results in the following two functional equations for $s_0(\alpha)$

$$s_0(\alpha) + s_0(\alpha+3\pi) = s_0(\alpha+\pi) + s_0(\alpha+4\pi) \tag{A4.3a}$$

and

$$s_0(\alpha) - s_0(\alpha+3\pi) = - s_0(\alpha+\pi) + s_0(\alpha+4\pi) \tag{A4.3b}$$

Taking the difference of these equations we obtain

$$s_0(\alpha+3\pi) = s_0(\alpha+\pi) \tag{A4.3c}$$

which implies that $s_0(\alpha)$ must have a period 2π with respect to α.

APPENDIX V

In this appendix we obtain the most general solution to the following system of functional equations which are equivalent to (11a)-(11d) with $\phi_0 = 3\pi/2$ and $\phi_1 = \pi/2$.

$$s(\alpha) = s(-\alpha+3\pi) \tag{A5.1a}$$

$$s_1(\zeta(\alpha)) = \frac{1}{2\tau(\alpha)}[s(\alpha)-s(-\alpha)] + \frac{1}{2}[s(\alpha)+s(-\alpha)] \tag{A5.1b}$$

$$s_1(\zeta(\alpha)) = s_1(\zeta(-\alpha-\pi)) \tag{A5.1c}$$

From (A5.1a) and (A5.1b) we obtain the following relations.

$$s(-\alpha) = s(\alpha+3\pi) \tag{A5.2a}$$

$$s(-\alpha-\pi) = s(\alpha+4\pi) \tag{A5.2b}$$

$$s_1(\zeta(\alpha)) = \frac{1}{2\tau}[s(\alpha)-s(\alpha+3\pi)] + \frac{1}{2}[s(\alpha)+s(\alpha+3\pi)]$$

$$= \frac{1}{2\tau}[(1+\tau)s(\alpha) - (1-\tau)s(\alpha+3\pi)] \tag{A5.2c}$$

$$s_1(\zeta(-\alpha-\pi) = \frac{1}{2\tau}[(1+\tau)s(\alpha+4\pi) - (1-\tau)s(\alpha+\pi)] \tag{A5.2d}$$

Substituting (A5.2c,d) in (A5.1c) we obtain

$$(1+\tau)s(\alpha) - (1-\tau)s(\alpha+3\pi) = (1+\tau)s(\alpha+4\pi) - (1-\tau)s(\alpha+\pi) \tag{A5.2e}$$

which upon dividing by $2(1+\tau)$ results in

$$\frac{1}{2}s(\alpha) + \sigma s(\alpha+3\pi) = \frac{1}{2}s(\alpha+4\pi) + \sigma s(\alpha+\pi) \tag{A5.2f}$$

where

$$\sigma = -\frac{1}{2}\frac{1-\tau}{1+\tau} = \text{cost} \tag{A5.2g}$$

Now we rewrite (A5.2a) and (A5.2f) which are equivalent to (A5.1a) - (A5.1c); as follows

$$s(\alpha) = s(3\pi-\alpha) \tag{A5.3a}$$

$$\frac{1}{2}\,[s(\alpha)-s(\alpha+4\pi)] = \sigma[s(\alpha+\pi)-s(\alpha+3\pi)] \tag{A5.3b}$$

We add and subtract $s(\alpha+2\pi)$ to the left hand side of the last equation and write

$$\frac{1}{2}\,[s(\alpha)-s(\alpha+2\pi)] + \frac{1}{2}\,[s(\alpha+2\pi)-s(\alpha+4\pi)] = \sigma[s(\alpha+\pi)-s(\alpha+3\pi)] \tag{A5.4}$$

Let

$$s(\alpha)-s(\alpha+2\pi) = g(\alpha) \tag{A5.5a}$$

so that (A5.4) becomes

$$g(\alpha)+g(\alpha+2\pi) = 2\sigma g(\alpha+\pi) \tag{A5.5b}$$

We divide (A5.5b) by $g(\alpha+\pi)$ and obtain

$$h(\alpha)+1/h(\alpha+\pi) = 2\sigma \tag{A5.6a}$$

where

$$h(\alpha) = g(\alpha)/g(\alpha+\pi) \tag{A5.6b}$$

Since σ is periodic, with period π, $h(\alpha)$ must be periodic of the same period, satisfying

$$h+1/h = 2\sigma \tag{A5.7a}$$

or

$$h^2 - 2\sigma h+1 = 0 \tag{A5.7b}$$

We may write the solution to h as

$$h = \sigma\pm\sqrt{\sigma^2-1} = \cos t\pm j\sin t = e^{\pm jt} \tag{A5.8}$$

Now

$$g(\alpha)/g(\alpha+\pi) = e^{\pm jt} \tag{A5.9}$$

We let

$$g(\alpha) = f(\alpha)e^{\pm j\alpha t/\pi} \tag{A5.10a}$$

so that

$$\frac{f(\alpha)e^{\mp j\alpha t/\pi}}{f(\alpha+\pi)e^{\mp j\alpha t/\pi \mp jt}} = e^{\pm jt} \tag{A5.10b}$$

and hence

$$f(\alpha) = f(\alpha+\pi) = F_\pi(\alpha) \tag{A5.10c}$$

Since (A5.5b) is linear in

$$g(\alpha) = F_\pi(\alpha)e^{-j\alpha t/\pi} + G_\pi(\alpha)e^{j\alpha t/\pi} = g_1(\alpha) + g_2(\alpha) \tag{A5.11}$$

Since (A5.5a) is linear, $s(\alpha)$ is the sum of the particular solutions of

$$s(\alpha) - s(\alpha+2\pi) = g_{1,2}, \tag{A5.12a,b}$$

and the homogeneous solution of

$$s(\alpha) - s(\alpha+2\pi) = 0 \tag{A5.12c}$$

i.e. $H_{2\pi}(\alpha)$, any function of period 2π.

Consider

$$s(\alpha) - s(\alpha+2\pi) = g_1(\alpha) \tag{A5.12a}$$

and let

$$s(\alpha) = p(\alpha)g_1(\alpha) \tag{A5.13a}$$

which gives

$$p(\alpha) - p(\alpha+2\pi)e^{-j2t} = 1 \tag{A5.13b}$$

Now we make the substitution

$$p(\alpha) = q(\alpha)/(1-e^{-j2t}) \tag{A5.14a}$$

which results in

$$q(\alpha) - q(\alpha+2\pi)e^{-j2t} = 1 - e^{-j2t} \qquad \text{(A5.14b}$$

Let

$$q(\alpha) = 1 + k(\alpha) \tag{A5.15a}$$

giving

$$1 + k(\alpha) - e^{-j2t}[1+k(\alpha+2\pi)] = 1-e^{-j2t} \tag{A5.15b}$$

and hence

$$k(\alpha) = k(\alpha+2\pi)e^{-j2t} \tag{A5.15c}$$

Now let

$$k(\alpha) = m(\alpha)e^{j\alpha t/\pi} \tag{A5.16a}$$

which gives

$$m(\alpha)e^{j\alpha t/\pi} = m(\alpha+2\pi)e^{j\alpha t/\pi} \tag{A5.16b}$$

whose solution is

$$m(\alpha) = m(\alpha+2\pi) = M_{2\pi}(\alpha) \tag{A5.16c}$$

Therefore

$$s(\alpha) = \frac{F_{\pi}(\alpha)[1+M_{2\pi}(\alpha)e^{j\alpha t/\pi}]}{(1-e^{-j2t})} e^{-j\alpha t/\pi} + \frac{G_{\pi}(\alpha)[1+N_{2\pi}(\alpha)e^{-j\alpha t/\pi}]}{(1-e^{j2t})} e^{j\alpha t/\pi} + H_{2\pi}(\alpha) \tag{A5.17}$$

which may be written as

$$s(\alpha) = J_{2\pi}(\alpha) + \bar{F}_{\pi}(\alpha)e^{-j\alpha t/\pi} + \bar{G}_{\pi}(\alpha)e^{j\alpha t/\pi} \quad \text{(A5.18a)}$$

where

$$J_{2\pi}(\alpha) = H_{2\pi}(\alpha) + \frac{F_{\pi}(\alpha)M_{2\pi}(\alpha)}{1-e^{-j2t}} + \frac{G_{\pi}(\alpha)N_{2\pi}(\alpha)}{1-e^{j2t}} \quad \text{(A5.18b)}$$

$$\bar{F}_{\pi}(\alpha) = F_{\pi}(\alpha)(1-e^{-j2t}) \quad \text{(A5.18c)}$$

and

$$\bar{G}_{\pi}(\alpha) = G_{\pi}(\alpha)(1-e^{+j2t}) \quad \text{(A5.18d)}$$

Since F_{π}, G_{π} are arbitrary functions with period π and $M_{2\pi}$, $N_{2\pi}$ are arbitrary functions of period 2π, $\bar{F}_{\pi}$, $\bar{G}_{\pi}$ are also arbitrary functions with period π, and $J_{2\pi}$ is an arbitrary function of period 2π. This follows from the fact that t is periodic in α with period π.

Equation (A5.18a) is the most general solution to (A5.3b). But $s(\alpha)$ must also satisfy (A5.3a), which means

$$J_{2\pi}(\alpha) + \bar{F}_{\pi}(\alpha)e^{-j\alpha t/\pi} + \bar{G}_{\pi}(\alpha)e^{j\alpha t/\pi}$$

$$= J_{2\pi}(3\pi-\alpha) + \bar{F}_{\pi}(-\alpha)e^{+j\alpha t/\pi-j3t} + \bar{G}_{\pi}(-\alpha)e^{-j\alpha t/\pi+j3t} \quad \text{(A5.19)}$$

so we require that

$$J_{2\pi}(\alpha) = J_{2\pi}(3\pi-\alpha) \quad \text{(A5.20a)}$$

$$\bar{F}_{\pi}(\alpha) = \bar{G}_{\pi}(-\alpha)e^{j3t} \quad \text{(A5.20b)}$$

$$\bar{G}_\pi(\alpha) = \bar{F}_\pi(-\alpha)e^{-j3t} \tag{A5.20c}$$

Equations (A5.20b) and (A5.20c) are self-consistent. Now substituting from $\bar{G}_\pi$ in terms of $\bar{F}_\pi$ from the last equation and letting

$$P_\pi(\alpha) = \bar{F}_\pi(\alpha)e^{-j3t/2} \tag{A5.21}$$

we obtain

$$s(\alpha) = J_{2\pi}(\alpha) + P_\pi(\alpha)e^{-j(\alpha t/\pi - 3t/2)} + P_\pi(-\alpha)e^{j(\frac{\alpha t}{\pi} - 3t/2)} \tag{A5.22}$$

which may be written as

$$s(\alpha) = J_{2\pi}(\alpha) + P_{e\pi}\cos(\alpha t/\pi - 3t/2) + P_{0\pi}\sin(\alpha t/\pi - 3t/2) \tag{A5.23a}$$

where

$$J_{2\pi}(\alpha) = J_{2\pi}(3\pi - \alpha) \tag{A5.23b}$$

and $P_{e\pi}$ is an arbitrary even function of α with period π

and $P_{o\pi}$ is an arbitrary odd function of α with period π.

APPENDIX VI

In this appendix we examine the branch cut integrals (35b) and show that, for the contribution from unshaded region to be zero, the functions $P_{e\pi}$ and $P_{0\pi}$ must identically vanish. From (35b) we have

$$2\pi j E_b(\rho,\phi) = \int_B s(\alpha) e^{-j\rho\cos(\alpha-\phi)} d\alpha \qquad \text{(A6.1a)}$$

where

$$B = \begin{cases} B_0 + B_\pi & 0 < \phi < \pi \\ B_\pi + B_{2\pi} & \pi < \psi < 3\pi/2 \end{cases} \qquad \text{(A6.1b)}$$

and

$$\begin{aligned} B_0 &= B_{0+} + B_{0-} \\ B_\pi &= B_{\pi+} + B_{\pi-} \\ B_{2\pi} &= B_{2\pi+} + B_{2\pi-} \end{aligned} \qquad \text{(A6.1c)}$$

As we note from Fig. A6.1, B_{0+}, $B_{\pi-}$ are in unshaded region for the range $0 < \phi < \pi$. In this range of ϕ we may write

$$2\pi j E_b(\rho,\phi) = E_{b_1}(\rho,\phi) + E_{b_2}(\rho,\phi) \qquad \text{(A6.2a)}$$

where

$$E_{b1}(\rho,\phi) = \int_{B_{0+}+B_{\pi-}} s(\alpha) e^{-j\rho\cos(\alpha-\phi)} d\alpha \qquad \text{(A6.2b)}$$

and

$$E_{b2}(\rho,\phi) \int_{B_{0-}+B_{\pi+}} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha \qquad \text{(A6.2c)}$$

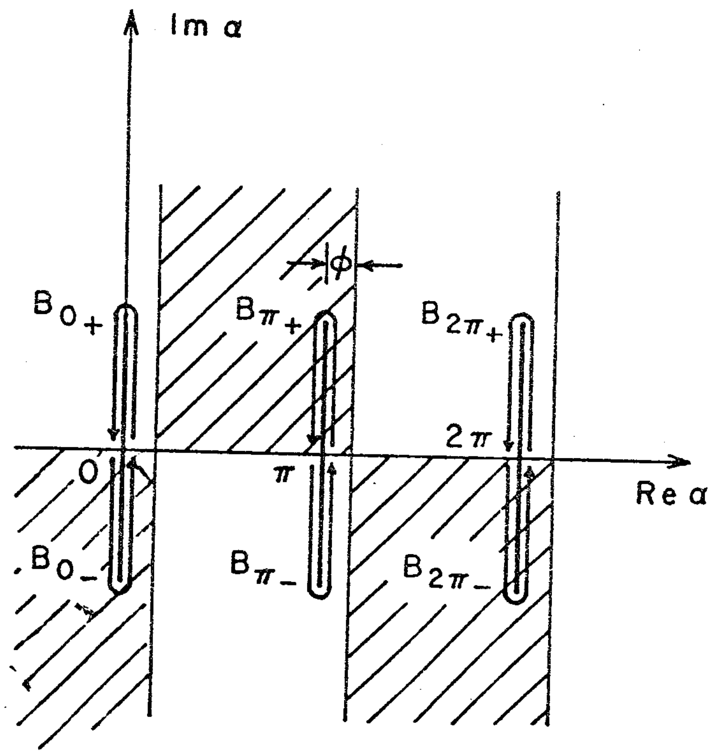

Figure A6.1 Branch cut contours B_{o+}, B_{o-}, $B_{\pi+}$. $B_{\pi-}$, $B_{2\pi+}$ and $B_{2\pi-}$.

The most general solution to $s(\alpha)$ is given by (48)

$$s(\alpha) = J_{2\pi}(\alpha) + P_{e\pi}(\alpha)\cos(\frac{\alpha t}{\pi} - \frac{3t}{2}) + P_{0\pi}(\alpha)\sin(\frac{\alpha t}{\pi} - \frac{3t}{2}) \qquad \text{(A6.3a)}$$

where $J_{2\pi}(\alpha)$ is any periodic function, of period 2π, satisfying the relation

$$J_{2\pi}(\alpha + 3\pi/2) = J_{2\pi}(-\alpha+3\pi/2) \tag{A6.3b}$$

and $P_{e\pi}$ and $P_{0\pi}$ are arbitrary even and odd periodic functions, respectively, of α of period π. t is defined by

$$\cos t = -\frac{1}{2}\,[1-\tau(\alpha)]/[1+\tau(\alpha)] \tag{A6.3c}$$

We may choose $J_{2\pi}(\alpha)$ to be free of branch points, so that it does not contribute to $E_b(\rho,\phi)$. Now let us examine the branch cut integrals in unshaded region which are given by (A6.2b), and write

$$\begin{aligned}
E_{b_1}(\rho,\phi) &= \int_{B_{0+}+B_{\pi-}} \left\{ P_{e\pi}\cos\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right) + P_{0\pi}\sin\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right)\right\} e^{-j\rho\cos(\alpha-\phi)}d\alpha \\
&= \int_{B_{0+}} \left\{ P_{e\pi}\cos\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right) + P_{0\pi}\sin\left(\frac{\alpha t}{\pi} - \frac{3t}{2}\right)\right\} e^{-j\rho\cos(\alpha-\phi)}d\alpha \\
&\quad - \int_{B_{0+}} \left\{ P_{e\pi}\cos\left(\frac{\alpha t}{\pi} + \frac{t}{2}\right) + P_{0\pi}\sin\left(\frac{\alpha t}{\pi} + \frac{t}{2}\right)\right\} e^{+j\rho\cos(\alpha+\phi)}d\alpha
\end{aligned} \tag{A6.4}$$

where we have made use of the periodicity of $P_{e\pi}$ and $P_{0\pi}$ and transformed the integral over $B_{\pi-}$ to an integral over B_{0+}.

In (A6.4) we do not see any possibility of the mutual cancellation between the two integrals because of the different nature of the exponential terms in the integrand. The only way in which $E_{b_1}(\rho,\phi)$ could vanish for all ρ and ϕ is that the non-exponential parts of the integrands be either identically zero or be free of branch point. Since t does have a branch point we conclude that the integrands must be identically zero which implies that $P_{e\pi}$ and $P_{0\pi}$ must separately vanish. We arrive at similar conclusions for the range $\pi < \phi < 3\pi/2$.

APPENDIX VII

In this appendix we find the most general solution to the following functional equation.

$$f(\phi) - f(\pi-\phi) - Rf(\pi+\phi) = 0 \tag{A7.1a}$$

where

$$R = [\sin\phi - (n^2-\cos^2\phi)^{\frac{1}{2}}]/[\sin\phi+(n^2-\cos^2\phi)^{\frac{1}{2}}] \tag{A7.1b}$$

Substituting for R and rearranging we obtain

$$f(\phi)[\sin\phi + (n^2-\cos^2\phi)^{\frac{1}{2}}] - f(\pi-\phi)[\sin\phi + (n^2-\cos^2\phi)^{\frac{1}{2}}]$$

$$+ f(\pi+\phi)[\sin(\pi+\phi) + (n^2-\cos^2(\pi+\phi))^{\frac{1}{2}}] = 0 \tag{A7.2}$$

Let

$$f(\phi) = g(\phi)[(n^2-\cos^2\phi)^{\frac{1}{2}} - \sin\phi] \tag{A7.3a}$$

then

$$g(\phi) - g(\pi-\phi) + g(\pi+\phi) = 0 \tag{A7.3b}$$

putting $-\phi$ for ϕ we obtain

$$g(-\phi) - g(\pi+\phi) + g(\pi-\phi) = 0 \tag{A7.3c}$$

Adding (7.3b) and (7.3c) we obtain

$$g(\phi) + g(-\phi) = 0 \tag{A7.4}$$

Hence g is an odd function of ϕ.

Let

$$g(\phi) = \int_0^\infty \sin\lambda\phi \; \bar{f}(\lambda)d\lambda \tag{7.5}$$

and substituting in (7.3b) we get

$$\int_0^\infty \bar{f}(\lambda) \; [\sin\lambda\phi - \sin\lambda(\pi-\phi) + \sin\lambda(\pi+\phi)] \; d\lambda = 0 \tag{A7.6}$$

which gives

$$\int_0^\infty f(\lambda)(1+2\cos\lambda\pi)\sin\lambda\phi d\lambda = 0 \tag{A7.7}$$

Hence $\cos\lambda\pi = -\frac{1}{2}; \; \lambda = 2n \pm 2/3$ (A7.8)

and

$$g(\phi) = \sum_0^\infty A_n \sin 2\phi(n+1/3) + \sum_0^\infty B_n \sin 2\phi(n-1/3) \tag{A7.9}$$

which, after combing the terms, may be rewritten as

$$g(\phi) = \cos\frac{2\phi}{3}\sum_0^\infty C_n \sin 2\phi + \sin\frac{2\phi}{3}\sum_0^\infty D_n \cos 2n\phi \tag{A7.10}$$

The series in (7.10) represent odd and even periodic functions of ϕ with period π. So that $g(\phi)$ is given by

$$g(\phi) = F_{e\pi}\sin\frac{2\phi}{3} + F_{0\pi}\cos\frac{2\phi}{3} \tag{A7.11a}$$

and

$$f(\phi) = [(n^2-\cos^2\phi)^{\frac{1}{2}} - \sin\phi]\left\{F_{e\pi}\sin\frac{2\phi}{3} + F_{0\pi}\cos\frac{2\phi}{3}\right\} \tag{A7.11b}$$

APPENDIX VIII

In this appendix we obtain a particular solution to equations (67a)-(67d) using the t-transform.

Defining $t(\alpha,\beta)$, $t_1(\alpha,\beta)$ to be the transforms of $s(\alpha)$ and $s_1(\zeta(\alpha))$ and continuing the representation $s(\alpha) \doteq t(\alpha,\beta)$ beyond the pole at $\alpha = \psi_0$, we obtain the following system of functional equations in t and t_1.

$$[t(\alpha+\pi/2,\beta) + e^{\beta\psi_0}]e^{-\beta\pi/2} = [t(-\alpha+\pi/2,-\beta) + e^{-\beta\psi_0}]e^{\beta\pi/2} \qquad (A8.1a)$$

$$t_1(\alpha,\beta) = -t_1(-\alpha,-\beta)e^{-\pi\beta} \qquad (A8.1b)$$

$$t(\alpha,\beta) - t(-\alpha,-\beta) = \tau(\alpha)[t_1(\alpha,\beta)-t_1(-\alpha,-\beta)] \qquad (A8.1c)$$

$$t(\alpha,\beta) + t(-\alpha,-\beta) = t_1(\alpha,\beta) + t_1(-\alpha,-\beta) \qquad (A8.1d)$$

The first of these equations may be transformed to

$$t(\alpha,\beta) = t(-\alpha+\pi,-\beta)e^{\pi\beta} + 2e^{\pi\beta/2}\ \mathrm{sh}[\beta(\pi/2-\psi_0)] \qquad (A8.2a)$$

Eliminating t_1 from the last three equations we obtain the following relationship between $t(\alpha,\beta)$ and $t(-\alpha,-\beta)$.

$$t(-\alpha,-\beta) = \frac{\lambda-e^{\pi\beta}}{1-\lambda e^{\pi\beta}}\, t(\alpha,\beta) \qquad (A8.2b)$$

where

$$\lambda = \cos t_2 = \frac{1-\tau(\alpha)}{1+\tau(\alpha)} \tag{A8.2c}$$

Combining (A8.2a) and (A8.2b) we obtain the following single functional equation for $t(\alpha,\beta)$.

$$t(\alpha,\beta) = \left\{ t(\alpha+\pi,\beta) + 2e^{\pi\beta/2}\mathrm{sh}[\beta(\psi_0-\pi/2)] \right\} \frac{e^{-\pi\beta}-\lambda}{\lambda-e^{\pi\beta}} \tag{A8.3}$$

which has the following periodic solution.

$$t(\alpha,\beta) = 2e^{\pi\beta/2}\ \mathrm{sh}[\beta(\psi_0-\pi/2)]\ \frac{p(\alpha,\beta)}{1-p(\alpha,\beta)} \tag{A8.4a}$$

where $p\,(\alpha,\beta) = (e^{-\pi\beta}-\lambda)/(\lambda-e^{\pi\beta})$ (A8.4b)

Upon simplification we obtain

$$t(\alpha,\beta) = \mathrm{sh}[\beta(\psi_0-\pi/2)]\ \frac{(1+\lambda)\mathrm{sh}(\beta\pi/2)-(1-\lambda)\mathrm{ch}(\beta\pi/2)}{\mathrm{ch}\pi\beta-\lambda} \tag{A8.5}$$

We obtain $s(\alpha)$ by inverting (A8.5)

$$s(\alpha) = \int_{-\infty}^{\infty} t(\alpha,\beta)e^{-\alpha\beta}d\beta$$

$$= \int_0^{\infty} [t(\alpha,\beta)+t(\alpha,-\beta)]\mathrm{ch}\alpha\beta d\beta - \int_0^{\infty}[t\ (\alpha,\beta)-t(\alpha,-\beta)]\mathrm{sh}\alpha\beta d\beta \tag{A8.6}$$

Substituting for $t(\alpha,\beta)$ we obtain

$$s(\alpha) = 2\int_0^\infty \frac{(1+\lambda)\,sh[\beta(\psi_0-\pi/2)]\,sh(\beta\pi/2)\,ch\alpha\beta+(1-\lambda)\,sh[\beta(\psi_0-\pi/2)]\,ch(\pi\beta/2)\,sh\alpha\beta}{ch\pi\beta-\lambda}\,d\beta$$

$$= 2\int_0^\infty \frac{sh[\beta(\psi_0-\pi/2)]\,sh[\beta(\pi/2+\alpha)]+\lambda sh[\beta(\psi_0-\pi/2)]\,sh[\beta(\pi/2-\alpha)]}{ch\pi\beta-\lambda}\,d\beta$$

$$= \int_0^\infty \frac{ch[\beta(\psi_0+\alpha)]-ch[\beta(\psi_0-\alpha-\pi)]}{ch\pi\beta-\lambda}\,d_\beta + \lambda\int_0^\infty \frac{ch[\beta(\psi_0-\alpha)]-ch[\beta(\psi_0+\alpha-\pi)]}{ch\pi\beta-\lambda}\,d\beta \tag{A8.7}$$

We make use of the following integral relationship [37]

$$\int_0^\infty \frac{chax}{ch\pi x-cost_2}\,dx = \frac{sin[\frac{a(\pi-t_2)}{\pi}]}{sint_2 sina} \tag{A8.8}$$

and obtain

$$s(\alpha) = \frac{1}{sint_2}\left\{\frac{sin[(\psi_0+\alpha)(\pi-t_2)/\pi]}{sin(\psi_0+\alpha)} + \frac{sin[(\psi_0-\alpha-\pi)(\pi-t_2)/\pi]}{sin(\psi_0-\alpha)}\right\}$$

$$+ \frac{cost_2}{sint_2}\left\{\frac{sin[(\psi_0-\alpha)(\pi-t_2)/\pi]}{sin(\psi_0-\alpha)} + \frac{sin[(\psi_0+\alpha-\pi)(\pi-t_2)/\pi]}{sin(\psi_0+\alpha)}\right\} \tag{A8.9}$$

$s_1(\zeta(\alpha))$ is obtained through the relation

$$s_1(\zeta(\alpha)) = \frac{1}{2}\left\{\frac{s(\alpha)-s(-\alpha)}{\tau(\alpha)} + s(\alpha) + s(-\alpha)\right\} \tag{A8.10}$$

APPENDIX IX

In this appendix we derive the functional equations (94a)-(94d). The branch cut structure in α-and ζ-planes is shown in Fig.3.15. We introduce a small amount of loss factor in the dielectric constant of the medium so that the branch points are slightly displaced from the real axis in the α-plane. When the branch cuts are defined as in Fig.3.15, $\zeta(\alpha)$ and $\tau(\alpha)$ satisfy the following relations.

$$\zeta(\alpha) = \zeta(-\alpha) \tag{A9.1a}$$

$$\zeta(\alpha+\pi) = -\zeta(\alpha)+\pi \tag{A9.1b}$$

$$\tau(\alpha) = -\tau(-\alpha) \tag{A9.1c}$$

$$\tau(\alpha+\pi) = -\tau(\alpha) \tag{A9.1d}$$

We define the fields E and E_1 as in (92a) and (92b), and write

$$2\pi jE(\rho,\phi) = \int_{\gamma_+ + \gamma_-} s(\alpha+\phi)e^{-j\rho\cos\alpha}d\alpha$$

$$= \int_{\gamma_+} [s(\alpha+\phi)-s(-\alpha+\phi)]e^{-j\rho\cos\alpha}d\alpha \tag{A9.2}$$

For the field $E_1(\rho,\phi)$ in the dielectric we write

$$2\pi jE_1(\rho,\phi) = \int_{\Gamma_+ + \Gamma_-} s_1(\zeta+\phi)e^{-j\rho n\cos\zeta}d\zeta$$

$$= \int_{\Gamma_+} [s_1(\zeta+\phi)-s_1(-\zeta+\phi)]e^{-j\rho n\cos\zeta}d\zeta$$

$$= \int_{\gamma_+} \tau(\alpha)[s_1(\zeta+\phi)-s_1(-\zeta+\phi)]e^{-j\rho\cos\alpha}d\alpha \tag{A9.3}$$

Now we obtain expressions for the ϕ-derivatives $E'(\rho,\phi)$ and $E_1'(\rho,\phi)$.

$$2\pi jE(\rho,\phi) = \int_{\gamma+\phi} s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha \tag{A9.4}$$

$$\left(\frac{-2\pi}{\rho}\right)E'(\rho,\phi) = \int_{\gamma+\phi} \sin(\alpha-\phi)s(\alpha)e^{-j\rho\cos(\alpha-\phi)}d\alpha$$

$$= \int_{\gamma} \sin\alpha s(\alpha+\phi)e^{-j\rho\cos\alpha}d\alpha$$

$$= \int_{\gamma+} \sin\alpha[s(\alpha+\phi)+s(-\alpha+\phi)]e^{-j\rho\cos\alpha}d\alpha \tag{A9.5}$$

$$2\pi jE_1(\rho,\phi) = \int_{\Gamma+\phi} s_1(\zeta)e^{-j\rho n\cos(\zeta-\phi)}d\zeta \tag{A9.6}$$

$$\left(\frac{-2\pi}{\rho}\right)E_1'(\rho,\phi) = \int_{\Gamma+\phi} n\sin(\zeta-\phi)s_1(\zeta)e^{-j\rho n\cos(\zeta-\phi)}d\zeta$$

$$= \int_{\Gamma} n\sin\zeta s_1(\zeta+\phi)e^{-j\rho n\cos\zeta}d\zeta$$

$$= \int_{\Gamma+} n\sin\zeta[s_1(\zeta+\phi)+s_1(-\zeta+\phi)]e^{-j\rho n\cos\zeta}d\zeta$$

$$= \int_{\gamma_+} \sin\alpha[s_1(\zeta+\phi)+s_1(-\zeta+\phi)]e^{-j\rho\cos\alpha}d\alpha \tag{A9.7}$$

where we have made use of the relation $\tau(\alpha) = \sin\alpha/n\sin\zeta$. Using the relations (A9.2), (A9.3), (A9.5), and (A9.7), and enforcing the boundary conditions at $\phi = 0, \pm\pi/2$ we obtain the desired functional equations. By equating the field $E(\rho,\phi)$ to zero at $\phi = \pi/2$ gives (94a), and by equating $E(\rho,0)$ to $E_1(\rho,0)$ and $E'(\rho,0)$ to $E_1'(\rho,0)$ we obtain (94c) and (94d). By equating $E_1'(\rho,-\pi/2)$ to zero we obtain

$$s_1(\zeta-\pi/2) = - s_1(-\zeta-\pi/2) \tag{A9.8}$$

which is equivalent to

$$s_1(\zeta) = -s_1(-\zeta-\pi) \tag{A9.9}$$

or

$$s_1(-\zeta) = -s_1(\zeta-\pi) \tag{A9.10}$$

Now we make use of (A9.1b) and write

$$s_1(-\zeta(\alpha)) = -s_1(-\zeta(\alpha+\pi)) \tag{A9.11}$$

which is same as (94b).

APPENDIX X

In this appendix we attempt to obtain a solution to the functional equations (94a)-(94d) which we reproduce here.

$$s(\alpha+\pi) = s(-\alpha) \tag{A10.1a}$$

$$s_1(-\zeta) = -s_1(-\zeta(\alpha+\pi)) \tag{A10.1b}$$

$$\frac{1}{\tau(\alpha)}[s(\alpha)-s(-\alpha)] = s_1(\zeta)-s_1(-\zeta) \tag{A10.1c}$$

$$s(\alpha)+s(-\alpha) = s_1(\zeta)+s_1(-\zeta) \tag{A10.1d}$$

From (A10.1c) and (A10.1d) we have

$$s_1(-\zeta) = \frac{1}{2\tau}[(1+\tau)s(-\alpha)-(1-\tau)s(\alpha)] \tag{A10.2a}$$

and

$$s_1(-\zeta(\alpha+\pi) = \frac{-1}{2\tau}[(1-\tau)s(-\alpha-\pi)-(1+\tau)s(\alpha+\pi)]$$

$$= \frac{-1}{2\tau}[(1-\tau)s(\alpha+2\pi)-(1+\tau)s(\alpha+\pi)] \tag{A10.2b}$$

where we have made use of (A9.1d) and (A10.1a). Substituting the last two equations in (A10.1b) we obtain

$$(1+\tau)s(\alpha+\pi)-(1-\tau)s(\alpha) = (1-\tau)s(\alpha+2\pi)-(1+\tau)s(\alpha+\pi) \tag{A10.3}$$

Dividing (A10.3) by $(1+\tau)$ and rearranging we obtain

$$\lambda s(\alpha)-s(\alpha+\pi) = s(\alpha+\pi)-\lambda s(\alpha+2\pi) \tag{A10.4a}$$

where

$$\lambda = (1-\tau)/(1+\tau) \tag{A10.4b}$$

Dividing (A10.4a) by $\sqrt{\lambda}$ we have

$$\sqrt{\lambda}\ s(\alpha) - \frac{1}{\sqrt{\lambda}} s(\alpha+\pi) = \frac{1}{\sqrt{\lambda}} s(\alpha+\pi) - \sqrt{\lambda}\ s(\alpha+2\pi) \tag{A10.5}$$

Noting that

$$\lambda(\alpha+\pi) = 1/\lambda \tag{A10.6a}$$

we have

$$P(\alpha)-P(\alpha+\pi) = P(\alpha+\pi)-P(\alpha+2\pi) \tag{A10.6b}$$

where

$$P(\alpha) = \sqrt{\lambda} s(\alpha) \tag{A10.6c}$$

Let

$$Q(\alpha) = P(\alpha)-P(\alpha+\pi) \tag{A10.7a}$$

then

$$Q(\alpha) = Q(\alpha+\pi) = Q_\pi(\alpha) \tag{A10.7b}$$

where $Q_\pi(\alpha)$ is an arbitrary function of α with period π.

Substituting (A10.7b) in (A10.7a) we have

$$P(\alpha)-P(\alpha+\pi) = Q_\pi(\alpha)$$

Therefore the most general solution to $P(\alpha)$ is given by

$$P(\alpha) = F_\pi(\alpha)+\alpha G_\pi(\alpha) \tag{A10.8a}$$

so that

$$s(\alpha) = [F_\pi(\alpha)+\alpha G_\pi(\alpha)]/\sqrt{\lambda} \tag{A10.8b}$$

where F_π, G_π are arbitrary periodic functions of α, with period π.

But $s(\alpha)$ must also satisfy (A10.1a). We decompose F_π and G_π into their even and odd parts and write

$$F_\pi(\alpha) = F_e(\alpha) + F_o(\alpha) \qquad \text{(A.10.9a)}$$

and

$$G_\pi(\alpha) = G_e(\alpha) + G_o(\alpha) \qquad \text{(A10.9b)}$$

Then

$$s(-\alpha) = \left\{F_e(\alpha) - F_o(\alpha) - \alpha G_e(\alpha) + \alpha G_o(\alpha)\right\} \sqrt{\lambda} \qquad \text{(A10.10a)}$$

and

$$s(\alpha+\pi) = \left\{F_e(\alpha) + F_o(\alpha) + \alpha[G_e(\alpha) + G_o(\alpha)] + \pi[G_e(\alpha) + G_o(\alpha)]\right\} \sqrt{\lambda} \qquad \text{(A10.10b)}$$

Now to satisfy (A10.1a) we must have

$$2F_0(\alpha)+2\alpha\, G_e(\alpha)+\pi[G_e(\alpha)+G_0(\alpha)] = 0 \qquad \text{(A10.11)}$$

which requires that

$$G_e(\alpha) = 0 \qquad \text{(A10.12a)}$$

and $$G_0(\alpha) = \frac{-2}{\pi} F_0(\alpha) \qquad \text{(A10.12b)}$$

so that

$$s(\alpha) = [F_e(\alpha)+(1-2\alpha/\pi)F_0(\alpha)]/\sqrt{\lambda} \qquad \text{(A10.13)}$$

which may be written in the following form,

$$s(\alpha) = [P_e(\alpha)+(\alpha-\pi/2)P_0(\alpha)]/\sqrt{\lambda} \qquad \text{(A10.14)}$$

where P_e and P_0 are arbitrary even and odd periodic functions of α with period π.

Now we must verify if the resulting functions $s_1(\zeta)$ and $s_1(-\zeta)$ are consistent.

From (A10.2a) we have

$$s_1(-\zeta) = \frac{1}{2\tau}\left\{\sqrt{1-\tau^2}\ [P_e+(\alpha+\pi/2)P_0]-\sqrt{1-\tau^2}\ [P_e+(\alpha-\pi/2)P_0]\right\}$$

$$= \frac{\pi(1-\tau^2)^{\frac{1}{2}}}{2\tau} P_0(\alpha) \tag{A10.15}$$

and

$$s_1(\zeta) = \frac{1}{2\tau}\left\{(1+\tau)s(\alpha)-(1-\tau)s(-\alpha)\right\}$$

$$= \frac{1}{2\tau}\left\{\frac{(1+\tau)^{3/2}}{(1-\tau)^{\frac{1}{2}}}[P_e+(\alpha-\pi/2)P_0]\right.$$

$$\left. - \frac{(1-\tau)^{3/2}}{(1+\tau)^{\frac{1}{2}}}\ [P_e+(\alpha+\pi/2)P_0]\right\}$$

$$= \frac{2}{\sqrt{1-\tau^2}}\ [P_e+\alpha P_0] - \frac{\pi}{2\tau}\,\frac{1+\tau^2}{(1-\tau^2)^{\frac{1}{2}}}\ P_0 \tag{A10.16}$$

Equations (A10.15) and (A10.16) are inconsistent and thus it appears that there is no solution to the equations (A10.1a) - (A10.1d).

AN ANALYSIS OF THE METHOD OF ALEKSANDROVA AND KHIZHNYAK FOR INVESTIGATING A SOLUTION TO THE PROBLEM OF DIFFRACTION BY A RECTANGULAR DIELECTRIC WEDGE

ABSTRACT

A detailed study of Aleksandrova and Khizhnyak's method of obtaining a solution to the problem of electromagnetic plane wave diffraction by a rectangular dielectric wedge is made. The spatial integrals involved in deriving the integral equation by their method are systematically carried out in full detail and the implied convergence criteria are thoroughly discussed. Besides pointing out some typographical and expository errors and a mix up on the sign convention in the time variation in their paper, it is shown that Aleksandrova and Khizhnyak omit some terms which, if properly accounted for, would lead to an integral equation that is not only different from the one they have solved but is also not amenable to solution using presently known standard techniques. A formulation using a modified contour of integration was attempted with a view to obtaining a singular integral equation for the weighting function that might be amenable to solution, but the attempt did not prove to be successful. It is concluded that the problem of wave diffraction by a rectangular dielectric wedge has not been solved by the method under review, and that this conclusion must hold also for the later work on arbitrary wedge angles using the identical method - though since no details are given no detailed analysis can be made.

TABLE OF CONTENTS

4.1 INTRODUCTION

The problem of wave diffraction by a perfectly conducting metallic wedge has been solved [1,2]. The analogous problem of diffraction by a wedge with impedance faces has also been solved [3-10]. A generalization of this solution to the case of dielectric wedge, either free or resting on a semi-infinite metal plate, entails serious mathematical difficulties.

Apart from being a classical boundary value problem the diffraction of electromagnetic waves by a dielectric wedge is of particular interest in the theory of dielectric wave guide matching [11,12], radio propagation over the earth [13,14], and in radar, for the effect of scattering by dielectric radomes [15]. An analogous problem to that of the dielectric wedge is encountered in the field of acoustics [16] and in seismological situations involving the behavior of Rayleigh waves at the boundary between the ocean and the earth [17,18]. It is not surprising, then, that considerable research effort has been directed towards the problem of diffraction by a dielectric wedge [19-30].

The problem of electromagnetic diffraction by a dielectric wedge, with or without a semi-infinite metal plate on one face, is a special case of scattering by sectoral media. The specific problem of the diffraction of an E-polarized plane wave by a right angled dielectric wedge, whose refractive index is limited to a certain range of values, has been solved theoretically by a number of authors. Radlow [19], Kuo and Plonus [20] and Kraut and Lehman [21] offer solutions to the problem by a generalization of the function theoretic method of the Wiener-Hopf technique [22] from one to two complex variables. However they do not simplify the final results and as remarked by Kuo and Plonus [20] "...the solutions are too complicated to be used practically". Kraut and Lehman [21] claim that Radlow's solution [19] is incorrect. Kurilko [23,24] obtains a solution in the form of a rather complicated system of Fredholm integral equations which have to be solved by numerical techniques. Latz's [25-27] final result ends up as an infinite system of Hilbert singular integrals which he states are suitable for numerical computation. However he does not actually obtain any explicit results of practical use.

Karp and Solfrey [14] have used an approximate technique known as the Raleigh-Gans-Born (R-G-B) approximation [28], to solve the problem of a dielectric wedge, whose refractive index is near unity, placed on a perfectly conducting infinite plane.

Rawlins [29] formulated the boundary value problem, of the diffraction of an E- or H-polarized electromagnetic line source by an arbitrary angled dielectric wedge, and obtained a solution in the form of a Fredholm integral equation. Using a standard perturbation technique he obtained a Neumann series solution, to the integral equation, which converges when $1 < n < \sqrt{2}$ where n is the refractive index of the dielectric wedge.

A number of years ago Zavadskii [30] proposed a method, which, he claimed, would give exact analytic solution to a class of two-dimensional wedge diffraction problems including the problem of diffraction by a rectangular dielectric wedge resting upon a semi-infinite perfectly conducting plate. However the solution obtained by his method contains branch cut integrals which give rise to waves "diverging" at infinity in complete violation of the radiation condition. In an earlier work [31] we made several attempts to modify Zavadskii's method so as to remove these drawbacks but did not meet with success.

Recently Aleksandrova and Khizhnyak [32], hereafter referred to as AK-1, claimed to have obtained a rigorous solution for the problem of plane-wave scattering by a rectangular dielectric wedge and in a later paper [33], hereafter referred to as AK-2, to scattering by a wedge of arbitrary angle. Their method seems to be quite appealing. They start with the integral form of Maxwell's equations and reduce the problem to a singular integral equation which lends itself to exact solution. However, in their first paper, on scattering by a rectangular dielectric wedge [AK-1], the authors omit numerous details, supposedly for the sake of brevity and in their later paper [AK-2], on scattering by a dielectric wedge of an arbitrary angle, they give only the final expressions for the total electromagnetic field with a statement that the method used is the same as in [AK-1]. Hence it was felt that, in order to make use of their results, a thorough understanding of their work on the rectangular dielectric wedge is essential. While doing so we found that, besides a number of typographical and expository errors, the authors' expression for the incident wave has the wrong sign in the exponent to start with. Further, in obtaining the singular integral equation the authors ignored some terms which, if properly accounted for, would make it impossible to obtain a solution. In this report we give a detailed exposure of these discrepancies. We looked at several possibilities of modifying their approach with a view to obtain a singular integral equation that can be solved using standard techniques [34,35]. Among these methods a seemingly promising approach, as described in this report, was to seek the diffracted fields in the wedge in the form of a weighted plane wave integral on the real axis of the complex t-plane. However this method also failed to give the desired singular integral equation for the weighting function.

In section 4.2 of this report we give a brief account of their method and point out the discrepancies in their evaluation of the integrals and the interpretation of the residues. In section 4.3 we start with the correct expression for the incident wave and systematically reduce the integrals involved to the desired form and show the missing terms in their integral equation. We conclude that if these terms are properly included the resulting equation is not amenable to solution using techniques known to us. The finer details involved in proper evaluation of the spatial integrals, which are not mentioned in [AK-1] are given in appendices. In section 4.4 we give a brief account of our modified approach to this problem. Conclusions are given in section 4.5. In our analysis we will make frequent reference to their paper on rectangular dielectric wedge [AK-1], and equation numbers quoted from it will be annotated with the letters 'AK'.

4.2 FORMULATION OF DIELECTRIC WEDGE PROBLEM USING THE INTEGRAL FORM OF MAXWELL'S EQUATIONS

4.2.1 The integral form of Maxwell's equations

Maxwell's equations, in integral form, give the electric field E and the magnetic field H everywhere via the equations

$$\bar{E}(\bar{r}) = \bar{E}_o(\bar{r}) + \frac{1}{4\pi}(\text{grad div} + k^2)\int_V (\varepsilon-1)\bar{E}(\bar{r}')f(|\bar{r}-\bar{r}'|)d\bar{r}'$$

$$+ \frac{ik}{4\pi}\,\text{curl}\int_V (\mu-1)\bar{H}(\bar{r}')f(|\bar{r}-\bar{r}'|)d\bar{r}' \tag{2.1a}$$

$$\bar{H}(\bar{r}) = \bar{H}_o(\bar{r}) + \frac{1}{4\pi}(\text{grad div} + k^2)\int_V (\mu-1)\bar{H}(\bar{r}')f(|\bar{r}-\bar{r}'|)d\bar{r}'$$

$$- \frac{ik}{4\pi}\,\text{curl}\int_V (\varepsilon-1)\bar{E}(\bar{r}')f(|\bar{r}-\bar{r}'|)d\bar{r}' \tag{2.1b}$$

$$f(|\bar{r}-\bar{r}'|) = \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \tag{2.1c}$$

where $k = \omega/c$; ε and μ are the relative dielectric and magnetic constants of the medium; $\bar{E}_o$ and $\bar{H}_o$ are the electric and the magnetic fields in the incident wave; V is the volume of the scattering body and a time dependence of $e^{-i\omega t}$ is assumed.

The first feature of the solution of (2.1) is that the internal field in the medium ($\bar{r} \in V$) is directly determined through the unperturbed field of the incident wave. The scattered field for ($\bar{r} \ni V$) is determined through the known internal field by the same relations (2.1).

The second feature is that a formal expression for the desired field in the medium is the sum of the fields of the unperturbed incident wave and of the waves formed by the integral terms [36]. In evaluating all integrations this term should yield a series of terms including a wave with a propagation constant coinciding with that of the unperturbed incident wave. In accordance with the Oseen-Ewald "extinction principle" [37-39] this wave must exactly cancel the unperturbed incident wave. By imposing this condition one obtains the amplitudes and the directions of propagation of the penetrating plane waves.

4.2.2 Plane wave scattering by a rectangular dielectric wedge

We now consider the specific problem treated by Aleksandrova and Khizhnyak i.e. the scattering of a plane electromagnetic wave

$$E_o(\bar{r}) = E_o\, e^{i(k_{20}y+k_{30}z)} \tag{2.2a}$$

$$H_o(\bar{r}) = H_o\, e^{i(k_{20}y+k_{30}z)} \tag{2.2b}$$

by a rectangular dielectric wedge (Fig.4.1) with a relative permittivity ε ($=n^2$), and a relative magnetic permeability of unity. The incident wave is polarized in the x-direction so that $\bar{E} = (E_x,0,0)$ and $\bar{H} = (0,H_y,H_z)$.

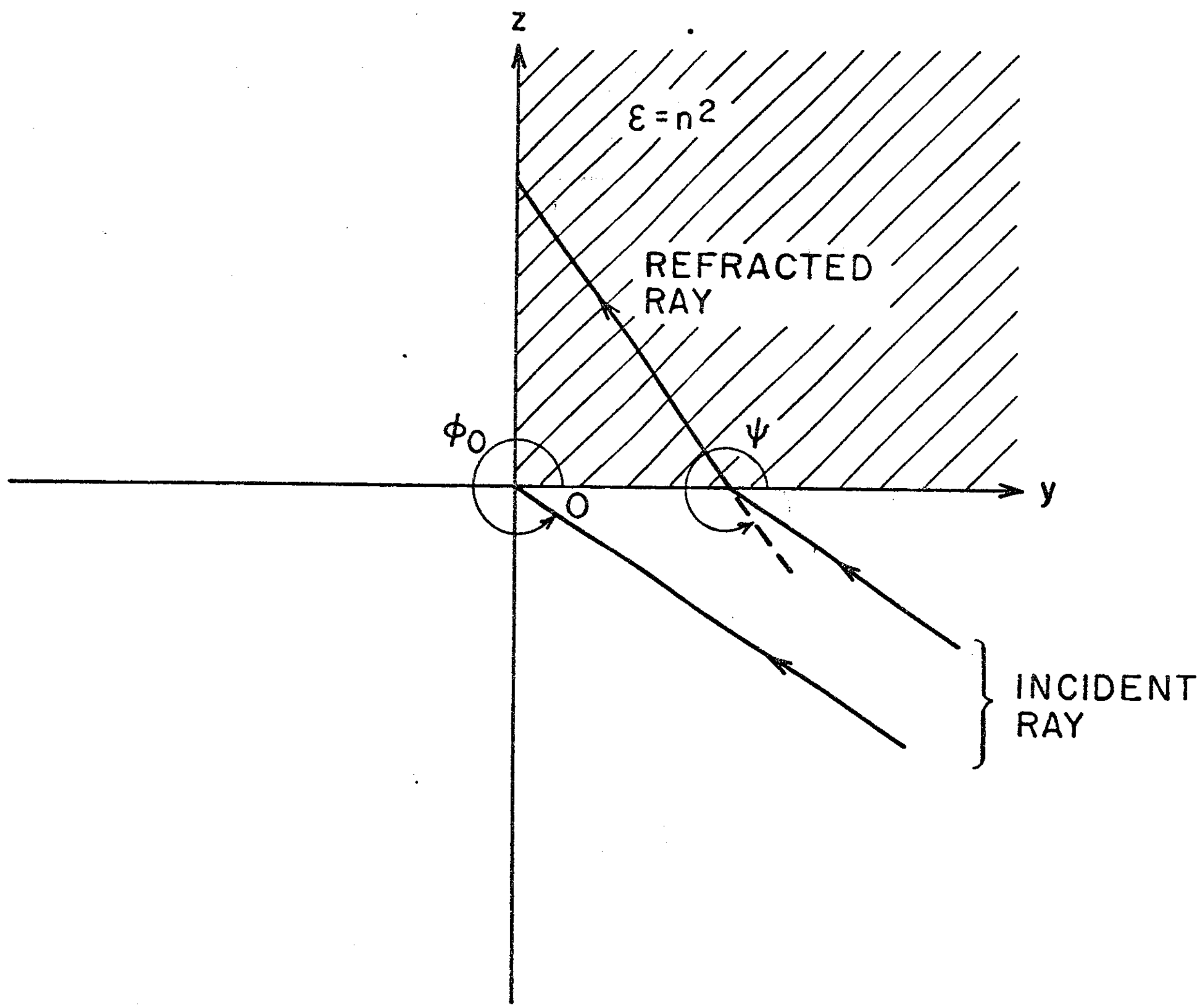

Fig. 4.1 Geometry of the rectangular dielectric wedge showing the incident ray and the ray refracted at the face z = 0.

From (2.1) we then have the following fundamental integral equation for the determination of the internal fields ($\bar{r} \varepsilon V$):

$$E_x(\bar{r}) = E_{ox}(\bar{r}) + \frac{k^2(\varepsilon-1)}{4\pi} \int_V E_x(\bar{r}') \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} d\bar{r}' \qquad (2.3)$$

The magnetic fields are given by

$$H_y(\bar{r}) = H_{oy}(\bar{r}) - \frac{ik(\varepsilon-1)}{4\pi} \frac{\partial}{\partial z} \int_V E_x(\bar{r}') \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} d\bar{r}' \qquad (2.4a)$$

$$H_z(\bar{r}) = H_{oz}(\bar{r}) + \frac{ik(\varepsilon-1)}{4\pi} \frac{\partial}{\partial y} \int_V E_x(\bar{r}') \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} d\bar{r}' \qquad (2.4b)$$

where the region of integration V is the first quadrant ($-\infty < x < \infty$, $y > 0$, $z > 0$).

A solution to (2.3) is sought in the following form, which consists of a superposition of a plane refracted wave and an unknown wave from the edge (AK5)*:

$$E_x(\bar{r}) = Ae^{i(k_2y+k_3z)} + \int_\alpha \frac{e^{ity+isz}}{s} \hat{f}(t)dt \qquad (2.5a)$$

---

* Aleksandrova and Khizhnyak use the same functional notation for the weighting functions in t- and η-planes which is potentially confusing. We denote the weighting function in t-plane by $\hat{f}(t)$ to distinguish it from $f(\eta)$.

where

$$k_2^2 + k_3^2 = \varepsilon k^2 \tag{2.5b}$$

$$t^2 + s^2 = \varepsilon k^2 \tag{2.5c}$$

and the contour α, as implicitly defined later, is shown in Fig.4.2. In this report we will only consider the case where the incident wave is in the fourth quadrant ($3\pi/2 < \phi_o < 2\pi$) and illuminates the face $\phi = 0$ of the wedge.

To evaluate the integral on the right-hand side of (2.3) the following representation for the Hankel function is used.

$$\int_{-\infty}^{\infty} \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} dx' = \pi i\, H_o^{(1)}\left(k\sqrt{(y-y')^2 + (z-z')^2}\right)$$

$$= i \int_{-\infty}^{\infty} \frac{\exp[i\omega|y-y'| + iv|z-z'|]}{v} dw \tag{2.6}$$

where

$$v = \sqrt{k^2 - \omega^2} \quad ; \quad \mathrm{Im}\, v > 0 \text{ when } w \text{ is real} . \tag{2.7}$$

In doing so it is assumed that the surrounding space is characterized by a small loss $[k = k_o(1 + i\delta);\ \delta > 0]$ which ensures convergence of the integrals at infinity $[\exp(iky) \to 0$ as $y \to \infty]$.

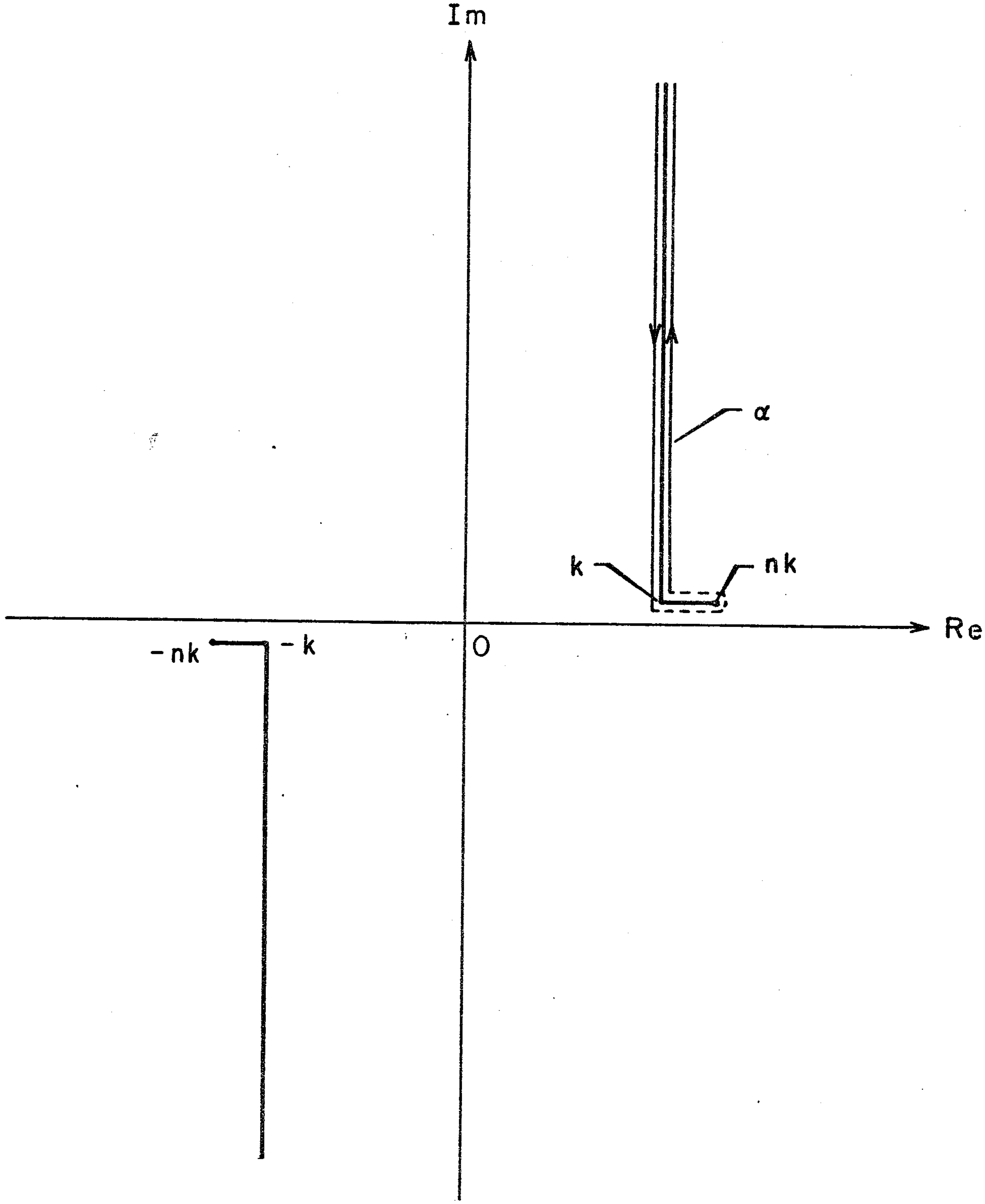

Fig. 4.2 The contour of integration α in the plane of the complex variable t.

We now write

$$\int E(\bar{r}') \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} d\bar{r}' = S_1(\rho,\phi) + S_2(\rho,\phi) \tag{2.8a}$$

where, after integration over V (Appendix I), we obtain

$$S_1 = iA \left\{ 2\int_{-\infty}^{\infty} \frac{e^{i\omega y+ik_3 z}}{(k_2-\omega)(k_3^2-v^2)} d\omega - \int_{-\infty}^{\infty} \frac{e^{i\omega y+ivz}}{v(k_2-\omega)(k_3-v)} d\omega \right\} \tag{2.8b}$$

$$S_2 = i \int_{\alpha} \frac{\hat{f}(t)dt}{s} \left\{ \int_{-\infty}^{\infty} \frac{2e^{i\omega y+isz}}{(t-\omega)(s^2-v^2)} d\omega - \int_{-\infty}^{\infty} \frac{e^{i\omega y+ivz}}{v(t-\omega)(s-v)} d\omega \right\} \tag{2.8c}$$

So far we are in full agreement with Aleksandrova and Khizhnyak. However it must be pointed out that the expressions for S_1 and S_2, as given by (2.8b) and (2.8c), are true only if the following conditions, (A1.4) and (A1.6), are met for ω real and $t \varepsilon \alpha$.

$$\mathrm{Im}\, k_2 > 0 \tag{2.9a}$$

$$\mathrm{Im}(\kappa_3 + v) > 0 \tag{2.9b}$$

$$\mathrm{Im}\, t > 0 \tag{2.9c}$$

$$\mathrm{Im}(s + v) > 0^* \tag{2.9d}$$

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* This condition is not met on α. However as shown in Appendix IV the expression (2.8c) for S_2 is still correct provided that the contour α is deformable, into a contour on which this condition is met, without crossing

4.2.3 Transformation to ξ- and η- planes

The following transformations are introduced which convert the ω- and t-plane integrations into contour integrals in ξ- and η- plane respectively,

$$\omega = k \cos \xi \tag{2.10a}$$

$$v = k \sin \xi \tag{2.10b}$$

$$t = nk \cos \eta \tag{2.11a}$$

$$s = nk \sin \eta \tag{2.11b}$$

giving equations (AK9) and (AK10)

$$S_1 = \frac{-iA}{k^2}\left[2\int_{F_1} \frac{\exp[ik\rho(\cos\xi\ \cos\phi + n\sin\phi\ \sin\psi)]\sin\xi\ d\xi}{(n\cos\psi - \cos\xi)(n^2\sin^2\psi - \sin^2\xi)}\right.$$

$$\left. - \int_{F_2} \frac{\exp[ik\rho\ \cos(\xi-\phi)]d\xi}{(n\cos\psi - \cos\xi)(n\sin\psi - \sin\xi)}\right] \tag{2.12a}$$

$$S_2 = \frac{2i}{k^2}\int_{G_o} f(\eta)d\eta\ e^{ink\rho\ \sin\eta\ \sin\phi}\int_{F_1}\frac{\exp[ik\rho\ \cos\xi\cos\phi\ ]\sin\xi\ d\xi}{(n\cos\eta - \cos\xi)(n^2\sin^2\eta - \sin^2\xi)}$$

$$- \frac{i}{k^2}\int_{G_o} f(\eta)d\eta \int_{F_2}\frac{\exp[ik\rho\ \cos\ (\xi-\phi)]d\xi}{(n\cos\eta - \cos\xi)(n\sin\eta - \sin\xi)} \tag{2.12b}$$ *

* The term $\sin\xi$ is missing from ξ-integral over F_1 in (AK10).

where

$$f(\eta) = \hat{f}(t) = \hat{f}(nk \cos \eta) \tag{2.12c}$$

and

$$k_2 = nk \cos \psi \tag{2.13a}$$

$$k_3 = nk \sin \psi \tag{2.13b}$$

$$y = \rho \cos \phi \tag{2.14a}$$

$$z = \rho \sin \phi \tag{2.14b}$$

The contours F_1, F_2, and G are shown in Figs. 3a and 3b and G_o is shown in Fig.4.4. The contours α and G_o, which depend on $G\begin{pmatrix}-\pi/2 + i\infty \\ \pi/2 - i\infty\end{pmatrix}$, are defined below (see Appendix II for a discussion about the choice of the contour G):

$$\xi_i = \cosh^{-1} (1/\cos \xi_r); \quad \sin \xi_r \sinh \xi_i \leq 0 \tag{2.15a}$$

for $\quad \xi_r + i\xi_i = \xi \varepsilon G$

$$\cos^{-1}\left(\frac{\cos \xi}{n}\right) = \eta \in G_o \quad \text{when } \xi \in G \tag{2.15b}$$

$$nk \cos \eta = t \in \alpha \quad \text{when } \eta \in G_o \tag{2.15c}$$

The shaded portions in Fig4.3a and Fig.4.3b correspond to the regions where $\text{Im} \cos \xi > 0$ and $\text{Im} \cos (\xi - \phi) > 0$ respectively.

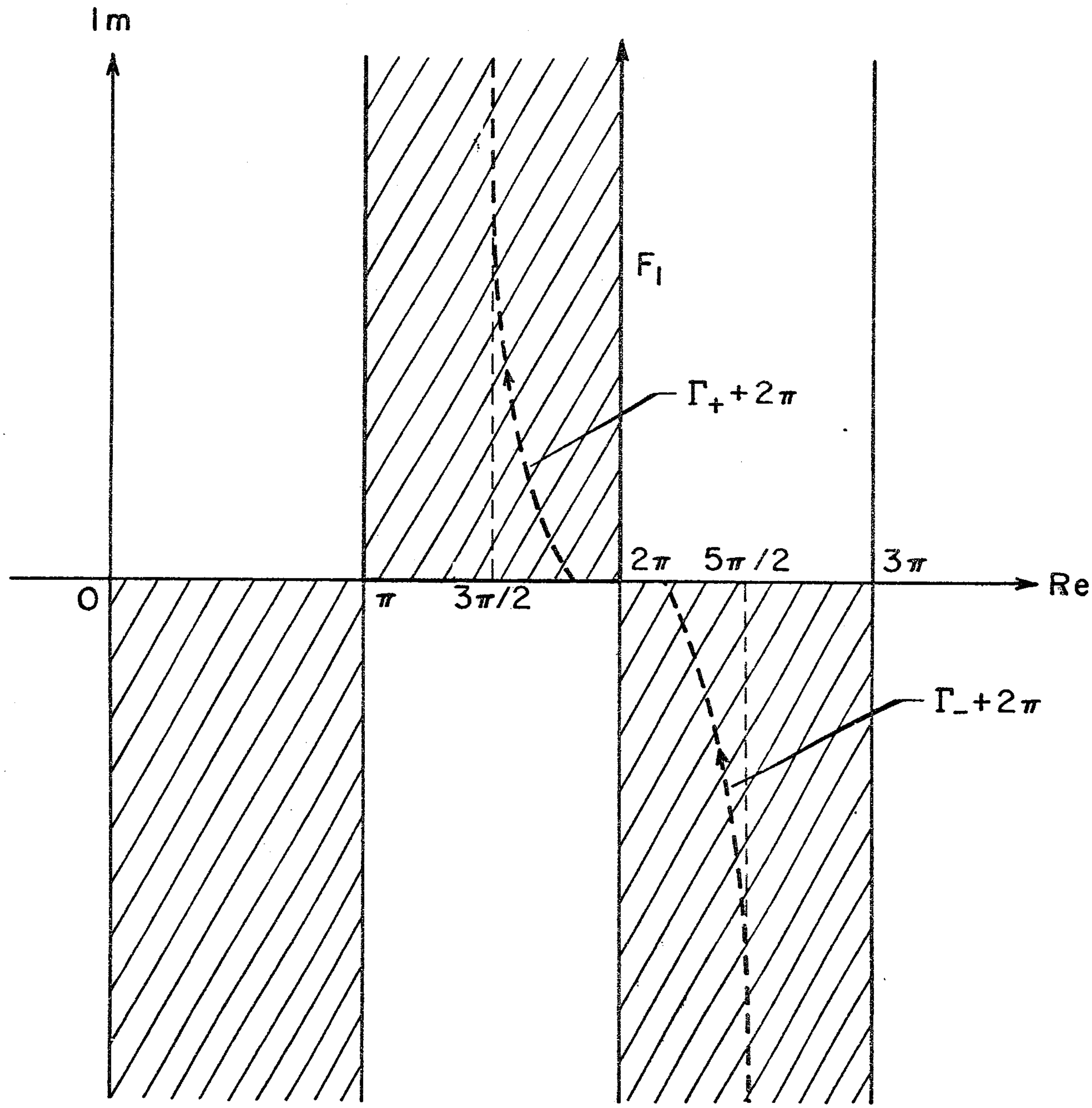

Fig. 4.3a. The contour of integration F_1 in the plane of the complex variable $\xi(\delta = 0)$.

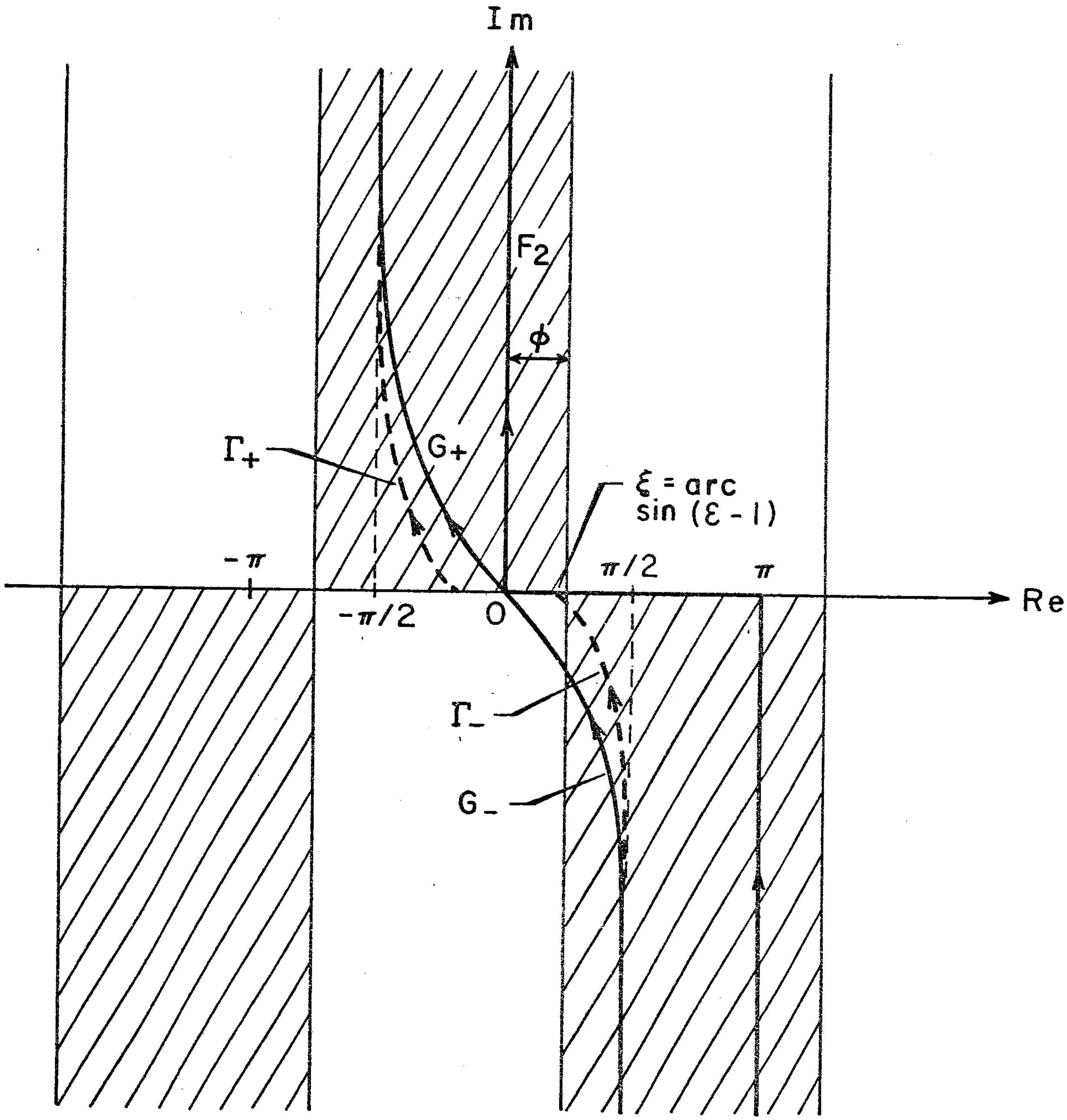

Fig. 4.3b. The contours of integration F_2 and G (G_+ + G_-) in the plane of the complex variable $\xi(\delta = 0)$.

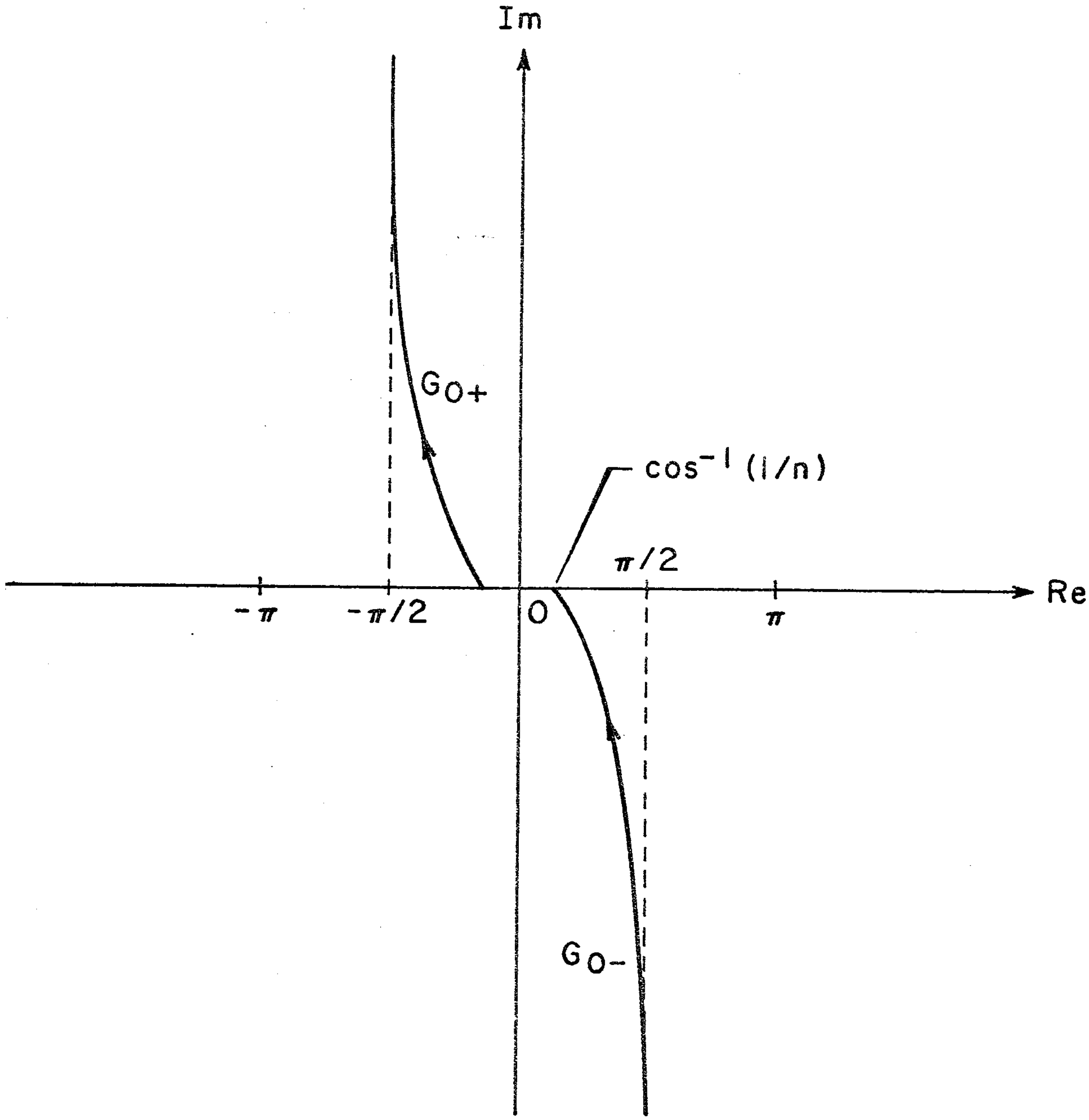

Fig. 4.4 The contour of integration $G_o(G_{o+} + G_{o-})$ in the plane of the complex variable $\eta(\delta = 0)$.

4.2.4 Deformation of contours in the ξ-plane

The crucial step, in Aleksandrova and Khizhnyak's method, of obtaining a singular integral equation, involves the closing of the contour F_1 at infinity and deforming F_2 onto G. In what follows we will make a careful examination of these steps.

4.2.4.1 Reduction of integrals in S_1

Let us consider S_1 as given by (2.12a). According to Aleksandrova and Khizhnyak the first term in S_1 gives rise to poles at $\cos \xi_1 = n \cos \psi$ and $\sin \xi_2 = n \sin \psi$ and the second term in S_1 gives rise to poles at $\cos \xi_1 = n \cos \psi$ and $\sin \xi_2 = n \sin \psi$. They seemingly, conclude that the residues, at $\sin \xi_2 = n \sin \psi$, in the first and the second terms of S_1 cancel each other, and write down the following expression (AK11) for S_1,

$$S_1 = \frac{2\pi A}{k^2} \left\{ \frac{2 \exp [ik\rho n \cos(\phi-\psi)]}{(n^2-1)} \right.$$

$$\left. - \frac{\exp[ik(ny \cos \psi + \sqrt{1-n^2 \cos^2\psi}\, z)]}{(n \sin \psi - \sqrt{1-n^2\cos^2\psi})\sqrt{1-n^2\cos^2\psi}} \right\}$$

$$+ i \int_G \frac{A \exp[ik\rho \cos(\phi-\xi)d\xi}{k^2(n \cos \psi - \cos \xi)(n \sin \psi - \sin \xi)} , \quad \frac{3\pi}{2} \le \phi_o \le 2\pi \tag{2.16}$$

where the first and second terms in (2.16) correspond to residues at $\cos \xi_1 = n \cos \psi$ in the first and second terms of (2.12a). Further, they claim that if A and ψ are given by [equations (AK18) and (AK14)]

$$A = \frac{2E_o \sin \phi_o}{n \sin \psi + \sin \phi_o} \tag{2.17a}$$

$$n \cos \psi = \cos \phi_o \tag{2.17b}$$

then the first term gives rise to a wave which is exactly equal to the refracted wave $A \exp[ik_2 y + ik_3 z]$ and the second term gives rise to a wave which exactly cancels the incident wave $E_o \exp[ik_{20} y + ik_{30} z]$. However there are several inconsistencies in their statements, as described below.

i) With the time dependence of the form $\exp[-i\omega t]$ the correct expression for the incident wave, in the fourth quadrant, is given by $\exp[-ik\rho \cos(\phi - \phi_o)]$ with $3\pi/2 < \phi_o < 2\pi$. Since from (2.7) $\text{Im}\, v > 0$, the square root expression $k\sqrt{1 - n^2 \cos^2 \psi}$, in (2.16) must be interpreted such that it has positive imaginary part. This in view of (2.17b) and the fact that $\sin \phi_o$ is negative implies that $k\sqrt{1 - n^2 \cos^2 \psi} = -k \sin \phi_o$. Consequently, the second term in (2.16) takes the form

$$\text{constant} \times \exp[+ik\rho \cos(\phi + \phi_o)]$$

which cannot cancel the incident wave.

ii) When k_2 and k_3 are given by (2.13) [equation AK8] the first term in (2.5a) [equation (AK5)] becomes $A \exp[+ink\rho \cos(\phi - \psi)]$ which is not the correct expression for the wave refracted at the face $\phi = 0$. The correct expression must have a negative sign in the exponent. This might lead us to suspect that the time dependence that they used is actually of the form $e^{+i\omega t}$. However this will not account for the discrepancy (i), regarding the extinction wave. Also the far field diffracted wave that results from their solution has a space variation of the form exp $[+ink\rho + i\pi/4]/\sqrt{nk\rho}$ which would be an incoming cylindrical wave if a time dependence of exp $[+i\omega t]$ is assumed. More importantly the type of Green's function used in (2.3) implies a time dependence of $e^{-i\omega t}$. Hence we rule out the possibility of the exp $[+i\omega t]$ time dependence and conclude that their expression for the refracted wave is incorrect.

Since they do not explicitly state the values of k_{20} and k_{30} one may still use (2.2) [equation AK2] to represent the incident wave provided that k_{20} and k_{30} are interpreted as

$$k_{20} = -k \cos \phi_o \tag{2.18a}$$

$$k_{30} = -k \sin \phi_o \tag{2.18b}$$

iii) Matters will not be resolved by simply taking k_2 and k_3 to be [contrary to equation AK8]

$$k_2 = -nk \cos \psi \tag{2.19a}$$

$$k_3 = -nk \sin \psi \tag{2.19b}$$

because, with ϕ_o and hence ψ being in the fourth quadrant, k_2 will now have a negative imaginary part which is in violation of a necessary condition (2.9a) for obtaining (2.8b). Nevertheless k_2 and k_3 must be given by (2.19) so that $E_x(\overline{r})$ as given by (2.5a) will contain the correct refracted wave.

Later in section 4.3 we show a way of getting around these difficulties, which involves the decomposition of the y-integral in S_1 into two integrals. We will find that the result would not only contain the correct refracted wave but also the correct extinction wave as well.

4.2.4.2 Reduction of integrals in S_2

Aleksandrova and Khizhnyak write down the following expression for S_2 [equation AK13] without any clear explanation, except saying that "We consider the singularities in S_2 in similar fashion...",

$$S_2 = \frac{-4\pi}{k^2(n^2-1)} \int_{G_o} \exp[ink\rho \cos(\eta-\phi)]f(\eta)d\eta$$

$$- \frac{i}{k^2} \int_G \exp[ik\rho \cos(\xi-\phi)d\xi \int_{G_o} \frac{f(\eta)d\eta}{(n \cos \eta - \cos \xi)(n \sin \eta - \sin \xi)} \qquad (2.20)^*$$

where in the last integral, as pointed out by them correctly, the order of integration is interchanged in accordance with a corollary to the Poincaré-Bertrand formula [34].

Their contention, seemingly, is that when F_1 is closed at infinity and F_2 is deformed into G the first double integral in (2.12b) gives rise to two residues at the poles n cos η = cos ξ and n sin η = sin ξ and the second double integral gives rise to a residue at the pole n sin η = sin ξ and a singular double integral which is the second term in (2.20). The first integral

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* The negative sign in front of 4π is missing in their paper, which we believe is a typographical error.

in (2.20) corresponds to the residue of the pole $n \cos \eta = \cos \xi$. Apparently they conclude that the residues due to poles at $n \sin \eta = \sin \xi$ in the first and the second terms of (2.12b) cancel each other. However a careful analysis, given later, shows that this is not true. Further, since G is a singular path ($\cos \xi = n \cos \eta$ when $\eta \epsilon G_o$ and $\xi \epsilon G$) one has to give due regard, to the pole $\cos \xi = n \cos \eta$, when deforming F_2 onto G. In point of fact one must add a term corresponding to half a residue, due to this pole, so that the remaining singular double integral may be legitimately interpreted as a Cauchy principal value integral.

In the next section, after obtaining the correct expression for S_1, we go on to consider the above mentioned points in detail and show that the expression for S_2 as given by (2.20) is incorrect. The correct expression for S_2 will be shown to contain additional terms one of which corresponds to half a residue, due to the pole $\cos \xi = n \cos \eta$, and the rest of the terms arising due to the non-cancellation of the residues corresponding to the poles $\sin \xi = \pm n \sin \eta$ in the first integral and the pole $\sin \xi = n \sin \eta$ in the second integral in (2.12b).

4.3 CORRECTED EVALUATION OF THE VOLUME INTEGRAL

Consider the volume integral (2.8a) which is reproduced here for convenience.

$$\int_V E(\bar{r}') \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} d\bar{r}' = S_1 + S_2 \tag{3.1}$$

where V is the volume of the dielectric wedge and $E(\bar{r}) = E_x(\bar{r})$ is given by (2.5a). S_1 and S_2 represent the contributions to the integral from the first and second terms of E_x in (2.5a). Hence

$$S_1 = A \int_V e^{i(k_2y'+k_3z')} \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} d\bar{r}' \tag{3.2a}$$

and

$$S_2 = \int_V \left\{ \int_\alpha \frac{e^{ity'+isz'}}{s} \hat{f}(t)dt \right\} \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} d\bar{r}' \tag{3.2b}$$

Now, we will consider S_1 and S_2 separately.

4.3.1 Derivation of an expression for S_1 containing the correct refracted and the extinction waves

Keeping the discussion in the previous section in mind let us reaffirm that, with an assumed time dependence of the form exp $[-i\omega t]$, and with $E_o(\bar{r})$ and $E_x(\bar{r})$ being given by (2.2a) and (2.5a) respectively, the constants k_{20}, k_{30}, k_2 and k_3 must satisfy equations (2.18) and (2.19) which are reproduced here,

$$k_{20} = -k \cos \phi_o \tag{3.3a}$$

$$k_{30} = -k \sin \phi_o \tag{3.3b}$$

$$k_2 = -nk \cos \psi \tag{3.4a}$$

$$k_3 = -nk \sin \psi \tag{3.4b}$$

where A and ψ are related to ϕ_o through (2.17), and with the implication that ψ as shown in Fig.4.1 is in the fourth quadrant ($3\pi/2 \leq \psi \leq 2\pi$).

Now, we write S_1 as

$$S_1 = A \int_0^\infty dy' \int_0^\infty dz' \; e^{ik_2y'+ik_3z'} \left\{ \int_{-\infty}^{\infty} \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} dx' \right\}$$

$$= A \int_0^\infty dy' \int_0^\infty dz' \; e^{ik_2y'+ik_3z'} \{\pi i \, H_o^{(1)} (k\sqrt{(y-y')^2+(z-z')^2})\} \tag{3.5}$$

which may be split into the following sum,

$$S_1 = S_{10} + S_{11} \tag{3.6a}$$

where

$$S_{10} = A \int_{-\infty}^\infty dy' \int_0^\infty dz' \; e^{ik_2y'+ik_3z'} \{\pi i \, H_o^{(1)} (k\sqrt{(y-y')^2+(z-z')^2})\} \tag{3.6b}$$

and

$$S_{11} = -A \int_{-\infty}^0 dy' \int_0^\infty dz' \; e^{ik_2y'+ik_3z'} \{\pi i \, H_o^{(1)} (k\sqrt{(y-y')^2+(z-z')^2})\} \tag{3.6c}$$

In Appendix III these integrals are shown to be given by

$$S_{10} = \frac{4\pi}{k^2(n^2-1)} \left\{ -E_o e^{-ik\rho \cos(\phi-\phi_o)} + A\, e^{-inkp \cos(\phi-\psi)} \right\} \tag{3.7a}$$

$$S_{11} = iA \left\{ 2 \int_{-\infty}^{\infty} \frac{e^{i\omega y + ik_3 z}\, d\omega}{(k_2-\omega)(k_3^2+\omega^2-k^2)} - \int_{-\infty}^{\infty} \frac{e^{i\omega y+ivz}}{v(k_2-\omega)(k_3-v)}\, d\omega \right\} \tag{3.7b}$$

Let us note the physical significance of decomposing S_1 into two parts namely S_{10} and S_{11}. The term S_{10} represents the integral over the half space $(z' > 0)$ and S_{11} represents the negative of the integral over the quarter space $(y' < 0,\ z' > 0)$. If we were to consider the problem of plane wave incidence on a dielectric filled half space the integral equation for the fields in the dielectric half space would still be the same as given by (2.3) with the range of integration, now, being the entire half space $(z' > 0)$. Since we know that the fields inside the dielectric are completely given by the single term $A \exp[ik_2 y + ik_3 z]$, with k_2, k_3, A, and ψ being given by (3.4) and (2.17), we expect that the following equation must hold.

$$\begin{aligned} A\, e^{ik_2 y + ik_3 z} &= E_o\, e^{-ik\rho \cos(\phi-\phi_o)} \\ &\quad + \frac{k^2(n^2-1)}{4\pi} \int_{z>0} A\, e^{ik_2 y' + ik_3 z'} \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\, d\bar{r}' \\ &= E_o\, e^{-ik\rho \cos(\phi-\phi_o)} + \frac{k^2(n^2-1)}{4\pi} S_{10} \end{aligned} \tag{3.8}$$

It is a trivial matter to verify this if we substitute for S_{10} from (3.7a) and note that

$$k_2 y + k_3 z = -nk\rho \cos(\phi - \psi) \tag{3.9}$$

Thus by decomposing S_1 we have simply separated the scattered fields corresponding to the half space problem.

Let us now consider S_{11} as given by (3.7b). The first integral on the right hand side may be evaluated by closing the ω-contour with a large semicircle in the upper half plane. In the second integral we may deform the ω-contour into a branch cut contour, $\bar{\alpha}$ as shown in Fig.4.5, with due consideration to the pole locations. The poles at $\omega_1 = k_2 = -k \cos \phi_o$ and $\omega_2 = -k\sqrt{(1-n^2)+\cos^2 \phi_o}$ lie below the real axis and do not contribute to the integrals. The pole at $\omega_3 = +k\sqrt{(1-n^2)+\cos^2 \phi_o}$ gives rise to equal and opposite residues and hence does not make any net contribution to S_{11}. Hence S_{11} may be simply written as

$$S_{11} = -iA \int_{\bar{\alpha}} \frac{e^{i\omega y + ivz}}{v(k_2-\omega)(k_3-v)}\, d\omega \tag{3.10}$$

which after transforming into the ξ-plane becomes

$$S_{11} = \frac{iA}{k^2} \int_G \frac{e^{ik\rho \cos(\xi-\phi)}}{(n \cos\psi + \cos\xi)(n \sin\psi + \sin\xi)}\, d\xi \tag{3.11}$$

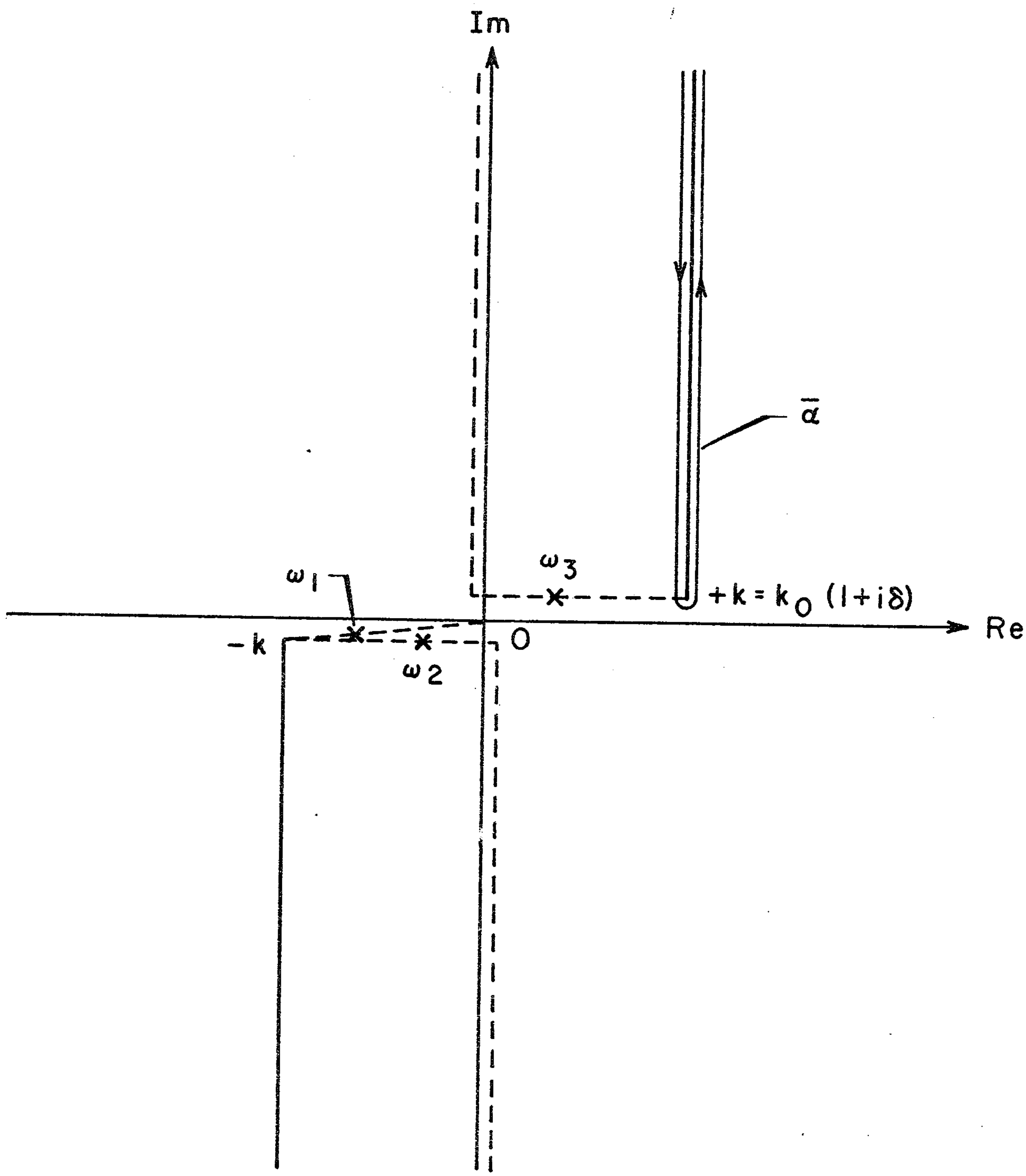

Fig. 4.5 Loci of the poles ω_1, ω_2, ω_3, as n is varied, and the contour $\bar{\alpha}$ in the complex ω-plane.

One may as well transform the ω-integrals in (3.7b) into the ξ-plane and then make proper contour deformations and arrive at the same result. In any event S_1 may finally be written as

$$S_1 = \frac{4\pi}{k^2(n^2-1)} \left\{ -E_o\, e^{-ik\rho\cos(\phi-\phi_o)} + A\, e^{-ink\rho\cos(\phi-\psi)} \right\}$$

$$+ \frac{iA}{k^2} \int_G \frac{e^{ik\rho\cos(\xi-\phi)}}{(n\cos\psi + \cos\xi)(n\sin\psi + \sin\xi)}\, d\xi \qquad (3.12)$$

4.3.2 Reduction of the volume integral S_2 and the derivation of the singular integral equation

In section 4.3.1 we obtained an expression for S_1 which contained the correct form of refracted wave and the extinction wave. First we noted that k_2 and k_3 must be given by (3.4) so that the first term, $A\exp[ik_2y + ik_3z]$, in (2.5a) would represent the correct refracted wave. Since k_2 as given by (3.4a) violates a necessary condition (2.9a) for obtaining (2.8b) we had to go through a different route which involved the decomposition of S_1 into two parts. This not only resolved the above difficulty but also resulted in the correct forms of refracted wave and the extinction wave. However the expression (2.8c) for S_2 is still correct as it stands (Appendix IV) because the α-contour is in the upper half plane of the complex variable t and hence the requirement Imt > 0 is always met. After transforming the integrations into ξ- and η-planes S_2 is given by (2.12b,c) which may be rewritten as

$$S_2 = S_{21} + S_{22} \tag{3.13a}$$

where

$$S_{21} = \frac{2i}{k^2}\int_{G_o} f(\eta)d\eta \; e^{ink\rho \sin\eta \sin\phi} \int_{F_1} \frac{\exp[ik\rho \cos\xi \cos\phi]\sin\xi \, d\xi}{(n\cos\eta - \cos\xi)(n^2\sin^2\eta - \sin^2\xi)} \tag{3.13b}$$

$$S_{22} = \frac{-i}{k^2}\int_{G_o} f(\eta)d\eta \int_{F_2} \frac{\exp[ik\rho \cos(\xi-\phi)]d\xi}{(n\cos\eta - \cos\xi)(n\sin\eta - \sin\xi)} \tag{3.13c}$$

$$f(\eta) = \hat{f}(t) = \hat{f}(nk\cos\eta) \tag{3.13d}$$

However a careful consideration of the loci of the poles $\cos\xi = n\cos\eta$ and $\sin\xi = \pm n\sin\eta$, as η is varied on G_o $(=G_{o+} + G_{o-})$, shows that the expression (2.22) for S_2 is incorrect. In Fig.4.3b, G_+ and G_- denote respectively the parts of G that lie in the upper and lower half planes of ξ. The contours G_{o+} and G_{o-} in Fig.4.4 have similar meaning with regard to G_o in the η-plane.

Let us first consider S_{21}. When $\eta \in G_{o+}$ the locus of the pole $\sin\xi = n\sin\eta$ is given by $\Gamma_+ + 2\pi$ in Fig.4.3a while for $\eta \in G_{o-}$ it is given by $\Gamma_- + 2\pi$. Since $\Gamma_- + 2\pi$ is outside the contour F_1 the only contribution to S_{21}, from the pole $\sin\xi = n\sin\eta$, comes when $\eta \in G_{o+}$. In a similar manner we note that the pole $\sin\xi = -n\sin\eta$ contributes to S_{21} only for $\eta \in G_{o-}$. Hence the total contribution to S_{21} from the term $(\sin^2\xi - n^2\sin^2\eta)$ is given by

$$\frac{-2\pi}{k^2}\int_{G_{o+}}\frac{\exp[ik\rho\ \cos(\zeta_1-\phi)]f(\eta)d\eta}{(\cos\ \zeta_1-n\ \cos\ \eta)\cos\ \zeta_1} - \frac{2\pi}{k^2}\int_{G_{o-}}\frac{\exp[ik\rho\ \cos(\zeta_1+\phi)]f(\eta)d\eta}{(\cos\ \zeta_1-n\ \cos\ \eta)\cos\ \zeta_1} \tag{3.14}$$

where $\zeta_1 = \sin^{-1}(n \sin \eta)$

However the pole $\cos \xi = n \cos \eta$ lies within F_1 for all $\eta \varepsilon G_o$. Hence when F_1 is closed we obtain the following result

$$\begin{aligned} S_{21} = \frac{-4\pi}{k^2(n^2-1)} & \int_{G_o} \exp[ink\rho\ \cos(\eta-\phi)]f(\eta)d\eta \\ & - \frac{2\pi}{k^2}\int_{G_{o+}}\frac{f(\eta)\exp[ik\rho\ \cos(\zeta_1-\phi)]}{(\cos\ \zeta_1-n\ \cos\ \eta)\cos\ \zeta_1}\ d\eta \\ & - \frac{2\pi}{k^2}\int_{G_{o-}}\frac{f(\eta)\exp[ik\rho\ \cos(\zeta_1+\phi)]d\eta}{(\cos\ \zeta_1-n\ \cos\ \eta)\cos\ \zeta_1} \end{aligned} \tag{3.15}$$

To consider S_{22} let us revert to Fig.4.3b. The locus of the pole $\sin \xi = n \sin \eta$ is given by Γ (= $\Gamma_+ + \Gamma_-$) for $\eta \varepsilon G_o$ and for the pole $\cos \xi = n \cos \eta$ it is given by G (= $G_+ + G_-$). When F_2 is deformed onto G only the lower segment Γ_- is crossed. Hence the contribution due to the pole $\sin \xi = n \sin \eta$ is given by

$$\frac{2\pi}{k^2}\int_{G_{o-}} \frac{\exp[ik\rho\ \cos(\zeta_1-\phi)]f(\eta)d\eta}{(\cos\ \zeta_1-n\ \cos\ \eta)\cos\ \zeta_1}$$

where ζ_1 is again given by (3.14).

Since G is the locus of the pole $\cos\ \xi = n \cos\ \eta$ we cannot bring F_2 into coincidence with G without considering this pole. We may first write S_{22} as

$$S_{22} = \frac{2\pi}{k^2}\int_{G_{o-}} \frac{\exp[ik\rho\ \cos(\zeta_1-\phi)]f(\eta)}{(\cos\ \zeta_1-n\ \cos\ \eta)\cos\ \zeta_1}\ d\eta$$

$$-\frac{i}{k^2}\int_{G_o} f(\eta)d\eta \int_{G+\Delta} \frac{\exp[ik\rho\ \cos(\xi-\phi)]}{(\cos\ \xi\ -n\ \cos\ \eta)(\sin\ \xi\ -n\ \sin\ \eta)}\ d\xi \tag{3.16}$$

where Δ is a very small but finite quantity. To convert the last term in (3.16) into a "Cauchy principal value" integral we must add a term equal to half a residue due to the pole* $\cos\ \xi = n\ \cos\ \eta$ resulting in

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* This was related to Aleksandrova and Khizhnyak in a private communication. To date there has been no response from them.

$$S_{22} = \frac{2\pi}{k^2} \int_{G_{o-}} \frac{\exp[ik\rho \cos(\zeta_1-\phi)]f(\eta)d\eta}{(\cos \zeta_1 - n \cos \eta)\cos \zeta_1}$$

$$- \frac{\pi}{k^2} \int_{G_o} \frac{\exp[ik\rho \cos(\zeta_2-\phi)]f(\eta)d\eta}{\sin \zeta_2(\sin \zeta_2 - n \sin \eta)}$$

$$- \frac{i}{k^2} \int_{G} \exp[ik\rho \cos(\xi-\phi)]d\xi \int_{G_o} \frac{f(\eta)d\eta}{(\cos \xi - n \cos \eta)(\sin \xi - n \sin \eta)} \tag{3.17a}$$

$$\zeta_2 = \cos^{-1}(n \cos \eta) \tag{3.17b}$$

where, in the last term, we have changed the order of integration which is permissible according to a corollary to the Poincaré-Bertrand formula. The η-integral, in the last term of (3.17a), may now be correctly interpretted as a "Cauchy principal value" integral. Combining (3.15) and (3.17) we arrive at the following expression for S_2.

$$S_2 = \frac{-4\pi}{k^2(n^2-1)} \int_{G_o} \exp[ink\rho \cos(\eta-\phi)]f(\eta)d\eta$$

$$- \frac{i}{k^2} \int_{G} \exp[ik\rho \cos(\xi-\phi)]d\xi \int_{G_o} \frac{f(\eta)d\eta}{(\cos \xi - n \cos \eta)(\sin \xi - n \sin \eta)}$$

$$- \frac{2\pi}{k^2} \int_{G_{o+}} \frac{\exp[ik\rho \cos(\zeta_1-\phi)]f(\eta)d\eta}{(\cos \zeta_1 - n \cos \eta)\cos \zeta_1} - \frac{2\pi}{k^2} \int_{G_{o-}} \frac{\exp[ik\rho \cos(\zeta_1+\phi)]f(\eta)d\eta}{(\cos \zeta_1 - n \cos \eta)\cos \zeta_1}$$

$$+ \frac{2\pi}{k^2} \int_{G_{o-}} \frac{\exp[ik\rho \cos(\zeta_1-\phi)]f(\eta)d\eta}{(\cos \zeta_1 - n \cos \eta)\cos \zeta_1} - \frac{\pi}{k^2} \int_{G_o} \frac{\exp[ik\rho \cos(\zeta_2-\phi)]f(\eta)}{\sin \zeta_2(\sin \zeta_2 - n \sin \eta)} d\eta \tag{3.18}$$

which has four additional terms as compared with the expression (AK13) obtained by Aleksandrova and Khizhnyak for S_2. If we substitute for S_1 and S_2, which determine the volume integral in (2.3), from (3.12) and (3.18) we obtain the following integral equation for $f(\eta)$.

$$\int_G \exp[ik\rho \cos(\xi - \phi)]d\xi \left\{ \frac{A}{(\cos\xi + n\cos\psi)(\sin\xi + n\sin\psi)} - \int_{G_o} \frac{f(\eta)d\eta}{(\cos\xi - n\cos\eta)(\sin\xi - n\sin\eta)} \right.$$

$$+ 2\pi i \int_{G_{o+}} \frac{\exp[ik\rho\cos(\zeta_1 - \phi)]f(\eta)d\eta}{(\cos\zeta_1 - n\cos\eta)\cos\zeta_1} + 2\pi i \int_{G_{o-}} \frac{\exp[ik\rho\cos(\zeta_1 + \phi)]f(\eta)d\eta}{(\cos\zeta_1 - n\cos\eta)\cos\zeta_1}$$

$$- 2\pi i \int_{G_{o-}} \frac{\exp[ik\rho\cos(\zeta_1 - \phi)]f(\eta)d\eta}{(\cos\zeta_1 - n\cos\eta)\cos\zeta_1}$$

$$+ \pi i \int_{G_o} \frac{\exp[ik\rho\cos(\zeta_2 - \phi)]f(\eta)d\eta}{\sin\zeta_2(\sin\zeta_2 - n\sin\eta)} = 0 . \tag{3.19}$$

where ζ_1 and ζ_2 are defined by (3.14) and (3.17b). If the last four terms were not present in (3.19) the resulting equation could be solved exactly, by using standard techniques [34,35], as has been done by Aleksandrova and Khizhnyak. We see no way of avoiding the presence of these additional terms and hence conclude that Aleksandrova and Khizhnyak's method fails to lead to a tractable integral euqation for the weighting function $f(\eta)$.

4.4 FORMULATION USING A MODIFIED CONTOUR

Aleksandrova and Khizhnyak seek the solution for the diffracted fields in the form of an integral over a branch cut contour α. It was felt that such a representation of the diffracted fields might not be complete. An integral over the real axis seemed to be more appropriate. With this idea in mind we could seek a solution in the following form

$$E_x(\bar{r}) = A\, e^{i(k_2 y + k_3 z)} + \int_C \frac{e^{ity+isz}}{s} \hat{f}(t)dt \tag{4.1a}$$

where, as in previous sections

$$k_2^2 + k_3^2 = n^2 k^2 \tag{4.2a}$$

$$t^2 + s^2 = n^2 k^2 \tag{4.2b}$$

$$k_2 = -nk \cos \psi \tag{4.3a}$$

$$k_3 = -nk \sin \psi \tag{4.3b}$$

$$\cos \psi = (\cos \phi_o)/n \tag{4.4a}$$

$$A = \frac{2E_o \sin \phi_o}{n \sin \psi + \sin \phi_o} \tag{4.4b}$$

It is assumed that the incident wave is in the fourth quadrant as given by (2.2) while k_{20} and k_{30} are defined by (3.3), and as shown in Fig4.6 the contour of integration C $\left({-\infty + j\Delta \atop \infty + j\Delta}\right)$ in the t-plane is chosen to lie just above the real axis ($\Delta > 0$). In the limit we let $\Delta \to 0$ with the understanding that any poles of $\hat{f}(t)$ that lie on the real axis of t are indented with clockwise semi-circles. We split the volume integral appearing in (2.3) into the sum $S_1 + S_2$ where S_1, S_2 are given by (3.2). In the sum $S_1 = S_{10} + S_{11}$ the expressions for S_{10} and S_{11} are given by (3.7) which are reproduced here

$$S_{10} = \frac{4\pi}{k^2(n^2-1)} \left\{ - E_o e^{-ik\rho \cos(\phi-\phi_o)} + A\, e^{-ink\rho \cos(\phi-\psi)} \right\} \tag{4.5}$$

$$S_{11} = iA \left\{ 2 \int_{-\infty}^{\infty} \frac{e^{i\omega y+ik_3 z}}{(k_2-\omega)(\omega^2-k^2+k_3^2)}\, d\omega - \int_{-\infty}^{\infty} \frac{e^{i\omega y+ivz}}{v(k_2-\omega)(k_3-v)}\, d\omega \right\} \tag{4.6}$$

The first integral in S_{11} may be evaluated by closing the ω-contour with a large semi-circle in the upper half plane and in the second integral the path of the integral may be shifted slightly above the real axis with the following result.

$$S_{11} = \frac{-2\pi A\, e^{i\sqrt{k^2-k_3^2}\, y+ik_3 z}}{(k_2-\sqrt{k^2-k_3^2})\sqrt{k^2-k_3^2}} - iA \int_C \frac{e^{i\omega y+ivz}}{v(k_2-\omega)(k_3-v)}\, d\omega \tag{4.7}$$

In writing (4.7) we note that the pole $\omega = k_2$ is below the real axis and hence does not contribute any residue. We choose Δ sufficiently close to zero such that the above deformation in the second integral is possible without crossing the poles at $\omega = \sqrt{k^2 - k_3^2}$. This is always possible except for the special case when $\phi_o = 3\pi/2$ which we may exclude from consideration here.

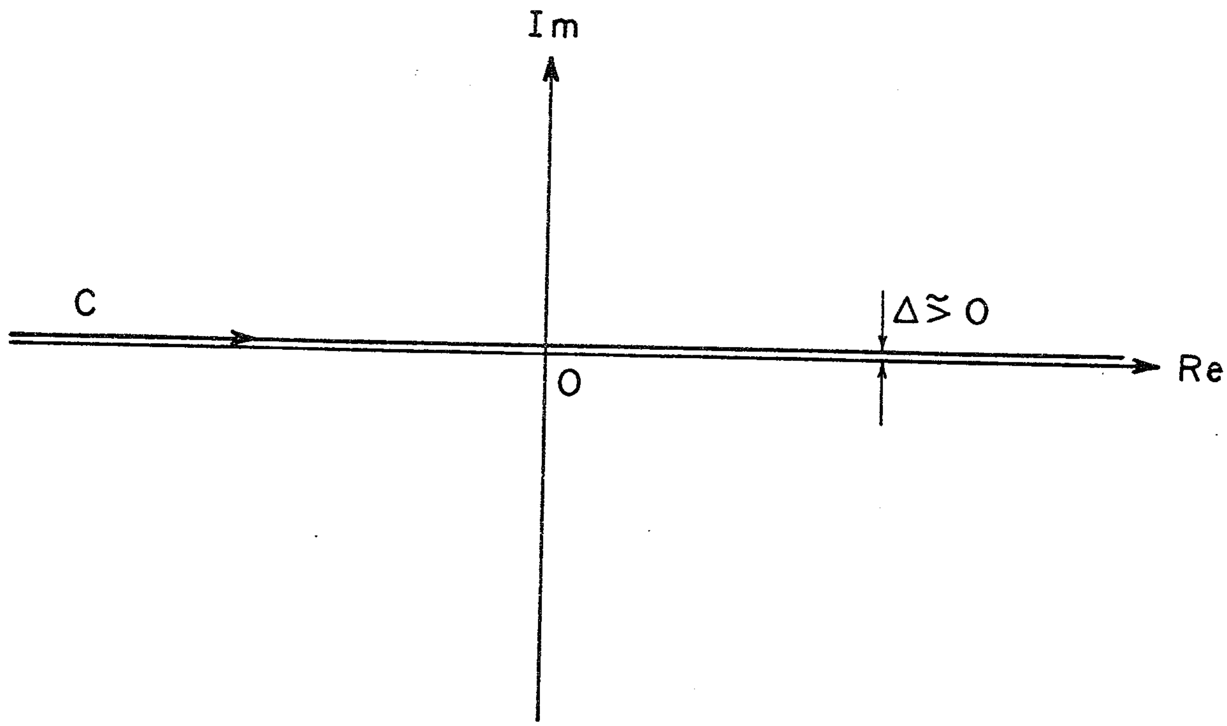

Fig. 4.6 The contour of integration C in the plane of the complex variable t.

Proceeding in a manner similar to that in Appendix I we obtain the following expression for S_2

$$S_2 = i \int_C \frac{\hat{f}(t)}{s} dt \left\{ \int_{-\infty}^{\infty} \frac{2e^{i\omega y + isz}}{(t-\omega)(s^2-v^2)} d\omega - \int_{-\infty}^{\infty} \frac{e^{i\omega y + ivz}}{v(t-\omega)(s-v)} d\omega \right\} \tag{4.8}$$

subject to the conditions

$$\mathrm{Im}\, t > 0 \tag{4.9a}$$

$$\mathrm{Im}(s + v) > 0 \tag{4.9b}$$

which are satisfied for any arbitrarily small $\Delta \gtrsim 0$.

The first ω-integral in (4.8) may be evaluated by closing the contour with a large semicircle and the second integral may be converted into a "Cauchy principle value integral" by properly accounting for a half residue corresponding to the pole at $\omega = t$. When this is done S_2 is given by

$$S_2 = \frac{4\pi}{k^2(n^2-1)} \int_C \frac{\hat{f}(t)}{s} e^{ity + isz} dt - 2\pi \int_C \frac{\hat{f}(t) e^{i\sqrt{k^2-s^2}\, y + isz}}{s(t-\sqrt{k^2-s^2})\sqrt{k^2-s^2}} dt$$

$$-\pi \int_C \frac{\hat{f}(t) e^{ity + i\sqrt{k^2-t^2}\, z}}{s(s-\sqrt{k^2-t^2})\sqrt{k^2-t^2}} dt - i \int_C \frac{e^{i\omega y + ivz}}{v} d\omega \int_C \frac{\hat{f}(t) dt}{s(t-\omega)(s-v)} \tag{4.10}$$

where the square root expression $\sqrt{k^2 - s^2}$ is to be interpreted such that

$$\mathrm{Im}\sqrt{k^2-s^2} > 0 , \quad t \in C \tag{4.11}$$

If we substitute for S_1 and S_2 in (2.3) we obtain the following integral equation for $\hat{f}(t)$.

$$\int_C \frac{e^{i\omega y + ivz}}{v} dw \left\{ \frac{A}{(k_2-\omega)(k_3-v)} + \int_C \frac{\hat{f}(t)dt}{s(t-\omega)(s-v)} \right\}$$

$$- \frac{2\pi i A\, e^{i\sqrt{k^2-k_3^2}\, y + ik_3 z}}{(k_2 - \sqrt{k^2-k_3^2})\sqrt{k^2-k_3^2}} - 2\pi i \int_C \frac{\hat{f}(t)e^{i\sqrt{k^2-s^2}\, y + isz}}{s(t-\sqrt{k^2-s^2})\sqrt{k^2-s^2}} dt$$

$$- \pi i \int_C \frac{\hat{f}(t)e^{ity + i\sqrt{k^2-t^2}\, z}}{s(s-\sqrt{k^2-t^2})\sqrt{k^2-t^2}} dt = 0 \qquad (4.12)$$

Because of the presence of the last three terms, which are not self-cancelling, it is not possible to solve (4.12) for the weighting function $\hat{f}(t)$ by using the standard technique that was employed by Aleksandrova and Khizhnyak to solve eqn (AK13). Thus, choosing the contour of integration along the real axis does not seem to avoid the difficulties underlying the method of Aleksandrova and Khizhnyak.

4.5 CONCLUSIONS

In this report we have made a detailed examination of Aleksandrova and Khizhnyak's method [32] of obtaining a solution to the problem of electromagnetic plane wave diffraction by a rectangular dielectric wedge. By carrying out the spatial integrals, involved in the problem in full detail and by a careful look at the subsequent contour deformations, we have shown that Aleksandrova and Khizhnyak, besides mixing up the sign convention in the time variation, omit some terms which, if properly accounted for, would lead to an integral equation that is not only different from what they have finally solved, but is also not amenable to solution by presently known techniques. We have attempted a formulation using a modified contour of integration in the t-plane but failed to obtain an integral equation that can be solved. We conclude that not only is the solution given by Aleksandrova and Khizhnyak incorrect but also that their method as presented is not capable of leading to a tractable integral equation for the unknown weighting function. Thus the problem of wave diffraction by a dielectric wedge remains, as yet, unsolved, even for the special case of a rectangular wedge, but for the solution of Radlow [19] which has been questioned by others [20,21].

The solution, of Aleksandrova and Khizhnyak, to the problem of diffraction by an arbitrary angled dielectric wedge [33], is also open to debate since the results are said to be based on a method identical to the one used in their earlier paper [32] which we have discussed in this report.

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APPENDIX I

EVALUATION OF THE VOLUME INTEGRAL (2.8a)

We write

$$S_1 = A \int_V e^{ik_2 y' + ik_3 z'} \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r} - \bar{r}'|} d\bar{r}' \tag{A1.1a}$$

where

$$V \; \varepsilon \, (-\infty < x < \infty, \quad y > 0, \quad z > 0).$$

Making use of (2.6) we write

$$S_1 = iA \int_{-\infty}^{\infty} \frac{d\omega}{v} \int_0^{\infty} \int_0^{\infty} e^{ik_2 y' + ik_3 z' + i\omega|y-y'| + iv|z-z'|} dy' dz'$$

$$= iA \int_{-\infty}^{\infty} \frac{d\omega}{v} \int_0^{\infty} dy' \, e^{ik_2 y' + i\omega|y-y'|} \int_0^{\infty} dz' \, e^{ik_3 z' + iv|z-z'|} \tag{A1.2}$$

We split the range of integration for y' into $y' < y$ and $y' > y$ (correspondingly for z' into $z' < z$ and $z' > z$) and write

$$S_1 = iA \int_{-\infty}^{\infty} \frac{d\omega}{v} \left\{ \int_0^y e^{ik_2 y' + i\omega(y-y')} dy' + \int_y^{\infty} e^{ik_2 y' + i\omega(y'-y)} dy' \right\} \times \left\{ \int_0^z e^{ik_3 z' + iv(z-z')} dz' + \int_z^{\infty} e^{ik_3 z' + iv(z'-z)} dz' \right\} \tag{A1.3}$$

For the integrand to vanish as $y' \to \infty$ and $z' \to \infty$ we require that

$$\text{Im } k_2 > 0 \tag{A1.4a}$$

and

$$\text{Im}(k_3 + v) > 0 \tag{A1.4b}$$

when ω is real.

Under these conditions S_1 may be simplified to

$$S_1 = iA\int_{-\infty}^{\infty} \frac{d\omega}{v} \left\{ \frac{2\omega e^{ik_2 y}}{i(k_2^2 - \omega^2)} - \frac{e^{i\omega y}}{i(k_2 - \omega)} \right\} \left\{ \frac{2ve^{ik_3 z}}{i(k_3^2 - v^2)} - \frac{e^{ivz}}{i(k_3 - v)} \right\} \tag{A1.5}$$

which reduces to (2.8b) if we note that the terms in the integrand, that are odd functions of ω, give zero after integration over the real axis.

Proceeding in a similar manner we may obtain the expression (2.8c) for S_2 subject to the following condition:

$$\mathrm{Im}\ t > 0 \tag{A1.6a}$$

$$\mathrm{Im}(s + v) > 0 \quad \text{when} \quad t \in \alpha, \quad \text{and} \quad \omega \in \text{real axis} \tag{A1.6b}$$

However since s takes on different signs on two sides of the branch cut the inequality (A1.6b) is violated on part of the contour α . To avoid this difficulty we do the following:

We deform α into another contour α' where, as shown in Fig.B 4.1, the latter coincides with α on the right side of the branch cut and on the left side it is defined by the equation $t't'' = \delta n^2 k_o^2$ where $t = t' + it''$ and $k = k_o(1 + i\delta)$. We may now carry out the spatial integration since the conditions (A1.6a,b) are satisfied for $t \in \alpha'$. After carrying out the spatial integration the contour α' may be replaced by α provided the weighting function $\hat{f}(t)$ meets certain requirements as discussed in Appendix IV.

APPENDIX II

DISCUSSION ON THE CHOICE OF THE CONTOUR G

As indicated in section 4.2 the contours of integration α and G_o in t- and η-planes are determined by the choice of the contour of integration G in ξ-plane through the following relations:

$$\cos^{-1}\left(\frac{\cos\xi}{n}\right) = \eta \in G_o \quad \text{when } \xi \in G \tag{A2.1a}$$

$$nk\cos\eta = t \in \alpha \quad \text{when } \eta \in G_o \tag{A2.1b}$$

Aleksandrova and Khiznyak do not explicitly state what G is except indicating that G is $\left(\begin{smallmatrix}-\pi/2 + i\infty \\ \pi/2 - i\infty\end{smallmatrix}\right)$ which only specifies the end points of G. In their later paper [33] on diffraction by an arbitrary dielectric wedge they explicitly define G_o in the η-plane (not the contour G in ξ-plane) to be a Sommerfeld contour $\left(\begin{smallmatrix}-\pi/2 + \phi + i\infty \\ \pi/2 + \phi - i\infty\end{smallmatrix}\right)$. This is quite different from choosing G as they have done in [32], to be a fixed (independent of ϕ) contour $\left(\begin{smallmatrix}-\pi/2 + i\infty \\ -\pi/2 - i\infty\end{smallmatrix}\right)$ which means that G_o will have two disjoint segments in the upper and the lower half planes of η with the end points as shown in Fig.4.4. Also G_o is independent of ϕ. Since the contours of integration in their two papers contradict each other we disregard their later paper and take G to be $\left(\begin{smallmatrix}-\pi/2 + i\infty \\ \pi/2 - i\infty\end{smallmatrix}\right)$ with the exact shape yet to be determined. We first consider G to be given by (2.15a) which leads to vertical branch cuts in the t-plane as shown in Fig.4.2. This necessitates some manipulations (Appendix IV) in obtaining the expression (2.8c) for S_2 so that the necessary convergence criteria are met and also places some restrictions on the singularities of $\hat{f}(t)$. If G is taken to consist of straight line segments and pass through the points $(-\pi/2 + i\infty)$, $(-\pi/2 + i0)$, $(\pi/2 + i0)$ and $(\pi/2 - i\infty)$, then the corresponding contour α in t-plane would be such that $\text{Im } s = 0$ for $t \in \alpha$ and $\text{Im } s > 0$ in the entire Riemann sheet. This necessitates a modification of the shape of branch cuts in the t-plane. Even though such a choice of branch cuts seems more natural for our problem our attempts at using such a modified contour also failed to give a desired singular integral equation for the weighting function $\hat{f}(t)$.

APPENDIX III

EVALUATION OF INTEGRALS S_{10} and S_{11}

In this appendix we reduce the expressions for S_{10} and S_{11}, which are given by (3.6), to the form (3.7).

We make use of the following form for the Hankel function ,

$$\pi i\, H_0^{(1)}\left(k\sqrt{(y-y')^2 + (z-z')^2}\right) = i \int_{-\infty}^{\infty} \frac{\exp[i\omega|z-z'| + iv|y-y'|]}{v}\, d\omega \qquad \text{(A3.1)}$$

and, after changing the order of integration, write S_{10} as

$$S_{10} = iA \int_{-\infty}^{\infty} \frac{d\omega}{v} \int_0^{\infty} dz'\, e^{i\omega|z-z'| + ik_3 z'} \int_{-\infty}^{\infty} dy'\, e^{iv|y-y'| + ik_2 y'} \qquad \text{(A3.2)}$$

Now divide the region of integration over y' into two ranges $(y < y'$ and $y > y')$ and similarly for the z'-integration and write

$$\begin{aligned} S_{10} = iA \int_{-\infty}^{\infty} \frac{d\omega}{v} &\left\{ \int_0^z dz'\, e^{i\omega(z-z') + ik_3 z'} + \int_z^{\infty} dz'\, e^{i\omega(z'-z) + ik_3 z'} \right\} \\ &\times \left\{ \int_{-\infty}^{y} dy'\, e^{iv(y-y') + ik_2 y'} + \int_y^{\infty} dy'\, e^{iv(y'-y) + ik_2 y'} \right\} \end{aligned} \qquad \text{(A3.3)}$$

The exponential integrals may now be evaluated to give

$$S_{10} = iA \int_{-\infty}^{\infty} \frac{d\omega}{v} \left\{ \frac{2\omega\, e^{ik_3 z}}{i(k_3^2 - \omega^2)} - \frac{e^{i\omega z}}{i(k_3 - \omega)} \right\} \left\{ \frac{2v\, e^{ik_2 y}}{i(k_2^2 - v^2)} \right\} \qquad \text{(A3.4)}$$

provided that the following inequalities are satisfied for real ω .

$$\operatorname{Im} k_3 > 0 \qquad \text{(A3.5a)}$$

$$\operatorname{Im} (k_2 - v) < 0 \qquad \text{(A3.5b)}$$

$$\operatorname{Im} (k_2 + v) > 0 \qquad \text{(A3.5c)}$$

Since k has a small positive imaginary part and ψ is in the fourth quadrant k_3 as given by (3.4b) will have a positive imaginary part and k_2 will have a negative imaginary part. Further, as mentioned in section 4.2, $\mathrm{Im}\, v > 0$. Thus conditions (A3.5a) and (A3.5b) are clearly satisfied. Using the binomial expansion for v we note that

$$\mathrm{Im}\, v \doteq \frac{\delta k_o}{\sqrt{1-\delta^2-(\omega/k_o)^2}} > \delta k_o \qquad \text{(A3.6a)}$$

and

$$\mathrm{Im}\, k_2 = -\delta k_o n \cos\psi = -\delta k_o \cos\phi_o \geq -\delta k_o \qquad \text{(A3.6b)}$$

Hence (A3.5c) is also met.

Noting that the part of the integrand which is an odd function of ω does not contribute to the integral we write

$$S_{10} = iA \int_{-\infty}^{\infty} \frac{2e^{i\omega z + ik_2 y}}{(k_3-\omega)(k_2^2-v^2)}\, d\omega \qquad \text{(A3.7)}$$

In view of (3.4) and (2.17) we note that

$$k_2^2 - v^2 = \omega^2 - k^2\sin^2\phi_o \qquad \text{(A3.8)}$$

which allows us to write

$$S_{10} = -iA \int_{-\infty}^{\infty} \frac{2\, e^{i\omega z + ik_2 y}}{(\omega + nk\sin\psi)(\omega + k\sin\phi_o)(\omega - k\sin\phi_o)}\, d\omega \qquad \text{(A3.9)}$$

Since $z > 0$ the ω-integral may now be evaluated, by closing the contour with a large semi-circle in the upper half plane, as the sum of two residues corresponding to the poles at $\omega = -nk\sin\psi$ and $\omega = -k\sin\phi_o$ both of which lie just above the real axis. Hence, S_{10} is now given by

$$S_{10} = \frac{4\pi A}{k^2(n^2-1)} \cdot e^{-ink\rho\cos(\phi-\psi)} - \frac{2\pi A}{k^2 \sin\phi_o (n\sin\psi - \sin\phi_o)} e^{-ik\rho\cos(\phi-\phi_o)} \tag{A3.10}$$

which in view of (2.17) may be put into the form (3.7a).

To evaluate S_{11} we make use of the following form for the Hankel function,

$$\pi i\, H_o^{(1)}\left(k\sqrt{(y-y')^2 + (z-z')^2}\right) = i\int_{-\infty}^{\infty} \frac{\exp[i\omega|y-y'| + iv|z-z'|]}{v}\, d\omega \tag{A3.11}$$

and write

$$S_{11} = -iA\int_{-\infty}^{\infty} \frac{d\omega}{v} \int_0^{\infty} dz' e^{iv|z-z'| + ik_3 z'} \int_{-\infty}^{0} dy'\, e^{i\omega|y-y'| + ik_2 y'} \tag{A3.12}$$

Since $y > 0$ and $y' < 0$, we need only to divide the region of z'-integration into two parts with the result

$$S_{11} = -iA\int_{-\infty}^{\infty} \frac{d\omega}{v} \left\{ \int_0^{z} dz'\, e^{iv(z-z') + ik_3 z'} + \int_z^{\infty} dz'\, e^{iv(z'-z)+ik_3 z'} \right\} \int_{-\infty}^{0} dy'\, e^{i\omega(y-y') + ik_2 y'} \tag{A3.13}$$

The exponential integrals may now be evaluated to give

$$S_{11} = -iA\int_{-\infty}^{\infty} \frac{d\omega}{v} \left\{ \frac{2v\, e^{ik_3 z}}{i(k_3^2 - v^2)} - \frac{e^{ivz}}{i(k_3 - v)} \right\} \frac{e^{i\omega y}}{i(k_2 - \omega)} \tag{A3.14}$$

subject to the following conditions

$$\mathrm{Im}(k_3 + v) > 0 \tag{A3.15a}$$

$$\mathrm{Im}\, k_2 < 0 \tag{A3.15b}$$

which are clearly satisfied. Equation (A3.14) is the same as (3.7b).

APPENDIX IV

EVALUATION OF THE INTEGRAL S_2

In this appendix we derive the expression (2.8c) for S_2 which is given by

$$S_2 = \int_V \frac{e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|} \left\{ \int_\alpha \frac{\hat{f}(t)}{s} e^{ity'+isz'} dt \right\} d\bar{r}' \qquad (A4.1)$$

where $V \in (-\infty < x < \infty,\ y > 0,\ z > 0)$ and α is shown in Fig. B4.1.

We note that, with the choice of α as shown in Fig. B4.1, s is complex and has opposite signs on the two sides of the branch cut. Hence the exponential term $\exp[ity' + isz']$ will diverge, as $z' \to \infty$, on one side of the branch cut if it is convergent on the other side and vice versa. To overcome this difficulty we replace α by α' on which $\mathrm{Im}\, s \geq 0$. This is permissible if the following conditions are satisfied.

i) $\hat{f}(t)$ has no singularities between α and α'

ii) $\hat{f}(t)$ takes a finite limit as $t \to \infty$.

The second condition is needed to ensure that the t-integral over the segment AA' is zero. Thus the final solution for $\hat{f}(t)$ must be checked against these conditions. It may be worth noting here that even though we believe and show in this report that the function $\hat{f}(t) = f(\eta)$ given in Aleksandrova and Khizhnyak's paper is obtained by solving a wrong integral equation, it does indeed meet the above two requirements.

Keeping the previous discussion in mind we deform the contour α in (B4.1) into α' and after changing the order of integration, write S_2 as

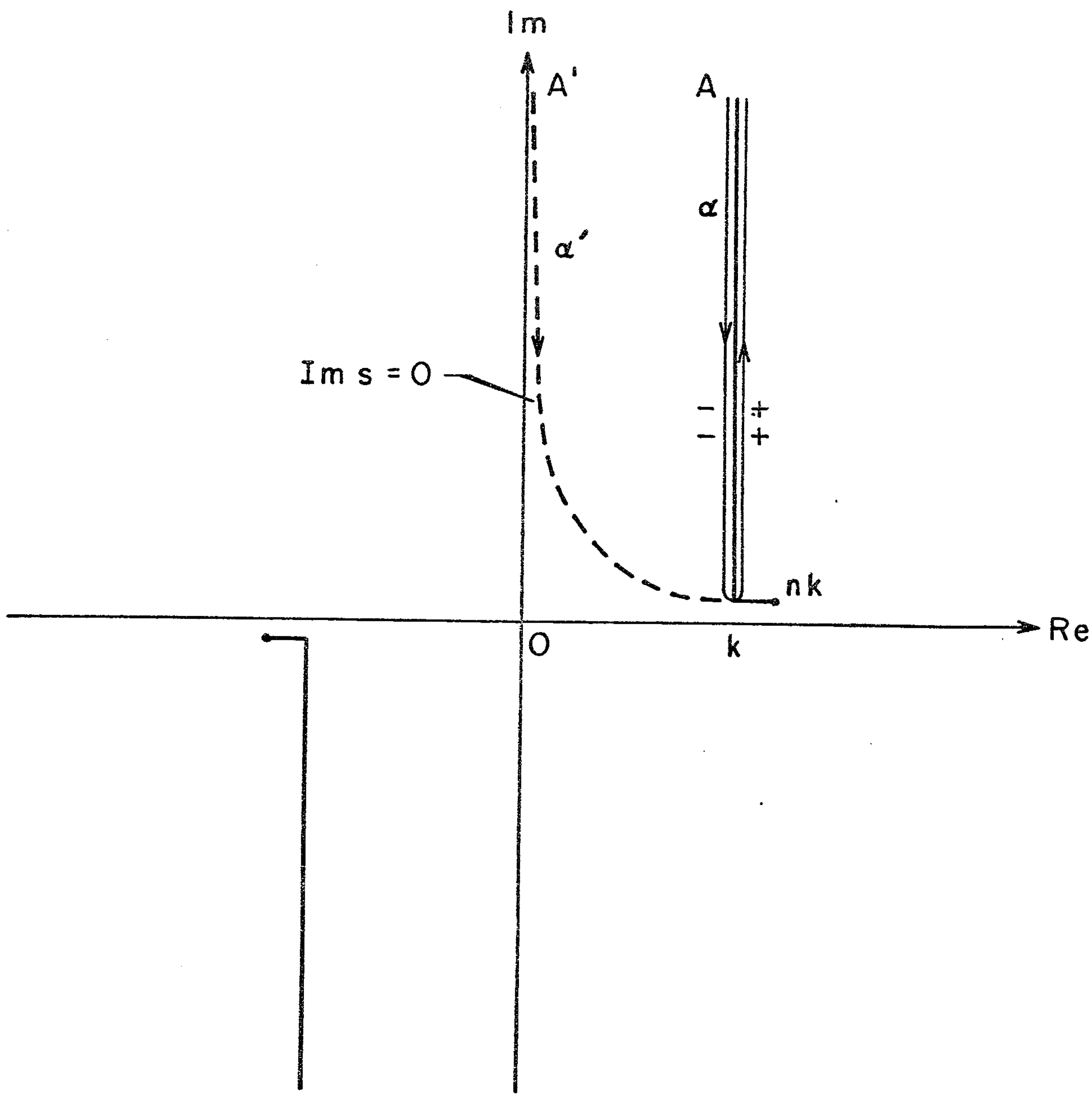

Fig. B4.1 The contours α and α' in the plane of the complex variable t.

$$S_2 = \int_{\alpha'} \frac{\hat{f}(t)}{s}\, dt \int_V \frac{\exp[ik|\bar{r}-\bar{r}'|}{|\bar{r}-\bar{r}'|}\, e^{ity'+isz'}\, d\bar{r}'$$

$$= i \int_{\alpha'} \frac{\hat{f}(t)}{s}\, dt \int_{-\infty}^{\infty} \frac{d\omega}{v} \int_0^{\infty} dy'\, e^{ity'+i\omega|y-y'|} \int_0^{\infty} dz'\, e^{isz'+iv|z-z'|} \tag{A4.2}$$

where we have made use of the Hankel function representation (2.6).

Now we may carry out the y' and z' integrals by splitting the range of y' into $y' < y$ and $y' > y$ and correspondingly the range of z' into $z' < z$ and $z' > z$ and obtain

$$S_2 = i \int_{\alpha'} \frac{\hat{f}(t)}{s}\, dt \int_{-\infty}^{\infty} \frac{d\omega}{v} \left\{ \frac{2\omega e^{ity}}{i(t^2-\omega^2)} - \frac{e^{i\omega y}}{i(t-\omega)} \right\} \left\{ \frac{2ve^{isz}}{i(s^2-v^2)} - \frac{e^{ivz}}{i(s-v)} \right\} \tag{A4.3}$$

provided

$$\mathrm{Im}\ t > 0 \tag{A4.4a}$$

$$\mathrm{Im}\ (s+v) > 0 \quad \text{for} \quad t \in \alpha' \quad \text{and} \quad \omega \quad \text{real} \tag{A4.4b}$$

The conditions (A4.4a,b) are obviously satisfied now. We note that the terms in the integrand that are odd functions of ω, give zero after integration over the real axis and hence S_2 may be written as

$$S_2 = i \int_{\alpha'} \frac{\hat{f}(t)}{s} dt \left\{ \int_{-\infty}^{\infty} \frac{2e^{i\omega y + isz}}{(t-\omega)(s^2-v^2)}\, d\omega - \int_{-\infty}^{\infty} \frac{e^{i\omega y + ivz}}{v(t-\omega)(s-v)}\, d\omega \right\} \tag{A4.5}$$

Now by a reasoning similar to that used before we may now deform α' into α and obtain (2.8c).