DIFFERENTIAL GEOMETRY OF PROJECTIVE LIMITS OF MANIFOLDS

by

MATTHEW GREGORY KRUPA

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has been approved for the Department of Mathematics

Dr. Markus Pflaum

Dr. Jeanne N. Clelland

Date____________

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The nascent theory of projective limits of manifolds in the category of locally $\mathbb{R}$-ringed spaces is expanded and generalizations of differential geometric constructions, definitions, and theorems are developed. After a thorough introduction to limits of topological spaces, the study of limits of smooth projective systems, called promanifolds, commences with the definitions of the tangent bundle and the study of locally cylindrical maps. Smooth immersions, submersions, embeddings, and smooth maps of constant rank are defined, their theories developed, and counter examples showing that the inverse function theorem may fail for promanifolds are provided along with potential substitutes. Subsets of promanifolds of measure 0 are defined and a generalization of Sard's theorem for promanifolds is proven. A Whitney embedding theorem for promanifolds is given and a partial uniqueness result for integral curves of smooth vector fields on promanifolds is found. It is shown that a smooth manifold of dimension greater than one has the final topology with respect to its set of C^1-arcs but not with respect to its C^2-arcs and that a particular class of promanifolds, called monotone promanifolds, have the final topology with respect to a class of smooth topological embeddings of compact intervals termed smooth almost arcs.

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Chapter 1

Introduction

A promanifold $((M, C_M^\infty), \mu_\bullet)$ is a projective limit in the category of commutative locally $\mathbb{R}$-ringed spaces of a projective system $\mathrm{Sys}_M = \left(\left(M_\bullet, C_{M_\bullet}^\infty\right)\mu_{ij}, \mathbb{N}\right)$ consisting smooth manifolds smooth and bonding maps that are smooth surjective submersions. A function $f : U \to \mathbb{R}$ from an open subset U of M is *smoothly locally cylindrical* at a point $m \in U$ if there exists some $i \in \mathbb{N}$, some $U_i \in \mathrm{Open}(M_i)$, and some function $f_i : U_i \to \mathbb{R}$ such that $m \in \mu_i^{-1}(U_i) \subseteq U$ and $f = f_i \circ \mu_i$ on $\mu_i^{-1}(U_i)$. The sheaf C_M^∞ of continuous real valued functions on M consists of all those continuous functions f defined on open subsets of M that are smoothly locally cylindrical at every point of their domain. As was done in [20], we will use the sheaf C_M^∞ of continuous real valued functions in lieu of a smooth atlas to extend many basic notions, constructions, and results from smooth manifolds to promanifolds.

Before initiating a study of the differential geometry of promanifolds, we provide a thorough introduction to limits of projective systems in the category Set and Top. In addition to containing a review of limits in Top, this introduction also contains many new examples and results. We find, for instance, sufficient conditions for a limit to be connected, locally connected (prop. 2.5.5), path-connected, and locally path-connected (prop. 2.5.12). The study of promanifolds then begins with a review of [20], which is the article that initiated the theory of the differential geometry on projective limits of manifolds. We formulate and prove a generalized Whitney embedding theorem for promanifolds (thm. 11.6.5). We define and study subsets of a promanifold that have measure 0, which then allows us to formulate and prove Sard's theorem (thm. 12.2.2) for promanifolds. Additionally, we show that a large class of finite-dimensional promanifolds have locally trivial tangent bundles (prop. 9.0.2).

We show that the usual inverse function theorem (theorem 13.0.1) fails to generalize from smooth manifolds to promanifolds. While investigating potential substitutes for the inverse function theorem, we are led to a particularly well-behaved class of promanifolds, called *monotone promanifolds* (def. 16.0.1), and to study the notion of *coherence* (def. A.5.1), where we say that a space is *coherent* with a collection of continuous maps into it if its topology is equal to the final topology induced by these maps. We prove that every monotone promanifold is coherent with its set of *smooth almost arcs at* 0, which are those smooth topological embeddings of $[0,1]$ whose first derivatives do not vanish on $]0,1]$ and all of derivatives vanish at 0. Knowing that monotone promanifolds are coherent with their smooth almost arcs at 0 allows us to prove theorem 16.5.1, which provides a simple sufficient condition for a smooth map between monotone promanifolds to be open.

We prove some substitute inverse function theorems with the first main result being theorem 13.2.1, which gives a version of the inverse function theorem where the requirement of having a diffeomorphism between open subsets has been relaxed to merely having a diffeomorphism between subpromanifolds. The second main result, theorem 13.4.1, is a characterization of when a smooth map into a monotone promanifold is, at some given point, a local diffeomorphism between open subsets. Theorem 13.4.2 leads to a conjecture about a version of the inverse function theorem for promanifolds that could potentially characterize local diffeomorphisms in terms of germs of vectors fields.

Notation and Terminology for Elementary Concepts

The following table lists notation for some the basic concepts that the reader is assumed to be familiar with. When using any of the following notation, we may omit writing a symbol if it is clear from context.

Let $x \in S \in \operatorname{Open}(X)$	"Let S be an open subset of X containing x." This notation will only be used when $x \in X$ is already known. We may also replace "Let" with "for some," or "for all," etc. or replace $\operatorname{Open}(X)$ with $\operatorname{Closed}(X)$, $\operatorname{Compact}(X)$, etc.
$L \underset{\text{def}}{=} R$	"L is by definition equal to R" or if the symbol L is free then it is shorthand for "let $L = R$."
$X \setminus Y$	Set subtraction: $X \setminus Y \underset{\text{def}}{=} \{x \in X \mid x \notin Y\}$.
$X - Y$	Minkowski set subtraction, where X and Y are subsets of some additive group: $X - Y \underset{\text{def}}{=} \{x - y \mid x \in X, y \in Y\}$.
$\{0\}^n$ (resp. $\{c\}^{\mathbb{N}}$, etc.)	For $n \in \mathbb{N}$ and c any object, $\{0\}^n \underset{\text{def}}{=} (0, \ldots, 0)$ (resp. $\{c\}^{\mathbb{N}} \underset{\text{def}}{=} (c, c, \ldots)$, etc.) is the n-tuple of 0's (resp. the constantly c sequence, etc.). $\{c\}^{\mathbb{N}}$ also denotes the singleton set of all maps $\mathbb{N} \to \{c\}$, which we identify with this tuple.
$(I, \leq)$	A set I with a partial order $\leq$ on I.
$(I, \leq^{\text{op}})$	The dual order of $(I, \leq)$, where for any $i, j \in I$, $i \leq^{\text{op}} j \iff j \leq i$.
$I^{\geq i_0}$, $I^{<i_0}$, etc.	Defined as $I^{\geq i_0} \underset{\text{def}}{=} \{i \in I \mid i \geq i_0\}$, where $i_0 \in I$. The sets $I^{<i_0}$, $I^{>i_0}$, and $I^{\leq i_0}$ are defined analogously.
$\varphi^{\leq j}$, $\varphi^{>j}$, etc	For $\varphi = (\varphi^i)_{i \in I}$, $\varphi^{\leq j} \underset{\text{def}}{=} (\varphi^i)_{\substack{i \in I \\ i \leq j}}$ is φ's first $j \in I$ coordinates. $\varphi^{>j}$, etc. are defined analogously. If $\varphi = (\varphi_i)_{i \in I}$ then we will instead write $\varphi_{\leq j}$, $\varphi_{>j}$, etc.

$\Pr_{I\to J} = \Pr_J$, $\Pr_{\le i_0}$, etc.	For $J \subseteq I$, $\Pr_J : \prod_{i\in I} S_i \to \prod_{j\in J} S_j$ is the canonical projection onto the coordinates in J defined by $(s_i)_{i\in I} \mapsto (s_j)_{j\in J}$. For $i_0 \in I$, $\Pr_{\le i_0} \underset{\text{def}}{=} \Pr_{I^{\le i_0}}$, $\Pr_{>i_0} \underset{\text{def}}{=} \Pr_{I^{>i_0}}$, etc.
Id_A	The identity morphism of an object A.
$\mathrm{In}_S^X = \mathrm{In}_S$	The natural inclusion $\mathrm{In}_S^X : S \to X$, where $S \subseteq X$.
$C^k(X\to Y)$	C^k-maps from X to Y, where $k \in \mathbb{Z}^{\ge 0} \cup \{\infty\}$. Similarly, $C^k((X,x)\to(Y,y))$ denotes the C^k-pointed maps from X to Y.
$C^k_{X\to Y}$ (resp. C^k_X)	Sheaf of C^k-maps from X into Y (resp. into $\mathbb{R}$).
$[\mathcal{G}]_x = \mathcal{G}_x$	The set of germs at $x \in X$, where $\mathcal{G}$ is a collection of maps defined on neighborhoods of x in X.
(X, τ_X)	A set X and a topology τ_X on X.
$\tau_X\big\vert_R$	For $R \subseteq X$, the subspace topology inherited from (X,τ_X) by R.
$\mathrm{Cl}_X(R) = \overline{R}$	The closure of R in X, where $R \subseteq X$.
$\mathrm{Int}_X(R)$	The interior of R in X, where $R \subseteq X$.
$\mathrm{Fr}_X(R)$	The frontier or topological boundary of R in X, where $R \subseteq X$.
$\dim_z Z$	The dimension of a (pro)manifold or vector space Z at $z \in Z$. If z is omitted then this indicates that $\dim_z Z$ is independent of $z \in Z$ and $\dim Z$ is this common value.
T_M	$\mathrm{T}_M : \mathrm{T}\,M \to M$ is the canonical projection from the tangent bundle $\mathrm{T}\,M$ of a smooth manifold (or promanifold) M onto M.
$\mathrm{diam}(S)$	The diameter of $S \subseteq M$ in a metric space (M,d). Defined as $\mathrm{diam}(S) \underset{\text{def}}{=} \sup_{s,\widehat{s}\in S} d(s,\widehat{s})$.
$d(S,T)$	Distance between $S \subseteq M$ and $T \subseteq M$: $d(S,T) \underset{\text{def}}{=} \inf_{s\in S, t\in T} d(s,t)$. And for $m_0 \in M$ we will write $d(m_0,T) \underset{\text{def}}{=} d(\{m_0\},T)$.
$\overline{\mathrm{B}}_r^d(m_0)$	Closed ball of radius $r > 0$ around $m_0 \in M$ in (M,d), where $\overline{\mathrm{B}}_r^d(m_0) \underset{\text{def}}{=} \{m \in M \mid d(m,m_0) \le r\}$ should not be confused with the notation $\overline{\mathrm{B}_r^d(m_0)}$ for the open ball's $\mathrm{B}_r^d(m_0)$ closure in M.

$\mathrm{B}_r^d(S)$	Open ball of radius $r > 0$ around $S \subseteq M$: $\mathrm{B}_r^d(S) \underset{\text{def}}{=} \underset{s \in S}{\cup} \mathrm{B}_r^d(s)$.
$f : D \subseteq X \to Y$	"f is a map on D with codomain Y where $D \subseteq X$." We may also write $D \in \operatorname{Open}(X)$ or $D \in \operatorname{Closed}(X)$ in place of $D \subseteq X$.
$f : (X, R) \to (Y, S)$	"f is a map $f : X \to Y$, $R \subseteq X$, $S \subseteq Y$, and $f(R) \subseteq S$." If $R = \{x\}$ or $S = \{y\}$ are singleton sets then we omit writing $\{\ \}$.
$\operatorname{co}(S)$	The convex hull of a subset S of some vector space over $\mathbb{R}$.
$\operatorname{carr} f$	The carrier of a map $f : X \to R$, where R is contained in some given additive group: $\operatorname{carr} f \underset{\text{def}}{=} \{x \in X \mid f(x) \neq 0\}$.
$\operatorname{supp} f$	The support of $f : X \to R$. Defined as $\operatorname{supp} f \underset{\text{def}}{=} \overline{\operatorname{carr}(f)}$.
$\operatorname{Im} f$	The image or range of a map f.
$f(R)$	$f(R) \underset{\text{def}}{=} \{f(x) \mid x \in \operatorname{Dom}(f) \cap R\}$, where R is any set.
$f\big\|_R : R \cap D \to S$	Restriction of $f : D \to Y$ to $D \cap R$ considered as a map with codomain S, where R and S are any sets such that $f(R \cap D) \subseteq S$.
$f\big\|_R$	Denotes $f\big\|_R : R \cap D \to Y$ where $f : D \to Y$ and R is any set.
Set, Top, Man	The category of sets (resp. topological spaces, smooth manifolds) and maps (resp. continuous maps, smooth maps).
$\mathbb{N}$, $\mathbb{Z}$	$\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$.
$\cup\mathcal{S}$ (resp. $\cap\mathcal{S}$)	For $\mathcal{S}$ a set of sets, $\cup\mathcal{S} \underset{\text{def}}{=} \underset{S \in \mathcal{S}}{\cup} S$ (resp. $\cap\mathcal{S} \underset{\text{def}}{=} \underset{S \in \mathcal{S}}{\cap} S$).
$\wp(X)$	The power set of a set X.

List of abbreviations:

LCTVS	Locally Convex Topological Vector Space
LHS (resp. RHS)	Left (resp. Right) Hand Side
resp.	respectively
TVS	Topological Vector Space

Definition, Notation, and Convention 1.1.1. Let P be a set, $S \subseteq P$, and $\leq$ be a binary relation on a set P, which we will identify the relation $\leq$ as a set of ordered pairs in the usual way. When we write $(S, \leq)$ then we mean the restricted relation $\left(S, \leq\big|_{S\times S}\right)$ and we call S an *ideal of* $(P, \leq)$ ([12, p. 36]) if for all $s \in S$ and $p \in P$, $p \leq s \implies p \in S$, or equivalently, if $S = \bigcup_{s\in S} P^{\leq s}$. The binary relation $\leq$ on P a *preorder (on P)* if it is reflexive and transitive and it is called a *partial order (on P)* if it is an antisymmetric (i.e. $p \leq q$ and $q \leq p$ implies $p = q$) preorder on P. If $S \subseteq P$ then an element $p \in P$ is called an *upper (resp. lower) bound of S (in P)* if $s \leq p$ (resp. $p \leq s$) for every $s \in S$ and we say that $\leq$ *directs* P and that P *is directed (by $\leq$)* if $\leq$ is a preorder and every pair of elements in P has an upper bound in P.

Notation and Convention 1.1.2. Unless indicated otherwise, $(I, \leq)$ and $(A, \leq)$ will henceforth denote partial orders. Elements of I will be denoted by h, i, j, and k while elements of A will be denoted by a, b, c, and possibly d, which will prevent the rise of any ambiguity from using the same symbol (i.e. $\leq$) to denote both order relations.

Definition 1.1.3. A map $\iota : A \to I$ is called *order-preserving* or an *order morphism (from $(A, \leq)$ to $(I, \leq)$)* if for all $a, b \in A$, whenever $a \leq b$ then $\iota(a) \leq \iota(b)$. We will say that the order-preserving map $\iota : A \to I$ is *cofinal* (resp. *strict* or *increasing*) if its image is cofinal in I (resp. if $a < b$ implies $\iota(a) < \iota(b)$).

Notation and Mnemonics 1.1.4. If the symbol ι (resp. α) represents a map between the sets I and A then its prototype will be $\iota : A \to I$ (resp. $\alpha : I \to A$) where the symbol ι (resp. α) was chosen so that one may immediately determine that a value $\iota(a)$ (resp. $\alpha(i)$) is an element of I (resp. A). Given two order-preserving maps $\iota : A \to I$ and $\alpha : I \to A$ and an element $a \in A$, we may write $\alpha\iota(a) \underset{\text{def}}{=} \alpha(\iota(a))$ to prevent an abundance of parentheses.

Remark 1.1.5. If $\iota : A \to I$ is a cofinal order morphism between two preorders then A being directed implies that I is directed.

Assumption 1.1.6. All categories will be assumed to be concrete categories. Each of Group, Top, Man, etc. will be paired with its usual forgetful functor.

Definition and Convention 1.1.7. If we say that M is a *manifold* then we mean that it is a Hausdorff second-countable locally-Euclidean topological space with a smooth structure. By using the canonical identification described in remark C.0.3, we will identify the smooth structure as either a smooth maximal atlas or equivalently by C_M^∞, its sheaf of smooth $\mathbb{R}$-valued functions.

Remark 1.1.8. Observe that our definition of a manifold does not require that it be connected nor that it have homogeneous dimension. This will be advantageous since even the study of projective systems of connected manifolds may require us to work with induced systems of disconnected manifolds whose dimensions diverge to infinity (e.g. if we wish to apply lemma 3.2.1(6)).

Definition 1.1.9. Suppose M is a d-dimensional manifold with no distinguished metric. Call a chart (U, φ) on M a *(coordinate) ball (resp. box, cube)* if $\varphi(U)$ is an open ball (in the Euclidean norm) in $\mathbb{R}^d$ (resp. a product of d-open intervals, a product of d-open intervals of the same length). If (U, φ) is a chart on M and $B \subseteq M$ then we will say that B is an *(proper) open ball (resp. box, cube) in* (U, φ) if $\mathrm{Cl}_M(B) = \overline{B} \subseteq U$ and $\left(B, \varphi\big|_B\right)$ is a coordinate ball (resp. box, cube). A subset $B \subseteq M$ will be called an *(proper) open ball (resp. box, cube) in* M if there exists some chart (U, φ) on M such that B is an open ball (resp. box, cube) in (U, φ). It should be clear what is meant if we replace the word "open" in "(proper) open ball (resp. box, cube)" with "closed" or if we add the words "centered at p", "of radius r", "with sides of length l", etc.

Convention and Remark 1.1.10. Although calling a map f *open* if it maps open sets to open sets is not controversial, one finds in the literature that some authors call a map $f : X \to Y$ open if it maps every open subset of X to an open subset of Y, which is the definition used in this paper (see def. A.0.6), while others require merely that these images be open in $\operatorname{Im} f$. To prevent this as well as other similar misunderstandings (and their consequences), we will often rewrite the map's prototype (e.g. we will usually write

"$f : X \to Y$ is open" or "$f : X \to \operatorname{Im} f$ is open" instead of simply "f is open") so that the reader may henceforth safely assume that the topological terms used in any definition (e.g. "maps open sets to open sets") are relative to the topologies of the domain and codomain that presented in the prototype. The analogous assumptions can also be safely made if X and Y are endowed with structures other than topologies, such as algebraic structures or sheaves.

Lifts, Factorizations, Fibrations, and Sequences

Definition 1.1.11. An indexed collection of objects $x_\bullet$ *has infinite range* or is *infinite-ranged* if $\{x_i : i \text{ is an index }\}$ is infinite, that it is *injective* if whenever i and j are distinct indices then $x_i \neq x_j$ and by a *net (resp. sequence) of distinct points* in a set X we mean an injective net (resp. sequence). If $x_\bullet$ is a net in a space X and $x \in X$ then "*$x_\bullet \to x$ is injective in X*" means that $x_\bullet$ is injective, $x \neq x_i$ for all indices i, and $x_\bullet \to x$ in X; the meanings of "let $x_\bullet \to x$ be injective in X" and "suppose $x_\bullet \to x$ has an injective subsequence in X" should be clear.

Let $f : X \to Y$ be a map between spaces and let $(y_i)_{i \in I}$ be a net in Y. If $(y_i)_{i \in I}$ is convergent in Y then by an *f-lift of* $(y_i)_{i \in I}$ we mean an I-directed convergent net $(x_i)_{i \in I}$ in X such that $f(x_i) = y_i$ for all $i \in I$. If there exists a net $(x_i)_{i \in I}$ in X that is an f-lift of a convergent net $(y_i)_{i \in I}$ then we'll say that f *lifts* $(y_i)_{i \in I}$ *to* $(x_i)_{i \in I}$, that $(y_i)_{i \in I}$ *is f-liftable*, and that f *can lift* $(y_i)_{i \in I}$. When we write $(x_i)_{i \in I} \to x$ *is an f-lift of* $(y_i)_{i \in I} \to y$ then we mean that $(y_i)_{i \in I}$ converges to y in Y, $f(x) = y$, and $(x_i)_{i \in I}$ is an f-lift of $(y_i)_{i \in I}$ that converges to x in X. ■

Definition 1.1.12 summarizes the terminology related to expressing a given morphism in terms of other morphisms. Most of the terminology is either based on or taken directly from the terminology found in [5].

Definition 1.1.12. Let E, B, X, and Z be objects and let $\pi : E \to B$ be a morphism (in some given category). If $f : Z \to B$ and $\widetilde{f} : Z \to E$ are morphisms such that $f = \pi \circ \widetilde{f}$ then we will call $\widetilde{f}$ a *π-lift of f (on Z to E)* and call f *the π-drop of $\widetilde{f}$ (on Z to B)*. If for some given morphism $f : Z \to B$ there exists some π-lift of f then we will say that f *is π-liftable (to E)* and that f *arises as a π-drop (from E) (defined on Z)*.

If $p : E \to X$ is a morphism for which there exists a morphism $\underset{\sim}{p} : B \to X$ such that $p = \underset{\sim}{p} \circ \pi$ then we will say that p *factorizes through π*, *p is factorized through π (by $\underset{\sim}{p}$)*, and that p *descends (through π) to $\underset{\sim}{p}$*.

$$\begin{tikzcd}
& E \arrow[d, "\pi"] \arrow[dr, "p"] & \\
Z \arrow[ur, "\widetilde{f}"] \arrow[r, "f"] & B \arrow[r, "\underset{\sim}{p}"] & X
\end{tikzcd}$$

■

It is apparent from the universal property of limits that the ability to find appropriate lifts of a maps can be useful when working with inverse systems and their limits. This naturally leads us to consider the homotopy lifting property, which we now generalize to a definition that is well-suited to inverse systems in the sense that it will make the statement of lemma 2.5.8 both concise and simple.

Definition 1.1.13. Let A, B, E, and Z be objects in a concrete category $\mathcal{C}$ such that $A \subseteq Z$ and the natural inclusion $\mathrm{In} : A \to Z$ is a morphism, and let $\pi : E \to B$ be a morphism. Say that π *has the extension lifting property from A to Z (in $\mathcal{C}$)* if for any morphism $f : A \to E$, whenever a morphism $H : Z \to B$ extends $\pi \circ f : A \to B$ to all of Z (i.e. $H \circ \mathrm{In} = \pi \circ f$) then there exists some π-lift, $\widetilde{H} : Z \to E$, of H extending f to Z (i.e. $\widetilde{H} \circ \mathrm{In} = f$):

$$\begin{tikzcd}
& & E \arrow[d, "\pi"] \\
A \arrow[r, hook, "\mathrm{In}"'] \arrow[urr, "f"] & Z \arrow[r, "H"'] \arrow[ur, dashed, "\widetilde{H}"] & B
\end{tikzcd}$$

In Top, if this is true with $Z = X \times [0,1]$ and $A = X \times \{0\}$ then we say that π has *the homotopy lifting property with respect to X* and π is called an *X-fibration*. If $r \in \mathbb{Z}^{\geq 0}$ then call

π an *r-fibration* if it is a Δ^n-fibration for all $0 \leq n \leq r$, where Δ^n is the standard n-simplex, and where in the particular case that π is a 0-fibration we will follow [34] and say that π has *the path-lifting property*. If π is an r-fibration for all positive integers r then we will call π a *weak fibration*, a *Serre fiber space*, or a *Serre fibration*. If $C \subseteq B$ then π *is a fibration over C* if $\pi|_{\pi^{-1}(C)} : \pi^{-1}(C) \to C$ is a fibration. ∎

Notation for Indexed Collections

The nature of inverse or direct systems regularly causes a proliferation of indices, where it is unfortunately often the case that the more plentiful the indices then the more difficult a proof or statement becomes to digest while simultaneously increasing the risk that the (usually relatively simple) ideas or intuition underlying it is obscured or even entirely missed. So to help avoid writing unnecessary indices we will now introduce some notation, conventions, and definitions.

Definition, Notation, and Convention 1.1.14. We will denote a collection $(M_i)_{i \in I}$ (resp. $(m^i)_{i \in I}$, $\left(H^i_{\alpha(i)}\right)_{i \in I}$, etc.) of objects indexed by some *indexing set* I by $M_\bullet$ (resp. $m^\bullet$, $H^\bullet_{\alpha(\bullet)}$, etc.), where for each index i, *the i^{th} component of $M_\bullet$* is the object M_i, which we may also denote by $(M_\bullet)_i$. If L is any set then $M_\bullet\big|_L \underset{\text{def}}{=} (M_i)_{i \in I \cap L}$ denotes *the restriction of $M_\bullet$ to L*. If f is a map and $M_\bullet$ are sets then by $f(M_\bullet)$ (resp. $f^{-1}(M_\bullet)$, etc.) we mean the collection $(f(M_i))_{i \in I}$ (resp. $(f^{-1}(M_i))_{i \in I}$, etc.).

If $S_\bullet = (S_l)_{l \in \Lambda}$ then by $S_\bullet \subseteq M_\bullet$ we mean that $S_i \subseteq M_i$ for all $i \in I \cap L$ whereas if we introduce $S_\bullet$ by saying "let $S_\bullet \subseteq M_\bullet$" without specifying $S_\bullet$'s indexing set then it should be assumed that $S_\bullet$ is indexed by $M_\bullet$'s indexing set. By $F_\bullet : M_\bullet \to N$ (resp. $G_\bullet : N \to M_\bullet$, $H^\bullet_{\alpha(\bullet)} : N_{\alpha(\bullet)} \to M_\bullet$, etc) we mean a collection of morphisms whose i^{th}-component has prototype $F_i : M_i \to N$ (resp. $G_i : N \to M_i$, $H^i_{\alpha(i)} : N_{\alpha(i)} \to M_i$, etc.) and if $S_\bullet = (S_l)_{l \in \Lambda} \subseteq M_\bullet$ then we'll use $F_\bullet(S_\bullet)$ (resp. $G_\bullet^{-1}(S_\bullet)$, $\left(H^\bullet_{\alpha(\bullet)}\right)^{-1}(S_\bullet)$, etc.) to denote $(F_\bullet(M_i))_{i \in I \cap \Lambda}$ (resp. $(G_i^{-1}(S_i))_{i \in I \cap \Lambda}$, $\left(\left(H^i_{\alpha(i)}\right)^{-1}(S_i)\right)_{i \in I \cap \Lambda}$, etc.). The meaning of $\cap M_\bullet$, $S_\bullet \cup M_\bullet$, and all other similar notation should now be easy to deduce. ∎

Mnemonical Notation and Convention 1.1.15. Given sets $Z_\bullet$ (resp. $Z^\bullet$) and some index λ, this index will always appear as a subscript (resp. superscript) to any element or subset of Z_λ (resp. Z^λ) (e.g. we will write "let $S_\lambda \subseteq Z_\lambda$" and never write "let $S \subseteq Z_\lambda$").

Remark 1.1.16. Although this convention does occasionally introduce an unnecessary index, by applying it consistently the net effect (in the author's opinion) will be to increase the overall readability of this paper *if* the reader adopts the perspective that this index is nothing more than a persistent reminder of which set (i.e. which component of $Z_\bullet$) this element or subset is contained within.

Definition and Convention 1.1.17. Given a collection of sets or maps $(S_\lambda)_{\lambda\in\Lambda}$ indexed by some subset $\Lambda \subseteq I$, if we define $S_i \underset{\text{def}}{=} \varnothing$ for all $i \in I \setminus \Lambda$ then we will call the I-indexed collection of sets $(S_i)_{i\in I}$ *the canonical $\varnothing$-extension (of $(S_\lambda)_{\lambda\in\Lambda}$) (to I)* and we may identify this I-indexed collection of sets with $(S_\lambda)_{\lambda\in\Lambda}$ and thereby denote both by $S_\bullet$. ■

Notation 1.1.18. We will henceforth always use $M_\bullet = (M_i)_{i\in J}$ and $\mu_\bullet = (\mu_i)_{i\in J}$ (resp. $N_\bullet = (N_a)_{a\in B}$ and $\nu_\bullet = (\nu_a)_{a\in B}$) to denote, respectively, a collection of objects and a collection of morphisms indexed by some subset $J \subseteq I$ (resp. $B \subseteq A$) where if the subset J (resp. B) is omitted or not clear from context then it is to be assumed that $J = I$ (resp. $B = A$). Furthermore, all $\mu_\bullet$ (resp. all $\nu_\bullet$) will share the same domain (usually denoted by M (resp. N)) and each μ_i (resp. ν_a) will have codomain M_i (resp. N_a).

Definition, Notation, and Convention 1.1.19. Given any map $\iota : A \to I$, by a *collection of sets (resp. morphisms, maps, etc.) indexed by ι* or an *ι-indexed collection of sets (resp. morphisms, maps, etc.)* we will mean the pair consisting of an A-indexed collection of sets (resp. morphisms, maps, etc.) together with ι, where if ι is understood then we may also refer to the A-indexed collection (rather than the pair) as an ι-indexed collection. If $\left(S^a_{\iota(a)}\right)_{a\in A}$ is an ι-indexed collection and ι is injective then we will drop the redundant index and instead write this collection as $S_{\iota(\bullet)} = \left(S_{\iota(a)}\right)_{a\in A}$ and furthermore, we may use ι to identify this A-indexed collection as an $(\operatorname{Im}\iota)$-indexed collection.

On the other hand, if ι is not injective, say $i \underset{\text{def}}{=} \iota(a_1) = \iota(a_2)$ for $a_1 \neq a_2$, then regardless of whether or not $S^{a_1}_{\iota(a_1)}$ and $S^{a_2}_{\iota(a_2)}$ are equal, we will still frequently write $S_{\iota(a_1)}$ (resp. $S_{\iota(a_2)}$) in place of $S^{a_1}_{\iota(a_1)}$ (resp. $S^{a_2}_{\iota(a_2)}$) since it is easy to deduce from the symbols present which of these sets we are referring to; however, we would not write "S_i" since there is no way to deduce from the symbols present if S_i is referring to $S^{a_1}_i$ or $S^{a_2}_i$. We further extend this convention to the notation used when introducing the collection such a collection in the following way: if it is the case (say in a proof or remark) that we will always be able to write $S_{\iota(a)}$ instead of $S^a_{\iota(a)}$ then rather than introducing this A-indexed collection by writing "let $S^{\bullet}_{\iota(\bullet)} = \left(S^a_{\iota(a)}\right)_{a \in A} \ldots$" we will instead write "let $S_{\iota(\bullet)} = \left(S_{\iota(a)}\right)_{a \in A} \ldots$".

Given any collection of sets $R_\bullet = (R_i)_{i \in I}$ indexed by I we will call an ι-indexed collection of sets $S_{\iota(\bullet)} = \left(S_{\iota(a)}\right)_{a \in A}$ an *(ι-indexed) collection of subsets (of $R_\bullet$)* if $S_{\iota(a)} \subseteq R_{\iota(a)}$ for all $a \in A$. If $S_{\iota(a)} = R_{\iota(a)}$ for all $a \in A$ then we will call $S_{\iota(\bullet)}$ a *subcollection* of $R_\bullet$. ■

Definition 1.1.20. If $\mathcal{O}$ is a collection of subsets of a set Z then if we say that $\mathcal{G}$ is *a presheaf of maps on* $\mathcal{O}$ we mean that $\mathcal{G}$ is presheaf on $\mathcal{O}$ that assigns to each $O \in \mathcal{O}$ a non-empty set of maps $\mathcal{G}(O)$ such that $\operatorname{Dom}\gamma = O$ for all $\gamma \in \mathcal{G}(O)$ and that $\mathcal{G}$'s restrictions are the canonical restrictions of maps. If $\mathcal{H}$ is a collection of maps defined on subsets of Z and if $\mathcal{O}$ is a collection of subsets of then say that $\mathcal{H}$ *is closed under restrictions to* $\mathcal{O}$ if for all $h \in \mathcal{H}$ and $O \in \mathcal{O}$, whenever $O \subseteq \operatorname{Dom} h$ then $h\big|_O \in \mathcal{H}$. If $\mathcal{H}$ is closed under restrictions to $\operatorname{Dom}\mathcal{H} \underset{\text{def}}{=} \{\operatorname{Dom} h \,|\, h \in \mathcal{H}\}$ then we will say that $\mathcal{H}$ *is closed under restrictions* and call $\mathcal{H}$ a *closed collection of maps (from subsets of Z).*

Remark and Convention 1.1.21. To every presheaf $\mathcal{G}$ of maps on a collection of subsets $\mathcal{O}$ of a set Z we can form the set of maps $\mathcal{H} \underset{\text{def}}{=} \bigcup_{O \in \mathcal{O}} \mathcal{G}(O)$ that is closed under restrictions while if $\mathcal{H}$ is a collection of maps defined on subsets of Z that is closed under restrictions then the assignment defined by sending $O \in \operatorname{Dom}\mathcal{H}$ to $\mathcal{G}(O) \underset{\text{def}}{=} \{\gamma \in \mathcal{H} \,|\, \operatorname{Dom}\gamma = O\}$ forms a presheaf of maps on $\operatorname{Dom}\mathcal{H}$. It is clear that above constructions are inverses of each other so we will henceforth identify collections of maps defined on subsets of Z that are closed under restrictions with presheaves of maps on collections of subsets of Z. Consequently, this identification allows us to treat any presheaf of maps on a collection of subsets of Z as if it was a closed collection of maps defined on subsets of Z and vice-versa, which we shall henceforth do without comment.

Definition 1.1.22. Let $h : Z \to M$ and $h_\bullet : Z \to M_\bullet$ be maps, $S \subseteq Z$, and $z \in Z$. By *the h-saturation of S* we mean the set $h^{-1}(h(S))$ and we'll say that *S is h-saturated* if $S = h^{-1}(h(S))$. Call S an *$h_\bullet$-fiber of z* or an *$h_\bullet$-fiber containing z* if there is some index i such that $S = h_i^{-1}(h_i(z))$.

Germs, and Submersions and Immersions of Germs

Many of the definitions below, including that of trace, generalizes that of Bourbaki ([11]).

Definition and Notation 1.1.23. Let M be a set, Φ a collection of M-valued maps, Z a space, $z \in Z$, $z_\bullet = (z_\lambda)_{\lambda \in \Lambda}$ a net, and let $\mathcal{R}$ and $\mathcal{S}$ be two collections of sets. The *trace of $\mathcal{R}$ in M* is $\operatorname{tr}_M(\mathcal{R}) = \{R \cap M : R \in \mathcal{R}\}$ while $\Phi\big|_{\mathcal{R}}$ will denote the set of all restrictions $\varphi\big|_R$ as φ ranges over Φ and R ranges over $\mathcal{R}$. We will denote the trace of $\operatorname{Nhd}_z(Z)$ in M by $[M]_z^Z$ and by $[z_\bullet]_z^Z$ we mean $[\{z_\lambda : \lambda \in \Lambda\}]_z^Z$, where we may omit Z from the notation if it is understood. If the domain of each map in Φ contains z and all of their values agree at z then we'll denote this common value by $\Phi(z)$ and call it *the value of Φ at z*. If m is any point then let $\Phi\big|_z$ (resp. $\Phi\big|_{z \to m}$) denote the set of all maps in Φ whose domains contain a neighborhood of z (resp. and map z to m). Say that *$\mathcal{R}$ is finer than $\mathcal{S}$* and that *$\mathcal{S}$ is coarser than $\mathcal{R}$* if for all $S \in \mathcal{S}$ there is some $R \in \mathcal{R}$ such that $R \subseteq S$.

Many of the following definitions consist of definitions from Bourbaki ([11]) or their generalizations.

Definition and Notation 1.1.24. Let Z be a set, $\mathcal{F}$ be a filter base on Z, $z \in Z$, $\mathcal{G}$ a set of maps, and m any object. If $R, S \subseteq Z$ then *R and S have the same germ (with respect to $\mathcal{F}$)* if there exists some $F \in \mathcal{F}$ such that $R \cap F = S \cap F$. This forms an equivalence relation on $\wp(Z)$, the power set of Z, and the equivalence class containing a set $R \subseteq Z$ is denoted by $[R]_{\mathcal{F}}$ and called the *germ of* R *(with respect to* $\mathcal{F}$*)* where if Z is a space and $\mathcal{F} = \operatorname{Nhd}_z(Z)$ then we will use the notation $[R]_z$ and call it *the germ of* R *at* z *in* Z.

Say that two maps γ *and* η *have the same germ with respect to* $\mathcal{F}$ *(*resp. *at* z*)* or that *they have the same* $\mathcal{F}$*-germ (*resp. *at* z*)* if there exists some $F \in \mathcal{F}$ such that $F \subseteq \operatorname{Dom}\gamma \cap \operatorname{Dom}\eta$ and $\gamma = \eta$ on F (resp. and $z \in F$), where if Z is a topological space and $\mathcal{F}$ is a filter bases for $\operatorname{Nhd}_S(Z)$ then we'll say that γ *and* η *have the same germ at* z *(in* Z*)*. This forms an equivalence relation on $\mathcal{G}$ and the equivalence class containing a map $g \in \mathcal{G}$, denoted by $[g]^{\mathcal{G}}_{\mathcal{F}}$, is called *the germ of* g *(in* $\mathcal{G}$*) (with respect to* $\mathcal{F}$*)* and the set of all germs in $\mathcal{G}$ will be denoted by $[\mathcal{G}]_{\mathcal{F}}$. If Z is a space, $g \in \mathcal{G}$, and $\operatorname{Dom} g$ is a neighborhood of z in Z then call the germ of $g \in \mathcal{G}$ with respect to $\operatorname{Nhd}_z(Z)$ *the* $\mathcal{G}$*-germ of* g *at* z *in* Z, denote it by $[g]^{\mathcal{G}}_z$, and let

$$[\mathcal{G}]_z \underset{\text{def}}{=} \left[\mathcal{G}\big|_z\right]_z \qquad \text{and} \qquad [\mathcal{G}]_{z\to m} \underset{\text{def}}{=} \left[\mathcal{G}\big|_{z\to m}\right]_z$$

If any of Z, $\mathcal{F}$, or $\mathcal{G}$ are understood then they may be omitted from the notation. ■

Remark 1.1.25. If $\mathcal{G}$ is a presheaf of maps instead of a set of maps, then the above definitions and notation related to sets of maps generalize immediately in the obvious way.

Definition 1.1.26. Let $F : (M, m) \to (N, n)$ be a pointed map, (Z, z) be a pointed space, and $\mathcal{G}$ (resp. $\mathcal{H}$) be a presheaf of M-valued (resp. N-valued) maps defined subsets of Z. If γ is an M-valued map then let $F_*(\gamma) \underset{\text{def}}{=} F \circ \gamma$. If we write $F_* : \mathcal{G} \to \mathcal{H}$ then we mean that $F_*(\gamma) = F \circ \gamma$ belongs to $\mathcal{H}$ for all $\gamma \in \mathcal{G}$, which then allows F to descend to the following map between germs at z:

$$\begin{aligned} F_* : [\mathcal{G}]_z &\longrightarrow [\mathcal{H}]_z \\ [\gamma]_z &\longmapsto [F \circ \gamma]_z \end{aligned}$$

which, by overloading notation, we will also write as $F : [\mathcal{G}]_z \to [\mathcal{H}]_z$ (so $F([\gamma]_z) \underset{\text{def}}{=} [F \circ \gamma]_z$).

More generally, for $\Phi \in [\mathcal{G}]_z$ and $\Psi \in [\mathcal{H}]_z$, if we write either $F_*(\Phi) = \Psi$ or $F(\Phi) = \Psi$ then we mean that there exists some $\gamma \in \Phi$ and some $\eta \in \Psi$ such that $F_*(\gamma) = F \circ \gamma$ and η have the same $\mathcal{H}$-germ at z, where in this case we will overload notation by letting both $F_*(\Phi)$ and $F(\Phi)$ denote this germ. If $\Phi \in [\mathcal{G}]_z$ and we write either $F_*(\Phi) \in [\mathcal{H}]_z$ or $F(\Phi) \in [\mathcal{H}]_z$ then we mean that there exists some $\Psi \in [\mathcal{H}]_z$ such that $F_*(\Phi) = \Psi$. If we write $F_* : [\mathcal{G}]_z \to [\mathcal{H}]_z$ or $F : [\mathcal{G}]_z \to [\mathcal{H}]_z$ then we mean that $F_*(\Phi) \in [\mathcal{H}]_z$ for all $\Phi \in [\mathcal{G}]_z$.

We will write and say that

(1) $F : [\mathcal{G}]_z \nwarrow [\mathcal{H}]_z$ *is a germ submersion at m (from $\mathcal{H}$ germs (at z)) (to $\mathcal{G}$ germs)* or that *F lifts $\mathcal{H}$ germs at z (through n) to $\mathcal{G}$ germs (at z) through m* if for all $\Psi \in [\mathcal{H}]_{z \to n}$ there exists some $\Phi \in [\mathcal{G}]_{z \to m}$ such that $\Psi = F_*(\Phi)$.

(2) $F : [\mathcal{G}]_z \nwarrow [\mathcal{H}]_z$ *can lift germs (at z) through n* if for all $\Psi \in [\mathcal{H}]_{z \to n}$ there exists some $\Phi \in [\mathcal{G}]_z$ (not necessarily though m) such that $F(\Phi) = \Psi$.

(3) $F : [\mathcal{G}]_z \to [\mathcal{H}]_z$ is a *germ immersion at m* if for all $\Phi, \widehat{\Phi} \in [\mathcal{G}]_{z \to m}$, $F(\Phi) = F\big(\widehat{\Phi}\big) \implies \Phi = \widehat{\Phi}$.

(4) $F : [\mathcal{G}]_z \to [\mathcal{H}]_z$ *is a germ bijection at m* if it is a germ immersion at m and $F : [\mathcal{G}]_z \nwarrow [\mathcal{H}]_z$ is a germ submersion at m.

where if we write $F : [\mathcal{G}]_z \to [\mathcal{H}]_z$ instead of $F : [\mathcal{G}]_z \nwarrow [\mathcal{H}]_z$ then we mean that in addition to satisfying that definition we also have $F : [\mathcal{G}]_z \to [\mathcal{H}]_z$. ∎

Remark 1.1.27. The notation $F : [\mathcal{G}]_z \nwarrow [\mathcal{H}]_z$ was chosen to emphasize that none of the above definitions of germ submersion require $F : [\mathcal{G}]_z \to [\mathcal{H}]_z$. These definitions were motivated by the situation where Z, M, and N are smooth manifolds and both $\mathcal{G}$ and $\mathcal{H}$ consist of various sets of continuous maps from neighborhoods of z in Z into M and N, respectively (e.g. say there are no additional restrictions on $\mathcal{G}$ while $\mathcal{H}$ consists solely of smooth topological embeddings).

Example 1.1.28 (Boman Theorem)**.** Recall ([1, p. 3], [27, cor. 3.14]) that one part of the Boman theorem states that a map $F : M \to N$ between two manifolds is smooth if and only if $F \circ \gamma : \mathbb{R} \to N$ is smooth for all smooth $\gamma : \mathbb{R} \to M$. It is easy to see that we can express its equivalent formulation in terms of germs of smooth curves as: $F : M \to N$ is smooth if and only if $F : [C^\infty(\mathbb{R} \to M)]_0 \to [C^\infty(\mathbb{R} \to N)]_0$, where $C^\infty(\mathbb{R} \to M)$ (resp. $C^\infty(\mathbb{R} \to N)$) denotes the set of all smooth curves into M (resp. N) with domain $\mathbb{R}$. The second part of the Boman theorem states that $F : M \to N$ is smooth if and only if $f \circ F : M \to \mathbb{R}$ is smooth for all smooth $f : N \to \mathbb{R}$.

Derivations

Definition 1.1.29. Let A be an algebra over a field F and let M be an A-bimodule. Then a map $D : A \to M$ is an *F-derivation into M* or a *derivation (over F) into M* if it is linear over F and satisfies the product rule:

$$D(fg) = D(f)g + fD(g) \quad \text{for all } f, g \in A.$$

If $\mathrm{ev} : A \to F$ is an F-algebra homomorphism then we may make F into an A-bimodule by defining

$$\begin{aligned} A \times F &\longrightarrow F \\ (a, \alpha) &\longmapsto \mathrm{ev}(a)\alpha \end{aligned}$$

with the right action of A on F defined analogously and we will denote the set of all F-derivations from A into F by $\mathrm{Der}_{\mathrm{ev}}(A \to F)$. In the particular case where the F-algebra A is a collection of either F-valued maps or equivalence classes of such maps, all of which may be evaluated at some point p, then we will let

$$\mathrm{Der}_p(A \to F) \underset{\mathrm{def}}{=} \mathrm{Der}_{\mathrm{ev_p}}(A \to F)$$

where $\mathrm{ev_p} : A \to F$ is the usual *evaluation at p* map (i.e. defined by $\mathrm{ev_p}(a) \underset{\mathrm{def}}{=} a(p)$) and we will call an element of $\mathrm{Der}_p(A \to F)$ an *(F-)derivation at p (on A) (into F).* ∎

Chapter 2

Limits in Set and Top

Just as a proper understanding of modern differential geometry would be made significantly more difficult without a well-developed intuitive understanding of the topology of Euclidean spaces and their construction from simpler spaces such as $\mathbb{R}$, so too would a proper understanding of promanifolds (i.e. projective limits of manifolds) be made more difficult without a well-developed intuition about their topology and construction from more basic spaces. For this reason the introduction to limits in Top that follows is written in a way so that, to the best of the author's ability, the statements and their proofs obscure as little of the underlying intuition that the author has about them. Furthermore, in addition to entirely new results and extensions of well-known results, where "well-known" means that they can be found in a standard reference on this subject such [11] or [12], even many of the well-known results in this chapter have proofs that, to the best of the author's knowledge, have not appeared elsewhere. Before continuing, it is recommended that the reader have a basic understanding of limits and colimits of systems where this can be obtained by reading Dugundji [12], which was the author's primary reference for this chapter, or Bourbaki [11].

Introduction to Systems, (Co)Cones, and (Co)Limits

Systems

As the name suggests, limits may be thought of as the objects that would result if one were able to "do a sequence of actions forever," where the objects being acted upon and the rules of these actions are encapsulated in the following definition of an inverse system. Example 2.1.8 may help clarify how, at least in the case when the system is ordered by $\mathbb{N}$, the definition of an inverse system captures these ideas. The definition of inverse system that we've adopted is based the definition given in [12].

Definition 2.1.1. An *inverse* or *projective system (over I) (in a concrete category $\mathscr{C}$)* is a quadruple $(M_\bullet, \mu_{ij}, I, \leq)$, which we may also denote by $(M_\bullet, \mu_{ij})$ or Sys_M, where (1) $(I, \leq)$ is a partial order, (2) $M_\bullet = (M_i)_{i \in I}$ with M_i an object in $\mathscr{C}$ for each index $i \in I$, (3) $\mu_{ij} : M_j \to M_i$ is a morphism for all $i, j \in I$ with $i \leq j$ with $\mu_{ii} = \operatorname{Id}_{M_i}$ if $i = j$, and where these morphisms satisfy *the compatibility condition*:

$$\mu_{ij} \circ \mu_{jk} = \mu_{ik} \text{ whenever } i \leq j \leq k$$

The morphisms μ_{ij} are called the *connecting maps* or the *bonding maps* of the system.

If $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, I, \leq)$ is a quadruple consisting of objects $M_\bullet$, morphisms μ_{ij}, and a partial order $(I, \leq)$, then call the quadruple

$$\operatorname{Sys}_M^{\mathrm{op}} \underset{\mathrm{def}}{=} (M_\bullet, \mu_{ij}, I, \leq^{\mathrm{op}})$$

the *dual* or *transpose of* Sys_M, where $\leq^{\mathrm{op}}$ represents the dual order of $(I, \leq)$. A *direct system* is a quadruple $(M_\bullet, \mu_{ij}, I, \leq)$ whose transpose $(M_\bullet, \mu_{ij}, I, \leq^{\mathrm{op}})$ is an inverse system. If Sys_M is a projective or direct system ordered by $(I, \leq)$ then we will say that Sys_M is:

- *directed* if its partial order $(I, \leq)$ is a directed set.

- *surjective* (resp. *injective*, etc.) if all connecting maps are surjective (resp. injective, etc.).
- *compact* (resp. *connected*, etc.) if all objects are *compact* (resp. *connected*, etc.) topological spaces.
- *smooth* (resp. *smooth submersive*) if all objects are smooth manifolds and all connecting maps are smooth (resp. smooth submersions).
- *pointed* if all objects and maps are pointed. ■

Remarks 2.1.2.

- Some authors (e.g. [5]) reserve the term bonding map for the particular case of $I = \mathbb{N}$ and then only for connecting maps of the form $\mu_{i,i+1}$.
- By viewing a partial order as a category in the usual way, a system may be viewed as a functor from a partial order $(I, \leq)$ into $\mathcal{C}$.
- The class of all inverse (resp. direct) systems in some given category will itself become a category if we use definition 3.0.1 to define its morphisms. The same holds true of the class of all inverse (resp. direct) systems when their orders (i.e. $(I, \leq)$) are required to belong to a certain category (e.g. systems indexed by directed partial orders, systems indexed by $\mathbb{N}$, etc).
- In the notation $(M_\bullet, \mu_{ij}, I, \leq)$, the symbol μ_{ij} in this tuple actually represents a collection of morphisms where there is one morphism for each pair of indices $i, j \in I$ such that $i \leq j$; this tuple should properly be written as $\left(M_\bullet, (\mu_{ij})_{(i,j) \in \leq}, I, \leq\right)$, where $\leq$ is viewed a collection of ordered pairs from $I \times I$. ■

Convention 2.1.3. Since we will only be working in concrete categories, whenever we refer to Sys_M as a system in Set then we are actually referring to the system that results from applying the category's forgetful functor to all of Sys_M's objects and connecting morphisms.

Assumption and Notation 2.1.4. Unless indicated otherwise, we will henceforth assume that $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, I, \leq)$ and $\operatorname{Sys}_N = (N_\bullet, \nu_{ab}, A, \leq)$ are projective systems where both $(I, \leq)$ and $(A, \leq)$ are partial orders. If we declare that Sys_M and Sys_N are direct systems then unless indicated otherwise, we will assume that these systems are the tuples $\operatorname{Sys}_M = (M^i, \mu_i^j, I, \leq)$ and $\operatorname{Sys}_N = (N^a, \nu_a^b, A, \leq)$. Whenever we write "$\mu_{pq}$" or "$\mu_p^q$" (resp. "$\nu_{pq}$" or "$\nu_p^q$") then unless indicated otherwise, it should be assumed the p and q are indices in I (resp. A) with $p \leq q$. We will also usually assume that all M_i and N_a are non-empty, where it should be clear from context when this assumption is or is not being made. ■

Example and Definition 2.1.5. Given any space Z and any partial order $(I, \leq)$ we can form the *constant* or *trivial system over* I by letting $Z_i = Z$ and $\mu_{ij} = \operatorname{Id}_Z$ for all $i \leq j$ in I. We will denote this system by $(Z, \operatorname{Id}_Z, I, \leq)$, $(Z_\bullet, \operatorname{Id}_Z, I, \leq)$, or simply $\operatorname{ConstSys}_Z$.

Example and Definition 2.1.6. If $J \subseteq I$ then *the restriction of* $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, I, \leq)$ *to* J, denoted by $\operatorname{Sys}_M\big|_J$ and called a *subsystem of* Sys_M, is the system $\left(M_\bullet\big|_J, \mu_{ij}, J, \leq\big|_{J\times J}\right)$ that consists of all those M_i and μ_{ij} for which all indices belong to J.

Example and Definition 2.1.7. If $J \subseteq I$, $i_0 \in I$, and $S_{i_0} \subseteq M_{i_0}$ then *the system induced by* S_{i_0} *and* J *(and* i_0*)* is

$$\operatorname{Sys}_M\big|_{J, S_{i_0}} \underset{\text{def}}{=} \left(S_j, \mu_{jk}\big|_{S_k}, J\right)$$

where for all $j, k \in J$ with $j \leq k$,

$$S_j \underset{\text{def}}{=} \begin{cases} \mu_{j_0,j}^{-1}(S_{j_0}) & \text{if } j \geq i_0 \\ M_j & \text{otherwise} \end{cases}$$

and $\mu_{jk}\big|_{S_k} : S_k \to S_j$. By *the system induced by* S_{i_0}, denoted by $\operatorname{Sys}_M\big|_{S_{i_0}}$, we mean the system induced by S_{i_0} and $J = I^{\geq i_0}$.

Example 2.1.8. Let $M_\bullet = (M_i)_{i\in\mathbb{N}}$ be any sequence of objects and let $\mu_{\bullet,\bullet+1} = (\mu_{i,i+1})_{i\in\mathbb{N}}$ be a sequence of morphisms where $\mu_{i,i+1} : M_{i+1} \to M_i$ for all $i \in \mathbb{N}$. This collection of objects and morphisms induces a projective system $\mathrm{Sys}_M \underset{\text{def}}{=} (M_\bullet, \mu_{ij}, \mathbb{N}, \leq)$ when we define $\mu_{ii} \underset{\text{def}}{=} \mathrm{Id}_{M_i}$ and

$$\mu_{ij} \underset{\text{def}}{=} \mu_{i,i+1} \circ \mu_{i+1,i+2} \circ \cdots \circ \mu_{j-1,j} : M_j \to M_i$$

for $i, j \in \mathbb{N}$ with $i+1 < j$. Similarly, if given sequences of objects $M^\bullet = (M^i)_{i\in\mathbb{N}}$ and morphisms $\mu_\bullet^{\bullet+1} = (\mu_i^{i+1})_{i\in\mathbb{N}}$ where $\mu_i^{i+1} : M^i \to M^{i+1}$ for all $i \in \mathbb{N}$ then define $\mu_i^i \underset{\text{def}}{=} \mathrm{Id}_{M^i}$ and

$$\mu_i^j \underset{\text{def}}{=} \mu_{j-1}^j \circ \cdots \circ \mu_{i+1}^{i+2} \circ \mu_i^{i+1} : M^i \to M^j$$

for $i, j \in \mathbb{N}$ with $i + 1 < j$ so as to obtain a direct system. It is clear that the above definitions of μ_{ii} and μ_{ij} (resp. μ_i^i and μ_i^j) are the only ones that would make $(M_\bullet, \mu_{ij}, \mathbb{N}, \leq)$ (resp. $\left(M^\bullet, \mu_i^j, \mathbb{N}, \leq\right)$) into an inverse (resp. direct) system.

Convention 2.1.9. Henceforth, we may define inverse (resp. direct) systems directed by $\mathbb{N}$ by specifying only the bonding maps $\mu_{i,i+1}$ (resp. μ_i^j) where it should then be immediately assumed that the bonding maps μ_{ij} (resp. μ_i^j) are defined as above for all $i \leq j$.

Example and Definition 2.1.10. Suppose that $R_\bullet = (R_j)_{j\in J}$ is a collection of sets indexed by some set J and give J the partial order induced by reverse set inclusion on $R_\bullet$ (i.e. $j \leq k \iff R_k \subseteq R_j$). For all $i \leq j$, let $\mathrm{In}_{ij} : R_j \to R_i$, $\mathrm{In}_i : \cap R_\bullet \to R_i$, and $\mathrm{In}^i : R_i \to \cup R_\bullet$ denote the natural inclusions. It is straightforward to verify that if $(J, \leq)$ is directed then $(\cap R_\bullet, \mathrm{In}_\bullet)$ is a limit of the inverse system $(R_\bullet, \mathrm{In}_{ij}, J, \leq)$ while if $(J, \leq^{\mathrm{op}})$ is directed then $(\cup R_\bullet, \mathrm{In}^\bullet)$ is a colimit of the direct system $(R_\bullet, \mathrm{In}_{ij}, J, \leq^{\mathrm{op}})$.

If J is directed or contains a minimum element then let $I = J$ where otherwise we will stipulate that J not contain the symbol $-\infty$ and then define $I = J \cup \{-\infty\}$. If $I \neq J$ and if there is also some distinguished set X that contains each R_j as a subset then let $R_{-\infty} = X$ and otherwise let $R_{-\infty} = \underset{j\in J}{\cup} R_j$, where in either case we also give I the partial order induced by reverse set inclusion on $(R_i)_{i\in I}$, which clearly extends J's original partial order and makes $-\infty$ into I's minimum. By *the (canonical) inverse system induced by $R_\bullet$ (and inclusions)* we mean the inverse system $\mathrm{Sys}_R \underset{\text{def}}{=} ((R_i)_{i\in I}, \mathrm{In}_{ij}, I, \leq)$ where each $\mathrm{In}_{ij} : R_j \to R_i$ is the natural inclusion.

The next definition generalizes the definition, found in [45, Sheaves], of a presheaf of objects of a category on a basis of a topology.

Example and Definition 2.1.11. Let $\mathscr{C}$ be some category, (Z, τ_Z) be a topological space, and $\mathcal{B}$ be a basis for Z. A *presheaf (of objects) in* $\mathscr{C}$ *on* $\mathcal{B}$ is a direct system $\operatorname{Sys}_{\mathcal{M}} = (\mathcal{M}(A), \mu_A^B, \mathcal{B}, \leq)$ in $\mathscr{C}$ where the partial order $(\mathcal{B}, \leq)$ is reverse set inclusion and where we write $\mathcal{M}(A)$ instead of $\mathcal{M}_A$ in order to conform with the standard notation for presheaves. If $\mathcal{B} = \tau_Z$ then we call $\operatorname{Sys}_{\mathcal{M}}$ a *presheaf in* $\mathscr{C}$ *on* Z and by a *morphism of presheaves* we mean a direct system morphism (def. 3.0.1) between presheaves.

A prototypical example of a presheaf is $C_{Z \to M}$ where for every open subset U of Z, $C_{Z \to M}(U) \underset{\text{def}}{=} C(U \to M)$ consists of all continuous maps from U into the given space M and where the connecting maps are the usual restrictions of domains of functions (i.e. $\mu_U^V(f) = f\big|_V$ for $V \subseteq U$ open in X). It should now be clear that a presheaf (and indeed any system) may be viewed as nothing more than an indexed collection of information (e.g. maps indexed by open subsets of Z) where information between different indices are related to each other in a consistent way (i.e. via the connecting morphisms, which satisfy the consistency condition). By taking this point of view, it is natural to extend the definition of a presheaf by considering direct system indexed by an arbitrary collection $\mathcal{B}$ of subsets of Z that is partially ordered by reverse set inclusion; we will call any such direct system a *presheaf on* $\mathcal{B}$. ■

Example 2.1.12. Let $(q_j)_{j=1}^{\infty}$ be a sequence of natural numbers greater than 1 and for all $j \in \mathbb{N}$, define

$$\begin{aligned} \rho_j : S^1 &\longrightarrow S^1 \subseteq \mathbb{C} \\ z &\longmapsto z^j \end{aligned}$$

and $\mu_{j,j+1} \underset{\text{def}}{=} \rho_{q_j}$, which gives us a projective system $\operatorname{Sys}_M \underset{\text{def}}{=} (S^1, \mu_{ij}, \mathbb{N})$. Suppose that for each $j \in \mathbb{N}$, we've written q_j as the product $q_j = \prod\limits_{l=1}^{\lambda(j)} p_{j,l}$ for some $\lambda(j) \in \mathbb{N}$ and some natural numbers $p_{j,1},\ldots,p_{j,\lambda(j)}$ greater than 1. Let $\mathrm{Sys}_{\widehat{M}} = (S^1, \widehat{\mu}_{ab}, \mathbb{N})$ denote the system that is defined just as Sys_M, except that the sequence $p_{1,1},\ldots,p_{1,\lambda(1)}$, $p_{2,1},\ldots,p_{2,\lambda(2)}$, $p_{3,1},\ldots$ is used in place of $q_1, q_2, \ldots$.

It is clear that Sys_M can be obtained from $\mathrm{Sys}_{\widehat{M}}$ by restricting $\mathrm{Sys}_{\widehat{M}}$ to some cofinal J subset of $\mathbb{N}$. Once the reader has knowledge of limits and lemma 2.1.37, this example will entail that if we replace $q_1, q_2, \ldots$ with $p_{1,1},\ldots,p_{1,\lambda(1)}, p_{2,1},\ldots,p_{2,\lambda(2)}, p_{3,1},\ldots$ then (up to a unique isomorphism) we would not have changed the limit of Sys_M. In particular, this implies that to understand the limits of systems of the form defined above, it suffices to understand the limits of such systems where every q_j is prime. ■

Example 2.1.13. Suppose that M is a smooth manifold and let $\mathrm{T}^0 M = M$. We can inductively define for every $k \in \mathbb{N}$, the k^{th}-order tangent bundle by $\mathrm{T}^k M \underset{\text{def}}{=} \mathrm{T}(\mathrm{T}^{k-1} M)$, where for all $k \in \mathbb{Z}^{\geq 0}$, we will denote the canonical projection by $\mathrm{T}_M^{k \leftarrow k+1} : \mathrm{T}^{k+1} M \to \mathrm{T}^k M$ or by $\mathrm{T}_{\mathrm{T}^k M \leftarrow \mathrm{T}^{k+1} M}$. This gives us the following projective system of manifolds whose bonding maps are smooth surjective submersions:

$$\mathrm{Sys}_{\mathrm{T}^\infty M} \underset{\text{def}}{=} \left(\mathrm{T}^k M, \mathrm{T}_M^{k \leftarrow k+1}, \mathbb{Z}^{\geq 0}\right).$$

Cones and Cocones

Definition 2.1.14. Let Sys_M be an inverse (resp. direct) system in some category $\mathscr{C}$ where $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, I, \leq)$ (resp. $\mathrm{Sys}_M = \left(M^\bullet, \mu_i^j, I, \leq\right)$), let Z is an object in $\mathscr{C}$, and let $h_i : Z \to M_i$ (resp. $h^i : M^i \to Z$) be a collection of morphisms indexed by I, which we will denote by $h_\bullet$ (resp. $h^\bullet$). We will say that $h_\bullet$ (resp. $h^\bullet$) is *compatible* or *consistent with* Sys_M if $\mu_{ij} \circ h_j = h_i$ (resp. $h^j \circ \mu_i^j = h^i$) whenever $i \leq j$, i.e. if the respective diagram commutes:

$$\begin{tikzcd} & Z \arrow[dl, "h_i"'] \arrow[dr, "h_j"] & \\ M_i & & M_j \arrow[ll, "\mu_{ij}"'] \end{tikzcd} \qquad \begin{tikzcd} & Z & \\ M^i \arrow[ur, "h^i"] \arrow[rr, "\mu_i^j"] & & M^j \arrow[ul, "h^j"'] \end{tikzcd}$$

cone (resp. cocone)

in which case we will call the pair $(Z, h_\bullet)$ a *cone (in $\mathscr{C}$) (from Z) into* Sys_M (resp. call $(Z, h^\bullet)$ a *cocone (in $\mathscr{C}$) from* Sys_M *(to Z)*). The object Z is called *the vertex* of the cone (resp. cocone) and we may also abuse terminology by referring to the collection of morphisms $h_\bullet$ (resp. $h^\bullet$), rather than the vertex and morphisms pair, as a cone (resp. cocone). Call a cone or cocone *epi* (resp. *mono.*, *surjective*, *injective*, etc.) if all $h_i : Z \to M_i$ (resp. all $h^i : M^i \to Z$) have that property. ■

Observation 2.1.15. If Sys_M is a system in Set and $h_\bullet = (h_i)_{i \in I}$ is any collection of maps valued in $M_\bullet$ with a common domain Z, then $(Z, h_\bullet)$ is a cone into Sys_M if and only if h_i is constant on each fiber of h_j for all $i \le j$ in I.

The proof of the following remark is straightforward.

Remark 2.1.16 (Extending a cone from a cofinal collection). Suppose that $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, I)$ is an inverse system, I is directed, and $J \subseteq I$ is cofinal in I. If $(h_j)_{j \in J}$ is a collection of morphisms compatible with $\mathrm{Sys}_M\big|_J$ then this collection can be uniquely extended to a collection of morphisms $(h_i)_{i \in I}$ compatible with Sys_M, which can be defined for each $i \in I$ by $h_i \underset{\text{def}}{=} \mu_{ij} \circ h_j$ for any choice of $j \in J$ such that $j \ge i$. Similarly, if $\mathrm{Sys}_M = \left(M^\bullet, \mu_i^j, I\right)$ is a directed direct system then any collection of morphisms $(h^j)_{j \in J}$ compatible with $\mathrm{Sys}_M\big|_J$ can be uniquely extended to a collection of morphisms $(h^i)_{i \in I}$ compatible with Sys_M, where for each $i \in I$, the i^{th} component of $h^\bullet$ can be defined by $h^i \underset{\text{def}}{=} h^j \circ \mu_i^j$ for any choice of $j \in J$ such that $j \ge i$.

In particular, if $(I, \le)$ has a greatest element $\gamma \in I$ (i.e. $i \le \gamma$ for all $i \in I$), which implies that $(I, \le)$ is directed, then in fact every cone $h_\bullet$ (resp. cocone $h^\bullet$) is completely determined by its γ^{th} component and the system's connecting morphisms. ■

Example 2.1.17. This original example shows that the extension in remark 2.1.16 may fail to exist if $(I, \leq)$ is not directed: Let L, C, R, and $\star$ be any non-zero distinct objects and define $Z = \{\star\}$, $M_L = \{L, C\}$, $M_R = \{C, R\}$, $M_0 = \{L, C, R\}$, $I = \{0, L, R\}$, and partially order I by $i \leq j \iff i = j$ or $i = 0$. For all $i \in I$, define $\mu_{ii} = \mathrm{Id}_{M_i}$ and let μ_{0L} and μ_{0R} be the natural inclusions (i.e. $\mu_{0L}(L) = L$, $\mu_{0L}(C) = C = \mu_{0R}(C)$, and $\mu_{0R}(R) = R$) where observe that $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, I, \leq)$ is a partially ordered inverse system in Set.

Let $J = \{L, R\}$ so J is cofinal in I and define

$$\begin{aligned} h_L : Z &\longrightarrow M_L \\ \star &\longmapsto L \end{aligned} \qquad \text{and} \qquad \begin{aligned} h_R : Z &\longrightarrow M_R \\ \star &\longmapsto R \end{aligned}$$

Observe that since $L \nleq R$ and $R \nleq L$, the partial order on J is $i \leq j \iff i = j$ so the maps (h_L, h_R) form a cone from Z into $\mathrm{Sys}_M\big|_J = (\{M_L, M_R\}, \{\mu_{LL}, \mu_{RR}\}, J)$ (and furthermore, once we define the limit of a system it will be clear that the limits of both Sys_M and $\mathrm{Sys}_M\big|_J$ are non-empty sets). Since $(\mu_{0L} \circ h_L)(\star) = \mu_{0L}(L) = L$ is not equal to $R = (\mu_{0R} \circ h_R)(\star)$, there is no map $h_0 : Z \to M_0$ that would allow for $\{h_0, h_L, h_R\}$ to be a cone from Z into Sys_M. ∎

Example 2.1.18. If $(I, \leq)$ has a greatest element $\gamma \in I$ (i.e. $i \leq \gamma$ for all $i \in I$) then $\mu_{\bullet\gamma} \underset{\text{def}}{=} (\mu_{i\gamma})_{i \in I}$ is a cone from M_γ into the inverse system $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, I)$. Similarly, $\mu_\bullet^\gamma = (\mu_i^\gamma)_{i \in I}$ would be a cocone from the direct system $\left(M^\bullet, \mu_i^j, I\right)$ into M_γ.

Example and Definition 2.1.19. If $(M, \mu_\bullet)$ is a cone into a system Sys_M then each morphism $h : Z \to M$ into M gives rise to a cone $(Z, \mu_\bullet \circ h)$, called *the (canonical) cone induced by h (and $\mu_\bullet$)*, where $\mu_\bullet \circ h \underset{\text{def}}{=} (\mu_i \circ h)_{i \in I}$. Similarly, if $(M, \mu^\bullet)$ is a cocone from Sys_M then each morphism $h : M \to Z$ into M gives rise to a cocone $(Z, h \circ \mu^\bullet)$, called *the (canonical) cocone induced by h (and $\mu^\bullet$).*

If the map $h : Z \to M$ from example 2.1.19 is a morphism in Set then it is possible for each $\mu_i \circ h : Z \to M_i$ to be surjective while $h : Z \to M$ fails to be surjective. This motivates the following definitions.

Definition 2.1.20 ($\mu_\bullet$-surjective, $\mu_\bullet$-open, etc.)**.** Let $(M, \mu_\bullet)$ be a cone into Sys_M, $h : Z \to M$ be a map, and $S \subseteq M$. Say that

- $h : Z \to M$ is *$\mu_\bullet$-surjective* (resp. *$\mu_\bullet$-open*, etc.) if the same is true of the canonical cone $(Z, \mu_\bullet \circ h)$, that is, if each $\mu_i \circ h : Z \to M_i$ is a surjective (resp. open, etc.) map.
- *S is $\mu_\bullet$-open* (resp. *$\mu_\bullet$-compact, $\mu_\bullet$-dense*, etc.) (*in $M_\bullet$*) if the same is true of $\mu_\bullet(S)$ in $M_\bullet$, that is, if each $\mu_i(S)$ is an open (resp. compact, dense, etc.) subset of M_i.
- *S is $\mu_\bullet$-surjective* if the same is true of the inclusion map $\operatorname{In}_S^M : S \to M$.

In particular, observe that S is $\mu_\bullet$-surjective if and only if $\mu_i(S) = M_i$ for every index i. ■

Observation 2.1.21. If Sys_M is a system in Top and $(Z, h_\bullet)$ is any cone of quotient maps into Sys_M then Sys_M is completely determined by the maps $h_\bullet$ and the partial order $\leq$.

Limits and Colimits

Definition 2.1.22 (Limit and Colimit)**.** Let Sys_M be a projective (resp. direct) system where $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, I, \leq)$ (resp. $\operatorname{Sys}_M = \left(M^\bullet, \mu_i^j, I, \leq\right)$). A cone $(M, \mu_\bullet)$ to (resp. cocone $(M, \mu^\bullet)$ from) Sys_M is said to be an *projective limit* or *limit* (resp. *direct limit* or *colimit*) *of* Sys_M if whenever $(Z, h_\bullet)$ (resp. $(Z, h^\bullet)$) is any other cone into (resp. cocone from) Sys_M then there exists a unique morphism $h : Z \to M$ (resp. $h : M \to Z$) such that $\mu_i \circ h = h_i$ (resp. $h \circ \mu^i = h^i$) for all indices i, which we may abbreviate by writing $\mu_\bullet \circ h = h_\bullet$ (resp. $h \circ \mu^\bullet = h^\bullet$). In this case, the (respective) following diagram will commute for all $i \leq j$ in I,

$$\begin{tikzcd}
 & Z \arrow[ddl, "h_i"'] \arrow[d, "h"] \arrow[ddr, "h_j"] & \\
 & M \arrow[dl, "\mu_i"] \arrow[dr, "\mu_j"'] & \\
M_i & & M_j \arrow[ll, "\mu_{ij}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & Z & \\
 & M \arrow[u, "h"'] & \\
M^i \arrow[uur, "h^i"] \arrow[ur, "\mu^i"'] \arrow[rr, "\mu_i^j"] & & M^j \arrow[uul, "h^j"'] \arrow[ul, "\mu^j"]
\end{tikzcd}$$

limit (resp. colimit)

and for each index i we will call μ_i (resp. μ^i) the i^{th} *projection* (resp. i^{th} *insertion*) *morphism*, write $(M, \mu_\bullet) = \lim \operatorname{Sys}_M$ (resp. $(M, \mu^\bullet) = \operatorname{colim} \operatorname{Sys}_M$), and call the unique morphism h *the limit of* $(Z, h_\bullet)$ *(resp. of* $(Z, h^\bullet)$*) (into* $(M, \mu_\bullet)$ *(resp. from* $(M, \mu^\bullet)$*)* or simply *into* M *(resp. from* M*)*. We may also call the limit $h : M \to Z$ of the cocone *the colimit of* $(Z, h^\bullet)$ *(from* $(M, \mu^\bullet)$ or *from* M). We will denote the limit of Sys_M by whichever of the following notations is most convenient,

$$\varprojlim \operatorname{Sys}_M, \quad \varprojlim M_\bullet, \quad \varprojlim_i M_i, \quad \varprojlim_I M_i, \quad \text{or} \quad \varprojlim(M_\bullet, \mu_{ij})$$

where for direct limits, we will reverse the arrow and possibly replace "lim" with "colim." We may also omit writing any arrow by agreeing to take lim to mean $\varprojlim$ and never $\varinjlim$. We will usually denote the vertex of this limit (resp. colimit) by M and if the projections are understood then we may also refer to the vertex of the limit cone (resp. cocone) (i.e. M) as the limit (resp. colimit) of Sys_M. Furthermore, we may use analogous notation (e.g. $\varprojlim h_\bullet$) for the limit of a cone (resp. cocone).

In the above situation, we will say that $(M_\bullet, \mu_\bullet, \mu_{ij}, I, \leq)$, $(\mu_\bullet, \operatorname{Sys}_M)$, or (if $(I, \leq)$ is understood) $(M_\bullet, \mu_\bullet, \mu_{ij})$ is *a projective representation of* M. Call a representation of M *epic* (resp. *mono., surjective, injective*, etc.) if this is true of the limit cone $(M, \mu_\bullet)$ and of all connecting morphisms $\mu_{ij} : M_j \to M_i$. ■

Remarks 2.1.23.

- If a limit (resp. colimit) exists then it is unique up to unique isomorphism so we will frequently abuse language by referring to a particular limit (resp. colimit) as *the* limit (resp. colimit) of a system.

- The i^{th} insertion morphism μ^i of a colimit is sometimes referred to as *the i^{th} projection* by some authors, despite it not necessarily being a component of any cone. In particular, this means that the morphisms of a cone and of a cocone are sometimes both called projections.

Notation and Convention 2.1.24. Unless indicated otherwise, we will henceforth assume that $(M, \mu_\bullet)$ (resp. $(N, \nu_\bullet)$) denotes a limit of Sys_M (resp. Sys_N), where recall from notation 2.1.4 that Sys_M (resp. Sys_N) denotes the system $(M_\bullet, \mu_{ij}, I)$ (resp. $(N_\bullet, \nu_{ab}, A)$).

More generally, if an inverse system is denoted by $(P_\bullet, \pi_{\alpha\beta}, \Omega, \leq)$ then unless indicated otherwise, by the symbol P (i.e. $P_\bullet$ with the subscript removed) we mean to denote the vertex of a limit cone of this system and similarly, we will use $\pi_\bullet = (\pi_\alpha)_{\alpha\in\Omega}$ to denote this limit cone's morphisms, where $\pi_\alpha : P \to P_\alpha$ for each $\alpha \in \Omega$. Conversely, if P is the vertex of the limit cone of some system then unless indicated otherwise, Sys_P should be assumed to denote this system and $P_\bullet$ should be assumed to denote this system's objects. Analogous notation will be used for direct systems and their colimits. ■

The following convention will allow us to talk about and write many important set equalities, such as the equality in corollary 2.3.10, in a natural way. It is well-known that intersections of sets can be defined as limits.

Example and Convention 2.1.25. Suppose that $R_\bullet = (R_j)_{j\in J}$ is a collection of sets indexed by some set J and let I, $R_{-\infty}$, and $\operatorname{Sys}_R \underset{\text{def}}{=} (R_\bullet, \operatorname{In}_{ij}, I, \leq)$ be as in definition 2.1.10. It is straightforward to verify that $\left(\bigcap\limits_{j\in J} R_j, \operatorname{In}_\bullet\right)$ is a limit of Sys_R where $\operatorname{In}_i : \bigcap\limits_{j\in J} R_j \to R_i$ is the natural inclusion for each $i \in I$.

If for each $j \in J$, R_j was defined as a subset of some set Z (e.g. as in corollary 2.3.10) and if we write $\varprojlim R_\bullet$ without it being clear from context what projective system we are taking the limit of, then it should be assumed that we are taking the limit of the above canonical system and furthermore, if we write $\varprojlim R_\bullet$ then unless explicitly stated otherwise we will mean the cone $\left(\bigcap_{j\in J} R_j, \mathrm{In}_\bullet\right)$, which consequently justifies writing $\varprojlim R_\bullet$ in place of $\bigcap_{j\in J} R_j$. ■

Characterizations of Limits

It is straightforward to verify the following well-known proposition.

Proposition 2.1.26. Let Sys_M be a projective system in Top and suppose that $(M, \mu_\bullet)$ is a limit of Sys_M in Set. If M is given the weakest topology τ making all $\mu_\bullet$ continuous then $((M,\tau), \mu_\bullet)$ is a limit of Sys_M in Top.

The following lemma shows that when working in the category Set, to prove that a particular cone is a limit of a system we need only to fix some singleton set $\{\star\}$ and then check for the existence of unique limits of cones with vertex $\{\star\}$. Similarly, it shows that when working in the category Group it suffices to show existence of unique limits of cones with vertex $(\mathbb{Z}, +)$. Except for the assertions (1) and (2), which are well-known, this lemma is otherwise original.

Lemma 2.1.27. Let Sys_M be an inverse system in Set (resp. Group), $(M, \mu_\bullet)$ be a cone into Sys_M in Set (resp. Group), and let $D = \{\star\}$ (resp. $D = (\mathbb{Z}, +)$) where $\{\star\}$ denotes an arbitrary singleton set. Then $(M, \mu_\bullet) = \varprojlim \mathrm{Sys}_M$ in the category Set (resp. Group) $\iff$ for all cones $(D, h_\bullet)$ into Sys_M there exists a unique map $h : D \to M$ satisfying $\mu_i \circ h = h_i$ for all $i \in I$. Furthermore, if this is the case then

(1) for all $m, \widehat{m} \in M$, $m = \widehat{m} \iff \mu_i(m) = \mu_i(\widehat{m})$ for all $i \in I$.

(2) $M = \varnothing \iff$ the only cone into Sys_M is the cone of empty maps (i.e. $(\varnothing, (\varnothing)_{i\in I})$).

Proof. In the category Set (resp. Group), for any set (resp. group) Z and any element $z \in Z$ define $c^z : D \to Z$ by $c^z(\star) = z$ (resp. $c^z(n) = nz$).

$(\implies)$ is immediate so to prove $(\impliedby)$ let $(Z, h_\bullet)$ be a cone into Sys_M and for all $z \in Z$ and $i \in I$ define $h_i^z \underset{\text{def}}{=} h_i \circ c^z : D \to M_i$ and observe that $(D, h_\bullet^z)$ is a cone into Sys_M in the category of Set (resp. Group). For each $z \in Z$ let $h^z : D \to M$ be such that $\mu_i \circ h^z = h_i^z$ for all $i \in I$ and define the map $h : Z \to M$ by $h(z) = h^z(\star)$ (resp. $h(z) = h^z(1)$). For all $z \in Z$ and $i \in I$, in the category Set

$$(\mu_i \circ h)(z) = \mu_i(h^z(\star)) = h_i^z(\star) = h_i(z)$$

while in the category Group the same equality holds when $\star$ is replaced by 1, which shows that $\mu_i \circ h = h_i$ for all $i \in I$. If $k : Z \to M$ is another map such that $\mu_i \circ k = h_i$ for all $i \in I$ then for all $i \in I$ and $z \in Z$ we have

$$\mu_i \circ k \circ c^z = h_i \circ c^z = h_i^z$$

where the uniqueness assumption now implies that $k \circ c^z = h \circ c^z$ so that $k(z) = h(z)$, which proves that $h = z$. This proves that $(M, \mu_\bullet)$ is the limit of Sys_M in Set.

If Sys_M is in the category Group then we must still show that h is a homomorphism so let $w, z \in Z$ and observe that for any index i,

$$\begin{aligned}
(\mu_i \circ c^{h(w)+h(z)})(1) &= \mu_i(h(w) + h(z)) = \mu_i(h(w)) + \mu_i(h(z)) \text{ since } \mu_i \text{ is a homomorphism} \\
&= (\mu_i \circ h)(w) + (\mu_i \circ h)(z) \\
&= (\mu_i \circ h \circ c^w)(1) + (\mu_i \circ h \circ c^z)(1) \\
&= (h_i \circ c^w)(1) + (h_i \circ c^z)(1) = h_i \circ (c^w + c^z) \\
&= (h_i \circ c^{w+z})(1) = (\mu_i \circ h^{w+z})(1)
\end{aligned}$$

which implies that $\mu_i \circ c^{h(w)+h(z)} = h_i \circ c^{w+z}$ so the uniqueness assumption now implies that $c^{h(w)+h(z)} = h^{w+z}$ where evaluating at 1 gives us $h(w) + h(z) = h(w + z)$.

If $(M, \mu_\bullet)$ is the limit of Sys_M in Set and $m, \widehat{m} \in M$ are such that $\mu_i(m) = \mu_i(\widehat{m})$ for all $i \in I$ then we obtain $m = \widehat{m}$ by considering the cone $(\{\star\}, h_\bullet)$ from the singleton set $\{\star\}$ into

Sys_M where for each $i \in I$, $h_i : \{\star\} \to M_i$ is defined by $h_i(\star) = \mu_i(m)$. (2) is immediate since $(M, \mu_\bullet)$ is a cone into Sys_M and maps with non-empty domains have non-empty images. ■

The following example is well-known.

Example 2.1.28. If $(I, \leq)$ has a greatest element $\gamma \in I$ then any inverse system $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, I, \leq)$ in a category $\mathcal{C}$ will have $(M_\gamma, \mu_{\bullet\gamma})$ as a limit, where $\mu_{\bullet\gamma} = (\mu_{i\gamma})_{i \in I}$. This is because if $(Z, h_\bullet)$ is any cone into Sys_M then $h_\gamma : Z \to M_\gamma$ is its unique limit morphism into $(M_\gamma, \mu_{\bullet\gamma})$. In particular, if $\left(\widehat{M}, \widehat{\mu}_\bullet\right)$ is any other limit of Sys_M in $\mathcal{C}$ then $\widehat{\mu}_\gamma : \widehat{M} \to M_\gamma$ is an isomorphism in $\mathcal{C}$.

The following proposition implies that in a category with limits, the vertex of a limit cone of any system Sys_M indexed by a set is canonically the vertex of limit cone of a canonical *totally ordered* system. Due to the simplicity of its statement, the author suspects that following proposition may have already been discovered, in which case the author claims merely to have discovered it independently.

Proposition 2.1.29. Let $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, I)$ be a partially ordered system with limit $(M, \mu_\bullet)$, let $\mathcal{I}$ denote the set of all non-empty ideals in I (def. 1.1.1), partially order $\mathcal{I}$ by set inclusion, and let A be a totally ordered subset of $\mathcal{I}$ such that $I = \bigcup_{a \in A} a$. For all $a \in A$, let $\mathrm{Sys}_{N_a} \underset{\text{def}}{=} \mathrm{Sys}_M\big|_a$ and suppose that $(N_a, \nu_{\bullet a} \underset{\text{def}}{=} (\nu_{ia})_{i \in a})$ is a limit of Sys_{N_a}. For all $a, b \in A$ with $a \leq b$, since both $\left(N_b, \nu_{\bullet b}\big|_a = (\nu_{ib})_{i \in a}\right)$ and $\left(M, \mu_\bullet\big|_a\right)$ form cones into Sys_{N_a}, we will let $\nu_{ab} : N_b \to N_a$ and $\nu_a : M \to N_a$ denote their respective limits. Then $\mathrm{Sys}_N \underset{\text{def}}{=} (N_\bullet, \nu_{ab}, A)$ defines a canonical inverse system, $(M, \nu_\bullet)$ defines a canonical cone into Sys_N, and $(M, \nu_\bullet) = \varprojlim \mathrm{Sys}_N$.

Proof. Note that by the universal property of limits of cones we have $\mu_{ij} \circ \nu_{ja} = \nu_{ia}$, $\nu_{ia} \circ \nu_{ab} = \nu_{ib}$, and $\nu_{ia} \circ \nu_a = \mu_i$ for all $i, j \in I$ with $i \leq j$ and $a, b \in A$ with $j \in a \leq b$. If $a, b, c \in A$ with $a \leq b \leq c$ then $\nu_{ab} \circ \nu_{bc} = \nu_{ac}$ since for all $i \in a$, $\nu_{ia} \circ (\nu_{ab} \circ \nu_{bc}) = \nu_{ib} \circ \nu_{bc} = \nu_{ic} = \nu_{ia} \circ \nu_{ab}$. Let $(Z, (h_a)_{a \in A})$ be a cone into Sys_N. For any $i \in I$, if $i \in a$ and $a \leq b$ then $\nu_{ia} \circ h_a = \nu_{ia} \circ \nu_{ab} \circ h_b = \nu_{ib} \circ h_b$ so that $h_i \underset{\text{def}}{=} \nu_{ia} \circ h_a$ is well-defined and independent of the choice of $a \in A$ that contains i. If $i \leq j$ are in I and j is contained in $a \in A$ then $\mu_{ij} \circ h_j = \mu_{ij} \circ \nu_{ja} \circ h_a = \nu_{ia} \circ h_a = h_i$, which shows that $\left(Z, h_\bullet \underset{\text{def}}{=} (h_i)_{i \in I}\right)$ is a cone into Sys_M. Let $h : Z \to M$ be the limit of $(Z, h_\bullet)$ so that $\mu_i \circ h = h_i$ for all $i \in I$. If $a \in A$ then by the uniqueness of limit morphisms we have $\nu_a \circ h = h_a$ since for all $i \in a$, $\nu_{ia} \circ (\nu_a \circ h) = \mu_i \circ h = h_i = \nu_{ia} \circ h_a$. If $H : Z \to M$ is a morphism such that $\nu_a \circ H = h_a$ for all $a \in A$ then for any $i \in a$, $\mu_i \circ H = \nu_{ia} \circ \nu_a \circ H = \nu_{ia} \circ h_a = h_i$ so that the uniqueness of limits of cones implies that $h = H$. ∎

The Canonical Limit

It is straightforward to verify that the following cone is always a limit of Sys_M in Set.

Definition 2.1.30 (Bourbaki [11, I.4.4])**.** (Canonical Limit): If $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, I, \leq)$ is a projective system in the category of sets then we will call the following limit of Sys_M *the canonical limit (of* Sys_M*) (in* Set*)*:

$$M \underset{\text{def}}{=} \left\{ m = (m_i)_{i \in I} \in \prod_{i \in I} M_i \,\middle|\, \mu_{ij}(m_j) = m_i \text{ for all } i \leq j \right\}$$

where the elements of M are called *threads* and where the projections are the restrictions to M of the canonical projections onto the i^{th} coordinate:

$$\begin{aligned} \mu_i \underset{\text{def}}{=} \mathrm{Pr}_i\big|_M : M &\longrightarrow M_i \\ m = (m_l)_{l \in I} &\longmapsto m_i \end{aligned}$$

If Sys_M is a projective system in the category Top then we will give M the weakest topology making all μ_i's continuous. For $i \in I$, by a *μ_i-subbasic open set (in M)* we mean a subset of M of the form $\mu_i^{-1}(U_i)$ for some $U_i \in \mathrm{Open}(M_i)$. By a *($\mu_\bullet$-)subbasic open set (in M)* we mean a subset of M that is a μ_i-subbasic open set in M for $i \in I$, where in the case that $(I, \leq)$ is directed then we will also call such a set a *($\mu_\bullet$-)basic open set (in M)*.

Suppose that Sys_M is a system in the category of semigroups (resp. topological semigroups) and write all groups multiplicatively. If for every $i \in I$ we define

$$\begin{aligned} h_i : M \times M &\longrightarrow M_i \\ (m, m') &\longmapsto \mu_i(m) \cdot \mu_i(m') \end{aligned}$$

then it is readily verified that $(Z, h_\bullet)$ is a cone in the category Set (resp. Top) from $M \times M$ into Sys_M whose limit map $h : M \times M \to M$ is a binary operation on M making M into a semigroup (resp. topological semigroup). It is clear that this binary operation on M makes every $\mu_\bullet$ into a morphism of semigroups (resp. continuous semigroups) and makes $(M, \mu_\bullet)$ into the limit of Sys_M in the category of semigroups (resp. topological semigroups).

Now suppose that in addition every M_j has a unique identity element, which we'll denote by 1_j, and that $\mu_{ij}(1_j) = 1_i$ for all $i \leq j$. Then there exists some element in M, which we'll denote by 1_M, such that $\mu_\bullet(1_M) = 1_\bullet$ where it is clear that 1_M is an identity element of M.

Suppose that Sys_M is a system in the category of groups (resp. topological groups). Then for any $m \in M$ observe that $\mu_{ij}((\mu_j(m))^{-1}) = (\mu_{ij}(\mu_j(m)))^{-1} = (\mu_i(m))^{-1}$ so that there is some element in M, which we will denote by m^{-1}, such that $\mu_\bullet(m^{-1}) = (\mu_\bullet(m))^{-1}$. It is clear that for all $m \in M$, this element m^{-1} is m's multiplicative inverse in M. This group structure on M makes every $\mu_\bullet$ into a homomorphism (resp. continuous homomorphism) and makes $(M, \mu_\bullet)$ into the limit of Sys_M in the category of groups (resp. topological groups). ■

Notation and Convention 2.1.31. If M is a pointed space with distinguished point $m^0 \in M$ then Sys_M will be identified with the pointed system whose pointed spaces are $(M_i, \mu_i(m^0))$, which then clearly makes all bonding maps and projections into pointed maps.

If each M_i has a uniformity $\mathcal{U}_i$ then we will henceforth assume that M has the weakest uniformity, call it $\mathcal{U}$, making all projections $\mu_i : M \to M_i$ uniformly continuous. This is just the relative uniformity on M induced from the product uniformity on $\prod_{i \in I} M_i$ so that if all M_i are Hausdorff then if all $\mathcal{U}_i$ are complete then so is $\mathcal{U}$. The uniform topology on M generated by $\mathcal{U}$ is the same as the weakest topology making all $\mu_\bullet$ continuous. Note that all μ_i will be uniformly continuous even if none of the bonding maps were but in case all bonding maps are uniformly continuous then we will say that the inverse system is a *uniform system.* If the index set is countable and all $M_\bullet$ are metrizable (resp. completely metrizable) then so too is M, where if d_i is the metric associated to M_i then we can associate to M the well-known metric

$$d(m, \widehat{m}) = \sum_{i=1}^{\infty} \frac{d_i(m_i, \widehat{m}_i)}{1 + d_i(m_i, \widehat{m}_i)}$$

where $m = (m_i)_{i \in \mathbb{N}}$ and $\widehat{m} = (\widehat{m}_i)_{i \in \mathbb{N}}$.

Example 2.1.32. Working in either Set or Top, if $(M, \mu_\bullet)$ is the canonical limit of Sys_M and $(Z, h_\bullet)$ is a cone into Sys_M then it is immediately verified that

$$\begin{aligned} h \underset{\text{def}}{=} (h_i)_{i \in I} : Z &\longrightarrow M \\ z &\longmapsto (h_i(z))_{i \in I} \end{aligned}$$

has range in M and is the limit of $(Z, h_\bullet)$.

Properties of Limit of Cones

The next proposition lists either well-known properties or simple observations that the reader may use to gain insight into how the components of a cone relate to its limit map.

Proposition 2.1.33. Let $(Z, h_\bullet)$ be a cone of continuous maps into Sys_M with limit h.

(1) Suppose $J \subseteq I$ and $S_\bullet = (S_j)_{j \in J}$ are sets such that $S_j \subseteq M_j$ for all $j \in J$. Then

$$h\left(\bigcap_{j \in J} h_j^{-1}(S_j) \right) = \bigcap_{j \in J} \mu_j^{-1}(S_j) \cap \operatorname{Im} h$$

so in particular, for any $i \in I$ and $S_i \subseteq M_i$, $h(h_i^{-1}(S_i)) = \mu_i^{-1}(S_i) \cap \operatorname{Im} h$.

(2) If M is the canonical limit of Sys_M then $M \cap \prod_{i \in I} h_i(S) = \bigcap_i \mu_i^{-1}(h_i(S))$ for any $S \subseteq Z$.

(3) If Sys_M is directed and $S \subseteq Z$ then $h(S)$ is a dense subset of $\bigcap_i \mu_i^{-1}(h_i(S))$ so that in particular,

- if in addition M is both Hausdorff and the canonical limit of Sys_M then $h(S)$ being compact will imply that $h(S) = M \cap \prod_{i \in I} h_i(S)$.
- if in addition $\operatorname{Im} h_i$ is dense in M_i for each index i then $\operatorname{Im} h$ is dense in M.

(4) If $\operatorname{Im} h_i$ is dense in M_i for some index i then $\operatorname{Im} \mu_i$ is dense in M_i and $M = \varnothing \iff Z = \varnothing \iff M_i = \varnothing$.

(5) If $S \subseteq Z$ and $V_i \underset{\text{def}}{=} h_i(Z \setminus S)$ for each i, then $V \underset{\text{def}}{=} \operatorname{Im} h \setminus \bigcap_i \mu_i^{-1}(V_i)$ is a subset of $h(S)$.

(6) For any index i and open subset U_i of M_i, $h(h_i^{-1}(U_i))$ is open in $\operatorname{Im} h$.

(7) If for some i, $h_i : Z \to M_i$ is injective (resp. a topological embedding) then for every $j \geq i$, the same is true of $h_j : Z \to M_j$, $h : Z \to M$, $\mu_{ij}\big|_{\operatorname{Im} h_j}$, and $\mu_j\big|_{\operatorname{Im} h}$ and furthermore, $\mu_{ij}\big|_{\operatorname{Im} h_j} = h_i \circ h_j^{-1}$ on $\operatorname{Im} h_j$, and $\mu_j\big|_{\operatorname{Im} h} = h_j \circ h^{-1}$ on $\operatorname{Im} h$.

(8) If $Z \subseteq M_i$ then $\mu_i \circ h = \mathrm{Id}_Z \iff \mu_{ij} \circ h_j = \mathrm{Id}_Z$ for each index $j \geq i$.

Proof. (1): Observe that

$$h^{-1}\left(\bigcap_{j\in J} \mu_j^{-1}(S_j)\right) = \bigcap_{j\in J} h^{-1}\left(\mu_j^{-1}(S_j)\right) = \bigcap_{j\in J} h_j^{-1}(S_j)$$

which implies the second of the following equalities:

$$\bigcap_{j\in J} \mu_j^{-1}(S_j) \cap \operatorname{Im} h = h\left(h^{-1}\left(\bigcap_{j\in J} \mu_j^{-1}(S_j)\right)\right) = h\left(\bigcap_{j\in J} h_j^{-1}(S_j)\right)$$

(2): Observe that $\mu_i^{-1}(h_i(S)) = M \cap \operatorname{Pr}_i^{-1}(h_i(S)) = M \cap \left(h_i(S) \times \prod_{\substack{l\in I \\ l\neq i}} M_i\right)$, where $h_i(S)$ is intended to be in the i^{th} position in the product, and so $\bigcap_i \mu_i^{-1}(h_i(S)) = M \cap \prod_{i \in I} h_i(S)$.

(3): If $S = \varnothing$ then each $h_i(S)$ is empty so $h(S) = \varnothing$ is a dense subset of $\underset{i}{\cap} \mu_i^{-1}(h_i(S)) = \varnothing$. Hence, assume that $S \neq \varnothing$. Let $m \in \underset{l \in I}{\cap} \mu_l^{-1}(h_l(S))$ and let $\mu_i^{-1}(U_i)$ be an arbitrary basic open neighborhood of m in M, where $i \in I$ and U_i is open in M_i so that in particular, $\mu_i(m) \in h_i(S) \cap U_i$. Since $\left(S, \left(h_k\big|_S\right)_{k \in I}\right)$ is a cone into M with limit $h\big|_S : S \to M$ we may apply (1) to deduce that

$$\mu_i^{-1}(U_i) \cap \operatorname{Im}\left(h\big|_S\right) = h\big|_S\left(h_i\big|_S^{-1}(U_i)\right) = h\left(S \cap h_i^{-1}(U_i)\right)$$

so that if $\mu_i^{-1}(U_i) \cap h(S) = h(S \cap h_i^{-1}(U_i))$ was empty then $S \cap h_i^{-1}(U_i) = \varnothing$, which would imply that $h_i(S) \cap U_i = \varnothing$ and hence contradict $\mu_i(m) \in h_i(S) \cap U_i$. Since by proposition 2.2.1 $\mu_i^{-1}(U_i)$ was an arbitrary non-empty basic open subset of M, we've thus shown that $h(S)$ is dense in $\underset{l \in I}{\cap} \mu_l^{-1}(h_l(S))$.

(4): Since $h(Z) \subseteq M$, $M = \varnothing$ implies $Z = \varnothing$ while if $Z = \varnothing$ then since $\varnothing = \operatorname{Im} h_i$ is dense in M_i, it follows that $M = \varnothing$. That $\operatorname{Im} \mu_i$ is dense in M_i follows from the fact that $h_i = \mu_i \circ h$ and $\operatorname{Im} h_i$ is dense in M_i.

(5): Let $v \in V$ so that there exists some $z \in Z$ such that $v = h(z)$. Since $v \notin \underset{i}{\cap} \mu_i^{-1}(S_i)$ there exists some index i such that $v \notin \mu_i^{-1}(S_i)$. This implies that $h_i(z) = \mu_i(h(z)) = \mu_i(v) \notin V_i = h_i(Z \setminus S)$ so that in particular we must have $z \in S$. Thus $v = h(z) \in h(S)$ so that $V \subseteq h(S)$.

(6): If U_i is an open subset of M_i then $h\left(h_j^{-1}(U_j)\right) = \mu_j^{-1}(U_j) \cap \operatorname{Im} h$ is open in $\operatorname{Im} h$.

(7): The claims follow immediately from lemma A.7.1 and the equalities $h_i = \mu_{ij}\big|_{\operatorname{Im} h_j} \circ h_j = \mu_i\big|_{\operatorname{Im} h} \circ h$ and $\mu_i\big|_{\operatorname{Im} h} = \mu_{ij}\big|_{\operatorname{Im} h_j} \circ \mu_j\big|_{\operatorname{Im} h}$.

(8): If $\mu_{ij} \circ h_j = \operatorname{Id}_Z$ for all $j \geq i$ then applying $\varprojlim_{j \geq i}$ to both sides and using the functoriality of inverse limits gives us $\mu_i \circ h = \operatorname{Id}_Z$. ∎

Remark 2.1.34. Examples 2.3.11 and 3.4.4 give surjective cones who limits fail to be surjective.

The following lemma is a list of observations that we will henceforth use without comment.

Lemma 2.1.35. Let $h : Z \to M$ be a continuous map from a space Z, let $h_\bullet = \mu_\bullet \circ h : Z \to M_\bullet$, $S = \operatorname{Im} h$, and $S_\bullet = \operatorname{Im} h_\bullet$. If i is any index then

(1) $h_i : Z \to M_i$ is injective (resp. an embedding) $\iff$ the same is true of $h_j : Z \to M_j$ for all $j \geq i$.

(2) $\mu_{ij}\big|_{S_j} : S_j \to M_i$ is injective (resp. an embedding) for all $j \geq i$ $\iff$ if the same is true of $\mu_j\big|_S : S \to M_j$ for all $j \geq i$.

(3) If the statements in (1) hold then so do the statements in (2) and furthermore $h : Z \to M$ is injective (resp. an embedding), $\mu_{ij}\big|_{S_j} = h_i \circ h_j^{-1}$ on S_j, and $\mu_j\big|_S = h_j \circ h^{-1}$ on S for all $j \geq i$.

(4) If $Z \subseteq M_i$ then $\mu_i \circ h = \mathrm{Id}_Z \iff \mu_{ij} \circ h_j = \mathrm{Id}_Z$ for each index $j \geq i$.

Proof. (1) and (2) follow immediately from lemma A.7.1 by observing that $h_i = \mu_{ij} \circ h_j$ and $\mu_i\big|_S = \mu_{ij}\big|_{S_j} \circ \mu_j\big|_S$ as does the fact that if any of the statements in (1) hold then so do the statements in (2). Since $h = \varprojlim h_i$, it follows from (1) that if any $h_j : Z \to M_j$ is injective (resp. an embedding) then so is $h : Z \to M$. For any $j \geq i$, it is clear from $h_i = \mu_i \circ h$ that $\mu_j : S \to S_j$ is injective we have that $\mu_i \circ \left(\mu_j\big|_S\right)^{-1} = \mu ij$ on S_j so that

$$h_i \circ h_j^{-1} = \mu_i \circ h \circ h^{-1} \circ \left(\mu_j\big|_S\right)^{-1} = \mu_i \circ \left(\mu_j\big|_S\right)^{-1} = \mu_{ij}\big|_{S_j}$$

That $\mu_j\big|_S = h_j \circ h^{-1}$ on S is apparent.

(4): If $\mu_{ij} \circ h_j = \mathrm{Id}_Z$ for all $j \geq i$ then applying $\varprojlim_{j \geq i}$ to both sides and using the functoriality of inverse limits gives us $\mu_i \circ h = \mathrm{Id}_Z$. ■

The Canonical Colimit

Definition 2.1.36 (Canonical Colimit)**.** Suppose that $\mathrm{Sys}_M = \left(M^i, \mu_i^j, I, \leq\right)$ is a directed direct system in Set and define an equivalence relation $\sim$ on the disjoint union $\bigsqcup_{i \in I} M^i$ by declaring $m^i \in M^i$ and $m^j \in M^j$ equivalent $\iff$ there exists some $k \geq i, j$ such that $\mu_i^k(m^i) = \mu_j^k(m^j)$. Let

$$[\cdot]_\sim : \underset{i \in I}{\sqcup} M^i \to \left(\underset{i \in I}{\sqcup} M^i\right) / \sim$$

denote the natural map onto the identification space $M \underset{\text{def}}{=} \left(\underset{i \in I}{\sqcup} M^i\right) / \sim$, which we may also write as $[\cdot]_{\text{Sys}_M}$ or simply $[\cdot]$, where recall that $\left(\underset{i \in I}{\sqcup} M^i\right) / \sim$ is the set of equivalence classes of $\sim$. Observe in particular that if $m \in M$ then $m = ([\cdot]_\sim)^{-1}(m)$, where this equality makes sense since every equivalence class of $\sim$ is a subset of $\underset{i \in I}{\sqcup} M^i$.

For each $i \in I$, let $\text{In}^i : M^i \to \underset{l \in I}{\sqcup} M^l$ denote the canonical inclusion and let $\mu^i : M^i \to M$ denote the composition

$$\mu^i : M^i \xrightarrow{\text{In}^i} \underset{l \in I}{\sqcup} M^l \xrightarrow{[\cdot]_\sim} M = \left(\underset{l \in I}{\sqcup} M^l\right) / \sim$$

which we may also denote by $[\cdot]_\sim^i$, $[\cdot]_{\text{Sys}_M}^i$, or simply $[\cdot]^i$.

We will call the cocone $(M, \mu^\bullet) = \left(\left(\underset{i \in I}{\sqcup} M^i\right) / \sim, (\mu^i)_{i \in I}\right)$ *the canonical colimit* or *the canonical direct limit (of* Sys_M*) (in the category* Set*)*. If $(Z, h^\bullet)$ is a cocone from Sys_M into Z then let $\underset{i \in I}{\sqcup} h^i$ denote the map

$$\begin{aligned} \underset{i \in I}{\sqcup} h^i : \underset{i \in I}{\sqcup} M^i &\longrightarrow Z \\ m_i \in M_i &\longmapsto h_i(m_i) \end{aligned}$$

It is straightforward to verify that for every $m \in M$ the image of the set $m = ([\cdot]_\sim)^{-1}(m)$ under this map is a (non-empty) singleton set and so we may denote the induced map by

$$h : M \to Z$$

which may equivalently be defined as the map sending each $\mu^i(m^i) \in M$ to $h_i(m^i)$. It is again straightforward to verify that $h : M \to Z$ is the limit of $(Z, h^\bullet)$ from $(M, \mu^\bullet)$ so that we will call this map *the canonical (co)limit (of* $(Z, h^\bullet)$*) (from* $(M, \mu^\bullet)$*)*, where we may write $h^\bullet$ or M in place of the respective cocone. For each $m \in M$, we may identify $h(m)$ with the singleton set $\left(\underset{i\in I}{\sqcup} h^i\right)(m) = \{h(m)\}$ whenever this would not cause confusion so that in particular, we would be justified in abusing the equality sign by writing $h(m) = \left(\underset{i\in I}{\sqcup} h^i\right)(m)$.

If Sys_M is also a direct system in Top then we will give $\underset{l\in I}{\sqcup} M^l$ the strongest topology making all $\mathrm{In}^i : M^i \to \underset{l\in I}{\sqcup} M_l$ continuous and we will give $M \underset{\text{def}}{=} \left(\underset{i\in I}{\sqcup} M^i\right)/\sim$ the strongest topology making $[\cdot]_\sim : \underset{i\in I}{\sqcup} M^i \to M$ continuous (i.e. the identification topology on M, which is the same as the strongest topology making all $\mu^i : M^i \to M$ continuous). Observe that a subset $S \subseteq \underset{l\in I}{\sqcup} M^l$ is open in $\underset{l\in I}{\sqcup} M^l \iff \left(\mathrm{In}^i\right)^{-1}(S)$ is open in M^i for all $i \in I$ and that a subset $T \subseteq M$ is open in $M \iff \left(\mu^i\right)^{-1}(T)$ is open in M^i for all $i \in I$. ■

Cofinal Subsystems

The following lemma establishes the well-known fact that if we restrict a directed system to a cofinal subset J of the indexing set then simply forgetting those μ_i whose index does not belong to J results in the limit of the restricted system.

Lemma 2.1.37. If $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, I)$ is directed inverse system, $J \subseteq I$ is cofinal in I, and $(M, \mu_\bullet = (\mu_i)_{i\in I})$ is a limit of Sys_M then $\left(M, (\mu_j)_{j\in J}\right) = \varprojlim \mathrm{Sys}_M\big|_J$.

Proof. Given a collection of morphisms $h_j : Z \to M_j$ indexed by $j \in J$ that are compatible with $\mathrm{Sys}_M\big|_J = (M_i, \mu_{ij}, J)$ we may extend this collection to a system indexed by $i \in I$ by defining, $h_i = \mu_{ij} h_j$ where $j \in J$ is any element such that $j \geq i$. By the universal property of the limit $(M, (\mu_i)_{i\in I})$ there exists a unique morphism $h : Z \to M$ such that $\mu_i \circ h = h_i$ for all $i \in I$ and hence for all $j \in J$. If $k : Z \to M$ is any other morphism such that $h_j = \mu_j \circ k$ for all $j \in J$ then for any $i \in I$, picking $j \geq i$ arbitrarily we have $\mu_i \circ k = \mu_{ij} \circ \mu_j \circ k = \mu_{ij} \circ h_j = h_i$ so that by the uniqueness property of h, we have that $h = k$. Thus $(M, (\mu_i)_{i\in J})$ is a limit of this subsystem. ■

Convention 2.1.38. If I is directed, $J \subseteq I$ is cofinal in I, and $(m_j)_{j \in J}$ are such that $m_j \in M_j$ and $\mu_{jk}(m_k) = m_j$ for all $j \leq k$ in J, then we will henceforth without comment identify this J-tuple $(m_j)_{j \in J}$ with the element of M that is contained in the singleton set $\bigcap_{j \in J} \mu_j^{-1}(m_j)$.

Remark 2.1.39. It is straightforward to verify that if $(I, \leq)$ is a countable directed partial order then I contains a cofinal subset that is order isomorphic to a subset of $\mathbb{N}$ so that if we are only concerned with a system's limit up to isomorphism then lemma 2.1.37 allows us replace a system indexed by $(I, \leq)$ with a subsystem indexed by a subset of $\mathbb{N}$.

The following example is original.

Example 2.1.40. Lemma 2.1.37 may fail if $(I, \leq)$ is not directed: Let Sys_M and J be as in example 2.1.17 and observe that the canonical limit of $\varprojlim \operatorname{Sys}_M$ is the singleton set $\{(C, C, C)\}$ while the canonical limit of $\operatorname{Sys}_M\big|_J$ is $M_L \times M_R$, a set of cardinality 4.

Examples

Smooth Functions

The following example is original.

Example 2.1.41. Smooth functions defined via limits: Let Ω be a convex open subset of $\mathbb{R}$, $a \in \Omega$, $M_0 = C(\Omega)$ be the algebra of continuous $\mathbb{R}$-valued functions on Ω, and let $M_i = \mathbb{R}^i \times M_0$ for all $i \geq 1$. For each $i \in \mathbb{Z}^{\geq 0}$ define:

$$\begin{aligned} \mu_{i,i+1} \underset{\text{def}}{=} \mu_{i,i+1}^a : M_{i+1} &\longrightarrow M_i \\ (p_0, \ldots, p_i, A) &\longmapsto \left(p_0, \ldots, p_{i-1},\ p_i + (x-a)\frac{1}{i+1}A(x)\right) \end{aligned}$$

where by $p_i + (x-a)\frac{1}{i+1}A(x)$ we mean the continuous function $x \mapsto p_i + (x-a)\frac{1}{i+1}A(x)$ defined on Ω. Observe that each $\mu_{i,i+1}$ is injective: if $\mu_{i,i+1}(p_0, \ldots, p_i, A) = \mu_{i,i+1}(q_0, \ldots, q_i, B)$ then from $p_i + (a-a)\frac{1}{i+1}A(x) = q_i + (a-a)\frac{1}{i+1}B(x)$ we can conclude that $p_i = q_i$ and since then

$(x-a)A(x) = (x-a)B(x)$ on $\Omega \setminus \{a\}$ we can conclude that $A = B$. Hence, by lemma 2.2.8 we can identify the limit of $\operatorname{Sys}_M \underset{\text{def}}{=} (M_\bullet, \mu_{ij}, \mathbb{Z}^{\geq 0})$ with $\bigcap\limits_{i=1}^{\infty} \operatorname{Im} \mu_{0i}$.

Now suppose that $f \in M_0$. Then $f \in \operatorname{Im} \mu_{01} \iff f = p_0 + (x-a)A_1(x)$ for some $p_0 \in \mathbb{R}$ and $A_1 \in M_0 = C(\Omega)$. If $(p_0, A_1) \in \operatorname{Im} \mu_{12}$ then $A_1(x) = p_1 + (x-a)\frac{1}{2}A_2(x)$ for some $p_1 \in \mathbb{R}$ and $A_2 \in C(\Omega)$ so that

$$\begin{aligned} f =& p_0 + (x-a)\left[p_1 + (x-a)\frac{1}{2}A_2(x)\right] \\ =& p_0 + p_1(x-a) + \frac{1}{2}(x-a)^2 A_2(x) \end{aligned}$$

If $(p_0, p_1, A_1) \in \operatorname{Im} \mu_{23}$ then $A_2(x) = p_2 + (x-a)\frac{1}{3}A_3(x)$ for some $p_2 \in \mathbb{R}$ and $A_3 \in C(\Omega)$ so that

$$\begin{aligned} f =& p_0 + p_1(x-a) + \frac{1}{2}(x-a)^2\left[p_2 + (x-a)\frac{1}{3}A_3(x)\right] \\ =& p_0 + p_1(x-a) + \frac{p_2}{2}(x-a)^2 + \frac{(x-a)^3}{3!}A_3(x) \end{aligned}$$

It is clear that this pattern continues so we may conclude from Taylor's theorem that for $k \in \mathbb{Z}^{\geq 0}$, f is k-times continuously differentiable $\iff f \in \operatorname{Im} \mu_{0k}$ and hence that f is smooth $\iff f \in \bigcap\limits_{k=1}^{\infty} \operatorname{Im} \mu_{0k}$, which we identified as the limit of Sys_M. Observe that the real numbers $p_0, p_1, p_2, \ldots$ are, respectively, the zeroth, first, second, etc. derivatives of f at a. It should also be clear that if Ω was a convex open subset of $\mathbb{R}^d$ for some $d \in \mathbb{N}$ (instead of simply a subset of $\mathbb{R}$) then by introducing more indices and appropriately modifying the bonding maps, we could extend this construction of C^k-functions to convex open subsets of $\mathbb{R}^d$.

Observe that this constriction shows that one need not have ever even seen the usual definition of the derivative (as a limit of divided differences) in order to define the C^k-functions on Ω for $k \in \{1, 2, \ldots, \infty\}$, where we used Taylor's theorem only to identify the set of maps constructed above as the usual set of all C^k-functions. To gain intuition for why these particular bonding maps produced the C^k-functions and for insight into the motivation that led to their definitions, consider an arbitrary map f taken from $C(\Omega)$. If f was a polynomial then it is often the case that one wishes to factor out some linear factor, say for instance that we would like to factor out $x - a$. However, in general $f(a)$ need not be 0 so if this is not possible then the next best alternative that one may reasonably hope for would instead be to vertically shift f to $f - f(a)$ and then attempt to factor out $x - a$. Generalizing this to the case where f is not necessarily a polynomial, let us say that we can "shift-factor out $x - a$ from f" if there exists some $g \in C(\Omega)$ such that $f(x) - f(a) = (x - a)g(x)$. The maps in the image of μ_{01} are exactly the continuous maps for which this is possible. Analogously, the more linear factors that one may "shift-factor out" of f then the higher its order of differentiability. Also, observe that if $\widehat{a} \in \Omega$ then $f \in \operatorname{Im} \mu_{01}^{a} \iff f$ is $C^1 \iff f \in \operatorname{Im} \mu_{01}^{\widehat{a}}$, which simply means that if it is possible to "shift-factor out" *some* linear term then it is possible to do so with *any* linear term.

The bonding maps of the above characterization of k-times continuous differentiability via limits shows that if we wish to study C^k-maps on Ω then it is only natural to try to somehow construct manifolds (or promanifolds) from the lists of constants $p_0^a, p_1^a, \ldots$ as a varies over Ω that hopefully would also induce in some way a natural map from this construction onto Ω. However, we would of course also prefer for any such construction to be diffeomorphism invariant. The search for such a construction leads naturally to the k^{th}-order jet bundle. ■

Topological Limits and Inverse Limits

When first introduced to limits and colimits, it may not be clear why although we write $\lim\limits_{n \to \infty} \frac{1}{n} = 0$ for this famous topological limit, we use the term "limit" to mean "inverse limit" rather than "direct limit," where in addition the arrow in the notation for a limit (i.e. $\varprojlim \operatorname{Sys}_M$) may appear to be "pointing in the wrong direction." In this subsection, it is hoped that the reader will conclude that it is natural have "limit" mean "inverse limit" and that the aforementioned discord stems from using the notation $\lim\limits_{n \to \infty} x_n$, rather than $\lim\limits_{\infty \leftarrow n} x_n$, for topological limits of sequences. We will first show in proposition 2.1.43 how topological limits can be described in terms of an inverse system and its limit.

Definition 2.1.42. By a *possibly non-proper filter* on a set X we mean a non-empty subset $\mathcal{F}$ of $\wp(X)$ that satisfies (1) and (3) of definition A.0.1. Clearly, every possibly non-proper filter on X is either a filter on X or otherwise the powerset of X so for any subset $\mathcal{S} \subseteq \wp(X)$, if $\mathcal{S} \neq \varnothing$ then there exists a unique smallest possibly non-proper filter on X containing $\mathcal{S}$, which we'll call *the possibly non-proper filter on X generated by $\mathcal{S}$*, while if $\mathcal{S} = \varnothing$ then this will refer to $\wp(X)$. For any $\mathcal{F} \subseteq \wp(X)$ and any $U \subseteq X$ call the set

$$\mathcal{F}\big|_U \underset{\text{def}}{=} \{F \in \mathcal{F} \,|\, F \subseteq U\}$$

the restriction of $\mathcal{F}$ to U.

The following proposition is original.

Proposition 2.1.43 (Topological Limits as Inverse Limits)**.** Let (X, τ_X) be a non-empty topological space, $x \in X$, I be the set of all neighborhoods of x in X, and $\mathcal{F}$ a filter on X. Partially order I by reverse set inclusion (i.e. $V \geq U \iff V \subseteq U$) and make $\wp^2(X) \underset{\text{def}}{=} \wp(\wp(X))$ into a category by defining for all objects $\mathcal{R}, \mathcal{S} \in \wp^2(X)$,

$$\operatorname{Mor}_{\wp^2(X)}(\mathcal{R}, \mathcal{S}) = \begin{cases} \operatorname{In}_{\mathcal{R}}^{\mathcal{S}} & \text{if } \mathcal{R} \subseteq \mathcal{S} \\ \varnothing & \text{otherwise} \end{cases}$$

For all $U \in I$, let $\mathcal{F}_U$ denote the possibly non-proper filter on X generated by $\mathcal{F}\big|_U$. Then $\mathcal{F} \subseteq \mathcal{F}_V \subseteq \mathcal{F}_U$ for all $U, V \in I$ with $V \geq U$, $\operatorname{Sys}_{\mathcal{F}} \underset{\text{def}}{=} \left(\mathcal{F}_\bullet, \operatorname{In}_{\mathcal{F}_V}^{\mathcal{F}_U}, I, \leq\right)$ is an inverse system in $\wp^2(X)$, and $\left(\mathcal{F}, \left(\operatorname{In}_{\mathcal{F}}^{\mathcal{F}_U}\right)_{U \in I}\right)$ is a cone into $\operatorname{Sys}_{\mathcal{F}}$. Furthermore, $\mathcal{F} \to x$ in (X, τ_X) if and only if $\left(\mathcal{F}, \operatorname{In}_{\mathcal{F}}^{\mathcal{F}_\bullet}\right) = \varprojlim \operatorname{Sys}_{\mathcal{F}}$ in the category $\wp^2(X)$.

Proof. For each $U \in I$, observe that $\mathcal{F}_U = \wp(X) \iff \varnothing = \mathcal{F}\big|_U \iff U$ has no element of $\mathcal{F}$ as a subset. Now if $\mathcal{F}_U \neq \wp(X)$ then there is some $F_0 \in \mathcal{F}$ such that $F_0 \subseteq U$ so that if $F \in \mathcal{F}$ then $F \cap F_0$, being an element of $\mathcal{F}$ that is a subset of U, is an element of $\mathcal{F}\big|_U$ where now $F_0 \cap F \subseteq F$ implies that $F \in \mathcal{F}_U$, which shows that $\mathcal{F} \subset \mathcal{F}_U$. But if $\mathcal{F}_U \neq \wp(X)$ then $\mathcal{F}\big|_U \neq \varnothing$ so that $\mathcal{F}_U$ is a filter that is contained in $\mathcal{F}$ from which it follows that $\mathcal{F} = \mathcal{F}_U$. Thus $\mathcal{F}_U \neq \wp(X) \iff \mathcal{F}_U = \mathcal{F}$, which in particular implies that $\mathcal{F} \subseteq \mathcal{F}_U$. Now if $U, V \in I$ and $V \subseteq U$ then since $\mathcal{F}\big|_V \subseteq \mathcal{F}\big|_U$ we have $\mathcal{F}_V \subseteq \mathcal{F}_U$. That $\operatorname{Sys}_{\mathcal{F}}$ is an inverse system in $\wp^2(X)$ and that $\left(\mathcal{F}, \operatorname{In}_{\mathcal{F}}^{\mathcal{F}_\bullet}\right)$ is a cone in $\operatorname{Sys}_{\mathcal{F}}$ is now obvious.

If $\mathcal{F} \to x$ in (X, τ_X) then $\mathcal{F}_U = \mathcal{F}$ for all $U \in I$ so that $\operatorname{Sys}_{\mathcal{F}}$ is just the constant system whose limit is consequently $\left(\mathcal{F}, \operatorname{In}_{\mathcal{F}}^{\mathcal{F}_\bullet}\right)$. So assume now that $\mathcal{F}$ does not converge to x in (X, τ_X) and pick a set $U \in I$ that contains no element of $\mathcal{F}$ as a subset. Since any $V \in I$ with $V \geq U$ also contains no element of $\mathcal{F}$ as a subset we have $\mathcal{F}_V = \wp(X)$ which implies that $\operatorname{Sys}_{\mathcal{F}}\big|_{I^{\geq U}}$ is just the constant system whose limit in the category $\wp(X)$ is $\left(\wp(X), \left(\operatorname{In}_{\wp(X)}^{\mathcal{F}_V}\right)_{V \in I^{\geq U}}\right)$. Since $I^{\geq U}$ is cofinal in I and I is directed it follows that $\wp(X)$ is the limit of $\operatorname{Sys}_{\mathcal{F}}$ in the category $\wp^2(X)$ so that, in particular, $\mathcal{F}$ is not a limit of $\operatorname{Sys}_{\mathcal{F}}$. ■

The following example, which is essentially a lemma for example 2.1.45, will be used to aid in the construction of some examples of promanifolds where by slightly varying the systems' objects or bonding maps, one can frequently use it as a basis for constructing counterexamples.

Example 2.1.44. Obtaining the limit of a system as a subset of a surjective system's limit: Let $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, I)$ be an inverse system in Set with limit $(M, \mu_\bullet)$ where we will consider the M_i's to be pairwise disjoint. For each $b \in A \underset{\text{def}}{=} I$, let $N_b \underset{\text{def}}{=} \bigsqcup_{h \leq b} M_h$ and let

$$\begin{aligned} \nu_{ab} : N_b &\longrightarrow N_a \\ m_h \in M_h &\longmapsto \begin{cases} m_h & \text{if } h < a \\ \mu_{bh}(m_h) & \text{otherwise} \end{cases} \end{aligned}$$

Let $\operatorname{Sys}_N = (N_\bullet, \nu_{ab}, A)$ and observe that this is an inverse system since the ν_{ab}'s are clearly continuous and if $a \leq b \leq c$ then for $m_h \in M_h$ with $b \leq h \leq c$ we have

$$\nu_{ac}(m_h) = \mu_{ah}(m_h) = \mu_{ab}(\mu_{bh}(m_h)) = \mu_{ab}(\nu_{bc}(m_h)) = \nu_{ab}(\nu_{bc}(m_h))$$

while the case of $h < b$ is immediate.

Let us specialize to the case of $I = \mathbb{N}$ and we will now "arrange the M_i's over the real line" in the following sense: for each $i \in \mathbb{N}$ identify M_i with $\left\{\frac{1}{i}\right\} \times M_i$ and M with $\{0\} \times M$ where this identification extends to these sets' elements (e.g. $(0, m)$ is identified with m). In particular, this means that N_a is identified with $\bigsqcup_{h \leq a} \left(\frac{1}{h} \times M_h\right)$. Let N be the set $(\{0\} \times M) \sqcup \left(\bigsqcup_{h} \left(\frac{1}{h} \times M_h\right)\right)$. For all $a \in \mathbb{N}$, define

$$\begin{aligned} \nu_a : N &\longrightarrow N_a \\ n &\longmapsto \begin{cases} \mu_a(n) & \text{if } n \in M \\ n & \text{if } n \in M_h \text{ with } h \leq a \\ \mu_{ah}(n) & \text{if } n \in M_h \text{ with } h > a \end{cases} \end{aligned}$$

Observe that under our identifications we have $M = \bigcap_{b \in \mathbb{N}} \nu_b^{-1}(M_b)$ while for all $a \in \mathbb{N}$ we have $M_a = \nu_{a+1}^{-1}(M_a)$. This construction is summarized in the following diagram:

$$
\begin{tikzcd}
(0 \times M) \sqcup \Big[\ \cdots \sqcup \arrow[dr, "\mu_a"'] & \left(\tfrac{1}{a} \times M_a\right) \sqcup \cdots \sqcup \arrow[d, "\mathrm{Id}"] & \left(\tfrac{1}{3} \times M_3\right) \sqcup \arrow[d, "\mathrm{Id}"] & \left(\tfrac{1}{2} \times M_2\right) \sqcup \arrow[d, "\mathrm{Id}"] & (1 \times M_1)\ \Big] \arrow[d, "\mathrm{Id}"] \\
& \left(\tfrac{1}{a} \times M_a\right) \sqcup \cdots \sqcup \arrow[dr, "\mu_{3a}"'] & \left(\tfrac{1}{3} \times M_3\right) \sqcup \arrow[d, "\mathrm{Id}"] & \left(\tfrac{1}{2} \times M_2\right) \sqcup \arrow[d, "\mathrm{Id}"] & (1 \times M_1) \arrow[d, "\mathrm{Id}"] \\
& & \left(\tfrac{1}{3} \times M_3\right) \sqcup \arrow[dr, "\mu_{23}"'] & \left(\tfrac{1}{2} \times M_2\right) \sqcup \arrow[d, "\mathrm{Id}"] & (1 \times M_1) \arrow[d, "\mathrm{Id}"] \\
& & & \left(\tfrac{1}{2} \times M_2\right) \sqcup \arrow[dr, "\mu_{12}"'] & (1 \times M_1) \arrow[d, "\mathrm{Id}"] \\
& & & & 1 \times M_1
\end{tikzcd}
$$

Figure 2.1: From the bottom row going up: the sets are N_1, N_2, N_3, N_a, and N while the maps are $\nu_{12}, \nu_{23}, \nu_{3a}$, and ν_a. This construction occurs in the category Set, not Top.

We will now show that $(N, \nu_a) = \varprojlim \mathrm{Sys}_N$ in Set. Let (Z, h_a) be an inverse cone into Sys_N so that $h_a = \nu_{ab} \circ h_b$ for all $a \leq b$, where $h_a : Z \to N_a$. Let $z \in Z$ and first suppose that there exists some $a \in \mathbb{N}$ such that $h_a(z) \in M_k$ with $k \neq a$. If for any $b \geq a$ we had $h_b(z) \in M_{k'}$ with $k' > a$ then

$$h_a(z) = \nu_{a,k'}(h_{k'}(z)) = \mu_{a,k'}(h_{k'}(z)) \in M_a$$

contradicting $h_a(z) \in M_h$ while if $k < k' \leq a$ then

$$h_k(z) = \nu_{k,k+1}(h_{k+1}(z)) = h_{k+1}(z) = \nu_{k,k'}(h_{k'}(z)) = h_{k'}(z) \in M_{k'}$$

contradicting $h_k(z) \in M_k$. Thus for all $b \geq a$, $h_b(z) \in M_k$ so we can define $h(z) = h_a(z) \in$

$M_a \subseteq N$ where for all $b \geq a$ we have $\nu_b(h(z)) = \nu_b(h_a(z)) = h_a(z) = h_b(z)$. Now suppose that there is no such $a \in \mathbb{N}$ so that $h_a(z) \in M_a$ for all $a \in \mathbb{N}$. Observe that $\mu_{ab}(h_b(z)) = \nu_{ab}(h_b(z)) = h_a(z)$ for all $a \leq b$ so that $h_\infty(z) = (h_1(z), h_2(z), \ldots) \in M$ is defined and we will write $h(z) \underset{\text{def}}{=} h_\infty(z) \in N$ to denote its consideration as an element of N, where observe that $\nu_a(h(z)) = \mu_a(h(z)) = h_a(z)$ for all $a \in \mathbb{N}$. We have thus defined the desired map $h : Z \to N$ that satisfies $\nu_a \circ h = h_a$, where the uniqueness of this map is immediate. If Sys_M was an inverse system in Top then by giving N the weak topology induced by the ν_a's it will become the limit of Sys_N in Top. ■

The following example ties together the usual geometric intuition underlying the notion of a sequence converging in a topology to the categorical definition of a limit. It also provides intuitive justification for using the word "limit" (rather than "colimit") to refer to inverse limits instead of direct limits by showing how inverse limits geometrically relate to the topological limit $\lim\limits_{n \to \infty} \dfrac{1}{n}$ in $\mathbb{R}$ of the sequence $\left(\frac{1}{n}\right)_{n \in \mathbb{N}}$.

Example 2.1.45. Let us now specialize the above example to the case where M_1 is a single point. For all $i \in \mathbb{N}$, let M_i be a distinct copy of M_1, say $M_i \underset{\text{def}}{=} \frac{1}{i} \times M_1$, which we will identify with M_1, and let $\mu_{ij} : M_j \to M_i$ be Id_{M_1} for $i \leq j$. Continuing with the notation from example 2.1.44, in this case we can identify each N_a with $\left\{1, \ldots, \frac{1}{a}\right\}$ and N with $\{0\} \cup \left\{1, \ldots, \frac{1}{a}, \ldots\right\}$ where explicitly,

$$\begin{aligned} \nu_{a,a+1} : N_{a+1} &\longrightarrow N_a \\ n_{a+1} &\longmapsto \begin{cases} \frac{1}{a} & \text{if } n_{a+1} = \frac{1}{a+1} \\ n_{a+1} & \text{otherwise} \end{cases} \end{aligned}$$

Since a basis neighborhoods of $0 \in N$ when N is given the weak topology (induced by the ν_a's) consists of sets of the form $\nu_a^{-1}\left(\frac{1}{a}\right) = \{0\} \cup \left\{\frac{1}{a}, \frac{1}{a+1}, \ldots\right\}$ it is clear that the subspace topology that $\mathbb{R}$ induces on N is the same as the weak topology on N induced by the ν_a's.

We have thus explicitly related the topological limit of the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ to categorical limits since, to summarize, we have shown that the limit in Top of the sets $N_a = \left\{1, \ldots, \frac{1}{a}\right\}$ (with the ν_{ab}'s as the connecting maps) is $\{0\} \cup \left\{1, \ldots, \frac{1}{a}, \ldots\right\}$, which is just the sequence $\left\{1, \ldots, \frac{1}{a}, \ldots\right\}$ together with the limit point (in $\mathbb{R}$) $\lim\limits_{a \to \infty} \dfrac{1}{a} = 0$. In fact, since each N_a is a manifold the space $N = \{0\} \cup \left\{1, \ldots, \frac{1}{a}, \ldots\right\}$ can be given the structure of a promanifold, which will make N a simple non-trivial example of a promanifold (however, it space of global sections is a subset of the restriction to each map in $C^\infty_{\mathbb{R}}(\mathbb{R})$ to N).

Let $S = \{\star\}$ be any singleton set and for each $a \in \mathbb{N}$ define

$$\begin{aligned} h_a : S &\longrightarrow N_a \\ \star &\longmapsto \frac{1}{a} \end{aligned}$$

It is clear that $(S, h_\bullet)$ is a cone into Sys_N whose limit map, $h_\infty : S \to N$, is $h_\infty(\star) = 0$. Intuitively, therefore, the cone of maps $h_a : S \to N_a$ makes rigorous the geometric notion of "following the sequence of points" $h_a(\star) = \frac{1}{a}$ to their topological limit $h_\infty(\star) = 0$.

Remark 2.1.46. Recall that the filter generated by $(1/i)_{i=1}^\infty$ is generated by the filter base $\left\{B_i \underset{\text{def}}{=} \{1/j \mid j \in \mathbb{N}^{\geq i}\} \,\middle|\, i \in \mathbb{N}\right\}$ and observe that each N_i in the above example was simply the complement of B_i in $\{1/j \mid j \in \mathbb{N}\}$. So the intuition given by the above example is in this sense more closely related to an intuitive understanding of what it means for a filter to converge than it is to the usual intuitive understanding of what it means for a sequence of points to converge (i.e. of "following a sequence of points to its limit").

Intersections of Sets

We can say more than what was already mentioned in example 2.1.25 about expressing the intersection of a collection of sets through limits.

Example 2.1.47. More on expressing intersections of sets via limits: Let J be any set that does not contain the symbol $-\infty$, let $I = J \cup \{-\infty\}$, and let $M_\bullet = (M_j)_{j \in J}$ be any collection of sets. Let $M_{-\infty}$ be any set that contains $\underset{i}{\cup} M_i$ where, in particular, if there is some M_{j_0} that contains $\underset{j\in J}{\cup} M_j$ then we may define $M_{-\infty} = M_{j_0}$. If these $M_\bullet$ are topological spaces then give $M_{-\infty}$ the indiscrete topology.

Partially order I by declaring $i \leq j \iff M_j \subseteq M_i$ and observe that $-\infty$ is a lower bound of $(I, \leq)$. Let $M = \underset{i\in I}{\cap} M_i$ and for all $i \leq j$ in I let $\operatorname{In}_{ij} : M_j \to M_i$ and $\operatorname{In}_i : M \to M_i$ denote the natural inclusions. Then $\operatorname{Sys}_M \underset{\text{def}}{=} (M_i, \operatorname{In}_{ij}, I, \leq)$ is an inverse system in Ens and if each M_i is a topological space, say with topology τ_i, and every In_{ij} is continuous then it is also an inverse system in Top. It is straightforward to verify that $(M, \operatorname{In}_\bullet) = \varprojlim \operatorname{Sys}_M$ in Set, while in Top, this will be true if M is given the topology τ_M generated by $\underset{i\in I}{\cup} \left(\tau_i\big|_M\right)$, where $\tau_i\big|_M$ is the subspace topology that M inherits from (M_i, τ_i). In particular, if all M_i are topological subspaces of some space (X, τ_X) then τ_M will simply be the subspace topology $\tau_X\big|_M$. Since the intersection of sets may result in the empty set, this example shows in particular that it is possible for the limit of an inverse system to be the empty set even if every object of this system is a non-empty set.

If it happens to be the case that $j \leq k \iff j = k$ for all $j, k \in J$ (i.e. $M_k \subseteq M_j \iff j = k$) then $\prod\limits_{j\in J} M_j = \varprojlim (M_i, \operatorname{In}_{ij}, J, \leq)$ which is generally drastically different from $M = \underset{j\in J}{\cap} M_j = \varprojlim (M_i, \operatorname{In}_{ij}, I, \leq)$. In particular, this shows that the introduction of the lower bound $M_{-\infty}$ into the inverse system may drastically change the limit. In contrast, if $(J, \leq)$ is directed, that is, for all $i, j \in J$ there is some $k \in J$ such that $M_k \subseteq M_i \cap M_j$, then it is easy to verify that the introduction of $M_{-\infty}$ is redundant since we could have defined I as $I \underset{\text{def}}{=} J$ and still have obtained $\underset{j\in J}{\cap} M_j$ as the limit of $(M_i, \operatorname{In}_{ij}, I = J, \leq)$. ■

Products

The following example and definition is well-known.

Example 2.1.48. The product of sets is an inverse limit: Let I be any set and let $M_\bullet = (M_i)_{i\in I}$ be any collection of sets. If we partially order I by declaring that $i \leq j \iff i = j$ then we obtain an inverse system $\operatorname{Sys}_M \underset{\text{def}}{=} (M_i, \operatorname{In}_{ij}, I, \leq)$ whose limit is $\left(\prod\limits_{i\in I} M_i, \operatorname{Pr}_\bullet\right)$, where each Pr_i is the canonical projections onto M_i.

The following lemma is an exercise that can be proved by using the universal property of inverse limits or by using the facts that products are limits and that limits commute.

Lemma 2.1.49 (The limit of the products is the product of the limits)**.** Let $\mathrm{Sys}_{M^\alpha} = (M_i^\alpha, \mu_{ij}^\alpha, I^\alpha)$ be a collection of inverse systems indexed by $\alpha \in A$ (where A is some indexing set) and for each $\alpha \in A$ let $(M^\alpha, \mu_i^\alpha) = \varprojlim \mathrm{Sys}_{M^\alpha}$. Let $I \underset{\text{def}}{=} \prod\limits_{\alpha\in A} I^\alpha$ have the usual product order and for all $i = (i^\alpha)_{\alpha\in A} \in I$ let $\mu_i \underset{\text{def}}{=} \prod\limits_{\alpha\in A} \mu_{i^\alpha}^\alpha : \prod\limits_{\alpha\in A} M^\alpha \to \prod\limits_{\alpha\in A} M_{i^\alpha}^\alpha$ and define the product system by

$$\mathrm{Sys}_{\prod M^\alpha} \underset{\text{def}}{=} \left(M_i \underset{\text{def}}{=} \prod_{\alpha\in A} M_{i^\alpha}^\alpha, \mu_{ij} \underset{\text{def}}{=} \prod_{\alpha\in A} \mu_{i^\alpha j^\alpha}^\alpha, I \right)$$

Then $\left(\prod\limits_{\alpha\in A} M^\alpha, \mu_\bullet = (\mu_i)_{i\in I} \right) = \varprojlim \mathrm{Sys}_{\prod M^\alpha}$. Furthermore, in the case where all I^α are order isomorphic then we can use $I = I^\alpha$ ($\alpha \in A$ arbitrary) in place of the product order.

The following example and definition is well-known.

Example and Definition 2.1.50 (Fibered Products and Coproducts)**.** Let I consist of the three distinct objects L, B, and R, and partially order I by declaring $i \leq j \iff i = j$ or $i = B$. Suppose that $\mu_{BL} : L \to B$ and $\mu_{BL} : R \to B$ are morphisms and let $\mu_{LL} = \mu_L^L$, $\mu_{BB} = \mu_B^B$, and $\mu_{RR} = \mu_R^B$ denote the identity morphisms on the objects L, B, and R, respectively. This makes $\mathrm{Sys}_M \underset{\text{def}}{=} (\{L, B, R\}, \mu_{ij}, I, \leq)$ into an inverse system that is frequently denoted by $L \xrightarrow{\mu_{BL}} B \xleftarrow{\mu_{BR}} R$. If $(M, \mu_\bullet)$ is a limit of this system then $(M, \{\mu_L, \mu_R\})$ is called a *pullback* or *fiber(ed) product of* $M_L \xrightarrow{\mu_{BL}} M_B \xleftarrow{\mu_{BR}} M_R$ and its vertex is often denoted by $L \times_B R$. If working in the category Set, Top, or Man then the reader may have seen $L \times_B R$ defined to be the set $\{(l, r) \in L \times R \,|\, \mu_{BL}(l) = \mu_{BR}(r)\}$, which is readily seen to just a simplification, tailored to this system, of the definition of the vertex of the canonical limit of Sys_M.

If we instead had morphisms $\mu_B^L : B \to L$ and $\mu_B^R : B \to R$ then $\left(\{L, B, R\}, \mu_i^j, I, \leq\right)$ is a direct system that is usually denoted by $L \xleftarrow{\mu_B^L} B \xrightarrow{\mu_B^R} R$. If $(M, \mu^\bullet)$ is a colimit of this system then $(M, \{\mu^L, \mu^R\})$ is called a *pushout* or *fibered coproduct of* $L \xleftarrow{\mu_B^L} B \xrightarrow{\mu_B^R} R$. ∎

The Space of Sequences

The following topological vector space will be an important frequently recurring example of a promanifold.

Example and Definition 2.1.51. For all $0 \leq i \leq j$, let $\operatorname{Pr}_{\leq i,j} : \mathbb{R}^j \to \mathbb{R}^i$ be the canonical projections of the first i coordinates and let $\operatorname{Sys}_{\mathbb{R}^{\mathbb{N}}} = (\mathbb{R}^i, \operatorname{Pr}_{\leq i,j}, \mathbb{N})$. If $I = \{i_1, i_2, \ldots\}$ is a (possibly finite) subset of $\mathbb{N}$ where $i_1 < i_2 < \cdots$ ($< i_{|I|}$ if I is finite) then $\operatorname{Sys}_{\mathbb{R}^{\mathbb{N}}}\big|_I = (\mathbb{R}^i, \operatorname{Pr}_{\leq i,j}, I)$ and the canonical limit M of this system consists of all elements of $\prod\limits_{\substack{i=0\\ i\in I}}^{\infty} \mathbb{R}^i$ of the form $m = ((m_1, \ldots, m_{i_1}), (m_1, \ldots, m_{i_1}, \ldots, m_{i_2}), \ldots)$ where the canonical limit's projection onto $\mathbb{R}^{i_l}$ maps this element to $(m_1, \ldots, m_{i_l})$. This repetition of information becomes unwieldy so for this system we will use the cone $\left(\prod\limits_{i=1}^{\sup I} \mathbb{R}, \operatorname{Pr}_{\leq i_\bullet} \underset{\text{def}}{=} (\operatorname{Pr}_{\leq i_l})_{l=1}^{\sup I}\right)$ as the canonical limit of $\operatorname{Sys}_{\mathbb{R}^{\mathbb{N}}}\big|_I$, where

$$\begin{aligned} \operatorname{Pr}_{\leq i_l} : \prod_{i=1}^{\sup I} \mathbb{R} &\longrightarrow \mathbb{R}^{i_l} \\ m = (m_i)_{i=1}^{\sup I} &\longmapsto (m_1, \ldots, m_{i_l}) \end{aligned}$$

The most important case for infinite-dimensional promanifolds occurs when $I = \mathbb{N}$, in which case $\mathbb{R}^{\mathbb{N}}$ is called *the space of real sequences*, which is described in more detail the appendix. ■

Solenoids

Example and Definition 2.1.52. Let $\operatorname{Sys}_M \underset{\text{def}}{=} (S^1, \mu_{ij}, \mathbb{N})$ be the system from example 2.1.12 where recall that for all $j \in \mathbb{N}$, $q_j > 1$ is an integer and $\mu_{j,j+1} : S^1 \to S^1$ is the map $\mu_{j,j+1}(z) = z^{q_j}$, where we consider unit circle S^1 as a subset of $\mathbb{C}$. The limit of such a system is called a *solenoid* and if p is a prime and it happens that all q_j are the same prime p then the limit is called the *p-adic solenoid*. By lemma 2.1.37 and example 2.1.12, for the limit $(M, \mu_\bullet)$ of Sys_M to be a p-adic solenoid it suffices that for all sufficiently large j, each q_j be some positive integral power of p. Since each S^1 is a topological group and each μ_{ij} is a continuous homomorphism, Sys_M is an inverse system in the category of topological groups so that M is also a topological group.

It is well-known that solenoids are compact, connected, metrizable spaces that are neither path-connected nor locally connected. Although the fact that solenoids are connected is a simple corollary to proposition 2.5.24, for the reader who has not yet encountered it we will now present the well-known standard proof that solenoids are connected; a proof that is highly specialized to solenoids and that differs drastically from the proof of proposition 2.5.24: Let $q_0 = 1$ and for each $j \in \mathbb{N}$ let $\gamma_j : \mathbb{R} \to S^1$ be $\gamma_j(\theta) = e^{\theta\sqrt{-1}/(q_{j-1}\cdots q_0)}$ and observe that $(\mathbb{R}, \gamma_\bullet)$ is a cone of continuous surjective homomorphisms into Sys_M whose limit, $\gamma : \mathbb{R} \to M$, is therefore a continuous homomorphism whose image is dense in M by proposition 2.1.33(3). Since the closure of a connected set is connected it follows that M is connected.

Let 1_M denote the multiplicative identity of M, let $\theta \in \mathbb{R}$ be such that $\gamma(\theta) = 1_M$, and let $n \in \mathbb{N}$ be such that $\theta = 2\pi n$. From $\gamma_\bullet(\theta) = 1$ it follows that $\theta = \frac{2\pi n}{q_0 \cdots q_j}$ for every $j \in \mathbb{N}$ which implies that $\theta = 0$ and thus that γ is injective. Also, γ is not surjective since, for instance, if all q_j are odd and we let $m_j \underset{\text{def}}{=} e^{\pi\sqrt{-1}(1-2/(q_0\cdots q_{j-1}))}$ for each $j \in \mathbb{N}$, then these $m_\bullet$ define an element $m \in M$ that is readily verified to not belong to the image of γ. Solenoids are among the simplest non-trivial examples of promanifolds that may be embedded in $\mathbb{R}^3$, which allows one to gain intuition about their construction. Furthermore, it will be clear the path γ is smooth when considered as a map into the promanifold M. ∎

Limits of Systems of Tori

Example 2.1.53 ([2]). Let $T^n = S^1 \times \cdots \times S^1$ denote the n-torus. If $k \in \mathbb{N}$ then any separable metrizable compact connected k-dimensional Abelian topological group G is isomorphic to the limit of an inverse system $\operatorname{Sys}_T = (T_i, \tau_{ij}, \mathbb{N})$ where each $T_i = T^k$ is the k-torus and each $\tau_{ij} : T_j \to T_i$ is a surjective continuous homomorphism, and conversely. Furthermore, either G is the k-torus or else $H^k(G)$ is isomorphic to an infinitely generated subgroup of $(\mathbb{Q}, +)$. In particular, if M is a regular orientable n-dimensional manifold and G is a compact connected topological group embedded in M with dimension n or $n-1$ then G is a torus.

Gluing

The following example illustrates why the operation of "gluing sets together" is often thought of in terms of colimits of a certain direct system.

Example 2.1.54 (Gluing and k-spaces)**.** Let $M^\bullet = (M^i)_{i\in I}$ be any collection of sets, let $M \underset{\text{def}}{=} \underset{i\in I}{\cup} M^i$, partially order I by declaring $i \leq j \iff M^i \subseteq M^j$, and for all $i \leq j$, let $\mathrm{In}_i^j : M^i \to M^j$ and $\mathrm{In}^i : M^i \to M$ be the natural inclusions. Observe that $\mathrm{Sys}_M \underset{\text{def}}{=} \left(M^\bullet, \mathrm{In}_i^j, I, \leq\right)$ defines a direct system in Set and that the partial order $(I, \leq)$ is not necessarily directed. Assume that at least one of the following is true: (1) $(I, \leq)$ is directed and/or (2) $\{M^i \,|\, i \in I\}$ forms a base for a topology on M. It is readily verified that the cocone $(M, \mathrm{In}^\bullet)$ is the colimit of Sys_M in Set, where the colimit of any cocone $(Z, h^\bullet)$ in Set from Sys_M is the map $h \underset{\text{def}}{=} \underset{i\in I}{\cup} h^i : M \to Z$. The assumption that at least one of (1) and (2) holds was included since with the help of proposition 2.1.55 it is not too difficult to come up with examples of $M^\bullet$ that satisfy neither (1) nor (2) and where the cocone $(M, \mathrm{In}^\bullet)$ fails to be a colimit of Sys_M in Set.

Suppose now that the M^i were endowed with topologies τ_{M^i} making Sys_M into a direct system in Top (i.e. making all $\mathrm{In}_i^j : (M^i, \tau_{M^i}) \to (M^j, \tau_{M^j})$ continuous). Define a topology τ_M on M by declaring a subsets $U \subseteq M$ to be open $\iff$ for each index i, $U \cap M^i$ is open in (M^i, τ_{M^i}). Then $((M, \tau_M), \mathrm{In}^\bullet)$ is the colimit of Sys_M in Top: since it is already know that $(M, \mu^\bullet)$ is the colimit of Sys_M in Set, to prove this assertion it suffices to show that (a) each $\mathrm{In}^i : (M^i, \tau_{M^i}) \to (M, \tau_M)$ is a morphism in Top (i.e. continuous) and (b) whenever $((Z, \tau_Z), h^\bullet)$ is a cocone in Top from Sys_M into (Z, τ_Z) whose limit morphism in the category of Set is the map $h : M \to Z$ then $h : (M, \tau_M) \to (Z, \tau_Z)$ is a morphism in Top. But (a) follows from the definition of τ_M and (b) is immediate from the continuity of each $h^i = h \circ \mathrm{In}^i$ and the fact that $M^i \cap h^{-1}(W) = \left(h \circ \mathrm{In}^i\right)^{-1}(W)$ for all $W \subseteq Z$.

Now suppose that τ is a topology on M and that in addition to the sets $M^\bullet$ satisfying at least one of (1) and (2) above, each (M^i, τ_{M^i}) is a topological subspace of (M, τ). Then $((M, \tau), \mathrm{In}^\bullet) = \operatorname{colim} \mathrm{Sys}_M$ in Top $\iff$ τ is coherent with $(M^i)_{i\in I}$. In particular, if $M^\bullet$

denotes the set of all compact subspaces of (M,τ) then this becomes a characterization of when (M,τ) is a k-space. Now, by simply translating the universal property of colimits we obtain the well-known result that (M,τ) is a k-space if and only if the following condition holds: whenever $f : M \to Z$ is a map into a space (Z,τ_Z) then $f : (M,\tau) \to (Z,\tau_Z)$ is continuous $\iff f\big|_K : \left(K, \tau\big|_K\right) \to (Z,\tau_Z)$ is continuous for all compact $K \subseteq M$. ∎

Disjoint Unions

The following proposition defines and characterizes a disjoint union of sets as colimits of a particular direct system, where this result is well-known since in terms of category theory, this system is essentially just the coproduct diagram expressed as a direct system.

Proposition and Definition 2.1.55 (Disjoint Union)**.** Let $M^\bullet = (M^i)_{i\in I}$ be any set of (not necessarily pairwise disjoint) sets, partially order I by declaring $i \leq j \iff i = j$, and let $\mu_i^i \underset{\text{def}}{=} \operatorname{Id}_{M^i} : M^i \to M^i$ for all $i \in I$. Then $\operatorname{Sys}_M \underset{\text{def}}{=} \left(M^\bullet, \mu_i^j, I, \leq\right)$ is a direct system in Set and every I-indexed collection of maps $h^\bullet : M^\bullet \to Z$ into a set M is a cocone in Set from Sys_M. Furthermore, if $(M,\mu^\bullet)$ is a cocone from Sys_M then $(M,\mu^\bullet) = \operatorname{colim} \operatorname{Sys}_M$ in Set $\iff$ (1) each $\mu^i : M^i \to M$ is injective, and (2) $(\operatorname{Im}\mu^i)_{i\in I}$ is a partition of M.

Call any cocone $(M,\mu^\bullet)$ that is a colimit of Sys_M, which clearly always exists, *a disjoint union of* $M^\bullet$, where if $\mu^\bullet$ is understood then we'll instead say that M is a disjoint union of $M^\bullet$. The vertex of a disjoint union will usually be denoted by $\underset{i\in I}{\sqcup} M^i$, $\underset{I}{\sqcup} M^i$, or $\sqcup M^\bullet$.

Proof. If $h^\bullet : M^\bullet \to Z$ is any collection of maps then it is immediately verified that $(Z,h^\bullet)$ forms a cocone from Sys_M.

($\Longrightarrow$) Suppose that $i \in I$ is such that $\mu^i : M^i \to M$ is not injective and let $m_1^i, m_2^i \in M^i$ be distinct elements of M^i such that $m \underset{\text{def}}{=} \mu^i(m_1^i) = \mu^i(m_2^i)$. Let $Z \underset{\text{def}}{=} \{m_1^i, m_2^i\}$ and for each $j \in I$ with $j \neq i$ let $h^j : M^j \to Z$ be arbitrary and define $h^i : M^i \to Z$ by $h^i \equiv m_2^i$ on $M^i \smallsetminus \{m_1^i\}$ and $h^i(m_1^i) = m_1^i$. Since $(Z,h^\bullet)$ is a cocone from Sys_M it has a limit $h : M \to Z$, which in particular satisfies $h^i = h \circ \mu^i$. But for $k = 1,2$ we have $m_k^i = h^i(m_k^i) = h(\mu^i(m_k^i)) = h(m) = h(m)$ so that $m_1^i = m_2^i$, a contradiction. Thus each μ^i is injective.

Now suppose that $i, j \in I$ are distinct indices for which exists some $m \in \operatorname{Im} \mu^i \cap \operatorname{Im} \mu^j$ and let $m^i \underset{\text{def}}{=} (\mu^i)^{-1}(m)$ and $m^j \underset{\text{def}}{=} (\mu^j)^{-1}(m)$. Let $h^\bullet : M^\bullet \to Z$ be any collection of maps such that $h^i(m^i) \neq h^j(m^j)$ and let $h : M \to Z$ be the limit map of $(Z, h^\bullet)$. Since $h^i(m^i) = h(\mu^i(m^i)) = h(m) = h(\mu^j(m^j)) = h^j(m^j)$, we have a contradiction.

Now suppose that $S \underset{\text{def}}{=} \underset{i \in I}{\cup} \operatorname{Im} \mu^i$ is not equal to all of M and let $c \in M \setminus S$. If $S = \varnothing$ then let $Z = \{z_1, z_2\}$ consist of two distinct elements and let z_1^M (resp. z_2^M) denote the constantly z_1 (resp. z_2) map on M. That each $\operatorname{Im} \mu^i = \varnothing$ implies that each $M^i = \varnothing$ so that we trivially have $z_1^M \circ \mu^i = z_2^M \circ \mu^i$ for each index i so that tje uniqueness of limit morphisms gives us $z_1^M = z_2^M$, a contradiction. Thus $S \neq \varnothing$ so that we may pick some $d \in S$ and define a map $h : M \to M$ by letting $h\big|_{M \setminus \{c\}}$ be the identity map and then letting $h(c) = d$. Clearly, Id_M is the limit of the cocone $(M, \mu^\bullet)$ but since $h \circ \mu^i = \operatorname{Id}_{M \setminus \{c\}} \circ \mu^i = \mu^i$ for each index i the map h is also a limit of $(M, \mu^\bullet)$ so that uniqueness of limit morphisms gives us $h = \operatorname{Id}_M$, a contradiction.

($\Longleftarrow$) Let $(Z, h^\bullet)$ be a cocone from Sys_M. Since $(\operatorname{Im} \mu^i)_{i \in I}$ partitions M for each $m \in M$, there exists a unique index $\iota(m) \in I$ such that $m \in \operatorname{Im} \mu^{\iota(m)}$, where observe that for all $i \in I$ and $m^i \in M^i$ we have $\iota(\mu^i(m^i)) = i$. Define the map $h : M \to Z$ by $h(m) \underset{\text{def}}{=} h^{\iota(m)}\Big(\big(\mu^{\iota(m)}\big)^{-1}(m) \Big)$, which is clearly well-defined. If $i \in I$ and $m^i \in M^i$ then

$$h(\mu^i(m^i)) = h^{\iota(\mu^i(m^i))}\Big(\Big(\mu^{\iota(\mu^i(m^i))}\Big)^{-1} \big(\mu^i(m^i)\big) \Big) = h^i\Big(\big(\mu^i\big)^{-1} \big(\mu^i(m^i)\big) \Big) = h^i(m^i)$$

so that for all $i \in I$, $h^i = h \circ \mu^i$.

Now suppose that $k : M \to Z$ is any map such that $k \circ \mu^i = h^i$ for all $i \in I$. Let $m \in M$, $i \underset{\text{def}}{=} \iota(m)$, and $m^i = (\mu^i)^{-1}(m)$, and observe that

$$k(m) = (k \circ \mu^i)(m^i) = h^i(m^i) = h^i\Big(\big(\mu^i\big)^{-1}(m) \Big) = h(m)$$

which shows that $k = h$, as desired. ■

Systems Having Properties Locally/Eventually/Cofinally

To motivate the introduction of the original definition 2.1.60, we will begin by describing some of the recurring issues that it was designed to solve.

Definition 2.1.56. Call a system in Top *monotone* if all of its connecting maps are monotone.,

Remark 2.1.57. One issue with definition 2.1.56 is that such a system may be equivalent to some other system that fails to have this property. By using example A.0.11 or example A.0.12 it is not difficult to construct a monotone system that is equivalent to a system that fails to be monotone. In particular, the requirement that *all* connecting maps be monotone may prevent this property from being dependent *only* on the equivalence class of Sys_M.

Definition 2.1.58. Say that a system $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, I)$ in Top is *compact* (resp. *locally compact*) if all $M_\bullet$ are compact (resp. locally compact) and that it is *proper* if all of its connecting maps are proper.

Remark 2.1.59. Despite how natural the above definitions are, an issue arises with them for if we wish to describe in terms of Sys_M when its limit M is locally compact then as proposition 2.5.25 shows, these definitions do not by themselves provide a satisfactory description. Similar problems occur when attempting to describe in terms of Sys_M when M (or its projections) has other "local" properties (e.g. locally connected, locally open map, etc.).

Many of the results in this paper relating to the limit M and its projections are proved under the assumption that *all* $M_\bullet$ or *all* connecting maps have some given properties. However, many of these results, in particular those describing local properties of M or its projections, do not require that all $M_\bullet$ or all connecting maps have these properties and instead weaker hypotheses would suffices for the conclusion to be true. Often, once we have proved some result under the hypotheses that all $M_\bullet$ or all connecting maps have certain properties then it will be readily seen from the proof or from lemma 2.1.37 that the same conclusion would still hold under the weaker hypotheses mentioned in the next definition. Remarks 2.1.57 and 2.1.59 and the definition and properties of locally cylindrical maps all motivated the following definitions.

Definition 2.1.60. Suppose that $\mathcal{P}$ is a property that an inverse system can posses (e.g. $\mathcal{P}$ may be the property that all of the system's connecting maps are surjective). If $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, I, \leq)$ is a directed inverse system with limit $(M, \mu_\bullet)$ then we will say that

(1) Sys_M *eventually has property* $\mathcal{P}$ or Sys_M *is eventually* $\mathcal{P}$ if there exists an index i_0 such that the subsystem indexed by $I^{\geq i_0}$ has property $\mathcal{P}$.

(2) Sys_M *cofinally has property* $\mathcal{P}$ or Sys_M *is cofinally* $\mathcal{P}$ if there exists a cofinal subset $J \subseteq I$ such that the subsystem indexed by J has property $\mathcal{P}$.

If $m \in M$ is given then we will say that

(3) Sys_M *eventually has property* $\mathcal{P}$ *at* m or Sys_M *is eventually* $\mathcal{P}$ *at* m if for every index $h \in I$ there exists an index $j_0 \geq h$ and some neighborhood N_{j_0} of $\mu_{j_0}(m)$ in M_{j_0} such that the inverse system induced by N_{j_0} and $I^{\geq j_0}$ has property $\mathcal{P}$.

(4) Sys_M *cofinally has property* $\mathcal{P}$ *at* m or Sys_M *is cofinally* $\mathcal{P}$ *at* m if for every index $h \in I$ there exists some subset $J \subseteq I^{\geq h}$ that is cofinal in I, some index $j_0 \in J$, and some neighborhood N_{j_0} of $\mu_{j_0}(m)$ in M_{j_0} such that the inverse system induced by N_{j_0} and J has property $\mathcal{P}$.

If we add the word "locally" to the above two definitions then (analogously to what it means to be "locally connected at a point" or "locally compact at a point", etc.) we mean that for every open neighborhood U of m in M, the index j_0 and neighborhood N_{j_0} in the above definition can be chosen so that $\mu_{j_0}^{-1}(N_{j_0}) \subseteq U$. To clarify this, we write the definitions explicitly:

(5) Sys_M *locally eventually has property* $\mathcal{P}$ *at* m or Sys_M *is locally eventually* $\mathcal{P}$ *at* m if for every $m \in U \in \mathrm{Open}(M)$ and for every index $h \in I$ there exists an index $j_0 \geq i$ and some neighborhood N_{j_0} of $\mu_{j_0}(m)$ in M_{j_0} such that the inverse system induced by N_{j_0} and $I^{\geq j_0}$ has property $\mathcal{P}$.

(6) Sys_M *locally cofinally has property* $\mathcal{P}$ *at* m or Sys_M *is locally cofinally* $\mathcal{P}$ *at* m if for every $m \in U \in \mathrm{Open}(M)$ and every index $h \in I$ there exists some cofinal subset $J \subseteq I$, some index $j_0 \geq h$ in J, and some neighborhood N_{j_0} of $\mu_{j_0}(m)$ in M_{j_0} such that $\mu_{j_0}^{-1}(N_{j_0}) \subseteq U$ and the inverse system induced by N_{j_0} and J has property $\mathcal{P}$.

If we omit mention of the point $m \in M$ in definition (5) or (6) then we mean that it holds for all $m \in M$. ■

Remark 2.1.61.

- Of course, if $\mathcal{P}$ consists solely of conditions dependent on the individual bonding maps or the individual spaces (and not, say, on some more complicated relationship between them) then the index h in the above definition is redundant and, along with the condition $j_0 \geq h$, can be ignored.

- Observe that $(1) \implies (2) \wedge (3)$, $(2) \vee (3) \implies (4)$, $(5) \implies (3) \wedge (4) \wedge (6)$ and that all of these imply (4).

- Recall that the property of being a monotone system is generally not solely dependent on the equivalence class of the system. However, clearly in some subcategories of inverse systems of Top the property of *cofinally* being a monotone system *is* dependent only on the equivalence class of the system.

Non-empty Limits and Surjectivity of Projections

Since limits in Set and Top always exist, when we say that the limit M of some system *exists* then we will mean that $M \neq \varnothing$. Although the empty set is important and has (often unintentionally) been studied extensively by many people, we will be primarily interested in studying non-empty limits, which are the main focus of this section.

The next proposition consists mainly of well-known results found in [12], slight generalizations of well-known results found in this reference, or straightforward observations. We provide their full proofs in this introductory section as they may be instructive to a reader who is uninitiated with limits in Top where in particular, the proposition's proof of 4a has not, to the author's knowledge, appeared elsewhere and is also written in manner that, in the author's opinion, is more amenable to gaining geometric intuition than the standard proof.

Proposition 2.2.1. Let $\mathrm{Sys}_M = ((M_\bullet, \tau_\bullet), \mu_{ij}, I, \leq)$ be an inverse system in Top and let $((M, \tau_M), \mu_\bullet)$ denote its canonical limit.

(1) If each $M_\bullet$ is, respectively, T_1, Hausdorff, regular, or completely regular then M will also have this property.

(2) If all $M_\bullet$ are Hausdorff then M is a closed subspace of $\prod_{i \in I} M_i$ so that in particular, if all $M_\bullet$ are compact and Hausdorff then so is M.

(3) If I is directed then the intersection of finitely many subbasic open sets is again a subbasic open set. In particular, this implies that

  (a) the collection of all subbasic open sets form a basis for τ_M.

  (b) a subset $S \subseteq M$ is dense (resp. nowhere dense) in M if and only if $\mu_i(S)$ is dense (resp. $\mu_i\big(M \setminus \overline{S}\big)$ dense) in $\operatorname{Im} \mu_i$ for all $i \in I$.

(4) If all $M_\bullet$ are non-empty compact Hausdorff spaces and I is directed then

  (a) M is non-empty.

  (b) $\operatorname{Im} \mu_i = \bigcap_{j \geq i} \operatorname{Im}(\mu_{ij})$ for each index i.

  (c) If $\mu_i(M) \subseteq U_i$ for some $i \in I$ and U_i is open in M_i then there exists some $j \geq i$ such that $\operatorname{Im} \mu_{ij} \subseteq U_i$.

(d) If in addition all $M_\bullet$ are connected then M is also connected.

(5) If I is directed and contains a cofinal countable subset and if all $M_\bullet$ are first (resp. second) countable then so is M.

Proof. (1): Observe that each of these properties is preserved under arbitrary products and subspaces so (1) follows immediately from the definition of the canonical limit.

(2): That M is closed in $\prod_{i\in I} M_i$ can be seen immediately by considering a net $(m^\alpha)_{\alpha\in A}$ contained in M that converges in $\prod_{i\in I} M_i$ to some $m^0 \in \prod_{i\in I} M_i$ and then observing that the canonical projections $\prod_{h\in I} M_h \to M_j$ and the connecting maps $\mu_{ij} : M_j \to M_i$ are continuous for all $i, j \in I$ with $i \leq j$. The compactness claim then follows from Tychonoff's theorem.

(3): A basic open subset of M is of the form $U = \mu_{i_1}^{-1}(U_{i_1}) \cap \cdots \cap \mu_{i_q}^{-1}(U_{i_q})$ for some indices $i_1, \ldots, i_q \in I$ and some open subsets U_{i_p} of M_{i_p} ($p = 1, \ldots, q$). Since $(I, \leq)$ is directed, we can pick $j \in I$ such that $j \geq i_p$ for all $p = 1, \ldots, q$. It follows from the continuity of each $\mu_{i_p,j}$ that

$$U = \bigcap_{p=1}^{q} \mu_{i_p}^{-1}(U_{i_p}) = \bigcap_{p} \left(\mu_{i_p,j} \circ \mu_j\right)^{-1}(U_{i_p}) = \bigcap_{p} \mu_j^{-1}\left(\mu_{i_p,j}^{-1}(U_{i_p})\right) = \mu_j^{-1}\left(\bigcap_{p} \mu_{i_p,j}^{-1}(U_{i_p})\right)$$

is a sub-basic open subset of M. Since every basic open set is also a sub-basic open set the characterization of when $S \subseteq M$ is dense in M follows immediately, which in turn implies the characterization of when S is nowhere dense in M since (by definition) S is nowhere dense in M if and only if $M \setminus \overline{S}$ is dense in M.

(4a): The proof of (4a) can be found in [12, pp. 428-429, 435], where it is shown that $(I, \leq)$ being directed leads to the following closed subsets of the compact space $\prod_{i\in I} M_i$

$$S_j \underset{\text{def}}{=} \left\{ (p_i)_{i\in I} \in \prod_{i\in I} M_i \,\middle|\, i \leq j \implies p_i = \mu_{ij}(p_j) \right\}, \text{ for } j \in I$$

having the finite intersection property, which results in their intersection being non-empty. However, by proving (4) with nets instead we obtain a simple proof that shows how an arbitrary element of $p = (p_i)_{i\in i} \in P \underset{\text{def}}{=} \prod_{i\in I} M_i$ can be used as a seed for obtaining an element of M. To construct a non-constant net in P from p it is only reasonable to use $(I,\leq)$ as the domain of our net and to consider the net:

$$\begin{aligned} I &\longrightarrow P \\ j &\longmapsto p^j = \left(p_i^j\right)_{i\in I} \end{aligned} \qquad \text{where} \quad p_i^j \underset{\text{def}}{=} \begin{cases} \mu_{ij}(p_j) & \text{if } i \leq j \\ p_i & \text{otherwise} \end{cases}$$

Since P is compact there exists a convergent subnet, meaning that there is a cofinal order morphism $\iota:(A,\leq)\to(I,\leq)$ from some directed set $(A,\leq)$ such that the net

$$\begin{aligned} A &\longrightarrow P \\ a &\longmapsto p^{\iota(a)} \end{aligned}$$

converges in P to some point $m = (m_i)_{i\in I} \in P$.

It is now easy to see that $m \in M$: the assumption that all connecting maps are continuous (and the fact that the canonical projections $\operatorname{Pr}_i : P \to M_i$ are always continuous) allows us to permute $\lim\limits_{a\in A}$ and with these maps while the Hausdorff assumption gives us uniqueness of limits of convergent nets, which allows us to show that the equalities necessary for m to belong to M hold. We will now fill in these details for the interested reader: For any $i \in I$, we have $m_i = \operatorname{Pr}_i(m) = \lim\limits_{a\in A} \operatorname{Pr}_i(p^{\iota(a)}) = \lim\limits_{a\in A} \mu_{i,\iota(a)}\left(p_{\iota(a)}\right)$ where the last equality holds since there exists some $a_0 \in A$ such that $i \leq \iota(a)$ for all $a \geq a_0$. For any $i \leq j$ in I we thus obtain

$$\mu_{ij}(m_j) = \lim_{a\in A} \mu_{ij}\left(\mu_{j,\iota(a)}\left(p_{\iota(a)}\right)\right) = \lim_{a\in A} \mu_{i,\iota(a)}\left(p_{\iota(a)}\right) = m_i.$$

(4b): This result is derived as corollary 2.2.11 but it can also be immediately proven as follows: To prove the non-trivial inclusion fix $i \in I$, assume that $m_i \in \bigcap\limits_{j\geq i} \operatorname{Im}\mu_{ij}$, and let $N_i = \{m_i\}$. For all $j > i$ observe that the set $N_j \underset{\text{def}}{=} \mu_{ij}^{-1}(m_i)$ is compact, Hausdorff, non-empty and that if $k \geq j$ then $\mu_{jk}(N_k) \subseteq N_j$ so that we can define $nu_{jk} \underset{\text{def}}{=} \mu_{jk}\big|_{N_k} : N_k \to N_j$. The system $\operatorname{Sys}_N \underset{\text{def}}{=} (N_j, \nu_{jk}, I^{\geq i})$ then satisfies the conditions of part (4a) of this theorem so that there exists some element n in the canonical limit of Sys_N. Since $I^{\geq i}$ is cofinal in I it follows that $n \in M$ and since $\mu_i(n) = \nu_i(n) \in N_i = \{m_i\}$, the result follows.

(4c): Assume that $C_i^j \underset{\text{def}}{=} \operatorname{Im}\mu_{ij} \smallsetminus U_i \neq \varnothing$ for each $j \geq i$. It is easy to see that $\left(C_i^j\right)_{j \in I^{\geq i}}$ has the finite intersection property so that the set $\underset{j \geq i}{\cap} C_i^j = \left(\underset{j \geq i}{\cap} \operatorname{Im}\mu_{ij}\right) \smallsetminus U_i = \operatorname{Im}\mu_i \smallsetminus U_i$ is non-empty. Alternatively, one could define for each $j \geq i$ the set $N_j = M_j \smallsetminus \mu_{ij}^{-1}(U_i)$, observe that $\mu_{jk}(N_k) \subseteq N_j$, and then show that defining ν_{jk} and Sys_N as in the proof of (4b) leads to existence of an impossible element of M.

(4d): Suppose that U and V are disjoint open subsets of M such that $U \cup V = M$. For each index $i \in I$, let U_i (resp. V_i) be the largest open subset of M_i such that $\mu_i^{-1}(U_i) \subseteq U$ (resp. $\mu_i^{-1}(V_i) \subseteq V$) and observe that $U = \underset{i \in I}{\cup} \mu_i^{-1}(U_i)$ (resp. $V = \underset{i \in I}{\cup} \mu_i^{-1}(V_i)$), $U_i \cap V_i = \varnothing$, and that $i \leq j$ implies $\mu_{ij}^{-1}(U_i) \subseteq U_j$ (resp. $\mu_{ij}^{-1}(V_i) \subseteq V_j$). For each $i \in I$, let $W_i = U_i \cup V_i$ and observe that $\left(\mu_i^{-1}(W_i)\right)_{i \in I}$ is an open cover of M so that we may pick a finite subcover indexed by $i_1, \ldots, i_q$. Let $j \in I$ be such that $j \geq i_p$ for all $p = 1, \ldots, q$ and observe that $\mu_{i_p,j}^{-1}\left(W_{i_p}\right) \subseteq W_j$ for all $p = 1, \ldots, q$, which implies that $M = \mu_j^{-1}(W_j)$. By (4c) there exists some $k \geq j$ such that $\operatorname{Im}\mu_{jk} \subseteq W_j$, which implies that $M_k = \mu_{jk}^{-1}(W_j) = \mu_{jk}^{-1}(U_j) \cup \mu_{jk}^{-1}(V_j)$. But $\mu_{jk}^{-1}(U_j) \subseteq U_k$ and $\mu_{jk}^{-1}(V_j) \subseteq V_k$ implies that $M_k = U_k \cup V_k$ where $U_k \cap V_k = \varnothing$ so that the connectedness of M_k implies that one of U_k and V_k is $\varnothing$, say $U_k = \varnothing$, so that $V_k = M_k$. But then $M = \mu_i^{-1}(V_k) = V$ so that $U = \varnothing$.

(5) follows immediately from (3) and the facts that I is countable and that the union of a countable collection of countable sets is again countable. ∎

Remark 2.2.2. Since the canonical limit M is a purely set theoretic construction (recall def. 2.1.30) so is the question of whether or not $M = \varnothing$. This means that proposition 2.2.1(4a) shows how the existence of certain topologies on the sets $M_\bullet$ can force a purely set theoretic result (i.e. $M \neq \varnothing$). Conversely, we can also use set theoretic results with proposition 2.2.1 to prove the non-existence of topologies with certain properties (e.g. as a simple example, does there exist *any* Hausdorff topology on $\mathbb{R}$ making all open intervals of the form $]a, b[$ with $a, b \in \mathbb{R}$ and $a < b$ compact?).

Example 2.2.3. The conclusion 4a of proposition 2.2.1 may fail without the Hausdorff condition: For each $i \in \mathbb{N}$, let $M_i \underset{\text{def}}{=}]0, 1/i[$ and form the inverse system directed by $\mathbb{N}$ where each M_i is given the indiscrete topology and the bonding maps are the canonical inclusions.

Example 2.2.4. Let $M_i \underset{\text{def}}{=}]0, \infty[^i$ for all $i \in \mathbb{N}$ and let $\mu_{i,i+1}$ be the canonical projection so that the limit is $M \underset{\text{def}}{=} \prod_{i=1}^{\infty}]0, \infty[$ with the canonical projections. Let $S = \prod_{i=1}^{\infty}]1/i, \infty[$ and observe that $\overline{S} = \prod_{i=1}^{\infty} [1/i, \infty[$. For any index i and any $(m_1, \ldots, m_i) \in M_i$ the element $m \underset{\text{def}}{=} (m_1, \ldots, m_i, \frac{1}{i+2}, 1, 1, \ldots)$ belongs to $\mu_i^{-1}(m_1, \ldots, m_i)$ but not to $\overline{S}$. Since $\overline{S}$ does not contain any fiber of any μ_i it follows from proposition 2.2.1(3) that $\overline{S}$, and hence $S = \prod_{i=1}^{\infty}]1/i, \infty[$, is nowhere dense in $M = \prod_{i=1}^{\infty}]0, \infty[$. In particular, observe that S is nowhere dense in M despite how the $]1/i, \infty[$'s become "increasingly larger in $]0, \infty[$". ■

Generalized Mittag-Leffler Lemma

The following generalization of the Mittag-Leffler Lemma (Bourbaki II, 3.5 thm. I) and its corollaries can be found in [35, pp. 310-313].The results of this subsection will not be used anywhere else in this paper.

Lemma 2.2.5 (Generalized Mittag-Leffler Lemma)**.** Suppose that each $(M_i, d_i)_{i \in \mathbb{N}}$ is a complete metric space and each $\mu_{i,i+1} : (M_{i+1}, d_{i+1}) \to (M_i, d_i)$ is non-expansive. Let $S_i \subseteq M_i$ be non-empty for every index $i \in \mathbb{N}$. If there exist positive $(\epsilon_i)_{i=1}^{\infty}$ such that for each $i \in \mathbb{N}$

(1) $r_i \underset{\text{def}}{=} \sum_{j=i}^{\infty} \epsilon_j < \infty$,

(2) $d_i(s_i, \mu_{i,i+1}(S_{i+1})) < \epsilon_i$ for every $s_i \in S_i$.

then the sets $B^{i_0} \underset{\text{def}}{=} \bigcap_{i=i_0}^{\infty} \bigcup_{s_i \in S_i} \mu_i^{-1}\left(\mathrm{B}_{r_i}^{d_i}(s_i)\right)$ are non-empty (subsets of M) and $d_i(s_i, \mu_i(B^i)) \leq r_i$ for all indices $i \in \mathbb{N}$ and $s_i \in S_i$. In particular, if $S_i \subseteq \overline{\mu_{i,i+1}(S_{i+1})}$ for all indices i then $S_i \subseteq \overline{\mu_i(M)}$ for each index i.

According to [35], the first Mittag-Leffler lemma is likely Theorem 2.4 of Arens [4]. We summarize Arens' theorem as the following corollary of lemma 2.2.5.

Corollary 2.2.6 (Mittag-Leffler Lemma [4, Thm. 2.4])**.** Suppose that all $M_1, M_2, \ldots$ are non-empty complete metric spaces and that each $\mu_{i,i+1}$ has an image that is dense in M_i. Then $(M, \mu_\bullet) = \varprojlim M_\bullet$ is a complete metric space and the image of each μ_i is dense in M_i.

Observe that the generalized Mittag-Leffler lemma 2.2.5 follows immediately from the next lemma, whose statement was obtained by inspection of the proof of lemma 2.2.5 in [35, pp. 310-313] but whose proof, which is original, was inspired by imagining the set $\overline{R_j}$ (from the proof) being "positioned above $\overline{R_i}$" and then "falling" (via μ_{ij}) onto the subset $\overline{\mu_{ij}(R_j)}$ of $\overline{R_i}$.

Lemma 2.2.7. Suppose that for each index i, (M_i, d_i) is a complete metric space, $\mu_{i,i+1} : (M_{i+1}, d_{i+1}) \to (M_i, d_i)$ is non-expansive, $S_i \subseteq M_i$ is a non-empty subset, and ϵ_i is such that $d_i(s_i, \mu_{i,i+1}(S_{i+1})) < \epsilon_i < \infty$ for all $s_i \in S_i$. Assume that $r_i \underset{\text{def}}{=} \sum_{j=i}^{\infty} \epsilon_j < \infty$ for some/all indices i. Then for any index i_0 and any $s_{i_0} \in S_{i_0}$ there exists some $m \in M \underset{\text{def}}{=} \varprojlim \operatorname{Sys}_M$ such that

(1) $d_{i_0}(\mu_{i_0}(m), s_{i_0}) < r_{i_0}$, and

(2) $d_i(\mu_i(m), \mu_{ij}(S_i)) < r_j$ for all $j \geq i \geq i_0$.

So in particular, $d_i(\mu_i(m), S_i) < r_i$ for all $i \geq i_0$.

Proof. Observe that for the conclusion to hold we need only to consider the subsystem $\operatorname{Sys}_M\big|_{I^{\geq i_0}}$ indexed by $I^{\geq i_0}$ so we will assume without loss of generality that $i_0 = 1$. Having picked $s_1, \ldots, s_i$ such that $s_l \in S_l$ for all $l = 1, \ldots, i$ and $d_{l-1}(s_{l-1}, \mu_{l-1,l}(s_l)) < \epsilon_l$ for all $l = 2, \ldots, i$ our assumption allows us to pick $s_{i+1} \in S_{i+1}$ such that $d_i(s_i, \mu_{i,i+1}(s_{i+1})) < \epsilon_i$.

Since all $\mu_{i,i+1}$ are non-expansive, it is clear that so too are all $\mu_{ij} = \mu_{i,i+1} \circ \cdots \circ \mu_{j-1,j}$. It is now straightforward to verify that for all $i < j$, $d_i(s_i, \mu_{ij}(s_j)) < \sum_{l=i}^{j-1} \epsilon_l$: Fix $i \in \mathbb{N}$, observe that it's true for $j := i+1$, and also that if it's true for $j \geq i+1$ then we have

$$\begin{aligned} d_i(s_i, \mu_{i,j+1}(s_{j+1})) &\leq d_i(s_i, \mu_{ij}(s_j)) + d_i(\mu_{ij}(s_j), \mu_{i,j+1}(s_{j+1})) \\ &\leq d_i(s_i, \mu_{ij}(s_j)) + d_j(s_j, \mu_{j,j+1}(s_{j+1})) \\ &< \left(\sum_{l=i}^{j-1} \epsilon_l\right) \qquad\quad + \epsilon_j. \end{aligned}$$

Since each μ_{ij} is non-expansive, whenever $i \leq j \leq k$ we have

$$d_i(\mu_{ij}(s_j), \mu_{ik}(s_k)) \leq d_j(s_j, \mu_{jk}(s_k)) < \sum_{l=j}^{k-1} \epsilon_l$$

so that for each index i the sequence $(\mu_{il}(s_l))_{l=i}^{\infty}$ is Cauchy and letting $k \to \infty$ we obtain $d_i\left(\mu_{ij}(s_j), \lim_{k\to\infty} \mu_{ik}(s_k)\right) < \sum_{l=j}^{\infty} \epsilon_l = r_j$, which becomes $d_i\left(s_i, \lim_{k\to\infty} \mu_{ik}(s_k)\right) < r_i$ when $i = j$.

For each index i, $R_i \underset{\text{def}}{=} \{\mu_{il}(s_l) \mid l \geq i\}$ is totally bounded and since (M_i, d_i) is complete the set $K_i \underset{\text{def}}{=} \overline{R_i}$ is compact. Observe that $\mu_{ij}(K_j) \subseteq K_i$ so proposition 2.2.1(4a) gives us that the vertex of the limit $(K, \mu_\bullet)$ of the inverse system $\left(K_i, \mu_{ij}\big|_{K_j}, \mathbb{N}\right)$ is compact and non-empty. Let $m \in K$ and observe that whenever $i \leq j$, the compactness of $K_j = \overline{R_j}$ gives us $\mu_{ij}\left(\overline{R_j}\right) = \overline{\mu_{ij}(R_j)}$ so that $\mu_j(m) \in K_j$ implies

$$\mu_i(m) \in \mu_{ij}(K_j) = \overline{\mu_{ij}(R_j)} = \overline{\{\mu_{il}(s_l) \mid l \geq j\}} = \left\{\lim_{l\to\infty} \mu_{il}(s_l)\right\} \cup \{\mu_{il}(s_l) \mid l \geq j\}$$

Since limits of convergent sequences are unique, we have $\bigcap_{j\geq i} \{\mu_{il}(s_l) \mid l \geq j\} \subseteq \left\{\lim_{l\to\infty} \mu_{il}(s_l)\right\}$ for each index i, which implies that

$$\mu_i(m) \in \bigcap_{j\geq i} \left[\left\{\lim_{l\to\infty} \mu_{il}(s_l)\right\} \cup \{\mu_{il}(s_l) \mid l \geq j\}\right] = \left\{\lim_{l\to\infty} \mu_{il}(s_l)\right\} \cup \bigcap_{j\geq i} \{\mu_{il}(s_l) \mid l \geq j\} = \left\{\lim_{l\to\infty} \mu_{il}(s_l)\right\}$$

Thus for each index i, $\mu_i(m)$ is the limit of the Cauchy sequence $(\mu_{il}(s_l))_{l=i}^{\infty}$ so that, as was shown above, $d_i(s_i, \mu_i(m)) < r_i$ and $d_i(\mu_{ij}(S_j), \mu_i(m)) < r_j$ for all $j \geq i$. ■

Surjectivity of Projections

One direction of the equivalence in the following lemma, as well as the statement following it, is well-known while the equivalence's other direction appears to have either gone unnoticed or unremarked upon for some reason.

Lemma 2.2.8. If $(M, \mu_\bullet) = \varprojlim(M_\bullet, \mu_{ij}, I)$ in Set (or in Top) and I is directed then the following are equivalent:

(1) $\mu_i : M \to M_i$ is surjective (resp. injective) for all indices i.

(2) $\mu_{ij}\big|_{\mathrm{Im}(\mu_j)} : \mathrm{Im}(\mu_j) \to M_i$ is surjective (resp. injective) for all $i \leq j$.

Let $\overline{M_i} = \mathrm{Im}\,\mu_i$, $\overline{\mu_{ij}} = \mu_{ij}\big|_{\overline{M_j}} : \overline{M_j} \to \overline{M_j}$, and $\overline{\mu_i} = \mu_i : M \to \overline{M_i}$. Then $\overline{\mathrm{Sys}_M} \underset{\mathrm{def}}{=} (\overline{M_\bullet}, \overline{\mu_{ij}}, I)$ is a surjective inverse system with surjective limit $(M, \overline{\mu_\bullet})$ in Set (or in Top). Furthermore, if $(M, \mu_\bullet)$ is the canonical limit of Sys_M then $(M, \overline{\mu_\bullet})$ is the canonical limit of $\overline{\mathrm{Sys}_M}$.

Proof. If (1) holds for surjectivity (resp. injectivity) then the surjectivity (resp. injectivity) of $\mu_{ij}|_{\mathrm{Im}(\mu_j)}$ follows from the surjectivity (resp. injectivity) of μ_i and the equality $\mu_i = \mu_{ij} \circ \mu_j = \mu_{ij}\big|_{\mathrm{Im}(\mu_j)} \circ \mu_j$. If (2) holds for surjectivity then

$$M_i = \mathrm{Im}\left(\mu_{ii}\big|_{\mathrm{Im}(\mu_i)}\right) = \mathrm{Im}\left(\mathrm{Id}_{M_i}\big|_{\mathrm{Im}(\mu_i)}\right) = \mathrm{Id}_{M_i}(\mathrm{Im}(\mu_i)) = \mathrm{Im}(\mu_i)$$

so (1) holds for surjectivity. Now assume that (2) holds for injectivity and that $m, \widetilde{m} \in M$ and $i \in I$ are such that $\mu_i(m) = \mu_i(\widetilde{m})$. If $j \geq i$ then $\mu_{ij}(\mu_j(m)) = \mu_i(m) = \mu_i(\widetilde{m}) = \mu_{ij}(\mu_j(\widetilde{m}))$ and the injectivity of $\mu_{ij}\big|_{\mathrm{Im}(\mu_j)} : \mathrm{Im}(\mu_j) \to M_i$ implies that $\mu_j(m) = \mu_j(\widetilde{m})$. Since $(I, \leq)$ is directed and $j \geq i$ was arbitrary, it follows that $m = \widetilde{m}$.

It is obvious that $\overline{\mathrm{Sys}_M}$ is an inverse system and lemma 2.1.27 makes it is straightforward to verify that $(M, \overline{\mu_\bullet})$ is a limit of $\overline{\mathrm{Sys}_M}$ in Set. If Sys_M was a system in Top, then observe that for any index i and any open subset U_i of M_i we have $\mu_i^{-1}(U_i) = \mu_i^{-1}(\mathrm{Im}(\mu_i) \cap U_i) = \overline{\mu_i}^{-1}\left(U_i \cap \overline{M_i}\right)$ so that the canonical sub-basic open subsets, and hence topologies, on M

induced by $\mu_\bullet$ and $\overline{\mu_\bullet}$ are identical, which shows that $(M, \overline{\mu_\bullet})$ is also the limit of $\overline{\mathrm{Sys}_M}$ in Top.

Now assume that $(M, \mu_\bullet)$ is the canonical limit of Sys_M and let $\overline{M}$ denote the object of the canonical limit of $\overline{\mathrm{Sys}_M}$. If $m = (m_i)_{i \in I}$ is in M then since $m_i = \mu_i(m)$ we have $m = (m_i)_{i \in I} = (\mu_i(m))_{i \in I} \in \prod_{i \in I} \mathrm{Im}(\mu_i)$ so that $m \in \overline{M}$. Since we clearly have $\overline{M} \subseteq M$ the equality of sets follows. It is now immediate from the definition of the canonical limit that $(M, \overline{\mu_i})$ is the canonical limit of $\overline{\mathrm{Sys}_M}$ in Set. ■

A Sufficient Condition for a Non-empty Limit

If an inverse system is directed by $\mathbb{N}$ and if we can find a sequence of elements $m \underset{\text{def}}{=} (m_i)_{i \in \mathbb{N}} \in \prod_{i \in \mathbb{N}} M_i$ such that $m_{i+1} \in \mu_{i,i+1}^{-1}(m_i)$ then m belongs to the canonical limit M so in particular, M is necessarily non-empty. It is for the construction of such a sequence of elements that the following lemmata will be useful. Another one of their uses is to give a sufficient condition that will, under certain conditions, allow us to replace one directed inverse system with an inverse system of subsets, all of whose connecting maps are surjective.

Lemma 2.2.9. Suppose $(M, \mu_\bullet) = \varprojlim \mathrm{Sys}_M$, where $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, I)$ is directed, and for each index i, let $R_i \underset{\text{def}}{=} \bigcap_{l \geq i} \mu_{il}(M_l)$. Then

(1) for any cofinal subset $J \subseteq I^{\geq i}$ we have $R_i = \bigcap_{j \in J} \mu_{ij}(M_j)$,

(2) each R_i contains both $\mathrm{Im}\, \mu_i$ and $\mu_{ij}(R_j)$, and if we consider all μ_i and $\mu_{ij}\big|_{R_j}$ as maps into R_i then $\left(R_i, \mu_{ij}|_{R_j}, I\right)$ is an inverse system whose limit is $(M, \mu_\bullet)$, and

(3) if $i \leq j$ and $r_i \in R_i$ then $\mu_{ij}^{-1}(r_i) \cap \mu_{jk}(M_k) \neq \varnothing$ for each $k \geq j$ and if in addition $\mu_{ij}^{-1}(r_i) \cap \bigcap_{l \geq j} \mu_{jl}(M_l) \neq \varnothing$ then $r_i \in \mu_{ij}(R_j)$.

In particular, if $i \leq j$ are indices such that the condition in (3) is satisfied for each $r_i \in R_i$ then $R_i = \mu_{ij}(R_j)$.

Proof. Fix i and let $K \subseteq I^{\geq i}$ be cofinal in $I^{\geq i}$. Let $R = \bigcap_{k \in K} \mu_{ik}(M_k)$ so that clearly $R_i \subseteq R$. Fix some $k \in K$ and observe that if $j \in I$ is such that $i \leq h \leq k$ then $\mu_{ik}(M_k) = \mu_{ij}(\mu_{jk}(M_k)) \subseteq \mu_{ij}(M_j)$ which implies $\mu_{ik}(M_k) \subseteq \bigcap_{\substack{i \leq j \leq k \\ j \in I}} \mu_{ij}(M_j)$. Since $k \in K$ was arbitrary it follows that for any $k_0 \in K$,

$$\bigcap_{\substack{k \in K \\ k \leq k_0}} \mu_{ik}(M_k) \subseteq \bigcap_{\substack{i \leq j \leq k_0 \\ j \in I}} \mu_{ij}(M_j)$$

where since every $\mu_{ik}(M_k)$ from the left hand side also occurs on the right hand side we in fact have equality. From here, we can now show that (1) follows:

$$R = \bigcap_{k \in K} \mu_{ik}(M_k) = \bigcap_{k_0 \in K} \bigcap_{\substack{k \in K \\ k \leq k_0}} \mu_{ik}(M_k) = \bigcap_{\substack{k \in K \\ k \leq k_0}} \bigcap_{\substack{i \leq j \leq k_0 \\ j \in I}} \mu_{ij}(M_j) = \bigcap_{\substack{j \geq i \\ j \in I}} \mu_{ij}(M_j) = R_i$$

For all $j \geq i$, applying (1) with $K = I^{\geq j}$ gives

$$\mu_{ij}(R_j) = \mu_{ij}\left(\bigcap_{l \geq j} \mu_{jl}(M_l)\right) \subseteq \bigcap_{l \geq j} \mu_{ij}(\mu_{jl}(M_l)) = \bigcap_{l \geq j} \mu_{il}(M_l) = R_i$$

Letting $\operatorname{Sys}_R = \left(R_i, \mu_{ij}|_{R_j}, I\right)$ then clearly $\varprojlim \operatorname{Sys}_R \subseteq M$ while if $m \in M$ then for each index i we have $\mu_i(m) \in \mu_{ij}(M_j)$ for any $j \geq i$ so that $\mu_i(m) \in \bigcap_{j \geq i} \mu_{ij}(M_j)$ for all i and thus $m \in \varprojlim \operatorname{Sys}_R$, which proves (2).

Fix $j_0 \geq i$. and let $s_i \in R_i$. Define $S \underset{\text{def}}{=} \mu_{i,j_0}^{-1}(s_i)$ and for all $l \geq j_0$, let $S^l = S \cap \mu_{j_0,l}(M_l)$. Observe that

$$s_i \in R_i = \bigcap_{j \geq i} \mu_{ij}(M_j) \subseteq \mu_{il}(M_l) = \mu_{i,j_0}(\mu_{j_0,l}(M_l))$$

so that $\mu_{i,j_0}^{-1}(s_i) \cap \mu_{j_0,l}(M_l) \neq \varnothing$. Note that

$$\bigcap_{l \geq j_0} S \cap \mu_{j_0,l}(M_l) = S \cap \left(\bigcap_{l \geq j_0} \mu_{j_0,l}(M_l)\right) = S \cap R_{j_0}$$

so that if $\mu_{i,j_0}^{-1}(s_i) \cap R_{j_0} \neq \varnothing$ then $S \cap R_{j_0} \neq \varnothing$ so we can conclude that $s_i \in \mu_{i,j_0}(R_{j_0})$. ∎

Proposition 2.2.10. Let $(M, \mu_\bullet) = \varprojlim \operatorname{Sys}_M$, where $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, I)$ is directed, and for each index i, let $R_i = \bigcap_{l \geq i} \mu_{il}(M_l)$. Let $\mathcal{R}$ be a subcategory of Set and suppose that for all $i \leq j$, M_i, $\operatorname{Im} \mu_{ij}$, and each fiber of μ_{ij} over an element in M_i belongs to $\mathcal{R}$. Partially order $\mathcal{R}$ by reverse set inclusion and assume that the intersection of two spaces in $\mathcal{R}$ belongs to $\mathcal{R}$ and that whenever $\iota : I \to \mathcal{R}$ is an order morphism with each $\iota(j) \neq \varnothing$ then $\cap \operatorname{Im} \iota \underset{\text{def}}{=} \bigcap_{i \in I} \iota(i)$ is non-empty. Then $\mu_{ij}(R_j) = R_i$ for any $i \leq j$ so that in particular, when $I = \mathbb{N}$ then all M_l are non-empty $\iff$ $M \neq \varnothing$.

Suppose that in addition whenever $\iota : I \to \mathcal{R}$ is an order morphism then $\operatorname{Im} \iota$ is well-ordered. Then for each i there exists some $j \geq i$ such that $R_i = \mu_{ij}(M_j)$ (so in particular, all R_l are non-empty $\iff$ all M_l are non-empty) and when $I = \mathbb{N}$ then we will also have $R_i = \operatorname{Im} \mu_i$.

Proof. Fix $j_0 \geq i$ and let $s_i \in R_i$. Define $S \underset{\text{def}}{=} \mu_{i,j_0}^{-1}(s_i)$ and for all $l \geq j_0$, let $S^l = S \cap \mu_{j_0,l}(M_l)$ so that our hypotheses imply that S and each S^l is a subset of M_{j_0} that belongs to $\mathcal{R}$. From $\mu_{j_0,l}(M_l) \subseteq \mu_{j_0,j}(M_j)$, where $j_0 \leq j \leq l$ we obtain $S^l \subseteq S^j$ so that the map

$$
\begin{aligned}
\iota : I^{\geq j_0} &\longrightarrow \mathcal{R} \\
i &\longmapsto \begin{cases} S^l & \text{if } i \geq j_0 \\ S^{j_0} & \text{otherwise} \end{cases}
\end{aligned}
$$

is an order morphism. By lemma 2.2.9 each S^l is a non-empty and to show that $s_i \in \mu_{i,j_0}(R_{j_0})$ it suffices to show that $\bigcap_{l \geq j_0} S^l \neq \varnothing$, but this holds by hypotheses. If $I = \mathbb{N}$ then since M is the limit of Sys_R, which is a surjective system, we have that $\mu_i|_M : M \to R_i$ is surjective so by lemma 2.2.9 all M_l are non-empty if and only if all M_l are non-empty, which happens if and only if $M \neq \varnothing$.

Now assume that whenever $\iota : I \to \mathcal{R}$ is an order morphism then $\operatorname{Im} \iota$ is well-ordered. Fix

$i \in I$ and define

$$\begin{aligned} \iota : I &\longrightarrow \mathcal{R} \\ l &\longmapsto \begin{cases} \mu_{il}(M_l) & \text{if } l \geq i \\ M_i & \text{otherwise} \end{cases} \end{aligned}$$

where by hypotheses, each $\iota(l)$ belongs to $\mathcal{R}$. Whenever $j \leq k$ with $j, k \in I^{\geq i}$ we have $\iota(k) \subseteq \iota(j)$ so that ι is an order morphism into $\mathcal{R}$. By assumption then, $\operatorname{Im} \iota = \{\mu_{il}(M_l) \mid l \geq i\}$ is well-ordered so there exists some $j \geq i$ such that $\mu_{ij}(M_j)$ is a least element of $\{\mu_{il}(M_l) \mid l \geq i\}$. By minimality, we have $\mu_{ij}(M_j) \subseteq \bigcap_{l \geq i} \mu_{il}(M_l) = R_i$ but since $j \geq i$ we also have the reverse inclusion, so that equality holds. It is now immediate that if $I = \mathbb{N}$ then $R_i = \operatorname{Im} \mu_i$. ■

Corollary 2.2.11. If Sys_M is a directed inverse system of compact Hausdorff sets and for each index i we define $R_i = \bigcap_{l \geq i} \mu_{il}(M_l)$, then $R_i = \mu_{ij}(R_j)$ for all $i \leq j$.

Proof. Apply proposition 2.2.10 with $\mathcal{R}$ being the category of all compact Hausdorff spaces.

■

Lemma 2.2.12. Let $(M, \mu_\bullet) = \varprojlim \operatorname{Sys}_M$ where $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, \mathbb{N})$. Suppose that all M_i are finite dimensional vector spaces and that all $\mu_{ij} : M_j \to M_i$ are (not necessarily surjective) affine linear maps. For each index i, let $R_i = \bigcap_{l \geq i} \mu_{il}(M_l)$. Then for each index i,

(1) for any increasing sequence of integers $(l_n)_{n=1}^{\infty}$ with $l_1 \geq i$ we have $R_i = \bigcap_{n \geq 1} \mu_{il_n}(M_{l_n})$.

- In particular, $R_i = \bigcap_{l \geq j} \mu_{il}(M_l)$ for each $j \geq i$.

(2) $\mu_{i,i+1}(R_{i+1}) = R_i = \operatorname{Im} \mu_i$.

(3) $M = \varprojlim \big(R_i, \mu_{ij}|_{R_j}, \mathbb{N}\big)$.

(4) $R_i \neq \varnothing$ (which implies that $M \neq \varnothing$).

Proof. Let $\mathcal{R}$ denote the category of all finite-dimensional affine linear spaces with affine linear maps and apply proposition 2.2.10. ■

For another corollary of these lemmata, see corollary 3.2.3.

Subsets of Inverse Systems

Notation 2.3.1. If $\mu_\bullet : M \to M_\bullet$ is an I-indexed collection of maps, $J \subseteq I$, and $S_\bullet = (S_j)_{j \in J}$ is a collection of sets such that $S_j \subseteq M_j$ for all $j \in J$, then by $\cup \mu_\bullet^{-1}(S_\bullet)$ and $\cap \mu_\bullet^{-1}(S_\bullet)$ we mean

$$\cup \mu_\bullet^{-1}(S_\bullet) \underset{\text{def}}{=} \bigcup_{j \in J} \mu_i^{-1}(S_j) \quad \text{and} \quad \cap \mu_\bullet^{-1}(S_\bullet) \underset{\text{def}}{=} \bigcap_{j \in J} \mu_i^{-1}(S_j)$$

Inverse Systems of Subsets

The following definition is based on the definition of inverse system of subsets that is found in [10].

Definition 2.3.2. Suppose that $J \subseteq I$and $S_\bullet = (S_j)_{j \in J}$ is a collection of sets such that $S_j \subseteq M_j$ for all $j \in J$. If $\mu_{ij}(S_j) \subseteq S_j$ for all $i, j \in J$ with $i \leq j$ then we will call

$$\operatorname{Sys}_S \underset{\text{def}}{=} \operatorname{Sys}_M \big|_{S_\bullet} \underset{\text{def}}{=} \left(S_\bullet, \mu_{ij}\big|_{S_j}, J \right)$$

the system (canonically) induced by $S_\bullet$ *(and* Sys_M*)* and say that $S_\bullet$ is an *inverse system of subsets (of* Sys_M*)* (or *of* $M_\bullet$*) (indexed by* J*)*, where for all $i \leq j$ in J, each $\mu_{ij}\big|_{S_j}$ has prototype $\mu_{ij}\big|_{S_j} : S_j \to S_i$. If Sys_M is a system in Top and if each S_j is a topological subspace of M_j then we will call $S_\bullet$ an *inverse system of subspaces of* Sys_M *(*or *of* $M_\bullet$*) indexed by* J while if $\mu_{jk}(S_k) = S_j$ for all $j \leq k$ in J then we will say that this system of subsets is *surjective.*

If $S_\bullet$ is a J-indexed inverse system of subsets of Sys_M and $(Z, h_\bullet)$ is a cone into Sys_M then by *the (*J*-indexed) cone (canonically) induced by* $S_\bullet$ *and* $h_\bullet$ we mean the cone

$$(Z, h_\bullet)\big|_{S_\bullet} \underset{\text{def}}{=} \left(\bigcap_{j \in J} h_j^{-1}(S_j),\ h_\bullet\big|_{\cap h_\bullet^{-1}(S_\bullet)} \right)$$

into $\operatorname{Sys}_M\big|_{S_\bullet}$, where each h_j has prototype $h_j : \cap h_\bullet^{-1}(S_\bullet) \to S_j$. If $(M, \mu_\bullet)$ is a limit of Sys_M

then we will call $(M, \mu_\bullet)\big|_{S_\bullet}$ *the limit of* $S_\bullet$ *(induced by* $\mu_\bullet$*)* where this terminology is justified by remark 2.3.3 and where, as usual, we may also use this terminology to describe this cone's vertex, which we will denote by

$$\varprojlim S_\bullet \underset{\text{def}}{=} \underset{i \in I}{\cap} \mu_i^{-1}(S_i)$$

or possibly just S.

If $L \subseteq M$ then we will say that L *arises as a (*$\mu_\bullet$*-)limit of an inverse system of subsets (of* Sys_M*)* if there exists an inverse system of subsets of Sys_M whose limit is L. ■

Remark 2.3.3. Suppose $S_\bullet = (S_i)_{i \in I}$ is an inverse system of subsets of Sys_M, $(M, \mu_\bullet)$ is a limit of Sys_M, and let $\mathrm{Sys}_S \underset{\text{def}}{=} \left(S_\bullet, \mu_{ij}\big|_{S_j}, I\right)$. Let $S \underset{\text{def}}{=} \underset{i \in I}{\cap} \mu_i^{-1}(S_i)$ and for each $i \in I$, consider $\mu_i\big|_S$ as the map $\mu_i\big|_S : S \to S_i$ where if M is a topological space then we will also assume that S and each S_i is a topological subspace of M and M_i, respectively. By a straightforward check of the universal mapping property of limits, it is immediately verified that $\left(S, \mu_\bullet\big|_S\right)$ is a limit of Sys_S in Set where if Sys_M is a system in Top then this is also true in Top.

Also observe that the Sys_S-subbasic open subsets of S are exactly the intersections with S of a $\mu_\bullet$-subbasic open subset of M: $\left(\mu_i\big|_S\right)^{-1}(U_i \cap S_i) = S \cap \mu_i^{-1}(U_i)$ for every $i \in I$ and open subset U_i of M_i. Furthermore, if $(M, \mu_\bullet)$ is the canonical limit of Sys_M then (as observed in [12]) it is immediately seen that the object of the canonical limit of Sys_S is

$$S \underset{\text{def}}{=} \left\{ s = (s_i)_{i \in I} \in \prod_{i \in I} S_i \;\middle|\; \mu_{ij}(s_j) = s_i \text{ for all } i \le j \right\} = M \cap \prod_{i \in I} S_i$$

while the canonical limit's morphisms are $\mu_\bullet\big|_S \underset{\text{def}}{=} \left(\mu_i\big|_S\right)_{i \in I} = \left(\mathrm{Pr}_{S_i}\big|_S\right)_{i \in I}$ where $\mu_i\big|_S : S \to S_i$ for each $i \in I$. In particular, this justifies writing $\varprojlim S_\bullet \subseteq M$. ■

Convention 2.3.4. Whenever $S_\bullet$ is an inverse system of subsets of Sys_M (resp. Sys_N) then by $\varprojlim S_\bullet$ we will mean the subset $\cap \mu_\bullet^{-1}(S_\bullet)$ of M (resp. the subset $\cap \nu_\bullet^{-1}(S_\bullet)$ of N).

Example and Definition 2.3.5. For any $S \subseteq M$, $S_\bullet \underset{\text{def}}{=} \mu_\bullet(S)$ forms a surjective inverse system of subsets of Sys_M (cf. [11, I.4.4]). This allows us to canonically associate to each

$S \subseteq M$ the canonical system induced by $\mu_\bullet(S)$ and Sys_M (def. 2.3.2), which we will call the *inverse system of subsets (of $M_\bullet$) (canonically) induced by S, ($\mu_\bullet$, and* Sys_M*)*. Although we always have $S \subseteq \varprojlim S_i$, it is possible that the reverse containment fails; lemma 2.3.9 provides a characterization of when equality holds.

Example and Notation 2.3.6. If $m \in M$ then $\{m\} = \varprojlim_{i\in I}\{\mu_i(m)\}$ where we may henceforth write such an equality as $m = \varprojlim_{i\in I} \mu_i(m)$ (or $m = \varprojlim \mu_i(m)$, or $m = \varprojlim \mu_\bullet(m)$, etc.)

Example 2.3.7. Suppose Sys_M is indexed by $I = \mathbb{N}$ and that $S_\bullet \subseteq M_\bullet$ be arbitrary subsets. Let $B_1 \underset{\text{def}}{=} S_1$ and inductively define $B_{i+1} \underset{\text{def}}{=} S_{i+1} \cap \mu_{i,i+1}^{-1}(B_i)$. Then $B_\bullet$ is an inverse system of subsets and $\bigcap_i \mu_i^{-1}(B_i) = \bigcap_i \mu_i^{-1}(S_i)$. Had I been an arbitrary partial order then the same conclusion would have held by defining $B_i \underset{\text{def}}{=} S_i \cap \bigcap_{h\leq i} \mu_{hi}^{-1}(S_h)$ for each index i.

Limit of a System of Subsets

The following well-known result, which is proved later, shows in particular that the closure of every subset of M arises as the limit of some (not necessarily unique) inverse system of subsets.

Lemma 2.3.8 (Bourbaki 4.4)**.** If $S \subseteq M$ then $\overline{S} = \bigcap_{i\in I} \mu_i^{-1}\left(\overline{\mu_i(S)}\right) = \varprojlim \overline{\mu_i(S)}$. In particular, if S is closed in M then $S = \varprojlim \mu_i(S) = \varprojlim \overline{\mu_i(S)}$.

The following lemma 2.3.9 contains a simple sufficient condition for a subset of M to arise as the limit of an inverse system of subsets whose mention seems to have been omitted from the literature (where observe in particular that hypotheses for $\Longleftarrow$ impose no conditions on the collection $S_\bullet$ of sets). In the last part of the proof of lemma 2.3.9 we demonstrate the use of some of the lemmata and definitions that will soon be introduced.

Lemma 2.3.9. If $S \subseteq M$ then S arises as the limit of some inverse system of subsets of Sys_M if and only if $S = \bigcap_{i\in I} \mu_i^{-1}(S_i)$ for some subsets $S_\bullet \subseteq M_\bullet$. Furthermore, if $S_\bullet$ is an inverse system of subsets of Sys_M then $\varprojlim S_\bullet = \bigcap_{i\in I} \mu_i^{-1}(S_i)$ and each S_i contains $\mu_i(S)$.

Proof. If $(S_i, \mu_{ij}|_{S_j})$ is an inverse system of subsets then recall that $\varprojlim(S_i, \mu_{ij}|_{S_j}) = \bigcap_{i\in I} \mu_i^{-1}(S_i)$ and it is clear that each S_i must contain $\mu_i(S)$. So suppose that $S = \bigcap_{i\in I} \mu_i^{-1}(S_i)$ for some subsets S_i where it is again clear that each S_i must contain $\mu_i(S)$. Note that although these $S_\bullet$ were not assumed to form an inverse system of subsets we may, by applying lemma 2.4.1, assume without loss of generality that these $S_\bullet$ are decreasing (def. 2.4.2) and as shown in lemma 2.4.5 below, $\left(S_\bullet, \mu_{ij}|_{S_j}\right)$ will then necessarily form an inverse system of subsets so that consequently, as was just shown, $\varprojlim(S_i, \mu_{ij}|_{S_j}) = \bigcap_i \mu_i^{-1}(S_i)$, which is just S. ∎

Surprisingly, the following corollary 2.3.10 seems to have been omitted from the literature.

Corollary 2.3.10. Let $(Z, h_\bullet)$ be a cone into Sys_M with limit $h : Z \to M$ and let $S_\bullet \subseteq M_\bullet$. Then $h^{-1}\left(\bigcap_i \mu_i^{-1}(S_i)\right) = \bigcap_i h_i^{-1}(S_i)$ and if in addition $S_\bullet$ forms an inverse system of subsets then $\mathrm{Sys}_Z \underset{\text{def}}{=} (h_\bullet^{-1}(S_\bullet), \mathrm{In}_{ij}, I)$, where each $\mathrm{In}_{ij} : h_j^{-1}(S_j) \to h_i^{-1}(S_i)$ is the natural inclusion, is an inverse system whose limit is $h^{-1}\left(\varprojlim S_\bullet\right)$ (with the natural inclusions as projections), where we can write this conclusion more provocatively as

$$\left(\varprojlim h_i\right)^{-1}\left(\varprojlim S_i\right) = \varprojlim\left(h_i^{-1}(S_i)\right)$$

Consequently,

(1) If a subset $S \subseteq M$ arises as the limit of some inverse system of subsets then $h^{-1}(S) = \cap h_\bullet^{-1}(\mu_\bullet(S))$. In particular, if $m \in M$ then $h^{-1}(m) = \cap h_\bullet^{-1}(\mu_\bullet(m))$.

(2) If $m \in M$ then $m \in \operatorname{Im} h \iff \varprojlim h_\bullet^{-1}(\mu_\bullet(m)) \neq \varnothing$.

Proof. For the first claim observe that for any $z \in Z$,

$$z \in h^{-1}\left(\bigcap_i \mu_i^{-1}(S_i)\right) \iff \forall i \in I,\ h_i(z) = \mu_i(h(z)) \in S_i \iff z \in \bigcap_i h_i^{-1}(S_i)$$

so assume that $S_\bullet$ forms an inverse system of subsets. That Sys_Z forms an inverse system is easy to see so let $S = \varprojlim S_\bullet$ and observe that $\varprojlim \mathrm{Sys}_Z = \bigcap_i h_i^{-1}(S_i)$. Since $S \underset{\text{def}}{=} \varprojlim S_\bullet = \bigcap_i \mu_i^{-1}(S_i)$ it follows that $h^{-1}(S) = \bigcap_i h_i^{-1}(S_i) = \varprojlim \mathrm{Sys}_Z$.

If S is a subset of M that arises as the limit of an inverse system of subsets then by lemmata 2.4.5 and 2.3.9 we have $S = \varprojlim \mu_i(S)$ from which the equality $h^{-1}(S) = \bigcap_i h_i^{-1}(\mu_i(S))$ follows immediately. ■

Example 2.3.11. Let $(\mathbb{R}^{\mathbb{N}}, \mathrm{Pr}_{\leq\bullet})$ be the limit of $\mathrm{Sys}_{\mathbb{R}^{\mathbb{N}}} = (\mathbb{R}^i, \mathrm{Pr}_{\leq ij}, \mathbb{N})$ and let S be the span of all $(1,0,0,\ldots), \ldots, (\{0\}^i, 1, 0, 0, \ldots), \ldots$. Clearly, $\mathrm{Pr}_{\leq i}(S) = \mathbb{R}^i$ for all $i \in \mathbb{N}$ so that S is dense in $\mathbb{R}^{\mathbb{N}}$ but S is not closed in $\mathbb{R}^{\mathbb{N}}$ since $(1,1,\ldots) \notin S$. In particular, the cone $\mathrm{Pr}_{\leq\bullet}\big|_S : S \to \mathbb{R}^{\bullet}$ that results from restricting the canonical projections to S is $\mathrm{Pr}_{\leq\bullet}$-surjective but its limit, which is the natural inclusion $\mathrm{In}_S^{\mathbb{R}^{\mathbb{N}}} : S \to \mathbb{R}^{\mathbb{N}}$, is not surjective.

We will now show that an open subset of a limit need not arise as the limit of any inverse system of subsets. One particularly important consequence of this is that we will occasionally have to treat subsets of M arising as inverse limits of subsets separately from open subsets of M.

Example 2.3.12. Let $M_i = \mathbb{R}^i$ for each $i \in \mathbb{N}$ so that $(\mathbb{R}^{\mathbb{N}}, \mathrm{Pr}_i) = \varprojlim(\mathbb{R}^i, \mathrm{Pr}_{ij}, \mathbb{N})$ where $\mathrm{Pr}_{ij} : \mathbb{R}^j \to \mathbb{R}^i$ and $\mathrm{Pr}_i : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^i$ the canonical projections onto the first i coordinates. Let $U = \mathbb{R}^{\mathbb{N}} - \{\{0\}^{\mathbb{N}}\}$, where $\{0\}^{\mathbb{N}} = (0,0,\ldots) \in \mathbb{R}^{\mathbb{N}}$, so that U is open in $\mathbb{R}^{\mathbb{N}}$. For each $i \in \mathbb{N}$, let $s^i \underset{\mathrm{def}}{=} (0,\ldots,0,1,1,\ldots) \in U$ where the first i elements are 0 and the rest are 1 and let $\{0\}^i = (0,\ldots,0)$ denote the 0 vector in $\mathbb{R}^i$. Then $\{0\}^i = \mathrm{Pr}_i(s^i) \in \mathrm{Pr}_i(U)$ for all i so that $\{0\}^{\mathbb{N}} \in \bigcap_i \mathrm{Pr}_i^{-1}(\mathrm{Pr}_i(U))$, which implies that $\bigcap_i \mathrm{Pr}_i^{-1}(\mathrm{Pr}_i(U)) = \mathbb{R}^{\mathbb{N}}$. If U did arise as an inverse system of sets, say $U = \bigcap_i \mathrm{Pr}_i^{-1}(S_i)$ for some subsets $S_i \subseteq M_i$, then by lemma 2.3.9 we'd necessarily have $U = \bigcap_i \mathrm{Pr}_i^{-1}(\mathrm{Pr}_i(U))$. Thus the open set U is not the limit of any inverse system of subsets of $\mathrm{Sys}_{\mathbb{R}^{\mathbb{N}}}$. ■

Example 2.3.13. We can greatly generalize the preceding example 2.3.12: Suppose that Sys_M is a surjective inverse system directed by $\mathbb{N}$ and let $S \subseteq M$ be any $\mu_\bullet$-surjective subset of M that does not necessarily equal to M. If S was the limit of an inverse system of subsets, say $S = \varprojlim\left(S_i, \mu_{ij}\big|_{S_j}\right)$, then by 2.3.9 $M_i = \mu_i(S) \subseteq S_i$ and $S = \bigcap_i \mu_i(S_i)$ so that $S = M$. Thus, such a set S is the limit of an inverse system of subsets $\iff S = M$. ■

Relationships Between a Subset and its Projections

This section is devoted to analyzing relationships between a subset $S \subseteq M$ and its projections $\mu_\bullet(S)$ and although a reader who skips this section will likely still be able to follow the development of the theory of promanifolds, many of this section's results, particularly corollary 2.4.8, will be used without comment throughout the rest of this paper.

Increasing and Decreasing Representations of Subsets

Reducing questions about the limit M down to questions about the $M_\bullet$ is one of primary methods of studying of limits. For these ends, the following lemma will be useful since it shows, in particular, that we can always express open and closed subsets of M in terms of subsets of $M_\bullet$ that are guaranteed to have certain properties.

Lemma 2.4.1. Suppose $U = \bigcup_{i \in I} \mu_i^{-1}(V_i)$ for some sets $V_\bullet \subseteq M_\bullet$ and let $U_i = \bigcup_{h \leq i} \mu_{hi}^{-1}(V_i)$ for each index i. Then for each index $i \in I$,

$$U = \cap \mu_\bullet^{-1}(U_\bullet) \quad \text{ and } \quad U_i = \bigcup_{h \leq i} \mu_{hi}^{-1}(U_i)$$

Similarly, if $C = \bigcap_i \mu_i^{-1}(D_i)$ for some sets $D_\bullet \subseteq M_\bullet$ and if we let $C_i = \bigcap_{h \leq i} \mu_{hi}^{-1}(D_i)$ for each index i then

$$C = \cap \mu_\bullet^{-1}(C_\bullet) \quad \text{ and } \quad C_i = \bigcap_{h \leq i} \mu_{hi}^{-1}(C_i)$$

Proof. A straightforward check. ∎

Definition 2.4.2. If $J \subseteq I$, $S_\bullet = (S_j)_{i \in J}$, and $(Z, h_\bullet)$ is a cone into Sys_M then we will call

$$\bigcap_J h_\bullet^{-1}(S_\bullet) \underset{\text{def}}{=} \bigcap_{j \in J} h_j^{-1}(S_j) \quad \left(\text{resp. } \bigcup_J h_\bullet^{-1}(S_\bullet) \underset{\text{def}}{=} \bigcup_{j \in J} h_j^{-1}(S_j)\right)$$

the $h_\bullet$*-intersection (resp.* $h_\bullet$*-union) induced by* $S_\bullet$ and we will say that $S_\bullet$ *is an* $h_\bullet$*-representation* of this resulting set. Any such subset of Z will be referred to as an $h_\bullet$*-intersection (resp.* $h_\bullet$*-union).* A subset of Z will be called $h_\bullet$*-representable* if it is either an $h_\bullet$-intersection or an $h_\bullet$-union. An $h_\bullet$-representation $S_\bullet$ of an $h_\bullet$-representable subset of Z will be called an *open/closed/compact/etc. representation (of the set)* if each set in $S_\bullet$ has this property.

If $S_\bullet = (S_j)_{j \in J}$ satisfies

$$S_j = \bigcup_{\substack{h \leq j \\ h \in J}} \mu_{hj}^{-1}(S_h) \qquad \text{(resp. } S_j = \bigcap_{\substack{h \leq j \\ h \in J}} \mu_{hj}^{-1}(S_h))$$

for all $j \in J$ then we will say that is $S_\bullet$ is (Sys_M-)*increasing* (resp. *decreasing*). If $(M, \mu_\bullet)$ is a limit of Sys_M and $S = \bigcup_J \mu_\bullet^{-1}(S_\bullet)$ (resp. $S = \bigcap_J \mu_\bullet^{-1}(S_\bullet)$) where $S_\bullet$ is an increasing (resp. decreasing) $\mu_\bullet$-representation of S then we may summarize this situation by saying that $S = \bigcup_J \mu_\bullet^{-1}(S_\bullet)$ *(resp.* $S = \bigcap_J \mu_\bullet^{-1}(S_\bullet)$*) is increasing (resp. decreasing)* or that it *is an increasing ($\mu_\bullet$-)union* (resp. *decreasing ($\mu_\bullet$-)intersection*).

We may omit writing $\mu_\bullet$ if it is understood or write Sys_M in its place while if the indexing set J is omitted in the notation $\bigcap_J \mu_\bullet^{-1}(S_\bullet)$ or $\bigcup_J \mu_\bullet^{-1}(S_\bullet)$ then the indexing set will either be clear from context or otherwise assumed to be all of I. ∎

Remark 2.4.3. It is straightforward to verify that the following are equivalent:

(1) $S_\bullet = (S_j)_{j \in J}$ is increasing (resp. decreasing).

(2) $(M_j \setminus S_j)_{j \in J}$ is decreasing (resp. increasing).

(3) For all $h, j \in J$, if $h \leq j$ then $\mu_{hj}^{-1}(S_h) \subseteq S_j$ (resp. $\mu_{hj}(S_j) \subseteq S_h$).

Example 2.4.4. Looking ahead to the definition of locally cylindrical maps (def. 6.1.5), if $F : M \to N$ is locally cylindrical on a promanifold M then $\operatorname{ODom}_\bullet F = (\operatorname{ODom}_i F)_{i=1}^\infty$ is an increasing open $\mu_\bullet$-representation of M.

If $S_\bullet$ is a $\mu_\bullet$-representation of a subset $S \subseteq M$ is then the next lemma shows that among the benefit of know that $S_\bullet$ is an increasing or decreasing representation of S is that, just as with directed inverse systems, we need only to consider cofinal collections of $S_\bullet$, which would not necessarily be possible otherwise (e.g. consider, for instance, $S = \bigcup_{i \in I} \mu_i^{-1}(S_i)$ with $S_{i_0} \cap \operatorname{Im} \mu_{i_0} \neq \varnothing$ for some index i_0 and all other S_i equal to $\varnothing$).

Lemma 2.4.5. If $U = \bigcup_{i \in I} \mu_i^{-1}(U_i)$ is an increasing union of sets then for all indices i and for any $J \subseteq I$ cofinal in I,

$$\mu_i^{-1}(U_i) = \bigcup_{h \leq i} \mu_h^{-1}(U_h) \quad \text{ and } \quad U = \bigcup_{j \in J} \mu_j^{-1}(U_j)$$

Similarly if $C = \bigcap_{i \in I} \mu_i^{-1}(C_i)$ is a decreasing intersection of sets then $C = \bigcap_{j \in J} \mu_j^{-1}(C_j)$ and $\mu_i^{-1}(C_i) = \bigcap_{h \leq i} \mu_h^{-1}(C_h)$ for every $i \in I$. Furthermore, $\mu_{ij}(C_j) \subseteq C_i$ for all $i \leq j$ in I so that in particular, the $C_\bullet$ form an inverse system of subsets.

Proof. A straightforward check. ■

Set Relations

To minimize redundant computations, we compile in the remainder of this subsection some facts and set relations, which were written in such a way that they may aid in applying part part 6 of proposition 3.2.1. Some of the relations (e.g. $\bigcap_i \mu_i^{-1}(\mu_i(C)) = C$) are well-known, most have at least one direction that is either well-known or immediately seen, while the remaining relations have longer proofs that the reader may verify.

Lemma 2.4.6. Let $i \leq j \leq k$ and $S_j \subseteq M_j$. Then $\mu_k\big(\mu_j^{-1}(S_j)\big) = \mu_{jk}^{-1}(S_j) \cap \operatorname{Im} \mu_k$ and $\mu_i\big(\mu_j^{-1}(S_j)\big) = \mu_{ij}(S_j \cap \operatorname{Im} \mu_j)$. Furthermore, for any $S \subseteq M$ we have $\mu_i(S) = \bigcap_{h \leq i} \mu_{hi}^{-1}(\mu_h(S))$.

Lemma 2.4.7. Suppose $U = \bigcup_i \mu_i^{-1}(U_i)$ (resp. $C = \bigcap_i \mu_i^{-1}(C_i)$) is an increasing union (resp. decreasing intersection) of sets. Then

$$U_i \cap \operatorname{Im} \mu_i = \bigcap_{i \leq j} \mu_{ij}(U_j \cap \operatorname{Im} \mu_j) = \bigcap_{i \leq j} \mu_{ij}(U_j) \cap \operatorname{Im} \mu_i \quad \text{ and } \quad C_i = \bigcup_{i \leq j} \mu_{ij}(C_j)$$

$$\mu_i(U) = \bigcup_{i\le j} \mu_{ij}(U_j \cap \operatorname{Im} \mu_j) \quad \text{but} \quad \mu_i(C) \subseteq \bigcap_{i\le j} \mu_{ij}(C_j) \quad \text{with equality if } C_\bullet = \mu_\bullet(C)$$

and also,

$$U = \bigcup_i \mu_i^{-1}(M_i \setminus \mu_i(M \setminus U)) \quad \text{and} \quad C = \bigcap_i \mu_i^{-1}(\mu_i(C))$$

where the intersection for C is decreasing and the union for U is increasing.

Furthermore, for any index j, the following are equivalent:

(1) $\mu_j(U) \subseteq U_j$

(2) $\mu_j(U) = \operatorname{Im} \mu_j \cap U_j$

(3) $U = \mu_j^{-1}(U_j)$

and in this case for all $k \geq j$ we'll have $U = \mu_k^{-1}(U_k) = \mu_k^{-1}(\mu_k(U))$ where if μ_k is surjective then we'll also have $\mu_k(U) = U_k = \mu_{jk}^{-1}(U_j)$.

Corollary 2.4.8. Suppose all $\mu_\bullet$ are surjective, all $M_\bullet$ are topological spaces, and let $S \subseteq M$.

(1) If all μ_{ij} are open maps then so too are all μ_i.

(2) If S can be expressed as $S = \bigcap_i \mu_i^{-1}(D_i)$ for some subsets $D_\bullet \subseteq M_\bullet$ (not necessarily closed) and each $\mu_i(S)$ is closed then S is necessarily closed.

(3) S is relatively compact in $M \iff$ each $\mu_i(S)$ is relatively compact in M_i.

(4) If all $\mu_\bullet$ are continuous open maps and $R_\bullet \subseteq M_\bullet$ consists of meager sets with all but countably many of these sets empty, then $\bigcup_i \mu_i^{-1}(R_i)$ is meager in M.

Proof. Statement (1) is immediate from $\mu_i(U) = \bigcup_{i\le j} \mu_{ij}(U_j)$, where $U = \bigcup_i \mu_i^{-1}(U_i)$ with U_i is an arbitrary open subset of M_i, while (2) is immediate from $\mu_i(C) \subseteq \bigcap_{i\le j} \mu_{ij}(\mu_j(C))$. If all $\mu_i(C)$'s are compact, they are all closed so C is closed. Since $C = \varprojlim \mu_i(C)$ is an inverse limit of compact sets C is compact so statement (3) follows immediately. Statement (4) follows immediately from the fact that the preimage of a nowhere dense subset under a continuous open map is nowhere dense. ∎

Remark 2.4.9. An explicit example where the conclusion of part (2) of corollary 2.4.8 due to there not being any $D_\bullet \subseteq M_\bullet$ such that $S = \bigcap_{i\in\mathbb{N}} \mu_i^{-1}(D_i)$ can be found in example 2.3.11.

Lemma 2.4.10. Let $(U^\alpha)_{\alpha\in A}$ be a collection of subsets of M indexed by A. Suppose that for each $\alpha \in A$, $U^\alpha = \bigcup_i \mu_i^{-1}(U_i^\alpha)$ is an increasing union. Then $\bigcup_{\alpha\in A} U^\alpha = \bigcup_i \mu_i^{-1}\left(\bigcup_{\alpha\in A} U_i^\alpha\right)$ is also an increasing union.

Proof. Since for each $\alpha \in A$ the U_i^α are increasing we have for each i, $U_i^\alpha = \bigcup_{h\leq i} \mu_{hi}^{-1}\left(U_h^\alpha\right)$. So

$$\bigcup_{h\leq i} \mu_{hi}^{-1}\left(\bigcup_\alpha U_h^\alpha\right) = \bigcup_{h\leq i}\bigcup_\alpha \mu_{hi}^{-1}(U_h^\alpha) = \bigcup_\alpha \bigcup_{h\leq i} \mu_{hi}^{-1}(U_h^\alpha) = \bigcup_\alpha U_i^\alpha$$

shows that the sets $\bigcup_{\alpha\in A} U_i^\alpha$ are increasing. ∎

Canonical Representation of Open and Closed Subsets of a Limit

We will now prove some additional topological properties of limits and find conditions that will allow all open (resp. closed) subsets of a limit to be expressed as the union (resp. intersection) of certain uniquely defined open (resp. closed) subsets system. We will describe some of the properties of these sets and as an application we will provide a characterization of regular open and regular closed subsets of a certain class of limits. Throughout this section we will assume that Sys_M is a surjective inverse system in Top over a directed index set $(I, \leq)$ with limit $(M, \mu_\bullet)$.

In general, any given open or closed subset of M has infinitely many $\mu_\bullet$-representation. From among all of these possible choices of $\mu_\bullet$-representations we will make the following choices that we will henceforth call canonical:

Definition 2.4.11. If U is an open (resp. C is a closed) subset of M then *the canonical open (resp. closed) $\mu_\bullet$-representation of* U (resp. C) is the I-indexed collection of sets

$$M_\bullet \setminus \overline{\mu_\bullet(M\setminus U)} \underset{\text{def}}{=} \left(M_i \setminus \overline{\mu_i(M\setminus U)}\right)_{i\in I} \quad \left(\text{resp.} \quad \overline{\mu_\bullet(C)} \underset{\text{def}}{=} \left(\overline{\mu_i(C)}\right)_{i\in I}\right)$$

The following lemma provides an expression for the closure of a set and also shows that for each index $i \in I$, $\overline{\mu_i(C)}$ is the smallest closed set that can be used in any expression of the form $C = \cap \mu_\bullet^{-1}(C_\bullet)$ with C_i closed in M_i and that $M_i \smallsetminus \overline{\mu_i(M \smallsetminus U)}$ is the largest open set that can be used in any expression of the form $U = \cup \mu_\bullet^{-1}(U_\bullet)$ with U_i open. It is in this sense that the canonical open (resp. closed) $\mu_\bullet$-representations of open (resp. closed) sets are the "best possible" open (resp. closed) sets that we can use in any expression of U (resp. C) of the form $U = \cup \mu_\bullet^{-1}(U_\bullet)$ (resp. $U = \cap \mu_\bullet^{-1}(C_\bullet)$) when attempting to deduce information about U (resp. C) from the properties of the $M_\bullet$ these sets.

Lemma 2.4.12. Assume that all $\mu_\bullet$'s are surjective, let $S \subseteq M$ be arbitrary and let $U, C \subseteq M$ be, respectively, open and closed subsets of M. Let $U_\bullet$ (resp. $C_\bullet$) be the canonical open (resp. closed) $\mu_\bullet$-representation of U (resp. C) (def. 2.4.11). Then

$$C = \bigcap_i \mu_i^{-1}\left(\overline{\mu_i(C)}\right) = \bigcap_i \mu_i^{-1}(\mu_i(C))$$

$$\overline{S} = \varprojlim \overline{\mu_i(S)} = \bigcap_i \mu_i^{-1}\left(\overline{\mu_i(S)}\right) = \bigcap_i \mu_i^{-1}\left(\mu_i\left(\overline{S}\right)\right) = \bigcap_i \mu_i^{-1}\left(\overline{\mu_i\left(\overline{S}\right)}\right)$$

$$U = \bigcup_i \mu_i^{-1}\left(M_i \smallsetminus \overline{\mu_i(M \smallsetminus U)}\right) = \bigcup_i \mu_i^{-1}(M_i \smallsetminus \mu_i(M \smallsetminus U))$$

with the intersections for C decreasing and the unions for U increasing. If $D_i \in \mathrm{Closed}(M_i)$ is such that $C \subseteq \mu_i^{-1}(D_i)$ then $C_i \subseteq D_i$ and if V_i is an open subset of M_i such that $\mu_i^{-1}(V_i) \subseteq U$ then $V_i \subseteq U_i$. Furthermore,

$$\overline{\mu_i(C)} = \overline{\bigcap_{i \le j} \mu_{ij}(C_j)} = \bigcap_{i \le j} \overline{\mu_{ij}(C_j)} \quad \text{and} \quad \overline{U_i} = M_i \smallsetminus \mathrm{Int}\left(\overline{\mu_i(M \smallsetminus U)}\right)$$

and letting $E_i = \mathrm{Int}(M_i \smallsetminus \mu_i(U)) = M_i \smallsetminus \overline{\mu_i(U)}$ gives us a representation of the exterior $\mathrm{Ext}(U) = \mathrm{Int}(M \smallsetminus U) = \bigcup_i \mu_i^{-1}(E_i)$ of U in M as an increasing union.

Proof. The equality $C = \bigcap_i \mu_i^{-1}(\mu_i(C))$ was proved in lemma 2.4.7 so

$$C = \overline{\bigcap_i \mu_i^{-1}(\mu_i(C))} \subseteq \bigcap_i \overline{\mu_i^{-1}(\mu_i(C))} \subseteq \bigcap_i \mu_i^{-1}\left(\overline{\mu_i(C)}\right)$$

where the last containment follows from the continuity of the $\mu_\bullet$. On the other hand, if $m \in \bigcap_i \mu_i^{-1}\left(\overline{\mu_i(C)}\right)$ then for any $i \in I$ and neighborhood $\mu_i(m) \in U_i \in \operatorname{Open}(M_i)$, we have $U_i \cap \mu_i(C) \neq \varnothing$ so $\mu_i^{-1}(U_i) \cap C \neq \varnothing$ and thus $m \in \overline{C} = C$. As is done in Bourbaki [11, I.4.4], this equality for C may also be written as $C = \varprojlim \overline{\mu_i(C)}$. The equalities for $\overline{S}$ and U now follow from this equality, lemma 2.4.7, and the fact that $\overline{\mu_i\left(\overline{S}\right)} = \overline{\mu_i(S)}$ for each i.

If $C = \bigcap_i \mu_i^{-1}(E_i)$ with all E_i closed (not necessarily in standard form) then $\mu_i(C) \subseteq E_i$ so that since E_i is closed, $\overline{\mu_i(C)} \subseteq E_i$. An analogous argument is used for the maximality of the U_i's. Note that

$$\overline{\mu_i(C)} \subseteq \overline{\bigcap_{i \leq j} \mu_{ij}(C_j)} \subseteq \bigcap_{i \leq j} \overline{\mu_{ij}(C_j)} = \bigcap_{i \leq j} \overline{\mu_{ij}\left(\overline{\mu_j(C)}\right)} = \bigcap_{i \leq j} \overline{\mu_{ij}(\mu_j(C))} = \bigcap_{i \leq j} \overline{\mu_i(C)} = \overline{\mu_i(C)}$$

so equality holds. Finally,

$$\operatorname{Int}\left(\overline{\mu_i(M \setminus U)}\right) = M_i \setminus \overline{M_i \setminus \overline{\mu_i(M \setminus U)}} = M_i \setminus \overline{U_i}$$

and

$$M_i \setminus \overline{\mu_i(M \setminus \operatorname{Int}(M \setminus U))} = M_i \setminus \overline{\mu_i(\overline{U})} = M_i \setminus \overline{\mu_i(U)} = E_i$$

which finishes the proof. ∎

Lemma 2.4.13. Let $U = \bigcup_{i \in I} \mu_i^{-1}(U_i)$ be increasing and suppose that all $\mu_\bullet$ are continuous open surjections. If $m \in M$ is such that for some index i, $\mu_i(m) \in \overline{U_i}$ then $m \in \overline{U}$. In particular,

$$U \subseteq \bigcup_i \mu_i^{-1}\left(\overline{U_i}\right) \subseteq \overline{U}$$

Proof. If $l \geq i$ then since $\mu_{il}(\mu_l(m)) = \mu_i(m) \in \overline{U_i}$ we have $\mu_l(m) \in \mu_{il}^{-1}\left(\overline{U_i}\right)$ and since

$\mu_{il}^{-1}(U_i) \subseteq U_l$ we have $\mu_l(m) \in \mu_{il}^{-1}\left(\overline{U_i}\right) = \overline{\mu_{il}^{-1}(U_i)} \subseteq \overline{U_l} \subseteq \overline{\mu_i(U)}$ so that $m \in \mu_l^{-1}\left(\overline{\mu_l(U)}\right)$. Since $I^{\geq i}$ is cofinal in I, we have $m \in \bigcap\limits_{l \geq i} \mu_l^{-1}\left(\overline{\mu_l(U)}\right) = \overline{U}$. ∎

Perfect Subsets and Isolated Points of Limits

For the following corollary of lemma 2.3.8, recall that in any space X, if $x \in X$ and $R \subseteq X$ then x being an isolated point of $\overline{R} = \mathrm{Cl}_X(R)$ implies that x is an isolated point of R, while the converse is always true if in addition X is T_1.

Corollary 2.4.14. Suppose Sys_M is T_1 and directed and $S \subseteq M$. If no $\mu_\bullet(S)$ has an isolated point (or equivalently no $S_\bullet \underset{\text{def}}{=} \mathrm{Cl}_{M_\bullet}(\mu_\bullet(S))$ has an isolated point) then neither does S and if there are cofinally many indices i such that S_i is a perfect subsets of M_i then $\mathrm{Cl}_M(S)$ is a perfect subset of M.

Proof. If no $\mu_\bullet(S)$ had an isolated point but m^0 was an isolated point of S then we can pick $i \in I$ and an open subset U_i of M_i such that $\{m^0\} = S \cap \mu_i^{-1}(U_i)$, which implies that $\mu_i(S) \cap U_i = \{\mu_i(m^0)\}$, giving a contradiction. Now if all S_i were perfect then no $\mu_\bullet(S)$ has an isolated point, which implies that neither S nor $\overline{S}$ has an isolated point. Since $\overline{S}$ is closed in M by lemma 2.3.8 and let $m^0 \in S$, it follows that $\overline{S}$ is perfect. ∎

The next example shows, in particular, that even if a subset $S \subseteq M$ is closed in M, it's still possible that no $\mu_i(S)$ is closed in M_i.

Example 2.4.15. A perfect subset S of $\mathbb{R}^{\mathbb{N}}$ such that each $\mathrm{Pr}_{\leq i}(S)$ is a non-closed subset of $\mathbb{R}^i$ with isolated points: Let $J = \left]-\frac{1}{2}, \frac{1}{2}\right[$, $f : J \to \mathbb{R}$ be $f(t) = \arctan\left(\frac{4}{\pi}t\right)$, and let $\Gamma(f) \subseteq \mathbb{R}^2$

denote the graph of f. For all $i > 1$, define $D_i \subseteq \mathbb{R}^i$ by

$$
\begin{aligned}
D_2 &= \Gamma(f) \\
D_3 &= (3) \times \Gamma(f) \underset{\text{def}}{=} \{(3, x, y) \mid (x, y) \in \Gamma(f)\} \\
D_4 &= (4, 0) \times \Gamma(f) \\
&\vdots \\
D_i &= (i, \{0\}^{i-3}) \times \Gamma(f) \underset{\text{def}}{=} \{(i, 0, \ldots, 0, x, y) \mid (x, y) \in \Gamma(f)\} \\
&\vdots
\end{aligned}
$$

and let $D^i = D_i \times \{0\}^{\mathbb{N}}$, where observe that D^i is closed in $\mathbb{R}^{\mathbb{N}}$. For each $i \in \mathbb{N}$, let $\operatorname{Pr}_{\leq i} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^i$ denote the canonical projection onto the first i coordinates. Since

$$
\operatorname{Pr}_1(D^l) = \begin{cases} J & \text{if } l = 2 \\ l & \text{if } l > 2 \end{cases}
$$

it follows that $(D^l)_{l=2}^{\infty}$ is locally finite and thus $S \underset{\text{def}}{=} \bigcup_{l=2}^{\infty} D^l$ is a closed subset of $\mathbb{R}^{\mathbb{N}}$.

Observe that $\operatorname{Pr}_1(S) = J \cup \mathbb{N}^{\geq 3}$ (resp. $\operatorname{Pr}_{\leq 2}(S) = D_2 \cup (3 \times J) \cup \{(j, 0) \mid j \geq 4\}$) is not closed in $\mathbb{R}^1$ (resp. $\mathbb{R}^2$). For each $i \geq 3$ and $l \in \mathbb{N}$, observe that

$$
\operatorname{Pr}_{\leq i}(D^l) = \begin{cases} \left\{l \times \{0\}^{i-1}\right\} & \text{if } l > i + 1 \\ (i+1) \times \{0\}^{i-2} \times J & \text{if } l = i + 1 \\ D_i & \text{if } l = i \\ D_l \times \{0\}^{l-i} & \text{if } l < i \end{cases}
$$

where all of these sets can be separated by pairwise disjoint open sets with $\operatorname{Pr}_{\leq i}(D^{i+1})$ not closed in $\mathbb{R}^i$, which implies that $\operatorname{Pr}_{\leq i}(S)$ is not closed in $\mathbb{R}^i$. Furthermore, observe that although S is perfect (since S is closed and each D^i is clearly perfect), no $\overline{\operatorname{Pr}_{\leq i}(S)}$ is perfect since each has infinitely many isolated points. ■

Regular Open and Regular Closed Subsets

Due to the frequency with which regular open and closed sets and their properties are used when studying smooth manifolds (often without remark) we will now provide an original characterization of when an open or closed subset of M is regular in terms of their canonical representations. We will henceforth use without comment the conclusions of lemma A.4.1, which describes some basic properties of continuous open maps.

Proposition 2.4.16. Assume that all bonding maps and projections are continuous open surjections. If $C \subseteq M$ is closed and $U \subseteq M$ is open then

(1) C is regular $\iff$ $\overline{\mu_i(C)} = \overline{\mu_i(\operatorname{Int}(C))}$ for each index i, in which case these sets are also regular.

(2) U is regular $\iff$ $U_i = M_i \setminus \overline{\mu_i(M \setminus U)}$ is regular for each index i.

Proof. (1): If C is regular then lemma A.4.1 shows that for all indices i, $\overline{\mu_i(C)} = \overline{\mu_i(\operatorname{Int}(C))}$ and that these are regular closed sets. Conversely, if $\overline{\mu_i(C)} = \overline{\mu_i(\operatorname{Int}(C))}$ for all indices i then the closure of $\operatorname{Int}(C)$ is $\overline{\operatorname{Int}(C)} = \bigcap_i \mu_i^{-1}\left(\overline{\mu_i(\operatorname{Int}(C))}\right) = \bigcap_i \mu_i^{-1}\left(\overline{\mu_i(C)}\right) = C$.

(2): Since U is a regular open subset of M if and only if $C \underset{\text{def}}{=} M \setminus U$ is a regular closed subset of M, we have that U is regular if for all indices i, $\overline{\mu_i(M \setminus U)} = \overline{\mu_i(\operatorname{Int}(M \setminus U))}$, which happens if and only if $M_i \setminus \overline{\mu_i(\operatorname{Int}(M \setminus U))} = U_i$ for all indices i. If U is regular then for all indices i,

$$\begin{aligned} M_i \setminus \overline{\mu_i(\operatorname{Int}(M \setminus U))} &= M_i \setminus \overline{\operatorname{Int}\left(\overline{\mu_i(M \setminus U)}\right)} \quad \text{by lemma A.4.1} \\ &= \operatorname{Int}\left(M_i \setminus \operatorname{Int}\left(\overline{\mu_i(M \setminus U)}\right)\right) = \operatorname{Int}\left(\overline{M_i \setminus \overline{\mu_i(M \setminus U)}}\right) = \operatorname{Int}\left(\overline{U_i}\right) \end{aligned}$$

so $U_i = M_i \setminus \overline{\mu_i(\operatorname{Int}(M \setminus U))} = \operatorname{Int}\left(\overline{U_i}\right)$, which shows that U_i is regular. Conversely, if all U_i

are regular then

$$M_i \setminus \overline{\mu_i(M \setminus U)} = U_i = \operatorname{Int}\left(\overline{U_i}\right) = M_i \setminus \overline{\operatorname{Int}\left(\overline{\mu_i(M \setminus U)}\right)}$$

so $\overline{\mu_i(M \setminus U)} = \overline{\operatorname{Int}\left(\overline{\mu_i(M \setminus U)}\right)}$, which says exactly that all $\overline{\mu_i(M \setminus U)}$ are regular closed sets and implies that $M \setminus U$, and hence U, is regular. ∎

Corollary 2.4.17. If all connecting maps of Sys_M are continuous open surjections then all $M_\bullet$ being semiregular (def. A.0.5((f))) implies that the same is true of $M = \varprojlim \operatorname{Sys}_M$.

Topologies of Limits of Directed Systems

The material in this section is not necessary for understanding promanifolds. We will prove in this section some additional results about the topologies of projective limits of directed systems in Top.

Connectedness and Local Connectedness

This section is devoted to finding an original sufficient condition for a limit to be connected and locally connected.

Example 2.5.1. A limit of (resp. path) connected spaces that's not (resp. path) connected: Let B denote the open unit ball in $\mathbb{R}^2$ and pick any sequence $p^\bullet$ of points in $B \cap (\{0\} \times \mathbb{R})$. For all for all $i \leq j$ in $\mathbb{N}$, let

$$M_i \underset{\text{def}}{=} \{(x,y) \in X \,|\, -1/i < x < 1/i\} \setminus \{p^1, \ldots, p^i\}$$

and let $\mu_{ij} : M_j \to M_i$ denote the natural inclusion. Then $\varprojlim \operatorname{Sys}_M$ is $(\{0\} \times]-1,1[) \setminus \{p^i \,|\, i \in \mathbb{N}\}$, which is not connected. Furthermore, if $p^\bullet$ enumerates the set $B \cap (\{0\} \times \mathbb{Q})$ then the resulting limit will not have *any* non-empty connected neighborhoods.

Observe that despite the above system's simplicity and the fact that all $M_\bullet$ were connected manifolds with all bonding maps smooth embeddings, this limit still failed to be connected.

Lemma 2.5.2. Let $K \subseteq M$ be a compact connected set and let $U_\bullet = (U_i)_{i\in I}$ be an increasing sequence of open subsets of $M_\bullet$ (def. 2.4.2) such that $U \underset{\text{def}}{=} \cup \mu_\bullet^{-1}(U_\bullet)$ contains K. Then there exists an index i_0 such that $\mu_i(K) \subseteq U_i$ for all $i \geq i_0$.

Proof. Pick indices $i_1 \leq \ldots \leq i_N$ such that $K \subseteq \mu_{i_1}^{-1}(U_{i_1}) \cup \cdots \cup \mu_{i_N}^{-1}(U_{i_N})$ and let $i_0 = i_N$. For any $m \in K$ pick an index i_a such that $\mu_{i_a}(m) \in U_{i_a}$ and note that $\mu_{i_0}(m) \in U_{i_0}$ since $\mu_{i_a,i_0}^{-1}(U_{i_a}) \subseteq U_{i_0}$. Thus $\mu_{i_0}(K) \subseteq U_{i_0}$ which implies that $\mu_i(K) \subseteq U_i$ for all $i \geq i_0$. ■

Corollary 2.5.3. Let $S \subseteq M$ be connected, $s^0 \in S$, and let $U_\bullet = (U_i)_{i\in I}$ be an increasing sequence of open subsets $M_\bullet$ such that $U \underset{\text{def}}{=} \cup \mu_\bullet^{-1}(U_\bullet)$ contains S. If some U_i contains $\mu_i(s^0)$ then let C_i denote the connected component of U_i containing it. Suppose that for all $s \in S$ there exists a compact connected subset $K_s \subseteq M$ containing s and s^0. Then $S \subseteq \underset{j\in J}{\cup} \mu_i^{-1}(C_j)$ for any cofinal subset $J \subseteq I$ and furthermore, if $s \in S$ is such that $\mu_i(s) \in C_i$ then $\mu_j(s) \in C_j$ for all $j \geq i$.

Proof. Let i_0 be any index such that $\mu_{i_0}(s^0) \in U_{i_0}$ For each $s \in S$ there exists some index $i(s) \geq i_0$ such that $\mu_j(K_s) \subseteq U_j$ for all $j \geq i(s)$ so that in particular $\mu_j(s) \in \mu_j(K_s) \subseteq C_j$ for all $j \in J$ with $j \geq i(s)$. Since $s \in S$ was arbitrary it follows that $S \subseteq \underset{j\in J}{\cup} \mu_i^{-1}(C_j)$. ■

Lemma 2.5.4. Let $(M, \mu_\bullet) = \varprojlim(M_\bullet, \mu_{ij}, \mathbb{N})$ in Top with $M_\bullet$ consisting of Hausdorff locally compact spaces. Let $K \subseteq M$ be compact and suppose there exists some index i_0 and some compact connected set $C_{i_0} \subseteq M_{i_0}$ containing $\mu_{i_0}(K)$. Assume that for all $i \geq i_0$, whenever $C_i \subseteq M_i$ is a compact connected set containing $\mu_i(K)$ and U_i is an open subset of M_i contains C_i then there exists some compact connected set that is contained in $\mu_{i,i+1}^{-1}(U_i)$ and contains $\mu_{i+1}(K)$. Then there exists a compact connected set C in M that contains K.

Proof. If $K = \varnothing$ then we're done so assume otherwise. Since C_{i_0} is compact we can pick a relatively compact open set U_{i_0} containing C_{i_0}. Suppose we've constructed for $k \geq i_0$

compact connected sets $C_{i_0}, \ldots, C_k$ and also relatively compact open sets $U_{i_0}, \ldots, U_k$ such that $\mu_{l-1,l}(U_l) \subseteq U_{l-1}$ for all $i_0 < l \leq k$ and $\mu_l(K) \subseteq C_l \subseteq U_l$ for all $i_0 \leq l \leq k$. Let C_{k+1} be a compact connected set containing $\mu_{k+1}(K)$ that is contained in $\mu_{k,k+1}^{-1}(U_k)$. Since M_{k+1} is locally compact we can select a relatively compact open set U_{k+1} containing C_{k+1} such that $\overline{U_{k+1}} \subseteq \mu_{k,k+1}^{-1}(U_k)$.

For each index i let D_i denote the closure of $\bigcup_{j \geq i} \mu_{ij}(C_j)$ in M_i. Observe that for all $i \leq j$, $\mu_{ij}(C_j)$ is a connected set contained in U_i with $\mu_i(K) \subseteq C_i \cap \mu_{ij}(C_j)$ so that $\bigcup_{j \geq i} \mu_{ij}(C_j)$, and hence its closure, is connected. Since D_i is contained in the compact set $\overline{U_i}$ it is also compact. By construction, $\mu_{ij}(D_j) \subseteq D_i$ for all $i \leq j$ so that we can take the limit $D \underset{\text{def}}{=} \varprojlim D_i$ of this inverse system of subsets, which by proposition 2.2.1 is a compact connected subset of M that contains K. ■

If one desires to find conditions that necessitate connectedness or local connectedness of a limit then in light of proposition 2.2.1(4d) it is only natural to consider the hypotheses of the following proposition.

Proposition 2.5.5. A sufficient condition for connectedness and local connectedness: Let $(M, \mu_\bullet) = \varprojlim(M_\bullet, \mu_{ij}, \mathbb{N})$ in Top with $M_\bullet$ consisting of Hausdorff locally compact and locally connected spaces. Assume that for each index $i \in \mathbb{N}$, whenever U_i is a connected open subset of M_i then the same is true of $\mu_{i,i+1}^{-1}(U_i)$ in M_{i+1}.

Then W_i is connected $\iff$ $\mu_i^{-1}(W_i)$ is connected, where $i \in \mathbb{N}$ and the open subset W_i of M_i are arbitrary. In particular, every point of M has a neighborhood basis consisting of connected $\mu_\bullet$-basic open sets and if all M_i are connected then so is M.

Remark 2.5.6. If every point of M has a neighborhood basis consisting of connected $\mu_\bullet$-basic open sets then of course M is locally connected but it is not clear that the converse is true. Hence, the above conclusion may be stronger than mere local connectedness. Also, the requirement on the bonding maps is weaker than requiring them to be monotone.

Proof. First recall ([12, p. 254]) that since each M_i is Hausdorff, locally connected, and locally compact then whenever m_i and m_i' belong to the same connected component of an open set U then there exists some compact connected subset of U containing them. For the non-trivial direction of the characterization assume that W_i is connected. Our hypotheses allows us to conclude that for all $m^0, m^1 \in \mu_i^{-1}(W_i)$, there is a compact connected set K containing m^0 and m^1 from which it follows that $\mu_i^{-1}(W_i)$ is connected. The rest of the statements in the proposition are immediate. ■

Corollary 2.5.7. Suppose $(M, \mu_\bullet) = \varprojlim (M_\bullet, \mu_{ij}, \mathbb{N})$ in Top with $M_\bullet$ consisting of Hausdorff locally compact and locally connected spaces and with each bonding map $\mu_{i,i+1}$ monotone and open. Then M is connected, locally connected with a basis of connected basic open sets, and for any $i \in \mathbb{N}$ and any open subset W_i of M_i, W_i is connected $\iff$ $\mu_i^{-1}(W_i)$ is connected.

Path-connectedness and Local Path-connectedness

This section is devoted to finding an original sufficient condition for a limit to be path-connected and path-locally connected.

Lemma 2.5.8. Let $(M, \mu_\bullet) = \varprojlim (M_\bullet, \mu_{ij}, \mathbb{N})$ in Top. If all μ_{ij} have the extension lifting property from A to Z then so do all $\mu_\bullet$. The converse is true if each μ_i can lift all morphisms on A.

Proof. Fix an index i_0 and let $f : A \to M$ and $H : Z \to M_{i_0}$ be morphisms such that $\mu_{i_0} \circ f = H\big|_A$. For all indices $j \geq i_0$, let $f_j = \mu_j \circ H : Z \to M_j$. Let $\widetilde{H}_{i_0} = H : Z \to M_{i_0}$. Having found morphisms $\widetilde{H}_{i_0}, \ldots, \widetilde{H}_k$, where $\widetilde{H}_i : Z \to M_i$, such that $f_i = \widetilde{H}_i\big|_A$ and $\mu_{ij} \circ \widetilde{H}_j = \widetilde{H}_i$ for all $i_0 \leq i \leq j \leq k$, let $\widetilde{H}_{k+1} : Z \to M_{k+1}$ be a $\mu_{k,k+1}$-lift of $\widetilde{H}_k$ extending $f_{k+1} : Z \to M_{k+1}$. By construction, $\left(Z, \widetilde{H}_\bullet\right)$ is an inverse cone into Sys_M so let $\widetilde{H} = \varprojlim \widetilde{H}_\bullet : Z \to M$. Since for all $i \geq i_0$ we have

$$\mu_i \circ f = f_i = \widetilde{H}_i\big|_A = \mu_i \circ \widetilde{H}\big|_A$$

it follows that $f = \widetilde{H}\big|_A$ so that $\widetilde{H}$ is an extension of f and since in particular $\mu_{i_0} \circ \widetilde{H} = H$ it is also a μ_{i_0}-lift of H. ■

Definition 2.5.9. Let Z be a space and $A \subseteq Z$. If each $\mu_{ij} : M_j \to M_i$ is a Serre fibration then we will say that Sys_M is *(Serre) fibrated* and if a $(M, \mu_\bullet)$ is the limit cone of some fibrated inverse system then we will say that $(M, \mu_\bullet)$ *is fibrated*. Similarly, we will say that an inverse system $(M_\bullet, \mu_{ij})$ *has the extension lifting property (from A to Z)* (resp. *is Z-fibrated*, *is r-fibrated*, *lifts maps from Z*) if all μ_{ij} have this property and we will also say that a limit cone $(M, \mu_\bullet)$ has this property if it is the limit of some inverse system with this property.

Even if we only consider projective systems whose objects consist of smooth connected manifolds with all bonding maps monotone smooth submersions and fibrations then there is no guarantee that, when given some map between the limits of such systems, that the inverse system of subsets induced by the range of this map will have these same properties. In particular, the image of a map that arises as the limit of a surjective inverse system morphism (def. 3.0.1) is only guaranteed to be a dense subset of its codomain. Following this next definition, we will provide sufficient conditions, weaker than those mentioned above, under which a limit will be locally path-connected.

Definition 2.5.10. Let $\pi : X \to Y$ be a map between topological spaces, $S \subseteq Y$, and $y_0 \in Y$. We will say that π *weakly preserves the*

- *path components of S* if for every open $V \in \operatorname{Open}(Y)$ that contains S and every path component P_S of S, $\pi^{-1}(P_S)$ is a subset of a path component of $\pi^{-1}(V)$.
- *path-connectedness of y_0* if π weakly preserves the path components of $\{y_0\}$.
- *path-connectedness of points* if π weakly preserves the path-connectedness of each $y \in Y$.

If we say that a system *weakly preserves the path-connectedness of points* then we mean that each of its connecting maps weakly preserves the path-connectedness of points. ■

Example 2.5.11. If $\pi : \mathbb{R}^2 \setminus (\mathbb{Z} \times \mathbb{Z}) \to \mathbb{R}$ is the canonical projection onto the x-axis then although π is not monotone, it does weakly preserves the path-connectedness of points.

Proposition 2.5.12. Suppose that $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, \mathbb{N})$ is 1-fibrated, weakly preserves the path-connectedness of points, and is first countable. If $U_i \in \mathrm{Open}(M_i)$ for some index i then $\mu_i^{-1}(U_i)$ is path-connected if and only if the same is true of U_i. In particular, if all M_i's are locally path-connected (resp. path-connected) then M has a neighborhood basis consisting of path-connected $\mu_\bullet$-basic open sets around each of its points (resp. M is path-connected).

Furthermore, if in addition all M_i are smooth manifolds and all μ_{ij} are smooth submersions then we can construct a topological embedding $\gamma : [0,1] \to M$ between any two distinct points of M such that all $\mu_i \circ \gamma\big|_{[0,1[} : [0,1[\to M_i$ are smooth and for any $0 < r < 1$ there exists some index i such that $\mu_i \circ \gamma\big|_{[0,r]} : [0,r] \to M_i$ is a smooth embedding.

Remark 2.5.13. As with proposition 2.5.5, the first of the above conclusions may be stronger than mere local path-connectedness.

Proof. For the non-trivial direction assume that U_i is path-connected and observe that whether or not $\mu_i^{-1}(U_i)$ is path-connected is not affected by replacing Sys_M's indexing set (i.e. $\mathbb{N}$) with $\mathbb{N}^{\geq i}$. Thus, we may assume without loss of generality that $i = 1$ and let i become a free symbol. Let $m^0, m^1 \in \mu_1^{-1}(U_1)$ and for all indices $l \in \mathbb{N}$, let $m_l^k = \mu_l(m^k)$ $(k = 0,1)$. Let $O_\bullet = (O_i)_{i \in \mathbb{N}}$ be a $\mu_\bullet$-neighborhood basis at m^1 and let $0 < r_1 < r_2 < \cdots < 1$ be any increasing sequence such that $\lim\limits_{j \to \infty} r_i = 1$. If all M_i are smooth manifolds then we will write the additional details needed to construct the claimed path with the additional properties in parentheses.

Let $\gamma_1 : [0,1] \to U_1$ be a (smooth) embedding from m_0^0 to m_1^0. Pick $t_1 \in]r_2, 1[$ such that $\gamma_i([t_1, 1]) \subseteq O_i$. Now suppose that we've defined (smooth) paths $\gamma_1, \ldots, \gamma_i$ and real numbers $t_1 < \cdots < t_i$ such that for all $h \leq l \leq i$

(1) $\gamma_h : [0,1] \to M_h$ is a (smooth) embedding from m_h^0 to m_h^1,

(2) $\mu_{hl} \circ \gamma_l\big|_{[0,t_h]} = \gamma_h\big|_{[0,t_h]}$,

(3) $r_{h+1} < t_h < 1$, and

(4) $\mu_{hl}(\gamma_l([t_h, 1])) \subseteq O_h$.

Since $\mu_{i,i+1}(m_{i+1}^0) = m_i^0$ and $\mu_{i,i+1}$ is a 1-fibration there exists a (smooth) $\mu_{i,i+1}$ lift of γ_i, call it $\eta : [0,1] \to M_{i+1}$ with $\eta(0) = m_{i+1}^0$. Observe that since γ_i is a (smooth) embedding so is η. Let W_{i+1} denote the path component of $\mu_{i,i+1}^{-1}(O_i)$ containing $\eta(1)$ and let $1 > \widehat{t_{i+1}} > \max\{r_{i+2}, t_i\}$ be such that $\eta\big([\widehat{t_{i+1}}, 1]\big) \subseteq W_{i+1}$ Since $\mu_{i,i+1}$ weakly preserves the path-connectedness of points, $\mu_{i,i+1}^{-1}(m_i^1)$ is contained in W_{i+1} (along with $\eta\big(\widehat{t_{i+1}}\big)$) so there exists some (smooth) embedding $\rho : [\widehat{t_{i+1}}, 1] \to \mu_{i,i+1}^{-1}(O_i)$ from $\eta\big(\widehat{t_{i+1}}\big)$ to m_{i+1}^1. Let

$$\begin{aligned} \gamma_{i+1} : [0,1] &\longrightarrow M_{i+1} \\ t &\longmapsto \begin{cases} \eta(t) & \text{if } 0 \leq t \leq \widehat{t_{i+1}} \\ \rho(t) & \text{if } \widehat{t_{i+1}} \leq t \leq 1 \end{cases} \end{aligned}$$

and observe that $\mu_{i,i+1}(\gamma_{i+1}([t_i, 1])) \subseteq \mu_{i,i+1}\big(\mu_{i,i+1}^{-1}(O_i)\big) \subseteq O_i$. If $h < i$ then since

$$\mu_{h,i+1} \circ \gamma_{i+1}\big|_{[t_h, t_i]} = \mu_{h,i+1} \circ \eta\big|_{[t_h, t_i]} = \mu_{hi} \circ \gamma_i\big|_{[t_h, t_i]}$$

it follows that

$$\begin{aligned} \mu_{h,i+1}(\gamma_{i+1}([t_h, 1])) &\subseteq \mu_{h,i+1}(\gamma_{i+1}([t_h, t_i])) \cup \mu_{h,i+1}(\gamma_{i+1}([t_i, 1])) \\ &\subseteq \mu_{hi}(\gamma_i([t_h, t_i])) \cup \mu_{hi}(O_i) \\ &\subseteq O_h \cup \mu_{hi}(O_i) \quad \text{by (4) with } l = i \end{aligned}$$

Since $O_i \subseteq \mu_{hi}^{-1}(O_h)$ (by definition of $O_\bullet$) it follow that $\mu_{h,i+1}(\gamma_{i+1}([t_h, 1])) \subseteq O_h$ for all $h = 1, \ldots, i$. Also, for $h \leq i$ we have

$$\mu_{h,i+1} \circ \gamma_{i+1}\big|_{[0, t_h]} = \mu_{hi} \circ \gamma_i\big|_{[0, t_h]} = \gamma_h\big|_{[0, t_h]}$$

Clearly, γ_{i+1} is a topological embedding (and it is a smooth embedding on $[0,1] \smallsetminus \left\{\widehat{t_{i+1}}\right\}$, however, it may not be smooth at $t = \widehat{t_{i+1}}$. In this case pick a chart in W_{i+1} centered at $\gamma_{i+1}\left(\widehat{t_{i+1}}\right)$ and let $0 < \epsilon < \frac{1}{4} \min\left\{\widehat{t_{i+1}} - t_i, 1 - \widehat{t_{i+1}}\right\}$ be sufficiently small so that, after defining $J = [\widehat{t_{i+1}} - \epsilon, \widehat{t_{i+1}} + \epsilon]$, the set $\gamma_{i+1}(J)$ is contained in this chart. We may replace $\gamma_{i+1}\big|_J$ with a smooth arc contained in W_i such that the resulting map, denoted by $\widehat{\gamma_{i+1}} : [0,1] \to M_{i+1}$, is a smooth arc that agrees with γ_{i+1} outside of J and is at every point a smooth embedding. Observe that $\widehat{\gamma_{i+1}}\left([\widehat{t_{i+1}}, 1]\right) \subseteq W_i$ so that $\mu_{i,i+1}(\gamma_{i+1}([t_i, 1])) \subseteq U_i$ and with the same computations as before we can conclude that $\mu_{h,i+1}(\widehat{\gamma_{i+1}}([t_h, 1])) \subseteq O_h$. Also, $\gamma_{i+1}\big|_{[\widehat{t_{i+1}}-\epsilon,1]}$ played no role in the last of the above computations so the same conclusion holds for $\widehat{\gamma_{i+1}}$. Thus, we may replace γ_{i+1} with $\widehat{\gamma_{i+1}}$ and continue with the proof.) Since γ_{i+1} is continuous and $\gamma_{i+1}(1) = m^1_{i+1}$ we can pick t_{i+1} such that $\widehat{t_{i+1}} < t_{i+1} < 1$ and $\gamma_{i+1}([t_{i+1}, 1]) \subseteq O_{i+1}$. This completes the inductive step.

Fix i and observe that if $i \leq j \leq k$ then $\mu_{ij} \circ \gamma_j \circ \big|_{[0,t_j]}$

$$\begin{aligned}
\mu_{ik} \circ \gamma_k \circ \big|_{[0,t_j]} &= \mu_{ij} \circ \mu_{jk} \circ \gamma_k\big|_{[0,t_j]} \\
&= \mu_{ij} \circ \gamma_j\big|_{[0,t_j]} \qquad \text{using } (j,k) \text{ for } (h,l) \text{ in } (2)
\end{aligned}$$

Thus for any index i, we can define a map $\eta_i : [0,1[= \bigcup_{j\geq i} [0, t_i] \to M_i$ by $\eta_i\big|_{[0,t_j]} \underset{\text{def}}{=} \mu_{ij} \circ \gamma_j\big|_{[0,t_j]}$ as j ranges over all integers greater than i. Given any $0 < t < 1$ pick an index $j \geq i$ such that $t < t_j$ and observe that $\eta_i\big|_{[0,t_j]} = \mu_{ij} \circ \gamma_j \circ \big|_{[0,t_j]}$ is continuous (and smooth) map so that η_i is continuous (and smooth since all μ_{ij} are smooth). Clearly, $\eta_\bullet$ is a cone into Sys_M so let $\eta : [0,1[\to M$ be its limit. Let h be any index and observe that if $t \in [t_h, 1[$ then by picking $l \geq h$ such that $t_j > t$ it follows from (4) that

$$\mu_h(\eta(t)) = \mu_{hl}(\eta_l(t)) = \mu_{hl}(\gamma_l(t)) \in O_h$$

so that $\eta([t_h, 1)) \subseteq \mu_h^{-1}(O_h)$. Since $O_\bullet$ is a neighborhood basis for M at m^1 it follows that if we extend η to $[0,1]$ by defining $\eta(1) = m^1$ then η will be continuous. Also, since all bonding maps are continuous it follows that if we extend each η_i to $[0,1]$ by defining $\eta_i(1) \underset{\text{def}}{=} \mu_i(\eta(1))$ then η_i will be continuous.

Fix $0 < r < 1$ and pick an index i such that $r < t_i$. By definition, we have $\eta_i\big|_{[0,t_i]} = \gamma_i \circ \big|_{[0,t_i]}$ which is a (smooth) embedding by (1) and since $\mu_i \circ \eta\big|_{[0,t_i]} = \eta_i\big|_{[0,t_i]}$ it follows that $\eta\big|_{[0,t_i]}$, and hence $\eta\big|_{[0,r]}$, is a topological embedding, which implies that $\eta\big|_{[0,1[}$ is a topological embedding. If $m^1 = \eta(r)$ for some $r \in [0,1[$ then let i be such that $r < t_i$ and observe that this implies that $m_i^1 = \mu_i(\eta(r)) = \gamma_i(r)$ so that γ_i was not injective, a contradiction. This shows that $\eta : [0,1] \to M$ is injective and thus a homeomorphism. ∎

Corollary 2.5.14. Let $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, \mathbb{N})$ be a projective system in Top whose spaces are first countable, Hausdorff, and locally path-connected and whose bonding maps are continuous, open, 1-fibrations all of whose fibers are path-connected. Then M is locally path-connected and a basic open subset $\mu_i^{-1}(U_i)$ of M is path-connected if and only if U_i is path-connected.

Example 2.5.15. Let $M_1 = \mathbb{R}$, $M_2 = M_1 \times \mathbb{R} \setminus \left(\frac{1}{2!}\right)(\mathbb{Z}^2)$, $M_3 = M_2 \times \mathbb{R} \setminus \left(\frac{1}{3!}\right)(\mathbb{Z}^3)$, and now inductively define all other M_i by letting $M_{i+1} = M_i \times \mathbb{R} \setminus \frac{1}{(i+1)!}(\mathbb{Z}^{i+1})$, where $\frac{1}{(i+1)!}(\mathbb{Z}^{i+1}) \underset{\text{def}}{=} \left\{\left(\frac{p_1}{(i+1)!}, \ldots, \frac{p_{i+1}}{(i+1)!}\right) \,\middle|\, p_1, \ldots, p_{i+1} \in \mathbb{Z}\right\}$. For each index i let $\mu_{i,i+1} : M_{i+1} \to M_i$ denote the canonical projection onto the first coordinate where observe that this is a smooth surjective submersion. By proposition 2.5.12 its limit M is path-connected, locally path-connected, and furthermore a basic open subset $\mu_i^{-1}(U_i)$ is path-connected if and only if U_i is path-connected.

Partial $\mu_\bullet$-Sections

The definitions and results in this section are original and were largely motivated by consideration of example 2.5.20. The definitions in this section will be used to find an original sufficient condition, found in proposition 2.5.22, for a limit to be connected.

Definition 2.5.16. By a *partial section of a map* $f : X \to Y$ we mean a continuous X-valued map σ defined on a subset Y that satisfies $f \circ \sigma = \mathrm{Id}_{\mathrm{Dom}\,\sigma}$ and by a *local section* we mean a partial section defined on an open subset of Y. When we say that a partial section is *through* a point $x \in X$ then we mean that x is contained in the image of this map.

Definition 2.5.17. Let $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, I)$ be a directed inverse system with limit $(M, \mu_\bullet)$. By a *direct system of subsets of* Sys_M we mean an a collection $D_\bullet \subseteq M_\bullet$ such that $D_i \subseteq \mu_{ij}(D_j)$ for all $i \leq j$ in I. By a *direct system of partial sections of* Sys_M we mean a direct system $\operatorname{Sys}_\sigma = (D_\bullet, \sigma_i^j, I)$ such that $D_\bullet$ is a direct system of subsets of Sys_M and $\sigma_i^j : D_i \to D_j$ is a partial section of $\mu_{ij} : M_j \to M_i$ for all $i \leq j$ in I in which case we may say that $\left(\sigma_i^j\right)$ (rather than $\operatorname{Sys}_\sigma$) is a direct system of partial sections of Sys_M. We will say that a direct system of partial sections $\left(\sigma_i^j\right)$ of Sys_M is *dense* if for each index i, $\bigcup_{i \leq j \leq k} \mu_{ij}\left(\operatorname{Dom} \sigma_j^k\right)$ is dense in M_i and that it is *connected* if every $\operatorname{Dom} \sigma_i^k$ is connected.

Observe that for each index i, $\sigma_i^\bullet = \left(\sigma_i^j\right)_{j \in I, j \geq i}$ forms a cone into $\operatorname{Sys}_M\big|_{J \geq i}$ whose limit map, which we'll denote by $\sigma_i \underset{\text{def}}{=} \varprojlim \sigma_i^\bullet : D_i \to M$, is a partial section of $\mu_i : M \to M_i$. We'll denote this induced collection of maps $\sigma_\bullet \underset{\text{def}}{=} (\sigma_i)_{i \in I}$ by $\varprojlim \operatorname{Sys}_\sigma$ and call them *the limit morphisms of* $\operatorname{Sys}_\sigma$ *into* $(M, \mu_\bullet)$ or *the partial* $\mu_\bullet$*-sections induced by* $\operatorname{Sys}_\sigma$. If in any of these definitions we omit writing the word "partial" or write "global" in its place then we mean that in addition, the domain of each σ_i^j is all of M_i. ∎

Definition 2.5.18. Suppose that $(M, \mu_\bullet)$ is a limit of a directed system $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, I)$, let $\sigma_\bullet = (\sigma_i)_{i \in \mathbb{N}}$ be a collection of M-valued maps such that $\operatorname{Dom} \sigma_\bullet \subseteq M_\bullet$, and for each index i let $\sigma_i^\bullet = \mu_\bullet \circ \sigma_i$. Say that $\sigma_\bullet$ is a *partial section of* $\mu_\bullet$ or a *partial* $\mu_\bullet$*-section* if σ_i is a partial section of $\mu_i : M \to M_i$ for every index i. Call $\sigma_\bullet$ a *cocone of partial sections of* $\mu_\bullet$ or a *cocone of partial* $\mu_\bullet$*-sections* if it is a partial section of $\mu_\bullet$, $\operatorname{Dom} \sigma_\bullet$ forms a direct system of subsets of Sys_M, and it satisfies any of the following (clearly) equivalent *consistency conditions*:

(1) $\sigma_j \circ \sigma_i^j = \sigma_i$ for all $i \leq j$,

(2) $\sigma_j^k \circ \sigma_i^j = \sigma_i^k$ for all $i \leq j \leq k$,

(3) the system $\operatorname{Sys}_\sigma \underset{\text{def}}{=} (\operatorname{Dom} \sigma_\bullet, \sigma_i^j, I)$, called *the canonical direct system induced by* $\sigma_\bullet$ *and* $\mu_\bullet$, forms a direct system of partial sections of Sys_M,

where if $I = \mathbb{N}$ then we may add to this list:

(4) $\sigma_i^j = \sigma_{j-1}^j \circ \cdots \circ \sigma_i^{i+1}$ for all $i < j$,

(5) $\sigma_i^{i+2} = \sigma_{i+1}^{i+2} \circ \sigma_i^{i+1}$ for all i.

We will say that a partial section σ of $\mu_\bullet$ is dense (resp. connected) if the same is true of $\operatorname{Sys}_\sigma$ and we will call $\sigma_\bullet$ *partial sectional coordinates of* $\mu_\bullet$ or *partial sectional* $\mu_\bullet$*-coordinates* if $\sigma_\bullet$ is a cocone of partial sections of $\mu_\bullet$ and $\operatorname{Dom}\sigma_\bullet$ forms an inverse system of subsets of Sys_M. If in any of these definitions we omit writing the word "partial" or write "global" in its place then we mean that in addition, $\operatorname{Dom}\sigma_\bullet = M_\bullet$.

Remark 2.5.19. The above definition's assignment $\sigma_\bullet \mapsto \operatorname{Sys}_\sigma$ clearly establishes a bijective correspondence between partial sections of $\mu_\bullet$ and direct systems of partial sections of Sys_M whose inverse is given by $\operatorname{Sys}_\sigma \mapsto \varprojlim \operatorname{Sys}_\sigma$.

Example and Definition 2.5.20. Let $\operatorname{Sys}_{\mathbb{R}^\mathbb{N}} = (\mathbb{R}^\bullet, \operatorname{Pr}_{\leq i,j}, \mathbb{N})$ be the canonical inverse system of projections and let $(\mathbb{R}^\mathbb{N}, \operatorname{Pr}_{\leq \bullet})$ be its canonical limit. For all $i \leq j$ in $\mathbb{N}$, let $\sigma_i^j : \mathbb{R}^i \to \mathbb{R}^j$ and $\sigma_i : \mathbb{R}^i \to \mathbb{R}^\mathbb{N}$ be the respective canonical insertions $\sigma_i^j(x) = (x, 0, \ldots, 0)$ and $\sigma_i(x) = (x, 0, 0, \ldots)$. Then $\operatorname{Sys}_\sigma \underset{\text{def}}{=} \left(\mathbb{R}^\bullet, \sigma_i^j, \mathbb{N}\right)$ is a direct system of partial sections of $\operatorname{Sys}_{\mathbb{R}^\mathbb{N}}$ and $\sigma_\bullet$ are sectional coordinates of $\operatorname{Pr}_{\leq \bullet}$, both of which we'll henceforth canonically associate to $\operatorname{Sys}_{\mathbb{R}^\mathbb{N}}$ and $(\mathbb{R}^\mathbb{N}, \operatorname{Pr}_{\leq \bullet})$. Clearly, $\sigma_\bullet$ are the partial $\operatorname{Pr}_{\leq \bullet}$-sections induced by $\operatorname{Sys}_\sigma$ and conversely, $\operatorname{Sys}_\sigma$ is the canonical direct system induced by $\sigma_\bullet$ and $\operatorname{Pr}_{\leq \bullet}$.

Example 2.5.21. Given any $m \in M$, the maps $\sigma_i : \{\mu_i(m)\} \to \{m\}$ form partial sectional coordinates of $\mu_\bullet$.

The following proposition gives basic properties of partial sections of limit cones and provides a sufficient condition for a limit to be connected.

Proposition 2.5.22. Let $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, I)$ be a directed inverse system in Top with a limit cone $(M, \mu_\bullet)$ and let $\sigma_\bullet$ be a partial section of $\mu_\bullet$.

(1) $\operatorname{Im}\sigma_i \subseteq \operatorname{Im}\sigma_j$ for all $i \leq j$ in I.

(2) If $\sigma_\bullet$ is dense in Sys_M then $\cup \operatorname{Im}\sigma_\bullet$ is dense in M.

(3) $\operatorname{Im}\sigma_i = \varprojlim_j \operatorname{Im}\sigma_i^j$ for each fixed index i (where the bonding maps are $\mu_{jk}\big|_{\operatorname{Im}\sigma_i^k} : \operatorname{Im}\sigma_i^k \to \operatorname{Im}\sigma_i^j$).

(4) If cofinally many $\operatorname{Dom}\sigma_i$ are connected then so is $\cup \operatorname{Im}\sigma_\bullet$ where if in addition $\sigma_\bullet$ is dense in Sys_M then M is connected.

Proof. (3) and (2) are immediate and (1) follows from definition 2.5.18(1). To prove (4), let J be cofinal in I, suppose that $\operatorname{Dom}\sigma_j$ is connected for all $j \in J$, and let $D = \bigcup_{i \in I} \operatorname{Im}\sigma_i$ where since I is directed, part (1) implies that $D = \bigcup_{j \in J} \operatorname{Im}\sigma_j$. Assume that $D \neq \varnothing$ so there exists some $j_0 \in J$ such that $\operatorname{Im}\sigma_j \neq \varnothing$. Observe that for all $j \in J$ the set $\operatorname{Im}\sigma_j$ is connected since each $\sigma_i : D_i \to M$ is an embedding so that $D = \bigcup_{\substack{j \in J \\ j \geq j_0}} \operatorname{Im}\sigma_j$ is an increasing union of connected sets and thus is connected, which implies that $\operatorname{Cl}_M(D)$ is connected. ■

The following well-known results follow immediately.

Corollary 2.5.23. Solenoids and $\mathbb{R}^{\mathbb{N}}$ are connected.

The following proposition establishes the notation that will henceforth be used when dealing partial sections of the limit cone of a system directed by $\mathbb{N}$ and lists the results of some common computations related to them that would otherwise be repeatedly rederived later.

Proposition and Notation 2.5.24. Suppose that $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, \mathbb{N})$ is a surjective system indexed by the integers. Let $D_\bullet$ be an $\mathbb{N}$-indexed collection of subsets of $M_\bullet$ and for each index i let $\sigma_i^{i+1} : D_i \to M_{i+1}$ be a morphism such that $\operatorname{Im}\sigma_i^{i+1} \subseteq D_{i+1}$ and $\mu_{i,i+1} \circ \sigma_i^{i+1} = \operatorname{Id}_{D_i}$. For all $h \leq i < j$ in I, define $\sigma_i^h = \mu_{hi}\big|_{D_i} : D_i \to M_h$ and

$$\sigma_i^j = \sigma_{j-1}^j \circ \cdots \circ \sigma_i^{i+1} : D_i \to M_j$$

Then $\left(\sigma_i^j\right)$ forms a direct system of sections of Sys_M so that $(\sigma_i^l)_{l=1}^{\infty}$ forms a cone into Sys_M whose limit we will denote by

$$\sigma_i \underset{\text{def}}{=} \varprojlim_l \sigma_i^l : D_i \to M$$

Furthermore,

(1) $\mu_{jk} \circ \sigma_i^k = \sigma_i^j$ and $\mu_j \circ \sigma_i = \sigma_i^j$ for all $i, j, k \in \mathbb{N}$ with $j \leq k$ (where observe that i is free).

- so in particular, $\mu_{ij} \circ \sigma_i^j = \mathrm{Id}_{D_i}$ and $\mu_h \circ \sigma_i = \mu_{hi}\big|_{D_i}$ for all $h \leq i \leq j$.

(2) for all $i, j, k \in \mathbb{N}$ with $i \leq j$,

$$\sigma_j \circ \sigma_i^j = \sigma_i \text{ and } \sigma_j^k \circ \sigma_i^j = \sigma_i^k$$

- so in particular, $\sigma_i \circ \mu_{ij} = \sigma_j$ on $\mathrm{Im}\, \sigma_i^j$ for $i \leq j$.

Proof. It is immediately verified that $\left(\sigma_i^j\right)$ forms a direct system of sections of Sys_M. Throughout the rest this proof h, i, j, and k will represent indices in I.

(1) If $i \leq j = k$ then $\sigma_i^j = \mathrm{Id}_{M_j} \circ \sigma_i^j = \mu_{jk} \circ \sigma_i^k$ is immediate and if it's been proved for k with $k \geq j \geq i$ then

$$\mu_{j,k+1} \circ \sigma_i^{k+1} = \mu_{jk} \circ \mu_{k,k+1} \circ \sigma_k^{k+1} \circ \sigma_i^k = \mu_{jk} \circ \sigma_i^k = \sigma_i^j$$

so that it's true for $k+1$ and hence $\sigma_i \underset{\text{def}}{=} \varprojlim_l \sigma_i^l : D_i \to M$ is well-defined. By the universal property of limits we have $\mu_j \circ \sigma_i = \sigma_i^j$ for all $i < j$ so that for all $h \leq i$

$$\mu_h \circ \sigma_i = \mu_{h,i+1} \circ \mu_{i+1} \circ \sigma_i = \mu_{h,i+1} \circ \sigma_i^{i+1} = \mu_{hi} \circ \mu_{i,i+1} \circ \sigma_i^{i+1} = \mu_{hi}\big|_{D_i} \underset{\text{def}}{=} \sigma_i^h$$

which shows $\mu_j \circ \sigma_i = \sigma_i^j$ regardless of whether $i \leq j$ or $j \leq i$. So for all i, j, k with $j \leq k$

$$\mu_{jk} \circ \sigma_i^k = \mu_{jk} \circ \mu_k \circ \sigma_i = \mu_j \circ \sigma_i = \sigma_i^j$$

regardless of whether, $i \leq j \leq k$, $j \leq i \leq k$, or $j \leq k \leq i$, which proves (1).

(2) We will assume through out this part of the proof that $i \leq j$. If $j \leq k$ then it is clear that $\sigma_j^k \circ \sigma_i^j = \sigma_i^k$ which shows that

$$\mu_k \circ \left(\sigma_j \circ \sigma_i^j\right) = \sigma_j^k \circ \sigma_i^j = \sigma_i^k = \mu_k \circ \sigma_i$$

so that since k was arbitrary, by uniqueness of limits of cones we must have $\sigma_i = \sigma_j \circ \sigma_i^j$. If however, $k \leq j$ then $\sigma_j^k = \mu_{kj}\big|_{D_j}$ so that by part (1)

$$\sigma_j^k \circ \sigma_i^j = \mu_{jk} \circ \sigma_i^j = \sigma_i^j$$

Now, $\sigma_i \circ \mu_{ij} \circ \sigma_i^j = \sigma_i \circ \mathrm{Id}_{D_i} = \sigma_j \circ \sigma_i^j$ so that $\sigma_i \circ \mu_{ij} = \sigma_j$ on $\operatorname{Im} \sigma_i^j$, which completes the proof of (2). ■

Local Compactness

Proposition 2.5.25. For any $m \in M$, the following are equivalent:

(1) M is locally compact at m.

(2) Sys_M is locally eventually compact at m.

(3) Sys_M is cofinally compact at m.

In particular, if each M_i is locally compact at $\mu_i(m)$ then M is locally compact at $m \iff \mathrm{Sys}_M$ is cofinally proper at m.

Proof. (1) $\implies$ (2): Pick any relatively compact $m \in U \in \operatorname{Open}(M)$ and then pick any index i and any subset $U_i \in \operatorname{Open}(M_i)$ such that $m \in \mu_i^{-1}(U_i) \subseteq U$. Observe that for any $j \geq i$ we have $\mu_j^{-1}\left(\mu_{ij}^{-1}(U_i)\right) = \mu_i^{-1}(U_i) \subseteq U$ so that $\mu_{ij}^{-1}(U_i) \subseteq \mu_j(\mu_i^{-1}(U_i)) \subseteq \mu_j(U)$, which implies that $N_j \underset{\text{def}}{=} \overline{\mu_j(U)} = \mu_j(\overline{U})$ is a compact set containing $\mu_{ij}^{-1}(U_i)$. Thus $\mathrm{Sys}_{N_i} \underset{\text{def}}{=} \left(N_j, \mu_{ij}\big|_{N_j}, I^{\geq i}\right)$ is the desired system.

(2) $\implies$ (3) is immediate so assume that (3) holds. Then by definition there exists some cofinal subset $J \subseteq I$, some index j in J, and some neighborhood N_j of $\mu_j(m)$ in M_j such that $\operatorname{Sys}_{N_j} \underset{\text{def}}{=} \left(N_j, \mu_{ij}\big|_{N_j}, J^{\geq j}\right)$, the inverse system induced by N_j and J, is compact where recall that $N_k \underset{\text{def}}{=} \mu_{jk}^{-1}(N_j)$ for all $k \in J^{\geq j}$. Consider $N \underset{\text{def}}{=} \varprojlim \operatorname{Sys}_{N_j}$ as a subset of M and let O_j denote the interior of N_j in M_j. Observe that N is a compact set containing $\mu_j^{-1}(O_j)$, which is an open subset of M containing m so that (1) holds. ■

Chapter 3

Inverse System Morphisms

Definition 3.0.1. If we write $(F_\bullet, \lambda) : \mathrm{Sys}_M \to \mathrm{Sys}_N$ then we mean that λ is an order morphism and $F_\bullet$ is an $\mathrm{Dom}(\lambda)$-indexed collection of morphisms where if Sys_M and Sys_N are inverse (resp. direct) systems then λ has prototype $\lambda : A \to I$ (resp. $\lambda : I \to A$) and for each $d \in \mathrm{Dom}\,\lambda$, F_d has prototype $F_d : M_{\lambda(d)} \to N_d$ (resp. $F_d : M_d \to N_{\lambda(d)}$).

If Sys_M and Sys_N are inverse (resp. direct) systems then we will say that $(F_\bullet, \lambda) : \mathrm{Sys}_M \to \mathrm{Sys}_N$ is a *morphism of inverse (resp. direct) systems*, an *inverse (resp. direct) system morphism*, or simply a *morphism (from* Sys_M *to* Sys_N*)* if the following condition, called *the compatibility* or *consistency condition*, is satisfied:

$$F_a \circ \mu_{\lambda(a)\lambda(b)} = \nu_{ab} \circ F_b \qquad \text{for all } a \leq b \text{ in } A$$

$$\text{(resp.,} \quad \nu_{\lambda(i)}^{\lambda(j)} \circ F_i = F_j \circ \mu_i^j \qquad \text{for all } i \leq j \text{ in } I\text{)}$$

i.e. the following respective diagram must commute:

$$\begin{tikzcd}
M_{\lambda(a)} \arrow[d, "F_a"'] & M_{\lambda(b)} \arrow[l, "\mu_{\lambda(a)\lambda(b)}"'] \arrow[d, "F_b"] \\
N_a & N_b \arrow[l, "\nu_{ab}"']
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
M_i \arrow[r, "\mu_i^j"] \arrow[d, "F_i"'] & M_j \arrow[d, "F_j"] \\
N_{\lambda(i)} \arrow[r, "\nu_{\lambda(i)}^{\lambda(j)}"] & N_{\lambda(j)}
\end{tikzcd}$$

Inverse System Morphism (resp. Direct System Morphism)

If $\mathrm{Sys}_P = (P_\bullet, \pi_{rs}, R)$ is another inverse (resp. direct) system and $(G_\bullet, \sigma) : \mathrm{Sys}_N \to \mathrm{Sys}_P$ is a morphism then the composition $(G_\bullet, \sigma) \circ (F_\bullet, \lambda) : \mathrm{Sys}_M \to \mathrm{Sys}_P$ consists of the order morphism $\lambda \circ \sigma : R \to I$ (resp. $\sigma \circ \lambda : I \to R$) and the R-indexed (resp. I-indexed) morphisms $G_r \circ F_{\sigma(r)} : M_{\lambda(\sigma(r))} \to P_r$ (resp. $G_{\lambda(i)} \circ F_i : M_i \to P_{\sigma(\lambda(i))}$). ■

Example 3.0.2. Let Sys_M be a system and let J be any (not necessarily cofinal) subset of I. If Sys_M is an inverse system then the identity maps Id_{M_j} indexed by $j \in J$ together with the natural inclusion $\iota \underset{\text{def}}{=} \mathrm{In}_J^I : J \to I$ form an inverse system morphism $(\mathrm{Id}_{M_j}, \mathrm{In}_J^I) : \mathrm{Sys}_M \to \mathrm{Sys}_M\big|_J$ from $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, I)$ to $\mathrm{Sys}_M\big|_J = (M_i, \mu_{ij}, J)$. If Sys_M is instead a direct system then these same maps will instead form a direct system morphism $(\mathrm{Id}_{M_j}, \mathrm{In}_J^I) : \mathrm{Sys}_M\big|_J \to \mathrm{Sys}_M$.

Example and Definition 3.0.3. Given any object Z and any collection $h_\bullet$ of morphisms $h_i : Z \to M_i$ indexed by I, we can form the Id_I-ordered collection of maps $(h_\bullet, \mathrm{Id}_I) : Z_\bullet \to M_\bullet$ that we will henceforth refer to as *canonical*, where $Z_\bullet = (Z_i)_{i \in I}$ is defined by $Z_i \underset{\text{def}}{=} Z$. Saying that this canonical collection of maps forms an inverse system morphism $(h_\bullet, \mathrm{Id}_I)$: $\mathrm{ConstSys}_Z \to \mathrm{Sys}_M$ is equivalent to saying that the $h_\bullet$ is compatible with Sys_M (def. 2.1.14), where $\mathrm{ConstSys}_Z = (Z_\bullet, \mathrm{Id}_Z, I)$ is the constant inverse system.

Limits of System Morphisms

Definition 3.1.1. If $(F_\bullet, \lambda) : \mathrm{Sys}_M \to \mathrm{Sys}_N$ is a morphism of inverse (resp. direct) systems then by *the canonical cone (resp. cocone) induced by (or associated with)* $(F_\bullet, \lambda)$ *(and* $\mu_\bullet$ *(resp.* $\nu^\bullet$*))* we mean the cone $\left(M, \left(F_a \circ \mu_{\lambda(a)}\right)_{a \in A}\right)$ from M into Sys_N (resp. the cocone $(N, (\nu^{\lambda(i)} \circ F_i)_{i \in I})$ from Sys_M into N).

Definition 3.1.2. Suppose that $(F_\bullet, \lambda)$: $\mathrm{Sys}_M \to \mathrm{Sys}_N$ is a morphism of inverse (resp. direct) systems and that Sys_M and Sys_N have limits (resp. colimits) $(M, \mu_\bullet)$ and $(N, \nu_\bullet)$ (resp. colimits $(M, \mu^\bullet)$ and $(N, \nu^\bullet)$). By *the limit (resp. colimit) (morphism) of* $(F_\bullet, \lambda)$ we mean the limit morphism of the canonical cone (resp. canonical cocone) induced by $(F_\bullet, \lambda)$

and $\mu_\bullet$ (resp. $\nu^\bullet$). We will denote this morphism with notation that is analogous to the notation used for limits of cones (resp. cocones):

$$\varprojlim(F_\bullet, \lambda), \quad \varprojlim_{a \in A}(F_a, \lambda), \quad \text{or} \quad \varprojlim_{a \in A} F_{\lambda(\bullet)}$$

where in the case that the order morphism and indexing set is clear we will also use the notation

$$\varprojlim F_a, \quad \varprojlim F_\bullet, \quad \text{or simply} \quad F$$

(resp. for direct systems, we will reverse the arrow and possibly replace "lim" with "colim"). We may also sometimes use "limit" rather than "colimit" to refer to the colimit of a direct system morphism.

When we say that a map $\widehat{F} : M \to N$ *arises as the limit of an inverse system morphism (resp. direct system morphism) (from* Sys_M *to* Sys_N*)* we mean that there exists an inverse system morphism (resp. direct system morphism) $\left(\widehat{F}_\bullet, \widehat{\lambda}\right) : \mathrm{Sys}_M \to \mathrm{Sys}_N$ such that $\widehat{F}$ is the limit of $\left(\widehat{F}_\bullet, \widehat{\lambda}\right)$.

The limit map is the unique map $F : M \to N$ satisfying the following *compatibility condition*:

$$\nu_a \circ F = F_a \circ \mu_{\lambda(a)} \text{ for all } a \in A = \operatorname{Dom} \lambda$$
$$\Big(\text{resp.} \quad F \circ \mu^i = \nu^{\lambda(i)} \circ F_i \text{ for all } i \in I = \operatorname{Dom} \lambda\Big)$$

where in this case the following diagram will commute for all $a \leq b$ in $\operatorname{Dom} \lambda$:

$$\begin{tikzcd}
M_{\lambda(a)} \arrow[d, "F_a"] & M_{\lambda(b)} \arrow[l, "\mu_{\lambda(a)\lambda(b)}"'] \arrow[d, "F_b"] & M \arrow[l, "\mu_{\lambda(b)}"'] \arrow[d, "F"] \\
N_a & N_b \arrow[l, "\nu_{ab}"'] & N \arrow[l, "\nu_b"']
\end{tikzcd}$$

(resp. for direct systems, reverse the horizontal arrows and appropriately raise and/or relabel the indices of the objects and connecting morphisms).

If M and N are the canonical limits then the limit of $(F_\bullet, \lambda)$ is the map

$$\begin{array}{rrcl} \varprojlim(F_\bullet, \lambda): & M & \longrightarrow & N \subseteq \prod_{a \in A} N_a \\ & m = (m_i)_{i \in I} & \longmapsto & \left(F_a(m_{\lambda(a)})\right)_{a \in A} \end{array}$$

which we may write more succinctly as $F_\bullet \circ \mu_{\lambda(\bullet)} = \left(F_a \circ \mu_{\lambda(a)}\right)_{a \in A}$. ■

Remarks 3.1.3.

- It is straightforward to verify that $\varprojlim$ is a functor from the category of inverse systems of sets (resp. topological spaces, groups, topological groups) into this same category.
- It was *not* necessary for $\operatorname{Im} \iota$ to be cofinal in I in order for $\varprojlim(F_\bullet, \iota)$ to be defined, although having $\operatorname{Im} \iota$ be cofinal in I is sometimes useful and even required by many authors.
- Let $(Z, h_\bullet)$ be a cone in Sys_M, let $\operatorname{ConstSys}_Z = (Z_\bullet, \operatorname{Id}_Z, I)$ be the constant system, and recall that by considering each h_i as the map $h_i : Z_i = Z \to M_i$ we may consider this cone as the inverse system morphism $(h_\bullet, \operatorname{Id}_I) : \operatorname{ConstSys}_Z \to \operatorname{Sys}_M$. Since $(Z, (\operatorname{Id}_Z)_{i \in I}) = \varprojlim \operatorname{ConstSys}_Z$ and since the limit h of this inverse system morphism satisfies $\mu_i \circ h = h_i \circ \operatorname{Id}_{Z_i} = h_i$ we see that h is just the limit of the cone $(Z, h_\bullet)$.

Properties of Inverse System Morphisms and Their Limits

Part (6) the following lemma gives a means of determining whether or not a particular element of the codomain of a limit morphism belongs to its range. In general, the limit of open maps is not necessarily open so that part (1) is provides a partial apology for this by showing that the image of certain opens sets are still guaranteed to be open. Also, although the limit in Top of epimorphisms is again an epimorphism it is not in general true that the limit of surjections is a surjection. However, the image of the limit morphism will be dense in N and it is in this sense that, for any subset S of M, part (3) below provides a way of "approximating" $F(S)$ without requiring any knowledge of what $F(S)$ actually is.

Surprisingly, part (6) of lemma 3.2.1 appears to have gone unnoticed in the literature and so is presumed to be new. Parts (1) and (2) of this lemma are observations whose statements the author could not find references for, parts (4) and (5) are well-known, while part (3) is a generalization of its own well-known corollary, which is mentioned immediately after part (3)'s statement.

Proposition 3.2.1. Suppose $(M, \mu_\bullet)$ and $(N, \nu_\bullet)$ are, respectively, limits in Top of Sys_M and Sys_N and $(F_\bullet, \iota) : \mathrm{Sys}_M \to \mathrm{Sys}_N$ is an inverse system morphism in Top. Then the limit map $F : M \to N$ of $(F_\bullet, \iota)$ is continuous and:

(1) If $a \in A$, $i \underset{\text{def}}{=} \iota(a)$, and $S_i \subseteq M_i$ then $F(\mu_i^{-1}(S_i)) \subseteq \nu_a^{-1}(F_a(S_i)) \cap \operatorname{Im} F$ with equality if S_i is F_a-saturated.

- So in particular, if U_i is an F_a-saturated open set in M_i and $F_a(U_i)$ is open then $F(\mu_i^{-1}(U_i))$ is open in $\operatorname{Im} F$.

(2) If $\operatorname{Im}\big(F_a \circ \mu_{\iota(a)}\big)$ is dense in N_a for some index a then $\operatorname{Im} \nu_a$ is dense in N_a and $N = \varnothing \iff M = \varnothing \iff N_a = \varnothing$.

(3) If Sys_N is directed and $S \subseteq M$ then $F(S)$ is dense in $\bigcap\limits_{a \in A} \nu_a^{-1}\big(F_a(\mu_{\iota(a)}(S))\big) = \bigcap\limits_{a \in A} \nu_a^{-1}(\nu_a(F(S)))$.

- If in addition $\operatorname{Im}\big(F_a \circ \mu_{\iota(a)}\big)$ is dense in N_a for all $a \in A$ then $\operatorname{Im} F$ is dense in N.

(4) If $\iota : A \to I$ is cofinal and each $F_\bullet$ is an injection (resp. topological embedding) then so is $F : M \to N$.

If $\iota : A \to I$ cofinal and Sys_M (and consequently Sys_N) is directed then also:

(5) If each $F_\bullet$ is a bijection (resp. homeomorphism) then so is $F : M \to N$.

(6) (The preimage under the limit map is the limit of the preimages):

Suppose that $S_\bullet = (S_a)_{a \in A}$ is an inverse system of subsets of Sys_N with limit $S \subseteq N$. For all $a \leq b$ in A, let $M_a^S = F_a^{-1}(S_a) \subseteq M_{\iota(a)}$ and define

$$\mu_{ab}^S = \mu_{\iota(a)\iota(b)}|_{M_b^S} : M_b^S \to M_a^S \qquad \text{and} \qquad \mu_a^S = \mu_{\iota(a)}|_{F^{-1}(S)} : F^{-1}(S) \to M_a^S$$

Then $\mathrm{Sys}_{M^S} \underset{\text{def}}{=} \left(M_\bullet^S, \mu_{ab}^S, A\right)$ is an inverse system with limit $\left(F^{-1}(S), \mu_\bullet^S\right)$. In particular,

- if each $F_\bullet$ is surjective with compact Hausdorff fibers then the same is true of $F : M \to N$.
- if $n \in N$, $n_\bullet \underset{\text{def}}{=} \nu_\bullet(n)$, and $\iota : I \to \operatorname{Im} \iota$ has an inverse $\alpha : \operatorname{Im} \iota \to A$ then

$$F^{-1}(n) = \varprojlim\left(F_i^{-1}\left(n_{\alpha(i)}\right),\ \mu_{ij}|_{F_j^{-1}(n_{\alpha(i)})},\ \operatorname{Im} \iota\right)$$

In short, this says that "the fiber of the limit (map) is the limit of the fibers."

Remark 3.2.2. Parts (1) and (6) as well as the portion of part (4) (resp. part (5)) dealing with injectivity (resp. bijectivity) all continue to hold if we work in the category Set instead of Top.

Proof. Since $\nu_a \circ F = F_a \circ \mu_{\iota(a)}$ is a composition of continuous maps for every $a \in A$, the preimage under F of every subbasic open subset of N is open in M so F is continuous.

(1) From $\nu_a(F(\mu_i^{-1}(S_i))) = F_a(\mu_i(\mu_i^{-1}(S_i))) \subseteq F_a(S_i)$ we have $F(\mu_i^{-1}(S_i)) \subseteq \nu_a^{-1}(F_a(S_i)) \cap \operatorname{Im} F$. Now suppose that S_i is F_a-saturated meaning that $S_i = F_a^{-1}(F_a(S_i))$. Let $n \in \nu_a^{-1}(F_a(S_i)) \cap \operatorname{Im} F$ and let $m \in M$ be such that $n = F(m)$. Since $F_a(\mu_i(m)) = \nu_a(F(m)) = \nu_a(n) \in F_a(S_i)$ we have $\mu_i(m) \in F_a^{-1}(F_a(S_i)) = S_i$. Hence $m \in \mu_i^{-1}(S_i)$ so that $n = F(m) \in F(\mu_a^{-1}(S_i))$, as desired. If this F_a-saturated set is such that $F_a(S_i)$ is open in N_a then $F(\mu_i^{-1}(S_i)) = \nu_a^{-1}(F_a(S_i)) \cap \operatorname{Im} F$ shows that $F(\mu_i^{-1}(S_i))$ is open in $\operatorname{Im} F$.

(2) and (3) are immediate from proposition 2.1.33.

(4) Suppose that ι is cofinal and let $m, \widehat{m} \in M$ be such that $F(m) = F(\widehat{m})$. Then $F_\bullet \circ \mu_{\iota(\bullet)}(m) = \nu_\bullet \circ F(m) = \nu_\bullet \circ F(\widehat{m}) = F_\bullet \circ \mu_{\iota(\bullet)}(\widehat{m})$ so that the injectivity of each F_a implies that $\mu_{\iota(a)}(m) = \mu_{\iota(a)}(\widehat{m})$ for all $a \in A$. Since $\operatorname{Im}\iota$ is cofinal in I, this implies that $\mu_i(m) = \mu_i(\widehat{m})$ for all $i \in I$ so that $m = \widehat{m}$. Furthermore, observe that for any $n \in \operatorname{Im} F$ and $a \in A$, $F_a(\mu_{\iota(a)}(F^{-1}(n))) = \nu_a(F(F^{-1}(n))) = \nu_a(n)$ so that $\nu_a(n) \in \operatorname{Im} F_a$ while the injectivity of F_a gives $\mu_{\iota(a)}(F^{-1}(n)) = F_a^{-1}(\nu_a(n))$.

Now suppose that all $F_\bullet$ are topological embeddings, C is a closed subset of M, and that $n \in \operatorname{Cl}_{\operatorname{Im} F}(F(C))$. Recall that since $\operatorname{Im}\iota$ is cofinal in the directed set I, we have $C = \bigcap_{a \in A} \mu_{\iota(a)}^{-1}\left(\overline{\mu_{\iota(\bullet)}(C)}\right)$ so to show that $F^{-1}(n) \in C$ it suffices to show that for all $a \in A$, $\mu_{\iota(a)}(F^{-1}(n)) \in \overline{\mu_{\iota(a)}(C)}$. So let $n^\bullet = (n^\alpha)_{\alpha \in \Lambda}$ be a net in $F(C)$ converging to n and let $a \in A$. Since $\nu_a(n^\bullet)$ converges to $\nu_a(n)$ with all of these points contained in the domain of the continuous function $F_a^{-1} : \operatorname{Im} F_a \to M_{\iota(a)}$, we have $\mu_{\iota(a)}(F^{-1}(n^\bullet)) = F_a^{-1}(\nu_a(n^\bullet)) \to F_a^{-1}(\nu_a(n)) = \mu_{\iota(a)}(F^{-1}(n))$. Applying $\mu_{\iota(a)}$ to both sides of $F^{-1}(n^\bullet) \subseteq C$, we can conclude that $\mu_{\iota(a)}(F^{-1}(n)) \in \overline{\mu_{\iota(a)}(C)}$. Thus, $n \in F(C)$ so that $F(C)$ is closed in $\operatorname{Im} F$.

Assume henceforth that Sys_M and Sys_N are directed and that ι is cofinal.

(5) Suppose that all $F_\bullet$ are bijections, let $n \in N$, and let $n_\bullet = \nu_\bullet(n)$. For all $a \leq c$ in A,

$$n_a = \nu_{ac}(n_c) = \nu_{ac}\left(F_c\left(F_c^{-1}(n_c)\right)\right) = F_a\left(\mu_{\iota(a)\iota(c)}\left(F_c^{-1}(n_c)\right)\right)$$

so that $F_a^{-1}(n_a) = \mu_{\iota(a)\iota(c)}(F_c^{-1}(n_c))$. Hence, for any $i \in I$ and $a, b \in A$ with $i \leq \iota(a), \iota(b)$, since A is directed we may pick $c \geq a, b$ to conclude that

$$\mu_{i\iota(a)}\left(F_a^{-1}(n_a)\right) = \mu_{i\iota(a)}\left(\mu_{\iota(a)\iota(c)}\left(F_c^{-1}(n_c)\right)\right) = \mu_{i\iota(c)}\left(F_c^{-1}(n_c)\right) = \mu_{i\iota(b)}\left(F_b^{-1}(n_b)\right)$$

This shows that the for any $i \in \operatorname{Im}\iota$ the element $m_i \underset{\text{def}}{=} \mu_{i\iota(a)}(F_a^{-1}(n_a))$ is independent of the index $a \in \iota^{-1}(i)$ chosen so that by using our convention, this defines an element of $m \in M$. Since for each $a \in A$, $\nu_a(F(m)) = F_a\left(m_{\iota(a)}\right) = F_a(F_a^{-1}(n_a)) = n_a = \nu_a(n)$ it follows that $F(m) = n$. If all $F_\bullet$ are also homeomorphisms then the bijection F is an embedding by (4) and is thus a homeomorphism.

(6) For any $a \leq b$ in A, if $m_b \in M_b^S = F_b^{-1}(S_b)$ then $F_a\left(\mu_{ab}^S(m_b)\right) = \nu_{ab}(F_b(m_b)) \in \nu_{ab}(S_b) \subseteq S_a$ so that $\mu_{ab}^S(m_b) \in F_a^{-1}(S_a) = M_a^S$, which shows that Sys_{M^S} is an inverse system. Let $(Z, h_\bullet = (h_a)_{a\in A})$ be a cone into Sys_{M^S}. Observe that if $a, b \in A$ are such that $\iota(a) = \iota(b)$ then pick $c \geq a, b$ and note that for all $z \in Z$,

$$h_a(z) = \mu_{ac}^S(h_c(z)) = \mu_{\iota(a)\iota(c)}(h_c(z)) = \mu_{\iota(b)\iota(c)}(h_c(z)) = \mu_{bc}^S(h_c(z)) = h_b(z)$$

so that for any $i \in \operatorname{Im}\iota$ the map $\widetilde{h}_i \underset{\text{def}}{=} h_{\iota(a)} : Z \to M_i$ is independent of the choice of $a \in \iota^{-1}(i)$. Clearly $\left(Z, \left(\widetilde{h}_i\right)_{i\in\operatorname{Im}\iota}\right)$ is a cone into $\operatorname{Sys}_M\big|_{\operatorname{Im}\iota}$ where since I is directed and ι is cofinal, this cone has a unique extension to a cone $\left(Z, \widetilde{h}_\bullet \underset{\text{def}}{=} \left(\widetilde{h}_i\right)_{i\in I}\right)$ into Sys_M. Let $h : Z \to M$ be the unique limit morphism of $\left(Z, \widetilde{h}_\bullet\right)$, which is continuous if all $h_\bullet$ are continuous. Once we show that $\operatorname{Im} h \subseteq F^{-1}(S)$ then it will be clear that $h : Z \to F^{-1}(S)$ is the unique morphism into $(F^{-1}(S), \mu_\bullet^S)$ such that $\mu_\bullet^S \circ h = h_\bullet$, thereby proving that $(F^{-1}(S), \mu_\bullet^S)$ is the limit of Sys_{M^S}. Observe that if $z \in Z$ then

$$(\nu_a \circ F)(h(z)) = (F_a \circ \mu_{\iota(a)} \circ h)(z) = \left(F_a \circ \widetilde{h}_{\iota(a)}\right)(z) = (F_a \circ h_a)(z) \in F_a\left(F_a^{-1}(S_a)\right) \subseteq S_a$$

so that $F(h(z)) \in \bigcap\limits_{a\in A} \nu_a^{-1}(S_a) = S$, as desired.

If all F_a are surjective with compact Hausdorff fibers then since Sys_{M^n} is an inverse system of non-empty compact Hausdorff spaces so it's limit exists, which implies that $F^{-1}(n) \neq \varnothing$. ■

Corollary 3.2.3. Let Sys_M and Sys_N be two inverse systems of finite-dimensional affine linear spaces with affine linear connecting maps. Let $(\Lambda_\bullet, \iota) : \operatorname{Sys}_M \to \operatorname{Sys}_N$ be an inverse system morphism with each $\Lambda_a : M_{\iota(a)} \to N_a$ affine linear and let $\Lambda = \varprojlim \Lambda_\bullet$.

(1) If all $\Lambda_a : M_{\iota(a)} \to N_a$ are surjective then so is $\Lambda : M \to N$ and the converse is true if all $\nu_a : N \to N_a$ are surjective.

(2) The kernel of Λ is

$$\ker \Lambda = \bigcap_a \ker(\nu_a \circ \Lambda) = \bigcap_a \ker(\Lambda_a \circ \mu_{\iota(a)})$$

where $a \leq b$ implies $\ker(\nu_b \circ \Lambda) \leq \ker(\nu_a \circ \Lambda)$.

(3) Λ is injective if and only if for each index a there exists some $b \geq a$ such that $\ker \Lambda_b \leq \ker \mu_{\iota(a)\iota(b)}$ (or equivalently, such that $\ker(\nu_b \circ \Lambda) \leq \ker \mu_{\iota(a)})$. In this case

$$\dim N_b \geq \dim(\operatorname{Im} \Lambda_b) \geq \dim\big(\operatorname{Im} \mu_{\iota(a),\iota(b)}\big)$$

where observe that if $\mu_{\iota(b)}$ is surjective then this is equivalent to the set equality $\operatorname{Im} \Lambda_b = \operatorname{Im}(\nu_b \circ \Lambda)$.

Proof. Let $n \in N$, $n_\bullet = \nu_\bullet(n)$, and recall that $\Lambda^{-1}(n) = \varprojlim\Big(\Lambda_a^{-1}(n_a), \mu_{\iota(a)\iota(b)}\big|_{\Lambda_b^{-1}(n_b)}, \mathbb{N}\Big)$. Since each $\Lambda_a^{-1}(n_a)$ is an affine linear subspace of M_i and since each $\mu_{\iota(a)\iota(b)}\big|_{\Lambda_b^{-1}(n_b)} : \Lambda_b^{-1}(n_b) \to \Lambda_a^{-1}(n_a)$ is affine linear it follows from lemma 2.2.12 that $\Lambda^{-1}(n) \neq \varnothing$. Let $m \in M$. Observe that $\Lambda(m) = 0$ if and only if $\nu_a(\Lambda(m)) = \nu_a(0) = 0$ for all indices a, which happens if and only if $m \in \bigcap_a \ker(\nu_a \circ \Lambda) = \bigcap_a \ker(\Lambda_a \circ \mu_{\iota(a)})$. The inequality $\dim(\operatorname{Im} \Lambda_b) \geq \dim\big(\operatorname{Im} \mu_{\iota(a),\iota(b)}\big)$ follows from the rank nullity theorem:

$$\dim M_{\iota(b)} - \dim(\operatorname{Im} \Lambda_b) = \dim \ker \Lambda_b \leq \dim \ker \mu_{\iota(a)\iota(b)} = \dim M_{\iota(b)} - \dim\big(\operatorname{Im} \mu_{\iota(a)\iota(b)}\big)$$

■

Characterization of Closed Vector Subspaces of $\mathbb{R}^{\mathbb{N}}$

The following corollary 3.2.4 of proposition 3.2.1 is original.

Corollary 3.2.4. Every closed vector subspace M of $\mathbb{R}^{\mathbb{N}}$ is complemented in $\mathbb{R}^{\mathbb{N}}$ and TVS-isomorphic to $\mathbb{R}^d$, where $d = \dim M$ if $\dim M < \infty$ and $d = \mathbb{N}$ otherwise.

Proof. By lemma B.1.8, it suffices to show that M is TVS-isomorphic to $\mathbb{R}^{\mathbb{N}}$. For all $i, j \in \mathbb{N}$

with $i \le j$ let $\Pr_{\le i} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^i$ be the canonical projection, $M_i = \Pr_{\le i}(M)$, $d_i = \dim M_i$, and let $\mu_{ij} = \Pr_{\le i,j}\big|_{M_j} : M_j \to M_i$ be the restriction of the canonical projection $\Pr_{\le i,j} : \mathbb{R}^j \to \mathbb{R}^i$. Note that since $M_i = \Pr_{\le i,i+1}(M_{i+1})$ and the kernel of $\Pr_{\le i,i+1} : \mathbb{R}^{i+1} \to \mathbb{R}^i$ has dimension 1, either $\mu_{i,i+1}$ is a TVS-isomorphism and $d_{i+1} = d_i$ or else $d_{i+1} = d_i + 1$ and $\ker \Pr_{\le i,i+1} \subseteq M_{i+1}$. Let $\Lambda_1 : M_1 \to \mathbb{R}^{d_1}$ be any TVS-isomorphism. Suppose that $j > 1$ and for all $1 \le i \le j$ we've constructed TVS-isomorphisms $\Lambda_1 : M_i \to \mathbb{R}^{d_i}$ such that for all $1 \le h \le i \le j$, $\Pr_{\le d_h,d_i} \circ \Lambda_i = \Lambda_h \circ \mu_{hi}$, where $\Pr_{\le d_h,d_i} : \mathbb{R}^{d_i} \to \mathbb{R}^{d_h}$ is the canonical projection. If $d_{i+1} = d_i$ then define the TVS-isomorphism $\Lambda_{i+1} : M_{i+1} \to \mathbb{R}^{d_{i+1}}$ by $\Lambda_{i+1} = \Lambda_i \circ \mu_{i,i+1}$ so suppose that $d_{i+1} = d_i + 1$. Then $(\{0\}^i, 1) \in M_{i+1}$ so define the TVS-isomorphism $\Lambda_{i+1} : M_{i+1} \to \mathbb{R}^{d_{i+1}}$ by $x = (x_1, \ldots, x_{d_i}, x_{d_{i+1}}) \mapsto ((\Lambda_i \circ \Pr_{\le d_i,d_{i+1}})(x), x_{d_{i+1}})$. It is clear that Λ_{i+1} satisfies the inductive hypotheses so that $\Lambda_\bullet : \mathrm{Sys}_M \to (\mathbb{R}^{d_\bullet}, \Pr_{\le d_a,d_{a+1}}, \mathbb{N})$ is an inverse system morphism consisting of TVS-isomorphisms whose limit map $\Lambda = \varprojlim \Lambda_\bullet : M \to \mathbb{R}^d$ is therefore a TVS-isomorphism. ∎

Equivalence Transformations

Definition 3.3.1. Let $(F_a, \iota) \to (M_\bullet, \mu_{ij}, I) \to (N_a, \nu_{ab}, A)$ and $(G_i, \alpha) : (N_\bullet, \nu_{ab}, A) \to (M_i, \mu_{ij}, I)$ be two inverse system morphisms, where as usual, if we are working in some given category then we require that all F_a and G_i be morphisms in this category. Then we will say that $(F_\bullet, \iota)$ and $(G_\bullet, \alpha)$ form an *equivalence transformation (of inverse systems)* if for all indices $i \in I$ and $a \in A$ we have

$$\mu_{i,\iota\alpha(i)} = G_i \circ F_{\alpha(i)} \quad \text{and} \quad \nu_{a,\alpha\iota(a)} = F_a \circ G_{\iota(a)}$$

i.e. if the following diagrams commute:

$$\begin{tikzcd}
M_i & & M_{\iota\alpha(i)} \arrow[ll, "\mu_{i,\iota\alpha(i)}"'] \arrow[dl, "F_{\alpha(i)}"] \\
& N_{\alpha(i)} \arrow[ul, "G_i"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
N_a & & N_{\alpha\iota(a)} \arrow[ll, "\nu_{a,\alpha\iota(a)}"'] \arrow[dl, "G_{\iota(a)}"] \\
& M_{\iota(a)} \arrow[ul, "F_a"] &
\end{tikzcd}$$

where recall from def. 1.1.3 that $\iota\alpha(i) \underset{\text{def}}{=} \iota(\alpha(i))$ and $\alpha\iota(a) \underset{\text{def}}{=} \alpha(\iota(a))$. In this case we will say that Sys_M and Sys_N are *equivalent* and the follow diagrams will necessarily commute:

$$\begin{tikzcd}
M_{\iota\alpha(i)} \arrow[d, "F_{\alpha(i)}"] \arrow[dd, bend right=60, "\mu_{i,\iota\alpha(i)}"'] & M_{\iota\alpha(j)} \arrow[l, "\mu_{\iota\alpha(i)\iota\alpha(j)}"'] \arrow[d, "F_{\alpha(j)}"] \arrow[dd, bend left=60, "\mu_{j,\iota\alpha(j)}"] \\
N_{\alpha(i)} \arrow[d, "G_i"] & N_{\alpha(j)} \arrow[l, "\nu_{\alpha(i)\alpha(j)}"'] \arrow[d, "G_j"] \\
M_i & M_j \arrow[l, "\mu_{ij}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
N_{\alpha\iota(a)} \arrow[d, "G_{\iota(a)}"] \arrow[dd, bend right=60, "\nu_{a,\alpha\iota(a)}"'] & N_{\alpha\iota(b)} \arrow[l, "\nu_{\alpha\iota(a)\alpha\iota(b)}"'] \arrow[d, "G_{\iota(b)}"] \arrow[dd, bend left=60, "\nu_{b,\alpha\iota(b)}"] \\
M_{\iota(a)} \arrow[d, "F_a"] & M_{\iota(b)} \arrow[l, "\mu_{\iota(a)\iota(b)}"'] \arrow[d, "F_b"] \\
N_a & N_b \arrow[l, "\nu_{ab}"']
\end{tikzcd}$$

so that in addition, the following diagram will also commute:

$$\begin{tikzcd}
N_{\alpha\iota(a)} \arrow[d, "G_{\iota(a)}"] \arrow[dd, bend right=60, "\nu_{a,\alpha\iota(a)}"'] & N_{\alpha\iota(b)} \arrow[l, "\nu_{\alpha\iota(a)\alpha\iota(b)}"'] \arrow[d, "G_{\iota(b)}"] & N \arrow[l, "\nu_{\alpha\iota(b)}"'] \arrow[d, "\varprojlim(G_\bullet, \alpha)"] \\
M_{\iota(a)} \arrow[d, "F_a"] & M_{\iota(b)} \arrow[l, "\mu_{\iota(a)\iota(b)}"'] \arrow[d, "F_b"] & M \arrow[l, "\mu_{\iota(b)}"'] \arrow[d, "\varprojlim(F_\bullet, \iota)"] \\
N_a & N_b \arrow[l, "\nu_{ab}"'] & N \arrow[l, "\nu_b"']
\end{tikzcd}$$

where note that $\varprojlim(F_\bullet, \iota) \circ \varprojlim(G_\bullet, \alpha) = \mathrm{Id}_N$.

If $(M, \mu_\bullet) = \varprojlim \mathrm{Sys}_M$ and $(M, \nu_\bullet) = \varprojlim \mathrm{Sys}_N$ (note that here we're assuming that $M = N$) then we will say that $(F_\bullet, \iota)$ and $(G_\bullet, \alpha)$ form an *equivalence transformation of (inverse) representations* (def. 2.1.22) if in addition to being an equivalence transformation of inverse systems we have

$$\mu_i = G_i \circ \nu_{\alpha(i)} \quad \text{and} \quad \nu_a = F_a \circ \mu_{\iota(a)}$$

for all $i \in I$ and $a \in A$ (i.e. if both $\varprojlim (F_\bullet, \iota)$ and $\varprojlim (G_\bullet, \alpha)$ are Id_M). ■

Remark 3.3.2. If all $M_\bullet$ and $N_\bullet$ are smooth manifolds and $F_\bullet$ and $G_\bullet$ are as above then if all μ_{ij} (resp. ν_{ab}) are smooth submersions then so too are all $F_\bullet$ (resp. $G_\bullet$).

The proof of the following lemma is a straightforward exercise.

Lemma 3.3.3. Let $(F_a, \iota) : (M_\bullet, \mu_{ij}) \to (N_a, \nu_{ab})$ and $(G_i, \alpha) : (N_\bullet, \nu_{ab}) \to (M_i, \mu_{ij})$ be two collections of arbitrary maps with $F_a : M_{\iota(a)} \to N_a$ and $G_i : N_{\alpha(i)} \to M_i$. Assume that $M = N$ and that for each $i \in I$ and $a \in A$ we the maps μ_i and ν_a are surjective, $\mu_i = G_i \circ \nu_{\alpha(i)}$, and $\nu_a = F_a \circ \mu_{\iota(a)}$. Then both $(F_\bullet, \iota)$ and $(G_\bullet, \alpha)$ are morphisms (in Set) of inverse systems, they form an equivalence transformation of inverse representations, and all $F_\bullet$ and $G_\bullet$ are surjective. If in addition we have that all $\mu_\bullet$ and $\nu_\bullet$ are quotient maps in Top then all $F_\bullet$ and $G_\bullet$ are morphisms in this category.

Equivalent Systems Have Isomorphic Limits

It is a well-known fact (with a straightforward proof) that if there exists an equivalence transformation between two inverse systems then their limits are isomorphic. We provide a proof of this in the case where the indexing sets are the natural numbers since in this case we can provide the following original proof that doesn't require checking the universal property of limits and that makes more apparent the reason why these two systems' limits are necessarily isomorphic.

Proposition 3.3.4. Let $(M_\bullet, \mu_{ij}, \mathbb{N})$ and $(N_\bullet, \nu_{ab}, \mathbb{N})$ be two inverse systems. If there exists an equivalence transformation between them then their limits are isomorphic.

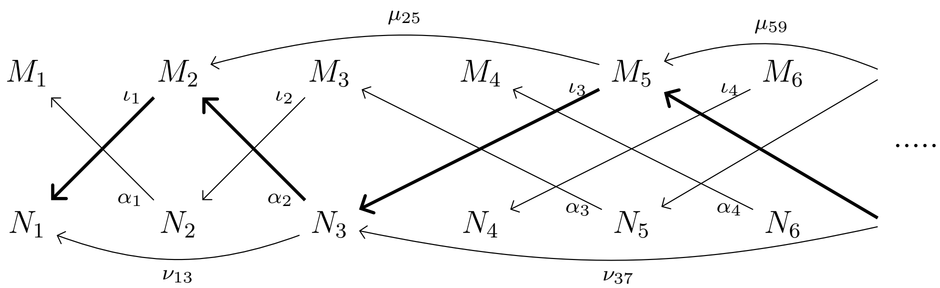

Figure 3.1: This diagram illustrates an example construction of Sys_{mix} (highlighted) whose simple construction may otherwise be hidden beneath its formal definition and which makes apparent why the existence of an equivalence transformation between systems directed by $\mathbb{N}$ forces (by lemma 2.1.37) the limits to be isomorphic. To reduce the quantity of symbols, we denoted each $F_a : M_{\iota(a)} \to N_a$ and each $G_i : N_{\alpha(i)} \to M_i$ by the index of its domain with the parentheses omitted. In the proof, Sys_{mixM} would consist of $M_2, M_5, M_9, \ldots$ and the morphisms $\mu_{25}, \mu_{59}, \ldots$ while Sys_{mixN} would consist of $N_1, N_3, N_7, \ldots$ and the morphisms $\nu_{13}, \nu_{37}, \ldots$.

Proof. Let $(F_a, \iota) : \mathrm{Sys}_M \to \mathrm{Sys}_N$ and $(G_i, \alpha) : \mathrm{Sys}_N \to \mathrm{Sys}_M$ be an equivalence transformation of inverse systems. If the order morphisms ι and α are the identity map on some cofinal subset of $\mathbb{N}$ then the result is immediate from lemma 2.1.37 and otherwise we may, by restricting to cofinal subsets of $\mathbb{N}$, assume without loss of generality that ι and α are strictly increasing. Let us construct an inverse system called Sys_{mix} as indicated by the diagram below:

$$N_1 \xleftarrow{F_1} M_{\iota(1)} \xleftarrow{G_{\iota(1)}} N_{\alpha\iota(1)} \xleftarrow{F_{\alpha\iota(1)}} M_{\iota\alpha\iota(1)} \xleftarrow{G_{\iota\alpha\iota(1)}} N_{\alpha\iota\alpha\iota(1)} \xleftarrow{F_{\alpha\iota\alpha\iota(1)}} \cdots$$

where the definition of an equivalence of systems guarantees that this does in fact form an inverse system.

We can now form two subsystems of Sys_{mix}, call them Sys_{mixM} and Sys_{mixN}, whose objects consist, respectively, of all $M_\bullet$'s and $N_\bullet$'s of Sys_{mix}. That is, Sys_{mixN} is the subsystem

$$N_1 \xleftarrow{\nu_{1,\alpha\iota(1)} = F_1 \circ G_{\iota(1)}} N_{\alpha\iota(1)} \xleftarrow{\nu_{\alpha\iota(1),\alpha\iota\alpha\iota(1)} = F_{\alpha\iota(1)} \circ G_{\iota\alpha\iota(1)}} N_{\alpha\iota\alpha\iota(1)} \xleftarrow{\nu_{\alpha\iota\alpha\iota(1),\alpha\iota\alpha\iota\alpha\iota(1)}} \cdots$$

and similarly for Sys_{mixM}.

Since Sys_{mixM} and Sys_{mixN} are both cofinal subsystems of Sys_{mix}, the limits of all three systems are isomorphic by lemma 2.1.37. Since Sys_{mixM} is cofinal in Sys_M it follows that the limit of Sys_M is isomorphic to that of Sys_{mix}. Similarly $\varprojlim \mathrm{Sys}_N \cong \varprojlim \mathrm{Sys}_{mix}$ so that the limits of all these systems are isomorphic to $\varprojlim \mathrm{Sys}_{mix}$. ∎

Examples

Canonical Limit Map into a Subsystem

Example and Definition 3.4.1 (Canonical Limit Map into a Subsystem)**.** Let $J \subseteq I$ and call the inverse system morphism $\left(\left(\mathrm{Id}_{M_j}\right)_{j \in J}, \mathrm{In}_J^I\right) : \mathrm{Sys}_M \to \mathrm{Sys}_M\big|_J$, which we will call *the canonical inverse system morphism from* Sys_M *into (the subsystem)* $\mathrm{Sys}_M\big|_J$. If $\left(\widetilde{M}, \left(\widetilde{\mu}_j\right)_{j \in J}\right)$ is a limit of the subsystem $\mathrm{Sys}_M\big|_J$ the limit map, $L : M \to \widetilde{M}$, of this inverse system morphism *the canonical limit map of M into $\widetilde{M}$*. Proposition 3.2.1 shows that if J is cofinal in I (resp. and if J is directed) then L is injective (resp. bijective) and in the category Top will represent an embedding (resp. homeomorphism) of M into (resp. onto) $\widetilde{M}$ so that we will call this limit morphism *the canonical embedding* (resp. *identification*) *of M into* (resp. *with*) $\widetilde{M}$.

If J is cofinal in I but is not directed then example 2.1.40 shows that although L is injective, it need not be surjective. If J is not cofinal in I then example 3.4.9 shows that L need not be injective nor surjective.

Suppose that $(F_\bullet, \iota) : (M_\bullet, \mu_{ij}, I) \to (N_\bullet, \nu_{ab}, A)$ is an inverse system morphism and $J \underset{\text{def}}{=} \iota(A)$ is not necessarily cofinal in I. Let $\widehat{\iota} : A \to J$ denote $\iota : A \to I$ considered as a map onto its image. Then $(F_\bullet, \iota)$ can be decomposed into the composition of the two inverse system morphisms

$$\left(\left(\mathrm{Id}_{M_j}\right)_{j \in J}, \mathrm{In}_J^I\right) : \mathrm{Sys}_M \to \mathrm{Sys}_M\big|_J \quad \text{and} \quad (F_\bullet, \widehat{\iota}) : \mathrm{Sys}_M\big|_J \to \mathrm{Sys}_N$$

so that since $\varprojlim$ is a functor, $\varprojlim (F_\bullet, \iota) : M \to N$ factors through $\widetilde{M} = \varprojlim \mathrm{Sys}_M\big|_J$. ∎

Remark and Definition 3.4.2. Fix an index i_0 for which $I^{\geq i_0}$ is cofinal in I and note that for all $i_0 \leq i \leq j$ the following diagram commutes:

$$\begin{tikzcd}
M_i \arrow[d, "\mu_{i_0 i}"] & M_j \arrow[l, "\mu_{ij}"'] \arrow[d, "\mu_{i_0 j}"] & M \arrow[l, "\mu_j"'] \arrow[d, "\mu_{i_0}"] \\
M_{i_0} & M_{i_0} \arrow[l, "\mathrm{Id}_{M_{i_0}}"'] & M_{i_0} \arrow[l, "\mathrm{Id}_{M_{i_0}}"']
\end{tikzcd}$$

so that the maps $\mu_{i_0 i} : M_i \to M_{i_0}$ form an inverse system morphism $\left(\mu_{i_0 i}, \mathrm{In}^{I}_{I^{\geq i_0}}\right) : \mathrm{Sys}_M \to \left(M_{i_0}, \mathrm{Id}_{M_{i_0}}, I^{\geq i_0}\right)$ whose limit is $\mu_{i_0} : M \to M_{i_0}$, where $\left(M_{i_0}, \mathrm{Id}_{M_{i_0}}, I^{\geq i_0}\right)$ is the constant system. This observation will allow us to apply the results pertaining to inverse system morphisms that follow to conclude, for instance, that if each μ_{ij} is, respectively, injective, bijective, an embedding, a homeomorphism, or (if all $M_\bullet$ are Hausdorff) a proper map then the same is true of all $\mu_\bullet$. We will refer to the above morphism of inverse systems as *the canonical morphism (of inverse systems) associated with (or induced by)* μ_{i_0}. ∎

Surjectivity Counter Examples

The following examples are original. Example 3.4.5 is another example of a surjective inverse system morphism whose limit map is not surjective.

Example 3.4.3. A surjective inverse system morphism with non-dense image: Let M be any (possibly empty) topological space disjoint from $\left]0,1\right[$ and for all $i \in \mathbb{N}$, let $M_i = M \cup \left]1 - \frac{1}{i}, 1\right[$, $N_i = [-1,1]$, and let both $\mu_{i,i+1}$ and $\nu_{i,i+1}$ be the natural inclusions. Note that $M = \varprojlim \mathrm{Sys}_M$ and $[-1,1] = \varprojlim \mathrm{Sys}_N$. For each $a \in \mathbb{N}$, let $F_a : M_a \to N_a$ be identically 0 on M and $F_a(x) = \sin\left(\arctan\left(\frac{\pi}{2}x\right)\right)$ on $\left]1 - \frac{1}{a}, 1\right[$. Observe that each F_a is a continuous surjection and that $(F_\bullet, \mathrm{Id}_\mathbb{N}) : \mathrm{Sys}_M \to \mathrm{Sys}_N$ is an inverse system morphism whose limit map $F : M \to [-1,1]$ is identically 0 (where if $M = \varnothing$ then $F = \varnothing$) so that $\mathrm{Im}\, F \subseteq \{0\}$ is not dense in $[-1,1] = \varprojlim \mathrm{Sys}_N$. In particular, this shows that "$\mathrm{Im}\left(F_a \circ \mu_{\iota(a)}\right)$ is dense in N_a" in proposition 3.2.1(3) cannot necessarily be replaced with "$\mathrm{Im}\, F_a$ is dense in N_a."

The next example shows that the containments in propositions 2.1.33(3) and 3.2.1(3) may be strict.

Example 3.4.4. For all $i \in \mathbb{N}$ let $\varphi_i : \mathbb{R} \to [0,1]$ be a smooth function on $\mathbb{R}$ such that $\varphi_i^{-1}(1) = [i, \infty[$ and $\varphi_i^{-1}(0) =]-\infty, 0]$. For all $a, b \in \mathbb{N}$ with $a \leq b$ let $N_a = [0,1]^a$, $N = [0,1]^{\mathbb{N}}$, and let $\nu_{ab} : N_b \to N_a$ and $\nu_a : N \to N_a$ be the canonical projections so that $(N, \nu_\bullet) = \varprojlim(N_\bullet, \nu_{ab}, \mathbb{N})$ in Top. For all $a \in \mathbb{N}$ let $h_a : \mathbb{R} \to N_a$ be $h_a(r) = (\varphi_1(r), \ldots, \varphi_a(r))$ and observe that $(\mathbb{R}, h_\bullet)$ is a cone into Sys_N with limit

$$\begin{aligned} h : \mathbb{R} &\longrightarrow N \\ r &\longmapsto (\varphi_1(r), \varphi_2(r), \ldots) \end{aligned}$$

Observe that for each $a \in \mathbb{N}$, if $r \in \mathbb{R}$ then $(\varphi_1(r), \ldots, \varphi_a(r)) = h_a(r) = \{1\}^a$ if and only if $r \geq a$, which implies that $\nu_a(\{1\}^{\mathbb{N}}) = \{1\}^a \in \operatorname{Im} h_a$ and that $h_a^{-1}(\{1\}^a) = [a, \infty[$. However, $\{1\}^{\mathbb{N}} \notin \operatorname{Im} h$ (even though N is compact) since if there was some $r \in \mathbb{R}$ for which $h(r) = \{1\}^{\mathbb{N}}$ then since $(\varphi_1(r), \ldots, \varphi_a(r)) = \nu_a(h(r)) = \{1\}^a$ for all $a \in \mathbb{N}$ we would necessarily have $r \geq a$ for all $a \in \mathbb{N}$. An alternative way to reach this same conclusion, which has added the benefit of allowing us to see geometrically why $\{1\}^{\mathbb{N}} \notin \operatorname{Im} h$, is to observe that by corollary 2.3.10 we have

$$h^{-1}\left(\{1\}^{\mathbb{N}}\right) = \bigcap_{a \in \mathbb{N}} h_a^{-1}(\{1\}^a) = \bigcap_{a \in \mathbb{N}} [a, \infty[= \varnothing$$

By considering the canonical inverse system morphism $(h_\bullet, \mathrm{Id}_{\mathbb{N}}) : \mathrm{ConstSys}_Z \to \mathrm{Sys}_N$ induced by this cone, we obtain an analogous counterexample to equality in proposition 3.2.1(3) for inverse system morphisms (in place of cones). ∎

Convergent Series in $\mathbb{R}^{\geq 0}$

The following example is original.

Example 3.4.5 (Convergent series in $\mathbb{R}^{\geq 0}$)**.** Let $M_1 = \mathbb{R}^{\geq 0}$ and for each $i \in \mathbb{N}$ let $M_i = \prod_{l=1}^{i} M_1$ and define

$$\begin{aligned} \mu_{i,i+1} : M_{i+1} &\longrightarrow M_i \\ (r_1, \ldots, r_{i+1}) &\longmapsto (r_1, \ldots, r_{i-1}, r_i + r_{i+1}) \end{aligned}$$

where $\mu_{12}(r_1, r_2) = r_1 + r_2$. Let $(M^1, \mu_\bullet)$ denote the canonical limit of this system in the category Set. For each $a \in \mathbb{N}$ let $F_a \underset{\text{def}}{=} \rho_{a,a+1} \underset{\text{def}}{=} \mathrm{Pr}_{\leq a} : M_{a+1} \to M_a$ and $\rho_a : M_1^{\mathbb{N}} \to M_a$ denote the canonical projections onto the first a coordinates, let $\mathrm{Sys}_P \underset{\text{def}}{=} (M_i, \rho_{a,a+1}, \mathbb{N})$, and let $\iota : \mathbb{N} \to \mathbb{N}$ be $\iota(a) \underset{\text{def}}{=} a + 1$ for all $a \in \mathbb{N}$. Then $(M_1^{\mathbb{N}}, \rho_\bullet) = \varprojlim \mathrm{Sys}_P$ and $(F_\bullet, \iota) : \mathrm{Sys}_M \to \mathrm{Sys}_P$ is an inverse system morphism since the following diagram commutes:

$$\begin{tikzcd} (r_1, \ldots, r_a, r_{a+1}, r_{a+2}) \arrow[r, "F_{a+1}"] \arrow[d, "\mu_{\iota(a),\iota(a+1)}"] & (r_1, \ldots, r_a, r_{a+1}) \arrow[d, "\rho_{a,a+1}"] \\ (r_1, \ldots, r_a, r_{a+1} + r_{a+2}) \arrow[r, "F_a"] & (r_1, \ldots, r_a) \end{tikzcd}$$

Let $F \underset{\text{def}}{=} F^{M_1} \underset{\text{def}}{=} \varprojlim F_{\iota(\bullet)} : M^1 \to M_1^{\mathbb{N}}$. Observe that once M_1 was chosen, the above constructions did not use any topological or order theoretic structures on $\mathbb{R}$.

Intuitively, elements of M^1 can be identified with infinite lists of equations of the form:

$$\begin{aligned} r =& r_1 + \widehat{r_2} \\ =& r_1 + r_2 + \widehat{r_3} \\ =& r_1 + r_2 + r_3 + \widehat{r_4} \\ &\vdots \\ =& r_1 + r_2 + r_3 + \cdots + r_{i-1} + \widehat{r_i} \\ &\vdots \end{aligned}$$

where r, all r_i, and all $\widehat{r_i}$ belong to M_1 (explicitly, the corresponding element of M^1 would be $((r), (r_1, \widehat{r_2}), (r_1, r_2, \widehat{r_3}), \ldots))$ and where the image under F (of this corresponding element of M^1) would just be the list $(r_1, r_2, \ldots)$ with r and all $\widehat{r_i}$ being forgotten. Observe that although each F_a is surjective (and in fact a smooth submersion) the limit morphism F is not surjective since, for instance, $\{1\}^{\mathbb{N}} \underset{\text{def}}{=} (1,1,1,\ldots) \in M_1^{\mathbb{N}}$ but $\{1\}^{\mathbb{N}} \notin \operatorname{Im} F$ since $\operatorname{Im} F$ is the set of all absolutely convergent series of non-negative real numbers. Indeed, as shown later in example 11.7.7, the image of F has empty interior in F's codomain.

Furthermore, F is not injective so although $\operatorname{Im} F$ consists of every convergent series of non-negative real numbers, we cannot use F to uniquely associate to such a series a unique element of M_1 (i.e. its sum). This should be expected since the above constructions did not make use of any distinguished order or topology on M_1.

Remark 3.4.6. Curiously, although the definition of when a series of non-negative real numbers converges depends explicitly on the topology of $\mathbb{R}$, the construction used in the above examples appears to be purely algebraic since only $\mathbb{R}^{\geq 0}$'s semigroup structure was used. The Euclidean topology does exist implicitly in the semigroup $\mathbb{R}^{\geq 0}$ due to the restriction of addition to this subset. But the above construction may clearly be immediately generalized to any subset of an arbitrary additive group (by replacing M_1 with this set) that contains the identity and satisfies the condition that whenever two of its elements sum to 0 then both elements are necessarily 0, which consequently allows us define purely algebraic generalizations of the notion of "absolutely converging series" to arbitrary groups. An investigation into such "generalized absolutely convergent series" would lead us too far afield and so will not be explored further. ■

Identity Maps and Limits

Example 3.4.7 (Obtaining the identity map as the limit of an inverse system morphism consisting of non-identity morphisms). Let $\operatorname{Sys}_N = (N_\bullet, \nu_{ab}, \mathbb{Z}^{\geq 0})$ be any (not necessarily surjective) inverse system with limit $(N, \nu_\bullet)$ and let Sys_M be the subsystem of Sys_N indexed by $\mathbb{N}$. Clearly, $M \underset{\text{def}}{=} N$ together with all projections ν_a indexed by $a \in \mathbb{N}$ is a limit of Sys_M. Let $\iota : \mathbb{Z}^{\geq 0} \to \mathbb{N}$ be $\iota(a) = a + 1$ and let $F_a \underset{\text{def}}{=} \nu_{a,a+1} : N_{a+1} \to N_a$. It is clear that

$(F_a, \iota) : \mathrm{Sys}_M \to \mathrm{Sys}_N$ is a morphism and note that for all $a \in \mathbb{Z}^{\geq 0}$ we have

$$F_a \circ \nu_{a+1} = \nu_{a,a+1} \circ \nu_{a+1} = \nu_a = \nu_a \circ \mathrm{Id}_N$$

so that $\mathrm{Id}_N = \varprojlim (F_\bullet, \iota)$. Thus $\varprojlim F_\bullet = \mathrm{Id}_N : M \to N$ is an isomorphism even though potentially none of the F_a were isomorphisms, injective, nor surjective, which shows that the conditions in lemma 3.2.1 for the limit morphism to be injective, surjective, or bijective, are sufficient but not necessary.

The following example shows that for every isomorphism between the limits of two inverse systems, there always exists some inverse system that allows this isomorphism to be expressed as the limit of identity maps.

Example 3.4.8 (An inverse system morphism consisting of identity maps whose limit is not the identity map)**.** Let N be any object, $(M, \mu_\bullet) = \varprojlim(M_\bullet, \mu_{ij}, I)$, and let $F : N \to M$ be any isomorphism. For each index i, let $\nu_i = \mu_i \circ F : N \to M_i$, $\nu_{ij} = \mu_{ij}$, and $N_i = M_i$. Since F is an isomorphism and inverse limits are unique up to unique isomorphism, it is immediate that $(N, \nu_i) = \varprojlim(M_\bullet, \mu_{ij})$. By definition, we have that $\mu_i \circ F = \mathrm{Id}_{M_i} \circ F$ so that $F = \varprojlim \mathrm{Id}_{M_i}$. As a cautionary note and to emphasize the affect that the projections $\nu_\bullet$ may have on the limit of an inverse system morphism, observe that if $N = M$ and F is not the identity morphism then although $\mathrm{Sys}_M = \mathrm{Sys}_N$ and each component of the above inverse system morphism is the identity morphism $\mathrm{Id}_{M_i} : M_i \to M_i$, its limit is not the identity morphism.

Injectivity and Cofinality of Order Morphisms

Example 3.4.9. Parts (4) and (5) of proposition 3.2.1 may fail if ι is not cofinal: For all $i \in I \underset{\text{def}}{=} \mathbb{N}$ let M_i denote the closed cube in $\mathbb{R}^i$ centered at the origin with sides of length $2 + \frac{2}{i}$ and let $\mu_{ij} : B_j \to B_i$ denote the restriction to B_j of the canonical projection $\mathbb{R}^j \to \mathbb{R}^i$. The limit of Sys_M is $\left(M \underset{\text{def}}{=} [-1, 1]^{\mathbb{N}}, \mu_\bullet\right)$ where each $\mu_i : M \to B_i$ is the canonical projection. If $J \subseteq \mathbb{N}$

is finite then the limit of the inverse system morphism $\left(\left(\mathrm{Id}_{M_j}\right)_{j\in J}, \iota \underset{\text{def}}{=} \mathrm{In}_J^I\right) : \mathrm{Sys}_M \to \mathrm{Sys}_M\big|_J$ is the canonical projection $M \to B_{\max J}$ onto the first $\max J$ coordinate,s which is neither injective nor surjective.

Chapter 4

The Canonical Sheaf

Notation 4.0.1. Given any function $F : M \to N$ and any map $g : N \to Z$ into any space Z, we will let $F^* g \underset{\text{def}}{=} g_* F \underset{\text{def}}{=} g \circ F$ and given a set $\mathcal{F}$ of maps into N we will let $h_*(\mathcal{F}) = \{h_* F \,|\, F \in \mathcal{F}\}$.

Assumption and Notation 4.0.2. We will henceforth work within the subcategory $\mathcal{C}$ of the category of commutative locally $\mathbb{R}$-ringed spaces $(M, \mathcal{M})$ satisfying:

(1) $\mathcal{M}$ is a subsheaf of C^0_M, the sheaf of algebras of continuous $\mathbb{R}$-valued functions whose connecting maps are the usual restrictions of maps,

(2) $\mathcal{M}$ contains the constant functions, and

(3) whenever $h \in C^\infty(\mathbb{R} \to \mathbb{R})$ then $h_*(\mathcal{M}(U)) \subseteq \mathcal{M}(U)$ for all $U \in \mathrm{Open}(M)$.

Remark and Notation 4.0.3. If $(M, \mathcal{M})$ and $(N, \mathcal{N})$ are two ringed spaces in this category and $(F, F^\sharp_\bullet) : (M, \mathcal{M}) \to (N, \mathcal{N})$ is a morphism of ringed spaces then recall ([48]) that this forces $(F, F^\sharp_\bullet) = (F, F^*_\bullet)$, where $(F, F^*_\bullet)$ is a morphism of locally $\mathbb{R}$-ringed spaces with and where the morphism of sheaves on N, $F^*_\bullet : \mathcal{N} \to f_* \mathcal{M}$, is defined by

$$\begin{array}{rcl} F^*_V : \mathcal{N}(V) & \longrightarrow & F_* \mathcal{M}(V) \underset{\text{def}}{=} \mathcal{M}(F^{-1}(V)) \\ f & \longmapsto & f \circ F|_{F^{-1}(V)} \end{array}$$

for each $V \in \operatorname{Open}(N)$. In particular, the morphism of sheaves on N is completely determined by the continuous map F. Conversely, if given a continuous map F then each of the maps F_V^* defined above produces an algebra of continuous $\mathbb{R}$-valued functions and the only addition requirement for $F_\bullet^*$ to be a morphism of ringed spaces is that the range of F_V^* be a subalgebra of $F_*\mathcal{M}(V)$.

This justifies abusing notation by writing F^* instead of $F_\bullet^*$ and it also justifies referring to $F : (M, \mathcal{M}) \to (N, \mathcal{N})$ or, if the shaves are understood, even $F : M \to N$ as a morphism of ringed spaces. ∎

Definition 4.0.4. Given any map $F : M \to N$ we will call F *smooth* if F is continuous and if $(F, F_\bullet^*) : (M, \mathcal{M}) \to (N, \mathcal{N})$ is a morphism of (locally $\mathbb{R}$-)ringed spaces. Explicitly, this latter conditions means that $g \circ F\big|_{F^{-1}(V)} \in \mathcal{M}(F^{-1}(V))$ for all $V \in \operatorname{Open}(N)$ and all $g \in \mathcal{N}(V)$. If $F : M \to N$ is a smooth bijection whose inverse $F^{-1} : N \to M$ is also smooth then we will call F a *diffeomorphism*.

Assumption and Notation 4.0.5. We will henceforth only consider directed inverse systems $\operatorname{Sys}_M \underset{\text{def}}{=} \big((M_i, \mathcal{M}_i), (\mu_{ij}, \mu_{ij,\bullet}^*), I\big)$ in the category $\mathcal{C}$. As before, $(M, \mu_\bullet)$ will denote a limit of $(M_\bullet, \mu_{ij}, I)$ in Top

Definition 4.0.6 (Canonical Sheaf)**.** For every $U \in \operatorname{Open}(M)$, let $\mathcal{M}(U)$ denote the set of all $\mathbb{R}$-valued continuous functions f such that for each $m \in M$ there exists some index $i \in I$, some $U_i \in \operatorname{Open}(M_i)$, and some $f_i \in \mathcal{M}_i(U_i)$ such that $m \in \mu_i^{-1}(U_i) \in \operatorname{Open}(U)$ and

$$f\big|_{\mu_i^{-1}(U_i)} = f_i \circ \mu_i\big|_{\mu_i^{-1}(U_i)}$$

It is immediately verified that $\mathcal{M}$ forms a sheaf, which we will refer to as *the canonical sheaf (on M) (induced by $\mu_\bullet$)* and that $(M, \mathcal{M})$ is a locally $\mathbb{R}$-ringed space, which we will call *the canonical limit of* Sys_M *(in the category of commutative locally $\mathbb{R}$-ringed spaces) (induced by $\mu_\bullet$).*

If we have commutative locally $\mathbb{R}$-ringed spaces $(S_\bullet, \mathcal{S}_\bullet)$ indexed by $\mathbb{N}$ then by the *canonical product sheaf on* $\prod_{l=1}^{\infty} S_l$ we mean the canonical sheaf on $\prod_{l=1}^{\infty} S_l$ induced by the canonical projections $\mathrm{Pr}_{\leq \bullet}$, where each projection map $\mathrm{Pr}_{\leq i} : \prod_{l=1}^{\infty} S_l \to M_i$ is the canonical projections onto the first i coordinates. ■

Proposition 4.0.7. If $\mathrm{Sys}_M = \big((M_i, \mathcal{M}_i), \big(\mu_{ij}, \mu_{ij,\bullet}^*\big), I\big)$ is a directed inverse system in $\mathcal{C}$, $(M, \mu_\bullet) = \varprojlim (M_\bullet, \mu_{ij}, I)$ in Top, and $\mathcal{M}$ is the canonical sheaf, then $\big((M, \mathcal{M}), \big(\mu_i, \mu_{i,\bullet}^*\big)_{i\in I}\big) = \varprojlim \mathrm{Sys}_M$ in the category $\mathcal{C}$ (which is defined in 4.0.2).

Proof. Exercise. ■

Corollary 4.0.8. If $(M, \mathcal{M}, (\mu_i)_{i\in I}) = \varprojlim (M_i, \mathcal{M}_i, \mu_{ij}, I)$ and $J \subseteq I$ is cofinal in I then $(M, \mathcal{M}, (\mu_i)_{i\in J}) = \varprojlim (M_i, \mathcal{M}_i, \mu_{ij}, J)$ in $\mathcal{C}$.

Corollary 4.0.9. Let Sys_M and Sys_N be two directed inverse systems in $\mathcal{C}$. If there exists an equivalence transformation between these two systems then their limits are diffeomorphic.

Remark 4.0.10. It follows from the universal property of inverse limits that a map $F : N \to M$ is smooth if and only if $\mu_i \circ F : N \to M_i$ is smooth for all indices i.

Example 4.0.11. Recall from example 2.1.51 that $(\mathbb{R}^{\mathbb{N}}, \mathrm{Pr}_{\leq \bullet}) = \varprojlim \big(\mathbb{R}^i, \mathrm{Pr}_{\leq ij}, \mathbb{N}\big)$ in Top where $\mathrm{Pr}_{\leq ij}$ and $\mathrm{Pr}_{\leq i}$ are the canonical projections, so that $\mathbb{R}^{\mathbb{N}}$ can be assigned its canonical sheaf $C^\infty_{\mathbb{R}^{\mathbb{N}}}$. However, since $\mathbb{R}^{\mathbb{N}}$ is also a Fréchet space we may also consider the sheaf $C^{\infty, TVS}_{\mathbb{R}^{\mathbb{N}}}$ of all infinitely Gâteaux-differentiable $\mathbb{R}$-valued functions (def. B.2.1), which raises the question of how these sheaves relate to each other. After developing some more tools, we will come back to this question and prove a theorem due to Abbati and Manià [1][thm. 14] that

$$C^\infty_{\mathbb{R}^{\mathbb{N}}} = C^{\infty, TVS}_{\mathbb{R}^{\mathbb{N}}}$$

so that in this important case, these two notions of smoothness coincide. ■

The Restriction Sheaf

Definition 4.1.1. Let $(M, \mathcal{M})$ and $(N, \mathcal{N})$ be two locally $\mathbb{R}$-ringed spaces of continuous $\mathbb{R}$-valued functions, let $C \subseteq M$ be arbitrary, and let $F : C \to N$ be a map. We will say that F is *smooth (on C)* if for all $c \in C$ there exists a *smooth local extension of F around c*, that is, a map $F_c : U_c \to N$ such that U_c is an open neighborhood of c in M, F_c is smooth, and $F_c = F$ on $U_c \cap C$.

For any subset $S \subseteq M$ and any $V \in \operatorname{Open}(S)$ let $\mathcal{M}\big|_S(V)$ denote the $\mathbb{R}$-algebra of all smooth $\mathbb{R}$-valued functions on V. It is clear $\mathcal{M}\big|_S$ with the canonical restrictions makes $\left(S, \mathcal{M}\big|_S\right)$ into a locally $\mathbb{R}$-ringed space that we will call *the restriction of $\mathcal{M}$ to S* or *the sheaf on S induced by $\mathcal{M}$*. If S is clear from context then we will call it *the restriction sheaf*, *the canonical sheaf*, or *the induced sheaf*. ■

Assumption 4.1.2. If $(M, \mathcal{M})$ is in $\mathcal{C}$ then given any $S \subseteq M$ we will assume, unless stated otherwise, that S is assigned the restriction sheaf $\mathcal{M}\big|_S$.

Definition 4.1.3. A smooth map $F : (M, \mathcal{M}) \to (N, \mathcal{N})$ will be called a *smooth embedding (resp. proper smooth embedding)* if $F : (M, \mathcal{M}) \to \left(\operatorname{Im} F, \mathcal{N}\big|_{\operatorname{Im} F}\right)$ is a diffeomorphism (resp. and $F : M \to N$ is a proper map). For any $m \in M$, say that F is *a smooth (local) embedding (resp. diffeomorphism) at m* if there exists an open set $m \in U \in \operatorname{Open}(M)$ such that

$$F\big|_U : \left(U, \mathcal{M}\big|_U\right) \to \left(F(U), \mathcal{N}\big|_{F(U)}\right)$$

is a diffeomorphism onto its image (resp. and $F(U)$ is an open subset of N). We will say that a smooth map is a *smooth local embedding (resp. local diffeomorphism)* if it is a smooth local embedding (resp. local diffeomorphism) at each point of its domain.

Remark 4.1.4. It is clear that for any $S \subseteq M$, the inclusion map $\operatorname{In} : \left(S, \mathcal{M}\big|_S\right) \to (M, \mathcal{M})$ is smooth. If $F : M \to N$ is any map then it is immediate that $F : (M, \mathcal{M}) \to (N, \mathcal{N})$ is smooth if and only if $F:(M,\mathcal{M})\to\left(\operatorname{Im}F,\mathcal{N}\big|_{\operatorname{Im}F}\right)$ is smooth. Furthermore, if $R\subseteq S\subseteq M$ then $\left(\mathcal{M}\big|_S\right)\big|_R=\mathcal{M}\big|_R$.

Since the image of $\mu_{ij}|_{\operatorname{Im}(\mu_j)}$ is $\operatorname{Im}(\mu_i)=\overline{M_i}$, we see immediately from lemma 2.2.8 that if we replace all M_i with $\operatorname{Im}(\mu_i)$ then we would have gained surjectivity of all maps while not even changing the object of the canonical limit and only changing the canonical projections trivially. We will now show that this replacement would also not change the canonical sheaf.

Proposition 4.1.5. Let $\overline{\mathcal{M}}_i=\mathcal{M}_i|_{\overline{M_i}}$ be the usual restriction of a sheaf. If $\overline{\mathcal{M}}$ denotes the canonical sheaf on M resulting from the system $\operatorname{Sys}_{\overline{M}}\underset{\text{def}}{=}(\overline{M_i},\overline{\mathcal{M}}_i,\overline{\mu_{ij}},I)$ (as defined in lemma 2.2.8) then $\overline{\mathcal{M}}=\mathcal{M}$.

Proof. Recall that $(M,\overline{\mu_i})=\varprojlim\operatorname{Sys}_{\overline{M}}$, where $\overline{\mu_i}:M\to\overline{M_i}$ are just the projections μ_i but considered as maps valued in $\overline{M_i}$, and observe that this implies that the maps μ_i and (μ_i) both induce the same (weak) topology on M.

Let $U\in\operatorname{Open}(M)$, $f:U\to\mathbb{R}$ be a continuous map. If $f\in\mathcal{M}(U)$ then for any $m\in U$ there exists some index i, some $U_i\in\operatorname{Open}(M_i)$ and some $f_i\in\mathcal{M}_i(U_i)$ such that $m\in\mu_i^{-1}(U_i)\subseteq U$ and $f=f_i\circ\mu_i$ on $\mu_i^{-1}(U_i)$. By definition of the sheaf $\overline{\mathcal{M}_i}$, the function $\overline{f_i}\underset{\text{def}}{=}f\big|_{\overline{M_i}}:U_i\cap\overline{M_i}\to\mathbb{R}$ belongs to $\overline{\mathcal{M}_i}\left(\overline{U_i}\right)$, where $\overline{U_i}\underset{\text{def}}{=}U_i\cap\overline{M_i}$, and since $\mu_i^{-1}(U_i)=\overline{\mu_i}^{-1}\left(\overline{U_i}\right)$ it follows that $\overline{f_i}\in\overline{\mathcal{M}_i}\left(\overline{U_i}\right)$ where $f=\overline{f_i}\circ\overline{\mu_i}$ on $\overline{\mu_i}^{-1}\left(\overline{U_i}\right)$. Thus $\mathcal{M}(U)\subseteq\overline{\mathcal{M}}(U)$.

Now suppose that $f\in\overline{\mathcal{M}}(U)$ and let $m\in U$. There some index i, some open subset $\overline{U_i}$ of $\overline{M_i}$ and some $\overline{f_i}\in\overline{\mathcal{M}_i}\left(\overline{U_i}\right)$ such that $m\in\overline{\mu_i}^{-1}\left(\overline{U_i}\right)\subseteq U$ and $f=\overline{f_i}\circ\overline{\mu_i}$ on $\overline{\mu_i}^{-1}\left(\overline{U_i}\right)$. Let $m_i=\mu_i(m)$ and let $U_i\in\operatorname{Open}(M_i)$ be such that $\overline{U_i}=U_i\cap\overline{M_i}$. By definition of $\overline{\mathcal{M}_i}\left(\overline{U_i}\right)$, there exists some open set $m_i\in V_i\in\operatorname{Open}(M_i)$ and some function $g_i\in\mathcal{M}(V_i)$ such that $g_i=\overline{f_i}$ on $V_i\cap\overline{U_i}$. Let $W_i=U_i\cap V_i$ and let $f_i=g_i\big|_{W_i}:W_i\to\mathbb{R}$ so that $f_i\in\mathcal{M}_i(W_i)$ and $f_i=\overline{f_i}$ on $W_i\cap\overline{M_i}$. Observe $m\in\mu_i^{-1}(W_i)$ and that since $\overline{M_i}=\mu_i(M)$ we have

$$\mu_i^{-1}(W_i)=\overline{\mu_i}^{-1}\left(W_i\cap\overline{M_i}\right)\subseteq\overline{\mu_i}^{-1}\left(\overline{U_i}\right)\subseteq U$$

while on $\mu_i^{-1}(W_i)$ we have

$$f_i \circ \mu_i = \overline{f_i}\big|_{W_i} \circ \mu_i = f$$

Since $m \in U$ was arbitrary it follows that $f \in \mathcal{M}(U)$ and thus $\mathcal{M}(U) = \overline{\mathcal{M}}(U)$. ∎

Bump Functions

Definition 4.2.1 ([27])**.** Recall that a continuous function $f : M \to \mathbb{R}$ is a *bump function for C with support in U* if $0 \leq f \leq 1$, $f \equiv 1$ on C, $\operatorname{supp}(f) \subseteq U$, and $U \in \operatorname{Open}(M)$ with $C \subseteq U$ where recall that $\operatorname{supp}(f) \underset{\text{def}}{=} \operatorname{Cl}_M(\operatorname{carr}(f))$. If M has a sheaf $\mathcal{M}$ of continuous functions then we will call a function $f : M \to \mathbb{R}$ a *smooth bump function* or a *$\mathcal{M}(M)$-bump function for C with support in U* if it is a bump function for C with support in U and $f \in \mathcal{M}(M)$. We will say that $(M, \mathcal{M})$ *admits smooth bump functions* or admits $\mathcal{M}(M)$-bump functions if for every closed subset C of M and every open subset U of M with $C \subseteq U$, there exists a smooth bump function for C with support in U.

The following proposition shows that under very general conditions, the space of global sections $\mathcal{M}(M)$ completely determines the sheaf $\mathcal{M}$. Of course, this proposition can be applied to smooth manifolds and, as we shall see later, to promanifolds (def. 5.0.2).

Proposition 4.2.2. Assume that all $\mu_\bullet$ are open surjections, $(M, \mathcal{M})$ admits $\mathcal{M}(M)$-bump functions, and that for all $U \in \operatorname{Open}(M)$, if $f : U \to M$ is a smooth function with support contained in U then its trivial 0-extension to M belongs to $\mathcal{M}(M)$. Then for all $U \in \operatorname{Open}(M)$ the space of sections $\mathcal{M}(U)$ is exactly the set of all continuous functions $f : U \to \mathbb{R}$ such that for all $m \in U$ there exists a $g \in \mathcal{M}(M)$ and an open neighborhood $m \in B \in \operatorname{Open}(U)$ for which $g|_B = f|_B$.

Proof. Let $U \in \operatorname{Open}(M)$. Let $\mathcal{S}$ denote the set of all continuous functions described in the statement of this proposition. Suppose $f \in \mathcal{M}(U)$ so that f is continuous and let $m \in U$.

Let $\phi = \phi^m : M \to \mathbb{R}$ be a $\mathcal{M}(M)$-bump function supported in U and identically 1 on some neighborhood B of m in M. Let g denote the trivial zero extension of $\phi|_U \cdot f$ to all of M. Then $g|_B = \phi|_B \cdot f|_B = f|_B$ and hence $f \in \mathcal{S}$.

Now suppose that $f \in \mathcal{S}$ so that f is continuous and let $m \in U$. Let $g \in \mathcal{M}(M)$ and $m \in B \in \operatorname{Open}(U)$ be such that $g|_B = f|_B$. Since $g \in \mathcal{M}(M)$ there exists some index i, some open set $C_i \in \operatorname{Open}(M_i)$, and some $g_i \in \mathcal{M}_i(C_i)$ such that $m \in \mu_i^{-1}(V_i) \subseteq B$ and $g|_{\mu_i^{-1}(C_i)} = g_i \circ \mu_i|_{\mu_i^{-1}(C_i)}$. Let $V = \mu_i^{-1}(V_i)$ so that $f|_V = g|_V = g_i \circ \mu_i|_V$ and thus $f \in \mathcal{M}(U)$. ■

Chapter 5

Profinite Dimensional Manifolds

Henceforth, unless indicated otherwise, any definition or result that doesn't appear in [20], [26], or [12], that isn't well-known, or that isn't in an appendix, is original, meaning that the author found the result independently.

We will be primarily interested in projective limits of smooth manifolds where all bonding maps are smooth surjective submersions. Although such systems have been frequently studied (e.g. p-adic solenoids), they were often only considered as systems in Top (with all smooth structures) since Man, the category of smooth manifolds, is not complete while Top is complete. In fact, the limits of such systems in Top are often not even locally path-connected, let alone locally homeomorphic to a TVS. This fact has, for obvious reasons, caused many mathematicians to only consider studying the limits of such systems in the category Top and has discouraged research into finding a "differential theory of projective limits of manifolds" since it is no longer clear how one should even define a "smooth structure" on the limit of a smooth system. The first step towards finding a reasonable definition of a "smooth structure" on the limit of a smooth system comes from the well-known observation that is described in remark C.0.3, which leads us to the following convention.

Convention 5.0.1. We will henceforth use the functor from remark C.0.3 to canonically identify the category of smooth manifolds as a full subcategory of the category of commutative locally $\mathbb{R}$-ringed spaces.

Using this identification, we may take the limit of a smooth system in the category of commutative locally $\mathbb{R}$-ringed spaces to obtain a commutative locally $\mathbb{R}$-ringed space $(M, \mathcal{M})$, where it is now only natural to interpret the sheaf $\mathcal{M}$ as being the "smooth structure" on M.

Definition 5.0.2. Say that a projective system $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, \mathbb{N})$ is a

- *smooth system* if it is a system in the category of manifolds (and smooth maps).
- *smooth submersive (resp. fibered, surjective, etc.) system* if it is a smooth system all of whose connecting maps are smooth submersions (resp. fiber bundles, surjective, etc.).
- *profinite system* if it is a surjective smooth submersive system direct by $\mathbb{N}$, that is, if it's a smooth system whose connecting maps are all smooth surjective submersions.
- *regular profinite system* or a *fibered profinite system* if it is a profinite system and every connecting map is a smooth fiber bundle.

Two smooth systems will be called *smoothly equivalent* if there exists an equivalence transformation (def. 3.3.1) between them consisting of smooth maps, where any such equivalence transformation will be called a *smooth equivalence transformation.* A smooth equivalence transformation between profinite systems will necessarily consist of surjective smooth submersions and so it will simply be called an *equivalence transformation of profinite systems.* If $(M, \mu_\bullet)$ is a limit in Top of a profinite system Sys_M then we will let C_M^∞ denote the canonical sheaf on M induced by $\mu_\bullet$ (def. 4.0.6), we will call the commutative locally $\mathbb{R}$-ringed space (M, C_M^∞) a *promanifold*, and we will say that $(\mu_\bullet, \operatorname{Sys}_M)$ is a *smooth projective representation* of this promanifold.

Two smooth projective representations of (M, C_M^∞) will be called *smoothly equivalent* if there exists a smooth equivalence transformation between them that is also an equivalence transformation of these representations (def. 3.3.1). Such an equivalence transformation will be called an *equivalence transformation of smooth projective representations*. A *pfd structure* for a promanifold (M, C_M^∞) is an equivalence class of smooth projective representations of (M, C_M^∞), which, for $(\mu_\bullet, \operatorname{Sys}_M)$ as above, will be denoted by $[(\mu_\bullet, \operatorname{Sys}_M)]$.

Remark 5.0.3. Example 6.2.7 shows that there exists at least two distinct pfd structures for the promanifold $\mathbb{R}^{\mathbb{N}}$.

A *profinite dimensional manifold* or *profinite manifold* is a pair consisting of a promanifold (i.e. a commutative locally $\mathbb{R}$-ringed space (M, C_M^∞)) together with a pfd structure defined on it. If there exists a regular smooth projective representation of a promanifold then we will call it *regular*. ■

Remark 5.0.4. Although often not an issue, it is stressed that for a system Sys_M to be a smooth submersive (resp. fibered, surjective, etc.) system we require this property of *all* connecting maps $\mu_{ij} : M_j \to M_i$, where $i \leq j$ in I, and not just of the bonding maps of the form $\mu_{i,i+1} : M_{i+1} \to M_i$.

Notation 5.0.5. If the topological space of a promanifold is denoted by Z then unless indicated otherwise, C_Z^∞ will denote its sheaf. We will consequently often abuse terminology by saying that "Z is a promanifold" instead of "(Z, C_Z^∞) is a promanifold."

Remark 5.0.6. Viewing a profinite dimensional manifold as a promanifold together with a choice of an equivalence class of a profinite system: It is clear that if Sys_M and Sys_N are profinite systems with limits $((M, C_M^\infty), \mu_\bullet)$ and $((N, C_N^\infty), \nu_\bullet)$ that are smoothly equivalent, say by $(F_\bullet, \iota)$ and $(G_\bullet, \alpha)$, then (M, C_M^∞) and (N, C_N^∞) are necessarily diffeomorphic via $F = \varprojlim F_\bullet : (M, C_M^\infty) \to (N, C_N^\infty)$. Note that if $M \neq N$ then $(\nu_\bullet, \operatorname{Sys}_N)$ does not belong to any pdf structure on (M, C_M^∞) but $(F \circ \nu_\bullet, \operatorname{Sys}_N)$ will be in this pdf structure. Furthermore, if $((\widehat{N}, C_{\widehat{N}}^\infty), \widehat{\nu}_\bullet)$ is another limit of Sys_N and if $\widehat{F} = \varprojlim F_\bullet : (M, C_M^\infty) \to (\widehat{N}, C_{\widehat{N}}^\infty)$ is the resulting limit map then $\left(\widehat{F} \circ \widehat{\nu}_\bullet, \operatorname{Sys}_N\right) = (F \circ \nu_\bullet, \operatorname{Sys}_N)$ so that this smooth projective representation is dependent only on Sys_N and not on the choice of a particular limit cone. The above discussion shows that the assignment $\mathrm{Sys}_N \mapsto (F \circ \nu_\bullet, \mathrm{Sys}_N)$ is a map from the equivalence class of all profinite systems that are smoothly equivalent to Sys_M into the pdf structure of (M, C^∞_M) containing $(\mu_\bullet, \mathrm{Sys}_M)$ where in addition, this assignment is clearly bijective. This shows that given (M, C^∞_M), the rule that assigns to a pfd structure $[(\mu_\bullet, \mathrm{Sys}_M]$ on (M, C^∞_M) the equivalence class $[\mathrm{Sys}_M]$ of smoothly equivalence profinite systems is a bijection. Thus we may view a profinite-dimensional manifold as being a promanifold (M, C^∞_M) together with a choice of pdf structure (as defined above, e.g. (M, C^∞_M) with a pdf structure $[(\mu_\bullet, \mathrm{Sys}_M])$ or as a promanifold together with a choice of an (appropriate) equivalence class of a profinite systems (e.g. (M, C^∞_M) together with $[\mathrm{Sys}_M]$) for which the promanifold is the vertex of some limit cone of one/all of these systems.

Assumption 5.0.7. If we say that M (resp. N) is a promanifold then it should be assumed that it has $(\mu_\bullet, \mathrm{Sys}_M)$ (resp. $(\nu_\bullet, \mathrm{Sys}_N)$) as a distinguished smooth projective representation and that its sheaf is the canonical sheaf (def. 4.0.6) induced by $\mu_\bullet$ (resp. $\nu_\bullet$).

Example 5.0.8. The the constant system canonically makes every smooth manifold into a promanifold.

The construction in the following example is sometimes useful for creating counterexamples.

Example 5.0.9. Describing a descending intersection of manifolds as a promanifold: Suppose that $M_\bullet$ is a countable decreasing sequence of manifolds where all M_i are of the same dimension d and M_j is smoothly embedded in M_i for $i \leq j$. For $i \leq j$ denote the natural inclusion of M_j into M_i by $\mathrm{In}_{ij} : M_j \to M_i$ so that we obtain $\mathrm{Sys}_M \underset{\text{def}}{=} (M_i, \mathrm{In}_{ij}, \mathbb{N})$ with $M \underset{\text{def}}{=} \bigcap_i M_i$ as its limit in Top and the natural inclusions $\mathrm{In}_i : M \to M_i$ as its projections.

For each $i \in \mathbb{N}$, let N_i denote the manifold $\bigsqcup_{h \leq i} M_h$ and denote the natural injections by $\mathrm{In}^i_h : M_h \to N_i$. Define bonding maps $\nu_{i,i+1} : N_{i+1} \to N_i$ by

$$\nu_{i,i+1}\big|_{N_i} \underset{\text{def}}{=} \mathrm{Id}_{N_i} \qquad \text{and} \qquad \nu_{i,i+1}\big|_{M_{i+1}} \underset{\text{def}}{=} \mathrm{In}_{i,i+1}$$

where observe that these are all smooth surjective submersions (where as the inclusions In_{ij} were not necessarily surjective). Let $\mathrm{Sys}_N = (N_i, \nu_{ij}, \mathbb{N})$, let $N = M \sqcup \left(\underset{i}{\sqcup} M_i \right)$, where we denote the natural injections by $\mathrm{In}_{M_i}^N : M_i \to N$ and $\mathrm{In}_M^N : M \to N$, and for all $a \in \mathbb{N}$ define

$$\begin{aligned} \nu_i : N &\longrightarrow N_i \\ n &\longmapsto \begin{cases} \mathrm{In}_i(n) & \text{if } n \in M \\ \nu_{ij}(n) & \text{if } n \in M_j \text{ with } i < j \\ \mathrm{In}_h^i(n) & \text{if } n \in M_h \text{ with } h \leq i \end{cases} \end{aligned}$$

Recall from example 2.1.44 that $(N, \nu_\bullet) = \varprojlim \mathrm{Sys}_N$ in Set.

Since M_j is smoothly embedded in M_i for $i \leq j$ the restrictions to M of $C_{M_j}^\infty$ and of $C_{M_i}^\infty$ are isomorphic so that we can define a sheaf $C_M^\infty \underset{\text{def}}{=} C_{M_i}^\infty\big|_M$ where $i \in \mathbb{N}$ is arbitrary. Since $M \subseteq M_i$ for each index i we can define $h_i \underset{\text{def}}{=} \mathrm{In}_i^i\big|_M : M \to N_i$ where recall that $\mathrm{In}_i^i : M_i \to N_i$ is the injection of M_i into $N_i = \underset{h \leq i}{\sqcup} M_h$. Clearly, each h_i is a smooth (in the sense of definition 4.0.4) topological embedding and $h_i = \nu_{i,i+1} \circ h_{i+1}$ for all indices i so that we obtain the smooth topological embedding $h \underset{\text{def}}{=} \varprojlim h_\bullet : (M, C_M^\infty) \to (N, C_N^\infty)$. By definition of $h_\bullet$ it is clear that $h = \mathrm{In}_M^N$. Observe that since $h = \mathrm{In}_M^N$ is a topological embedding we now also have the option of assigning to M the sheaf induced by $C_N^\infty\big|_{h(M)}$. We've shown above that h is smooth and by identifying each M_h as a submanifold of N_i (for $h \leq i$) it is not too difficult to see from the definition of $C_N^\infty\big|_{h(M)}$ that $h : (M, C_M^\infty) \to \left(h(M), C_N^\infty\big|_{h(M)} \right)$ is in fact a diffeomorphism. ∎

Some Basic Properties of Promanifolds

Lemma 5.1.1. Every promanifold M has the following properties:

(1) M is a Polish space and so is, in particular, also strongly Baire.

(2) If there is any M_i without isolated points then M has no isolated points.

(3) If there is any M_i whose components all have non-zero dimension then M is perfect. In particular, if all M_i are connected and $\sup_i \dim M_i = \infty$ then for all i and all properly embedded submanifolds $S_i \subseteq M_i$ the set $\mu_i^{-1}(S_i)$ is perfect.

(4) If C is a connected component of M that is also an open subset of M then C is a promanifold and the limit of the inverse system of subsets $(\mu_i(C))_{i\in\mathbb{N}}$, where each C_i is an open submanifold of M_i.

Proof. (1): Since countable products and closed subspaces of Polish spaces are Polish then the fact that every manifold is a Polish space implies that every promanifold is a Polish space.

(2): If $m \in M$ is an isolated point (i.e. $\{m\}$ is open in M) then each $\mu_i(m)$ is an open subset of M_i i.e. it is an isolated point. Hence if any M_i has no isolated points then M has no isolated points.

(3): Since M is a Polish space by the Cantor-Bendixson theorem M can be written as the disjoint union of a perfect set P and a countable open set U. So for all indices i, $\mu_i(U)$ is a countable open subset which is only possible $\dim M_i = 0$. Since there is some i for which $\dim M_i \neq 0$ this implies that $U = \varnothing$ so that $M = P$ is perfect. If $m \in M$ is an isolated point (i.e. $\{m\}$ is open in M) then each $\mu_i(m)$ is an open subset of M_i i.e. it is an isolated point. Hence, if any M_i has no isolated points then M has no isolated points. If $\sup_i \dim M_i = \infty$ and $S_i \subseteq M_i$ is a properly embedded submanifold it is in particular closed so that $\mu_i^{-1}(S_i)$ is closed and since $\mu_i^{-1}(S_i)$ is also a promanifold with at least one $\mu_{ij}^{-1}(S_i)$ having non-zero dimension, it has no isolated points so that $\mu_i^{-1}(S_i)$ is perfect.

(4): Since the connected component C of M is open in M and since the $\mu_\bullet$ are open maps we have that each $\mu_i(C)$ is open in M_i and hence a submanifold of M_i. Since C is also closed in M we have that $C = \bigcap_{i\in\mathbb{N}} \mu_i^{-1}(\mu_i(C)) = \varprojlim \mu_i(C)$ and since it is clear that each $\mu_{ij}|_{\mu_j(C)} : \mu_j(C) \to \mu_i(C)$ is a smooth surjective submersion, it follows that C is a promanifold.

∎

Lemma 5.1.2. Fix $k = 0, 1, \ldots, \infty$ and let $(M, C_M^k, \mu_i) = \varprojlim(M_\bullet, C_{M_i}^k, \mu_{ij})$. Then M's topology (as the limit of $(M_\bullet, \mu_{ij})$ in Top) is equal to the weak-topology that C_M^k induces on M.

Proof. Let $\overleftarrow{\tau}$ denote the topology on M that M has from being $\varprojlim(M_\bullet, \mu_{ij})$ and let τ_w be the weakest topology on M making all functions in C_M^k continuous. Since all functions in C_M^k are continuous we have $\tau_w \subseteq \overleftarrow{\tau}$. Let $U = \mu_i^{-1}(U_i)$ be a basic open set for some index i and some $U_i \in \operatorname{Open}(M_i)$. Let $f_i : M_i \to \mathbb{R}$ be non-negative function in $C_{M_i}^k(M_i)$ such that $f_i \equiv 0$ on $M_i \setminus U_i$ and $f_i > 0$ on U_i. Let $f = f_i \circ \mu_i : M \to \mathbb{R}$ so $f \in C_M^k$. Note that

$$f^{-1}((0,\infty)) = \mu_i^{-1}\big(f_i^{-1}((0,\infty))\big) = \mu_i^{-1}(U_i) = U$$

so that $U \in \tau_w$. Thus $\overleftarrow{\tau} = \tau_w$. ∎

Subpromanifolds

The next definition is from [20, p. 10].

Definition 5.2.1. Let M be a promanifold and let $S \subseteq M$. Call S a *subpromanifold (of M)* if there exists a smooth representation of (M, C_M^∞) such that the inverse system of subsets induced by S is a surjective inverse system of manifolds whose limit in Top is S where, as usual, this includes the requirement that the system's bonding maps are smooth surjective submersions. Explicitly, this means that there exists some smooth representation $(\mu_\bullet, \operatorname{Sys}_M = (M_\bullet, \mu_{ij}))$ of (M, C_M^∞) such that

(1) $S_i \underset{\text{def}}{=} \mu_i(S)$ is an embedded submanifold of M_i for all i,

(2) $\mu_{ij}|_{S_j} : S_j \to S_i$ is a surjective smooth submersion for all $i \le j$, and

(3) $S = \varprojlim \operatorname{Sys}_S$ in Top, where $\operatorname{Sys}_S = \big(S_i, \mu_{ij}|_{S_j}, \mathbb{N}\big)$ is the inverse system induced by S.

Definition 5.2.2. Since the above definition is dependent not only on the ringed space (M, C_M^∞) but also on the entire profinite structure $(M, C_M^\infty, \mu_\bullet, \operatorname{Sys}_M)$ we may also call a subpromanifold of M a Sys_M*-submanifold (of M)*.

Remark 5.2.3. If for some given representation $(\mu_i, \operatorname{Sys}_M)$ there exists a sequence $(l_i)_{i \in \mathbb{N}} \subseteq \mathbb{N}$ such that the system $\left(\mu_{l_i}, M_{l_i}, \mu_{l_i, l_j}\right)$ satisfies the above condition then by using this subrepresentation in place of $(\mu_i, \operatorname{Sys}_M)$ we see that S is a subpromanifold of M.

We now generalize various definitions of submanifolds to promanifolds that are dependent only on the ringed space (M, C_M^∞). Recall that for any $S \subseteq M$, the restriction sheaf is denoted by $C_M^\infty\big|_S$.

Definition 5.2.4. Let M be a promanifold and let $S \subseteq M$. Call S a *(smoothly) embedded subpromanifold* if there exists some smooth embedding $F : N \to M$ from some promanifold (N, C_N^∞).

Definition 5.2.5. Call a subset $S \subseteq M$ an *immersed subpromanifold (of M)* if it is the image of some injective smooth local embedding.

Neighborhood Basis at a Point

The following definition will be used frequently.

Definition 5.3.1. Suppose $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, I)$ is a directed inverse system in Top with limit $(M, \mu_\bullet)$, $m \in M$, $J \subseteq I$, and $U_\bullet = (U_j)_{j \in J}$. Call $U_\bullet$ a *topological $\mu_\bullet$-neighborhood basis at m* if J is cofinal in I and for all $j \in J$,

(1) U_j is a neighborhood of $\mu_j(m)$ in M_j,

(2) $\mu_{jk}(U_k) \subseteq \operatorname{Int}(U_j)$ whenever $k \geq j$ with $k \in J$,

(3) for all $\mu_j(m) \in O_j \in \operatorname{Open}(M_j)$ there exists some $k \in J$ with $k \geq j$ such that $\mu_{jk}(U_k) \subseteq O_j$,

If we say "let $U_\bullet$ topological $\mu_\bullet$-neighborhood basis at m" without specifying J then it should be assumed that $J = I$. ■

Remarks 5.3.2.

- If a topological $\mu_\bullet$-neighborhood basis $U_\bullet = (U_j)_{j\in J}$ at m exists then the sets $\left(\mu_j^{-1}(U_j)\right)_{j\in J}$ will clearly form a neighborhood basis at m in M.
- It is easy to see that a topological $\mu_\bullet$-neighborhood basis at m exists whenever I is countable (and directed) and each M_i has a countable neighborhood basis at $\mu_i(m)$.
- The following illustrates a typical use of a topological $\mu_\bullet$-neighborhood basis $(U_i)_{i\in I}$ at m:

  We somehow inductively construct a cofinal subset J of I while simultaneously somehow picking $m^j \in \mu_j^{-1}(U_j)$ for each $j \in J$. The definition of "topological $\mu_\bullet$-neighborhood basis at m" then guarantees that for every $i \in I$, the net $(\mu_i(m^j))_{j\in J}$ converges to $\mu_i(m)$ in M_i from which we conclude that $(m^j)_{j\in J} \to m$ in M. ■

Since we will most often be working with a system whose bonding maps are smooth surjective submersion between manifolds, we will want to work with topological $\mu_\bullet$-neighborhood bases have even more properties that those described in definition 5.3.1.

Definition 5.3.3. Let $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, \mathbb{N})$ be profinite system with limit $(M, \mu_\bullet)$, let $U_\bullet = (U_i)_{i\in\mathbb{N}}$ be an indexed collection of sets, and let $m \in M$. Say that $U_\bullet$ is a *$\mu_\bullet$-neighborhood basis at m* if

(1) $U_\bullet$ forms a topological $\mu_\bullet$-neighborhood basis at m in M,

(2) whenever $i \leq j < k$ then

$$\overline{\mu_{ik}(U_k)} \subseteq \mu_{ij}(\operatorname{Int}(U_j))$$

where the closure (resp. interior) is taken in M_i (resp. M_j),

- so in particular, whenever $i < j$ then $\mu_{ij}\left(\overline{U_j}\right) \subseteq \operatorname{Int}(U_i)$,

(3) each U_i is relatively compact in M_i, and

(4) for each index i, there exists a chart $(\operatorname{Dom}\varphi_i, \varphi_i)$ centered at $\mu_i(m)$ whose domain contains $\overline{U_i}$ and such that each $\mu_{i,i+1}$'s coordinate representation is the canonical projection.

Say that a $\mu_\bullet$-neighborhood basis at m is *fast descending* if in addition, for all $i \in \mathbb{N}$,

(5) there is some $r_i > 0$ such that $\varphi_i(U_i) =]-r_i, r_i[^{\dim_{\mu_i(m)} M_i}$, and

(6) $r_{i+1} < \frac{r_i}{(i+2)^{i+2}}$ with $r_1 < 1$.

If $W_\bullet \subseteq M_\bullet$ then we'll say that $U_\bullet$ *is subordinate to* $W_\bullet$ if $\overline{U_\bullet} \subseteq W_\bullet$. ■

Assumption 5.3.4. If the sets $\mathcal{U}_\bullet$ in lemma 5.3.6 below are not given then we will define $\mathcal{U}_\bullet$ as follows: (1) if we are not given coordinate charts $(\operatorname{Dom}\varphi_i, \varphi_i)$ centered at $\mu_i(m)$ then inductively pick coordinate charts $(\operatorname{Dom}\varphi_i, \varphi_i)$ on M_i centered at $\mu_i(m)$ such that $\mu_{i,i+1}\left(\overline{\operatorname{Dom}\varphi_{i+1}}\right) \subseteq \operatorname{Dom}\varphi_i$, $\varphi_i : \operatorname{Dom}\varphi_i \to \mathbb{R}^{\dim M_i}$ is a diffeomorphism from the relatively compact open set $\operatorname{Dom}\varphi_i \in \operatorname{Open}(M_i)$ onto an open cube in $\mathbb{R}^{\dim M_i}$, and $\mu_{i,i+1}$'s coordinate representation with respect to φ_{i+1} and φ_i is the canonical projection. (2) For each index i, define $\mathcal{U}_i$ as the set of all $\varphi^{-1}\left(\widetilde{U}_l\right)$, where $\widetilde{U}_l$ is the open cube in $\mathbb{R}^{\dim M_i}$ centered at the origin with sides of length $l > 0$.

Remark 5.3.5. Lemma 5.3.9 provided the motivation for the definition of a "fast descending $\mu_\bullet$-neighborhood basis" that is given in definition 5.3.3, where such a $\mu_\bullet$-neighborhood basis will be used to conclude that all derivatives of certain inductively constructed maps will vanish at a particular point.

Lemma 5.3.6 (Existence of $\mu_\bullet$-neighborhood bases)**.** Suppose Sys_M is a profinite dimensional system and fix $m \in M$. Let $\mathcal{U}_\bullet = (\mathcal{U}_i)_{i=1}^{\infty}$ where for each index i, $\mathcal{U}_i$ consists of some collection of (not necessarily open) neighborhoods of $\mu_i(m)$ in M_i that form a neighborhood basis at $\mu_i(m)$ for M_i. Let $(W_i)_{i\in\mathbb{N}}$ be any collection of sets such that for each index i, W_i is a neighborhood of $\mu_i(m)$ in M_i.

(1) If $U_\bullet = (U_i)_{i\in\mathbb{N}}$ is a $\mu_\bullet$-neighborhood basis at m then $\left(\mu_i^{-1}(U_i)\right)_{i\in\mathbb{N}}$ is neighborhood basis for M at m.

(2) There exists a topological $\mu_\bullet$-neighborhood basis $U_\bullet = (U_i)_{i\in\mathbb{N}}$ at m subordinate to $W_\bullet$ such that $U_i \in \mathcal{U}_i$ for each index i.

(3) There exists a fast descending $\mu_\bullet$-neighborhood basis $U_\bullet = (U_i)_{i\in\mathbb{N}}$ at m subordinate to $W_\bullet$.

(4) If $U_\bullet$ is a fast descending $\mu_\bullet$-neighborhood basis at m and $m^\bullet \subseteq M$ is a sequence such that $m^i \in \mu_i^{-1}(U_i)$ for all $i \in \mathbb{N}$ then $m^\bullet$ converges fast to m in M (def. 15.2.1).

Proof. Let $m_\bullet = \mu_\bullet(m)$ and $d_\bullet = \dim_{m_\bullet} M_\bullet$.

(1) is obvious and (4) is immediate from the definition of fast convergence.

(2): Observe that we may replace $\mathcal{U}_i$ with the set of all $U_i \in \mathcal{U}_i$ such that U_i is relatively compact whose closures are contained in some coordinate chart centered at m_i such that statement (1) (although not necessarily statement (2)) in assumption 5.3.4 is satisfied. This will guarantee that (6) in definition 5.3.3 holds (note that this will not limit us to considering only $U_i \in \mathcal{U}_i$ that are open coordinate boxes (which may not even belong to this set) since (6) in def. 5.3.3 merely requires that each U_i be contained in such a coordinate box).

Let $U_1^\bullet = (U_1^l)_{l=1}^\infty \subseteq \mathcal{U}_i$ be any neighborhood basis for M_1 at m_1 such that $\overline{U_1^l} \subseteq \operatorname{Int}(U_1^{l+1})$ for all l and $\overline{U_1^1} \subseteq \operatorname{Int}(W_1)$. Suppose we've defined $U_1^\bullet = (U_1^l)_{l=1}^\infty, \ldots, U_j^\bullet = \left(U_j^l\right)_{l=1}^\infty$ where for all $1 \le i \le j$,

(a) $U_i^\bullet \subseteq \mathcal{U}_i$ is a neighborhood bases for M_i at m_i,

(b) $\overline{U_i^1} \subseteq \operatorname{Int}(W_i)$,

(c) $\overline{U_i^l} \subseteq \operatorname{Int}(U_i^{l+1})$ for each $l \in \mathbb{N}$, and

(d) $h \leq i < j$ implies $\overline{\mu_{hj}\left(U_j^l\right)} \subseteq \mu_{hi}(\operatorname{Int}(U_i^{l+1}))$ for each $l \in \mathbb{N}$.

Let $\left(V_{j+1}^l\right)_{l=1}^{\infty}$ be any neighborhood basis for M_{j+1} at m_{j+1} consisting of open sets such that $\overline{V_{j+1}^l} \subseteq V_{j+1}^{l+1}$ for all l and $\overline{V_{j+1}^1} \subseteq \operatorname{Int}(W_{j+1})$. Let $U_{j+1}^1 \in \mathcal{U}_{j+1}$ be a neighborhood of m_{j+1} whose closure is contained in $V_{j+1}^1 \cap \mu_{j,j+1}^{-1}\left(\operatorname{Int}\left(U_j^{1+1}\right)\right)$ and such that for each $h = 1, \ldots, j$,

$$\overline{\mu_{h,j+1}\left(U_{j+1}^1\right)} \subseteq \bigcap_{h \leq i \leq j} \mu_{hi}\left(\operatorname{Int}\left(U_i^{1+1}\right)\right)$$

which is possible since each μ_{hj} is an open map so that the above intersection is an open neighborhood of m_h. Now inductively pick for each $l \geq 2$ a neighborhood $U_{j+1}^l \in \mathcal{U}_i$ of m_{j+1} in M_{j+1} whose closure is contained in $V_{j+1}^l \cap \mu_{j,j+1}^{-1}\left(\operatorname{Int}\left(U_j^{l+1}\right)\right) \cap \operatorname{Int}\left(U_{j+1}^{l-1}\right)$ and such that for each $h = 1, \ldots, j$,

$$\overline{\mu_{h,j+1}\left(U_{j+1}^l\right)} \subseteq \bigcap_{h \leq i \leq j} \mu_{hi}\left(\operatorname{Int}\left(U_i^{l+1}\right)\right)$$

Note that (a) is satisfied since $U_j^l \subseteq \overline{U_j^l} \subseteq V_j^l$ for all $l \in \mathbb{N}$ while (b), (c), and (d) are satisfied by construction. This completes the construction of $U_{j+1}^{\bullet} \underset{\text{def}}{=} \left(U_{j+1}^l\right)_{l=1}^{\infty}$. For all $i \in \mathbb{N}$, let $U_i = U_i^i$. For any $i \leq j < l$ in $\mathbb{N}$ observe that by using (j, j, l) in place of (h, i, j) in hypothesis (d) of the above construction of $U_l^{\bullet}$ we get

$$\overline{\mu_{jl}(U_l^l)} \subseteq \mu_{jj}\left(\operatorname{Int}\left(U_j^{l+1}\right)\right) = \operatorname{Int}\left(U_j^{l+1}\right)$$

so that

$$\overline{\mu_{il}(U_l)} = \mu_{ij}\left(\overline{\mu_{jl}\left(U_l^l\right)}\right) = \overline{U_l^l} \subseteq \mu_{ij}\left(\operatorname{Int}\left(U_j^{l+1}\right)\right) \subseteq \mu_{ij}\left(\operatorname{Int}\left(U_j^j\right)\right) = \mu_{ij}(\operatorname{Int}(U_j))$$

Similarly, by using (i, i, l) in place of (h, i, j) (where we still have $i < l$) we obtain

$$\mu_{il}\left(\overline{U_l}\right) \subseteq \overline{\mu_{il}(U_l^l)} \subseteq \mu_{ii}\left(\operatorname{Int}\left(U_i^{l+1}\right)\right) = \operatorname{Int}\left(U_i^{l+1}\right) \subseteq \operatorname{Int}\left(U_i^i\right) = \operatorname{Int}(U_i)$$

so that $\overline{U_l} \subseteq \mu_{il}^{-1}(\operatorname{Int}(U_i))$. Since $l > i$ was arbitrary and since $(U_i^l)_{l=i}^{\infty}$ is a neighborhood basis of M_i at m_i it follows that the same is true of $(\mu_{il}(U_l))_{l=i}^{\infty}$.

(3) Let (U_1, φ_1) be any chart on M_1 centered at m_i such that $\varphi_1(U_1) =]-r_1, r_1[^{d_1}$ for some $0 < r_1 < 1$ and $\overline{U_1} \subseteq W_1$. It is clear that in the above construction, upon removing the requirement that $U_i^l \in \mathcal{U}_i$ for all $l \in \mathbb{N}$, we could have added the new requirement that there be $\varphi_2, \varphi_3, \ldots$ such that for all $i \geq 2$, $\mu_{i-1,i}$'s coordinate representation with respect to φ_{i-1} and φ_i is the canonical projection, $\overline{U_i^l} \subseteq \operatorname{Dom} \varphi_i^l$, $\varphi_i(U_i) =]-r_i, r_i[^{d_i}$, and $r_{i+1} < \frac{r_i}{(i+2)^{i+2}}$. ∎

Definition 5.3.7 (Peano differentiability)**.** Recall ([15]) that for $e, n \in \mathbb{Z}^{\geq 0}$, a map $f : \mathbb{R}^e \to \mathbb{R}$ is n-times Peano differentiable at $\{0\}^e \in \mathbb{R}^e$ if there exists an n^{th}-degree polynomial $p(x_1, \ldots, x_e)$ in e-variables (i.e. the n^{th}-degree Taylor polynomial) such that $f(x) - p(x) = o(|x|^n)$, $x \to \{0\}^e$, where we say this of a $\mathbb{R}^d$ valued map if it is true of each of its coordinate. When we say that the all of *f's Peano derivatives vanish at* $\{0\}^e$ then we mean that for all $n \in \mathbb{N}$, $f(x) = o(|x|^n)$, $x \to \{0\}^e$ where since this property is clearly invariant under diffeomorphisms of the codomain that map $\{0\}^e \to \{0\}^e$, it should clear how to generalize this definition to maps going from subsets of $\mathbb{R}^e$ into smooth manifolds.

Example 5.3.8 (Peano smooth but not C^1)**.** It is known ([15, ex. 1.2]) that for any $m \in \mathbb{N}$, the map $f : \mathbb{R} \to \mathbb{R}$ given by $f(0) = 0$ and $f(x) = x^{m+1} \sin(x^{-m})$ is smooth on $\mathbb{R} \smallsetminus \{0\}$, m-times Peano differentiable at 0 (which implies that f differentiable at 0), but that f' is not continuous. That a curve f being smooth off of $\{0\}$ and having all of its Peano derivatives at 0 (rather than having only $m \in \mathbb{N}$ of them as in the previous example) does not even imply that f is C^1 (a likely already known fact for which the author could find no reference) is demonstrated by the following (original) example: let $f : \mathbb{R} \to \mathbb{R}^2$ be identically 0 on $]-\infty, 0]$ and let $f(x) = e^{-1/x} e^{\sqrt{-1}\, e^{1/x}}$ for $x > 0$.

Lemma 5.3.9. Let e be a non-negative integer, $m^0 \in M$, $\{0\}^e \in O \in \operatorname{Open}(\mathbb{R}^e)$, $O^* \underset{\text{def}}{=} O \smallsetminus \{\{0\}^e\}$, $\eta : (O, \{0\}^e) \to (M, m^0)$ a map such that $\eta\big|_{O^*} : O^* \to M$ is smooth, and $(U_i)_{i=1}^{\infty}$ a fast descending $\mu_\bullet$-neighborhood basis of m^0 with smooth surjective charts $\varphi_i : U_i \to]-r_i, r_i[^{d_i}$ centered at $\mu_i(m^0)$, where $d_i \underset{\text{def}}{=} \dim_{\mu_i(m^0)} M_i$. For all $r > 0$, let B_r denote the open ball of radius r in $\mathbb{R}^e$ (normed by some norm $\|\cdot\|$) centered at $\{0\}^e$. Suppose that for all $n \in \mathbb{N}$, $(\mu_n \circ \eta)\big(O^* \cap B_{1/n}\big) \subseteq U_n$. Then η is continuous and for all $i \in \mathbb{N}$, $\mu_i \circ \eta$ is everywhere infinitely Peano differentiable with all Peano derivatives vanishing at $\{0\}^e$, so that if for all $k, i \in \mathbb{N}$ there is some B_r such that $\mathrm{T}^k\big(\mu_i \circ \eta\big|_{O^*}\big)(B_r)$ is relatively compact in $\mathrm{T}^k M_i$ then η is a smooth map, all of whose derivatives vanishes at $\{0\}^e$.

Proof. Clearly, our hypotheses imply that $\eta : O \to M$ is continuous at $\{0\}^e$ and thus continuous everywhere. If $e = 0$ then there is nothing to prove so we may assume that $e > 0$. For all $l \in \mathbb{Z}^{\geq 0}$, let $m_\bullet^l = \mu_\bullet(m^l)$ and for all $l \neq 0$ consider $\mathbb{R}^l$ as a normed space normed by the maximum norm $\|\cdot\|_l$. For all $h \in \mathbb{N}$, let $\mu_{h,h+1}\big|_{U_{h+1}}$'s coordinate representation be $\widehat{\mu_{h,h+1}} : \operatorname{Im} \varphi_{h+1} \to \operatorname{Im} \varphi_h$, where this is the canonical projection. Since $U_\bullet$ is fast descending we have that for all $h \in \mathbb{N}$, $r_{h+1} < \frac{r_h}{(h+2)^{h+2}}$ and $\varphi_h(U_h) =]-r_h, r_h[^{d_h}$. In particular, observe that for all $m_h \in U_h$, $(h+1)^{h+1} \cdot \widehat{\mu_{h-1,h}}(\widehat{m_h}) \in \varphi_{h-1}(U_{h-1})$, where $(h+1)^{h+1}\cdot$ is scalar multiplication in $\mathbb{R}^{d_{h-1}}$ and where we denoted $\varphi_h(m_h)$ by $\widehat{m_h}$.

To show that all of η's Peano derivatives exist and vanish at 0, let $\eta_\bullet = \mu_\bullet \circ \eta$, and fix an index $i \in \mathbb{N}$. We must show that all of η_i's derivatives exist and vanish at 0 and for this, it suffices to show that the same is true of $\varphi_i \circ \eta_i\big|_{\eta_i^{-1}(U_i)}$. Let $p \in \mathbb{N}$ and $\epsilon > 0$. Since $(r_l)_{l=1}^\infty$ monotone converges to 0 we may pick $j > i$ such that $r_j < \epsilon$. Let $z \in O^* \cap B_{1/(j+p+1)}$ be arbitrary and define $m \underset{\text{def}}{=} \eta(z)$ and $m_\bullet \underset{\text{def}}{=} \mu_\bullet(m)$. Pick $g \in \mathbb{N}$ such that $\frac{1}{g+1} \leq \|z\| < \frac{1}{g}$, where we will necessarily have $g > j + p + 1$. Since $\|z\| < \frac{1}{g}$ we have by assumption that $m_g = \mu_g(\eta(z)) \in U_g$. Thus for all $h = 1, \ldots, g, \ldots, \iota(g)$, $m_h \in \mu_{h,\iota(g)}(U_{\iota(g)}) \subseteq U_h$ so that $\widehat{m}_h \underset{\text{def}}{=} \varphi_h(m_h)$ is well-defined and since for any $1 \leq d \leq h$, $\widehat{\mu}_{dh}$ is the canonical projection we have that $r \cdot \widehat{m}_d = r \cdot \widehat{\mu_{dh}}(\widehat{m}_h) = \widehat{\mu_{dh}}(r \cdot \widehat{m}_h)$ for any $r \in \mathbb{R}$ for which the above quantities are defined (i.e. for any r such that $r \cdot \widehat{m}_h \in \varphi_h(U_h)$).

Since $(g+1)^{g+1} r_g < r_{g-1}$ we have $(g+1)^{g+1} \widehat{\mu}_{g-1,g}(\widehat{m}_g) \in \varphi_{g-1}(U_{g-1})$, where the convexity of $\varphi_{g-1}(U_{g-1})$ implies that $(g+1)^p \widehat{m}_{g-1} \in \varphi_{g-1}(U_{g-1})$ (since $(g+1)^p \leq (g+1)^{g+1}$). From $\varphi_{g-1}(U_{g-1}) = \;]-r_{g-1}, r_{g-1}[^{d_{g-1}}$ we conclude that $\|(g+1)^p \widehat{m}_{g-1}\|_{g-1} < r_{g-1}$, where recall that

$\|\cdot\|_{g-1}$ is the max norm, and since $\widehat{\mu_{i,g-1}}$ is the canonical projection we have

$$(g+1)^p \|\widehat{m}_i\|_i = \|(g+1)^p \widehat{\mu_{i,g-1}}(\widehat{m}_{g-1})\|_i = \|\widehat{\mu_{i,g-1}}((g+1)^p \widehat{m}_{g-1})\|_i \leq \|(g+1)^p \widehat{m}_{g-1}\|_{g-1} < r_{g-1} < \epsilon$$

which implies that

$$\frac{\|\varphi_i(\eta_i(z))\|_i}{\|z\|^p} = \frac{1}{\|z\|^p} \|\widehat{m_i}\|_i \leq (g+1)^p \|\widehat{m_i}\|_i < \epsilon$$

Thus for all $p \in \mathbb{N}$, the p^{th}-Peano derivative of η_i at 0 exists and vanishes. ■

Products of Promanifolds

The next lemma will allow us to prove a Whitney embedding theorem for promanifolds and to provide an alternative construction of the canonical sheaf on the canonical limit M in terms of the canonical product sheaf.

Remark 5.4.1. If $\mu : N \to M$ is a smooth map and $\gamma : N \to M \times N$ is $\gamma(n) = (n, \mu(n))$ then $\gamma : N \to \Gamma(\mu) \underset{\text{def}}{=} \operatorname{Im} \gamma$ is a diffeomorphism onto the graph of μ so that if we consider μ as a map on $\Gamma(\mu)$ (i.e. $\mu \circ \gamma^{-1}$) then μ is just the restriction to $\Gamma(\mu)$ of the canonical projection $M \times N \to M$. As was done in [38] for complete metric spaces, this idea can be extended to systems by showing that we may always view the limit of a countable directed system as arising from an equivalent system that consists of subspaces of products of the original spaces with all bonding maps being canonical projections and with the equivalence transformation consisting of homeomorphisms. The following lemma extends the construction of [38] to profinite systems and describes it in the greater detail that is necessary for theorem 11.6.5 and corollary 5.4.4.

Lemma 5.4.2. Let $((M, C_M^\infty), \mu_\bullet)$ denote the canonical limit of $\operatorname{Sys}_M = (M_\bullet, \mu_{ij}, \mathbb{N})$ with its canonical sheaf. Let $\prod^a M_\bullet$ denote $\prod_{i=1}^a M_i$ with the standard smooth product manifold structure, let $\prod M_\bullet$ denote $\prod_{i=1}^{\infty} M_i$, and let each of the following maps

$$\operatorname{Pr}_{\le ab} : \prod^{b} M_\bullet \to \prod^{a} M_\bullet \qquad \operatorname{Pr}_{\le a} : \prod M_\bullet \to \prod^{a} M_\bullet$$
$$\operatorname{Pr}_{ab} : \prod^{b} M_\bullet \to M_a \qquad \operatorname{Pr}_{a} : \prod M_\bullet \to M_a$$

denote the canonical projection onto its codomain. Let $\operatorname{Sys}_{\prod M_\bullet} = \left(\prod^{a} M_\bullet, \operatorname{Pr}_{\le ab}, \mathbb{N}\right)$ and let $C^\infty_{\prod M_\bullet}$ denote the canonical product sheaf on $\prod M_\bullet$.

For each $b \in \mathbb{N}$, let $((P_b, \mathcal{P}_b), \pi_{\bullet b})$ be the canonical limit of $\operatorname{Sys}_{P_b} \underset{\text{def}}{=} (M_i, \mu_{ij}, \{1, \ldots, b\})$ with the canonical sheaf. Define for all $a, b, i \in \mathbb{N}$ with $a \le b$ the maps

$$\rho_{ab} = \operatorname{Pr}_{\le ab}\big|_{P_b} : P_b \to P_a \qquad \rho_a = \operatorname{Pr}_{\le a}\big|_{M} \quad : M \to P_a$$
$$F_a = (\mu_{1a}, \ldots, \mu_{aa}) : M_a \to P_a \qquad G_i \underset{\text{def}}{=} \pi_{ii} = \operatorname{Pr}_{ii}\big|_{P_i} : P_i \to M_i$$

so that for $a > 1$, $F_a(m_a) = (\mu_{1a}(m_a), \ldots, \mu_{a-1,a}(m_a), m_a)$. Then

(1) The canonical sheaf $\mathcal{P}_a$ is exactly the restriction to P_a of $\prod^{a} M_\bullet$'s sheaf of smooth $\mathbb{R}$-valued functions.

(2) $F_a : M_a \to \prod^{a} M_\bullet$ is a proper smooth embedding with image P_a and smooth inverse $G_a : P_a \to M_a$.

(3) $\operatorname{Sys}_P \underset{\text{def}}{=} (P_\bullet, \rho_{ab}, \mathbb{N})$ forms a smooth surjective inverse systems of manifolds with each ρ_{ab} a surjective smooth submersion between manifolds and $(M, \rho_\bullet) = \varprojlim \operatorname{Sys}_P$. Furthermore, the canonical sheaves on M induced by $\rho_\bullet$ and $\mu_\bullet$ are equal.

(4) $(F_\bullet, \operatorname{Id}_{\mathbb{N}}) : \operatorname{Sys}_M \to \operatorname{Sys}_P$ and $(G_\bullet, \operatorname{Id}_{\mathbb{N}}) : \operatorname{Sys}_P \to \operatorname{Sys}_M$ are inverse system morphisms, both of whose limits are the identity map $\operatorname{Id}_M : M \to M$. Furthermore, they form an equivalence transformation of profinite systems.

(5) The inclusion maps $\operatorname{In}_{P_a} : P_a \to \prod^{a} M_\bullet$ form an inverse system morphism $(\operatorname{In}_{P_\bullet}, \operatorname{Id}_{\mathbb{N}})$:

$\mathrm{Sys}_P \to \mathrm{Sys}_{\prod M_\bullet}$ consisting of proper smooth embeddings whose limit map is the natural inclusion $\mathrm{In} : M \to \prod M_\bullet$.

(6) $C_M^\infty = C_{\prod M_\bullet}^\infty\big|_M$.

Remark 5.4.3. Recall that the canonical product sheaf on $\prod M_\bullet$ (def. 4.0.6) is defined to be the canonical sheaf induced by $\mathrm{Pr}_{\le a}$ and the profinite system $\mathrm{Sys}_{\prod M_\bullet}$. Also recall that by definition of the canonical limit, $\mu_\bullet \underset{\text{def}}{=} \mathrm{Pr}_\bullet\big|_M$ and $\pi_{ab} \underset{\text{def}}{=} \mathrm{Pr}_{ab}\big|_{P_b} : P_b \to M_a$ for all $a \le b$.

Proof. That $(P_a, \mathcal{P}_a)$ is a smooth manifold follows from example 2.1.28 and the fact that $(P_a, \mathcal{P}_a)$ is, by construction, the canonical limit of $(M_i, \mu_{ij}, \{1, \ldots, a\})$ in the category of commutative locally $\mathbb{R}$-ringed spaces. Since the statements (3) and (6) assert set equalities (and not merely the existence of diffeomorphisms) we will begin our proof by showing how the canonical limit P_a may be identified with the graph of a smooth function. If $\eta \underset{\text{def}}{=} (\mu_{1a}, \ldots, \mu_{a-1,a}) : M_a \to \prod^{a-1} M_\bullet$ then it is well-known that its graph

$$\Gamma(\eta) = \{(m_a, \eta(m_a)) \,|\, m_a \in M_a\}$$

is a properly embedded smooth submanifold of $M_a \times \prod_{i=1}^{a-1} M_i$ and the map

$$\begin{aligned} M_a &\longrightarrow M_a \times \prod^{a-1} M_\bullet \\ m_a &\longmapsto (m_a, \eta(m_a)) \end{aligned}$$

is a proper smooth embedding onto $\Gamma(\eta)$. Now observe that for any $(m_1, \ldots, m_a) \in \prod^{a} M_\bullet$,

$$\begin{aligned} (m_a, m_1, \ldots, m_{a-1}) \in \Gamma(\eta) &\iff (m_1, \ldots, m_{a-1}) = \eta(m_a) \\ &\iff m_i = \mu_{ij}(m_j) && \text{for all } 1 \le i \le j \le a \\ &\iff (m_1, \ldots, m_{a-1}, m_a) \in P_a && \text{by definition of } P_a \end{aligned}$$

that is, the set P_a is just the graph of η with the positions of the domain and codomain elements switched: $P_a = \{(\eta(m_a), m_a) \,|\, m_a \in M_a\}$. So if we let $\mathcal{S}_a$ denote the restriction to P_a of sheaf of smooth $\mathbb{R}$-valued functions on $\prod^a M_\bullet$ then $(P_a, \mathcal{S}_a)$ becomes a properly embedded smooth submanifold of the product manifold $\prod^a M_\bullet$ and $F_a = (\mu_{1a}, \ldots, \mu_{1a}) : M_a \to \prod^a M_\bullet$ is a proper smooth embedding onto P_a. It should now be clear that $\pi_a : (P_a, \mathcal{S}_a) \to M_a$ is the smooth inverse of $F_a : M_a \to (P_a, \mathcal{S}_a)$.

Even through $(P_a, \mathcal{S}_a)$ and $(P_a, \mathcal{P}_a)$ are diffeomorphic, since the sheaves $\mathcal{S}_a$ and $\mathcal{P}_a$ are defined differently it is not entirely clear that these smooth structure are identical (i.e. that the identity map $\mathrm{Id}_{P_a} : (P_a, \mathcal{S}_a) \to (P_a, \mathcal{P}_a)$ is a diffeomorphism) so we will now show that $\mathcal{S}_a = \mathcal{P}_a$, which will consequently also make statement (2) unambiguous. This is straightforward: The map $\mathrm{Id}_{P_a} : (P_a, \mathcal{S}_a) \to (P_a, \mathcal{P}_a)$ is smooth by the universal property of limits since for any index $i \leq a$, the composition $\pi_{ia} \circ \mathrm{Id}_{P_a} = \mathrm{Pr}_{ia}\big|_{P_a}$ is smooth as a map from $(P_a, \mathcal{S}_a)$ into $(M_i, C^\infty_{M_i})$. Hence for all $W \in \mathrm{Open}(P_a)$, $\mathcal{P}_a(W) \subseteq \mathcal{S}_a(W)$. To see that the inclusion map $\mathrm{In}_{P_a} : (P_a, \mathcal{P}_a) \to \left(\prod^a M_\bullet, C^\infty_{\prod^a M_\bullet}\right)$ is smooth, note that this map is simply $\mathrm{In}_{P_a} = (\pi_{1a}, \ldots, \pi_{aa})$, where the fact that each coordinate $\pi_{ia} : (P_a, \mathcal{P}_a) \to (M_i, C^\infty_{M_i})$ is smooth implies that $(\pi_{1a}, \ldots, \pi_{aa})$ is a smooth map into the product manifold. The smoothness of this inclusion implies that for all $W \in \mathrm{Open}(P_a)$, $\mathcal{S}_a(W) \subseteq \mathcal{P}_a(W)$. Observe that (2) follows immediately from this equality of sheaves.

(3): Observe that $\rho_{ab} \circ F_b = \varphi_a \circ \mu_{ab}$ where since F_b and F_a are diffeomorphisms and μ_{ab} is a smooth submersion we it follows that $\rho_{ab} = F_a \circ \mu_{ab} \circ F_b^{-1}$ is a smooth submersion. Recalling that

$$P_a \underset{\text{def}}{=} \left\{ (m_1, \ldots, m_a) \in \prod^a M_\bullet \,\middle|\, \mu_{ij}(m_j) = m_i \right\}$$

we see that since each μ_{ij} is surjective then the same is true of each ρ_{ab}. Let Z be any space and let $h_a = (h_a^1, \ldots, h_a^a) : Z \to P_a \subseteq \prod^a M_\bullet$ be any collection of morphisms compatible with Sys_P where $h_a^i : Z \to M_i$ for all $i = 1, \ldots, a$. Since

$$\left(h_b^1, \ldots, h_b^a\right) = \mathrm{Pr}_{ab}\left(h_b^1, \ldots, h_b^b\right) = \rho_{ab} \circ h_b = h_a = \left(h_a^1, \ldots, h_a^a\right)$$

we have that $h^i_a = h^i_b$ for all indices $i \leq a, b$ so that $h_a = (h^1_a, \ldots, h^a_a) = (h^1_1, \ldots, h^a_a)$ for all a. Let $h = (h^1_1, h^2_2, \ldots) : Z \to M$ where for each $z \in Z$, $h(z)$ belongs to M since if $i \leq j$ then $h_j(z) \in P_j$ implies that $\mu_{ij}(h^j_j(z)) = h^i_j(z) = h^i_i(z)$. Note that $\rho_a \circ h = (h^1_1, \ldots, h^a_a) = h_a$ for all indices a and if $k = (k^1, k^2, \ldots) : Z \to M$ is any map such that $\rho_a k = h_a$ for all a then

$$\left(h^1_1, \ldots, h^a_a\right) = h_a = \rho_a k = \left(k^1, \ldots, k^a\right)$$

so that $k^a = h^a_a$ for all a, which implies that $k = h$. Thus $(M, \rho_\bullet)$ is a limit of Sys_P. That the canonical sheaves on M induced by $\rho_\bullet$ and $\mu_\bullet$ are equal follows immediately from the equalities $\mu_i = G_i \circ \rho_i$ and $F_a \circ \pi_a = \mathrm{Id}_{P_a}$ and the fact that both π_a and F_a are diffeomorphisms.

(4): It is immediate that $F_a \circ \mu_{ab} = \rho_{ab} \circ F_b$ so that $(F_\bullet, \mathrm{Id}_\mathbb{N})$ is an inverse system morphism and likewise the π_a's form an inverse system morphism where the fact that F_i and G_i are inverses immediately implies that they form an equivalence transformation of profinite systems and that their limits are diffeomorphisms that are inverses of each other. If $m = (m_1, m_2, \ldots) \in M$ then since

$$(\mu_i \circ \mathrm{Id}_M)(m) = m_i = G_i(m_1, \ldots, m_i) = (G_i \circ \rho_i)(m)$$

it follows that $\mathrm{Id}_M = \varprojlim \pi_\bullet$.

(5) is immediate.

(6): The proof that $C^\infty_M = C^\infty_{\prod M_\bullet}\big|_M$ is analogous to the above proof that $\mathcal{S}_a = \mathcal{P}_a$. Alternatively, in light of (3), (4), and (5) one may apply theorem 11.6.1 to conclude that the identity map $\mathrm{Id}_M : (M, C^\infty_M) \to \left(M, C^\infty_{\prod M_\bullet}\big|_M\right)$ is a diffeomorphism. ■

Corollary 5.4.4. If $(M, \mu_\bullet)$ denotes the canonical limit of $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, \mathbb{N})$ then the canonical sheaf on M induced by $\mu_\bullet$ is just the restriction to M of the canonical product sheaf $C^\infty_{\prod M_\bullet}$ on $\prod\limits_{i=1}^{\infty} M_i$. Equivalently, the inclusion map $\mathrm{In} : (M, C^\infty_M) \to \left(\prod\limits_{i=1}^{\infty} M_i, C^\infty_{\prod M_\bullet}\right)$ is a smooth embedding, which is also proper since M is closed in $\prod\limits_{i=1}^{\infty} M_i$.

Chapter 6

Smooth and Locally Cylindrical Maps

Example 6.0.1 (A non-smooth inclusion map of a promanifold into $\mathbb{R}$). For each $i \in \mathbb{N}$, let $M_i \underset{\text{def}}{=} \left\{\frac{1}{n} \,\middle|\, 1 \leq n \leq i\right\}$ and let $\mu_{i,i+1} : M_{i+1} \to M_i$ be defined by $\mu_{i,i+1}\left(\frac{1}{n}\right) = \frac{1}{n}$ if $n \leq i$ and $\mu_{i,i+1}\left(\frac{1}{1+i}\right) = \frac{1}{i}$. Recall from example 2.1.45 that this system's limit in Top is $(M, \mu_\bullet)$ where $M \underset{\text{def}}{=} \{0\} \cup \left\{\frac{1}{n} \,\middle|\, n \in \mathbb{Z}\right\}$ has the subspace topology induced by $\mathbb{R}$ and for each $i \in \mathbb{N}$, $\mu_i(1/n) = 1/n$ of $n \leq i$ and $\mu_i(1/n) = 1/i$ otherwise. Observe that M's sheaf consists of all real-valued maps that take on only finitely many values: Let $f : M \to \mathbb{R}$ be smooth. Since it is smooth at 0 so there exists some $i \in \mathbb{N}$ and some $\mu_i(0) = \frac{1}{i} \in U_i \in \operatorname{Open}(M_i)$ such that $f = f_i \circ \mu_i$ on $\mu_i^{-1}(U_i)$, where $\mu_i^{-1}(U_i)$ contains $\mu_i^{-1}\left(\frac{1}{i}\right) = \left\{\frac{1}{n} \,\middle|\, n \geq i\right\}$. Hence, for every $n \geq i$, $f\left(\frac{1}{n}\right) = f_i\left(\mu_i\left(\frac{1}{n}\right)\right) = f_i\left(\frac{1}{i}\right)$. This implies, in particular, that the natural inclusion $F \underset{\text{def}}{=} \operatorname{In}_M^{\mathbb{R}} : M \to \mathbb{R}$ is continuous but not smooth since, for instance, $f(x) \underset{\text{def}}{=} e^{-x^2}$ is a smooth function on $\mathbb{R}$ such that $f \circ F : M \to \mathbb{R}$ fails to be smooth. However, the identity map from the restriction sheaf $\left(M, C_{\mathbb{R}}^{\infty}\big|_M\right)$ into (M, C_M^{∞}) is smooth. ∎

Locally Cylindrical Maps

Definition and Notation 6.1.1. Let $(M, \mu_\bullet) = \varprojlim(M_\bullet, \mu_{ij})$, $R \subseteq M$, $F : R \to N$ be a into a set N, and let i be any index. We will denote by $\operatorname{Dom}_i F$ the set of all $m_i \in \mu_i(R)$ such that F is constant on $R \cap \mu_i^{-1}(m_i)$ and we will denote the interior of $\operatorname{Dom}_i F$ in M_i by $\operatorname{ODom}_i F$.

We will denote the map that F induces on $\operatorname{Dom}_i F$ by affixing the index i as a subscript to the symbol F, that is, we will use F_i to the denote the induced map

$$\begin{aligned} F_i : \operatorname{Dom}_i F &\longrightarrow N \\ m_i &\longmapsto F\left(\mu_i^{-1}(m_i)\right) \end{aligned}$$

Given any $S_i \subseteq M_i$, if we say

$$\text{“}F = G_i \circ \mu_i \text{ on } \mu_i^{-1}(S_i)\text{”}$$

then we mean that G_i is an N-valued map, $S_i \subseteq \operatorname{Dom}_i F \cap \operatorname{Dom} G_i$, and that $G_i\big|_{S_i} = F_i\big|_{S_i}$. ■

Convention 6.1.2. Since open subsets of manifolds are again manifolds, it will often be convenient to redefine F_i to be the above map's restriction to $\operatorname{ODom}_i F$. If it may not be clear from context whether F_i is meant to denote a map defined on $\operatorname{Dom}_i F$ or on $\operatorname{ODom}_i F$ then we will indicate what its domain is by writing its prototype, i.e. by writing either $F_i : \operatorname{Dom}_i F \to N$ or $F_i : \operatorname{ODom}_i F \to N$.

Remark 6.1.3. The set $\operatorname{Dom}_i F$ and the map $F_i : \operatorname{Dom}_i F \to N$ are unique and maximal in the sense that if $G_i : D_i \to N$ is any map defined on a subset D_i of $\mu_i(R)$ such that $F\big|_{\mu_i^{-1}(D_i)\cap R} = G_i \circ \mu_i\big|_{\mu_i^{-1}(D_i)\cap R}$ then $D_i \subseteq \operatorname{Dom}_i F$ and $F_i = G_i$.

The key point of the following lemma 6.1.4 is that when $F : M \to N$ is a continuous map between promanifolds then $\operatorname{Dom}_i F$ is necessarily a closed subset of M_i for each indices i. This implies, in particular, that if we wish to define a continuous map on M by inductively defining continuous maps $F_1, F_2, \ldots$ on sets $S_1, S_2, \ldots$ (where such a construction can be viewed as a the promanifold analogue of a “piecewise construction” of a continuous map) then we must necessarily be able to continuously extend each F_i to $\overline{S_i}$. And if this map is to be locally cylindrical then the each $\operatorname{Int}_{M_i}(S_i)$ must be a regular open subset of M_i.

Lemma 6.1.4. Let $(M, \mu_\bullet) = \varprojlim(M_\bullet, \mu_{ij})$ be a profinite manifold, N be any space, $R \subseteq M$, and $F : R \to N$ be any map. Then for all $j \geq i$,

(1) if $S_i \subseteq \mathrm{Dom}_i F$ is such that $\mu_i\big|_{R \cap \mu_i^{-1}(S_i)} : R \cap \mu_i^{-1}(S_i) \to S_i$ is a quotient map, then $F\big|_{\mu_i^{-1}(S_i)} : \mu_i^{-1}(S_i) \to N$ is continuous $\iff$ $F_i\big|_{S_i} : S_i \to N$ is continuous.

(2) if $\mu_i\big|_R : R \to \mu_i(R)$ is an open map and $F : R \to N$ is continuous then D_i is closed in $\mu_i(R)$.

(3) $\mu_{ij}^{-1}(\mathrm{Dom}_i F) \subseteq \mathrm{Dom}_j F$, $\mu_{ij}^{-1}(\mathrm{ODom}_i F) \subseteq \mathrm{ODom}_j F$, and $F_j = F_i \circ \mu_{ij}$ on $\mu_{ij}^{-1}(\mathrm{Dom}_i F)$.

Proof. Parts (1) and (2) follow from lemma A.4.3. To prove (3), observe that the equality

$$F\big|_{\mu_j^{-1}\left(\mu_{ij}^{-1}(D_i)\right)} = F\big|_{\mu_i^{-1}(D_i)} = F_i \circ \mu_i\big|_{\mu_i^{-1}(D_i)} = F_i \circ \mu_{ij} \circ \mu_j\big|_{\mu_j^{-1}\left(\mu_{ij}^{-1}(D_i)\right)}$$

implies that $\mu_{ij}^{-1}(D_i) \subseteq D_j$ and that $F_j\big|_{\mu_{ij}^{-1}(D_i)} = F_i \circ \mu_{ij}\big|_{\mu_{ij}^{-1}(D_i)}$. ■

Definition 6.1.5. Let $R \subseteq M$ and $F : R \to N$ be a map where M is a promanifold and N is a set. If $m \in M$ and there exists an index i, an open set $\mu_i(m) \in U_i \in \mathrm{Open}(M_i)$, and a map $F_i : U_i \to N$ such that

$$F|_{\mu_i^{-1}(U_i) \cap R} = F_i \circ \mu_i|_{\mu_i^{-1}(U_i) \cap R}$$

then we will say that F is *roughly locally cylindrical at* m, where if in addition F is smooth (resp. continuous) then we'll also require that F_i be smooth (resp. continuous) and then instead say that F *is locally cylindrical at* m. If there exists an index i such that $\mathrm{Dom}_i F = \mu_i(R)$ then we will call F *local*, *cylindrical*, or *trivially cylindrical*. We will say that F is *roughly cylindrical at* $m \in M$ if there exists an index i such that $\mu_i(m) \in \mathrm{Dom}_i F$, i.e.

$$F|_{\mu_i^{-1}(\mu_i(m)) \cap R} = F_i \circ \mu_i|_{\mu_i^{-1}(\mu_i(m)) \cap R}$$

If F is locally (resp. roughly) cylindrical at every point of some set then we will say that F *is locally (resp. roughly) cylindrical* on that set. If F is locally (resp. roughly) cylindrical at every point of its domain then we will say that F *is locally (resp. roughly) cylindrical.* ■

Remark 6.1.6. Suppose $F : R \subseteq M \to N$ is a locally cylindrical map on a subset R of M. If $U = \bigcup_i \mu_i^{-1}(U_i)$, where $\mu_{i,i+1}^{-1}(U_i) \subseteq U_{i+1}$, then $F(U) = \bigcup_i F_i(\mu_i(R) \cap U_i)$ so that in particular, if $R = M$ then $F : M \to N$ is an open map if and only if all $F_i : \mathrm{ODom}_i\, F \to N$ are open maps.

Canonical Maps Induced by a Map Between Promanifolds

Definition 6.1.7. If $F : R \subseteq M \to N$ is a map between promanifolds and $F^\bullet \underset{\text{def}}{=} \nu_\bullet \circ F$ then for all $i, a \in \mathbb{N}$, unless specified otherwise, by F_i^a we will mean the unique map

$$F_i^a : \mathrm{Dom}_i(F^a) \to N_a$$

from definition 6.1.1 induced by $F^a = \nu_a \circ F : R \to N_a$. We will call F_i^a *the i^{th} canonical map associated with (or induced by) F and ν_a.*

Thus, every map $F : R \to N$ from a subset $R \subseteq M$ of a promanifold into another promanifold induces an $\mathbb{N} \times \mathbb{N}$-indexed collection of morphisms $(F_i^a)_{(a,i) \in \mathbb{N} \times \mathbb{N}}$ that we will henceforth refer to as *the canonical ($\mathbb{N} \times \mathbb{N}$-indexed) collection of maps associated with (or induced by) F (from Sys_M) (to Sys_N).* ■

Remark 6.1.8. Suppose M and N are promanifolds, $R \subseteq M$, and $F : R \to N$ is a map such that $R \subseteq \bigcup_i \mu_i^{-1}(\mathrm{Dom}_i\, F^a)$ for each index a where $F^\bullet \underset{\text{def}}{=} \nu_\bullet \circ F$. If $V = \bigcup_a \nu_a^{-1}(V_a)$ is a subset of N, where $\nu_{a,a+1}^{-1}(V_a) \subseteq V_{a+1}$, then

$$F^{-1}(V) = R \cap \bigcup_{i,a \in \mathbb{N}} \mu_i^{-1}\big((F_i^a)^{-1}(V_a)\big) = R \cap \bigcup_{i,a \in \mathbb{N}} \left(F_i^a \circ \mu_i\big|_{\mu_i^{-1}(\mathrm{Dom}_i\, F^a)} \right)^{-1} (V_a)$$

Lemma 6.1.9. If $a \leq b$, and $i \leq j$, and $F : R \subseteq M \to N$ is a roughly cylindrical map between promanifolds then

$$\nu_{ab} \circ F_j^b = F_i^a \circ \mu_{ij} \quad \text{and} \quad F_j^a = \nu_{ab} \circ F_i^b \circ \mu_{ij}$$

on $\mathrm{Dom}_j\, F^b \cap \mu_{ij}^{-1}(\mathrm{Dom}_i\, F^a)$.

Proof. Let $D_j = \mathrm{Dom}_j F^b \cap \mu_{ij}^{-1}(\mathrm{Dom}_i F^a)$. For all $m \in \mu_j^{-1}(D_j) \cap R$ we have

$$\left(\nu_{ab} \circ F_j^b\right)(\mu_j(m)) = (\nu_{ab} \circ \nu_b \circ F)(m) = (\nu_a \circ F)(m) = F^a(m) = F_i^a(\mu_i(m)) = (F_i^a \circ \mu_{ij})(\mu_j(m))$$

and similarly, $F_j^a(\mu_j(m)) = \nu_a(F(m)) = \nu_{ab}(\nu_b(F(m))) = \nu_{ab}(F_i^b(\mu_i(m)))$. ∎

Smoothness and Local Cylindricity

The following proposition 6.1.10 shows that any smooth map from a promanifold into a manifold can, around any point of its domain, be written in the form $F = F_i \circ \mu_i$ on some open set of the form $\mu_i^{-1}(U_i)$ with $F_i : \mu_i^{-1}(U_i) \to N$ smooth.

Proposition 6.1.10. If $F : R \to N$ be a smooth map from a subset $R \subseteq M$ of the promanifold M into a manifold N then F is locally cylindrical.

Proof. Fix $m \in R$, let (V, ϕ) be a chart of N at $n \underset{\text{def}}{=} F(m)$, and let $\phi = (\phi^1, \ldots, \phi^d)$ where $d \underset{\text{def}}{=} \dim N$. For each $k = 1, \ldots, d$ the map $\phi^k \circ F|_{F^{-1}(V)} : R \cap F^{-1}(V) \to \mathbb{R}$ is smooth at m so there exists some index i_k, some open $\mu_{i_k}(m) \in U_{i_k} \in \mathrm{Open}(M_{i_k})$, and some smooth $F_{i_k}^k \in C^\infty_{M_{i_k}}(U_{i_k})$ such that

$$\phi^k \circ F|_{\mu_{i_k}^{-1}(U_{i_k})} = F_{i_k}^k \circ \mu_{i_k}|_{R \cap \mu_{i_k}^{-1}(U_{i_k})}$$

Let $i = \max\{i_1, \ldots, i_d\}$ and let $U_i = \bigcap_{k=1}^{d} \mu_{i_k i}^{-1}(U_{i_k})$ so that $\mu_i(m) \in U_i \in \mathrm{Open}(M_i)$. Let $F_i^k \underset{\text{def}}{=} F_{i_k}^k \circ \mu_{i_k i}|_{U_i}$ and $U \underset{\text{def}}{=} \mu_i^{-1}(U_i)$ so that $\phi^k \circ F|_U = F_i^k \circ \mu_i|_U$ and $F_i^k \in C^\infty_{M_i}(U_i)$. Let

$F_i \underset{\text{def}}{=} \phi^{-1} \circ (F_i^1, \ldots, F_i^d)$ so that $F_i : U_i \to V \subseteq N$ is smooth. Note that

$$\begin{aligned}
\phi \circ F_i \circ \mu_i|_{R \cap U} &= (F_i^1, \ldots, F_i^d) \circ \mu_i|_{R \cap U} \\
&= \left(F_i^1 \circ \mu_i|_{R \cap U}, \ldots, F_i^d \circ \mu_i|_{R \cap U}\right) \\
&= \left(F_{i_1}^k \circ \mu_{i_1 i}|_{U_i} \circ \mu_i|_{R \cap U}, \ldots, F_{i_d}^k \circ \mu_{i_d i}|_{U_i} \circ \mu_i|_{R \cap U}\right) \\
&= \left(F_{i_1}^k \circ \mu_{i_1}|_{R \cap U}, \ldots, F_{i_d}^k \circ \mu_{i_d}|_{R \cap U}\right) \\
&= \left(\phi^k \circ F|_{R \cap U}, \ldots, \phi^k \circ F|_{R \cap U}\right) \\
&= \phi \circ F|_{R \cap U}
\end{aligned}$$

so that $F_i \circ \mu_i|_{R \cap U} = F|_{R \cap U}$, as desired. ∎

Proposition 6.1.11. Let $F : M \to N$ be a locally cylindrical smooth map between promanifolds. Then for all indices i, the map $F_i : \operatorname{Dom}_i F \to N$ is smooth on the closed set $\operatorname{Dom}_i F$ in the sense that at every point of its domain there is a smooth local extension of F_i to a neighborhood of that point.

Proof. Fix i and assume that $D_i \underset{\text{def}}{=} \operatorname{Dom}_i F$ is not empty. Let $m_i^0 \in D_i$, let $m^0 \in \mu_i^{-1}(m_i^0)$ be arbitrary, and $m_j^0 \underset{\text{def}}{=} \mu_j(m^0)$. Since F is smooth there exists some $j \geq i$ such that $\mu_j(m^0) \in \operatorname{ODom}_j F$. Since $\mu_{ij} : M_j \to M_i$ is a surjective smooth submersion there exists a smooth local section $\sigma_i^j : W_i \to M_j$ of μ_{ij} such that $\sigma_i^j(m_i^0) = m_j^0$ where by shrinking W_i we can assume that the range of σ_i^j is contained in $\operatorname{ODom}_j F$. Note that if $m_i \in W_i \cap D_i$ then since $m_j \underset{\text{def}}{=} \sigma_i^j(m_i) \in \operatorname{ODom}_j F$ we have

$$F_j(m_j) = F_i(\mu_{ij}(m_j)) = F_i(\mu_{ij}(\sigma_i^j(m_i))) = F_i(m_i)$$

so that $F_j \circ \sigma_i^j : W_i \to N$ is a smooth map that agrees with F_i where their domains overlap. ∎

The following lemma implies, in particular, that each projection $\mu_\bullet$ has the universal property of quotient maps in the category of promanifolds.

Lemma 6.1.12 (Each $\mu_\bullet$ has the universal property of quotient maps)**.** Let $F : M \to N$ be any map between promanifolds, i any index, and $U_i \in \text{Open}(M_i)$ any non-empty open set. Suppose that $F_i : U_i \to N$ is a map such that $F|_{\mu_i^{-1}(U_i)} = F_i \circ \mu_i|_{\mu_i^{-1}(U_i)}$. Then F is smooth on $\mu_i^{-1}(U_i) \iff F_i$ is smooth.

Proof. Assume that $F|_{\mu_i^{-1}(U_i)}$ is smooth. Since $\mu_i\big|_{\mu_i^{-1}(U_i)} : \mu_i^{-1}(U_i) \to U_i$ is an open and continuous map, it is a quotient map so that the continuity of $F|_{\mu_i^{-1}(U_i)}$ implies the continuity of F_i. So assume that $F|_{\mu_i^{-1}(U_i)}$ is smooth. Suppose first that N is a manifold. Let $u_i \in U_i$ and pick $u \in \mu_i^{-1}(U)$. Since $F|_{\mu_i^{-1}(U_i)}$ is smooth there exists an index $j \geq i$, an open set $V_j \in \text{Open}(M_j)$, and a continuous (resp. smooth) $F_j : V_j \to N$ such that $F|_{\mu_i^{-1}(V_i)} = F_j \circ \mu_j|_{\mu_j^{-1}(V_j)}$ and $u \in \mu_j^{-1}(V_j)$. Since $\mu_i(u) \in U_i$, $u \in \mu_{ij}^{-1}(U_i)$ so that by replacing V_j with $V_j \cap \mu_{ij}^{-1}(U_i)$ we may assume that $V_j \subseteq \mu_{ij}^{-1}(U_i)$. Hence, $\mu_j^{-1}(V_j) \subseteq \mu_j^{-1}\big(\mu_{ij}^{-1}(U_i)\big) = \mu_i^{-1}(U_i)$. Note that

$$(F_i|_{\mu_{ij}(V_j)} \circ \mu_{ij}|_{V_j}) \circ \mu_j|_{\mu_j^{-1}(V_j)} = F_i \circ \mu_i|_{\mu_j^{-1}(V_j)} = F|_{\mu_j^{-1}(V_j)} = F_j \circ \mu_j|_{\mu_j^{-1}(V_j)}$$

so that by the surjectivity of $\mu_j|_{\mu_j^{-1}(V_j)}$ we have $F_i|_{\mu_{ij}(V_j)} \circ \mu_{ij}|_{V_j} = F_j$. Note that $\mu_i\big|_{\mu_i^{-1}(V_i)} : \mu_i^{-1}(V_i) \to V_i$ is a surjective submersion so that since F_j is smooth we have that $F_i|_{\mu_{ij}(V_j)}$ is also smooth. Since $u_i = \mu_i(u) = \mu_{ij}(\mu_j(u)) \in \mu_{ij}(V_j)$ with $\mu_{ij}(V_j) \in \text{Open}(U_i)$, f_i is smooth.

Suppose now that (N, ν_a) is a profinite manifold. Since $F|_{\mu_i^{-1}(U_i)}$ is smooth so are all $\nu_a \circ F|_{\mu_i^{-1}(U_i)} = (\nu_a \circ F_i) \circ \mu_i|_{\mu_i^{-1}(U_i)}$ so that since the codomain of $\nu_a \circ F_i$ is a manifold, we have that each $\nu_a \circ F_i$ is smooth. By the universal property of limits, $F_i : U_i \to N$ is smooth. ∎

Observation 6.1.13. Continuing from observation 2.1.21, lemma 6.1.12 implies that any profinite system Sys_M is completely determined by the limit cone's projections $\mu_\bullet$.

Smoothness at a Point

Only definition 6.1.14 and remark 6.1.15 will be used elsewhere in this paper.

Definition 6.1.14. Let $F : M \to N$ be a map between promanifolds and let $m^0 \in M$. If N is a manifold then say that F *is smooth at* m^0 if there exists some index $i \in \mathbb{N}$ such that $\mu_i(m^0) \in \operatorname{ODom}_i F$ and some open neighborhood $\mu_i(m^0) \in U_i \in \operatorname{Open}(\operatorname{ODom}_i F)$ such that $F_i\big|_{U_i} : U_i \to N$ is smooth. If N is an arbitrary promanifold then say that F *is smooth at* m^0 if $F^a : M \to N_a$ is smooth at m^0 for all $a \in \mathbb{N}$.

Remark 6.1.15. It is easy to see from this definition that a map from an open subset of a promanifold into another promanifolds is smooth $\iff$ it is smooth at every point of its domain.

Remark 6.1.16. Suppose that $F : M \to N$ is a map between smooth manifolds embedded in Euclidean spaces and $m^0 \in M$. Just as there is a concept of what it means for F to be continuous *at* the single point m^0 without being continuous on $M \setminus \{m^0\}$, Peano differentiability (def. 5.3.7) is a concept of differentiability that allows for F to be (Peano) smooth *at* the single point m^0 without F needing to be smooth on $M \setminus \{m^0\}$. However, it is possible for a map F to be everywhere infinitely Peano differentiable without the map even being once continuously differentiable. Now while the definition of "smoothness of a map at a point" given in definition 6.1.14 does not suffer from this issue, it does permit paradoxical maps, such as the map constructed in example 6.1.17, to exist.

Example 6.1.17. A map that is smooth at one point and discontinuous everywhere else: Let $B_i(r)$ denote the open box with sides of length r in $M_i \underset{\text{def}}{=} \mathbb{R}^i$. We will define an inverse system morphism $F_\bullet : \operatorname{Sys}_M \to \operatorname{Sys}_M$ where $F_i : \mathbb{R}^i \to \mathbb{R}^i$ for all $i \in \mathbb{N}$ and we begin by letting $F_i\big|_{B_i(1/i)} \underset{\text{def}}{=} \operatorname{Id}_{B_i(1/i)}$ for all i. Let $E_i \underset{\text{def}}{=} \mathbb{R}^i \setminus B_i(1/i)$ for each index i. For $i = 1$ define F_1 on E_1 to be any bijection of E_1 onto itself that is discontinuous at every point of its domain. Let $b : \mathbb{R} \to \mathbb{R}$ be any bijection that is discontinuous at every point.

Suppose we've finished defining bijections $F_i : \mathbb{R}^i \to \mathbb{R}^i$ such $F_i\big|_{E_i} : E_i \to E_i$ is a bijection and discontinuous at every point of C_i and where if $i \geq 2$ then $\operatorname{Pr}_{i-1,i} \circ F_i = F_{i-1} \circ \operatorname{Pr}_{i-1,i}$. Let

$S_i = \mathbb{R}^i \smallsetminus B_i(1/(i+1))$. For every $m_i = (r_1, \ldots, r_i) \in S_i$ define F_{i+1} on $\operatorname{Pr}_{i,i+1}^{-1}(m_i)$ by

$$F_{i+1}(r_1, \ldots, r_i, r_{i+1}) = (F_i(r_1, \ldots, r_i), b(r_{i+1}))$$

Observe that we have defined a bijection from $\operatorname{Pr}_{i,i+1}^{-1}(S_i) \cup B_{i+1}(1/(i+1))$ onto itself so let C_{i+1} denote the complement in $\mathbb{R}^{i+1}$ of this set.

We will define F_{i+1} on C_{i+1} in a similar manner as was just done above, except that now we must deal with some additional details so as to avoid (mistakenly re-)defining F_{i+1} on $B_{i+1}(1/(i+1))$. For any $m_i = (r_1, \ldots, r_i) \in \operatorname{Pr}_{i,i+1}(C_{i+1})$, observe that $\operatorname{Pr}_{i,i+1}^{-1}(m_i) \cap C_{i+1} = \{m_{i+1} \,|\, m_{i+1} = (m_i, r_{i+1}), |r_{i+1}| \geq 1/(i+1)\}$ so let

$$b_{i+1} : \mathbb{R} \smallsetminus \left]-\frac{1}{i+1}, \frac{1}{i+1}\right[\;\to\; \mathbb{R} \smallsetminus \left]-\frac{1}{i+1}, \frac{1}{i+1}\right[$$

denote any bijection that is continuous at no point of its domain. Define F_{i+1} on $\operatorname{Pr}_{i,i+1}^{-1}(m_i) \cap C_{i+1}$ by mapping $m_{i+1} = (m_i, r_{i+1})$ to $(m_i, b_{i+1}(r_{i+1})$. Observe that $\operatorname{Pr}_{i,i+1} \circ F_{i+1} = F_i \circ \operatorname{Pr}_{i,i+1}$, which completes the construction.

Let $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ denote the limit of $F_\bullet$ and let $\{0\}^k = (0, 0, \ldots) \in \mathbb{R}^{\mathbb{N}}$ denote the zero of $\mathbb{R}^k$ for each $k = 1, 2, \ldots, \mathbb{N}$. Note that since $F = \varprojlim F_\bullet$ we have $\operatorname{ODom}_i F^a = \mathbb{R}^i$ for all $i \geq a$ and for any index a. Using $i = a$, defining $U_i = B_i(1/i)$, and observing that $F^a = \operatorname{Pr}_a$ on the neighborhood $\operatorname{Pr}_i^{-1}(U_i))$ (where $F^a \underset{\text{def}}{=} \operatorname{Pr}_a \circ F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^a$) shows that F^a is continuous and smooth at $\{0\}^{\mathbb{N}}$. Since $a \in \mathbb{N}$ was arbitrary it follows (by definition) that F is smooth at $\{0\}^{\mathbb{N}}$.

Let $m = (r_1, r_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$ be any element distinct from $\{0\}^{\mathbb{N}}$. Let $l \in \mathbb{N}$ be such that $r_l \neq 0$ and let $a \in \mathbb{N}$ be such that $1/a < |r_l|$. Observe that $m_a \underset{\text{def}}{=} \operatorname{Pr}_a(m) = (r_1, \ldots, r_a)$ does not belong to $B_a(1/a)$. If there was any index $j \geq a$ such that $F_a \circ \operatorname{Pr}_{aj}$ was continuous at $\operatorname{Pr}_j(m)$ then since Pr_{aj} is an open continuous map this would force F_a to be continuous at m_a, contradicting the construction of F_a. Thus there exists no neighborhood of m on which the restriction of F^a would be continuous, which implies that F is neither continuous nor smooth at m. ■

Nowhere Roughly Cylindrical Maps

Remark 6.1.18. Let $F : M \to N$ be a map between promanifolds. If i and j are indices such that $\dim M_i < \dim M_j$ then since $\mu_{ij} : M_j \to M_i$ is a smooth submersion it follows that for any $d_j \in \mu_{ij}^{-1}(d_i)$ there exist neighborhoods of d_i in M_i and d_j and M_j such that μ_{ij} can be represented in coordinates as the canonical projection so that in particular $\mu_{ij}^{-1}(d_i)$ is infinite. Hence $\mu_i^{-1}(d_i) = \mu_j^{-1}\big(\mu_{ij}^{-1}(d_i)\big)$ is infinite and since $\mu_i^{-1}(d_i) \subseteq F^{-1}(F_i(d_i))$ it follows that the fiber of F over $F_i(d)$ is also infinite (where $F_i = \mu_i \circ \mu_i$ on $\operatorname{Dom}_i F$). In particular, if $\sup_i \dim \mathrm{T}_{m_i} M_i = \infty$ and F is injective then $\operatorname{Dom}_i F = \varnothing$ for all indices i so that if we are dealing we embeddings or diffeomorphisms from an infinite-dimensional promanifold into some other promanifold then $\operatorname{Dom}_i F$ is necessarily empty for all indices i. ■

The following example shows that there are continuous real-valued functions $f : M \to \mathbb{R}$ for which all $\operatorname{Dom}_i f$ are empty.

Example 6.1.19. Let $\psi : \mathbb{R} \to \mathbb{R}$ be a smooth non-constant function such that $-1 \leq \psi \leq 1$ on $\mathbb{R}$. Let $\psi_n \to \mathbb{R}^n \to \mathbb{R}$ be $\psi_n(x_1, \ldots, x_n) = \psi(x_1) \cdot \cdots \cdot \psi(x_n)$ and let $F^n \underset{\text{def}}{=} \frac{1}{2^n} \psi_n \circ \operatorname{Pr}_n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$. Let $S^n : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ be the partial sum $S^n = \sum_{l=1}^{n} F^l$. If $n > m$ then

$$|S^n - S^m| \leq \sum_{l=m+1}^{n} \left| \frac{1}{2^l} \psi_l \circ \operatorname{Pr}_l \right| \leq \sum_{l=m+1}^{n} \frac{1}{2^l}$$

so that the partial sums converge uniformly to some continuous function $S : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$. Note that for any positive integer N, if $x = (x_1, \ldots, x_N, 0, 0, \ldots)$ then for all $l \geq N$, $(\psi_l \circ \operatorname{Pr}_l)(x) = (\psi_N \circ \operatorname{Pr}_N)(x)$ so that

$$\sum_{l=N}^{\infty} \frac{1}{2^l} \psi_l(\operatorname{Pr}(x)) = \sum_{l=N}^{\infty} \frac{1}{2^l} \psi_N(\operatorname{Pr}_N(x)) = \psi_N(\operatorname{Pr}_N(x)) C_N$$

where $C_N = \sum_{l=N}^{\infty} \frac{1}{2^l}$ so

$$S(x) = \sum_{l=1}^{\infty} \frac{1}{2^l} \psi_l(\operatorname{Pr}_l(x)) = \sum_{l=1}^{N-1} \frac{1}{2^l} \psi_l(\operatorname{Pr}_l(x)) + \psi_N(\operatorname{Pr}_N(x)) C_N$$

Observe that if $\varphi(x_0) = 0$ for some $x_0 \in \mathbb{R}$ then for any $x_1, x_2, \ldots \in \mathbb{R}$ and $k \in \mathbb{N}$ we have $S(x_1, \ldots, x_{k-1}, x_0, x_{k+1}, \ldots) = \sum_{n=1}^{k-1} \frac{1}{2^n} \psi_n(x_1, \ldots, x_n)$ (where if $k = 1$ then the RHS is 0), which would imply that $(x_1, \ldots, x_{k-1}, x_0) \in \operatorname{Dom}_k S$ and so to obtain the desired counter-example we must assume in addition that ψ is never 0. Now suppose that there was some index i such that $D_i \underset{\text{def}}{=} \operatorname{Dom}_i S \neq \varnothing$. Let $d_i = (x_1, \ldots, x_{i+1}) \in D_i$ and let x_{i+1} and v_{i+1} be any distinct real numbers such that $\psi(x_{i+1}) \neq \psi(v_{i+1})$. Let $x = (x_1, \ldots, x_i, x_{i+1}, 0, 0, \ldots)$ and $v = (x_1, \ldots, x_i, v_{i+1}, 0, 0, \ldots)$ so that in particular $x, v \in D \underset{\text{def}}{=} \mu_i^{-1}(D_i)$. Note that $\operatorname{Pr}_l(x) = \operatorname{Pr}_l(v)$ for all $l = 1, \ldots, i$ and $\psi_{i+1}(\operatorname{Pr}_{i+1}(x)) \neq \psi_{i+1}(\operatorname{Pr}_{i+1}(v))$ so that by taking $N = i + 1$ in the above formula we see that $S(x) \neq S(v)$. But this gives a contradiction since

$$S(x) = S_i(\operatorname{Pr}_i(x)) = S_i(x_1, \ldots, x_i) = S_i(\operatorname{Pr}_i(v)) = S(v)$$

Thus $S : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a continuous real-valued function for which all $\operatorname{Dom}_i S$ are empty. ■

A Sufficient Condition for Rough Cylindricity

The following theorem suggests that studying promanifolds in terms of their smooth almost arcs may be fruitful since it establishes a direct link between smooth almost arcs and cylindricity of real valued functions, where this latter concept is fundamental to promanifolds. Theorem 6.1.20 will not be used anywhere else in this paper and it is recommended that the reader be familiar with theorem 16.1.7 before reading this theorem's proof.

Theorem 6.1.20. Let M be a monotone promanifold, $U \in \operatorname{Open}(M)$, and $m^0 \in U$ with $\dim_{m^0} M = \infty$. Let $f : U \to \mathbb{R}$ be a continuous function such that $f \circ \gamma : [0,1] \to \mathbb{R}$ is differentiable at 0 with $(f \circ \gamma)'(0) = 0$ whenever $\gamma : ([0,1], 0) \to (U, m^0)$ is a smooth almost arc vanishing at 0. Then f is roughly cylindrical at m^0.

Proof. It suffices to prove this under the assumption that $f(m^0) = 0$, Sys_M is monotone, and U is connected. Let $m^0_\bullet = \mu_\bullet(m^0)$ and let $R^i = \mu_i^{-1}(m^0_i)$ for all $i \in \mathbb{N}$. Suppose that f was not roughly cylindrical at m^0. Observe that every $f(R^\bullet)$ must be a non-degenerate interval in $\mathbb{R}$ since each $R^\bullet$ is connected and no R^i is contained in $f^{-1}(0)$. Since f is continuous, every $f^{-1}(]-1/n, 1/n[)$ contains some R^i, which implies that $\cap f(R^\bullet) = \{0\}$ so we may pick a strictly monotone sequences $(i_l)_{l=1}^\infty \subseteq \mathbb{N}$ and $(s_l)_{l=1}^\infty \subseteq [-1,1]$ such that $s_\bullet \to 0$ and $s_l \in f(R^{i_l}) \smallsetminus f(R^{i_{l+1}})$. By replacing f with $-f$, we may assume without loss of generality that all $s_\bullet$ are positive. For all $l \in \mathbb{N}$, pick $m^l \in f^{-1}(s_l) \cap R^{i_l}$ so that $f(m^l) = s_l$ and $m^l \notin R^{i_{l+1}}$. By lemma 15.4.3, there exists a smooth topological embedding $\gamma : ([0,1], 0) \to (U, m^0)$ such that $\gamma'(t)$ vanishes if and only if $t = 0$ and $\gamma(s_l) = m^l$ for all $l \in \mathbb{N}$. We thus obtain the contradiction

$$(f \circ \gamma)'(0) = \lim_{l \to \infty} \frac{f(\gamma(s_l))}{s_l} = \lim_{l \to \infty} \frac{f(m^l)}{s_l} = \lim_{l \to \infty} \frac{s^l}{s_l} \neq 0$$

∎

Sufficient Conditions for Local Cylindricity

The only result in this subsection that will be used elsewhere in this paper is theorem 6.1.23, which is due to [1]. While all results in this subsection other than theorem 6.1.23 are new, the proofs of proposition 6.1.27 and lemma 6.1.24 use some ideas that the author first found in [1]. It is recommended that the reader be familiar with the tangent bundle of a promanifold before reading this section.

Note that if $F : M \to N$ is any map, then even if all $\mathrm{Dom}_i\, F$ are non-empty and dense in M_i there is in general no reason to expect for there to exist any index i such that $\mathrm{ODom}_i\, F$ is not empty. So we now produce the following proposition 6.1.21, which provides a sufficient condition for the existence of a non-empty $\mathrm{ODom}_i\, F$.

Proposition 6.1.21. Let $F : M \to N$ be a continuous map and suppose that $S \subseteq M$ is a non-meager (in M) subset such that for each $m \in S$ there exists some index $i(m) \in \mathbb{N}$ such that $\mu_{i(m)}(m) \in \operatorname{Dom}_i F$. Then there exists some index i such that $\operatorname{ODom}_i F \neq \varnothing$. If in addition S is comeager in M (i.e. its complement is meager in M) then $\underset{i \in \mathbb{N}}{\cup} \mu_i^{-1}(\operatorname{ODom}_i F)$ is a dense open subset of M.

Proof. Let $D = \underset{i \in \mathbb{N}}{\cup} \mu_i^{-1}(\operatorname{Dom}_i F)$ and suppose that each $\operatorname{Dom}_i F$ has empty interior in M_i. Since all μ_i are open and each $\operatorname{Dom}_i F$ is closed, this implies that $\mu_i^{-1}(\operatorname{Dom}_i F)$ is a closed nowhere dense subset of M. But $S \subseteq D$ by assumption, which contradicts the fact that S is non-meager. Now assume that S is comeager in M and let $O = \underset{i \in \mathbb{N}}{\cup} \mu_i^{-1}(\operatorname{ODom}_i F)$. Suppose $\overline{O} \neq M$ and pick and index i and a non-empty open set $U_i \in \operatorname{Open}(M_i)$ such that $U \underset{\text{def}}{=} \mu_i^{-1}(U_i) \subseteq M \setminus \overline{O}$. Since the intersection of a meager set with an open set is meager, it follows that $U \cap S$ is comeager in U and since M is a Baire space S is dense in M it also follows that $U \cap S$ is dense in U. Also, observe that if $\mu_h(m) \in \operatorname{Dom}_h F$ for some $h \in \mathbb{N}$ then $\mu_j(m) \in \operatorname{Dom}_j F$ for all $j \geq h$. Now apply the first part of this proposition with $F\big|_U$ in place of F, $S \cap U$ in place of S, and the inverse system of subsets $\left(\mu_{ij}^{-1}\left(\mu_{ij}^{-1}(U_i)\right)\right)_{j \geq i}$ in place of Sys_M to obtain a contradiction. ∎

Theorem 6.1.22. Let M be a monotone promanifold, $U \in \operatorname{Open}(M)$, and suppose that $\dim_m M = \infty$ for all $m \in U$. Let $f : U \to \mathbb{R}$ be continuous and suppose that whenever $\gamma : [0,1] \to U$ is a smooth topological embedding such that $\gamma'(t)$ vanishes $\iff$ $t = 0$, then $f \circ \gamma : [0,1] \to \mathbb{R}$ is differentiable at 0 and $(f \circ \gamma)'(0) = 0$. Then f is roughly cylindrical at every point of U and there exists a dense (in U) open subset $O \subseteq U$ such that f is locally cylindrical at every point of O. If in addition $f \circ \gamma$ is smooth for every smooth $\gamma : \mathbb{R} \to O$ then $f\big|_O : O \to \mathbb{R}$ is smooth.

Proof. Combine theorem 6.1.20 and proposition 6.1.21 for the first part. Assume that $f \circ \gamma$ is smooth for every smooth $\gamma : \mathbb{R} \to O$ and let $i \in \mathbb{N}$. Every smooth curve $\gamma_i : \mathbb{R} \to \operatorname{ODom}_i f$ has a smooth μ_i-lift $\gamma : \mathbb{R} \to O$ so that $f_i \circ \gamma_i = f \circ \gamma$ is smooth, which implies that $f_i : \operatorname{ODom}_i f \to \mathbb{R}$

is smooth by the Boman theorem. ■

Recall that $(\mathbb{R}^{\mathbb{N}}, \mathrm{Pr}_{\leq i}) = \varprojlim\left(\mathbb{R}^i, \mathrm{Pr}_{\leq ij}, \mathbb{N}\right)$ in Top where $\mathrm{Pr}_{\leq i,j}$ and $\mathrm{Pr}_{\leq i}$ are the canonical projections. Note that since the promanifold $\mathbb{R}^{\mathbb{N}}$ is also a Fréchet topological vector space, there already exist well-established notations of Gâteaux and Fréchet differentiability (def. B.2.1). The following theorem is due to Abbati and Manià [1][thm. 14] and can be viewed as providing some additional justification for studying the canonical sheaf since it implies that in the case of the TVS $\mathbb{R}^{\mathbb{N}}$, the notions of promanifold continuous differentiability (as defined by the canonical sheaf) and Gâteaux continuous differentiability coincide.

Theorem 6.1.23 (Abbati and Manià)**.** Let $U \in \mathrm{Open}(\mathbb{R}^{\mathbb{N}})$ and let $f : U \to \mathbb{R}$ be a function. If f is Gâteaux continuously differentiable then it is locally cylindrical.

We will now generalize the above theorem 6.1.23 of Abbati and Manià to a larger class of promanifolds by generalizing and combining the proofs of some of their theorems. It is recommended that the reader be familiar with the tangent bundle of a promanifold before continuing with the remainder of this subsection.

Lemma 6.1.24. Let $f : M \to \mathbb{R}$ be any map such that for all smooth $\eta : I \to M$, where $I \in \mathrm{Open}(\mathbb{R})$, the map $f \circ \eta : J \to \mathbb{R}$ is smooth. Let $m^0 \in M$, $i \in \mathbb{N}$, and $\mu_i(m^0) \in U_i \in \mathrm{Open}(M_i)$. Suppose that for all $m_i \in U_i$, all $m, \widehat{m} \in \mu_i^{-1}(m_i)$, and all $\epsilon > 0$ there exists a map $\gamma : [0,1] \to M$ (not necessarily continuous) from m to $\widehat{m}$ such that

(1) $f \circ \gamma : [0,1] \to \mathbb{R}$ is absolutely continuous (so that $f \circ \gamma$ is differentiable a.e.) and

(2) $\left\|(f \circ \gamma)'\right\|_\infty < \epsilon$, where $\|\cdot\|_\infty$ is the $L^\infty([0,1])$ norm.

Then $f : M \to \mathbb{R}$ is locally cylindrical at m^0 and in fact $U_i \subseteq \mathrm{Dom}_i f$. If in addition there is some $j \geq i$ and some $\mu_j(m^0) \in O_j \in \mathrm{Open}(M_j)$ such that every smooth curve $\eta_j : J \to O_j$ defined on an open subset of $\mathbb{R}$ has some smooth μ_j-lift into M then f is smooth on $\mu_i^{-1}(U_i) \cap \mu_j^{-1}(O_j)$.

Remark 6.1.25. Observe that for any $m_i \in U_i$, if m_i is to belong to $\operatorname{Dom}_i f$ then the satisfaction, for all $m, \widehat{m} \in \mu_i^{-1}(m_i)$ and $\epsilon > 0$, of conditions (1) and (2) by some γ is also necessary.

Proof. Fix $m_i \in U_i$ and let $m, \widehat{m} \in \mu_i^{-1}(m_i)$. Let $\epsilon > 0$ so that by assumption there is a map $\gamma : [0,1] \to M$ from m to $\widehat{m}$ that satisfies (1) and (2) above. Since $f \circ \gamma : [0,1] \to \mathbb{R}$ is absolutely continuous we have

$$\begin{aligned}
|f(\widehat{m}) - f(m)| &= |(f \circ \gamma)(1) - (f \circ \gamma)(0)| \\
&= \left| \int_0^1 (f \circ \gamma)'(t)\, dt \right| \\
&< \int_0^1 \epsilon\, dt \quad \text{since} \quad \left\| (f \circ \gamma)' \right\|_\infty < \epsilon \\
&= \epsilon
\end{aligned}$$

so that since $\epsilon > 0$ was arbitrary we must have $f(\widehat{m}) = f(m)$. Since the $m, \widehat{m} \in \mu_i^{-1}(m_i)$ were arbitrary the set $f(\mu_i^{-1}(m_i))$ is singleton so that we may define the map

$$\begin{aligned}
f_i : U_i &\longrightarrow \mathbb{R} \\
m_i &\longmapsto f\big(\mu_i^{-1}(m_i)\big)
\end{aligned}$$

which satisfies $f = f_i \circ \mu_i$ on $\mu_i^{-1}(U_i)$. Assume that there is some $\mu_j(m^0) \in O_j \in \operatorname{Open}(M_j)$ such that every smooth curve $\eta_i : J \to O_j$ defined on an open subset of $\mathbb{R}$ has some smooth μ_j-lift into M. Let $U_j = O_j \cap \mu_{ij}^{-1}(U_i)$ and let $f_j = f_i \circ \mu_{ij}\big|_{U_j} : U_j \to \mathbb{R}$ so that $f = f_j \circ \mu_j$ on $\mu_j^{-1}(U_j)$. Let $\gamma_j : \mathbb{R} \to U_j$ be any smooth map and let $\gamma : \mathbb{R} \to M$ be any smooth μ_j-lift of γ_j. By assumption, $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is smooth and since $\operatorname{Im}(\mu_j \circ \gamma) = \operatorname{Im} \gamma_j \subseteq U_j$ we have that $f_j \circ \gamma_j = f_j \circ \mu_j \circ \gamma = f \circ \gamma$ is also smooth. Since $\gamma_j : \mathbb{R} \to U_j$ was an arbitrary smooth curve into the manifold U_j it follows from Boman's theorem that f_j is smooth on U_j, which implies that f is smooth on $\mu_j^{-1}(U_j) = \mu_i^{-1}(U_i) \cap \mu_j^{-1}(O_j)$. ■

Lemma 6.1.26. Assume that all tangent vectors in $\mathrm{T}\,M$ are kinematic. Let $f : M \to \mathbb{R}$ be any map such that

(1) for all smooth $\gamma : I \to M$, where $I \in \mathrm{Open}\,(\mathbb{R})$, the map $f \circ \gamma : I \to \mathbb{R}$ is smooth,

(2) whenever $\gamma : I \to M$ and $\eta : J \to M$, where I and J are open neighborhoods of 0 in $\mathbb{R}$, are smooth maps such that $\gamma'(0) = \eta'(0)$ then $(f \circ \gamma)'(0) = (f \circ \eta)'(0)$.

Then the map $\mathrm{T}\,f : \mathrm{T}\,M \to \mathbb{R}$ defined by sending $\mathbf{v} \in \mathrm{T}\,M$ to

$$\mathrm{T}\,f(\mathbf{v}) \underset{\text{def}}{=} (f \circ \gamma)'(0)$$

where γ is any smooth curve in M with $\gamma'(0) = \mathbf{v}$, is well-defined and $\mathrm{T}\,f(r\mathbf{v}) = r\,\mathrm{T}\,f(\mathbf{v})$ for all $r \in \mathbb{R}$ and all $\mathbf{v} \in \mathrm{T}\,M$.

Proof. That $\mathrm{T}\,f$ is defined on all of $\mathrm{T}\,M$ follows from the assumption that all tangent vectors are kinematic and $\mathrm{T}\,f$ is well-defined by assumption (2). Let $r \in \mathbb{R}$, $\mathbf{v} \in \mathrm{T}\,M$. Let $\gamma : \mathbb{R} \to M$ be a smooth curve such that $\mathbf{v} = \gamma'(0)$. Define the smooth curve $\eta : \mathbb{R} \to M$ by $\eta(t) = \gamma(rt)$ where observe that $\eta'(0) = r\gamma'(0) = r\mathbf{v}$ so that

$$\mathrm{T}\,f(r\mathbf{v}) = (f \circ \eta)'(0) = \left.\frac{d}{dt}\right|_{t=0} ((f \circ \gamma)(rt)) = r(f \circ \gamma)'(r \cdot 0) = r\,\mathrm{T}\,f(\mathbf{v})$$

as desired. ∎

The the following proposition provides a sufficient condition for determining that a given arbitrary function from a promanifold M into $\mathbb{R}$ is locally cylindrical. Also observe that Abbati and Manià's theorem 6.1.23 follows immediately from the following proposition.

Proposition 6.1.27. Assume that all tangent vectors in $\mathrm{T}\,M$ are kinematic. Let $f : M \to \mathbb{R}$ be any map that satisfies the conditions of lemma 6.1.26 and let $\mathrm{T}\,f : \mathrm{T}\,M \to \mathbb{R}$ be the induced map defined in that same lemma. Assume that for all $m^0 \in M$ and all $i_0 \in \mathbb{N}$ there exists some index $i \geq i_0$ and some $\mu_i(m^0) \in U_i \in \mathrm{Open}\,(M_i)$ such that for all $m_i \in U_i$ and all

$m, \widehat{m} \in \mu_i^{-1}(m_i)$ there exists a continuous map $\gamma : [0,1] \to \mu_i^{-1}(m_i)$ from m to $\widehat{m}$ that is smooth as a map into M (e.g. this condition is satisfied if, for instance, Sys_M is monotone, or more generally, that is smooth on some open subset of $[0,1]$ whose complement (in $[0,1]$) has Lebesgue measure 0. If $\mathrm{T} f : \mathrm{T} M \to \mathbb{R}$ is continuous then $f : M \to \mathbb{R}$ is locally cylindrical and if in addition every $\mu_\bullet$ has the smooth path lifting property then f is smooth.

Proof. Fix $m^0 \in M$. Since $\mathrm{T} f : \mathrm{T} M \to \mathbb{R}$ is continuous and $\mathrm{T} f(\mathbf{0}_{m^0}) = 0$, where $\mathbf{0}_{m^0}$ is the zero of $\mathrm{T}_{m^0} M$, there exists some $i_0 \in \mathbb{N}$ and some open set $W_{i_0}^0 \in \operatorname{Open}(\mathrm{T} M_{i_0})$ such that $\mathbf{0}_{m^0} \in (\mathrm{T} \mu_{i_0})^{-1}(W_{i_0}^0) \subseteq (\mathrm{T} f)^{-1}(]-1,1[)$. Pick $i \geq i_0$ and $U_i \in \operatorname{Open}(M_i)$ as in the statement of this theorem and observe that by replacing i_0 with i and $W_{i_0}^0$ by $(\mathrm{T} \mu_{i_0,i})^{-1}(W_{i_0})$ we may assume without loss of generality that $i_0 = i$. Since $(\mathrm{T}_{M_i} \circ \mathrm{T} \mu_i)(\mathbf{0}_{m^0}) = \mu_i(m^0)$ we may pick $\mu_i(m^0) \in O_i \in \operatorname{Open}(M_i)$ such that $O_i \subseteq U_i \cap \mathrm{T}_{M_i}(W_i^0)$ and by shrinking W_i^0 and O_i as necessary, we may also assume that $\mathrm{T}_{M_i}(W_i^0) = O_i$ and that W_i^0 contains the zero vector of $\mathrm{T}_{m_i} M_i$ for each $m_i \in O_i$.

Fix $m_i \in O_i$ and let $m, \widehat{m} \in \mu_i^{-1}(m_i)$. By assumption there exists a map $\gamma : [0,1] \to \mu_i^{-1}(m_i)$ from m to $\widehat{m}$ and some open subset I of $]0,1[$ such that $[0,1] \setminus I$ has Lebesgue measure 0 and $\widetilde{\gamma} \underset{\text{def}}{=} \gamma\big|_I : I \to M$ is smooth. Since $\operatorname{Im}(\mu_i \circ \widetilde{\gamma}) \subseteq \mu_i(\operatorname{Im} \gamma) \subseteq \mu_i^{-1}(\mu_i^{-1}(m_i)) = \{m_i\}$ we have that $(\mu_i \circ \gamma)' \equiv \mathbf{0}$ on I. For any $t_0 \in I$ and any $r \in \mathbb{R}$ we have

$$\mathrm{T} \mu_i(r \widetilde{\gamma}'(t_0)) = r(\mu_i \circ \widetilde{\gamma})'(t_0) = r \cdot \mathbf{0} = \mathbf{0}$$

where since $m_i \in U_i$ we have $\mathrm{T} \mu_i(r \widetilde{\gamma}'(t_0)) = \mathbf{0} \in W_i$. Hence

$$r \widetilde{\gamma}'(t_0) \in (\mathrm{T} \mu_i)^{-1}(W_i) \subseteq (\mathrm{T} f)^{-1}(]-1,1[)$$

so that $r \mathrm{T} f(\widetilde{\gamma}'(t_0)) = \mathrm{T} f(r \widetilde{\gamma}'(t_0)) \in]-1,1[$, which implies that $\mathrm{T} f(\widetilde{\gamma}'(t_0)) = 0$ since $r \in \mathbb{R}$ was arbitrary. Letting $\eta(t) = \widetilde{\gamma}(t + t_0)$ we have by definition of $\mathrm{T} f$

$$0 = \mathrm{T} f(\widetilde{\gamma}'(t_0)) = \mathrm{T} f(\eta'(0)) = (f \circ \eta)'(0) = (f \circ \widetilde{\gamma})'(t_0)$$

Thus $(f \circ \gamma)' \equiv 0$ on I. Since $[0,1] \setminus I$ has measure 0 it follows that $(f \circ \gamma)'$ is measurable, integrable, and $\left\|(f \circ \gamma)'\right\|_\infty = 0$ while from the differentiability of $f \circ \gamma$ on I we obtain the absolute continuity of $f \circ \gamma$. The conclusion now follows from lemma 6.1.24. ■

Trivially Cylindrical Maps and Inverse System Morphisms

Cylindricity and Compactness

The following lemma implies, in particular, that every locally cylindrical map from a compact promanifold is trivially cylindrical.

Lemma 6.2.1. Let $K \subseteq M$ be compact and let $F : K \to N$ be a roughly locally cylindrical map (def. 6.1.5) from a promanifold into a set N. Then there exists some index i and some $U_i \in \operatorname{Open}(M_i)$ such that $F = F_i \circ \mu_i$ on $\mu_i^{-1}(U_i)$ and $K \subseteq \mu_i^{-1}(U_i)$.

Proof. Since $F : K \to N$ is a continuous locally cylindrical map, for all $m \in K$ there exists some index $\iota(m) \in \mathbb{N}$ and some $U^m_{\iota(m)} \in \operatorname{Open}\left(M_{\iota(m)}\right)$ containing $\mu_{\iota(m)}(m)$ such that $F = F_{\iota(m)} \circ \mu_{\iota(m)}$ on the open subset $U^m \underset{\text{def}}{=} \mu^{-1}_{\iota(m)}\left(U^m_{\iota(m)}\right) \cap K$ of K. Pick a finite subcover $U^{m_1}, \ldots, U^{m_L}$ of K, let $i = \max\{\iota(m_1), \ldots, \iota(m_L)\}$, and for each $h = 1, \ldots, L$, let $U^h_i = \mu^{-1}_{\iota(m_h),i}\left(U^{m_h}_{\iota(m_h)}\right)$. Since $F = F_{\iota(m)} \circ \mu_{\iota(m)} = F_{\iota(m)} \circ \mu_{\iota(m),i} \circ \mu_i$ on $U^{m_h} = \mu_i^{-1}(U^h_i)$, it follows that $\operatorname{Dom}_i F$ is equal to $\mu_i(K) = \operatorname{Dom}_i F$ and $\operatorname{Dom}_i F \subseteq U^1_i \cup \cdots \cup U^L_i$. ■

Corollary 6.2.2. Let $F : M \to N$ be a roughly locally cylindrical map (def. 6.1.5) from a promanifold into a set N. For any compact $K \subseteq M$, there exists some index i and some $U_i \in \operatorname{Open}(M_i)$ such that $F = F_i \circ \mu_i$ on $\mu_i^{-1}(U_i)$ and $K \subseteq \mu_i^{-1}(U_i)$. In particular, if M is compact and $F : M \to N$ is a locally cylindrical map into a set N then F is trivially cylindrical.

As shown in the following lemma, if M is compact then F can be obtained as the limit of some subcollection of $F_\bullet^\bullet$ that forms an inverse system morphism.

Lemma 6.2.3. Let $F : M \to N$ be a locally cylindrical map between promanifolds, let $F^{\bullet} = \nu_{\bullet} \circ F$, and denote by $F^{\bullet}_{\bullet}$ the canonical collection of maps induced by F. If M is compact then there exists a strictly increasing $\iota : \mathbb{N} \to \mathbb{N}$ such that $\left(F^{\bullet}_{\iota(\bullet)}, \iota\right) : \mathrm{Sys}_M \to \mathrm{Sys}_N$ is an inverse system morphism whose limit is F where if in addition F is continuous (resp. smooth) then so are all $F^{\bullet}_{\iota(\bullet)} = \left(F^{a}_{\iota(a)}\right)_{a=1}^{\infty}$. In particular, if M is compact then F arises as the limit of some inverse system morphism.

Proof. For all indices a and i, let $O^a_i = \mathrm{ODom}_i\, F^a$. For any $a \in \mathbb{N}$ the open sets $(\mu_i^{-1}(O^a_i))_{i \in \mathbb{N}}$ form an increasing open cover of M so that there exists some $i_0 \in \mathbb{N}$ such that $i \geq i_0$ implies $M = \mu_i^{-1}(O^a_i)$, which in turn implies that $O^a_i = M_i$. Pick such an index for $a = 1$ and call it $\iota(1)$ and then inductively pick such indices satisfying the additional property that $\iota(a+1) > \iota(a)$. For each index a let $F_a \underset{\text{def}}{=} F^a_{\iota(a)} : M_{\iota(a)} \to N_a$. It is clear that $(F_{\bullet}, \iota)$ is the desired inverse system morphism. ■

Smooth Maps that are Not Limits of Inverse System Morphisms

The following examples show that if M is not compact then there may exist smooth maps that do not arise as the limit of any inverse system morphism from Sys_M into Sys_N where in particular, example 6.2.6 shows that this may even be true of a diffeomorphism from $\mathbb{R}^{\mathbb{N}} \cong\,]0, \infty[^{\mathbb{N}}$ onto itself, which indicates that, as tools, inverse system morphisms are not very well suited for proving an inverse function theorem for promanifolds.

Example 6.2.4. A locally cylindrical smooth map that is not trivially cylindrical: Let $M_i =]0,1[^i$, $M =]0,1[^{\mathbb{N}}$, and let $\mathrm{Pr}_{i,i+1}$ and Pr_i denote the canonical projections onto M_i so that $(M, \mathrm{Pr}_{\bullet})$ is the limit of this inverse system. For each index $i \in \mathbb{N}$ define

$$\begin{aligned} f_i : M_i \quad &\longrightarrow\ (0,1) \\ (m_1, \ldots, m_i) \ &\longmapsto \left(\frac{1}{1+2} - m_1\right)^2 + \cdots + \left(\frac{1}{i+2} - m_i\right)^2 \end{aligned}$$

let $B^i =]0, 1/i[$ and let $\varphi^i :]0,1[\to [0,1]$ be a smooth bump function with carrier B^i and such that $\varphi^i \equiv 1$ on $]0, 1/(i+1)]$. Let

$$\begin{aligned} F : M &\longrightarrow \mathbb{R} \\ m = (m_1, m_2, \dots) &\longmapsto \sum_{i=1}^{\infty} f_i(m_1, \dots, m_i)\varphi^i(m_1) \end{aligned}$$

Observe that if $m_1 \in [1/i, 1[$ then for all $j \geq i$ we have $\varphi^j(m_1) = 0$ so that for each $m \in M$ the above sum is finite.

F is smooth and locally cylindrical: fix $m^0 = (m_1^0, m_2^0, \dots) \in M$, pick $i \in \mathbb{N}$ such that $m_1^0 \in [1/i, 1[$, and let $U = \mathrm{Pr}_i^{-1}(]1/i, 1[)$. Observe that for any $m \in U$,

$$F(m) = \sum_{l=1}^{\infty} f_l(m_1, \dots, m_l)\varphi^l(m_1) = \left(\sum_{l=1}^{i-1} (f_l \circ \mathrm{Pr}_{li}) \cdot \left(\varphi^l \circ \mathrm{Pr}_{1i}\right)\right)(\mathrm{Pr}_i(m))$$

so that F is locally cylindrical and smooth at m^0.

F is not the limit of any inverse system morphism: suppose that there existed some inverse system morphism $(F_\bullet, \iota) : \mathrm{Sys}_M \to \mathrm{Sys}_\mathbb{R}$, where $\mathrm{Sys}_\mathbb{R}$ is the trivial system, such that $F = \varprojlim F_{\iota(\bullet)}$. Let $i = \iota(1)$ and pick any $m_1^0 \in \left]\frac{1}{i+2}, \frac{1}{i+1}\right[$ so that for all $l \geq i+2$, $\varphi^l(m_1^0) = 0$ while $\varphi^l(m_1^0) = 1$ for all $l \leq i+1$. For any $0 < r < 1$ let

$$m_r^0 \underset{\text{def}}{=} \left(m_1^0, \frac{1}{2+2}, \dots, \frac{1}{i+2}, r, \frac{1}{(i+2)+2}, \frac{1}{(i+3)+2}, \dots\right)$$

where the r is in the $i+1^{\text{th}}$ position. By definition of F,

$$\begin{aligned} F(m_r^0) &= \sum_{l=1}^{\infty} f_l(\mu_l(m_r^0))\varphi^l(m_1^0) \\ &= \sum_{l=1}^{i+1} f_l(\mu_l(m_r^0))(1) + \sum_{l=i+2}^{\infty} 0 \\ &= \left[\sum_{l=1}^{i} f_l\left(m_1^0, \frac{1}{2+2}, \dots, \frac{1}{l+2}\right)\right] + f_{i+1}\left(m_1^0, \frac{1}{2+2}, \dots, \frac{1}{i+2}, r\right) \end{aligned}$$

Observe that if $r \neq r'$ then $f_{i+1}(m_1^0, \frac{1}{2+2}, \ldots, \frac{1}{i+2}, r) \neq f_{i+1}(m_1^0, \frac{1}{2+2}, \ldots, \frac{1}{i+2}, r')$ so that $F(m_r^0) \neq F(m_{r'}^0)$. But, since $F = F_1 \circ \mu_i$ on M for any $m = (m_1, m_2, \ldots) \in M$ the value of $F(m)$ can only depend on $\mu_i(m) = (m_1, \ldots, m_i)$, which gives us a contradiction. ∎

Example 6.2.5. Another smooth function that is not trivially cylindrical: Let $\kappa :]0,\infty[\to \mathbb{R}$ be a smooth cutoff function such that $\kappa^{-1}(1) =]0,1]$, $\kappa^{-1}(0) = [2,\infty[$, and $\kappa' < 0$ on $]1,2[$. Let $\beta :]0,\infty[\to \mathbb{R}$ be a smooth function such that $\beta^{-1}(1) = [1,\infty[$, $\beta'(t) > 0 \iff t \in]0,1[$, and $\lim_{t \searrow 0} \beta(t) = 0$. For each $i \in \mathbb{N}$ let $M_i \underset{\text{def}}{=}]0,\infty[^i$, $M \underset{\text{def}}{=}]0,\infty[^{\mathbb{N}}$ and for all $i \leq j$ let $\mathrm{Pr}_{ij} : M_j \to M_i$ denote the canonical projection onto the first i coordinates. Define

$$\begin{aligned} F^1 : M &\longrightarrow \mathbb{R} \\ \mathbf{r} = (r_1, r_2, \ldots) &\longmapsto r_1 + \beta(r_1)[\kappa(r_2) + \kappa(2r_2 + r_3) + \kappa(3r_2 + r_3 + r_4) + \cdots] \end{aligned}$$

where the p^{th} term in the above series is $\kappa(pr_2 + r_3 + \cdots r_{p+1})$.

F^1 is well-defined: Fix $\mathbf{r}^0 = (r_1^0, r_2^0, \ldots) \in M$. Since $r_2^0 > 0$ there exists some smallest $n \in \mathbb{N}$ such that $nr_2^0 > 2$. Observe that

$$F^1(r^0) = r_1^0 + \beta(r_1^0)\left[\kappa(r_2^0) + \kappa(2r_2^0 + r_3^0) + \kappa((n-1)r_2^0 + \cdots + r_{n-1}^0) + 0 + 0 + \cdots\right]$$

since $nr_2^0 + \cdots r_{n-1}^0 > 2$ so that F^1 is well-defined.

Let $\epsilon_2 = r_2^0 - \frac{2}{n}$ and observe that $\epsilon_2 > 0$ and $2 = nr_2^0 - n\epsilon_2$. Let $\epsilon = \frac{1}{2}\min\{\epsilon, r_1^0, \ldots, r_{n-1}^0\}$, $U_{n-1} = (r_1^0, \ldots, r_{n-1}^0) +]-\epsilon,\epsilon[^{n-1}$ (i.e. U_{n-1} is the open cube centered at $(r_1^0, \ldots, r_{n-1}^0)$ with sides of length 2ϵ), and let $U = \mathrm{Pr}_{\leq n-1}^{-1}(U_{n-1})$.

Observe that for any $\mathbf{r} = (r_1, r_2, \ldots) \in M$, $|r_2 - r_2^0| < \frac{1}{2}\epsilon_2 < \epsilon_2$ so that $r_2^0 - \epsilon_2 < r_2$, which implies that $2 = nr_2^0 - n\epsilon_2 < nr_2$. Thus for any $l \leq n$, $\kappa(lr_2 + r_3 + \cdots + r_l) = 0$ since $lr_2 + r_3 + \cdots + r_l > 2$ so that

$$F^1(\mathbf{r}) = r_1 + \beta(r_1) c_{n-1}(r_1, \ldots, r_{n-1})$$

where $c_{n-1}(r_1, \ldots, r_{n-1}) \underset{\text{def}}{=} \kappa(r_2) + \kappa(2r_2 + r_3) + \cdots + \kappa((n-1)r_2 + \cdots r_{n-1})$. Since $F^1\big|_U$ is dependent only on $r_1, \ldots, r_{n-1}$ it is locally cylindrical at $\mathbf{r}^0$ and since the map

$$\begin{aligned} F^1_{n-1} : U_{n-1} &\longrightarrow \mathbb{R} \\ (r_1, \ldots, r_{n-1}) &\longmapsto r_1 + \beta(r_1) c_{n-1}(r_1, \ldots, r_{n-1}) \end{aligned}$$

is smooth on U_{n-1} it follows that F^1 is smooth at $\mathbf{r}^0$. Thus F^1 is a smooth map on M but since for any $l \in \mathbb{N}$ it is possible to choose $r_2, \ldots, r_l > 0$ such that $1 < l r_2 + r_3 + \cdots + r_l < 2$ (in which case $\kappa(l r_2 + r_3 + \cdots + r_l) \neq 0$), F^1 can not be trivially cylindrical. Observe that for any $v^1 \in \mathbb{R}$,

$$\mathrm{T}_{(r_1,\ldots,r_{n-1})} F^1_{n-1}(v^1, 0, \ldots, 0) = v^1 \big[1 + \beta'(r_1) c_{n-1}(r_1, \ldots, r_{n-1}) v^1 \big]$$

so that $\mathrm{T}_{(r_1,\ldots,r_{n-1})} F^1_{n-1}(v^1, 0, \ldots, 0) = 0 \iff v^1 = 0$. Hence the tangent map of

$$\begin{aligned} F^{n-1}_{n-1} : U_{n-1} &\longrightarrow \ ]0, \infty[^{n-1} \\ (r_1, \ldots, r_{n-1}) &\longmapsto \big(F^1_{n-1}(r_1, \ldots, r_{n-1}), r_2, \ldots, r_{n-1} \big) \end{aligned}$$

is at every Pointwise isomersive so that H is clearly a local diffeomorphism. ∎

Example 6.2.6. A smooth diffeomorphism $M \to M$ that is not the limit of any inverse system morphism: Let us continue using the notation and definitions from example 6.2.5. Let $M = \mathbb{R}^{\mathbb{N}}$ and for all $a \in \mathbb{N}^{\geq 2}$, define

$$\begin{aligned} F^a : M &\longrightarrow \mathbb{R} \\ \mathbf{r} = (r_1, r_2, \ldots) &\longmapsto r_a \end{aligned}$$

and let $F \underset{\text{def}}{=} (F^1, F^2, F^3, \ldots) : M \to M$ so that F is smooth since each coordinate is smooth. If F arose as the limit of some inverse system morphism $(F_\bullet, \iota)$ then in particular, $F_1 = \mathrm{Pr}_1 \circ F = F^1_{\iota(1)} \circ \mu_{\iota(1)}$ so that F^1 is trivially cylindrical (since $F^1(\mathbf{r} = F^1(r_1, \ldots, r_{\iota(1)}, r_{\iota(1)+1}, \ldots)$ is dependent only on $r_1, \ldots, r_{\iota(1)}$), which gives us a contradiction. Thus F is not the limit of any inverse system morphism in the category of Set. For any $c \geq 0$ define

$$\begin{aligned} g_c : \mathbb{R}^{>0} &\longrightarrow \mathbb{R} \\ z &\longmapsto z + c\beta(z) \end{aligned}$$

and observe that g_c is a strictly increasing function with image $\mathbb{R}^{>0}$ since it is the sum of the strictly increasing function $\mathrm{Id}_{\mathbb{R}^{>0}}$ and the non-decreasing function $z \mapsto c\beta(z)$. Since $g_c' > 0$ it follows that $g_c \mathbb{R}^{>0} \mathbb{R}^{>0}$ is a diffeomorphism. For every $\mathbf{s} = (s_1, s_2, \ldots) \in M$ we define $c(\mathbf{s}) \underset{\text{def}}{=} \kappa(r_2) + \kappa(2r_2 + r_3) + \kappa(3r_2 + r_3 + r_4) + \cdots$ and observe that the map

$$\begin{aligned} M &\longrightarrow \mathbb{R} \\ \mathbf{s} &\longmapsto c(\mathbf{s}) \end{aligned}$$

is smooth.

F is injective: Suppose $F(\mathbf{r}) = F(\mathbf{s})$ where $\mathbf{r} = (r_1, r_2, \ldots)$ and $\mathbf{s} = (s_1, s_2, \ldots)$. Then $r_1 = F^a(\mathbf{r}) = F^a(\mathbf{s}) = s_a$ for all $a \in \mathbb{N}^{\geq 2}$ so that $c \underset{\text{def}}{=} c(\mathbf{r})$ equals $c(\mathbf{s})$. This implies that

$$g_c(r_1) = r_1 + \beta(r_1)c = F^1(\mathbf{r}) = F^1(\mathbf{s}) = s_1 + \beta(s_1)c = g_c(s_1)$$

so the injectivity of g_c implies that $r_1 = s_1$ and thus that $\mathbf{r} = \mathbf{s}$.

F is surjective: Let $\mathbf{s} = (s_1, s_2, \ldots) \in M$, for all $i \in \mathbb{N}^{\geq 2}$ let $r_i \underset{\text{def}}{=} s_i$, and let $c \underset{\text{def}}{=} c(\mathbf{r})$. Since $c \geq 0$ we can let $r_1 \underset{\text{def}}{=} g_c^{-1}(s_1)$ and then define $\mathbf{r} \underset{\text{def}}{=} (r_1, r_2, \ldots)$ so that $F^1(\mathbf{r}) = r_1 + \beta(r_1)c = g_c(r_1) = s_1$, which implies that $F(\mathbf{r}) = \mathbf{s}$. We have thus shown that the inverse of F is

$$\begin{aligned} F^{-1} : M &\longrightarrow M \\ \mathbf{s} = (s_1, s_2, \ldots) &\longmapsto \left(g_{c(\mathbf{s})}^{-1}(s_1), s_2, s_3, \ldots \right) \end{aligned}$$

To see that $F^{-1} : M \to M$ is smooth, fix $\mathbf{s}^0 = (s_1^0, s_2^0, \ldots) \in M$, let $\mathbf{r}^0 = F^{-1}(\mathbf{s}^0)$ and define $n \in \mathbb{N}$, $\epsilon > 0$, $(r_1^0, \ldots, r_{n-1}^0) \in U_{n-1} \in \mathrm{Open}(M_{n-1})$, U, F_{n-1}^1, $c_{n-1}(r_1, \ldots, r_{n-1})$, and F_{n-1}^{n-1}

exactly as in example 6.2.5 above. Let $(r_1^0,\ldots,r_{n-1}^0) \in O_{n-1} \in \mathrm{Open}(U_{n-1})$ be such that $F_{n-1}^{n-1}\big|_{O_{n-1}} : O_{n-1} \to F_{n-1}^{n-1}(O_{n-1})$ is a diffeomorphism onto the open (in $]0,\infty[^{n-1}$) subset $V_{n-1} \underset{\text{def}}{=} F_{n-1}^{n-1}(O_{n-1})$. For any $i \le n-1$ observe that the map

$$\begin{aligned} F_i^i : O_{n-1}\times\,]0,\infty[^{i-n+1} &\longrightarrow V_{n-1}\times\,]0,\infty[^{i-n+1} \\ (r_1,\ldots,r_i) &\longmapsto \big(F_{n-1}^1(r_1,\ldots,r_{n-1}), r_2,\ldots,r_i\big) \end{aligned}$$

is a diffeomorphism. If the inverse of $F_{n-1}^{n-1}\big|_{O_{n-1}}$ is the map

$$\begin{aligned} H_{n-1}^{n-1} : V_{n-1} &\longrightarrow O_{n-1} \\ (s_1,\ldots,s_{n-1}) &\longmapsto (h(s_1,\ldots,s_{n-1}), s_2,\ldots,s_{n-1}) \end{aligned}$$

where $h : V_{n-1} \to \mathbb{R}$ is some smooth map then the inverse of F_i^i is the map

$$\begin{aligned} H_i^i : V_{n-1}\times\,]0,\infty[^{i-n+1} &\longrightarrow O_{n-1}\times\,]0,\infty[^{i-n+1} \\ (s_1,\ldots,s_i) &\longmapsto (h(s_1,\ldots,s_{n-1}), s_2,\ldots,s_i) \end{aligned}$$

If $\mathbf{s} \in \mathrm{Pr}_{\le n-1}^{-1}(V_{n-1})$ then it is straightforward to verify that

$$(h(s_1,\ldots,s_{n-1}), s_2,\ldots,s_{n-1}) \in O_{n-1} \quad \text{and} \quad F(h(s_1,\ldots,s_{n-1}), s_2,\ldots) = \mathbf{s}$$

so that $h(s_1,\ldots,s_{n-1}) = g_{c(\mathbf{s})}^{-1}(s_1)$ and $F^{-1}(\mathbf{s}) \in \mathrm{Pr}_{\le n-1}^{-1}(O_{n-1})$ from which it is immediate that $\mu_i \circ F^{-1}(\mathbf{s}) = H_i^i \circ \mathrm{Pr}_{\le i}(\mathbf{s})$. Since h is smooth it follows that $H \underset{\text{def}}{=} F^{-1}$ is locally cylindrical and smooth at $\mathbf{s}^0$. Thus $F : M \to M$ is a diffeomorphism that does not arise that the limit of any inverse system morphism in Set. ■

Example 6.2.7 ($\mathbb{R}^{\mathbb{N}}$ admits distinct pfd structures)**.** Let $\mathrm{Sys}_{\mathbb{R}^{\mathbb{N}}} = (\mathbb{R}^{\bullet}, \mathrm{Pr}_{\le i,j}, \mathbb{N})$ and $(\mathbb{R}^{\mathbb{N}}, \mathrm{Pr}_{\le\bullet}) = \varprojlim \mathrm{Sys}_{\mathbb{R}^{\mathbb{N}}}$ be as in example 2.1.51, let $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the diffeomorphism from example 6.2.6, and let $\nu_\bullet = \mathrm{Pr}_{\le\bullet} \circ F$. Observe that $(\mathbb{R}^{\mathbb{N}}, \nu_\bullet)$ is a limit of $\mathrm{Sys}_{\mathbb{R}^{\mathbb{N}}}$ so that both $(\mathrm{Pr}_{\le\bullet}, \mathrm{Sys}_{\mathbb{R}^{\mathbb{N}}})$ and $(\nu_\bullet, \mathrm{Sys}_{\mathbb{R}^{\mathbb{N}}})$ are smooth projective representations for $\mathbb{R}^{\mathbb{N}}$. However, there does not exist any smooth equivalence transformation between these two representations since $F : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ does not arise as the limit of any inverse system morphism. In particular, this shows that there exists at least two distinct pfd structures for $\mathbb{R}^{\mathbb{N}}$ (def. 5.0.2).

A Characterization of Smooth Maps that Arise as Limits of Inverse System Morphisms

The results in this subsection will not be used anywhere else in this paper.

Lemma 6.2.8. Suppose $F : M \to N$ is a smooth map from a promanifold M into a manifold N. Consider the following statements:

(1) F is trivially cylindrical.

(2) $F^* f = f \circ F : M \to \mathbb{R}$ is trivially cylindrical for all $f \in C_N^\infty(N)$.

(3) For all $V \in \operatorname{Open}(N)$ and $f \in C_N^\infty(V)$, $F^* f : F^{-1}(V) \to \mathbb{R}$ is trivially cylindrical and defined on a basic open subset of M.

Then (1) $\implies$ (2) $\implies$ (3) and if in addition $\sup_{n \in N} \dim_n N < \infty$ and some N_a has at most finitely many isolated points then (3) $\implies$ (1).

Proof. (1) $\implies$ (2) is immediate.

(2) $\implies$ (3): Let $V \in \operatorname{Open}(N)$, $f \in C_N^\infty(V)$, $U \underset{\text{def}}{=} F^{-1}(V)$. Let $\beta : N \to [0,1]$ be a smooth function such that $\beta^{-1}(0) = N \smallsetminus V$ and let $g \underset{\text{def}}{=} f \cdot \beta$. Pick an index i for which there exists some g_i and β_i in $C_{M_i}^\infty(M_i)$ such that $g \circ f = g_i \circ \mu_i$ and $\beta \circ F = \beta_i \circ \mu_i$. Since $U \subseteq \mu_i^{-1}(\mu_i(U))$ is always true, we need only to show the reverse containment: If $m \in \mu_i^{-1}(\mu_i(U))$ then $\mu_i(m) \in \mu_i(U)$ so that

$$\beta(F(m)) = \beta_i(\mu_i(m)) \in \beta_i(\mu_i(U)) = \beta(U) = \beta(F(U)) \subseteq \beta(V)$$

In particular, this implies that $\beta(F(m)) \neq 0$ so that $F(m) \in N \smallsetminus \beta^{-1}(0) = V$ and thus $m \in F^{-1}(V) = U$, as desired. We've thus shown that $U = \mu_i^{-1}(U_i)$ is a basic open subset of

M, where $U_i \underset{\text{def}}{=} \mu_i(U)$. Observe that if $m_i \in U_i$ then $\beta_i(m_i) \in \beta_i(\mu_i(U)) = \beta(F(U)) \subseteq \beta(V) \subseteq]0,1[$ so that $\beta_i(m_i) \neq 0$ and

$$\begin{aligned} f_i : U_i &\longrightarrow \mathbb{R} \\ m_i &\longmapsto \frac{g_i(m_i)}{\beta_i(m_i)} \end{aligned}$$

is well-defined and smooth. If $m \in U$ then

$$(f_i \circ \mu_i)(m) = \frac{g_i(\mu_i(m))}{\beta_i(\mu_i(m))} = \frac{(f \cdot \beta)(F(m))}{\beta(F(m))} = f(F(m))$$

Thus $f \circ F = f_i \circ \mu_i$ on the basic open set $\mu_i^{-1}(U_i) = F^{-1}(V)$.

(3) $\implies$ (1): Suppose $d \underset{\text{def}}{=} \sup_{n \in N} \dim_n N < \infty$ and for each $l = 0, \ldots, d$ let C^l denote the unions of the connected components of N of dimension l. Since each C^l has Lebesgue covering dimension l, for each $l = 0, \ldots, d$ there exist $l+1$ charts that cover C^l. Suppose that $(V^1, \psi^1), \ldots, (V^K, \psi^K)$ are these charts so that they, in particular, form a finite open cover of N. For each l, let $G^l = \psi^l \circ F$ and let $U^l = F^{-1}(V^l)$. Since each coordinate of each ψ^l is trivially cylindrical and each U^l is a basic open set, it follows that each ψ^l is trivially cylindrical and each U^l is a basic open set. That there are only finitely many of these charts allows us to pick an index i such that for each l there exists some $G_i^l \in C_{M_i}^\infty(U_i)$ such that $\psi^l \circ F = G_i^l \circ \mu_i$ on $\mu_i^{-1}(U_i^l)$ where $U_i^l \underset{\text{def}}{=} \mu_i(U^l)$. If $m \in U^p \cap U^q$ then $F(m) \in V^p \cap V^q$ so since $(V^p \psi^p)$ and $(V^q \psi^q)$ are charts we have $(\psi^p)^{-1}(\psi^p \circ F)(m) = (\psi^q)^{-1}(\psi^q \circ F)(m)$ so that $(\psi^p)^{-1}(G_i^p \circ \mu_i)(m) = (\psi^q)^{-1}(G_i^q \circ \mu_i)(m)$. This shows in particular that if $m_i \in U_i^p \cap U_i^q$ then $\left((\psi^p)^{-1} \circ G_i^p\right)(\mu_i^{-1}(m_i))$ is a singleton set that is independent of choice of p; call this unique element $F_i(m_i)$. Observe that if $m \in \mu_i^{-1}(m_i)$ then $m \in \mu_i^{-1}(U_i^p)$ so that

$$F_i(m_i) = F_i(\mu_i(m)) = \left((\psi^p)^{-1} \circ G_i^p\right)(\mu_i(m)) = (\psi^p)^{-1}(G^p(m)) = (\psi^p)^{-1}((\psi^p \circ F)(m)) = F(m)$$

We have thus defined a map $F_i : M_i \to N$ such that $F = F_i \circ \mu_i$ on M where note that the smoothness of F implies the smoothness F_i. ∎

A particular consequence of the following characterization is that although inverse system morphisms would be well-suited to our needs if we only needed to deal with trivially cylindrical functions, they are not the appropriate tool to be used with promanifolds since smooth ($\mathbb{R}$-valued) functions on promanifolds need not be trivially cylindrical. Indeed, this observation was the original motivation for definition 10.2.10 and the other definitions in that subsection.

Theorem 6.2.9. Assume that $\sup_{n_a \in N_a} \dim_{n_a} N_a < \infty$ is finite for each $a \in \mathbb{N}$ and that some N_a has at most finitely many isolated points (e.g. if all $N_\bullet$ are connected). If $F : M \to N$ is a smooth map between promanifolds then the following are equivalent:

(1) F arises as the limit of an inverse system morphism from Sys_M to Sys_N.

(2) For all trivially cylindrical $f \in C^\infty_N(N)$, $f \circ F : M \to \mathbb{R}$ is also trivially cylindrical.

(3) For all trivially cylindrical $f \in C^\infty_N(V)$ defined on a basic open subset V of N, the pullback $F^* f : F^{-1}(V) \to \mathbb{R}$ is trivially cylindrical and defined on a basic open subset of M.

Proof. (1) $\implies$ (2) $\implies$ (3) is immediate.

(3) $\implies$ (1): By repeatedly applying lemmata 6.2.8 and 6.1.4, we may inductively pick an increasing sequence of indices $\iota(1) < \iota(2) < \cdots$ such that for each index a, $\nu_a \circ F = F^a_{\iota(a)} \circ \mu_{\iota(a)}$ for some smooth $F^a_{\iota(a)} : M_{\iota(a)} \to N_a$. If $a < b$ then $i \underset{\text{def}}{=} \iota(a) < j \underset{\text{def}}{=} \iota(b)$ and for any $m \in M$,

$$\left(\nu_{ab} \circ F^b_j\right)(\mu_j(m)) = (\nu_{ab} \circ \nu_b \circ F)(m) = (\nu_a \circ F)(m) = (F^a_i \circ \mu_i)(m) = (F^a_i \circ \mu_{ij})(\mu_j(m))$$

so that the surjectivity of μ_j implies that $\nu_{ab} \circ F^b_j = F^a_i \circ \mu_{ij}$. It is immediately seen that F is the limit of this inverse system morphism. ∎

Smooth Partitions of Unity

The following definitions are taken from [27, def. 16.1].

Definition 6.3.1. Suppose X is topological space and $\mathcal{S} \subseteq C(X \to \mathbb{R})$ is a subalgebra of $C(X \to \mathbb{R})$. If $U^\bullet = (U^\lambda)_{\lambda \in \Lambda}$ is an open cover of X and $\phi^\bullet = (\phi^\lambda)_{\lambda \in \Lambda}$ be a collection of continuous $\mathbb{R}$-valued functions on X then we will say that $\phi^\bullet$ is a *smooth* or *$\mathcal{S}$-partition of unity subordinate to $\mathcal{U}$* if each ϕ^λ belongs to $\mathcal{S}$ and they form a partition of unity subordinate to $U^\bullet$. We will say that X (or $(X, \mathcal{S})$)

(1) is *smoothly normal* or *$\mathcal{S}$-normal* if for any two disjoint closed sets $A, B \subseteq X$ there exists a smooth $f : X \to \mathbb{R}$ such that $f\big|_A = 0$ and $f\big|_B = 1$.

(2) *admits smooth partitions of unity* and that it is *$\mathcal{S}$-paracompact* if every open cover of X admits a $\mathcal{S}$-partition of unity subordinate to it.

We may replace the word "smooth" with "$\mathcal{S}$" in these definitions.

Lemma 6.3.2. Suppose that $e : (S, \mathcal{S}) \to (M, \mathcal{M})$ is a smooth map and a topological embedding onto a closed subspace of M where $\mathcal{S}$ (resp, $\mathcal{M}$) are sheaves of continuous $\mathbb{R}$-valued functions on S (resp. M). If $(M, \mathcal{M})$ is smoothly normal (resp. smoothly paracompact) then so is $(S, \mathcal{S})$.

Proof. If $(M, \mathcal{M})$ is smoothly normal then for any disjoint closed subsets A_0 and A_0 of M we may find a smooth $f \in \mathcal{M}(M)$ such that $f_i\big|_{e(A_i)} = i$ for $i = 0, 1$ so that the map $f \circ e : S \to \mathbb{R}$ belongs to $\mathcal{S}(S)$ and satisfies $f_i \circ e\big|_{A_i} = i$ for $i = 0, 1$. Now suppose that $(M, \mathcal{M})$ is smoothly paracompact and let $U^\bullet = (U^\lambda)_{\lambda \in \Lambda}$ be an open cover of S. Let $V^0 = M \setminus \operatorname{Im}(e)$ and for all $\lambda \in \Lambda$ let V^λ denote an open subset of M such that $e(U^\lambda) = V^\lambda \cap \operatorname{Im} e$. Pick a $\mathcal{M}(M)$-partition of unity $\{\phi^0, \phi^\bullet\}$ subordinate to $\{V^0\} \cup \{V^\bullet\}$ and observe that the Λ-indexed collection of maps $\phi^\bullet \circ e$ forms a $\mathcal{S}(S)$-partition of unity subordinate to $U^\bullet$. ∎

Of course a necessary condition for $(M, \mathcal{M})$ to admit smooth partitions of unity is for M to be paracompact.

Example 6.3.3. Let $C^{\infty, TVS}(\mathbb{R}^{\mathbb{N}})$ denote the set of all smooth (def. B.2.1) real-valued functions on the Fréchet space $\mathbb{R}^{\mathbb{N}}$ and let $C^\infty_{\mathbb{R}^{\mathbb{N}}}$ denote the canonical sheaf when $\mathbb{R}^{\mathbb{N}}$ is considered as a promanifold. Since $\mathbb{R}^{\mathbb{N}}$ is a nuclear Fréchet space, we can conclude from [27, thm. 16.10] that $\mathbb{R}^{\mathbb{N}}$ is $C^{\infty,TVS}(\mathbb{R}^{\mathbb{N}})$-paracompact. But theorem 6.1.23 established that all Gâteaux continuously differentiable $\mathbb{R}$-valued functions are locally cylindrical, which makes it easy to see that $C^{\infty,TVS}(\mathbb{R}^{\mathbb{N}}) = C^{\infty}_{\mathbb{R}^{\mathbb{N}}}(\mathbb{R}^{\mathbb{N}})$. We may thus conclude that $\mathbb{R}^{\mathbb{N}}$ is $C^{\infty}_{\mathbb{R}^{\mathbb{N}}}(\mathbb{R}^{\mathbb{N}})$-paracompact and furthermore, it is easy to see how one may use smooth bump functions to conclude that $C^{\infty}_{\mathbb{R}^{\mathbb{N}}} = C^{\infty,TVS}_{\mathbb{R}^{\mathbb{N}}}$, which proves the claim that was made in example 4.0.11.

Theorem 6.3.6 will allow us to apply proposition 6.3.4 and theorem 6.3.5 to a promanifold (M, C^{∞}_{M}) with M and $C^{\infty}_{M}(M)$ in place of X and $\mathcal{S}$, respectively.

Proposition 6.3.4 ([27, prop. 16.2, pp. 153, 165 - 166])**.** Let X be a Hausdorff space and let $\mathcal{S} \subseteq C(X \to \mathbb{R})$ be a subalgebra of $C(X \to \mathbb{R})$. Assume that for all $h \in C^{\infty}(\mathbb{R} \to \mathbb{R})$ one has $h_*(\mathcal{S}) \subseteq \mathcal{S}$ and that whenever a function $f : X \to \mathbb{R}$ belongs locally to the presheaf on X defined by $\mathcal{S}(U) \underset{\text{def}}{=} \left\{ f\big|_U \,\middle|\, f \in \mathcal{S} \right\}$ (i.e. for all $x \in X$ there is some $x \in U \in \operatorname{Open}(X)$ such that $f\big|_U \in \mathcal{S}(U)$) then $f \in \mathcal{S}$.

Consider the following statements:

(1) X is $\mathcal{S}$-normal.

(2) For any two closed disjoint subsets $A_0, A_1 \subseteq X$ there is a function $f \in \mathcal{S}$ with $f|_{A_0} = 0$ and $0 \notin f(A_1)$.

(3) Every locally-finite open covering admits S-partitions of unity subordinated to it.

(4) For any two disjoint zero-sets A_0 and A_1 of continuous functions there exists a function $g \in \mathcal{S}$ with $g|_{A_j} = j$ for $j = 0, 1$ and $g(X) \subseteq [0, 1]$.

(5) For any continuous function $f : X \to \mathbb{R}$ there exists a function $g \in \mathcal{S}$ with $f^{-1}(0) \subseteq g^{-1}(0) \subseteq f^{-1}(\mathbb{R} \setminus \{1\})$.

(6) The set $\mathcal{S}$ is dense in the algebra of continuous functions on X with respect to the topology of uniform convergence.

(7) The set of all bounded functions in $\mathcal{S}$ is dense in the algebra of continuous bounded functions on X with respect to the supremum norm.

(8) The bounded functions in $\mathcal{S}$ separate points in the Stone-Čech-compactification βX of X.

Then (1) - (3) are equivalent, (4) - (8) are equivalent, and if X is metrizable then they are all equivalent. Furthermore, X is $\mathcal{S}$-paracompact if and only if it is paracompact and $\mathcal{S}$-normal.

Theorem 6.3.5 ([27, thm. 16.15])**.** Let X and $\mathcal{S}$ be as in proposition 6.3.4. If X is metrizable then the following are equivalent:

(1) X is $\mathcal{S}$-paracompact i.e. admits $\mathcal{S}$-partitions of unity.

(2) X is $\mathcal{S}$-normal.

(3) The topology of X has a basis which is a countable union of locally finite families of carriers of smooth functions.

(4) There is a homeomorphic embedding $i : X \to c_0(A)$ for some A (with image in the unit ball) such that $ev_a \circ i$ is smooth for all $a \in A$.

Theorem 6.3.6. Promanifolds admit smooth partitions of unity and are smoothly normal.

Proof. We will prove that (3) of theorem 6.3.5 holds with $X \underset{\text{def}}{=} M$ and $\mathcal{S} \underset{\text{def}}{=} C^\infty_M(M)$. For each $i \in \mathbb{N}$, M_i has a basis $\mathcal{B}_i$ that is a countable union of locally finite families of carriers of smooth (i.e. $C^\infty_{M_i}(M_i)$) functions, say $\mathcal{B}_i = \bigcup\limits_{n=1}^{\infty} \mathcal{B}_i^n$. For any $B_i \in \mathcal{B}_i^n$ there exists some smooth function $f_{B_i} : M_i \to \mathbb{R}$ such that $\operatorname{carr}(f_{B_i}) = B_i$ so that $f_{B_i} \circ \mu_i : M \to \mathbb{R}$ is a smooth function such that $\operatorname{carr}(f_{B_i} \circ \mu_i) = \mu_i^{-1}(B_i)$. Thus, every open set $\mu_i^{-1}(B_i) \in \mu_i^{-1}(\mathcal{B}_i) = \bigcup\limits_{n=1}^{\infty} \mu_i^{-1}(\mathcal{B}_i^n)$ is the carrier of some smooth function in $C^\infty_M(M)$ and furthermore, note that since each $\mathcal{B}_i^n$ each is a locally finite family of open sets the same is true of $\mu_i^{-1}(\mathcal{B}_i^n)$. Thus $\bigcup\limits_{i=1}^{\infty} \mu_i^{-1}(\mathcal{B}_i) = \bigcup\limits_{i=1}^{\infty} \bigcup\limits_{n=1}^{\infty} \mu_i^{-1}(\mathcal{B}_i^n)$ is a basis for M that is a countable union of locally finite families of carriers of smooth functions. ∎

We now present an alternative proof of theorem 6.3.6 that uses the generalized Whitney embedding theorem for promanifolds.

Proof. Let (M, C^∞_M) be a promanifold so that it is paracompact and metrizable. By theorem 11.6.5 there exists a smooth map $j : M \to \mathbb{R}^{\mathbb{N}}$ that is a topological embedding onto a closed subspace of $\mathbb{R}^{\mathbb{N}}$. Recall that $\mathbb{R}^{\mathbb{N}}$ is $C^\infty_{\mathbb{R}^{\mathbb{N}}}(\mathbb{R}^{\mathbb{N}})$-paracompact so that $\mathbb{R}^{\mathbb{N}}$ is $C^\infty_{\mathbb{R}^{\mathbb{N}}}(\mathbb{R}^{\mathbb{N}})$-paracompact. Now lemma 6.3.2 implies that M is smoothly-paracompact. ∎

The following example shows that not every closed subset of a promanifold is the support of a smooth bump functions and explains the absence of their mention in theorem 6.3.6.

Example 6.3.7. There are no smooth $\mathbb{R}$-valued functions on $\mathbb{R}^{\mathbb{N}}$ with non-empty compact support: suppose that $f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is a smooth function and $r \underset{\text{def}}{=} f(m)$ is non-zero for some $m \in \mathbb{R}^{\mathbb{N}}$. Since f is smooth, it is locally cylindrical so there is some index $i \in \mathbb{N}$ and some smooth $f_i : \operatorname{ODom}_i f \to \mathbb{R}$ such that $\operatorname{Pr}_{\leq i}(m) \in \operatorname{ODom}_i f$ and so $f^{-1}(r) = \operatorname{Pr}^{-1}_{\leq i}(f_i^{-1}(r)) = f_i^{-1}(r) \times \prod_{l=i+1}^{\infty} \mathbb{R}$.

Partial Generalization of the Boman Theorem for Promanifolds

The following is a partial generalization of the Boman Theorem to promanifolds.

Theorem 6.4.1 (Partial Generalization of the Boman Theorem)**.** Let M and N be a promanifolds.

(1) A curve $\gamma : \mathbb{R} \to M$ is smooth $\iff$ $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is smooth for all smooth $f : M \to \mathbb{R}$.

(2) A map $F : Z \to N$ from a manifold Z is smooth $\iff$ $F \circ \gamma : \mathbb{R} \to N$ is smooth whenever $\gamma : \mathbb{R} \to M$ is a smooth curve.

Suppose that $F : M \to N$ is a map such that for each $m \in M$, there are cofinally many $a \in \mathbb{N}$ such that $\mu_i(m) \in \operatorname{ODom}_i F^a$ for some i. If there for each index i, $\mu_i : M \to M_i$ can lift any germ of any smooth curve on $\mathbb{R}$ in M_i to some germ of a smooth curve in M then

(3) $F : M \to N$ is smooth $\iff$ $F \circ \gamma : \mathbb{R} \to N$ is smooth whenever $\gamma : \mathbb{R} \to M$ is a smooth curve.

Proof. (1): For the non-trivial direction, let $\gamma : \mathbb{R} \to M$ is be a curve and suppose that for all $f \in C^\infty_M(M)$ the composition $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is smooth and fix an index i. For any $f_i \in C^\infty_{M_i}(M_i)$, the function $f \underset{\text{def}}{=} f_i \circ \mu_i : M \to \mathbb{R}$ belongs to $C^\infty_M(M)$ so that our assumption implies that $f_i \circ (\mu_i \circ \gamma)$ is smooth. Since $\mu_i \circ \gamma : \mathbb{R} \to M_i$ is a map between manifolds and f_i was arbitrary we have by Boman's theorem that $\mu_i \circ \gamma$ is smooth. Since the index i was arbitrary it follows that $\gamma : \mathbb{R} \to M$ is smooth.

(2): Now assume that for all smooth curves $\gamma : \mathbb{R} \to Z$ the composition $F \circ \gamma : \mathbb{R} \to N$ is smooth, where Z is a manifold. Fix an index i and observe that $\nu_i \circ F \circ \gamma$ is smooth for all smooth curves $\gamma : \mathbb{R} \to Z$ since ν_i and $F \circ \gamma$ are smooth. Since Z and N_i are manifolds it follows from Boman's theorem that $\nu_i \circ F : Z \to N_i$ is smooth. Since the index i was arbitrary it follows that $F : \mathbb{R} \to N$ is smooth.

(3): Fix $m \in M$ and let a be any index for which there exists an i such that $\mu_i(m) \in O_i \underset{\text{def}}{=} \operatorname{ODom}_i F^a$, where by increasing i we may assume that i is an index such that every germ of a smooth open curve in M_i has a μ_i-lift to a germ of some smooth open curve. Let $\gamma_i : \mathbb{R} \to O_i$ be any smooth curve and fix $t^0 \in \mathbb{R}$. Let $\gamma :]a, b[\to M$ be a smooth curve whose domain contains t^0 such that $\mu_i \circ \gamma = \gamma_i$ on $]a, b[$. By assumption, $F \circ \gamma :]a, b[\to N$ is smooth so that

$$\nu_a \circ F \circ \gamma = F^a_i \circ \mu_i \circ \gamma = F^a_i \circ \gamma_i\big|_{]a,b[}$$

is smooth. Since $t^0 \in \mathbb{R}$ was arbitrary it follows that $F^a_i \circ \gamma_i$ is smooth. Since $\gamma_i : \mathbb{R} \to O_i$ was an arbitrary smooth curve it follows from Boman's theorem that $F^a_i : O_i \to N_a$ is smooth. Thus $\nu_a \circ F$ is smooth at m where since a was one of cofinally many indices it follows that

F is smooth at m. ■

Chapter 7

The Tangent Space at a Point

Note that when M is a manifold then the definition 7.0.1 is consistent with its usual definition.

Definition 7.0.1 ([20]). Define the *tangent space at* $m \in M$ to be

$$\mathrm{T}_m\, M \underset{\text{def}}{=} \mathrm{Der}_m\big(C^\infty_{M,m} \to \mathbb{R}\big)$$

which is the space of all $\mathbb{R}$-derivations from $C^\infty_{M,m}$ into $\mathbb{R}$, (def. 1.1.29) where $C^\infty_{M,m} = [C^\infty_M]_m$ denotes the stalk of smooth functions at m. We will call elements of $\mathrm{T}_m\, M$ *(analytic) tangent vectors (of M at m).*

Given any index i, $m_i \in M_i$, and $m_i \in U_i \in \mathrm{Open}\,(M_i)$, recall that since the smooth manifold M_i admits smooth bump function, we are able define the following canonical isomorphism

$$\begin{array}{ccc} \mathrm{T}_{m_i}\, M_i \underset{\text{def}}{=} \mathrm{Der}_{m_i}\big(C^\infty_{M_i,m_i} \to \mathbb{R}\big) & \longrightarrow & \mathrm{Der}_{m_i}\big(C^\infty_{M_i}(U_i) \to \mathbb{R}\big) \\ v & \longmapsto & \big[f_i \mapsto v\big([f_i]_{m_i}\big)\big] \end{array}$$

By an argument that is completely analogous to the proof that the above map is an isomorphism, one may use the following lemma 7.0.2 to prove for any $m \in M$ and $m \in U \in$

Open(M), the canonical map

$$\begin{aligned} \mathrm{T}_m M \underset{\text{def}}{=} \mathrm{Der}_m\left(C^\infty_{M,m} \to \mathbb{R}\right) &\longrightarrow \mathrm{Der}_m(C^\infty_M(U) \to \mathbb{R}) \\ v &\longmapsto [f \mapsto v([f]_m)] \end{aligned}$$

is also an isomorphism, which we will henceforth use to identify $\mathrm{T}_m M$ with $\mathrm{Der}_m(C^\infty_M(U) \to \mathbb{R})$.

Lemma 7.0.2. Let $f, g \in C^\infty_M(U)$ for some $U \in$ Open(M) and fix $m \in U$. If $x \in \mathrm{Der}_m\left(C^\infty_{M,m}(U) \to \mathbb{R}\right)$ and if $f \equiv g$ on some neighborhood of m then $x(f) = x(g)$.

Proof. Since both f and g are smooth at m, we may pick an index i, $V_i \in$ Open(M_i), and $f_i, g_i : V_i \to \mathbb{R}$ such that $m \in \mu_i^{-1}(V_i) \subseteq U$, $f \equiv g$ on $\mu_i^{-1}(V_i)$, and $f = f_i \circ \mu_i$ and $g = g_i \circ \mu_i$ on $\mu_i^{-1}(V_i)$. Let $\phi_i : M_i \to \mathbb{R}$ be a smooth bump function such that $\mu_i(m) \in \mathrm{Int}_{M_i}(\phi_i^{-1}(0))$, $\phi_i^{-1}(0) \subseteq V_i$, and $\phi_i \equiv 1$ on $M_i \smallsetminus V_i$. Let $\phi \underset{\text{def}}{=} \phi_i \circ \mu_i\big|_U : U \to \mathbb{R}$. Note that $f - g = \phi \cdot (f - g)$ on all of U so by using the fact that x is a derivation it becomes easy to see that $x(f - g) = 0$, as desired. ∎

The Tangent Map at a Point

The following definition extends to promanifolds the notion of the tangent map at a point.

Definition 7.1.1 ([20])**.** Let $F : M \to N$ be a smooth map between promanifolds and let $m \in M$. Define *the tangent map (of F) at m* to be

$$\begin{aligned} \mathrm{T}_m F : \mathrm{T}_m M &\longrightarrow \mathrm{T}_{F(m)} N \\ x &\longmapsto x \circ F^* \end{aligned}$$

where recall that $F^* g = g \circ F$. Explicitly,

$$\begin{aligned} \mathrm{T}_m F x \underset{\text{def}}{=} x \circ F^* : \mathrm{Der}_{F(m)}\left(C^\infty_{N,F(m)} \to \mathbb{R}\right) &\longrightarrow \mathbb{R} \\ [g]_{F(m)} &\longmapsto x(g \circ F) \end{aligned}$$

Furthermore, for any given $x \in \mathrm{T}_m M$ and $y \in \mathrm{T}_{F(m)} N$ if $\mathrm{T}_m F x = y$ then we will say that x *and* y *are* F*-related (at* m*)* and that x *an* F*-lift of* y where if x is an F-lift of $0 \in \mathrm{T}_{F(m)} N$ then we'll call x *is* F*-vertical* (or simply *vertical* if F is clear from context). If $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_{F(m)} N$ is a (continuous) surjection then we will call m a *regular point (of* F*)* and a *critical point (of* F*)* otherwise. A *critical value (of* F*)* is the image of a critical point and a *regular value (of* F*)* is any point of the codomain that is not a critical value of F. Given a vector $y \in \mathrm{T}_{F(m)} N$, we will say that y *is tangent to* F *at* m if y is contained in the image of $\mathrm{T}_m F$. ■

Remark 7.1.2. This assignment of $(M, m) \mapsto \mathrm{T}_m M$ and $F \mapsto \mathrm{T}_m F$ behaves just as it does with manifolds: For $F : M \to N$ and $G : N \to P$ smooth we clearly have that $\mathrm{T}_m(G \circ F) = \mathrm{T}_{F(m)} G \circ \mathrm{T}_m F$ and $\mathrm{T}_m \mathrm{Id}_M = \mathrm{Id}_{\mathrm{T}_m M}$. If $F : M \to \mathbb{R}$ is real-valued then we will as usual identify $(\mathrm{T}_m F x) \underset{\text{def}}{=} x \circ F^*$ with the real number $(x \circ F^*)(\mathrm{Id}_{\mathbb{R}}) = x(F)$.

In addition, if $F : M \to N^1 \times N^2$ is $F = (F^1, F^2)$ then $\mathrm{T}_m F = (\mathrm{T}_m F^1, \mathrm{T}_m F^2)$ where we will use the usual identification of $(x^1, x^2) \in \mathrm{T}_{F^1(m)} N^1 \times \mathrm{T}_{F^2(m)} N^2$ as an element of $\mathrm{T}_m(N^1 \times N^2)$ by giving it the canonical action on $f \in C^\infty_{N^1 \times N^2}(N^1 \times N^2)$ defined by $(x^1, x^2) f = x^1(f \circ \mathrm{In}_1) + x^2(f \circ \mathrm{In}_2)$ where $\mathrm{In}_j : N^j \to N^1 \times N^2$ are the usual inclusions at $F(m) = (F^1(m), F^2(m))$ defined by $n^1 \mapsto (n^1, F^2(m))$ and $n^2 \mapsto (F^1(m), n^2)$. ■

Definition 7.1.3. Say that a smooth map $F : M \to N$ between promanifolds is a *point(wise) submersion* (resp. *immersion*, *isomersion*) *at* $m \in M$ if $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_{F(m)} N$ is surjective (resp. injective, an isomorphism of TVSs). If we don't mention the point m then we mean that this is true at every point of M.

Dimension of a Promanifold at a Point

Since $\mathrm{T}_m M \underset{\text{def}}{=} \mathrm{Der}_m\big(C^\infty_{M,m} \to \mathbb{R}\big)$ is a vector space over $\mathbb{R}$ dependent only on (M, C^∞_M), the following definition is independent of the any smooth projective representation that (M, C^∞_M) belongs to.

Definition 7.2.1. For any $m \in M$, the *dimension of* (M, C_M^∞) *at* $m \in M$ will be defined as

$$\dim_m M \underset{\text{def}}{=} \dim \mathrm{T}_m M$$

If $\dim_m M$ is independent of $m \in M$ then we will denote this common value by $\dim M$ and call M a $\dim M$*-dimensional promanifold*. If M is a d-dimensional promanifold for some $d \in \mathbb{Z}^{\geq 0}$ then we will call M a *finite-dimensional promanifold*.

Let $m^0 \in M$. For any $d \in \{0, 1, \ldots, \infty\}$, if there exists a neighborhood $m^0 \in U \in \operatorname{Open}(M)$ such that $\dim_m M = d$ (resp. $\dim_m M < \infty$) for all $m \in U$ then we will say that M is *d-dimensional (resp. finite-dimensional) around* m^0 and that M *has (locally) constant (resp. finite) dimension around* m^0. We will say that M *has locally constant (resp. finite) dimension* if M has constant (resp. finite) dimension around every $m \in M$. ■

Lemma 7.2.2. For each $d \in \mathbb{Z}$ let $M^{\leq d} \underset{\text{def}}{=} \{m \in M \,|\, \dim_m M \leq d\}$, $M^{=d} \underset{\text{def}}{=} \{m \in M \,|\, \dim_m M = d\}$ and define $M^{\geq d}$, $M^{<d}$ and $M^{>d}$ analogously. Then

(1) $M^{\leq d}$ is closed in M, $M^{\geq d}$ is open in M, and $M^{=d}$ is locally closed in M.

(2) $M^\infty \underset{\text{def}}{=} \{m \in M \,|\, \dim_m M = \infty\}$ is a G_δ and F_σ-set in M.

(3) If C is a connected component of M then for all $c, \hat{c} \in C$, $\dim_c M = \dim_{\hat{c}} M$ and $\dim_{\mu_i(c)} M_i \leq d$.

(4) If $m \in M$ and $\{m\} \in \operatorname{Open}(M)$ then $\dim_m M = 0$.

Proof. Let $m \in M$ and let $m_i = \mu_i(m)$ for all i. It is clear that $\dim_m M \underset{\text{def}}{=} \dim_m \mathrm{T}_m M = \sup i \in \mathbb{N} \dim_{m_i} \mathrm{T}_{m_i} M_i = \sup i \in \mathbb{N} \dim_{m_i} M_i$.

(1): Suppose that m was in the closure of $M^{\leq d}$ in M. Let i be arbitrary, $m_i = \mu_i(m)$, and let $m_i \in U_i \in \operatorname{Open}(M_i)$ be an open ball in M_i of dimension $d_i \underset{\text{def}}{=} \dim_{m_i} M_i$. Since $\mu_i^{-1}(U_i)$ is an open set containing m, there exists some $c \in M^{\leq d} \cap \mu_i^{-1}(U_i)$ so $c_i = \mu_i(c)$ and m_i belong the same d_i-dimensional ball so that $d \geq \dim_{c_i} M = \dim_{m_i} M$. Since i was arbitrary, we have

$\dim_m M \le d$ so $m \in M^{\le d}$. Since $M^{\ge d} = M \setminus M^{\le d-1}$ it follows that $M^{\ge d}$ is open in M and that $M^{=d} = M^{\le d} \cap M^{\ge d}$ is locally closed in M.

(3): Since C is connected so are all $\mu_i(C)$ so that $c_i = \mu_i(c)$ and $\hat{c}_i = \mu_i(\hat{c})$ belong the same connected component of the manifold M_i and hence $\dim_{c_i} M_i = \dim_{\hat{c}_i} M_i$ for all i.

(2) follows immediately from (1) and (3).

(4): If $\{m\} \in \operatorname{Open}(M)$ then pick an index i and an open set $U_i \in \operatorname{Open}(M_i)$ such that $m \in \mu_i^{-1}(U_i) \subseteq \{m\}$. This implies that for any $j \ge i$, $\{\mu_j(m)\}$ is an open and closed subset of the manifold M_j, which is only possible if $\dim_{\mu_j(m)} M_j = 0$. Since $\dim_m M = \sup j \in \mathbb{N} \dim_{\mu_j(m)} M_j$ it follows that $\dim_m M = 0$. ∎

Corollary 7.2.3. For all $d \in \mathbb{Z}$, $M^{\ge d}$ is a subpromanifold of M.

Canonical Identifications of the Tangent Space at a Point

The following lemma 7.3.1 details a result from [20].

Lemma and Definition 7.3.1. Fix $m \in M$ and let $m_\bullet = \mu_\bullet(m)$.

(1) For any $x \in \mathrm{T}_m M$ and any index i,

$$x^i \underset{\text{def}}{=} \mathrm{T}_m \mu_i(x) \underset{\text{def}}{=} x \circ \mu_i^* : C^\infty_{M_i}(M_i) \to \mathbb{R}$$

is a derivation at $m_i = \mu_i(m)$ on $C^\infty_{M_i}(M_i)$ where its defining property is that $x^i(f_i) = x(f_i \circ \mu_i)$ for all $f_i \in C^\infty_{M_i}(M_i)$. Furthermore, $\mathrm{T}_{\mu_j(m)} \mu_{ij}(x^j) = x^i$ for all $i \le j$ so that $x \circ \mu_\bullet^* \underset{\text{def}}{=} (x \circ \mu_i^*)_{i \in \mathbb{N}}$ defines an element $\varprojlim x \circ \mu_\bullet^*$ of $\varprojlim \mathrm{T}_{\mu_\bullet(m)} M_\bullet$.

(2) For any $x \in \varprojlim \mathrm{T}_{\mu_i(m)} M_i$, we will identify x with a derivation at m on C^∞_M by giving it the following (henceforth canonical) well-defined action on $f \in C^\infty_M(M)$: if for some index i, some $U_i \in \operatorname{Open}(M_i)$, and some $f_i \in C^\infty_{M_i}(U_i)$ we have $m \in \mu_i^{-1}(U_i)$ and $f = f_i \circ \mu_i$ on $\mu_i^{-1}(U_i)$ then

$$xf \underset{\text{def}}{=} x^i(f_i)$$

where $x^\bullet \underset{\text{def}}{=} \mathrm{T}_m \, \mu_\bullet(x)$.

This forms an isomorphism of vector spaces

$$\begin{array}{lcl} \mathrm{T}_m \, M & \longrightarrow & \varprojlim \mathrm{T}_{\mu_\bullet(m)} \, M_\bullet \\ x & \longmapsto & \varprojlim (x \circ \mu_\bullet^*) \end{array}$$

that we will henceforth use to identify the two space.

Proof. (2): Suppose that $x \in \varprojlim \mathrm{T}_{m_\bullet} \, M_\bullet$ and let $x^\bullet \underset{\text{def}}{=} \mathrm{T}_m \, \mu_\bullet(x)$. Let f, i, U_i, and f_i be as above and suppose in addition that $j \geq i$, U_j, and f_j also satisfy analogous conditions. Since $j \leq i$ we may replace U_j with $U_j \cap \mu_{ij}^{-1}(U_i)$ so that $m_j \in U_j \subseteq \mu_{ij}^{-1}(U_i)$ and $U \underset{\text{def}}{=} \mu_i^{-1}(U_i) \cap \mu_j^{-1}(U_j) = \mu_j^{-1}(U_j)$ with $m \in U$. Note that $f_j \circ \mu_j = f|_U = f_i \circ \mu_i|_U = (f_i \circ \mu_{ij}) \circ \mu_j$ on U so that $f_j|_{U_j} = f_i \circ \mu_{ij}|_{U_j}$. Since $x^i = \mathrm{T}_{m_j} \, \mu_{ij}(x^j)$ we have

$$x^i f_i = \left(\mathrm{T}_{m_j} \, \mu_{ij}(x^j)\right) f_i = (x^j \circ \mu_{ij}^*)(f_i) = x^j(f_i \circ \mu_{ij}) = x^j(f_j)$$

so that $x(f) = x^i(f_i)$ is well-defined. If $g \in C_M^\infty(M)$ is any other function then by increasing i and shrinking U_i we can write $g|_U = g_i \circ \mu_i|_U$ for some $g_i \in C_{M_i}^\infty(U_i)$ so that $x(f \cdot g) = f(m)x(f) + g(m)x(g)$ is immediate.

(1): If $v \in \mathrm{T}_m \, M$ then it is easy to that each $v^i \underset{\text{def}}{=} v \circ \mu_i^*$ is a derivation at m_i on $C_{M_i}^\infty(M_i)$. Since the usual tangent map between manifolds $\mathrm{T}_{m_j} \, \mu_{ij} : \mathrm{T}_{m_j} \, M_j \to \mathrm{T}_{m_i} \, M_i$ is defined by $\mathrm{T}_{m_j} \, \mu_{ij}(v^i) = v^i \circ \mu_{ij}^*$ we see that $\mathrm{T}_{m_j} \, \mu_{ij}(v^j) = v^i$ so that $\varprojlim v^\bullet$ is in fact an element of $\varprojlim \mathrm{T}_{\mu_i(m)} \, M_i$. If f, i, U_i, and f_i are as above then

$$\left(\varprojlim v^\bullet\right)(f) = v^i(f_i) = (v \circ \mu_i^*)(f_i) = v(f_i \circ \mu_i) = v(f)$$

Furthermore, since the operations on the vector space $\varprojlim \mathrm{T}_{\mu_i(m)} \, M_i$ are defined componentwise, the assignment $v \mapsto \varprojlim (v \circ \mu_\bullet^*)$ is clearly a vector space homomorphism.

Now suppose that $x \in \varprojlim \mathrm{T}_{m_\bullet} \, M_\bullet$ and $x^\bullet \underset{\text{def}}{=} \mathrm{T}_m \, \mu_\bullet(x)$ are before and let v denote the derivation at m on $C_M^\infty(M)$ induced by x as described above. Then for each index i and all $f_i \in C_{M_i}^\infty(M_i)$ we have

$$(v \circ \mu_i^*)(f_i) = v(f_i \circ \mu_i) = x^i(f_i)$$

so that $x^i = v \circ \mu_i^*$. We've thus shown that the map $v \mapsto \varprojlim(v \circ \mu_\bullet^*)$ is a vector space-isomorphism. ■

Assumption 7.3.2. The vector space isomorphism from 7.3.1 allows us to place on $\mathrm{T}_m M$ the TVS topology of $\varprojlim \mathrm{T}_{\mu_\bullet(m)} M_i$, which will make $\mathrm{T}_m M$ into a nuclear Fréchet space that is TVS-isomorphic to $\mathbb{R}^d$ where $d = \sup \dim \mathrm{T}_{\mu_i(m)} M_i$.

Remark 7.3.3. The vector space $\mathrm{T}_m M \underset{\text{def}}{=} \mathrm{Der}_m\left(C_{M,m}^\infty \to \mathbb{R}\right)$ is dependent only on the ringed space (M, C_M^∞) and as the following proposition shows, even the limit topology of $\mathrm{T}_m M$ is completely determined by the ringed space (M, C_M^∞) (indeed, it shows that it is determined entirely by the sheaf's space of global sections). In particular, if $d = \dim \mathrm{T}_m M$ then $\mathrm{T}_m M$ is necessarily TVS-isomorphic to $\mathbb{R}^d$ where if $d = \infty$ then it will necessarily be TVS-isomorphic to $\mathbb{R}^{\mathbb{N}}$ while if $d < \infty$ then it has the unique finite-dimensional Hausdorff TVS-topology

Proposition 7.3.4. Fix $m \in M$ and let τ_w be the weakest topology on $\mathrm{T}_m M$ making all $\mathrm{T}_m f : \mathrm{T}_m M \to \mathrm{T}_{f(m)} \mathbb{R} \cong \mathbb{R}$ continuous, where $f \in C_M^\infty(M)$. Then τ_w is the limit topology on $\mathrm{T}_m M$ induced by $(\mathrm{T}_m M, \mathrm{T}_m \mu_\bullet) = \varprojlim \mathrm{Sys}_{T_m M}$, where $m_\bullet \underset{\text{def}}{=} \mu_\bullet(m)$ and $\mathrm{Sys}_{T_m M} \underset{\text{def}}{=} \left(\mathrm{T}_{m_\bullet} M_\bullet, \mathrm{T}_{m_j} \mu_{ij}, \mathbb{N}\right)$.

Proof. Let τ be the limit topology on $\mathrm{T}_m M$ induced by $(\mathrm{T}_m M, \mathrm{T}_m \mu_\bullet) = \varprojlim \mathrm{Sys}_{T_m M}$. If $f \in C_M^\infty$ then pick i, $\mu_i(m) \in U_i \in \mathrm{Open}(M_i)$, and $f_i \in C_{M_i}^\infty(U_i)$ such that $f = f_i \circ \mu_i$ on $\mu_i^{-1}(U_i)$. The tangent map $\mathrm{T}_m f = \mathrm{T}_{m_i} f_i \circ \mathrm{T}_m \mu_i$ is also continuous since $\mathrm{T}_m \mu_i : \mathrm{T}_m M \to \mathrm{T}_{m_i} M_i$ is τ-continuous and $\mathrm{T}_{m_i} f_i : \mathrm{T}_{m_i} M_i \to \mathbb{R}$ is always continuous. Since $f \in C_M^\infty$ was arbitrary we've shown that $\tau_w \subseteq \tau$. Now let $i \in \mathbb{N}$, $d = \dim_{m_i} M_i$ identify $\mathrm{T}_{m_i} M_i$ with $\mathbb{R}^d$. Let $W_i \in \mathrm{Open}(\mathrm{T}_{m_i} M_i)$ so that $W \underset{\text{def}}{=} (\mathrm{T}_m \mu_i)^{-1}(W_i)$ is a τ-basic open set in $\mathrm{T}_m M$ and suppose $w \in W$. Clearly, we may find $f_i^1, \ldots, f_i^d \in C_{M_i}^\infty(M_i)$ such that $w_i \underset{\text{def}}{=} \mathrm{T}_m \mu_i(w) \in$

$\bigcap\limits_{l=1}^{d} \left(\mathrm{T}_{m_i} f_i^l\right)^{-1}(\mathbb{R}^{>0}) \subseteq W_i$. For all l, $f^l \underset{\text{def}}{=} f_i^l \circ \mu_i : M \to \mathbb{R}$ is smooth so that $\bigcap\limits_{l=1}^{d} \left(\mathrm{T}_m f^l\right)^{-1}(\mathbb{R}^{>0}) = \bigcap\limits_{l=1}^{d} \left(\mathrm{T}_m \mu_i\right)^{-1}\left(\left(\mathrm{T}_{m_i} f_i^l\right)^{-1}(\mathbb{R}^{>0})\right)$ is a τ_w-open subset of $\mathrm{T}_m M$ contained in W, which shows that $\tau \subseteq \tau_w$. ∎

Properties of the Tangent Map at a Point

Proposition 7.4.1. Let $F : (M, m) \to (N, n)$ be a smooth map between promanifolds, let $F^\bullet = \nu_\bullet \circ F$, and let $m_\bullet = \mu_\bullet(m)$. Then

(1) $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_n N$ is a TVS-homomorphism (def. B.1.1) with a closed image in $\mathrm{T}_n N$ that splits in $\mathrm{T}_n N$ (def. B.1.5).

(2) If $\iota : \mathbb{N} \to \mathbb{N}$ is an order morphism such that $\mu_{\iota(a)}(m) \in \mathrm{ODom}_{\iota(a)} F^a$ for all $a \in \mathbb{N}$ then the ι-indexed collection of maps $\mathrm{T}_{m_{\iota(a)}} F^a_{\iota(a)} : \mathrm{T}_{m_{\iota(a)}} M_{\iota(a)} \to \mathrm{T}_{\nu_a(n)} N_a$ form an inverse system morphism from $\mathrm{Sys}_{\mathrm{T}_m M}$ to $\mathrm{Sys}_{\mathrm{T}_n N}$ whose limit is the tangent map $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_n N$.

Proof. Let $n_\bullet \underset{\text{def}}{=} \nu_\bullet(n)$. To show that the linear map $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_n N$ is continuous we must show, by the universal property of limits, that $\mathrm{T}_{F(m)} \nu_i \, \mathrm{T}_m F = \mathrm{T}_m(\nu_i \circ F) : \mathrm{T}_m M \to \mathrm{T}_{\nu_i(F(m))} N_i$ is continuous for all indices i. It thus suffices to prove continuity for the case where N is a manifold. Since $F : M \to N$ is smooth and N is finite-dimensional, we have $F = F_i \circ \mu_i$ on $\mu_i^{-1}(U_i)$ where $m \in \mu_i^{-1}(U_i)$. Note that $\mathrm{T}_{m_i} F_i$ is continuous since it is a linear map between finite-dimensional vector spaces and that $\mathrm{T}_m \mu_i$ is continuous by definition of the topology on $\mathrm{T}_m M$ so that $\mathrm{T}_m F = \mathrm{T}_{m_i} F_i \circ \mathrm{T}_m \mu_i$ is continuous.

If $a \leq b$ are such that $\mu_{\iota(a)}(m) \in \mathrm{ODom}_{\iota(a)} F^a$ and $\mu_{\iota(b)}(m) \in \mathrm{ODom}_{\iota(b)} F^b$ then

$$\nu_{ab} \circ F^b_{\iota(b)} = F^a_{\iota(a)} \circ \mu_{\iota(a),\iota(b)} \text{ on } \mathrm{ODom}_{\iota(b)} F^b \cap \mu^{-1}_{\iota(a),\iota(b)}\left(\mathrm{ODom}_{\iota(a)} F^a\right)$$

so that by the functoriality of the tangent map it is immediate that the tangent maps $\Lambda_a \underset{\text{def}}{=} \mathrm{T}_{m_{\iota(a)}} F^a_{\iota(a)}$ form an inverse system morphism. By assumption $m \in M$ is such that

$\mu_{\iota(a)}(m)$ belongs to $\operatorname{ODom}_{\iota(a)} F^a$ for infinitely many $a \in \mathbb{N}$ and for any such index a we have by definition of $F^a_{\iota(a)}$ that $\nu_a \circ F = F^a_{\iota(a)} \circ \mu_{\iota(a)}$ on $\mu^{-1}_{\iota(a)}\left(\operatorname{ODom}_{\iota(a)} F^a\right)$ so that by the functoriality of the tangent map we see that $\mathrm{T}_m F$ satisfies the characteristic property of the limit map. Pick w in the closed set $W \underset{\text{def}}{=} \bigcap_{a \in \mathbb{N}} \left(\mathrm{T}_n \nu_a\right)^{-1}\left(\operatorname{Im} \mathrm{T}_{m_{\iota(a)}} F^a_{\iota(a)}\right)$, which contains $\operatorname{Im} \mathrm{T}_m F$. For all $a \in \mathbb{N}$ let $V^a = \left(\mathrm{T}_{m_{\iota(a)}} F^a_{\iota(a)}\right)^{-1}(w)$ and recall that $\left(\mathrm{T}_m F\right)^{-1}(w) = \varprojlim\left(V^\bullet, \mathrm{T}_{m_{\iota(b)}} \mu_{\iota(b),\iota(b)}, \mathbb{N}\right)$, which is not empty. Thus $w \in \operatorname{Im} \mathrm{T}_m F$, which shows that $\operatorname{Im} \mathrm{T}_m F = W$ is closed in the Fréchet space $\mathrm{T}_n N$ so that by corollary 3.2.4, it splits in $\mathrm{T}_n N$. That $\mathrm{T}_m F$ is a TVS-homomorphism now follows from the open-mapping theorem. ∎

Remark 7.4.2. Example 2.3.11 shows $\mathbb{R}^{\mathbb{N}}$ contains a non-closed dense vector subspace S such that $\operatorname{Pr}_{\leq i}(S) = \mathbb{R}^i$ for each $i \in \mathbb{N}$ so the fact that $\operatorname{Im} \mathrm{T}_m F$ is closed in $\mathrm{T}_n N$ could have failed to be true had $\mathrm{T}_m F$ not arisen as the limit of an inverse system morphism.

Corollary 7.4.3. If $x \in \mathrm{T}_m M$ and $w \in \mathrm{T}_{F(m)} N$ then $\mathrm{T}_m F x = w$ if and only if for all $a \in \mathbb{N}$ there exists an index i such that $\mu_i(m) \in \operatorname{ODom}_i F^a$ and $\mathrm{T}_{\mu_i(m)} F^a_i x^i = w^a$, where $x^\bullet \underset{\text{def}}{=} \mathrm{T}_m \mu_\bullet(x)$ and $w^\bullet \underset{\text{def}}{=} \mathrm{T}_{F(m)} \nu_\bullet(w)$. Furthermore, this remains true if we replace "there exists an index i" with "whenever i is an index".

Proposition 7.4.4. Let $F : (M, m) \to (N, n)$ be smooth, $w \in \mathrm{T}_n N$, and $w^\bullet \underset{\text{def}}{=} \mathrm{T}_n \nu_\bullet w$. Then $w \in \operatorname{Im}(\mathrm{T}_m F)$ if and only if $w^a \in \operatorname{Im} \mathrm{T}_m(\nu_a \circ F)$ for all indices a

Proof. If $w \in \operatorname{Im}(\mathrm{T}_m F)$ then $w^i = \mathrm{T}_n \nu_i w \in \mathrm{T}_n \nu_i(\operatorname{Im}(\mathrm{T}_n F)) = \operatorname{Im} \mathrm{T}_m(\nu_i \circ F)$. Note that the image of $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_n N$ splits in $\mathrm{T}_n N$ by proposition 7.4.1 and that $w^a \in \operatorname{Im} \mathrm{T}_a(\nu_a \circ F)$ for all indices a. Let $R^a = \left(\mathrm{T}_n \nu_a\right)^{-1}(w^a)$ so that $a \leq b$ implies $R^b \subseteq R^a$ and $\bigcap_{a \in \mathbb{N}} R^a = \{w\}$. Let $C^i = (\operatorname{Im} \mathrm{T}_m F) \cap R^i$. The assumption that $w^i \in \operatorname{Im} \mathrm{T}_m(\nu_i \circ F)$ means exactly that $C^i \neq \varnothing$ for each index i. Since $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_n N$ is a continuous linear map whose image splits, $\operatorname{Im}(\mathrm{T}_m F)$ is a closed subspace of $\mathrm{T}_n N$ and since both $\mathrm{T}_m M$ and $\mathrm{T}_n N$ are both F-spaces we have by the open mapping theorem that $\mathrm{T}_m F : \mathrm{T}_m M \to \operatorname{Im}(\mathrm{T}_m F)$ is a TVS-isomorphism. Thus each C^i is a closed subset of $\mathrm{T}_n N$. Since $C^i \subseteq R^i$ we have that $\bigcap_i C^i \subseteq \bigcap_i R^i = \{w\}$ so that $\bigcap_i C^i$ is either the empty set or $\{w\}$. Since C^i is a decreasing sequence of closed subsets whose diameters (in any complete metric on $\mathrm{T}_n N$) goes to 0 we have that $\bigcap_i C^i \neq \varnothing$ so that $\bigcap_i C^i = \{w\}$. In particular, $w \in \operatorname{Im} \mathrm{T}_m F$ and since $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_n N$ is a TVS-isomorphism we can identify $\mathrm{T}_m M$ as a TVS subspace of $\mathrm{T}_n N$. ■

Identifying Tangent Spaces via Smooth Maps

Kinematic Tangent Vectors

Definition 7.5.1. Given any smooth curve $\gamma : J \to M$, where $J \subseteq \mathbb{R}$ is a non-degenerate interval, and any $t_0 0 \in J$,define *the derivative of* γ *at* t_0 to be the derivation

$$\gamma'(t_0) \underset{\text{def}}{=} \mathrm{T}_{t_0} \gamma \left(\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=t_0} \right)$$

Definition 7.5.2. Let $m \in M$ and $v \in \mathrm{T}_m M$. Call v a *kinematic tangent vectors (at m)* and say that v *arises as the derivative of a curve* if there exists a smooth curve $\gamma : (I, 0) \to (M, m)$ such that $\gamma'(0) = v$.

Lemma 7.5.3. If $F : (P, p) \to (M, m)$ is a smooth map where P is a manifold, then every element of $\operatorname{Im}(\mathrm{T}_p F)$ is a kinematic tangent vector.

Proof. Let $v \in \operatorname{Im}(\mathrm{T}_p F)$ and pick $w \in \mathrm{T}_p P$ be such that $\mathrm{T}_p F w = v$. Let $\eta : (I, 0) \to (P, p)$ be a smooth curve from an open interval I such that $\eta'(0) = w$. Since $(F \circ \eta)'(0) = \mathrm{T}_p F(\eta'(0)) = \mathrm{T}_p F w = v$, v is a kinematic tangent vector. ■

Lemma 7.5.4. Let $m \in M$ and consider $M_\bullet$ as being the pointed spaces $(M_i, \mu_i(m))$ with all of Sys_M's connecting maps being pointed. If each connecting map is a pointed 1-fibration then every tangent vector at m is a kinematic tangent vector.

Proof. Let $v \in \mathrm{T}_m M$ be arbitrary and let $v^\bullet = \mathrm{T}_m \mu_\bullet v$. If $v = \mathbf{0}$ then we're done so let i_0 be any index such that $v^{i_0} \neq 0$ and let $\gamma_{i_0} : ([-1, 1], 0) \to (M_{i_0}, m_{i_0})$ be any smooth embedding such that $\gamma'_{i_0}(0) = v^{i_0}$. Suppose we've constructed smooth embeddings $\gamma_i : ([-1, 1], 0) \to (M_i, m_i)$

for $i = i_0, \ldots, k$ such that $\gamma_i'(0) = v^i$ and $\mu_{ij} \circ \gamma_j = \gamma_i$ for all $i_0 \leq i \leq j \leq k$. Since $\mu_{k,k+1}$ is a pointed 1-fibration there exists a smooth map $\eta_{k+1} : ([-1,1], 0) \to (M_{k+1}, m_{k+1})$ that is $\mu_{k,k+1}$-lift of γ_k so that η_{k+1} is necessarily a smooth embedding. By applying theorem C.4.2 we obtain a smooth map $\gamma_{k+1} : ([-1,1], 0) \to (M_{k+1}, m_{k+1})$ that is also a $\mu_{k,k+1}$-lift of γ_k but with $\gamma_{k+1}'(0) = v^{k+1}$. Letting $\gamma = \varprojlim \gamma_i$ gives us a smooth map that satisfies $\gamma'(0) = v$. ■

Identifying Linear Independence

Lemma 7.6.1. Let $m \in M$ be such that $\dim \mathrm{T}_m M \neq 0$, let $x_1, \ldots, x_n \in \mathrm{T}_m M$, and let $x_l^\bullet = \mathrm{T}_m \mu_\bullet (x_l)$ for all $l = 1, \ldots, n$. Then $x_1, \ldots x_n$ are linearly independent $\iff$ there exists some index i such that $x_1^i, \ldots, x_n^i \in \mathrm{T}_{m_i} M_i$ are linearly independent, in which case

(1) $x_1^j, \ldots, x_n^j$ are linearly independent for all $j \geq i$, and

(2) for any $h \leq i$ the vectors $x_1^h, \ldots, x_n^h$ are linearly independent $\iff$ $\ker \mathrm{T}_{m_i} \mu_{hj} \cap \operatorname{span}\{x_1^i, \ldots, x_n^i\} = 0$.

Proof. If $x_1, \ldots x_n$ are linearly dependent then there are constants $c^1, \ldots, c^n$ such that $c^1 x_1 + \cdots + c^n x_n = 0$ then since each $\mathrm{T}_m \mu_i : \mathrm{T}_m M \to \mathrm{T}_{m_i} M_i$ is linear it follows that $c^1 x_1^i + \cdots + c^n x_n^i = 0$ for all indices i. Now assume that $x_1, \ldots x_n$ are linearly independent.

Remark 7.6.2. If for each index i the vectors $x_1^i, \ldots, x_n^i \in \mathrm{T}_{m_i} M_i$ failed to be linearly independent then we could obtain constants $c_i^1, \ldots, c_i^n$ (dependent on i) such that $c_i^1 x_1^i + \cdots + c_i^n x_n^i = 0$. However, to conclude that $x_1, \ldots x_n$ are linearly dependent we need remove the dependency of these constants on i and this is the reason why the proof of this converse is more complicated than one may have initially suspected.

Observe that if i is such that $x_1^i, \ldots, x_n^i$ are linearly independent and $j \geq i$ then since $\mathrm{T}_{m_j} \mu_{ij} : \mathrm{T}_{m_j} M_j \to \mathrm{T}_{m_i} M_i$ is linear it follows that $x_1^j, \ldots, x_n^j$ are also linearly independent. For all $l \in \mathbb{N}$, let $K_l = \{(c^1, \ldots, c^n) \in \mathbb{R}^n \mid c^1 x_1^l + \cdots + c^n x_n^l = 0\}$ and note that K_l is a vector space and that if $j \leq l$ then $K_l \subseteq K_j$. Let $K = \bigcap_{l \in \mathbb{N}} K_l$ so that K is a finite-dimensional vector subspace of $\mathbb{R}^n$. Observe that $\dim K = \inf_{l \in \mathbb{N}} \dim K_l$ so that $K = \{\mathbf{0}\}$ if and only if there exists some $l \in \mathbb{N}$ for which $K_l = \{\mathbf{0}\}$. If $K \neq \{\mathbf{0}\}$ then pick any non-zero $\mathbf{c} = (c^1, \ldots, c^n) \in K$. Then since $\mathbf{c} \in K_l$ for all $l \in \mathbb{N}$ we have that $c^1 x_1^l + \cdots + c^n x_n^l = 0$ for all $l \in \mathbb{N}$ so that $c^1 x_1 + \cdots + c^n x_n = 0$, a contradiction. Thus $K = \{\mathbf{0}\}$ so that there is some $i \in \mathbb{N}$ such that $K_i = \{\mathbf{0}\}$, which means exactly that $x_1^i, \ldots, x_n^i$ are linearly independent. Let $S_l = \operatorname{span}\{x_1^i, \ldots, x_n^i\}$ for all $l \in \mathbb{N}$ and fix $h \leq i$. Let $N_i = \ker \mathrm{T}_{m_i} \mu_{hj}$. To prove (2), first suppose that $x_1^h, \ldots, x_n^h$ are linearly dependent so that there exists a non-zero $\mathbf{c} \in K_h$. Let $v_i = c^1 x_1^i + \cdots + c^n x_n^i$ and note that v_i is non-zero and $\mathrm{T}_{m_i} \mu_{hi} v_i = c^1 x_1^h + \cdots + c^n x_n^h = 0$ so that $v_i \in S_i \cap N_i$. Conversely, if $v_i \in S_i \cap N_i$ is a non-zero vector then we can write $v_i = c^1 x_1^i + \cdots + c^n x_n^i$ where not all $c^1, \ldots, c^n$ are 0. Then by the linearity of $\mathrm{T}_{m_i} \mu_{hi}$, we have $0 = \mathrm{T}_{m_i} \mu_{hi} v_i = c^1 x_1^h + \cdots + c^n x_n^h$ so that $x_1^h, \ldots, x_n^h$ are linearly dependent. ∎

As shown by the following corollary, lemma 7.6.1 implies that if $V = \operatorname{span}\{x_1, \ldots, x_n\}$ with $x_1, \ldots, x_n$ linearly independent then the index i from the above lemma is actually independent of the vectors $x_1, \ldots, x_n$ and depends only on the vector space V and $\mu_\bullet$.

Corollary and Definition 7.6.3. Let V be a finite-dimensional vector subspace of $\mathrm{T}_m M$ and let $m_\bullet = \mu_\bullet(m)$. There exists a (unique) smallest index i such that $\dim(\mathrm{T}_m \mu_i(V)) = \dim V$ or equivalently, such that $\mathrm{T}\,\mu_i$ is injective on V. Furthermore, for any $v_1, \ldots, v_n \in V$, the vectors $v_1, \ldots, v_n$ are linearly independent (resp. form a basis for V) if and only if $v_1^i, \ldots, v_n^i$ are linearly independent (resp. form a basis for $\mathrm{T}_{m_i} \mu_i(V)$).

Definition 7.6.4. Call this unique smallest index i *the* $\mu_\bullet$*-index of* V or *the index of* V *(in* Sys_M*)* and denote it by $\operatorname{Ind}_{\mathrm{Sys}_M}(V)$, $\operatorname{Ind}_{\mu_\bullet}(V)$, or simply $\operatorname{Ind}(V)$ if Sys_M or $\mu_\bullet$ is understood. If $W \leq \mathrm{T}_m M$ is an infinite-dimensional vector space then let $\operatorname{Ind} W = \infty$.

Corollary 7.6.5. For any vector subspace V of $\mathrm{T}_m M$, $\operatorname{Ind}(V) = i \in \mathbb{N} \iff \dim V < \infty$, in which case $\mathrm{T}_m \mu_i\big|_V : V \to \mathrm{T}_{\mu_i(m)} M_i$ is injective and for all $1 \leq h < i$, $\mathrm{T}_m \mu_h\big|_V : V \to \mathrm{T}_{\mu_h(m)} M_h$ is *not* injective.

Corollary 7.6.6. Let $F : (M, m) \to (N, n)$ be a smooth map that is a pointwise immersion at m and let $i \in \mathbb{N}$. Then there exists some vector subspace V of $\mathrm{T}_m M$ and some index $a \in \mathbb{N}$ such that both $\mathrm{T}_m \mu_i\big|_V : V \to \mathrm{T}_{\mu_i(m)} M_i$ and $\mathrm{T}_m(\nu_a \circ F)\big|_V : V \to \mathrm{T}_{\nu_a(F(m))} N_a$ are injective.

Chapter 8

The Tangent Bundle

The following definitions of tangent bundle and tangent map are from [20].

Definition 8.0.1. By *the (total space of the) tangent bundle of M* we mean the set

$$\mathrm{T}\, M \underset{\text{def}}{=} \bigsqcup_{m \in M} \mathrm{T}_m\, M$$

and we will call the map

$$\begin{aligned} \mathrm{T}_M : \mathrm{T}\, M \quad & \longrightarrow \; M \\ x \in \mathrm{T}_m\, M \; & \longmapsto \; m \end{aligned}$$

the *canonical projection (of* $\mathrm{T}\, M$ *onto* M*)*.

Remark 8.0.2. We use the terminology "tangent bundle" since $\mathrm{T}\, M$ is defined analogously to the tangent bundle of a manifold but it is emphasized that there are promanifolds M where $(\mathrm{T}\, M, \mathrm{T}_M, M)$ is not even a fiber bundle (def. 9.0.1).

Definition 8.0.3. Define *the tangent map of* (or *induced by) F*, denoted by $\mathrm{T}\, F$ or possibly by F_*, to be the map

$$\begin{aligned} \mathrm{T}\, F : \mathrm{T}\, M \; & \longrightarrow \; \mathrm{T}\, N \\ x \quad & \longmapsto \; \mathrm{T}_m\, F x \quad \text{where} \quad m = \mathrm{T}_M\, x \end{aligned}$$

where recall that $\mathrm{T}_m Fx$ was defined in definition 7.1.1. If $n \in N$ is the image of some critical point of F then n will be called a *critical value (of F)* and we will call it a *regular value (of F)* otherwise.

The following proposition details a result from [20].

Proposition 8.0.4. The limit in Top of the profinite system $(\mathrm{T}\, M_\bullet, \mathrm{T}\, \mu_{ij})$ is

$$(\mathrm{T}\, M, \mathrm{T}\, \mu_\bullet) = \varprojlim (\mathrm{T}\, M_\bullet, \mathrm{T}\, \mu_{ij})$$

where $\mathrm{T}\, M$ has the weak topology induced by the $\mathrm{T}\, \mu_\bullet$. Furthermore, the canonical projections $\mathrm{T}_{M_\bullet} : \mathrm{T}\, M_\bullet \to M_\bullet$ form an inverse system morphism whose limit morphism is the canonical projection defined above

$$\begin{tikzcd}
\mathrm{T}\, M_i \arrow[d, "\mathrm{T}_{M_i}"] & \mathrm{T}\, M_j \arrow[l, "\mathrm{T}\, \mu_{ij}"'] \arrow[d, "\mathrm{T}_{M_j}"] & \mathrm{T}\, M \arrow[l, "\mathrm{T}\, \mu_j"'] \arrow[d, "\mathrm{T}_M"] \\
M_i & M_j \arrow[l, "\mu_{ij}"'] & M \arrow[l, "\mu_j"']
\end{tikzcd}$$

and the assignment $(M, \mu_i) \mapsto (\mathrm{T}\, M, \mathrm{T}\, \mu_\bullet)$ and $F \mapsto \mathrm{T}\, F$ is a covariant functor from the category of promanifolds into itself. If (N_i, ν_{ij}) is another system then the canonical isomorphisms $\mathrm{T}(M_i \times N_i) \cong \mathrm{T}\, M_i \times \mathrm{T}\, N_i$ form an inverse system morphism whose limit is the diffeomorphism

$$\begin{aligned}
\mathrm{T}\, M \times \mathrm{T}\, N &\longrightarrow \mathrm{T}(M \times N) \\
(\mathbf{v}, \mathbf{w}) &\longmapsto \varprojlim \left(v^i + w^i\right)_{i \in \mathbb{N}} \qquad \text{where} \quad v^\bullet \underset{\text{def}}{=} \mathrm{T}\, \mu_\bullet v \quad \text{and} \quad w^\bullet \underset{\text{def}}{=} \mathrm{T}\, \nu_\bullet w
\end{aligned}$$

and where we used the canonical identification from lemma 7.3.1. We will use this diffeomorphism to canonically identify these promanifolds and also abuse notation by writing $\mathrm{T}_M \times \mathrm{T}_N = \mathrm{T}_{M \times N}$.

Proof. In the category of manifolds, the assignment of the tangent map to a map is a covariant functor so that $(\mathrm{T}\, M_\bullet, \mathrm{T}\, \mu_{ij})$ forms an inverse system, which is surjective since all μ_{ij} are smooth submersions. It suffices to show that $(\mathrm{T}\, M, \mathrm{T}\, \mu_\bullet) = \varprojlim(\mathrm{T}\, M_\bullet, \mathrm{T}\, \mu_{ij})$ in the category of sets so let Z be a set and let $h_\bullet : Z \to \mathrm{T}\, M_\bullet$ be a family of compatible maps. By lemma 2.1.27, we may assume without loss of generality that $Z = \{z\}$ is a singleton set. Now the points $m_i^z \underset{\text{def}}{=} \mathrm{T}_{M_i} h_i(z)$ form an element $m^z \underset{\text{def}}{=} \varprojlim m_\bullet^z$ in M so that the $h_i(z) \in \mathrm{T}_{m_i^z} M_i$ form $h(z) \underset{\text{def}}{=} \varprojlim h_\bullet(z) \in \mathrm{T}_{m^z} M$, where $m_\bullet^z \underset{\text{def}}{=} (m_i^z)_{i \in \mathbb{N}}$ and $h_\bullet(z) \underset{\text{def}}{=} (h_i(z))_{i \in \mathbb{N}}$. That the map $h : Z \to \mathrm{T}\, M$ is the unique map satisfying the compatibility condition now follows from lemma 7.3.1.

Observe that the projective limit topology on $\mathrm{T}_m M = \varprojlim(\mathrm{T}\, M_\bullet, \mathrm{T}\, \mu_{ij})$ is the same as the subspace topology that $\mathrm{T}_m M$ inherits from $\mathrm{T}\, M$ since the same is true of each $\mathrm{T}_{m_i} M_i$ with respect to $\mathrm{T}\, M_i$. That the diagram commutes and that the limit morphism of $\varprojlim \mathrm{T}_{M_\bullet}$ is just the map T_M from definition 8.0.1 is now apparent. If $F : M \to N$ is smooth then for any index $a \in \mathbb{N}$ we may pick some index $i \in \mathbb{N}$ such that $\mu_i(m) \in O_i \underset{\text{def}}{=} \mathrm{ODom}_i(\nu_a \circ F)$ so that $\nu_a \circ F = F_i^a \circ \nu_a$ on $\mu_i^{-1}(O_i)$. Applying the tangent map gives $\mathrm{T}(\nu_a \circ F) = (\mathrm{T}\, F_i^a) \circ \mathrm{T}\, \nu_a$ on the open (in M) set $(\mathrm{T}\, \mu_i)^{-1}\big(\mathrm{T}_{M_i}^{-1}(O_i)\big)$ so that the continuity and smoothness of $\mathrm{T}\, F_i^a$ implies the continuity and smoothness of $\mathrm{T}(\nu_a \circ F)$. Since the index a was arbitrary, it follows that $\mathrm{T}\, F$ is continuous and smooth. It is easily verified that T forms a covariant functor.

Recall that $\varprojlim (M_i \times N_i, \mu_{ij} \times \nu_{ij}, \mathbb{N}) \cong \left(\varprojlim \mathrm{Sys}_M\right) \times \left(\varprojlim \mathrm{Sys}_N\right)$ and that for each index i the map $\theta_i : \mathrm{T}\, M_i \times \mathrm{T}\, N_i \to \mathrm{T}(M_i \times N_i)$ defined on each $\mathrm{T}_{m_i} M_i \times \mathrm{T}_{n_i} N_i$ by $(v, w) \mapsto \mathrm{T}_{m_i} \mathrm{In}_1 v + \mathrm{T}_{n_i} \mathrm{In}_2 w$, where In_1 and In_2 are the usual inclusions, is a diffeomorphism. It is straightforward to check that these $\theta_\bullet$ from an inverse system morphism from $(\mathrm{T}\, M_i \times \mathrm{T}\, N_i, \mathrm{T}\, \mu_{ij} \times \nu_{ij}, \mathbb{N})$ to $(\mathrm{T}(M_i \times N_i), \mathrm{T}(\mu_{ij} \times \nu_{ij}), \mathbb{N})$ so that under the $\theta_\bullet$'s identifications, we see that $\varprojlim \theta_\bullet : \mathrm{T}\, M \times \mathrm{T}\, N \cong \mathrm{T}\,(M \times N)$ is the diffeomorphism described above. ■

Corollary 8.0.5. Suppose that for all $i \in \mathbb{N}$, $X_i : M_i \to \mathrm{T}\, X_i$ is a (possibly non-continuous) right inverse of the canonical projection $\mathrm{T}_{M_i} : \mathrm{T}\, M_i \to M_i$ and that these $X_\bullet$ form an inverse system morphism. Then the limit morphism (in Set) $X \underset{\text{def}}{=} \varprojlim X_i : M \to \mathrm{T}\, M$ is a right inverse of $\mathrm{T}_M : \mathrm{T}\, M \to M$. If cofinally many $X_\bullet$'s are continuous or smooth then so is X.

Proof. For all i, we have $\mathrm{T}_{M_i} \circ X_i = \mathrm{Id}_{M_i}$ so that taking the limit of both sides gives us $\mathrm{T}_M \circ X = \mathrm{Id}_M$. ∎

Lemma 8.0.6. Let i be any index, $W_i \subseteq \mathrm{T}\, M_i$, $W \underset{\text{def}}{=} (\mathrm{T}\, \mu_i)^{-1} (W_i) \subseteq \mathrm{T}\, M$, and $\mathrm{T}_{M_i} : \mathrm{T}\, M_i \to M_i$ denote the canonical projection. Then $\mathrm{T}_M (W) = \mu_i^{-1}(\mathrm{T}_{M_i}(W_i))$ so that in particular, $\mathrm{T}_M : \mathrm{T}\, M \to M$ is an open map.

Proof. If $x \in W = (\mathrm{T}\, \mu_i)^{-1} (W_i)$ then let $m = \mathrm{T}_M\, x$, $m_\bullet \underset{\text{def}}{=} \mu_\bullet(m)$, and $x^\bullet = \mathrm{T}_m\, \mu_\bullet x$. Then $x^i \in W_i \cap \mathrm{T}_{m_i} M_i$ and $\mathrm{T}_{M_i}(x^i) = m_i$ so that $\mu_i(\mathrm{T}_M\, x) = m_i = \mathrm{T}_{M_i}(x^i) \in \mathrm{T}_{M_i}(W_i)$ and hence $\mathrm{T}_M(W) \subseteq \mu_i^{-1}(\mathrm{T}_{M_i}(W_i))$. Now let $m \in \mu_i^{-1}(\mathrm{T}_{M_i}(W_i))$ so that there exists some $x^i \in W_i$ such that $m_i = \mathrm{T}_{M_i}(x^i)$ and let $m_\bullet \underset{\text{def}}{=} \mu_\bullet(m)$. Proceeding by induction on $j \geq i$, since each connecting map is a submersion we may pick some $x^{j+1} \in \mathrm{T}_{m_{j+1}} M_{j+1}$ such that $\mathrm{T}_{m_{j+1}}\, \mu_{j,j+1} x^{j+1} = x^j$. By lemma 7.3.1, $(x^j)_{j=i}^{\infty}$ forms an element $x \in \mathrm{T}_m\, M$ such that $\mathrm{T}_m\, \mu_j x = x^j$ for all $j \geq i$, which in particular implies that $\mathrm{T}\, \mu_i(x) = x^i \in W_i$. Since $x \in (\mathrm{T}\, \mu_i)^{-1} (W_i) = W$ and $m = \mathrm{T}_M\, x$, we have $m \in \mathrm{T}_M(W)$ and thus $\mathrm{T}_M(W) = \mu_i^{-1}(\mathrm{T}_{M_i}(W_i))$. Since a basic open subset of $\mathrm{T}\, M$ is of the form $W = (\mathrm{T}\, \mu_i)^{-1} (W_i)$ where $W_i \in \mathrm{Open}\, (\mathrm{T}\, M_i)$ so that $\mathrm{T}_{M_i}(W)$ being open in M_i implies that $\mathrm{T}_M(W) = \mu_i^{-1}(\mathrm{T}_{M_i}(W_i))$ is open in $\mathrm{T}\, M$. ∎

Lemma 8.0.7. Let $F : M \to N$ be a smooth map such that $\mathrm{T}_m\, F = 0$ for all $m \in M$. Then F is constant on each connected component of M and if N is a manifold then F is locally constant.

Proof. Assume first that N is a manifold. For each index i let $O_i = \mathrm{ODom}_i\, F$ so that $F = F_i \circ \mu_i$ on $O^i \underset{\text{def}}{=} \mu_i^{-1} (O_i)$. Then for all $m_i \in O_i$ pick any $m \in O^i$ with $m_i = \mu_i(m)$ so that we have $0 = \mathrm{T}_m\, F = \mathrm{T}_{m_i}\, F_i \circ \mathrm{T}_m\, \mu_i$ where the surjectivity of $\mathrm{T}_m\, \mu_i : \mathrm{T}_m\, M \to \mathrm{T}_{m_i}\, M_i$ now implies that $\mathrm{T}_{m_i}\, F_i = 0$. It follows that $F_i : O_i \to N$ is constant on the connected components of O_i so that if V_i is a connected component of U_i, which is itself an open subset of M_i, it follows that $F = F_i \circ \mu_i$ is constant on the open set $\mu_i^{-1}(V_i)$, from which it follows that each fiber of F is open in M. Since O_i has at most countably many connected components it follows that $\operatorname{Im} F_i$ consists of countably many points. Every $m \in M$ is contained in some O^i and since there are only countably many O^i it follows that $\operatorname{Im} F = \bigcup_{i \in \mathbb{N}} \operatorname{Im} F_i$ consists of countably many points. Now if C is a connected component of M then $F(C)$ is a countable connected metric space, which is only possible if $F(C)$ is singleton.

Now assume that N is an arbitrary promanifold and let C be a connected component of M. For all indices a we have that $\mathrm{T}_m(\nu_a \circ F) = \mathrm{T}_{F(m)}\, \nu_a \circ \mathrm{T}_m F = 0$ for all $m \in M$ where since $\mu_a \circ F : M \to N_a$ is a smooth map into a manifold it follows from the above case that $\nu_a \circ F$ is constant on C. Since $F(C) \subseteq \bigcap_{a \in \mathbb{N}} \nu_a^{-1}(\nu_a(F(C)))$ where the right hand side is a singleton set it follows that $F(C)$ is a single point. ■

Corollary 8.0.8. If $F : M \to N$ is a smooth map into a promanifold N where M is either a connected promanifold or a connected manifold with boundary, then F is constant on $M \iff$ its tangent map vanishes at every point of M.

The next example shows that if N is not a manifold then the map F from lemma 8.0.7 may fail to be locally constant.

Example 8.0.9. Smooth map with vanishing tangent map that is not everywhere locally constant: Using $\mathbb{R}$ for every M_i and $\mathrm{Id}_{\mathbb{R}}$ in place of μ_{ij} in example 2.1.44, let Sys_N and N denote the be the new system that was constructed in that example (where $N_a = \bigsqcup_{h \leq a} \left(\frac{1}{h} \times M_h\right)$ for all $a \in \mathbb{N}$). Let $\mathrm{Sys}_P = \left(P_i \underset{\text{def}}{=} \left\{1, \ldots, \frac{1}{a}\right\}, \pi_{ij}, \mathbb{N}\right)$ be the system constructed in example 2.1.45. For all $i \in \mathbb{N}$, let $F_i : N_i \to P_i$ be defined to be the map that sends each $\frac{1}{h} \times M_h$ (where $h \leq i$) to $\frac{1}{h}$. Then the limit morphism of $F_\bullet : N \to \left\{0, 1, \ldots, \frac{1}{a}\right\}$ is smooth with a tangent map that vanishes everywhere but $F^{-1}(0)$ is not open in N.

Remark 8.0.10. One may generalize this construction by replacing P_i and N_a with $P_a \underset{\text{def}}{=} \{0,1\}^i$ and $N_a \underset{\text{def}}{=} \bigsqcup_{p \in P_a} (p \times M_h)$, respectively, with the obvious bonding maps to get a smooth map between finite-dimensional promanifolds that is *nowhere* locally constant despite its tangent map vanishing everywhere.

Vector Fields

Definition 8.1.1. Let M and N be promanifolds, $U \in \operatorname{Open}(M)$, $F : N \to M$, and $\sigma : U \to N$. Then we will call σ a *rough section (of F) (on U)* if it is a right inverse for F in the category of sets (i.e. $F \circ \sigma = \mathrm{Id}_U$). If $n \in N$ and $\sigma : U \to N$ is a rough local section such that $n \in \operatorname{Im}(\sigma)$ then we will say that σ *is a rough local section through* n and that F *has a rough local section through* n.

In the above definitions, if we write "*global*" in place of "local" then we mean that $U = M$ and if we write "*continuous*" (resp. "*smooth*") in place of "rough" then we mean that σ is continuous (resp. smooth). If we omit mention of whether a local (resp. global) section of F is rough/continuous/smooth then it should be assumed to be a continuous local (resp. global) section of F. In the case that $F = \mathrm{T}_M : \mathrm{T}\, M \to M$ is the canonical projection then we will may replace the word "section" with *vector field*. ■

Notation 8.1.2. Let $k \in \mathbb{Z}^{\geq 0}$, and $F : M \to N$ be a C^k-map. The set of all C^k-section of F over an open set $V \in \operatorname{Open}(N)$ will be denoted by $\Gamma^k(F; V)$ and this sheaf will be denoted by $\Gamma^k(F; \bullet)$ while the space of global sections will be denoted by $\Gamma^k(F) \underset{\text{def}}{=} \Gamma^k(F; M)$. The set of all C^k-sections of F will be denoted by

$$\Gamma^k_{loc}(F) \underset{\text{def}}{=} \bigcup_{V \in \operatorname{Open}(N)} \Gamma^k(F; V)$$

If $n \in N$ then the set of all C^k-local sections σ of F such that $\operatorname{Dom} \sigma$ is a neighborhood of n in N will be denoted by $\Gamma^k(F; n)$ while its stalk at n will be denoted either by $\Gamma^k_n(F)$ or $[\Gamma^k(F; n)]_n$. In the case where $F = \mathrm{T}_M : \mathrm{T}\, M \to M$ is the canonical projection then we will also use the notation $\mathscr{X}(U) \underset{\text{def}}{=} \Gamma_U(\mathrm{T}_M)$, $\mathscr{X}_{loc}(M) \underset{\text{def}}{=} \Gamma_{loc}(\mathrm{T}_M)$, and $\mathscr{X}_m(M) \underset{\text{def}}{=} \Gamma_m(\mathrm{T}_M)$.

Definition 8.1.3. If M is a promanifold, J is an open interval, and $\gamma : J \to M$ is a smooth curve then a *rough vector field along* γ is a map $X : J \to \mathrm{T}\, M$ such that $X(t) \in \mathrm{T}_{\gamma(t)}\, M$ for all $t \in J$. A *vector field along* γ is a vector field along γ that is smooth. The space of all smooth vector fields along γ will be denoted by $\mathscr{X}(\gamma)$.

Let $U \in \operatorname{Open}(M)$, $X : U \to \mathrm{T}\, M$ be a rough vector field, and $X^{\bullet} = \mathrm{T}\, \mu_{\bullet} \circ X$. Recall that for all $m \in U$, the vector X_m acts on all smooth $f : V \to \mathbb{R}$ for which $m \in V$ by $X_m(f) = X_m^i(f_i)$ where i, $U_i \in \operatorname{Open}(M_i)$, and $f_i \in C^{\infty}_{M_i}(U_i)$ are any triple for which $f|_{\mu_i^{-1}(U_i)} = f_i \circ \mu_i|_{\mu_i^{-1}(U_i)}$ and $m \in \mu_i^{-1}(U_i)$. So even through X is not necessarily even continuous, for a given smooth $f : V \to \mathbb{R}$ we can define the following function (whose definition is taken from [20, pp. 21-23]):

Definition 8.1.4. Let $U, V \in \operatorname{Open}(M)$, $X : U \to \mathrm{T}\, M$ be a rough vector field, and $f : V \to \mathbb{R}$ be smooth. Then define Xf by

$$\begin{aligned} Xf : U \cap V &\longrightarrow \mathbb{R} \\ m &\longmapsto X_m f \end{aligned}$$

or equivalently, if $[f]_m$ represents the equivalence class of f in the stalk $C^{\infty}_{M,m}$ then

$$(Xf)_m \underset{\text{def}}{=} X_m([f]_m)$$

The following lemma is a generalization of [20, pp. 19-23], where note that it does not initially require that any $X^{\bullet}$ be locally cylindrical.

Lemma 8.1.5. Let $U \in \operatorname{Open}(M)$, $X : U \to \mathrm{T}\, M$ be a rough vector field, and $X^{\bullet} = \mathrm{T}\, \mu_{\bullet} \circ X$. If $i \leq j$, $V_i \in \operatorname{Open}(M_i)$, and $f_i \in C^{\infty}_{M_i}(V_i)$ then

$$X_m\left(f_i \circ \mu_i\big|_{\mu_i^{-1}(V_i)}\right) = X_m^i(f_i) = X_m^j\left(f_i \circ \mu_{ij}\big|_{\mu_{ij}^{-1}(V_i)}\right)$$

for all $m \in \mu_i^{-1}(V_i) \cap U$ (where note that we do not require X to be in any way related to i, j, V_i, or f_i).

Furthermore, the following are equivalent:

(1) X is smooth.

(2) $\mathrm{T}\,\mu_i \circ X : U \to \mathrm{T}\,M_i$ is smooth for all i.

(3) For every $f \in C_M^\infty(U)$ the map $Xf : U \to \mathbb{R}$ is smooth.

(4) For every $V \in \operatorname{Open}(M)$ and $f \in C_M^\infty(V)$ the map $Xf : U \cap V \to \mathbb{R}$ is smooth.

Proof. That (1) $\iff$ (2) is the universal property of limits while (4) $\implies$ (3) is immediate. The claim that $X_m\left(f_i \circ \mu_i\big|_{\mu_i^{-1}(V_i)}\right) = X_m^i(f_i) = X_m^j\left(f_i \circ \mu_{ij}\big|_{\mu_{ij}^{-1}(V_i)}\right)$ for all $m \in \mu_i^{-1}(V_i) \cap U$ follows from lemma 7.3.1(2) and by going into coordinates.

(3) $\implies$ (4): Let $V \in \operatorname{Open}(M)$, $f \in C_M^\infty(V)$, and $m \in V \cap U$. Pick $i \in \mathbb{N}$, $W_i \in \operatorname{Open}(M_i)$, and $f_i \in C_{M_i}^\infty(W_i)$ such that $f = f_i \circ \mu_i$ on $\mu_i^{-1}(W_i)$ with $m \in \mu_i^{-1}(W_i) \subseteq V \cap U$. Let $\phi : M_i \to \mathbb{R}$ be a smooth bump function that equals 1 on some open neighborhood B_i of $m_i \underset{\text{def}}{=} \mu_i(m)$ with support contained in W_i. The product $\phi_i \cdot f_i$ is then defined and smooth on all of M_i so that $(\phi_i \cdot f_i) \circ \mu_i\big|_U \in C_M^\infty(U)$ so that (3) implies that the function $X((\phi_i \cdot f_i) \circ \mu_i|_U)$ is smooth on U. By lemma 7.0.2, for every $m \in \mu_i^{-1}(B_i)$ we have

$$X_m\left((\phi_i \cdot f_i) \circ \mu_i\big|_U\right) = X_m\left((\phi_i \cdot f_i) \circ \mu_i\big|_{\mu_i^{-1}(B_i)}\right) = X_m\left(f_i \circ \mu_i\big|_{\mu_i^{-1}(B_i)}\right) = X_m f$$

so that the restriction to $\mu_i^{-1}(B_i)$ of the smooth map $X((\phi_i \cdot f_i) \circ \mu_i|_U) : U \to \mathbb{R}$ is just $(Xf)|_{\mu_i^{-1}(B_i)}$. We've thus shown that Xf is smooth in a neighborhood around each point in its domain.

(1) $\implies$ (4): Fix $V \in \operatorname{Open}(M)$, $f \in C_M^\infty(V)$, and $m^0 \in U \cap V$. Pick i and $U_i \in \operatorname{Open}(M_i)$ so that $m^0 \in \mu_i^{-1}(U_i) \in \operatorname{Open}(U \cap V)$ and $f = f_i \circ \mu_i$ on $\mu_i^{-1}(U_i)$. Since $X^i \underset{\text{def}}{=} \mathrm{T}\,\mu_i \circ X : U \to \mathrm{T}\,M_i$, there exist $j > i$, $U_j \in \operatorname{Open}(M_j)$, and $X_j^i \in C_{M_j}^\infty(U_j \to \mathrm{T}\,M_j)$ such that $X^i = X_j^i \circ \mu_j$ on $\mu_j^{-1}(U_j)$ with $m^0 \in \mu_j^{-1}(U_j) \subseteq \mu_i^{-1}(U_i)$, where note that this implies that $U_j \subseteq \mu_{ij}^{-1}(U_i)$. By replacing U_i with $\mu_{ij}(U_j)$ and then restricting both X_j^i and f_i we may assume that $U_j = \mu_{ij}^{-1}(U_i)$. Now, since X_j^i is a smooth vector field along the manifold U_j and $f_i : U_i \to \mathbb{R}$ is a smooth map on

$U_i = \mu_{ij}(U_j)$ we know that the function

$$\begin{aligned} X^i_j f_i : U_j &\longrightarrow \mathbb{R} \\ m_j &\longmapsto (X^i_j)_{m_j}(f_i) \end{aligned}$$

is smooth (which can be seen by going into coordinates). By lemma 7.3.1(2) we have that for all $m \in \mu_j^{-1}(U_j)$, $X\big|_m f = X^i_m f_i$ so that $(X^i_j f_i)(\mu_i(m)) = (Xf)(m)$, which shows that Xf equals the smooth function $(X^i_j f_i) \circ \mu_j$ on the neighborhood $\mu_j^{-1}(U_j)$ of m^0, as desired.

(4) $\implies$ (2): Fix an index i and an element $m \in U$. Note that for any $j \geq i$ we have $X^i = \mathrm{T}\,\mu_i \circ X = \mathrm{T}\,\mu_{ij} \circ \mathrm{T}\,\mu_j \circ X = \mathrm{T}\,\mu_{ij} \circ X^j$ so that since $\mathrm{T}\,\mu_{ij}$ is smooth if suffices to prove that X^j is smooth. Thus by increasing i as necessary we may assume there is some $U_i \in \mathrm{Open}(M_i)$ such that $m \in \mu_i^{-1}(U_i) \subseteq U$. Let $d = \dim M_i$, let (W_i, ψ) be a chart in M_i centered at $m_i \underset{\text{def}}{=} \mu_i(m)$ and let $\psi = (\psi_i^1, \ldots, \psi_i^d) = (y^1, \ldots, y^d)$ where $\psi^l : W_i \to \mathbb{R}$ for all l. By replacing both U_i and W_i with $U_i \cap W_i$ we may assume without loss of generality that $W_i = U_i$. And since smoothness is a local property, it suffices to prove smoothness under the assumption that $U = \mu_i^{-1}(U_i)$. For each $l = 1, \ldots, d$, let $\mu_i^l \underset{\text{def}}{=} y^l \circ \mu_i|_U$ be the l^{th}-coordinates of $\mu_i|_U$ so that $\mu_i|_U = (\mu_i^1, \ldots, \mu_i^d)$. Recall that $\mathrm{T}\,\mu_i^l \circ X = X(\mu_i^l)$ so that under the canonical identification,

$$\begin{aligned} X^i &= \mathrm{T}\,\mu_i \circ X \\ &= \mathrm{T}\big(\mu_i^1, \ldots, \mu_i^d\big) \circ X \\ &= \big(\mathrm{T}\,\mu_i^1, \ldots, \mathrm{T}\,\mu_i^d\big) \circ X \\ &= \big(\mathrm{T}\,\mu_i^1 \circ X, \ldots, \mathrm{T}\,\mu_i^d \circ X\big) \\ &= \big(X(\mu_i^1), \ldots, X(\mu_i^d)\big) \end{aligned}$$

expresses, at each point $m \in U$, the vector X^i_m in terms of its components. However, since each $\mu_i^l : U \to \mathbb{R}$ is a smooth real-valued function our assumption gives us that each $X(\mu_i^l)$ is smooth so that X^i is smooth. ■

The following result is due to [20, pp. 21-23].

Corollary 8.1.6. For any open subset $U \in \operatorname{Open}(M)$, there is a bijective correspondence between $\mathscr{X}(U)$ and $\operatorname{Der}(C^\infty_M(U) \to C^\infty_M(U))$ given by

$$(X_m)f = (Xf)(m)$$

Explicitly, a smooth vector field X on U defines the derivation $f \mapsto Xf$ and a derivation $D : C^\infty_M(U) \to C^\infty_M(U)$ defines on U the smooth vector field that sends $m \in U$ to $X_m = [f \mapsto (D(f))(m)]$.

Proof. Given a smooth vector field X on U, let $D_X = D : C^\infty_M(U) \to C^\infty_M(U)$ be $D(f) = Xf$. Observe that $D(fg) = fDg + gDf$ for any $m \in U$ since

$$(D(fg))(m) = X_m(fg) = f(m)X_m g + g(m)X_m f = (fDg + gDf)(m)$$

so that D is a derivation. If given a derivation $D : C^\infty_M(U) \to C^\infty_M(U)$ then define a rough vector field $X^D = X : C^\infty_M(U) \to \mathbb{R}$ pointwise on U by $X_m f = (D(f))(m)$, where it is easy see that X_m is a derivation at m. For all $f \in C^\infty_M(U)$ we have by definition of Xf that $Xf = D(f)$ so that in particular Xf is smooth which implies that X is a smooth vector field on U. It is clear that these constructions are inverses of each other. ■

Definition 8.1.7 ([20])**.** For any $U \in \operatorname{Open}(M)$ and any vector fields X and Y on U define *the Lie bracket of X and Y* to be the vector field on U corresponding to the derivation

$$\begin{aligned} [X, Y] : C^\infty_M(U) &\longrightarrow C^\infty_M(U) \\ f &\longmapsto V(Wf) - W(Vf) \end{aligned}$$

Remark 8.1.8. The Lie bracket $[\cdot,\cdot] : \mathscr{X}(U) \times \mathscr{X}(U) \to \mathscr{X}(U)$ turns $\mathscr{X}(U)$ into a Lie algebra.

Definition 8.1.9. Suppose $F : M \to N$ is a smooth map, Y is a rough vector field defined on some $V \subseteq N$, and $U \subseteq M$ is such that $F(U) \subseteq V$. Then we will say that Y *is pointwise tangent to F on U* if for all $m \in U$, $V_{F(m)}$ is tangent to F at m (i.e. $V_{F(m)} \in \operatorname{Im} \mathrm{T}_m F$) where if U is omitted then it should be assumed that $U = F^{-1}(V)$. If Y is pointwise tangent to F on U and if in addition for each $m \in U$, $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_{F(m)} N$ is injective then we can define a rough vector field on U called *the pullback of Y onto U (by F)* or *the F-pullback of Y onto U* by

$$X_m \underset{\text{def}}{=} (\mathrm{T}_m F)^{-1}\big(Y_{F(m)}\big), \quad m \in U$$

■

Definition 8.1.10. Let $F : M \to N$ be a smooth map between promanifolds and suppose that X and Y are rough vector fields on M and N, respectively. If for all $m \in M$ we have $\mathrm{T}_m F(X_m) = Y_{F(m)}$ then we will say that *X and Y are F-related* and call Y *the pushforward of X by F*.

Lemma 8.1.11. With F, X, and Y as above, it is immediately seen that the following are equivalent:

(1) X and Y are F-related.

(2) for all $V \in \operatorname{Open}(N)$ and all $g \in C_N^\infty(V)$, $X(g \circ F) = (Yg) \circ F$.

(3) $X(g \circ F) = (Yg) \circ F$ for all $g \in C_N^\infty(N)$.

Example 8.1.12. If $F : M \to N$ is a diffeomorphism, $U \in \operatorname{Open}(M)$, and $X \in \mathscr{X}(U)$ then *the pushforward of X by F* is defined to be the vector field $(F_* X)_n \underset{\text{def}}{=} \mathrm{T}_{F^{-1}(n)} F X_{F^{-1}(n)}$ on $V = F(U)$. Equivalently, it is the unique vector field on V satisfying $X(g \circ F) = ((F_* X) g) \circ F$ for all $g \in C_B^\infty(V)$.

Lemma 8.1.13. Let $m^0 \in M$ and let $m^\bullet = (m^l)_{l=1}^\infty$ and $\widehat{m}^\bullet = (\widehat{m}^l)_{l=1}^\infty$ be sequences in M such that $\{m^l \,|\, m \in \mathbb{N}\} \cap \{\widehat{m}^l \,|\, l \in \mathbb{N}\} = \varnothing$ and both $m^\bullet \to m^0$ and $\widehat{m}^\bullet \to m^0$ are injective. If $\dim_{m^0} M = \infty$ then there exists a smooth vector field $X \in \mathscr{X}(M)$ and an increasing sequence $(l_k)_{k=1}^\infty \subseteq \mathbb{N}$ such that $X(m^0) = \mathbf{0}$ and for each $k \in \mathbb{N}$, $X(m^{l_k}) = \mathbf{0}$ while $X(\widehat{m}^{l_k}) \neq \mathbf{0}$.

Proof. For all $l, i \in \mathbb{N}$ let $m_i^l \underset{\text{def}}{=} \mu_i(m^l)$, $m_i^0 \underset{\text{def}}{=} \mu_i(m^0)$, $\widehat{m_i}^l \underset{\text{def}}{=} \mu_i(\widehat{m}^l)$, and $d_i \underset{\text{def}}{=} \dim_{m_i^0} M_i$. Our assumptions allow us to inductively pick a strictly increasing sequence $(i_l)_{l=1}^\infty \subseteq \mathbb{N}$ such that for each $l \in \mathbb{N}$, all of $m_{i_l}^0, m_{i_l}^1, \ldots, m_{i_l}^l, \widehat{m_{i_l}}^1, \ldots, \widehat{m_{i_l}}^l$ are distinct and $d_{i_l} < d_{i_{l+1}}$. By replacing Sys_M with its restriction to $\{i_l \,|\, l \in \mathbb{N}\}$ we may assume without of generality that $(d_i)_{i=1}^\infty$ is strictly increasing that for each $i \in \mathbb{N}$, all of $m_i^0, m_i^1, \ldots, m_i^i, \widehat{m_i}^1, \ldots, \widehat{m_i}^i$ are distinct.

Let $(U_i)_{i=1}^\infty$ be a Sys_M-nhood basis at m^0 such that for each $i \in \mathbb{N}$, $\mathrm{Cl}_{M_i}(U_i)$ does not contain any of $m_i^1, \ldots, m_i^i, \widehat{m_i}^1, \ldots, \widehat{m_i}^i$. Let $l_1 \underset{\text{def}}{=} 1$, $\iota(1) \underset{\text{def}}{=} 1$, and pick $\widehat{m}_1^1 \in O_1 \in \mathrm{Open}(M_1)$ such that $\overline{O_1}$ is disjoint from $\overline{U_1} \cup \{m_1^1\}$. Having picked l_k pick an integer $l_{k+1} > l_k$ such that $l \geq l_{k+1} \implies m^l, \widehat{m}^l \in \mu_{l_k}^{-1}(U_{l_k})$. For all $k \in \mathbb{N}^{\geq 2}$, pick $\widehat{m}_{l_k}^{l_k} \in O_{l_k} \in \mathrm{Open}(M_{l_k})$ such that $\overline{O_{l_k}}$ is disjoint from $\overline{U_{l_k}} \cup \{m_{l_k}^{l_k}\}$ and also contained in $\mu_{l_{k-1}, l_k}^{-1}(U_{l_{k-1}})$. Observe that sets $\mu_{l_k}^{-1}(\overline{O_{l_k}})$ are pairwise disjoint since for $j < k$, $\overline{O_{l_k}} \subseteq \mu_{l_j, l_k}^{-1}(U_{l_j})$ and $\overline{O_{l_j}} \cap U_{l_j} = \varnothing$. Also, observe that $m^{l_k} \notin \bigcup_{p \in \mathbb{N}} \mu_{i_p}^{-1}(\overline{O_{i_p}})$ for each $k \in \mathbb{N}$.

Let X^1 be any smooth vector field on M_1 such that $\operatorname{supp} X^1 = \overline{O_1}$. For all $k \in \mathbb{N}^{geq2}$ let X^{l_k} be any smooth vector field on M_{l_k} such that $\operatorname{supp} X^{l_k} = \overline{O_{l_k}}$ and X^{l_k} is μ_{l_{k-1}, l_k}-vertical. For all $k \in \mathbb{N}$ let X_k be a smooth vector field on M such that $\mathrm{T}\,\mu_{l_k} \circ X_k = X^{l_k} \circ \mu_{l_k}$. Since the closed sets $\mu_{l_k}^{-1}(\overline{O_{l_k}})$ are pairwise disjoint it follows that for each $m \in M$ the sum $X_m \underset{\text{def}}{=} \sum_{k=1}^{\infty} X_k$ has at most one non-zero term and thus is well-defined. The carrier of X is contained in $\bigcup_{l \in \mathbb{N}} \mu_{l_k}^{-1}(\overline{O_{l_k}})$, which does not contain m^0 nor any m^{l_k}, it follows that $X(m^0) = \mathbf{0}$ and $X(m^{l_k}) = \mathbf{0}$ for all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, we have that $X(\widehat{m}^{l_k}) \neq \mathbf{0}$ since $\widehat{m}^{l_k}$ belongs to $\mu_{l_k}^{-1}(O_{l_k})$ but not to the interior (in M) of $\overline{\mu_{l_k}^{-1}(O_{l_k})}$.

Fix $k \in \mathbb{N}$ and let $p > k$. Since X^{l_p} is μ_{l_k, l_p}-vertical it follows that X_p is μ_{l_k}-vertical, which implies that $\mathrm{T}\,\mu_{l_k} \circ X$ can only be non-zero on a subset of $\mu_{l_1}^{-1}(\overline{O_{l_1}}) \cup \ldots \cup \mu_{l_k}^{-1}(\overline{O_{l_k}})$. In particular, $\mathrm{T}\,\mu_{l_k} \circ X$ is identically $\mathbf{0}$ on $\mu_{i_k}^{-1}(U_{i_k})$ so we've thus shown that each $\mathrm{T}\,\mu_{l_k} \circ X$

is smooth at m^0, which implies that X is smooth at m^0. Observe that for all $m \in M \setminus \{m^0\}$ there exists a neighborhood U of m that intersects at most one of the $\mu_{l_k}^{-1}\left(\overline{O_{l_k}}\right)$'s and from here it is easy to see that $X\big|_U$ is smooth, which implies that X is smooth on $M \setminus \{m^0\}$. Thus X is smooth on all of M. ∎

Integral Curves

The results of this subsection will not be used anywhere else in this paper.

Definition 8.2.1. Let M be a promanifold, $U \in \operatorname{Open}(M)$, $X \in \mathscr{X}_U$, and J be an interval containing 0. Say that a smooth curve $\gamma : (J, 0) \to (M, m)$ is an *integral curve of* X *(starting at* m*)* if $\gamma'(t) = X_{\gamma(t)}$ for all $t \in J$.

Lemma 8.2.2. Let $\gamma : (J, 0) \to (M, m)$ be a smooth curve, $X \in \mathscr{X}(M)$, $\gamma_\bullet \underset{\text{def}}{=} \mu_\bullet \circ \gamma : (J, 0) \to (M_\bullet, \mu_\bullet(m))$, and $X^\bullet \underset{\text{def}}{=} \mathrm{T}\,\mu_\bullet \circ X : M \to \mathrm{T}\,M_\bullet$. Then γ is an integral curve of X if and only if $\gamma_\bullet' = X^\bullet \circ \gamma$ (i.e. $\gamma_i'(t) = X^i\big|_{\gamma(t)}$ for all indices i and all $t \in J$).

Proof. Note that $\gamma_i'(t) = (\mu_i \circ \gamma)'(t) = \left(\mathrm{T}_{\gamma(t)}\,\mu_i\right)(\gamma'(t))$ and $\left(\mathrm{T}_{\gamma(t)}\,\mu_i\right)\left(X_{\gamma(t)}\right) = X^i_{\gamma(t)}$. So it is clear that $\gamma'(t) = X_{\gamma(t)}$ if and only if $\gamma_i'(t) = (X^i \circ \gamma)(t)$ for all $t \in J$ and all indices i. ∎

Proposition 8.2.3. A sufficient condition for uniqueness of an integral curve: Let X be a smooth vector field on M, $m^0 \in M$, and suppose that $\gamma : I \to M$ and $\eta : J \to M$ are integral curves of X starting at m^0. If $X_{m^0} \neq 0$ then there exists a non-degenerate closed interval $D \subseteq I \cap J$ containing 0 such that $\gamma\big|_D = \eta\big|_D$.

Proof. Let $\gamma_\bullet = \mu_\bullet \circ \gamma$, $X^\bullet = \mathrm{T}\,\mu_\bullet \circ X$, and $m^0_\bullet = \mu_\bullet\left(m^0\right)$, and for all $i \in \mathbb{N}$ let $Z_i : M_i \to \mathrm{T}\,M_i$ be the zero section of $\mathrm{T}_{M_i} : \mathrm{T}\,M_i \to M_i$. Since $X_{m^0} \neq \mathbf{0}$ there is some index h such that $X^h_{m^0} \neq \mathbf{0}$, which implies that $\eta_h'(0) = X_{m^0} = \gamma_h'(0) \neq \mathbf{0}$ so that there exists some non-degenerate compact interval $K \subseteq I \cap J$ containing 0 such that neither γ_h' nor η_h' vanish anywhere on K. Replace γ and η with their restrictions to K so that we may assume without loss of generality that $I = J = K$ is compact. Since $\gamma : I \to M$ and $\eta : I \to M$ are smooth immersions from a compact manifold there exists some index $h_0 \geq h$ such that $\gamma_{h_0} : I \to M_{h_0}$ and $\eta_{h_0} : I \to M_{h_0}$ are smooth embeddings. Replace h with h_0 so that we may henceforth assume that both γ_h and η_h are smooth embeddings from the compact interval I into M_h.

Fix an index $i \geq h$. We must show that $\gamma_i = \eta_i$ but we do not yet even know whether or not one of $\operatorname{Im}\gamma_i$ and $\operatorname{Im}\eta_i$ is a subset of the other, so we will now construct a smooth vector field Y^i such that γ_i and η_i are both integral curves of Y^i starting at the same point. Let $D_\bullet = \operatorname{Im}\gamma_\bullet$ and observe that since $\operatorname{Im}\gamma \cup \operatorname{Im}\eta$ is a compact subset of M, there exists some $j \geq i$, $U_j \in \operatorname{Open}(M_j)$, and smooth $X^i_j : U_j \to \mathrm{T}\, M_i$ such that $\operatorname{Im}\gamma \cup \operatorname{Im}\eta \subseteq \mu_j^{-1}(U_j)$ and $X^i_j \circ \mu_j = X^i$ on $\mu_j^{-1}(U_j)$. Since $\mu_{ij}\big|_{D_j} : D_j \to D_i$ is a diffeomorphism and $\widehat{\sigma}^j_i \underset{\text{def}}{=} \left(\mu_{ij}\big|_{D_j}\right)^{-1} : D_i \to D_j$ satisfies $\mu_{ij} \circ \widehat{\sigma}^j_i = \operatorname{Id}_{D_i}$, by lemma C.1.3, there exists an open set $V_j \in \operatorname{Open}(U_j)$ containing D_j and a smooth local section $\sigma^j_i : V_i \to M_j$ of μ_{ij} extending $\widehat{\sigma}^j_i$, where $V_i \underset{\text{def}}{=} \mu_{ij}(V_j)$ and $\operatorname{Im}\sigma^j_i \subseteq V_j$. Let $Y^i \underset{\text{def}}{=} X^i_j \circ \sigma^j_i : V_i \to \mathrm{T}\, M_i$, which is a smooth map. Observe that for all $m_j \in \operatorname{Im}\sigma^j_i$, if $m \in \mu_j^{-1}(m_j)$ then $m_i \underset{\text{def}}{=} \mu_{ij}(m_i) \in V_i$, $m_j = \sigma^j_i(m_i)$, and $\sigma^j_i(\mu_i(m)) = \sigma^j_i(m_i) = m_j = \mu_j(m)$ so that

$$\left(Y^i \circ \mu_i\right)(m) = \left(X^i_j \circ \sigma^j_i \circ \mu_i\right)(m) = \left(X^i_j \circ \mu_j\right)(m) = X^i(m) = (\mathrm{T}_m\, \mu_i \circ X)(m)$$

where since $m_j \in \operatorname{Im}\sigma^j_i$ and $m \in \mu_j^{-1}(m_j)$ where arbitrary, it follows that $Y^i \circ \mu_i = \mathrm{T}_m\, \mu_i \circ X$ on $\mu_j^{-1}\left(\operatorname{Im}\sigma^j_i\right)$. Recalling that $\mathrm{T}_{M_i} \circ \mathrm{T}\, \mu_i = \mu_i \circ \mathrm{T}_M$, it follows that on $\mu_j^{-1}\left(\operatorname{Im}\sigma^j_i\right)$,

$$\mathrm{T}_{M_i} \circ Y^i \circ \mu_i = \mathrm{T}_{M_i} \circ \mathrm{T}_m\, \mu_i \circ X = \mu_i \circ \mathrm{T}_M \circ X = \mu_i \circ \operatorname{Id}_M = \mu_i$$

For any $m_i \in V_i$, if $m \in \mu_j^{-1}\left(\sigma^j_i(m_i)\right)$ then $\mu_i(m) = m_i$ and

$$\left(\mathrm{T}_{M_i} \circ Y^i\right)(m_i) = \left(\mathrm{T}_{M_i} \circ X^i_j \circ \sigma^j_i\right)(m_i) = \left(\mathrm{T}_{M_i} \circ X^i_j\right)(\mu_j(m)) = \mu_{ij}\left(\mu_j(m)\right) = m_i$$

so that $\mathrm{T}_{M_i} \circ Y^i = \operatorname{Id}_{V_i}$, which shows that Y_i is a vector field on V_i. Fix $t \in D$, let $m \underset{\text{def}}{=} \gamma(t)$

and note that

$$\gamma_i'(t) = \mathrm{T}_m\,\mu_i(\gamma'(t)) = \mathrm{T}_m\,\mu_i(X_m) = (\mathrm{T}_m\,\mu_i \circ X)(m)$$

Since $\operatorname{Im}\gamma_j = \operatorname{Im}\big(\widehat{\sigma}_i^j\big) \subseteq D_j$ it follows that $m \in \mu_j^{-1}(D_j)$ so

$$\gamma_i'(t) = (\mathrm{T}_m\,\mu_i \circ X)(m) = \big(Y^i \circ \mu_i\big)(m) = Y^i_{\gamma_i(t)}$$

and thus γ_i is an integral curve of Y^i. Now let $\widehat{m} = \eta(t)$, $\widehat{m}_\bullet = \mu_\bullet(\widehat{m})$, $s_j = \sigma_i^j(\widehat{m}_i)$, and let $s \in \mu_j^{-1}(s_j)$ so that $\mu_i(s) = \mu_{ij}(s_j) = \widehat{m}_i$. Since $s \in \mu_j^{-1}\big(\operatorname{Im}\sigma_i^j\big)$ we have $(Y^i \circ \mu_i)(s) = (\mathrm{T}_m\,\mu_i \circ X)(s)$ and so

$$\eta_i'(t) = X^i_{\eta(t)} = (\mathrm{T}_{\widehat{m}}\,\mu_i \circ X)(s) = \big(Y^i \circ \mu_i\big)(s) = Y^i(\widehat{m}_i) = \big(Y^i \circ \mu_i\big)(\widehat{m}) = \big(Y^i \circ \mu_i\big)(\eta(t)) = Y^i_{\eta_i(t)}$$

Thus both γ_i and η_i are integral curves of Y^i defined on D starting at m_i^0 so that $\gamma_i = \eta_i$. Since $i \geq h$ was arbitrary and $K = \operatorname{Dom}\gamma$ is independent of i it follows that $\gamma = \eta$. ∎

Corollary 8.2.4. Let X be a smooth vector field on M. If for every $m \in M$ there exist integral curves starting at m then for all $m \in M$ such that $X_m \neq 0$ there exists a unique maximal integral curve $\gamma : I \to M$ of X starting at m and furthermore, X never vanishes on $\operatorname{Im}\gamma$.

Lemma 8.2.5. For each index i, let $X_i^i : M_i \to \mathrm{T}\,M_i$ be a smooth vector field on M_i and assume that these vector fields form an inverse system morphism $\mathrm{Sys}_M \to \mathrm{Sys}_{\mathrm{T}\,M}$ (i.e. $\mathrm{T}\,\mu_{ij} \circ X_j^j = X_i^i \circ \mu_{ij}$ for all $i \leq j$) whose limit is $X : M \to \mathrm{T}\,M$. Let $m \in M$, for each $i \in \mathbb{N}$ let $\gamma_i : J_i \to M_i$ be the maximal integral curve of X_i^i starting at $m_i = \mu_i(m)$, and let $J = \bigcap_i J_i$. Then

(1) X is a smooth vector field on M and for all $i \leq j$, $J_j \subseteq J_i$ and $\mu_{ij} \circ \gamma_j = \gamma_i\big|_{J_j}$.

(2) $(J, \gamma_\bullet)$ is an inverse cone into Sys_M whose limit we will denote by $\gamma : J \to M$.

(3) X has an integral curve $\eta : I \to M$ through m if and only if J is a non-degenerate

interval, in which case $I \subseteq J$ and $\eta = \gamma\big|_I$ so that $\gamma\big|_{\mathrm{Int}_{\mathbb{R}}(J)} : \mathrm{Int}_{\mathbb{R}}(J) \to M$ is the unique maximal integral curve starting at m.

- Remark: even if J is non-degenerate, it is not being claimed that J is an open interval.

(4) If $X_m = \mathbf{0}$ then $J = \mathbb{R}$ and $\gamma \equiv m$ is the unique integral curve of X through m.

In particular, X has a complete integral curve through m (i.e. defined on all of $\mathbb{R}$) if and only if each X_i^i has a complete integral curve through $\mu_i(m)$.

Proof. Since γ_j is an integral curve of X_j^j we have

$$(\mu_{ij} \circ \gamma_j)' = \mathrm{T}_{\gamma_j}\, \mu_{ij} \circ \gamma' = \mathrm{T}_{\gamma_j}\, \mu_{ij} \circ X_j^j \circ \gamma_j = X_i^i \circ \mu_{ij} \circ \gamma_j$$

so that $\mu_{ij} \circ \gamma_j : J_j \to M_i$ is an integral curve of X_i^i starting at $\mu_{ij}(\gamma_j(0)) = \mu_{ij}(\mu_j(\gamma(0)) = \mu_i(m)$. By uniqueness of integral curves, we must have $J_j \subseteq J_i$ and $\gamma_i\big|_{J_j} = \mu_{ij} \circ \gamma_j$ from which (2) follows.

If $\eta : I \to M$ is an integral curve of X starting at m then

$$(\mu_i \circ \eta)' = \mathrm{T}_i\, \mu_i \circ \eta' = \mathrm{T}_i\, \mu_i \circ X \circ \eta = X_i^i \circ \mu_i \circ \eta$$

so that $\mu_i \circ \eta : I \to M_i$ is an integral curve of X_i^i starting at $\mu_i(m)$ which shows that $\mu_i \circ \eta = \gamma_i\big|_I$. Since $i \in \mathbb{N}$ was arbitrary it follows that $I \subseteq \bigcap\limits_i J_i = J$ and $\gamma\big|_I = \eta$. In particular, J is a non-degenerate interval. Conversely, if J is a non-degenerate interval then it is clear that the restriction of γ to $\mathrm{Int}_{\mathbb{R}}(J)$ is an integral curve of X through m. Note that $X_m = \mathbf{0}$ if and only if $0 = \mathrm{T}_m\, \mu_i X_m = X_i^i(m)$ for all $i \in \mathbb{N}$, in which case $J_i = \mathbb{R}$ and $\gamma_i \equiv \mu_i(m)$ is the constant curve. Hence $J = \mathbb{R}$ and $\gamma \equiv m$ is the constant curve at m. ■

Example 8.2.6. A non-vanishing vector field with a point without an integral curve: Let $I =]-1, 1[$ and for all $i \in \mathbb{N}$ let $\mathrm{Pr}_{i,i+1} : I^{i+1} \to I^i$ denote the canonical projection onto the first

i coordinates. Let $C_1 = \{-1, 1\}$ and having defined C_{i-1} let

$$C_i = \operatorname{Pr}_{i-1,i}^{-1}(C_{i-1}) \cup \{\left(-\frac{1}{i}, 0, \ldots, 0\right), \left(\frac{1}{i}, 0, \ldots, 0\right)\}$$

For all $i \in \mathbb{N}$, let $M_i = I^i \setminus C_i$ so that M_i is an open subset of I^i (where $M_1 = I$, $M_2 = M_1 \times I$ with 2 points removed, etc.) and define the smooth surjective submersion $\mu_{i,i+1} \underset{\text{def}}{=} \operatorname{Pr}_{i,i+1}\big|_{M_{i+1}} : M_{i+1} \to M_i$. Clearly, $M = I^{\mathbb{N}} \setminus \bigcup_{i \in \mathbb{N}} \operatorname{Pr}_i^{-1}(C_i)$ together with the restrictions of the canonical projections is the limit of Sys_M (and in fact, M is even an open subset of $I^{\mathbb{N}}$ since $\{\operatorname{Pr}_i^{-1}(C_i) \mid i \in \mathbb{N}\}$ is locally finite). Let $X_i^i : M_i \to \operatorname{T} M_i$ denote the constant vector field $\frac{\partial}{\partial x^1}$ on M_i and since these maps clearly form an inverse system morphism we can define the smooth vector field $X \underset{\text{def}}{=} \varprojlim_{i \in \mathbb{N}} X_i^i$.

For every $i \in \mathbb{N}$, the map $\gamma_i : \left]-\frac{1}{i}, \frac{1}{i}\right[\to M_i$ defined by $\gamma_i(t) = (t, 0, \ldots, 0)$ is the integral curve of X_i^i starting at $(0, \ldots, 0) \in M_i$. However, $J \underset{\text{def}}{=} \bigcap_{i \in \mathbb{N}} \left]-\frac{1}{i}, \frac{1}{i}\right[= \{0\}$ so by lemma 8.2.5 there is no integral curve of X starting at $m \underset{\text{def}}{=} (0, 0, \ldots) \in M$. Moreover, the smooth constant map $\gamma \equiv m : J \to M$, which is the limit of the cone $(J, \gamma_i\big|_J)$, does not even satisfy $\gamma'(0) = X(\gamma(0))$ since $\gamma'(0) = \mathbf{0} \neq X(m)$. ∎

Example 8.2.7. For all $i \in \mathbb{N}$, let $X_i^i : \mathbb{R}^i \to \operatorname{T} \mathbb{R}^i$ be the vector field defined by

$$X_i^i(x_1, \ldots, x_i) = \sum_{l=1}^{i} x_l^2 \frac{\partial}{\partial x_l}$$

so that the integral curve γ_i of X_i^i starting at $(1, 2, \ldots, i)$ is $\gamma_i(t) = \left(\frac{1}{1/1-t}, \frac{1}{1/2-t}, \ldots, \frac{1}{1/i-t}\right)$ with domain $\bigcap_{l=1}^{i} \left]-\infty, 1/l\right[= \left]-\infty, 1/i\right[$. These vector fields form an inverse system morphism so let $X : \mathbb{R}^{\mathbb{N}} \to \operatorname{T} \mathbb{R}^N = \mathbb{R}^N \times \mathbb{R}^N$ be their limit. By lemma 8.2.5, the integral curve of X starting at $(1, 2, 3, \ldots)$ is $\gamma(t) = \left(\frac{1}{1/1-t}, \frac{1}{1/2-t}, \ldots, \frac{1}{1/i-t}, \ldots\right)$ with domain $\bigcap_{l=1}^{\infty} \left]-\infty, 1/l\right[= \left]-\infty, 0\right]$. There does not exist any integral curve of X starting at $m \underset{\text{def}}{=} (1, -1, 2, -2, 3, -3, \ldots)$ for lemma 8.2.5 implies that if it did exist then its domain would be the degenerate interval $\bigcap_{l=1}^{\infty} \left]-\infty, 1/l\right[\cap \left]-1/l, \infty\right[= \{0\}$.

Example: Infinite and Higher Order Tangent Bundles

The example described in this section will not be used anywhere else in this paper.

Definition and Notation 8.3.1. For any promanifold M, let $\mathrm{T}^0 M = M$ and then inductively define for each $k \in \mathbb{N}$ *the k^{th}-order tangent bundle* by $\mathrm{T}^{k+1} M \underset{\text{def}}{=} \mathrm{T}(\mathrm{T}^k M)$ where, as usual, the canonical projection from the tangent space $\mathrm{T}(\mathrm{T}^k M)$ onto $\mathrm{T}^k M$ is denoted by

$$\mathrm{T}_{\mathrm{T}^k M} : \mathrm{T}^{k+1} M \to \mathrm{T}^k M$$

which we may also denote by $\mathrm{T}_{\mathrm{T}^k M \leftarrow \mathrm{T}^{k+1} M}$ or by $\mathrm{T}_M^{k \leftarrow k+1}$. Fix $k \in \mathbb{N}$ and for any $l > k$ define the map $\mathrm{T}_M^{k \leftarrow l} = \mathrm{T}_{\mathrm{T}^k M \leftarrow \mathrm{T}^l M} : \mathrm{T}^l M \to \mathrm{T}^k M$ to be the composition

$$\mathrm{T}^k M \xleftarrow{\mathrm{T}_{\mathrm{T}^k M}} \mathrm{T}^{k+1} M \xleftarrow{\mathrm{T}_{\mathrm{T}^{k+1} M}} \cdots\cdots \xleftarrow{\mathrm{T}_{\mathrm{T}^{l-2} M}} \mathrm{T}^{l-1} M \xleftarrow{\mathrm{T}_{\mathrm{T}^{l-1} M}} \mathrm{T}^l M$$

i.e. $\mathrm{T}_M^{k \leftarrow l} \underset{\text{def}}{=} \mathrm{T}_{\mathrm{T}^k M \leftarrow \mathrm{T}^l M} \underset{\text{def}}{=} \mathrm{T}_{\mathrm{T}^k M} \circ \cdots \circ \mathrm{T}_{\mathrm{T}^{l-1} M}$ and let $\mathrm{T}_M^{k \leftarrow k} \underset{\text{def}}{=} \mathrm{T}_{\mathrm{T}^k M \leftarrow \mathrm{T}^k M}$ denote the identity map on $\mathrm{T}^k M$. Call

$$\mathrm{Sys}_{\mathrm{T}^\bullet M} \underset{\text{def}}{=} \left(\mathrm{T}^k M, \mathrm{T}_M^{k \leftarrow l}, \mathbb{Z}^{\geq 0}\right)$$

the canonical inverse system of the k^{th}-order tangent bundles on M and we will denote its canonical limit by $(\mathrm{T}^\infty M, \mathrm{T}_M^{\bullet \leftarrow \infty})$ where we may also denote its canonical projections by

$$\mathrm{T}_{\mathrm{T}^k M \leftarrow \mathrm{T}^\infty M} \underset{\text{def}}{=} \mathrm{T}_M^{k \leftarrow \infty} : \mathrm{T}^\infty M \to \mathrm{T}^k M$$

If M is clear from context then we may write $\mathrm{T}^{k \leftarrow l}$ in place $\mathrm{T}_M^{k \leftarrow l}$ for any $k, l \in \{0, 1, \ldots, \infty\}$ with $k \leq l$.

If $F : M \to N$ is a smooth map between promanifolds then let $\mathrm{T}^0 F \underset{\text{def}}{=} F$ and inductively define for each $k \in \mathbb{N}$ *the k^{th}-order tangent map (induced by F)* by

$$\mathrm{T}^k F \underset{\text{def}}{=} \mathrm{T}\big(\mathrm{T}^{k-1} F\big) : \mathrm{T}^k M \to \mathrm{T}^k N$$

it is clear that $\mathrm{T}^\bullet F \underset{\text{def}}{=} \big(\mathrm{T}^k F\big)_{k \geq 0} : \mathrm{Sys}_{\mathrm{T}^\infty M} \to \mathrm{Sys}_{\mathrm{T}^\infty N}$ forms a morphism of inverse systems (indexed by the identity map on $\mathbb{Z}^{\geq 0}$) so we will denote it's limit morphism by

$$\mathrm{T}^\infty F \underset{\text{def}}{=} \varprojlim \mathrm{T}^\bullet F : \mathrm{T}^\infty M \to \mathrm{T}^\infty M$$

which will make the following diagram commute:

$$\begin{tikzcd}
\mathrm{T}^\infty N \arrow[d, "\mathrm{T}_N^{k\leftarrow\infty}"'] & \mathrm{T}^\infty M \arrow[l, "\mathrm{T}^\infty F"'] \arrow[d, "\mathrm{T}_M^{k\leftarrow\infty}"] \\
\mathrm{T}^l N \arrow[d, "\mathrm{T}_N^{k\leftarrow l}"'] & \mathrm{T}^l M \arrow[l, "\mathrm{T}^l F"'] \arrow[d, "\mathrm{T}_M^{k\leftarrow l}"] \\
\mathrm{T}^k N & \mathrm{T}^k M \arrow[l, "\mathrm{T}^k F"']
\end{tikzcd}$$

If $\gamma : J \to M$ is a smooth curve then recall that we identify $\mathrm{T}\gamma : \mathrm{T} J \to \mathrm{T} M$ with the map $\gamma^{(1)} : J \to \mathrm{T} M$ defined by $\gamma^{(1)}(t) = \mathrm{T}\gamma\left(\frac{\partial}{\partial t}\big|_t\right)$ for all $t \in J$ and that we inductively define $\gamma^{(k+1)} = \left(\gamma^{(k)}\right)^{(1)}$. These maps form a cone $\left(J, \left(\gamma^{(k)}\right)_{k \in \mathbb{N}}\right)$ into $\mathrm{Sys}_{\mathrm{T}^\infty M}$ whose limit map we'll denote by $\gamma^{(\infty)} : J \to \mathrm{T}^\infty M$. ∎

Observation 8.3.2. If $\gamma : J \to M$ is a smooth curve and $t \in J$, then $\gamma^{(\infty)}$ vanishes at t if and only if all of γ's derivatives vanish at t. So if $M = \mathbb{R}^d$ then for some $d \in \mathbb{N}$, then $\gamma^{(\infty)}$ vanishes at t if and only if all of γ's Taylor coefficients vanish at t. This suggests that $\mathrm{T}^\infty M$ may be a setting that is well suited for the study the Taylor series of smooth maps.

Observe that every profinite system $\mathrm{Sys}_M = (M_\bullet, \mu_{ij}, \mathbb{N})$ naturally induces the spaces and morphisms shown below

$$\begin{tikzcd}
\mathrm{T}^{\infty} M_1 \arrow[d, "\mathrm{T}^{k\leftarrow\infty}_{M_1}"'] & \mathrm{T}^{\infty} M_i \arrow[l, "\mathrm{T}^{\infty}\mu_{1i}"'] \arrow[d, "\mathrm{T}^{k\leftarrow\infty}_{M_i}"] & \mathrm{T}^{\infty} M_j \arrow[l, "\mathrm{T}^{\infty}\mu_{ij}"'] \arrow[d, "\mathrm{T}^{k\leftarrow\infty}_{M_j}"] & \mathrm{T}^{\infty} M \arrow[l, "\mathrm{T}^{\infty}\mu_{j}"'] \arrow[d, "\mathrm{T}^{k\leftarrow\infty}_{M}"] \\
\mathrm{T}^{l} M_1 \arrow[d, "\mathrm{T}^{k\leftarrow l}_{M_1}"'] & \mathrm{T}^{l} M_i \arrow[l, "\mathrm{T}^{l}\mu_{1i}"'] \arrow[d, "\mathrm{T}^{k\leftarrow l}_{M_i}"] & \mathrm{T}^{l} M_j \arrow[l, "\mathrm{T}^{l}\mu_{ij}"'] \arrow[d, "\mathrm{T}^{k\leftarrow l}_{M_j}"] & \mathrm{T}^{l} M \arrow[l, "\mathrm{T}^{l}\mu_{j}"'] \arrow[d, "\mathrm{T}^{k\leftarrow l}_{M}"] \\
\mathrm{T}^{k} M_1 \arrow[d, "\mathrm{T}^{0\leftarrow k}_{M_1}"'] & \mathrm{T}^{k} M_i \arrow[l, "\mathrm{T}^{k}\mu_{1i}"'] \arrow[d, "\mathrm{T}^{0\leftarrow k}_{M_i}"] & \mathrm{T}^{k} M_j \arrow[l, "\mathrm{T}^{k}\mu_{ij}"'] \arrow[d, "\mathrm{T}^{0\leftarrow k}_{M_j}"] & \mathrm{T}^{k} M \arrow[l, "\mathrm{T}^{k}\mu_{j}"'] \arrow[d, "\mathrm{T}^{0\leftarrow k}_{M}"] \\
M_1 & M_i \arrow[l, "\mu_{1i}"'] & M_j \arrow[l, "\mu_{ij}"'] & M \arrow[l, "\mu_{j}"']
\end{tikzcd}$$

where it can be readily checked that this diagram commutes.

Notation 8.3.3. So as to more systematically refer to the maps above we will let $M_\infty \underset{\text{def}}{=} M$ and also introduce the following notation (whose pattern should be clear), where each of the maps below denotes the appropriate map from the diagram above:

$$\begin{tikzcd}
\mathrm{T}^{\infty} M_i \arrow[d, "\mathrm{T}^{l\leftarrow\infty}_{ii}"'] & \mathrm{T}^{\infty} M_j \arrow[l, "\mathrm{T}^{\infty\leftarrow\infty}_{ij}"'] \arrow[d, "\mathrm{T}^{l\leftarrow\infty}_{jj}"] & \mathrm{T}^{\infty} M_\infty \arrow[l, "\mathrm{T}^{\infty\leftarrow\infty}_{j\infty}"'] \arrow[d, "\mathrm{T}^{l\leftarrow\infty}_{\infty\infty}"] \\
\mathrm{T}^{l} M_i \arrow[d, "\mathrm{T}^{k\leftarrow l}_{ii}"'] & \mathrm{T}^{l} M_j \arrow[l, "\mathrm{T}^{l\leftarrow l}_{ij}"'] \arrow[d, "\mathrm{T}^{k\leftarrow l}_{jj}"] & \mathrm{T}^{l} M_\infty \arrow[l, "\mathrm{T}^{l\leftarrow l}_{j\infty}"'] \arrow[d, "\mathrm{T}^{k\leftarrow l}_{\infty\infty}"] \\
\mathrm{T}^{k} M_i \arrow[d, "\mathrm{T}^{0\leftarrow k}_{ii}"'] & \mathrm{T}^{k} M_j \arrow[l, "\mathrm{T}^{k\leftarrow k}_{ij}"'] \arrow[d, "\mathrm{T}^{0\leftarrow k}_{jj}"] & \mathrm{T}^{k} M_\infty \arrow[l, "\mathrm{T}^{k\leftarrow k}_{j\infty}"'] \arrow[d, "\mathrm{T}^{0\leftarrow k}_{\infty\infty}"] \\
M_i & M_j \arrow[l, "\mathrm{T}^{0\leftarrow 0}_{ij}"'] & M_\infty \arrow[l, "\mathrm{T}^{0\leftarrow 0}_{j\infty}"']
\end{tikzcd}$$

where, as usual, we let $\mathrm{T}^{k\leftarrow k}_{ii} \underset{\text{def}}{=} \mathrm{Id}_{\mathrm{T}^k M_i}$ denote the identity map. Since the above diagram is commutative we will let

$$\mathrm{T}^{k\leftarrow l}_{ij} : \mathrm{T}^l M_j \to \mathrm{T}^k M_i$$

denote any composition of maps from the above diagram that defines a map from $\mathrm{T}^l M_j$ to $\mathrm{T}^k M_i$, where $i \leq j \leq \infty$ and $0 \leq k \leq l \leq \infty$ vary in all possible ways. Furthermore, in the case where the domain is $\mathrm{T}^\infty M$ and $(i,k) \neq (\infty,\infty)$ then we may also use the notation

$$\mathrm{T}^\infty_{\mathrm{T}^k M_i} \underset{\mathrm{def}}{=} \mathrm{T}^{k \leftarrow \infty}_{i\infty} : \mathrm{T}^\infty M \to \mathrm{T}^k M_i$$

so that, just as with canonical projection of the tangent bundle, the subscript indicates the codomain of the canonical projection ■

Definition and Notation 8.3.4. Let I denote Sys_M's index set (which is a subset of $\mathbb{Z}$, usually either $\mathbb{N}$ or $\mathbb{Z}^{\geq 0}$, whichever is most convenient for the problem at hand) and give

$$\left(I \times \mathbb{Z}^{\geq 0}\right)^* \underset{\mathrm{def}}{=} \{(i,k) \,|\, i \in I \cup \{\infty\}, k \in \{0,1,\ldots,\infty\}, \text{ and } (i,k) \neq (\infty,\infty)\}$$

the partial order defined by $(i,k) \leq (j,l)$ if and only if $i \leq j$ and $k \leq l$. Observe that

$$\mathrm{Sys}_{\mathrm{T}^\infty M, max} \underset{\mathrm{def}}{=} \left(\mathrm{T}^k M_i, \mathrm{T}^{k \leftarrow l}_{ij}, \left(I \times \mathbb{Z}^{\geq 0}\right)^*\right)$$

is an inverse system, which we will call *the largest canonical inverse system of* $\mathrm{T}^\infty M$ *(induced by* Sys_M*)*, and that its limit cone is the promanifold $\mathrm{T}^\infty M$ together with the maps

$$\mathrm{T}^{\bullet \leftarrow \infty}_{\bullet \infty} \underset{\mathrm{def}}{=} \left(\mathrm{T}^{k \leftarrow \infty}_{i\infty}\right)_{(i,k) \in (I \times \mathbb{Z}^{\geq 0})^*}, \text{ which is equal to } \mathrm{T}^\infty_{\mathrm{T}^\bullet M_\bullet} \underset{\mathrm{def}}{=} \left(\mathrm{T}^\infty_{\mathrm{T}^k M_i}\right)_{(i,k) \in (I \times \mathbb{Z}^{\geq 0})^*}$$

Finally, define *the canonical diagonal projective system of* $\mathrm{T}^\infty M$ *(induced by* Sys_M*)* by

$$\mathrm{Sys}_{\mathrm{T}^\infty M} \underset{\mathrm{def}}{=} \left(\mathrm{T}^i M_i, \mathrm{T}^{i \leftarrow j}_{ij}, I^{\geq 0}\right)$$

where the canonical limit cone is $\left(\mathrm{T}^\infty M, \left(\mathrm{T}^\infty_{\mathrm{T}^i M_i}\right)_{i \in I^{\geq 0}}\right)$.

■

Chapter 9

Fiber Bundles

The results of this chapter will not be used anywhere else in this paper.

Definition 9.0.1. Let E, N, and F be promanifolds, $n \in N$, and $p \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$. Let $\pi : E \to N$ be a continuous surjection where if $p \neq 0$ then we also require that it be a C^p-submersion. Say that the triple (E, π, N) (or more informally, the map $\pi : E \to N$) is

- a *trivial C^p-fiber bundle modeled on F* if there exists a C^p-isomorphism $\tau : E \to N \times F$, called a *(global) C^p-trivialization of* π, such that $\operatorname{Pr}_N \circ \tau = \pi$.

- a *trivial C^p vector bundle* (abbreviated *VB* or *V.B.*) *modeled on* F if F is a TVS and there exists a C^p-trivialization $\tau : E \to N \times F$ of (E, π, N), called a *(global) linear C^p-trivialization of* π, such that $\operatorname{Pr}_F \circ \tau\big|_{\pi^{-1}(n)} : \pi^{-1}(n) \to F$ is a TVS-isomorphism for all $n \in N$.

- *locally a trivial C^p fiber (resp. vector) bundle at (* or *around) n* if there exists some $n \in V \in \operatorname{Open}(N)$ and some promanifold (resp. TVS) F' such that $\pi\big|_{\pi^{-1}(V)} : \pi^{-1}(V) \to V$ is a trivial C^p fiber (resp. vector) bundle modeled on F', in which case we'll say that π *is a trivial C^p fiber (resp. vector) bundle on V* and call any global (resp. linear) C^p-trivialization of $\pi\big|_{\pi^{-1}(V)} : \pi^{-1}(V) \to V$ a *local (resp. local linear) C^p-trivialization of* (E, π, N) *(at/around n).*

- a *(locally trivial) C^p fiber (resp. vector) bundle* if (E, π, N) is locally a trivial fiber (resp. vector) bundle at each point of N.

Proposition 9.0.2 (Finite-dimensional promanifolds have locally trivial tangent bundles)**.** Let $i \in \mathbb{N}$, (U_i, φ_i) be a coordinate chart on M_i with $\varphi_i = (\varphi_i^1, \ldots, \varphi_i^d) : U_i \to \mathbb{R}^d$ (where $d \in \mathbb{Z}^{\geq 0}$), $U \in \operatorname{Open}(\mu_i^{-1}(U_i))$ be non-empty, $\mathrm{T}\,U \underset{\text{def}}{=} \mathrm{T}_M^{-1}(U)$, and $\mathrm{T}\,U_i \underset{\text{def}}{=} \mathrm{T}_{M_i}^{-1}(U_i)$. For any $\mathbf{v} \in \mathrm{T}\,U$ let $\mathbf{v}^i \underset{\text{def}}{=} \mathrm{T}\big(\mu_i\big|_U\big)\mathbf{v}$ and consider $\mathbf{v}$ (resp. $\mathbf{v}^i$) as being the derivation $\mathbf{v} : C_M^\infty(U) \to \mathbb{R}$ (resp. $\mathbf{v}^i : C_{M_i}^\infty(U_i) \to \mathbb{R}$). For any $\mathbf{v} \in \mathrm{T}\,U$ define the map $\mathbf{v} : C_M^\infty(U \to \mathbb{R}^d) \to \mathbb{R}^d$, also denoted by $\mathbf{v}$, by sending a smooth map $F = (F^1, \ldots, F^d) : U \to \mathbb{R}^d$ to $\mathbf{v}(F) \underset{\text{def}}{=} (\mathbf{v}(F^1), \ldots, \mathbf{v}(F^d))$ and for any $\mathbf{v}^i \in \mathrm{T}\,U_i$ define $\mathbf{v}^i : C_{M_i}^\infty(U_i \to \mathbb{R}^d) \to \mathbb{R}^d$ analogously. Recall that if $\mathrm{T}(\operatorname{Im}\varphi) = (\operatorname{Im}\varphi_i) \times \mathbb{R}^d$ then the map

$$\begin{aligned} \tau_i : \mathrm{T}\,U_i &\longrightarrow U_i \times \mathbb{R}^d \\ \mathbf{v}^i &\longmapsto \big(\mathrm{T}_{M_i}\mathbf{v}^i, \mathbf{v}^i(\varphi_i)\big) \end{aligned}$$

is a global linear trivialization of $\mathrm{T}_{U_i} \underset{\text{def}}{=} \mathrm{T}_{M_i}\big|_{\mathrm{T}\,U_i} : \mathrm{T}\,U_i \to U_i$.

Analogously, if for all $m \in U$ we have $\dim_m M = d$ then the map

$$\begin{aligned} \tau : \mathrm{T}\,U &\longrightarrow U \times \mathbb{R}^d \\ \mathbf{v} &\longmapsto \big(\mathrm{T}_M(\mathbf{v}), \mathbf{v}\big(\varphi_i \circ \mu_i\big|_U\big)\big) \end{aligned}$$

is a linear trivialization of $\mathrm{T}_U \underset{\text{def}}{=} \mathrm{T}_M\big|_{\mathrm{T}\,U} : \mathrm{T}\,U \to U$ and furthermore,

$$\tau_i \circ \mathrm{T}\big(\mu_i\big|_U\big) = \big(\mu_i\big|_U \times \mathrm{Id}_{\mathbb{R}^d}\big) \circ \tau$$

Proof. That the first coordinates of $\tau_i \circ \mathrm{T}\big(\mu_i\big|_U\big)$ and of $\big(\mu_i\big|_U \times \mathrm{Id}_{\mathbb{R}^d}\big) \circ \tau$ are equal is immediate so to show their equality it remains to show that the second coordinates are equal. Let $\mathbf{v} \in \mathrm{T}\,U$ and recall that for any smooth $f_i : U_i \to \mathbb{R}$ we have $\mathbf{v}^i(f_i) \underset{\text{def}}{=} \big(\mathrm{T}\big(\mu_i\big|_U\big)\mathbf{v}\big)(f_i) = \mathbf{v}\big(f_i \circ \mu_i\big|_U\big)$, which immediately implies that $\mathbf{v}^i(\varphi_i) = \mathbf{v}\big(\varphi_i \circ \mu_i\big|_U\big)$, as desired. For all $m \in U$, the map $\mathrm{T}_m\big(\mu_i\big|_U\big) : \mathrm{T}_m M \to \mathrm{T}_{\mu_i(m)} M_i$ is a vector space isomorphism since $\dim_m M = d = \dim_{\mu_i(m)} M_i$

so the equality that was just proved implies that $\tau\big|_{\mathrm{T}_m M} : \mathrm{T}_m M \to \mathbb{R}^d$ is also a vector space isomorphism, which in turn allows one to readily deduce the bijectivity of τ. The above equality also shows that the second coordinate of τ is smooth and since its first coordinate is also smooth, it follows that τ is smooth.

It remains to show that $\tau^{-1} : U \times \mathbb{R}^d \to \mathrm{T}\,U$ is smooth so fix $(m, \mathbf{r}) \in U \times \mathbb{R}^d$, let $m_\bullet \underset{\text{def}}{=} \mu_\bullet(m)$, and let $j \geq i$. We must show that $\mathrm{T}\,\mu_j \circ \tau^{-1} : U \times \mathbb{R}^d \to \mathrm{T}\,M_j$ is smooth at $(m, \mathbf{r})$. Since μ_{ij} has rank d at m_j there exists some $m_j \in O_j \in \operatorname{Open}\left(\mu_{ij}^{-1}(U_i)\right)$ such that $\theta \underset{\text{def}}{=} \mu_{ij}\big|_{U_j} : U_j \to \mu_{ij}(U_j)$ is a diffeomorphism. Let $O = \mu_j^{-1}(O_j)$ and observe that since θ is a diffeomorphism, to show that τ^{-1} is smooth at $(m, \mathbf{r})$ it suffices to show that $\theta \circ \mathrm{T}\,\mu_j \circ \tau^{-1}\big|_{O \times \mathbb{R}^d} : O \times \mathbb{R}^d \to \mathrm{T}\,M_i$ is smooth. But on $O \times \mathbb{R}^d$ we have

$$\theta \circ \mathrm{T}\,\mu_j \circ \tau^{-1} = \mathrm{T}\left(\mu_i\big|_U\right) \circ \tau^{-1} = \tau_i^{-1} \circ \left(\mu_i\big|_U \times \mathrm{Id}_{\mathbb{R}^d}\right)$$

where the RHS is a composition of smooth maps, which completes the proof. ■

Corollary 9.0.3. A promanifold M has finite and locally constant dimension at a point $m \in M \iff$ the tangent bundle is locally a trivial finite-dimensional vector bundle around m.

Corollary 9.0.4. A promanifold has everywhere finite and locally constant dimension $\iff$ the tangent bundle a finite-dimensional vector bundle (where different fibers may have different dimensions).

Proposition 9.0.5. Let $F : U \to V$ be a smooth pointwise isomersion, where $U \in \operatorname{Open}(M)$ and $V \in \operatorname{Open}(N)$, and let $\mathrm{T}\,U = \mathrm{T}_M^{-1}(U)$ and $\mathrm{T}\,V = \mathrm{T}_N^{-1}(V)$. If $\mathrm{T}_V \underset{\text{def}}{=} \mathrm{T}_N\big|_{\mathrm{T}\,V} : \mathrm{T}\,V \to V$ has a smooth linear trivialization $\tau = (\mathrm{T}_V, \tau_2) : \mathrm{T}\,V \to V \times \mathbb{R}^d$ then $\xi \underset{\text{def}}{=} (\mathrm{T}_U, \tau_2 \circ \mathrm{T}\,F) : \mathrm{T}\,U \to U \times \mathbb{R}^d$ is a smooth linear trivialization of $\mathrm{T}_U \underset{\text{def}}{=} \mathrm{T}_M\big|_{\mathrm{T}\,U} : \mathrm{T}\,U \to U$.

Remark 9.0.6. Observe that F was not required to be a local smooth embedding or even locally injective.

Proof. It is clear that $\xi\big|_{\mathrm{T}_m M} : \mathrm{T}_M^{-1}(m) \to \{F(m)\} \times \mathrm{T}_{F(m)} N$ is a TVS-isomorphism for all $m \in U$ and that ξ is a bijective smooth pointwise isomersion. Let $v \in \mathrm{T}\, U$ and $v_\bullet \subseteq \mathrm{T}\, U$ be a sequence such that $\xi(v_\bullet) \to \xi(v)$ in $U \times \mathbb{R}^d$. We will show that $v_\bullet \to v$ in $\mathrm{T}\, U$ thereby proving that ξ is a homeomorphism so that theorem 11.6.1 will then imply that τ is a diffeomorphism. Let $m^\bullet = \mathrm{T}_M(v_\bullet)$, $m = \mathrm{T}_M(v)$, $n^\bullet = F(m^\bullet)$, and $n = F(m)$ and observe that $m^\bullet \to m$ in U. Pick $i \in \mathbb{N}$ and a smooth chart (U_i, φ_i) on M_i centered at $\mu_i(m)$ such that $\mu_i^{-1}(U_i) \subseteq U$. Use proposition 11.5.1 to find indices $a \in \mathbb{N}$ and $j \geq i$ and smooth charts (U_j, φ_j) and (V_a, ψ_a) on M_j and N_a centered at $\mu_j(m)$ and $\nu_a(n)$, respectively, such that $\mu_{ij}\left(\overline{U_j}\right) \subseteq U_i$, $F_j^a(U_j) \subseteq V_a$, $\overline{U_j} \subseteq \mathrm{ODom}_j\, F^a$ and the coordinate representations of μ_{ij} and F_j^a, denoted by $\widehat{\mu_{ij}}$ and $\widehat{F_j^a}$, satisfy

$$\begin{array}{ccc} (p_1, \ldots, p_d, \ldots, p_r \ldots, p_{d_j}) & \xrightarrow{\widehat{F_j^a}} & (p_1, \ldots, p_d, \ldots, p_r) \\ \Big\downarrow{\scriptstyle \widehat{\mu_{ij}}} & & \\ (p_1, \ldots, p_d) & & \end{array}$$

where $d = \dim_{\mu_i(m)} M_i$, $d_j = \dim_{\mu_j(m)} M_j$, and $r = \dim_{\nu_a(n)} N_a$. Consequently, the tangent maps of these coordinate representations satisfy

$$\begin{array}{ccc} (p_1, \ldots, p_d, \ldots, p_r \ldots, p_{d_j}) \times \left(x_1, \ldots, x_d, \ldots, x_r, \ldots, x_{d_j}\right) & \xrightarrow{\mathrm{T}\,\widehat{F_j^a}} & (p_1, \ldots, p_d, \ldots, p_r) \times (x_1, \ldots, x_d, \ldots, x_r) \\ \Big\downarrow{\scriptstyle \mathrm{T}\,\widehat{\mu_{ij}}} & & \\ (p_1, \ldots, p_d) \times (x_1, \ldots, x_d) & & \end{array}$$

for all $(p_1, \ldots, p_d, \ldots, p_r \ldots, p_{d_j}) \times \left(x_1, \ldots, x_d, \ldots, x_r, \ldots, x_{d_j}\right) \in \operatorname{Im} \varphi_j \times \mathbb{R}^{d_j} = \mathrm{T}\left(\operatorname{Im} \varphi_j\right)$.

Let $L \in \mathbb{N}$ be such that $l \geq L$ implies $m^l \in \mu_j^{-1}(U_j)$ and observe that this implies that $v_l \in (\mathrm{T}\,\mu_j)^{-1}(\mathrm{T}\, U_j)$ for all $l \geq L$. Since $\mathrm{T}\, F^a(v_\bullet) \to \mathrm{T}\, F^a v$ and v and eventually all $v^\bullet$ belong to $(\mathrm{T}\,\mu_j)^{-1}(\mathrm{T}\, U_j)$, the commutativity of the above diagram implies that $\mathrm{T}\,\mu_i(v_\bullet) \to \mathrm{T}\,\mu_i(v)$. Since i was arbitrary, it follows that $v_\bullet \to v$ in $\mathrm{T}\, U$, as desired. ∎

Corollary 9.0.7. Let $F : M \to N$ be a smooth pointwise isomersion. If $\mathrm{T}_N : \mathrm{T}\, N \to N$ is a smooth vector bundle then so is $\mathrm{T}_M : \mathrm{T}\, M \to M$.

Corollary 9.0.8. Let $m \in M$ and suppose that $\mathrm{T}_M : \mathrm{T}\, M \to M$ is *not* locally a smooth vector bundle around m. Then *every* smooth map $F : U \to N$ from an open neighborhood U of m into a promanifold N for which $\mathrm{T}_N : \mathrm{T}\, N \to N$ is a smooth vector bundle necessary fails to be a smooth pointwise isomersion.

Corollary 9.0.9. Let M and N be promanifolds of equal, constant, and finite dimension (i.e. $\dim_m M = \dim_n N < \infty$ for all $(m, n) \in M \times N$) and suppose that $\mathrm{T}_N : \mathrm{T}\, N \to N$ is a smooth vector bundle but that $\mathrm{T}_M : \mathrm{T}\, M \to M$ fails to locally a smooth vector bundle around a point $m \in M$. Then *every* smooth map $F : U \to N$ from an open neighborhood U of m has a critical point.

Chapter 10

Generalized Cones and Generalized Inverse System Morphisms

All of the material in this chapter is original and none of it will be needed anywhere else in this paper. In this chapter, we will take the view of inverse system morphisms as mathematical tools that allow one to define or study maps defined on all of $M = \varprojlim \operatorname{Sys}_M$ via maps defined on the system's objects (i.e. $M_\bullet$). These tools will allow us to concisely state and prove theorems 10.2.12 and 10.3.1. We will begin in one of the most general settings possible since this generality is needed for theorem 10.2.12 and also since, in the author's opinion, removing all unnecessary assumptions actually ends up resulting in the simplest notation and definitions necessary to write this theorem's statement.

Assumption 10.0.1. Throughout this section we will assume that $(I, \leq)$ and $(A, \leq)$ are partial orders, all $M_\bullet$ and $N_\bullet$ are pairwise disjoint, Q is an arbitrary non-empty set, and both $F_\bullet = (F_q)_{q \in Q}$ and $h_\bullet = (h_q)_{q \in Q}$ are arbitrary collections of maps.

Remarks 10.0.2.

- The picture that one should have throughout these sections is of having the canonical collection $F_\bullet^\bullet \underset{\text{def}}{=} (F_i^a)_{(i,a) \in \mathbb{N} \times \mathbb{N}}$ of smooth maps induced by a smooth map $F : R \to N$, where $R \subseteq M$ and M and N are promanifolds, and then, while using *only* the maps

$F_\bullet^\bullet$, attempting to reconstruct in a pointwise manner not only $F : R \to N$, but also any unique smooth extension that F may have.

- One should keep in mind that it's possible, for instance, that $\widehat{F} : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ smoothly extends a smooth map $F :]0,1[^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ but that $\operatorname{Dom}_i \widehat{F}^a$ neither contains, nor is contained in, $\operatorname{Dom}_i F^a$. This and other technicalities are part of the reason for the seemingly complicated constructions below.

- The main idea behind these constructions is that for any given point, we determine whether or not the given collection of maps can be used to associate this point with a unique element of N.

Generalized Cones

Before generalizing inverse system morphisms and their limits we will generalize the notion of cones and their limits, which will, for instance, be used with generalized cones. We will start our investigation by asking, for an arbitrary collection of maps $h_\bullet = (h_q)_{q \in Q}$, how can we determine those $z \in \cup \operatorname{Dom} h_\bullet \underset{\text{def}}{=} \underset{q}{\cup} \operatorname{Dom} h_q$ for which $h_\bullet$ can be used to define an element of N in a manner that does not require choice and that is similar in nature to the construction of a limit map of a cone into Sys_N. The most immediate observation is that for any given $z \in \cup \operatorname{Dom} h_\bullet$, we must select those maps whose domains and ranges could even potentially be used to define an element of N in terms Sys_N and z. These maps the index set will be denoted by $Q(z, h_\bullet)$ and we also introduce the following notation.

Notation 10.1.1. For any $z \in \cup \operatorname{Dom} h_\bullet$ let $(Q \times A)(z, h_\bullet)$ denote the set of all $(q, a) \in Q \times A$ such that $z \in \operatorname{Dom} h_q$ and $h_q(z) \in N_a$. If $\operatorname{Pr}_Q : Q \times A \to Q$ denotes the canonical projection then let

$$Q(z, h_\bullet) \underset{\text{def}}{=} \operatorname{Pr}_Q((Q \times A)(z, h_\bullet))$$

which is just the set of all $q \in Q$ with $z \in \operatorname{Dom} h_q$ and for which there exists some $a \in A$ such that $h_q(z) \in N_a$ where note that since all $N_\bullet$ are pairwise disjoint this $a \in A$ is unique and so will be denoted by $\alpha_{(z,h_\bullet)}(q)$. Equivalently, if $q \in Q(z, h_\bullet)$ then $\alpha_{(z,h_\bullet)}(q)$ denotes the unique index in A such that $\big(q, \alpha_{(z,h_\bullet)}(q)\big) \in (Q \times A)(z, h_\bullet)$. We thus have the following induced map:

$$\alpha_{(z,h_\bullet)} : Q(z, h_\bullet) \to A$$

Finally, for any $a \in A$ let

$$Q(z, h_\bullet, \geq a) \underset{\text{def}}{=} \left\{q \in Q(z, h_\bullet) \,\middle|\, \alpha_{(z,h_\bullet)}(q) \geq a\right\}$$

As usual, we may omit writing $h_\bullet$ or z if they are clear from context. ■

Definition 10.1.2. Fix $z \in \cup \operatorname{Dom} h_\bullet$ and write α for $\alpha_{(z,h_\bullet)}$. For any $q, \widehat{q} \in Q$ we will say that *h_q and $h_{\widehat{q}}$ are consistent at z* and if they satisfy the following *consistency condition at z*: for every $a \in A$, if $q, \widehat{q} \in Q(z, h_\bullet, \geq a)$ then

$$\nu_{a,\alpha(q)}(h_q(z)) = \nu_{a,\alpha(\widehat{q})}(h_{\widehat{q}}(z))$$

We will say that *$h_\bullet$ is strongly consistent at z* if h_q and $h_{\widehat{q}}$ are consistent at z for all $q, \widehat{q} \in Q$. Observe that for any $a \in A$, if $h_\bullet$ is strongly consistent at z and if there exists any q in $Q(z, h_\bullet, \geq a)$ then the element $\nu_{a,\alpha(q)}(h_q(z))$, which we will henceforth denote suggestively by $(h_\bullet(z))_a$, is independent of the choice of $q \in Q(z, h_\bullet, \geq a)$.

Say that $h_\bullet$ *satisfies the strong definability condition at z* or that *the limit of $h_\bullet$ is strongly definable at z* if

(1) $\operatorname{Im} \alpha_{(z,h_\bullet)}$ is cofinal in A, and

(2) $h_\bullet$ is strongly consistent at z.

It is easy to see that when the above two conditions are satisfied then as a ranges over $\operatorname{Im} \alpha_{(z,h_\bullet)}$ the elements $(h_\bullet(z))_a$ define an element of N, which we will henceforth denote by

$\left(\operatorname{Str}\varprojlim h_\bullet\right)(z)$ or by $\left(\varprojlim h_\bullet\right)(z)$. ■

Definition and Notation 10.1.3. Let $\operatorname{StrDom}(h_\bullet)$ denote the set of all $z \in \cup \operatorname{Dom} h_\bullet$ such that $h_\bullet$ satisfies the strong definability condition at z. By *the strong limit of* $h_\bullet$ we will mean the induced map

$$\operatorname{Str}\varprojlim h_\bullet : \operatorname{StrDom}(h_\bullet) \to N$$

and we will say that this map *arises as a strong limit (from* Sys_M *to* Sys_N*).*

Remark 10.1.4. Definition 10.1.3 clearly generalizes the construction of the limit map that is found in the definition of the limit of a cone. In particular, if $(Z, h_\bullet) : Z \to \operatorname{Sys}_N$ is a cone then the two limits maps obtained from definition 10.1.3 and definition 2.1.22 agree.

Definition 10.1.5. Let Z be a set and say that a Q-indexed collection of maps $h_\bullet = (h_q)_{q \in Q}$ is a *collection of maps from subsets of* Z *into* $N_\bullet = (N_a)_{a \in A}$ if for each $q \in Q$ there exists $\alpha(q) \in A$ such that

(1) $\operatorname{Dom} h_q \subseteq Z$, and

(2) $\operatorname{Codom} F_q \subseteq N_{\alpha(q)}$

We will indicate this by writing $h_\bullet : \subseteq Z \to N_\bullet$ where if we wish to specify the map $\alpha : Q \to A$ then we may write $(h_\bullet, \alpha) : \subseteq Z \to N_\bullet$ or $h_\bullet : \subseteq Z \to N_{\alpha(\bullet)}$. ■

Constructing Maps Between Subsets of Limits

Strongly Defined Limit

We will now generalize the definitions and construction from the above subsection to inverse system morphisms. We again start our investigation by asking, for this arbitrary collection of maps $F_\bullet$, how can we determine those $m \in M$ for which $F_\bullet$ can be used to define an element of N in a manner that does not require choice and that is similar in nature to the construction of a limit map from an inverse system morphism. The most immediate observation is that for any given $m \in M$, we must select those maps whose domains and ranges could even potentially be used to define an element of N in terms of Sys_M, Sys_N, and m so we introduce the following notation.

Notation 10.2.1. For any $m \in M$ let $(Q \times I \times A)(m, F_\bullet)$ denote the set of all $(q, i, a) \in Q \times I \times A$ such that $\mu_i(m) \in \operatorname{Dom} F_q$ and $F_q(\mu_i(m)) \in N_a$ and let

$$Q(m, F_\bullet) \underset{\text{def}}{=} \Pr{}_Q((Q \times I \times A)(m, F_\bullet))$$

where $\Pr_Q : Q \times I \times A \to Q$ is the canonical projection. Since all $M_\bullet$ and $N_\bullet$ are pairwise disjoint, for each $q \in Q(m, F_\bullet)$ there are unique $\iota_{(m,F_\bullet)}(q) \in I$ and $\alpha_{(m,F_\bullet)}(q) \in A$ such that $\left(q,\, \iota_{(m,F_\bullet)}(q),\, \alpha_{(m,F_\bullet)}(q)\right) \in (Q \times I \times A)(m, F_\bullet)$, so we have the following induced maps

$$\iota_{(m,F_\bullet)} : Q(m, F_\bullet) \to I \quad \text{ and } \quad \alpha_{(m,F_\bullet)} : Q(m, F_\bullet) \to A$$

Finally, for any $a \in A$ let

$$Q(m, F_\bullet, \geq a) \underset{\text{def}}{=} \left\{ q \in Q(m, F_\bullet) \,\middle|\, \alpha_{(m,F_\bullet)}(q) \geq a \right\}$$

As usual, we may omit writing $F_\bullet$ or m if they are clear from context. ■

The following definition is motivated by lemma 6.1.9.

Definition 10.2.2. Fix $m \in M$, write α for $\alpha_{(m,F_\bullet)}$, and write ι for $\iota_{(m,F_\bullet)}$. For any $q, \widehat{q} \in Q$ we will say that F_q *and* $F_{\widehat{q}}$ *are consistent at* m if they satisfy the following *consistency condition at* m: for every $a \in A$, if $q, \widehat{q} \in Q(m, F_\bullet, \geq a)$ then

$$\nu_{a,\alpha(q)}\left(F_q\left(\mu_{\iota(q)}(m)\right)\right) = \nu_{a,\alpha(\widehat{q})}\left(F_{\widehat{q}}\left(\mu_{\iota(\widehat{q})}(m)\right)\right)$$

We will say that $F_\bullet$ *is strongly consistent at* m if F_q and $F_{\widehat{q}}$ are consistent at m for all $q, \widehat{q} \in Q$. Observe that for any $a \in A$, if $F_\bullet$ is strongly consistent at m and if there exists any q in $Q(m, F_\bullet, \geq a)$ then the element $\nu_{a,\alpha(q)}\big(F_q\big(\mu_{\iota(q)}(m)\big)\big)$, which we will henceforth denote suggestively by $(F_\bullet(m))_a$, is independent of the choice of $q \in Q(m, F_\bullet, \geq a)$.

Say that $F_\bullet$ *satisfies the strong definability condition at* m or that *the limit of* $F_\bullet$ *is strongly definable at* m if

(1) $\operatorname{Im} \alpha_{(m,F_\bullet)}$ is cofinal in A, and

(2) $F_\bullet$ is strongly consistent at m.

It is easy to see that when the above two conditions are satisfied then as a ranges over $\operatorname{Im} \alpha_{(m,F_\bullet)}$ the elements $(F_\bullet(m))_a$ define an element of N, which we will henceforth denote by $\left(\varprojlim F_\bullet\right)(m)$. ∎

Definition and Notation 10.2.3. Let $\operatorname{StrDom}_M(F_\bullet)$ denote the set of all $m \in M$ such that $F_\bullet$ satisfies the strong definability condition at m. Of course, the map that sends $m \in \operatorname{StrDom}_M(F_\bullet)$ to $\left(\varprojlim F_\bullet\right)(m)$ will be denoted by $\varprojlim F_\bullet$, called *the strong limit of* $F_\bullet$, and we will say that this map *arises as a strong limit (from* Sys_M *to* Sys_N*)*.

Remark 10.2.4. Definition 10.2.3 clearly generalizes the construction of the limit map that is found in the definition of the limit of an inverse system morphism. In particular, if $(F_\bullet, \iota) : \operatorname{Sys}_M \to \operatorname{Sys}_N$ is an inverse system morphism then the two limits maps obtained from definition 10.2.3 and definition 3.1.2 agree.

The following proposition shows that these definitions suffice to construct any map from any subset of M into N.

Proposition 10.2.5. Let $D \subseteq M$ and let $F : D \to N$ be any map. Then there exists a set Q and a Q-indexed collection of maps $F_\bullet$ such that $\operatorname{StrDom}_M(F_\bullet) = D$ and $F = \varprojlim F_\bullet$.

Proof. Let $Q \underset{\text{def}}{=} D \times I \times A$ and for all $(d, i, a) \in Q$ define

$$\begin{aligned} F_{(d,i,a)} : \mu_i(d) &\longrightarrow N_a \\ \mu_i(d) &\longmapsto \nu_a(F(d)) \end{aligned}$$

Observe that for all $d \in D$, $(Q \times I \times A)(d, F_\bullet) \underset{\text{def}}{=} \{((d, i, a), i, a) \,|\, (d, i, a) \in Q\}$ so that $Q(d, F_\bullet) = \{d\} \times I \times A$ and that these sets are empty for $d \notin D$ so that $\operatorname{StrDom}_D F_\bullet \subseteq D$. For all $d \in D$, write α_d for $\alpha_{(d,F_\bullet)}$ and ι_d for $\iota_{(d,F_\bullet)}$ and observe that for all $q = (d, i, a) \in Q$, $\iota_d(q) = i$ and $\alpha_d(q) = a$ so that all elements $(q = (d, i, a), i, a)$ of $(Q \times I \times A)(d, F_\bullet)$ can be rewritten as $(q, \iota_d(q), \alpha_d(q))$.

Let $d \in D$, $a \in A$, and $q, \widehat{q} \in Q(d, F_\bullet, \geq a)$ so $\alpha(q) \geq a$, $\alpha(\widehat{q}) \geq a$, and $q, \widehat{q} \in Q(m, F_\bullet)$. Observe that

$$\left(\nu_{a,\alpha_d(q)} \circ F_q\right)\left(\mu_{\iota_d(q)}(d)\right) = \nu_{a,\alpha_d(q)}\left(\nu_{\alpha_d(q)}(F(m))\right) = \nu_a(F(m))$$

and similarly $\left(\nu_{a,\alpha_d(\widehat{q})} \circ F_{\widehat{q}}\right)\left(\mu_{\alpha_d(\widehat{q})}(d)\right) = \nu_a(F(m))$. We've thus shown that F_q and $F_{\widehat{q}}$ are consistent at d and since $\operatorname{Im} \alpha_d = A$ it follows that $d \in \operatorname{StrDom}_M F_\bullet$. It remains to show that $\left(\varprojlim F_\bullet\right)(d) = F(d)$ so let $a \in A$ be arbitrary. Pick and $i \in I$ and define $q \underset{\text{def}}{=} (d, i, a)$ so that by definition, $\nu_a\left(\left(\varprojlim F_\bullet\right)(d)\right) = \left(\left(\varprojlim F_\bullet\right)(d)\right)_a = \nu_a(F(m))$, as desired. ∎

Remark 10.2.6. The construction in the proof of proposition 10.2.5 would be impractical in almost any situation and its only purpose was to show that the definitions we have thus far would suffice to construct, in a manner similar to limits of inverse system morphisms, any map that we may encounter. However, we will later be primarily interested in locally cylindrical maps on promanifolds and under these circumstances we will be able to find much more practical collections of maps.

Lemma 10.2.7. Let $m \in M$, suppose that $(I, \leq)$ and $(A, \leq)$ are directed, and write α for $\alpha_{(m,F_\bullet)}$ and ι for $\iota_{(m,F_\bullet)}$. Then $F_\bullet$ satisfies the strong definability condition at $m \iff$

(1) $\operatorname{Im}\alpha$ is cofinal in A, and

(2) for all $a \in A$, $q, \widehat{q} \in Q(m, F_\bullet, \geq a)$, $\alpha(q) \leq \alpha(\widehat{q})$ implies

$$\text{Condition } \star(q): \quad F_q\big(\mu_{\iota(q)}(m)\big) = \nu_{\alpha(q),\alpha(\widehat{q})}\big(F_{\widehat{q}}\big(\mu_{\iota(\widehat{q})}(m)\big)\big)$$

Proof. The ($\Longrightarrow$) implication is obvious so to prove the less obvious implication fix $a \in A$ and let $q, \widehat{q} \in Q(m, F_\bullet, \geq a)$ so that $\alpha(q) \geq a$, $\alpha(\widehat{q}) \geq a$, and $q, \widehat{q} \in Q(m, F_\bullet)$. Since $\operatorname{Im}\alpha$ is cofinal in A we can pick $r \in Q(m, F_\bullet)$ such that $\alpha(r) \geq \alpha(q)$ and $\alpha(r) \geq \alpha(\widehat{q})$. Since $\star(q)$ holds we have

$$\big(\nu_{a,\alpha(q)} \circ F_q\big)\big(\mu_{\iota(q)}(m)\big) = \big(\nu_{a,\alpha(q)} \circ \nu_{\alpha(q),\alpha(r)} \circ F_r\big)\big(\mu_{\iota(r)}(m)\big) = \big(\nu_{a,\alpha(r)} \circ F_r\big)\big(\mu_{\iota(r)}(m)\big)$$

Similarly $\star(\widehat{q})$ implies $\big(\nu_{a,\alpha(\widehat{q})} \circ F_{\widehat{q}}\big)\big(\mu_{\iota(\widehat{q})}(m)\big) = \big(\nu_{a,\alpha(r)} \circ F_r\big)\big(\mu_{\iota(r)}(m)\big)$ so that the desired conclusion now follows. ∎

Example 10.2.8. Let M and N be promanifolds, $R \subseteq M$, and let $F : R \to N$ be a map between promanifolds such that each $\nu_a \circ F$ is locally cylindrical. Let ι and α be the canonical projections of $Q \underset{\text{def}}{=} \mathbb{N} \times \mathbb{N}$ onto its first and second coordinates Pr_2, and let $F_\bullet^\bullet \underset{\text{def}}{=} (F_i^a)_{(i,a) \in Q}$ where $F_i^a : \operatorname{Dom}_i F^a \to N_a$ for each $(i,a) \in Q$. Then $R \subseteq \operatorname{StrDom}_M(F_\bullet^\bullet) \subseteq M$ and $\varprojlim F_\bullet^\bullet\big|_R = F$.

The following example shows that these constructions may be used to extend maps beyond their original domains in a way that is well-suited for promanifolds.

Example 10.2.9. Let M, N, R, $F : R \to N$, Q, ι, α, and $F_\bullet^\bullet$ be as in the above example 10.2.8 and suppose that for all $a \in \mathbb{N}$, the increasing union $\underset{i \in \mathbb{N}}{\cup} \mu_i^{-1}(\operatorname{Dom} F_i^a)$ equals M. It is easy to see that $\operatorname{StrDom}_M(F_\bullet^\bullet) = M$ and that if $F : R \to N$ has a smooth extension to all of M then this limit map $\varprojlim F_\bullet^\bullet : M \to N$ would be it.

Weakly Defined Limit

The strong definability condition requires that the strong consistency condition be satisfied for each pair of elements in Q, which is unfortunate since the situation may arise that, for some given $m \in M$, if we simply ignored a select few F_q's for which the consistency condition failed then we would still be able to define an element of n. We remedy this with the following definition.

Definition and Notation 10.2.10. For any $m \in M$, say that $F_\bullet$ *satisfies the (weak) definability condition at* m or that $\varprojlim F_\bullet$ *is (weakly) definable at* m if

(1) there is some $P_0 \subseteq Q$ such that $F_\bullet\big|_{P_0} \underset{\text{def}}{=} (F_q)_{q \in P_0}$ satisfies the strong definability condition at m, and

(2) for all $P, \widehat{P} \subseteq Q$, whenever both $F_\bullet\big|_P$ and $F_\bullet\big|_{\widehat{P}}$ satisfy the strong definability condition at m then $\left(\varprojlim F_\bullet\big|_P\right)(m) = \left(\varprojlim F_\bullet\big|_{\widehat{P}}\right)(m)$.

and in this case we will denote the value $\left(\varprojlim F_\bullet\big|_{P_0}\right)(m)$ by $\left(\varprojlim F_\bullet\right)(m)$ or by any of the usual notation used for limits of inverse system morphisms, where by (2), this value is independent of the choice of $P_0 \subseteq Q$ satisfying the strong definability condition at m.

For any $D \subseteq M$, let $\operatorname{Dom}_D F_\bullet$ denote the set of all $m \in D$ such that $F_\bullet$ satisfies the definability condition at m, where we may write $\operatorname{Dom} F_\bullet$ in place of $\operatorname{Dom}_M F_\bullet$. We thus have an induced map $\varprojlim F_\bullet : \operatorname{Dom} F_\bullet \to N$ that we will call *the limit (map) of* $F_\bullet$ *over (a subset of)* M *into* N *(relative to* $(\mu_\bullet, \operatorname{Sys}_M)$ *and* $(\nu_\bullet, \operatorname{Sys}_N)$*)* or simply *the limit of* $F_\bullet$ and we will say that this map *arises as a (weak) limit (from* $(\mu_\bullet, \operatorname{Sys}_M)$ *to* $(\nu_\bullet, \operatorname{Sys}_N)$*) of* Q*-indexed maps.* ■

Remark 10.2.11. Observe that even if $F_\bullet$ was to fail to satisfy the strong definability condition at every $m \in M$ then the above definition may still allow us to define a map on a non-empty subset of M. Also, the first condition in definition 10.2.10 guarantees that there exists some element of N that can be defined from $F_\bullet$ in a way reminiscent of inverse system morphisms while the second condition guarantees that there at most one choice for such an element.

The following theorem shows, in particular, how we may use definitions 10.2.10 and proposition 11.5.1 to reconstruct a diffeomorphism F's inverse by using *only* the canonical maps $F_\bullet^\bullet \underset{\text{def}}{=} (F_i^a)_{(i,a)\in Q}$, where note that the collection of maps needed for this construction is potentially uncountable. Of course, if we knew that $F^{-1} : \operatorname{Im} F \to M$ was smooth then we could use F^{-1} $\mathbb{N} \times \mathbb{N}$-indexed canonical collection of maps, but this is not assumed.

Theorem 10.2.12. Let $F : M \to N$ be an injective $\nu_\bullet$-regular smooth immersion and let Q denote a set of tuples $q = (m, i, j, a, U_i, \varphi_i, U_j, \varphi_j, V_a, \psi_a)$ where $m \in M$ and $i, j, a, U_i, \varphi_i, U_j, \varphi_j, V_a, \psi_a$ are as in proposition 11.5.1 (with m in place of m^0) and $a \geq i$ and observe that the properties in proposition allows us to us the commutative diagram 11.1 to define a smooth surjective submersion $G_q : F_j^a(U_j) \to \mu_{ij}(U_j)$ that is the canonical projection with respect to the coordinates φ_i and ψ_a. Denote each $q = (m, i, j, a, U_i, \varphi_i, U_j, \varphi_j, V_a, \psi_a) \in Q$ by $(m_q, i_q, j_q, a_q, U_{i_q}, \varphi_{i_q}, U_{j_q}, \varphi_{j_q}, V_{a_q}, \psi_{a_q})$ and for each $m \in M$, let $Q(m) = \{q \in Q \,|\, m_q = m\}$. If for each $m \in M$, both $\{i_q \,|\, q \in Q(m)\}$ and $\{a_q \,|\, q \in Q(m)\}$ are cofinal in $\mathbb{N}$ then $\operatorname{Im} F \subseteq \operatorname{Dom}_N G_\bullet$ and $\varprojlim G_\bullet = F^{-1}$ on $\operatorname{Dom} G_\bullet$.

Remark 10.2.13. Despite each G_q is a smooth submersion, even if $F : M \to N$ was a surjective pointwise isomersion, it is not clear whether or not one can deduce that $F^{-1} : N \to M$ is smooth since it's not clear that each $\mu_\bullet \circ F^{-1} : N \to M_\bullet$ is necessarily locally cylindrical. However, one might still be able to use propositions 6.1.21 and 6.1.27 to prove that the $\mu_\bullet \circ F^{-1} : N \to M_\bullet$ are locally cylindrical on some open subset of M.

Proof. Let $n \in \operatorname{Im} F$ and suppose that $P \subseteq Q$ is such that $G_\bullet\big|_P$ satisfies the strong definability condition at $n \in \operatorname{Im} F$, where at least one such subset exists since $\{i_q \,|\, q \in Q(F^{-1}(n))\}$ is cofinal in $\mathbb{N}$. Let $\widehat{m} = \varprojlim G_\bullet\big|_P(n)$. Let $P(n) = P\left(n, G_\bullet\big|_P\right)$ be as in notation 10.2.1. Let $a \in \mathbb{N}$. Note that by definition of $\widehat{m}$, for all $q \in P(n)$, $\mu_{i_q}(\widehat{m}) \in \mu_{i_q, j_q}(U_{j_q}) = \operatorname{Codom} G_q$, $\nu_{a_q}(n) \in F_{j_q}^{a_q}\left(U_{j_q}\right)$, and $\mu_{i_q}(\widehat{m}) = G_q(\nu_{a_q}(n))$. This and the fact that $\{i_q \mid q \in Q(m)\}$ is cofinal in $\mathbb{N}$ means that there is some $q \in Q$ such that $i \underset{\text{def}}{=} i_q > a$ and $\mu_{i_q}(\widehat{m}) \in \operatorname{ODom}_{i_q} F^a$. By definition of $G_q(\nu_{a_q}(n))$, there is some $m_{j_q} \in U_{j_q}$ such that $F_{j_q}^{a_q}(m_{j_q}) = \nu_{a_q}(n)$ and $\mu_{i,j_q}(m_{j_q}) = \mu_i(\widehat{m})$. Note that $a_q \geq a$ so

$$\nu_a(F(\widehat{m})) = F_i^a(\mu_i(m)) = F_i^a(\mu_{i,j_q}(m_{j_q})) = F_{j_q}^a(m_{j_q}) = \nu_{a,a_q}(F_{j_q}^{a_q}(m_{j_q})) = \nu_{a,a_q}(\nu_{a_q}(n)) = \nu_a(n)$$

Since $a \in \mathbb{N}$ was arbitrary, it follows that $F(\widehat{m}) = n$ and since F is injective, this implies that $\widehat{m} = F^{-1}(n)$. Thus by definition, $n \in \operatorname{Dom}_N F_\bullet$ and $\varprojlim F_\bullet(n) = F^{-1}(n)$. ∎

Generalized Inverse System Morphisms

Definition 10.2.14. Say that a Q-indexed collection of maps $F_\bullet = (F_q)_{q \in Q}$ is a *collection of maps from subsets of* $M_\bullet = (M_i)_{i \in I}$ *into* $N_\bullet = (N_a)_{a \in A}$ if for each $q \in Q$ there exists $\iota(q) \in I$ and $\alpha(q) \in A$ such that

(1) $\operatorname{Dom} F_q \subseteq M_{\iota(q)}$, and

(2) $\operatorname{Codom} F_q \subseteq N_{\alpha(q)}$

where these indices are necessarily unique since all $M_\bullet$ and $N_\bullet$ are pairwise disjoint. We will indicate this by writing $F_\bullet : \subseteq M_\bullet \to N_\bullet$ where if we wish to specify the maps $\alpha : Q \to A$ and $\iota : Q \to I$ then we may write $(F_\bullet, \iota, \alpha) : \subseteq M_\bullet \to N_\bullet$ or $F_\bullet : \subseteq M_{\iota(\bullet)} \to N_{\alpha(\bullet)}$. ∎

Definition 10.2.15. If $D \subseteq M$ then by a (resp. *strong*) *generalized inverse system morphism* (or *generalized inverse system morphism from* Sys_M *to* Sys_N *over* D we mean a Q-indexed collection of maps $F_\bullet : \subseteq M_{\iota(\bullet)} \to N_{\alpha(\bullet)}$ (for some Q) such that $D \subseteq \operatorname{Dom}_M F_\bullet$ (resp. $D \subseteq \operatorname{StrDom}_M F_\bullet$). We will abbreviate this situation by saying that "$(F_\bullet, \iota, \alpha) : \operatorname{Sys}_M \to \operatorname{Sys}_N$ is a (resp. strong) generalized inverse system morphism over D." If we omit mention of D then it should be assumed that $D = \operatorname{Dom}_M F_\bullet$ (resp. $D = \operatorname{StrDom}_M F_\bullet$).

If we ever write "$F_\bullet : \mathrm{Sys}_M \to \mathrm{Sys}_N$ is a (resp. strong) generalized inverse system morphism" then we mean that this is true of $(F_\bullet, \iota, \alpha)$ where $Q = I \times A$, $\iota(i,a) = i$, $\alpha(i,a) = a$, while if we write "$F_{\iota(\bullet)} : \mathrm{Sys}_M \to \mathrm{Sys}_N$ is a (resp. strong) generalized inverse system morphism" then we mean that this is true of $\left(F_{\iota(\bullet)}, \iota, \alpha\right)$ where $Q \subseteq A$ is cofinal in A, $\iota : Q \to I$ is some order morphism, and $\alpha = \mathrm{Id}_Q$. ■

Example 10.2.16. Clearly, every inverse system morphism $(F_\bullet, \iota) : \mathrm{Sys}_M \to \mathrm{Sys}_N$ is a strong generalized inverse system morphism over M and $\mathrm{StrDom}_M(F_\bullet, \iota) = M$. Conversely, if $F_\bullet$ is an A-indexed collection of maps for which there exists some $\iota : A \to I$ such that each component of $F_\bullet$ has the prototype $F_a : M_{\iota(a)} \to N_a$ then $(F_\bullet, \iota)$ is a generalized inverse system morphism $\iff$ it is an inverse system morphism.

Although the definition of an inverse system morphism (did not require knowledge of the limit of Sys_M, the definition of the definability condition does. So we to remedy this we now describe conditions on the maps $F_\bullet$ that may be checked in a manner analogous to the definition of an inverse system morphism and that are better suited to use with definition 10.2.10. The following definition 10.2.17 is motivated by lemma 6.1.9 and essentially makes rigorous what is meant when one says that "the equality $\nu_{ab} \circ F_r(m_j) = F_q \circ \mu_{ij}(m_j)$ holds whenever both sides are defined."

Definition 10.2.17. Let $F_\bullet : \subseteq M_{\iota(\bullet)} \to N_{\alpha(\bullet)}$ be a Q-indexed collection of maps. If $q, r \in Q$ and $m_{\iota(r)} \in M_{\iota(r)}$ then we will say that F_q *and* F_r *are consistent at* $m_{\iota(r)}$ if, after letting $i = \iota(q)$ and $j = \iota(r)$, they satisfy the following condition, which we will call *the consistency condition (at* m_j*)*:

(1) $i \leq j$, $\alpha(q) \leq \alpha(r)$, and $m_j \in \mathrm{Dom}\, F_r \cap \mu_{ij}^{-1}(\mathrm{Dom}\, F_q)$ implies

$$\nu_{\alpha(q),\alpha(r)} \circ F_r(m_j) = F_q \circ \mu_{ij}(m_j)$$

(2) $i \leq j$, $\alpha(q) \geq \alpha(r)$, and $m_j \in \operatorname{Dom} F_r \cap \mu_{ij}^{-1}(\operatorname{Dom} F_q)$ implies

$$F_r(m_j) = \left(\nu_{\alpha(r),\alpha(q)} \circ F_q\right)(\mu_{ij}(m_j))$$

where observe that these two conditions are the same if $\alpha(q) = \alpha(r)$.

We will say that F_q *is consistent with* F_r *(on* $M_{\iota(r)}$*)* if they are consistent at every $m_{\iota(r)} \in M_{\iota(r)}$ and that F_q *and* F_r *are consistent* if F_q is consistent with F_r and F_r is consistent with F_q. If F_q and F_r are consistent for all $q, r \in Q$ then we will call $F_\bullet$ *strongly consistent.* ∎

A Smooth Map is "Almost" the Limit of an Inverse System Morphism

Corollary 10.3.3 of following proposition essentially states that every smooth map F between promanifolds is "almost" the limit of an inverse system morphism in the sense that except for on a comeager measure 0 subset of M, F is the limit of a ι-index generalized inverse system morphism for some order-morphism $\iota : A \to I$.

Proposition 10.3.1. Let $F : M \to N$ be a map between promanifolds such that each $F^a \underset{\text{def}}{=} \nu_a \circ F$ is locally cylindrical, let $F_\bullet^\bullet$ be the canonical $\mathbb{N} \times \mathbb{N}$-indexed collection of maps induced by F, and let $(K^l)_{l=1}^\infty$ be any collection of compact subsets of M. There exists an increasing order morphism $\iota : \mathbb{N} \to \mathbb{N}$ such that

(1) $\bigcup_{l \in \mathbb{N}} K^l \subseteq \operatorname{StrDom} F_{\iota(\bullet)}^\bullet$ and $\operatorname{Str} \varprojlim F_{\iota(\bullet)}^\bullet = F$ on $\operatorname{StrDom} F_{\iota(\bullet)}^\bullet$.

(2) For all $a \in \mathbb{N}$, $\mu_{\iota(a)}^{-1}\left(\operatorname{ODom}_{\iota(a)} F^a\right)$ contains $K^1 \cup \cdots \cup K^a$.

(3) $\operatorname{StrDom} F_{\iota(\bullet)}^\bullet$ is a $\mu_\bullet$-surjective subset of M (i.e. its image under each μ_i is all of M_i).

Remark 10.3.2. Observe that none of the above maps are required to be continuous.

Proof. By replacing each K^a with $K^1 \cup \cdots \cup K^a$ we may assume without loss of generality that $K^a \subseteq K^{a+1}$. Pick an index $\iota(1)$ such that $\mu_i(K^1) \subseteq \operatorname{ODom}_i F^1$. Let $E^{\bullet}_{\iota(1)} = \left(E^l_{\iota(1)}\right)_{l=1}^{\infty}$ be an increasing exhaustion of $M_{\iota(1)}$ by relatively compact open sets such that $\mu_{\iota(1)}(K^{\bullet}) \subseteq E^{\bullet}_{\iota(1)}$. Suppose that for $a_0 \geq 1$ we've defined $\iota(1) < \cdots < \iota(a_0)$ and $E^{\bullet}_{\iota(1)} = \left(E^l_{\iota(1)}\right)_{l=1}^{\infty}, \ldots, E^{\bullet}_{\iota(a_0)} = \left(E^l_{\iota(a_0)}\right)_{l=1}^{\infty}$ where each $E^{\bullet}_{\iota(b)}$ $(1 \leq b \leq a_0)$ is an increasing exhaustion of $M_{\iota(b)}$ by relatively compact open sets such that

(a) $\mu_{\iota(b)}(K^{\bullet}) \subseteq E^{\bullet}_{\iota(b)}$ and $\mu_{\iota(b)}(K^b) \subseteq \operatorname{ODom}_{\iota(b)} F^b$ for $1 \leq b \leq a_0$,

(b) $\overline{E^{\bullet}_{\iota(b)}} \subseteq \mu_{\iota(b),\iota(c)}\left(E^{\bullet}_{\iota(c)}\right)$ for $1 \leq b < c \leq a_0$,

(c) $\overline{E^{c}_{\iota(b)}} \subseteq \mu_{\iota(b),\iota(c)}\left(\operatorname{ODom}_{\iota(c)} F^c\right)$ for $1 \leq b \leq c \leq a_0$.

Let $i > \iota(a_0)$ be such that $\mu_i(K^{a_0+1}) \subseteq \operatorname{ODom}_i F^{a_0+1}$ where by increasing i we may also assume that $\overline{E^{a_0+1}_{\iota(b)}} \subseteq \mu_{\iota(b),i}(\operatorname{ODom}_i F^{a_0+1})$ for each $b = 1, \ldots, a_0$. Let $E^{\bullet}_i = (E^l_i)_{l=1}^{\infty}$ be an increasing exhaustion of M_i by relatively compact open subsets of M_i such that $\mu_i(K^{\bullet}) \subseteq E^{\bullet}_i$, $\overline{E^{\bullet}_{\iota(a_0)}} \subseteq \mu_{\iota(a_0),i}(E^{\bullet}_i)$, and $\overline{E^{a_0+1}_i} \subseteq \operatorname{ODom}_i F^{a_0+1}$. Let $\iota(a_0+1) \underset{\text{def}}{=} i$ and observe that (a) - (c) hold for $a_0 + 1$ in place of a_0.

Note that (b) implies that for each $b \in \mathbb{N}$, $\left(\mu_{\iota(b),\iota(c)}\left(E^c_{\iota(c)}\right)\right)_{c \geq b}$ is an exhaustion of $M_{\iota(b)}$ while (a) implies that $K^a \subseteq \mu^{-1}_{\iota(a)}\left(E^a_{\iota(a)}\right)$ for each $a \in \mathbb{N}$. For any $a \leq b$ we have

$$\mu_{\iota(a)}(K^a) \subseteq E^a_{\iota(a)} \subseteq \mu_{\iota(a),\iota(b)}\left(E^b_{\iota(b)}\right) \subseteq \mu_{\iota(a),\iota(b)}\left(\operatorname{ODom}_{\iota(b)} F^b\right)$$

Since the consistency condition is obviously satisfied at every $m \in K^a$ we have that $K^a \subseteq \operatorname{StrDom} F^{\bullet}_{\iota(\bullet)}$ for each index a, so that $\cup K^{\bullet} \subseteq \operatorname{StrDom} F^{\bullet}_{\iota(\bullet)}$. By construction, $\mu^{-1}_{\iota(a)}\left(\operatorname{ODom}_{\iota(a)} F^a\right)$ contains K^a for each $a \in \mathbb{N}$.

Given an index i and $m_i \in M_i$ let a be such that $i \leq \iota(a)$ and let $m_{\iota(a)} \in \mu^{-1}_{i,\iota(a)}(m_i)$ be arbitrary. To prove that $\operatorname{StrDom} F^{\bullet}_{\iota(\bullet)}$ is a $\mu_{\bullet}$-surjective subset of M it suffices to show that $m_{\iota(a)} \in \mu_{\iota(a)}\left(\operatorname{StrDom} F^{\bullet}_{\iota(\bullet)}\right)$. Hence, we may assume without loss of generality that $i = \iota(a)$. Since $\left(\mu_{\iota(a),\iota(b)}\left(E^b_{\iota(b)}\right)\right)_{b \geq a}$ is an exhaustion of $M_{\iota(a)}$ we may pick $b \geq a$

such that $m_i \in \mu_{i,\iota(b)}\left(E^b_{\iota(b)}\right)$. By (b), we may inductively pick $m_{\iota(c)} \in E^c_{\iota(c)}$ such that $\mu_{\iota(c-1),\iota(c)}(m_{\iota(c)}) = m_{\iota(c-1)}$, where now $\left(m_{\iota(c)}\right)_{c \geq b}$ defines an element $m \in M$. By (c), whenever $c \geq b$ then $\mu_{\iota(c)}(m) = m_{\iota(c)} \in E^c_{\iota(c)} \subseteq \operatorname{ODom}_{\iota(c)} F^c$ and since the consistency condition at m is clearly satisfied we have that $m \in \operatorname{StrDom} F^{\bullet}_{\iota(\bullet)}$. Since $m_{\iota(a)} = \mu_{\iota(a)}(m)$ we have our desired conclusion. ■

Corollary 10.3.3. Let $F : M \to N$ be a smooth map between promanifolds that are both limits of profinite systems satisfying 12.1.3. Then there exists an increasing $\iota : \mathbb{N} \to \mathbb{N}$ such that if we let $F_a \underset{\text{def}}{=} F^a_{\iota(a)} : \operatorname{ODom}_{\iota(a)} F^a \to N_a$ for all $a \in \mathbb{N}$ then $M \smallsetminus \operatorname{StrDom} F_{\iota(\bullet)}$ is a meager measure 0 subset of M and $\operatorname{Str} \varprojlim F_{\iota(a)} = F$ on the $\mu_\bullet$-surjective set $\operatorname{StrDom} F_{\iota(\bullet)}$.

Chapter 11

Submersions, Immersions, and Isomersions

Definition 11.0.1. Let $F : M \to N$ be a smooth map between promanifolds and let $m \in M$. Say that

- F is a *point(wise) submersion* (resp. *immersion*, *isomersion*) *at* m if $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_{F(m)} N$ is surjective (resp. injective, an isomorphism of TVSs).
- F *has full rank at* m if F is a pointwise submersion or a pointwise immersion at m. If $S \subseteq M$ and $r \in \mathbb{Z}^{\geq 0}$ then we will say that F *has finite constant rank* r *on* S if F has finite rank r at every $s \in S$.
- F is a *(smooth) sectional submersion at* m if there exists a smooth local section of F through m,

As usual, if we omit mention of any point then we mean that F has the property at every point of M. ∎

Remark 11.0.2. Clearly, every smooth sectional submersion at a point has full rank at that point.

Pointwise Immersions from Smooth Manifolds

Example 11.1.1. If $F : M \to N$ is a smooth map that is a smooth embedding at $m \in M$ then it is immediate from lemma 7.0.2 that F is a smooth pointwise immersion at m.

We will now find conditions under which the converse of the above statement is true.

Lemma 11.1.2. Let $F : (M, m) \to (N, n)$ be a smooth map where M is a manifold, let $F^{\bullet} = \nu_{\bullet} \circ F$, $S_{\bullet} = \operatorname{Im} F^{\bullet}$, and let a be an index. If $\nu_a \circ F : M \to N_a$ is a smooth immersion at m then so is $F^b = \nu_b \circ F : M \to N_b$ for all $b \geq a$ and if $S_b = \operatorname{Im} F^b$ is an immersed submanifold of N_a then $\nu_{ab}\big|_{S_b} : S_b \to N_a$ is a smooth immersion. If $F^a : M \to N_a$ is a smooth embedding (resp. proper smooth embedding) then so are both $F^b : M \to N_b$ and $\nu_{ab}\big|_{S_b} : S_b \to N_a$ for all $b \geq a$ and $F : M \to N$ is a smooth topological embedding (resp. proper topological embedding).

Proof. Since $\mathrm{T}_m F^a = \mathrm{T}_m(\nu_{ab} \circ F^b) = \mathrm{T}_{F^b(m)} \nu_{ab} \circ \mathrm{T}_m F^b$ is injective it follows that $\mathrm{T}_m F^b$ is also injective so that $F^b : M \to N_b$ is a smooth immersion. If in addition S_b is an immersed submanifold of N_b then $\nu_{ab}|_{S_b} : S_b \to N_a$ is a smooth map between manifolds so that the injectivity of $\mathrm{T}_m F^a = \mathrm{T}_{F^b(m)}(\nu_{ab}|_{S_b}) \circ \mathrm{T}_m F^b$ and the fact that $\operatorname{Im}(\mathrm{T}_m F^b) = \mathrm{T}_{F^b(m)} S_b$ implies the injectivity of $\mathrm{T}_{F^b(m)} \nu_{ab}|_{S_b}$. By lemma 2.1.33(7), we have that F^b and $\nu_{ab}|_{S_b}$ are topological embeddings for all $b \geq a$ and since $F = \varprojlim_a \nu_a \circ F$ is a limit of topological embeddings, F is also a topological embedding. If in addition F^a is proper then since $F^a = \nu_{ab} \circ F^b$ and F^b is a topological embedding it follows that F^b and $\nu_{ab}|_{S_b} : S_b \to N_a$ are also proper and since the inverse limit of proper maps is proper, F is also proper. ∎

Corollary 11.1.3. Let $F : M \to N$ be a smooth map where M is a manifold. Let U be the set of all $m \in M$ such that $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_{F(m)} N$ is injective. Then U is open in M and for each $m \in U$,

(1) If a is an index such that $\mathrm{T}_m F^a : \mathrm{T}_m M \to \mathrm{T}_{F^a(m)}$ is injective then there exists $m \in$

$V \in \operatorname{Open}(U)$ such that $F^a\big|_V : V \to N_a$ is a smooth topological embedding and for any such V the map $F|_V : V \to N$ will be a smooth topological embedding and for each $b \geq a$ the map $F^b\big|_V : V \to N_b$ is a smooth embedding.

(2) There exists an index a such that $\mathrm{T}_m F^a : \mathrm{T}_m M \to \mathrm{T}_{F^a(m)}$ is injective.

Proof. Let $d = \dim M$ and note that the conclusion is clear if $d = 0$ so assume that $d \geq 1$. Assume that $U \neq \varnothing$, let $m \in U$, $n = F(m)$, and $F^a = \nu_a \circ F : M \to N_a$ for all indices a. Let $\{v_1, \ldots, v_d\}$ be a basis for $\mathrm{T}_m M$ and let $x_l = \mathrm{T}_m F v_l$ for all $l = 1, \ldots, d$. Since all x_l are distinct and non-zero, there exists some index a such that all $x_l^a = \mathrm{T}_n \nu_a x_l$ are non-zero and non-zero. For all $l = 1, \ldots, d$

$$\mathrm{T}_m F^a v_l = \mathrm{T}_n \nu_a (\mathrm{T}_m F(v_l)) = \mathrm{T}_n \nu_a x_l = x_l^a$$

so that $\mathrm{T}_m F^a : \mathrm{T}_m M \to \mathrm{T}_{F^a(m)} N_a$ has full rank i.e. is injective. Since $F^a : M \to N_a$ is a smooth map between manifolds there exists some open neighborhood V of m such that $F^a : V \to N_a$ has full rank. By shrinking V we may assume that $F^a|_V : V \to N_a$ is a topological embedding so that $F|_V : V \to N$ is also a topological embedding. ∎

Remark 11.1.4. Let $(F_\bullet, \iota) : \operatorname{Sys}_M \to \operatorname{Sys}_N$ be an inverse system morphism with limit $F : M \to N$ and let $m \in M$. Although corollary 3.2.3 makes it is clear that each F_a being an immersion at $\mu_{\iota(a)}(m)$ guarantees that F will be a pointwise immersion at m, this is not a necessary condition.

Proposition 11.1.5. Let $F : M \to N$ be a smooth map from a manifold M into a promanifold N Let $S = \operatorname{Im} F$ and suppose that F is a topological embedding and a smooth pointwise immersion. Let $C_N^\infty\big|_S$ denote the restriction of N's sheaf to S and let C_S^∞ denote the sheaf defined by:

for each $W \in \operatorname{Open}(S)$, let $C_S^\infty(W)$ be the $\mathbb{R}$-algebra of all maps of the form $f \circ \left(F\big|_{F^{-1}(W)}\right) : W \to \mathbb{R}$ where $f \in C_M^\infty(F^{-1}(W))$.

Then $C_N^\infty\big|_S = C_S^\infty$.

Definition 11.1.6. Consequently, we can call a smooth map $F : M \to N$ from a manifold M into a promanifold N a *smooth embedding* (cf. def. 4.1.3) if it is a topological embedding and a smooth pointwise immersion.

Proof. Fix $W \in \operatorname{Open}(S)$, let $U = F^{-1}(W)$, $F^\bullet = \nu_\bullet \circ F$, and let $g : W \to \mathbb{R}$ be continuous. Suppose that $g \in C_N^\infty\big|_S(W)$. Recall that since $F : M \to (N, C_N^\infty)$ is smooth so is $F\big|_U : U \to (W, C_N^\infty\big|_W)$ so that $f \underset{\text{def}}{=} g \circ F\big|_U : U \to \mathbb{R}$ is smooth. Since $g = \left(g \circ F\big|_U\right) \circ \left(F\big|_U\right)^{-1} = f \circ F\big|_U$ we have that $g \in C_S^\infty(W)$. Now suppose that $g \in C_S^\infty(W)$ and let $f \in C_M^\infty(U)$ be such that $g = f \circ \left(F\big|_U\right)^{-1} : U \to \mathbb{R}$. Let $s \in W$, $m = F^{-1}(s)$. Let $Y = \operatorname{Im} \mathrm{T}_m F$ and let $a = \operatorname{Ind} Y$ so that $\mathrm{T}_s\, \nu_a\big|_Y : Y \to \mathrm{T}_{s_a} N_a$ is injective and hence $\mathrm{T}_m(\nu_a \circ F) : \mathrm{T}_m M \to \mathrm{T}_{s_a} N_a$ is injective. Since $F^a = \nu_a \circ F : M \to N_a$ is a smooth immersion between manifolds there exists some open set $m \in U^0 \in \operatorname{Open}(U)$ such that $F^a\big|_{U^0} : U^0 \to N_a$ is a smooth embedding onto the submanifold $R_a \underset{\text{def}}{=} F^a(U^0)$. Let $g_a : V_a \to \mathbb{R}$ be any smooth extension of the smooth map $f \circ \left(F^a\big|_{U^0}\right)^{-1} : R_a \to \mathbb{R}$ to some open set $V_a \in \operatorname{Open}(N_a)$. Observe that for any $n \in F(U_0)$, $(\nu_a \circ F)(F^{-1}(n))) = F^a(F^{-1}(n)) \in R_a$ so that

$$g_a(\nu_a(n)) = g_a\big(F^a(F^{-1}(n))\big) = \left(f \circ \left(F^a\big|_{U^0}\right)^{-1}\right)\big(F^a(F^{-1}(n)\, n)\big) = f(F^{-1}(n)) = g(n)$$

Thus, $g_a \circ \nu_a\big|_{\nu_a^{-1}(V_a)} \in C_N^\infty(\nu_a^{-1}(V_a))$ is a smooth map whose restriction to $F(U_0)$, an open set in S containing s, is $g\big|_{F(U_0)}$. So by definition of the restriction sheaf, $g \in C_N^\infty\big|_S(W)$. ∎

Corollary 11.1.7. A smooth pointwise immersion $F : M \to N$ from a manifold M into a promanifold N is locally a smooth embedding. Furthermore, for each $m \in M$, $a \underset{\text{def}}{=} \operatorname{Ind}(\operatorname{Im} \mathrm{T}_m F)$ (see def. 7.6.3) is the unique smallest index for which there exists an open set $m \in U \in \operatorname{Open}(M)$ such that $\nu_a \circ F\big|_U : U \to N_a$ is a smooth embedding. And if $K \subseteq M$ is compact then exists a unique smallest index for which there exists an open set $K \subseteq U \in \operatorname{Open}(M)$ such that $\nu_a \circ F\big|_U : U \to N_a$ is a smooth embedding.

Proof. Let $n = F(m)$ and let $Y = \operatorname{Im} \mathrm{T}_m F$. By definition of $a \underset{\text{def}}{=} \operatorname{Ind} Y$, this is the unique smallest index such that $\mathrm{T}_n \nu_a\big|_Y : Y \to \mathrm{T}_n \nu_a(Y)$ is injective, which is necessary for $\nu_a \circ F\big|_U$ to be an immersion at m and conversely, for any $b \geq a$ the map $\nu_b \circ F : M \to N_b$ is a smooth immersion at m and hence a local embedding. The rest of this follows from lemma 11.1.2. For every $m \in K$ we can pick an open neighborhood $U(m)$ of m such that $\nu_{a(m)} \circ F\big|_{U(m)} : U(m) \to N_{a(m)}$ is a smooth embedding, where $a(m) \underset{\text{def}}{=} \operatorname{Ind}(\operatorname{Im} \mathrm{T}_m F)$. Picking a finite subcover $U(m_1), \ldots, U(m_p)$ of K we can define $a = \max\{a(m_1), \ldots, a(m_p)\}$ and $U = U(m_1) \cup \cdots \cup U(m_p)$ so that $\nu_a \circ F\big|_U : U \to N_a$ is a smooth embedding. It is clear that this index a is in fact also the smallest index for which such an open neighborhood of K exists. ∎

Example 11.1.8. For all $a \in \mathbb{N}$, let $S_a = S^1$ denote a copy of S^1, let $T_a = S_1 \times \cdots \times S_a$ be the a-torus, and let $T^\infty = T = \prod_{a \in \mathbb{N}} S_a$ Let $\tau_{ab} : T_b \to T_a$ and $\tau_a : T \to T_a$ be the canonical projections onto the first a coordinates: $\tau_{ab}(z_1, \ldots, z_j) = (z_1, \ldots, z_a)$ and $\tau_a(z_1, z_2, \ldots) = ((z_1, \ldots, z_a))$. Then $(T, \tau_a) = \varprojlim \operatorname{Sys}_T$ where $\operatorname{Sys}_T = (T_a, \tau_{ab}, \mathbb{N})$. Let $\alpha_1, \alpha_2, \ldots$ be irrational real numbers, all of which are rationally independent and for each index a let

$$\begin{aligned} h_a : \mathbb{R} &\longrightarrow T_a \\ t &\longmapsto \left(e^{2\pi i \alpha_1, t}, \ldots, e^{2\pi i \alpha_a, t}\right) \end{aligned}$$

Then $(\mathbb{R}, h_a)$ forms a cone into Sys_T whose limit is

$$\begin{aligned} h : \mathbb{R} &\longrightarrow T \\ t &\longmapsto \left(e^{2\pi i \alpha_1, t}, e^{2\pi i \alpha_2, t}, \ldots\right) \end{aligned}$$

Since all $h_\bullet$ are smooth injective immersions with each $\operatorname{Im} h_a$ dense in T_a, it follows that h is a smooth, injective, pointwise immersion whose image is dense in N. ∎

Sectional Submersions

The lemma and example in this section will not be needed anywhere else in this paper.

Lemma 11.2.1. Let $n \in N$, a be an index, and let $n_a = \nu_a(n)$. The following are equivalent:

(1) There exists a smooth manifold M and a smooth map $F : (M, m) \to (N, n)$ such that $\nu_a \circ F : M \to N_a$ has full rank at m.

(2) There exists a smooth map $F : (V_a, n_a) \to (N, n)$, where $V_a \in \operatorname{Open}(N_a)$, such that $\nu_a \circ F : V_a \to N_a$ has full rank at n_a.

(3) There exists a smooth local section $\sigma_a : V_a \to N$ of ν_a through n.

Proof. (3) $\implies$ (2) $\implies$ (1) are immediate.

(1) $\implies$ (2): Since $F^a = \nu_a \circ F : (M, m) \to (N_a, n_a)$ is a submersion at m there exists a smooth local section $\sigma_a : (V_a, n_a) \to (M, m)$ of F^a so that $G \underset{\text{def}}{=} F \circ \sigma_a : (V_a, n_a) \to (N, n)$ is the map necessary for (2).

(2) $\implies$ (3): Since $\nu_a \circ F : (V_a, n_a) \to (N_a, n_a)$ is a smooth map of full rank at n_a there exists a smooth local section $\eta : (W_a, n_a) \to (V_a, n_a)$ of $\nu_a \circ F$ through n_a where we can take $W_a \subseteq V_a$. If $\sigma_a \underset{\text{def}}{=} F \circ \eta : (W_a, n_a) \to (N, n)$ then $\operatorname{Id}_{W_a} = \nu_a \circ F \circ \eta = \nu_a \circ \sigma_a$ so that σ_a is a smooth local section of ν_a through n. ∎

The following example shows that it is possible for a smooth map to be a pointwise submersion at a point without admitting any smooth local section through that point.

Example 11.2.2. Let $N_1 = (0, 1)$ and let $N_2 = N_2^1 \cup N_2^2$ be two disjoint copied of the interval $(0, \frac{3}{4})$, say $N_2^1 = (0, \frac{3}{4})$ and $N_2^2 = (1, 1 + \frac{3}{4})$, and let $N_3 = N_3^1 \cup N_3^2 \cup N_3^3 \cup N_3^4$ be four disjoint copies of the interval $(0, (3/4)^2)$, and so forth. Define $\nu_{12} : N_2 \to N_1$ by $\nu_{12}|_{N_2^1} = \operatorname{Id}_{N_2^1}$ and $\nu_{12}|_{N_2^2} = \operatorname{Id}_{N_2^2} - \frac{3}{4}$ (i.e. this is a "downward shift by 3/4" of $N_2^2 = (1, 1 + \frac{3}{4})$ onto $(\frac{1}{4}, 1)$). To define $\nu_{23} : N_3 \to N_2$, let ν_{23} be defined on $N_3^1 \cup N_3^2 = (0, (3/4)^2) \cup (1, 1 + (3/4)^2)$ in the same way as ν_{12} as defined on $N_2 = N_2^1 \cup N_2^2$ with that $\nu_{23}(N_3^1) = (0, (3/4)^2)$, $\nu_{23}(N_3^2) = \left(\frac{3}{4} - (3/4)^2, \frac{3}{4}\right)$ (so that

$\nu_{23}(N_3^1 \cup N_3^2) = N_2^1)$ and then repeat this on $N_3^3 \cup N_3^4$ so that $\nu_{32}(N_3^3 \cup N_3^4) = (1, 1 + \frac{3}{4}) = N_2^2$. Continue defining $\nu_{i,i+1}$ inductively in this manner and then let $\nu_{ij} = \nu_{i,i+1} \circ \cdots \circ \nu_{j-1,j}$ for all $i \leq j$. Note that since all $\nu_{i,i+1}$ are smooth surjective submersions, so too are all ν_{ij}. Let $(N, \nu_i) = \varprojlim (N_i, \nu_{ij})$. A non-empty basic open subset of N is of the form $\nu_i^{-1}(V_i)$ for some non-degenerate open interval $V_i \subseteq N_i$, which since ν_i is surjective implies that the cardinality of $\nu_i^{-1}(V_i)$ is at least that of V_i so that in particular N does not have the discrete topology.

Let $M_i = \mathbb{R} \times N_i$, $M_i^k = \mathbb{R} \times N_i^k$, and $\mu_{ij} = \operatorname{Id}_{\mathbb{R}} \times \nu_{ij}$ so that

$$(M, \mu_\bullet) \underset{\text{def}}{=} \varprojlim (M_\bullet, \mu_{ij}) = \varprojlim (\mathbb{R} \times N_i, \operatorname{Id}_{\mathbb{R}} \times \nu_{ij}) = (\mathbb{R} \times N, \operatorname{Id}_{\mathbb{R}} \times \nu_i)$$

Note that by construction, for an arbitrary open ball B_1 in M_1 there is some index i_{B_1} such that for all $i \geq i_{B_1}$, B_1 is not contained in the range of $\mu_{1i}|_{M_i^k}$ for any $k = 1, \ldots, 2^{i_{B_1}}$.

Now pick $m \in M$ and suppose that there existed some $\mu_1(m) \in U_1 \in \operatorname{Open}(M_1)$ and some local section $\sigma : U_1 \to M$ of μ_1 so that $\mu_1 \circ \sigma = \operatorname{Id}_{U_1}$. By shrinking U_1 we may assume that U_1 is an open ball. For all indices i, let $\sigma^i : U_1 \to M_i$ be $\sigma^i = \mu_i \circ \sigma$. Then $\operatorname{Id}_{U_1} = \mu_1 \circ \sigma = \mu_{1i} \sigma^i$ so that σ^i is a section of μ_{1i}. Since $\operatorname{Im}(\sigma^i) \subseteq \mathbb{R}^2$ is diffeomorphic to the open ball U_1 we have by invariance of domain that $\operatorname{Im}(\sigma^i)$ is an open connected set in M_i and hence contained in M_i^k for some $k = k_i \in \{1, \ldots, 2^i\}$. In particular, U_1 is contained in the range of $\mu_{1i}|_{M_i^k}$ but since i was arbitrary, this gives us a contradiction.

Let $m = (t_0, n) \in M = \mathbb{R} \times N$ and let $\frac{\partial}{\partial t}$ denote the canonical tangent vector of $\operatorname{T}_{t_0} \mathbb{R}$ at t_0. Let $n_i = \nu_i(n)$ and $m_i = \mu_i(m) = (t_0, n_i)$ for each index i. Let $\gamma^i : \mathbb{R} \to M_i$ be $\gamma^i(s) = (t_0 + s, \nu_i(n))$ so that $(\gamma^i)'(0) = \left(\frac{\partial}{\partial t}, 0\right) \in \operatorname{T}_{t_0} \mathbb{R} \times \operatorname{T}_{n_i} N_i$. Since these γ^i's are compatible with the μ_{ij}'s this defines a smooth map $\gamma : \mathbb{R} \to M = \mathbb{R} \times N$ with $\gamma'(0) = \left(\frac{\partial}{\partial t}, 0\right)$, which shows that some tangent vectors of $\operatorname{T} M$ arise as derivatives of curves.

In contrast, consider $x_i \underset{\text{def}}{=} \left(0, \frac{\partial}{\partial t}|_{m_i}\right)$ which are clearly consistent with all $\operatorname{T}_{\mu_j(m)} \mu_{ij}$'s and hence define a tangent vector $x \in \operatorname{T}_m M$. If there was some smooth curve $\gamma : \mathbb{R} \to M$ with $\gamma'(0) = x$ then letting $\gamma^i = \mu_i \circ \gamma$ we'll have $(\gamma^i)'(0) = \left(0, \frac{\partial}{\partial t}|_{m_i}\right)$. By continuity of $(\gamma^1)'$, there is some open interval in $\mathbb{R}$ around 0 on which γ' does not vanish and by picking a sufficiently small interval we may also assume that γ^1 is an embedding on this interval. By definition of the maps μ_{ij} this implies that all γ^i (when restricted to this same interval) are embeddings into M_i. Since all μ_{ij}'s are local isometries and since $\operatorname{Im}\gamma^1$ is not contained in $\mathbb{R} \times \{m_1\}$ we obtain a contradiction in a manner analogous to the last contradiction that was obtained above. This shows that the tangent vectors that arise as derivatives of curves may be a non-trivial subset of $\mathrm{T}\,M$.

Note that we can also define a smooth vector field where each component is defined by $X_i(n) \underset{\text{def}}{=} \left(0, \frac{\partial}{\partial t}\big|_{n_i}\right)$ for all $n \in M$. As before, at each point $n \in M$ there is not smooth curve at n with $X(n)$ as its tangent vector so in particular this smooth vector field on M that has no integral curves at any point. ∎

Rank

Observe that the following definitions reduce to the usual definitions in the case where both promanifolds are manifolds.

Definition 11.3.1. If $\Lambda : X \to Y$ is a linear map between two vector spaces then define *the rank (resp. nullity) of* Λ as $\operatorname{rank}\Lambda \underset{\text{def}}{=} \dim(\operatorname{Im}\Lambda)$ (resp. $\operatorname{nullity}\Lambda \underset{\text{def}}{=} \dim(\ker\Lambda)$). If $F : M \to N$ is a smooth map between promanifolds and $m \in M$ then define *the rank (resp. nullity) of F at m* as $\operatorname{rank}_m F \underset{\text{def}}{=} \operatorname{rank}(\mathrm{T}_m F)$ (resp. $\operatorname{nullity}_m F \underset{\text{def}}{=} \operatorname{nullity}(\mathrm{T}_m F)$), the rank (resp. nullity) of the tangent map $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_{F(m)} N$. Denote the induced maps on M by

$$\begin{aligned} \operatorname{rank} F : M &\longrightarrow \mathbb{Z}^{\geq 0} \cup \{\infty\} \\ m &\longmapsto \operatorname{rank}_m F \end{aligned} \qquad \text{and} \qquad \begin{aligned} \operatorname{nullity} F : M &\longrightarrow \mathbb{Z}^{\geq 0} \cup \{\infty\} \\ m &\longmapsto \operatorname{nullity}_m F \end{aligned}$$

If $S \subseteq M$ then say that F *has constant rank (resp. nullity) on S* if $\operatorname{rank} F$ (resp. $\operatorname{nullity} F$) is constant on S where if $d \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$ and $\operatorname{rank} F$ (resp. $\operatorname{nullity} F$) is identically d on S then we'll say that F *has constant rank (resp. nullity) d on S*. Call a point $m \in M$ *a rank* *(resp. nullity) regular point of* F and say that F *has locally constant rank (resp. nullity) at (*or *around)* m if there exists some neighborhood of m in M on which F has constant rank (resp. nullity).

Remark 11.3.2. It is clear that whether or not m is a constant rank point of F is independent of Sys_M and Sys_N and is in fact a diffeomorphism invariant (i.e. it only depends on the locally ringed spaces (M, C_M^∞) and (N, C_N^∞)).

Proposition 11.3.3. If $F : M \to N$ is a smooth map between promanifolds then the rank map $\operatorname{rank} F : M \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ is lower-semicontinuous.

Proof. Fix $m^0 \in M$, $n^0 = F(m^0)$, $m^0_\bullet \mu_\bullet(m^0)$, and $n^0_\bullet = \nu_\bullet(n^0)$. Let $r \in \mathbb{Z}^{\geq 0}$ be any integer such that $r \leq \operatorname{rank}_{m^0} F$ and pick r-dimensional vector spaces $Y \leq \operatorname{Im} \mathrm{T}_{m^0} F$ and $X \leq \mathrm{T}_{m^0} M$ such that $\mathrm{T}_{m^0} F\big|_X : X \to Y$ is a vector space isomorphism. Let $X_\bullet = (\mathrm{T}_{m^0} \mu_\bullet) X$ and $Y_\bullet = (\mathrm{T}_{n^0} \nu_\bullet) Y$. Pick an index a such that $\mathrm{T}_{n^0} \nu_a\big|_Y : Y \to Y_a$ is a vector space isomorphism and then pick $i \in \mathbb{N}$ such that $m_i^0 \in \operatorname{ODom}_i F^a$ and $\mathrm{T}_{m^0} \mu_i\big|_X : X \to X_i$ is a vector space isomorphism. Since $\operatorname{rank} F_i^a : \operatorname{ODom}_i F^a \to \mathbb{Z}^{\geq 0} \cup \{\infty\}$ is lower-semicontinuous and $\operatorname{rank}_{m_i^0} F_i^a \geq r$, there exists some $m_i^0 \in U_i \in \operatorname{Open}(\operatorname{Dom}_i^a F)$ such that $\operatorname{rank}_{m_i} F_i^a \geq \operatorname{rank}_{m_i^0} F_i^a \geq r$ for all $m_i \in U_i$. But then for any $m \in \mu_i^{-1}(U_i)$, since $\mathrm{T}_m \mu_i : \mathrm{T}_m M \to \mathrm{T}_{\mu_i(m)} M_i$ is surjective it follows that $\dim \mathrm{T}_m(\nu_a \circ F) \geq \operatorname{rank}_{m_i^0} F_i^a$, which implies that $\operatorname{rank}_m F = \dim \mathrm{T}_m F \geq r$. ∎

Example 11.3.4 (nullity(F) may fail to be upper-semicontinuous)**.** For each $i \in \mathbb{N}$, let $J_i = \left]-\frac{1}{i}, -\frac{1}{2i+1}\right[\cup \left]\frac{1}{2i+1}, \frac{1}{i}\right[$, $K_i = \left]-\frac{1}{i+1}, \frac{1}{i+1}\right[$, and define

$$O_i = \{i\} \times J_i \times \mathbb{R} \quad \text{and} \quad M_i = O_1 \cup \cdots \cup O_i \cup K_i$$

where observe that $J_{i+1} \cup K_{i+1} = K_i$ since $\frac{1}{2(i+1)+1} < \frac{1}{i+2}$. For each $i \in \mathbb{N}$, define $\mu_{i,i+1} : M_{i+1} \to M_i$ by letting $\mu_{i,i+1} = \mathrm{Id}_{O_1 \cup \cdots \cup O_i \cup K_{i+1}}$ on $O_1 \cup \cdots \cup O_i \cup K_{i+1}$ and sending $(i+1, r, s) \in O_{i+1}$ to r. Clearly, $\mu_{i,i+1}$ is a smooth submersion and since $\mu_{i,i+1}(O_{i+1} \cup K_{i+1}) = J_{i+1} \cup K_{i+1} = K_i$, it is also surjective. Let $(M, \mu_\bullet)$ be a limit of Sys_M. Observe that $\mu_{i,i+1}(0) = 0$ for all $i \in \mathbb{N}$ so that there exists some $m^0 \in M$ such that $\mu_i(m^0) = 0$ for all $i \in \mathbb{N}$ and note that $\mathrm{T}_{m^0} M \cong \mathbb{R}$ since $\mathrm{T}_0 M_i \cong \mathbb{R}$ for all $i \in \mathbb{N}$. Suppose that $m \in M$ is such that $m \neq m^0$ and let $m_\bullet \underset{\text{def}}{=} \mu_\bullet(m)$. Since $m \neq m^0$ there exists some $h \in \mathbb{N}$ such that $m_h \neq 0$. If $m_h \notin K_i$ then let $i \underset{\text{def}}{=} h$ and otherwise pick $i \geq h$ such that $|m_h| \geq \frac{1}{i+1}$, which implies that $m_i \notin K_i$. Since $\dim_{m_j} M_j = 2$ for all $j \geq i$ we have $\dim_m M = 2$.

Now if $F : M \to \mathbb{R}$ is the (necessarily smooth) constant 0 map then for any $m \in M$,

$$\operatorname{nullity}_m F = \dim_m M = \begin{cases} 1 & \text{if } m = m^0 \\ 2 & \text{otherwise} \end{cases}$$

In particular, for any $m^0 \in U \in \operatorname{Open}(M)$, $\operatorname{nullity}_m F = 2 > 1 = \operatorname{nullity}_{m^0} F$ for all $m \in U \setminus \{m^0\}$ and since m^0 is clearly not an isolated point in M, it follows that $\operatorname{nullity} F : M \to \mathbb{Z}^{\geq 0}$ is not upper-semicontinuous at m^0. ∎

$\nu_\bullet$-Constant Rank

We will now give a useful alternative generalization of the notion of "constant rank point" to smooth maps $F : M \to N$ between arbitrary promanifolds. While $m \in M$ being a constant rank point of F is a "local property" in the sense that it requires the existence of a neighborhood of m in M with a certain property (i.e. that $\operatorname{rank} F$ be constant on it), the following property can, in contrast, be though of as an "infinitesimal property."

Definition and Notation 11.4.1. Let $F : M \to N$ be a smooth map between promanifolds and suppose that $(M, \mu_\bullet)$ and $(N, \nu_\bullet)$ are the limits of the profinite systems Sys_M and Sys_N, respectively, and fix $m \in M$. Let $\operatorname{RankInd}_{Reg}(m, F, \mu_\bullet, \nu_\bullet)$ denote all those $a \in \mathbb{N}$ for which there exists some index $i \in \mathbb{N}$ such that

Const1: $\mu_i(m) \in \operatorname{ODom}_i(\nu_a \circ F)$, and

Const2: $\mu_i(m)$ is a constant rank point of $F_i^a : \operatorname{ODom}_i(\nu_a \circ F) \to N_a$.

Say that m is a *(cofinally) constant rank* or *(cofinally) rank regular* (resp. *singular*) *point (of F) (with respect to $\mu_\bullet$ and $\nu_\bullet$)* if $\operatorname{RankInd}_{Reg}(m, F, \mu_\bullet, \nu_\bullet)$ is (resp. is not) cofinal in $\mathbb{N}$ and denote the set of such points by $\operatorname{Rank}_{Reg}(F, \mu_\bullet, \nu_\bullet)$ (resp. $\operatorname{Rank}_{Sing}(F, \mu_\bullet, \nu_\bullet)$).

If $\operatorname{RankInd}_{Reg}(m, F, \mu_\bullet, \nu_\bullet)$ contains all but finitely many positive integers then we will say that m is an *eventually constant rank (or rank regular) point (of F) (with respect to $\mu_\bullet$ and $\nu_\bullet$)* or that F *eventually has constant rank at m*, and we will denote the set of such points by $\operatorname{EvRank}_{Reg}(F, \mu_\bullet, \nu_\bullet)$. Any of F, $\mu_\bullet$, or $\nu_\bullet$ may be omitted from this notation if they are clear from context. ∎

Remark 11.4.2. If M and N are finite-dimensional promanifolds then it is clear that m is a constant rank point of F (as defined in 11.3.1) if and only if $m \in \operatorname{Rank}_{Reg}(F, \mu_\bullet, \nu_\bullet)$ so that definitions 11.3.1 and 11.4.1 are consistent. In fact, in this case it is easy to see that $\operatorname{RankInd}_{Reg}(m, F, \mu_\bullet, \nu_\bullet)$ will contain all but finitely many integers.

Lemma 11.4.3. Let $F : M \to N$ be any smooth map between promanifolds and let $(P, \pi_\alpha) = \varprojlim(P_\alpha, \pi_{\alpha\beta}, \mathbb{N})$ be a promanifold.

(1) If $a \in \operatorname{RankInd}_{Reg}(m, F)$ and $i \in \mathbb{N}$ is an index for which Const1 and Const2 are satisfied for a then each $j \geq i$ also satisfies Const1 and Const2 for a.

(2) If $E : (P, C^\infty_P) \to (M, C^\infty_M)$ is a diffeomorphism then

$$\operatorname{RankInd}_{Reg}(m, F, \mu_\bullet, \nu_\bullet) = \operatorname{RankInd}_{Reg}\left(E^{-1}(m), F \circ E, \pi_\bullet, \nu_\bullet\right)$$

In particular, m is a cofinally (resp eventually) constant rank point of F with respect to $\mu_\bullet$ and $\nu_\bullet$ $\iff$ $E^{-1}(m)$ is a cofinally (resp eventually) constant rank point of $F \circ E : P \to N$ with respect to $\pi_\bullet$ and $\nu_\bullet$.

Proof. (1): Since $F^a_j = F^a_i \circ \mu_{ij}$ on $\mu^{-1}_{ij}(\operatorname{ODom}_i F^a)$ and μ_{ij} is a submersion it is immediate that if F^a_i has constant rank on U_i then F^a_j has constant rank on $U_j \underset{\text{def}}{=} \mu^{-1}_{ij}(U_i)$ where $\mu_j(m) \in U_j \subseteq \operatorname{ODom}_j F^a$.

(2): Let $p = E^{-1}(m)$ and $G = F \circ E$ and observe that it suffices to prove only $\subseteq$ since then we may apply this result with E^{-1} and G in place of E and F to obtain the reverse containment. So assume that $A \underset{\text{def}}{=} \operatorname{RankInd}_{Reg}(m, F, \mu_\bullet, \nu_\bullet)$ is infinite, let $a \in A$, pick an index i such that Const1 and Const2 are satisfied for a and let $\mu_i(m) \in U_i \in \operatorname{Open}(M_i)$ be an open set such that F_i^a has constant rank on U_i. Pick an index j such that $\pi_j(p) \in \operatorname{ODom}_j E^i$, where $E^i \underset{\text{def}}{=} \mu_i \circ E : P \to M_i$ and let $O_j = \operatorname{ODom}_j E^i \cap \left(E_j^i\right)^{-1}(U_i)$ where observe that $\mu_i(m) = (\mu_i \circ E)(p) = E^i(p) = E_j^i(\pi_j(p)) \in \operatorname{Im} E_j^i$ so that $\pi_j(p) \in O_j$. Since E is a diffeomorphism the map $E_j^i : \operatorname{ODom}_j E^i \to M_i$ has full rank everywhere on its domain so that $F_i^a \circ E_j^i : O_j \to N_a$ has constant rank on O_j. Thus $a \in \operatorname{RankInd}_{Reg}(p, G, \pi_\bullet, \nu_\bullet)$, as desired. ■

Notation 11.4.4. Lemma 11.4.3 shows that whether or not m is a cofinally (resp eventually) constant rank point of F with respect to $\mu_\bullet$ and $\nu_\bullet$ is actually independent of $\mu_\bullet$ so the notation $\operatorname{Rank}_{Reg}(F, \nu_\bullet)$, $\operatorname{EvRank}_{Reg}(F, \nu_\bullet)$, and $\operatorname{RankInd}_{Reg}(m, F, \nu_\bullet)$, which we will henceforth use, is unambiguous. Furthermore, if N is a manifold or if Sys_N is understood then we may even write $\operatorname{Rank}_{Reg} F$ (resp. $\operatorname{EvRank}_{Reg} F$) in place of $\operatorname{Rank}_{Reg}(F, \nu_\bullet)$ (resp. $\operatorname{EvRank}_{Reg}(F, \nu_\bullet)$.).

Remark 11.4.5. If the manifold N is the limit of some smooth system $\nu_\bullet$ that for whatever reason is not the constant system $\operatorname{ConstSys}_N$, whose limit is $(N, (\operatorname{Id}_N)_{i \in \mathbb{N}})$, then it is easy to see that $\operatorname{RankInd}_{Reg}(F, \nu_\bullet) = \operatorname{RankInd}_{Reg}(F, (\operatorname{Id}_N)_{i \in \mathbb{N}})$.

Definition 11.4.6. For any smooth map $F : M \to N$ and any $m \in M$, say that

- F *cofinally (resp. eventually) has $\nu_\bullet$-constant rank at m* and that *m is a cofinally (resp. eventually) $\nu_\bullet$-constant rank point of F* if $m \in \operatorname{Rank}_{Reg}(F, \nu_\bullet)$ (resp. $m \in \operatorname{EvRank}_{Reg}(F, \nu_\bullet)$).

- F *is $\nu_\bullet$-regular at m* and that *m is a $\nu_\bullet$-regular point of F* if F cofinally has $\nu_\bullet$-constant rank at m.

- F is a *$\nu_\bullet$-regular immersion at m* if F is a smooth pointwise immersion at m and m is a $\nu_\bullet$-regular point of F.

If we don't mention a point m then we mean that it this is true of F at every $m \in M$.

Example 11.4.7. If $F : M \to N$ is a smooth pointwise submersion at $m \in M$ then F eventually has $\nu_\bullet$-constant rank at m.

Lemma 11.4.8. Let $F : M \to N$ be any smooth map between promanifolds.

(1) If N is a manifold then $\operatorname{Rank}_{Reg} F$ is a dense open subset of M.

(2) For each $a \in \mathbb{N}$, $\bigcup_{i \in \mathbb{N}} \mu_i^{-1}(\operatorname{Rank}_{Reg} F_i^a)$ is a dense open subset of M and

$$\operatorname{Rank}_{Reg}(F, \nu_\bullet) = \bigcap_{a \in \mathbb{N}} \bigcup_{\substack{b \geq a, \\ i \in \mathbb{N}}} \mu_i^{-1}(\operatorname{Rank}_{Reg} F_i^a)$$

In particular, $\operatorname{Rank}_{Reg}(F, \nu_\bullet)$ is a dense comeager G_δ subset of M.

Proof. (1): Assume N is a manifold and let S_i denote the rank singular points of $F_i : \operatorname{ODom}_i F \to N$. By part (1) of lemma 11.4.3 we have $i \leq j$ implies $\mu_{ij}^{-1}(S_i) \subseteq S_j \cap \mu_i j^{-1}(\operatorname{ODom}_i F)$ where recall ([31, p. 3]) that S_i^a is a closed nowhere dense subset of $\operatorname{ODom}_i F^a$. Let i be any index and let $O^i \underset{\text{def}}{=} \mu_i^{-1}(\operatorname{ODom}_i F)$. Then $O^i \cap \bigcap_{j \geq i} \mu_j^{-1}(S_j)$ is a decreasing sequence of closed (in O^i) and nowhere dense sets and hence closed in O^i. Furthermore, it is clear that an element in M belongs to $O^i \cap \operatorname{Rank}_{Sing}(F, \operatorname{Sys}_N)$ if and only if it belongs to $O^i \cap \bigcap_{j \geq i} \mu_j^{-1}(S_j)$. Hence, each $O^i \cap \operatorname{Rank}_{Sing}(F, \nu_\bullet)$ is closed and nowhere dense in O^i and since $(O^i)_{i \in \mathbb{N}}$ forms an open cover of M it follows that $\operatorname{Rank}_{Sing}(F, \nu_\bullet)$ is closed and nowhere dense in M.

(2): Let $R = \operatorname{Rank}_{Reg}(F, \nu_\bullet)$ and for all $(i, a) \in \mathbb{N} \times \mathbb{N}$ let R_i^a denote the set of all constant rank points of F_i^a. Note that R_i^a is a dense open subset of $\operatorname{ODom}_i F^a$ and hence open in M_i. For all $a \in \mathbb{N}$ let $R^a = \bigcup_{b \geq a} \bigcup_{i \in \mathbb{N}} \mu_i^{-1}(R_i^b)$ and observe that $\bigcup_{i \in \mathbb{N}} \mu_i^{-1}(R_i^b)$, and hence R^a, is dense in M since for every $m \in M$ there exists some index i such that $\mu_j(m) \in \operatorname{ODom}_j F^a$ for all $j \geq i$ so that any basic open subset of M containing m must intersect some $\mu_i^{-1}(R_i^a)$. Note that an element $m \in M$ fails to belong to R if and only if there exists some index $a(m)$ such that for all $b \geq a(m)$ and all $i \in \mathbb{N}$, $\mu_i(m) \notin R_i^b$ or equivalently $m \notin \bigcup_{b \geq a(m)} \bigcup_{i \in \mathbb{N}} \mu_i^{-1}(R_i^b) = R^{a(m)}$.

Thus $M \smallsetminus R = \bigcup_{a\in\mathbb{N}} M \smallsetminus R^a$ so $R = \bigcap_{a\in\mathbb{N}} R^a$. Since M is a Baire space and each R^a is a dense open subset of M it follows that R is dense and comeager in M. ■

$\nu_\bullet$-Regular Immersions

The Canonical Form at a Point of a $\nu_\bullet$-Regular Immersion

Proposition 11.5.1. Let $F : (M, m^0) \to (N, n^0)$ be a smooth map, $m^0_\bullet = \mu_\bullet(m^0)$, i be an index, $d_\bullet = \dim_{m^0_\bullet} M_\bullet$ and let $X \le \mathrm{T}_{m^0} M$ be any vector space such that $\mathrm{T}_{m^0} \mu_i\big|_X : X \to \mathrm{T}_{m^0_i} M_i$ is an isomorphism. Suppose a and $j \ge i$ are such that $\mu_j(m^0) \in \mathrm{ODom}_j F^a$ is a regular point of $F^a_j : \mathrm{ODom}_j F^a \to N_a$ of rank r, and $\mathrm{T}_{m^0} F^a\big|_X : X \to \mathrm{T}_{\nu_a(n^0)} N_a$ is injective. Given any chart (U_i, φ_i) centered at $\mu_i(m^0)$ there exist charts (U_j, φ_j) and (V_a, ψ_a) centered respectively at $\mu_j(m^0)$ and $\nu_a(n^0)$, such that

(1) $\mu_{ij}\big(\overline{U_j}\big) \subseteq U_i$, $F^a_j(U_j) \subseteq V_a$, $\overline{U_j} \subseteq \mathrm{ODom}_j F^a$, and both $\overline{U^j}$ and $\overline{V_a}$ are compact.

(2) F^a_j has constant rank $r \ge d_i$ on a neighborhood of $\overline{U_j}$.

(3) The coordinate representations of μ_{ij} and F^a_j, denoted by $\widehat{\mu_{ij}}$ and $\widehat{F^a_j}$, have the following form, which we will call the *canonical* or *standard representation*:

$$\begin{array}{ccc} (p_1, \ldots, p_{d_i}, \ldots, p_r \ldots, p_{d_j}) & \xrightarrow{\widehat{F^a_j}} & (p_1, \ldots, p_{d_i}, \ldots, p_r, 0, \ldots, 0) \\ \Big\downarrow{\scriptstyle \widehat{\mu_{ij}}} & & \\ (p_1, \ldots, p_{d_i}) & & \end{array}$$

Figure 11.1: The canonical coordinate representation.

That is, $\varphi_j^{\le r} = \psi_a^{\le r} \circ F^a_j\big|_{U_j}$ and $\varphi_j^{\le d_i} = \varphi_i \circ \mu_{ij}\big|_{U_j}$, where for any $l \in \mathbb{N}$, $\varphi_j^{\le l}$ (resp. $\psi_a^{\le l}$) represent the first l coordinates of φ_j (resp. ψ_a).

(4) Both (U_j, φ_j) and (V_a, ψ_a) are coordinate boxes where φ_j and ψ_a may be extended to charts around $\overline{U_j}$ and $\overline{V_a}$, respectively, which again have all of the properties mentioned in this proposition.

(5) Whenever a point $\psi_a^{-1}(p_1, \ldots, p_r, \ldots, p_{\dim_{\nu_a(n^0)} N_a})$ belongs to V_a then the point $\psi_a^{-1}(p_1, \ldots, p_r, 0, \ldots, 0)$ belongs to $F_j^a(U_j)$.

Furthermore, if F is a $\nu_\bullet$-regular immersion at m^0 then there are cofinally many indices a and $j \geq i$ that satisfy the above hypotheses and if $\widetilde{U}_j$ and $\widetilde{V}_a$ are any given neighborhoods of $\mu_j(m^0)$ and $\nu_a(n^0)$, respectively, then U_j and V_a can be chosen so that in addition $\overline{U_j} \subseteq \widetilde{U}_j$ and $\overline{V_a} \subseteq \widetilde{V}_a$.

Proof. That such an index a can be found follows from corollary 7.6.3. If F is a $\nu_\bullet$-regular immersion at m^0 then the existence of cofinally many indices a and $j \geq i$ that satisfy the above hypotheses is the very definition of what it means for F to be $\nu_\bullet$-regular at m^0. Let $n_\bullet^0 = \nu_\bullet(n^0)$ and $X_\bullet = \mathrm{T}_{m_\bullet^0}(X)$. Since $F^a = F_i^a \circ \mu_j$ our hypotheses imply that $\mathrm{T}_{m_j^0} F^a\big|_{X_j} : X_j \to \mathrm{T}_{n_a^0} N_a$ is injective, $\mathrm{T}_{m_j^0} \mu_{ij}\big|_{X_j} : X_j \to \mathrm{T}_{m_i^0} M_i$ is bijective, and $d \leq r$, so that the result is now just the statement of lemma C.2.1. ∎

Smoothness of Maps Composed on the Right

Theorem 11.5.2. Let $F : (M, m^0) \to (N, n^0)$ be a smooth map, let $p^0 \in P$ where $(P, \rho_\bullet) = \varprojlim(P_\alpha, \rho_{\alpha\beta}, \mathbb{N})$ is a promanifold, and let $G : (P, p^0) \to (M, m^0)$ be any map that is continuous at p^0. If F is a smooth $\nu_\bullet$-regular immersion at $G(p^0)$ then $G : P \to M$ is smooth at p^0 (def. 6.1.14) $\iff$ $F \circ G : P \to N$ is smooth at p^0.

Proof. For the non-trivial direction, assume that $H \underset{\text{def}}{=} F \circ G$ is smooth at p^0 and let $m_\bullet^0 = \mu_\bullet(m)$, $n_\bullet^0 = \nu_\bullet(n^0)$, and $d_\bullet = \dim_{m_\bullet^0} M_\bullet$. Let $U = \mu_i^{-1}(U_i)$ be an arbitrary basic open neighborhood of m^0. Pick indices a and $j \geq i$ and coordinate charts U_i, U_j, and V_a centered, respectively, at m_i^0, m_j^0, and n_j^0 such that $U_j \subseteq \mu_{ij}^{-1}(U_i)$ and the coordinate representations of μ_{ij} and F_j^a are

$$\begin{array}{ccc}(p_1,\ldots,p_{d_i},\ldots,p_{d_j}) & \xrightarrow{\widehat{F_j^a}} & (p_1,\ldots,p_{d_i},\ldots,p_r,0,\ldots,0)\\ \Big\downarrow{\scriptstyle\widehat{\mu_{ij}}} & & \\ (p_1,\ldots,p_{d_i}) & & \end{array}$$

where r is the rank of F_j^a. Since G is continuous at p^0 and $H^a \underset{\text{def}}{=} \nu_a \circ H$ is smooth at p^0 there exist $\alpha \in \mathbb{N}$ and $W_\alpha \in \operatorname{Open}(\operatorname{ODom}_\alpha H^a)$ such that $p^0 \in \rho_\alpha^{-1}(W_\alpha) \subseteq (\mu_j \circ G)^{-1}(U_j)$ and $H_\alpha^a\big|_{W_\alpha} : W_\alpha \to V_a$ is smooth. Let $W = \rho_\alpha^{-1}(W_\alpha)$ so $p^0 \in W \in \operatorname{Open}(P)$. Given any $p_\alpha \in W_\alpha$, H^a maps the fiber $Q = \rho_\alpha^{-1}(p_\alpha)$ to the point in V_a whose coordinates are $(H_1^a(p_\alpha),\ldots,H_r^a(p_\alpha),0,\ldots,0)$ so that G^i maps every point of Q to the (single) point in U_i whose coordinates are $(H_1^a(p_\alpha),\ldots,H_i^a(p_\alpha))$. Thus $p_\alpha \in \operatorname{Dom}_\alpha G^i$ where since $p_\alpha \in W_\alpha$ was arbitrary we have $W_\alpha \subseteq \operatorname{ODom}_\alpha G^i$. In particular, we've shown that G^i is locally cylindrical at p^0 and for all $p_\alpha \in W_\alpha$, $G_\alpha^i(p_\alpha)$ has coordinate representation $(H_1^a(p_\alpha),\ldots,H_i^a(p_\alpha))$. But each $H_1^a,\ldots,H_i^a$ is smooth on W_α so that $G_\alpha^i\big|_{W_\alpha}$ is smooth. Thus G^i is smooth at p^0 where since i was arbitrary we have shown that G is smooth at p^0. ■

Corollary 11.5.3. Continuous lifts of smooth maps by $\nu_\bullet$-regular immersions are smooth: Suppose that $F : M \to N$ is a smooth $\nu_\bullet$-regular immersion and $G : P \to M$ is any continuous map on the promanifold P. Then $G : P \to M$ is smooth if and only if $F \circ G : P \to N$ is smooth.

Using Composition to Determine Equality of Continuous Maps

Theorem 11.5.4. Let $F : M \to N$ be a smooth $\nu_\bullet$-regular immersion and let $\gamma, \eta : Z \to M$ be continuous maps from a non-empty connected space Z. Then

$$\gamma = \eta \iff F \circ \gamma = F \circ \eta \text{ and } \gamma(z) = \eta(z) \text{ for some } z \in Z.$$

Proof. For the non-trivial direction, assume that $F \circ \gamma = F \circ \eta$ and that $\{z \in Z \,|\, \gamma(z) = \eta(z)\}$ is not empty. Fix an index i, let $d = \dim M_i$, and let $\gamma_i = \mu_i \circ \gamma$, $\eta_i \underset{\text{def}}{=} \mu_i \circ \eta : Z \to M_i$. Observe that since i was arbitrary it is necessary and sufficient to show that $\gamma_i = \eta_i$ where since $E_i \underset{\text{def}}{=} \{z \in Z \,|\, \gamma_i(z) = \eta_i(z)\}$ is non-empty and closed (since M is Hausdorff), it suffices to show that E_i is open in Z. Let $z^0 \in E_i$, $m^0 = \gamma(z^0)$, $n^0 = F(m^0)$, and let $m_j^0 = \mu_j(m^0)$ and $n_a^0 = \nu_a(m^0)$ for all a and j. Let $j \geq i$ and a be indices for which there exist charts U_i, U_j, and V_a centered at m_i^0, m_j^0, and n_a^0, respectively, for which the coordinate representations of μ_{ij} and F_j^a are

$$\begin{tikzcd}
(p_1, \ldots, p_d, \ldots, p_{\dim M_j}) \arrow[r, "\widehat{F_j^a}"] \arrow[d, "\widehat{\mu_{ij}}"'] & (p_1, \ldots, p_d, \ldots, p_r, 0, \ldots, 0) \\
(p_1, \ldots, p_d) &
\end{tikzcd}$$

where r is the rank of F_j^a and $U_j \subseteq \mu_{ij}^{-1}(U_i) \cap \operatorname{ODom}_j F^a$.Let $W = \gamma_j^{-1}(U_j) \cap \eta_j^{-1}(U_j)$ so that $z^0 \in W \in \operatorname{Open}(Z)$. Observe that for any $z \in W$, from the equality $(F^a \circ \gamma)(z) = (F^a \circ \eta)(z)$ and the above diagram we can conclude that $\gamma_i(z) = \eta_i(z)$ so that $z \in E_i$. Thus $W \subseteq E_i$ which shows that E_i is open in Z, as desired. ∎

Corollary 11.5.5. Let $F : M \to N$ be a smooth $\nu_\bullet$-regular immersion and let $\eta : Z \to N$ be continuous maps. For $l = 1, 2$, let $\gamma^l : Z^l \to M$ be a map from a subset Z^l of Z such that $F \circ \gamma = \eta\big|_{Z^l}$. If C is a connected subset of $Z^1 \cup Z^2$ on which such that $\gamma^1\big|_C$ and $\gamma^2\big|_C$ are continuous and there is some $w \in C$ such that $\gamma^1(w) = \gamma^2(w)$ and then $\gamma^1 = \gamma^2$ on C.

Components of Fibers of $\nu_\bullet$-Regular Immersions

Proposition 11.5.6. If $F : (M, m^0) \to (N, n^0)$ is a $\nu_\bullet$-regular immersion at m^0 then the connected component of the fiber $F^{-1}(F(m^0))$ containing m^0 is a singleton set. Furthermore, for each index i there exist cofinally many indices $j \in \mathbb{N}^{\geq i}$ and $a \in \mathbb{N}$ for which there exist open sets $\mu_j(m^0) \in U_j \in \operatorname{Open}(\operatorname{ODom}_j F^a)$ and $\nu_a(n^0) \in V_a \in \operatorname{Open}(N_a)$ such that if C is any connected component of $F^{-1}(\nu_a^{-1}(V_a))$ that intersects $\mu_j^{-1}(U_j)$ then $\mu_i(C)$ is a point i.e. $C \subseteq \mu_i^{-1}(\mu_i(c))$, where $c \in C$ is arbitrary.

Proof. Let C^0 denote the connected component of $F^{-1}(n^0)$ containing m^0. Let i be an arbitrary index, $d \underset{\text{def}}{=} \dim M_i$, and let (U_i, φ_i) be any chart centered at $\mu_i(m^0)$. By proposition

11.5.1, we can find indices a and $j \geq i$ and charts (U_j, φ_j) and (V_a, ψ_a) centered respectively at $\mu_j(m^0)$ and $\nu_a(n^0)$, such that $\mu_j(m^0) \in \overline{U_j} \subseteq \operatorname{ODom}_j F^a$, F_j^a has constant rank r on a neighborhood of $\overline{U_j}$, $\mu_{ij}(U_j) \subseteq U_i$, $F_j^a(U_j) \subseteq V_a$, and where the coordinate representations of μ_{ij} and F_j^a has the form:

$$\begin{tikzcd}
(p_1, \ldots, p_d, \ldots, p_{\dim M_j}) \arrow[r, "\widehat{F_j^a}"] \arrow[d, "\widehat{\mu_{ij}}"'] & (p_1, \ldots, p_d, \ldots, p_r, 0, \ldots, 0) \\
(p_1, \ldots, p_d) &
\end{tikzcd}$$

Observe that $\nu_a(n^0)$ belongs to V_a so it has a coordinate representation, say $\left(n_1^0, \ldots, n_{\dim N_a}^0\right)$, but since $\nu_a(n^0) = F_j^a(m^0)$ it follows that the last $\dim N_a - r$ of these coordinates 0. Let $e = \dim M_j$. If $m \in \mu_j^{-1}(U_j)$ is any point that belong to $F^{-1}(n^0)$ and if $(m_1, \ldots, m_e)$ is the coordinate representation of $\mu_j(m)$ then since $F_j^a(\mu_j(m)) = \nu_a(F(m)) = \nu_a(n^0)$ it follows from the coordinate representation that we necessarily have $m_1 = n_1^0, \ldots, m_r = n_r^0$ so that since $r \geq d$, it follows that $\mu_i(m)$ will necessarily have the coordinate representation $(n_1^0, \ldots, n_r^0)$. In particular, $\mu_j(m^0)$ has the coordinate representation $(n_1^0, \ldots, n_r^0, m_{r+1}^0, \ldots, m_e^0)$ for some $m_{r+1}^0, \ldots, m_e^0 \in \mathbb{R}$ and $\mu_i(m^0) = (n_1^0, \ldots, n_r^0)$. Thus $\mu_j^{-1}(U_j) \cap F^{-1}(n^0) \subseteq \mu_i^{-1}(\mu_i(m^0))$.

Now let C denote any connected component of $F^{-1}(n^0)$ that intersects $\mu_j^{-1}(U_j)$ and suppose that C was not contained in $\mu_i^{-1}(\mu_i(m^0))$. Then $\mu_j(C)$ is a connected set not contained in $\mu_{ij}^{-1}(m_i^0)$ so there exists some $m \in C$ such that $\mu_j(m) \in U_j \setminus \mu_{ij}^{-1}(m_i^0)$. But since $m \in F^{-1}(n^0)$ this implies that $m \in \mu_j^{-1}(U_j) \cap F^{-1}(n^0) \subseteq \mu_i^{-1}(\mu_i(m^0))$ so $\mu_i(m) = \mu_i(m^0)$, a contradiction. Thus $C \subseteq \mu_i^{-1}(\mu_i(m^0))$ so in particular $C^0 \subseteq \mu_i^{-1}(\mu_i(m^0))$. Observe that if $n \in \nu_a^{-1}(V_a)$ and C is any connected component of $F^{-1}(n)$ that intersects $\mu_j^{-1}(U_j)$ then this same argument, with n^0 and m^0 replaced respectively with n and any element of $C \cap \mu_j^{-1}(U_j)$, would work to show that $\mu_i(C)$ is a singleton set. Finally, since i was arbitrary it follows that $C^0 \subseteq \bigcap_{i \in \mathbb{N}} \mu_i^{-1}(\mu_i(m^0)) = \{m^0\}$. ■

Corollary 11.5.7. Every fiber of a smooth $\nu_\bullet$-regular immersion is totally disconnected (def. A.0.5), i.e. such maps are light (def. A.0.6(b)). In particular, each fiber of a smooth pointwise isomersion is totally disconnected.

Note that corollary 11.5.7 may also be proven more directly by using proposition 11.5.1 to show that for all $m^0 \in M$, if C is the connected component of $F^{-1}(F(m^0))$ containing m^0 then for all $i \in \mathbb{N}$, $\mu_i^{-1}(\mu_i(m^0)) \cap C$ is a closed and open subset of C.

Corollary 11.5.8. A smooth $\nu_\bullet$-regular immersion is injective if and only if it is monotone. In particular, a smooth pointwise isomersion is injective if and only if it is monotone.

Continuity of Maps Composed on the Right

The results of this subsection will not be used anywhere else in this paper.

Theorem 11.5.9. Let $F : M \to N$ be a smooth $\nu_\bullet$-regular immersion, Z be any space, $\gamma : Z \to M$ be *any* map, and for all $i \in \mathbb{N}$, let C^i (resp. D^i, C) be the set of all $z \in Z$ such that $\mu_i \circ \gamma : Z \to M_i$ (resp. $\nu_i \circ F \circ \gamma : Z \to N_i$, $\gamma : Z \to M$) is continuous at z. If every $D^\bullet$ is a neighborhood of C in Z then the same is true of every $C^\bullet$ and $C = \cap C^\bullet$ is a G_δ-subset of Z.

Proof. That $C = \cap C^\bullet$ is always true and obvious. Let $\eta = F \circ \gamma$, $\gamma_\bullet = \mu_\bullet \circ \gamma$, $\eta_\bullet = \nu_\bullet \circ \eta$. Let $i \in \mathbb{N}$, $z^0 \in C$, $m^0 = \gamma(z)$, $n^0 = F(m)$, $m_\bullet^0 = \mu_\bullet(m)$, $n_\bullet^0 = \nu_\bullet(n^0)$, and $d_\bullet = \dim_{m_\bullet^0} M_\bullet$. Pick integers a and $j \geq i$ and coordinate charts (U_i, φ_i), (U_j, φ_j), and (V_a, ψ_a) as in proposition 11.5.1 so that in particular, the coordinate representations of μ_{ij} and F_j^a are

$$\begin{tikzcd}
(p_1, \ldots, p_{d_i}, \ldots, p_{d_j}) \arrow[r, "\widehat{F_j^a}"] \arrow[d, "\widehat{\mu_{ij}}"'] & (p_1, \ldots, p_{d_i}, \ldots, p_r, 0, \ldots, 0) \\
(p_1, \ldots, p_{d_i}) &
\end{tikzcd}$$

where r is the rank of F_j^a. Since D^a is a neighborhood of z in Z and γ_j is continuous at z, there exists some $z \in W \in \operatorname{Open}(Z)$ such that $W \subseteq D^a \cap \gamma_j^{-1}(U_j)$. The canonical coordinate representation now implies that γ_i is continuous at every point of W: for let $z_\bullet = (z_\alpha)_{\alpha \in A} \to z$ be any convergent net in W (with $z \in W$) and note that $F_j^a(\gamma_j(z_\bullet)) = \eta_a(z_\bullet) \to \psi_a(\eta_a(z))$

since η_a is continuous on W. Since $\psi_a(F_j^a(\gamma_j(z_\bullet))) \to \psi_a(\eta_a(z))$, regardless of whether or not the last $(d_j - r)$-coordinates of $(\varphi_j(\gamma_j(z_b))_{b \in A}$ converge, the commutativity of the above diagram implies that $\varphi_i(\gamma_i(z_\bullet)) \to \varphi_i(\gamma_i(z))$. ■

Corollary 11.5.10 (Rough lifts of continuous maps are continuous on G_δ-subsets)**.** Let $F : M \to N$ be a smooth $\nu_\bullet$-regular immersion and let $\gamma : Z \to M$ be *any* map from any space Z such that $F \circ \gamma : Z \to N$ is continuous. Then the set of all points at which $\gamma : Z \to M$ is continuous is a G_δ-subset of Z.

Corollary 11.5.11 (Inverses of injective $\nu_\bullet$-regular immersions are continuous on G_δ-subsets)**.** If $F : M \to N$ is an injective smooth $\nu_\bullet$-regular immersion then the set of points at which $F^{-1} : \operatorname{Im} F \to M$ is continuous is a G_δ-subset of $\operatorname{Im} F$.

Smooth Embeddings of Promanifolds

Theorem 11.6.1. A smooth $\nu_\bullet$-regular (def. 11.4.6) map $F : M \to N$ is a smooth embedding if and only if it is a topological embedding and a pointwise immersion.

Proof. One direction follows from example 11.1.1 so assume that F is a topological embedding and a pointwise immersion. Let $U \in \operatorname{Open}(M)$, $F^\bullet = \nu_\bullet \circ F$, $f \in C_M^\infty(U)$, and $n^0 \in F(U)$ be arbitrary. Define $S \underset{\text{def}}{=} \operatorname{Im} F$, make S into a locally $\mathbb{R}$-ringed space by giving it the restriction $\left(S, C_N^\infty\big|_S\right)$, let F^{-1} denote the homeomorphism $F^{-1} : S \to M$, let $m^0 \underset{\text{def}}{=} F^{-1}(n^0)$, and for all indices i and a let $m_i^0 \underset{\text{def}}{=} \mu_i(m^0)$ and $n_a^0 \underset{\text{def}}{=} \nu_a(n^0)$. Pick an index i_0, open coordinate box $U_{i_0} \in \operatorname{Open}(M_{i_0})$, and a map $f_{i_0} \in C_{M_{i_0}}^\infty(U_{i_0})$ such that $m^0 \in \mu_{i_0}^{-1}(U_{i_0}) \in \operatorname{Open}(U)$ and $f = f_{i_0} \circ \mu_{i_0}$ on $\mu_{i_0}^{-1}(U_{i_0})$. Define $d = \dim M_{i_0}$ and apply proposition 11.5.1 to obtain indices $i \geq i_0$ and a and open coordinates cubes (U_i, φ_i) and (V_a, ψ_a) such that $\mu_{i_0,j}(U_j) \subseteq U_{i_0}$, $F_i^a(U_i) \subseteq V_a$, where the coordinate representations of $\mu_{i_0,j}$ and F_j^a, denoted by $\widehat{\mu_{ij}}$ and $\widehat{F_j^a}$, have the following form:

$$\begin{array}{ccc} (p_1,\ldots,p_d,\ldots,p_{\dim M_i}) & \xrightarrow{\widehat{F_i^a}} & (p_1,\ldots,p_d,\ldots,p_r,0,\ldots,0) \\ \Big\downarrow{\scriptstyle \widehat{\mu_{i_0,i}}} & & \\ (p_1,\ldots,p_d) & & \end{array}$$

where $\overline{U_i} \subseteq \operatorname{ODom}_i F^a$ and where whenever a point $(p_1,\ldots,p_r,\ldots,p_{\dim N_a})$ belongs to V_a then the point $(p_1,\ldots,p_d,\ldots,p_r,0,\ldots,0)$ is the preimage under F_j^a of some point in U_j. Since $\mu_{i_0,i}$ is an open map we may replace U_{i_0} with the coordinate box $\mu_{i_0,i}(U_i)$. It is necessary and sufficient to show that there exists some index a, some open set $\nu_a(n^0) \in V_a \in \operatorname{Open}(N_a)$, and some $g_a \in C^\infty_{N_a}(V_a)$ such that $\nu_a^{-1}(V_a) \cap S \subseteq F(U)$ and $f \circ F^{-1}\big|_{V \cap S} = g_a \circ \nu_a$ on $\nu_a^{-1}(V_a) \cap S$. Observe that it suffices to prove the above claim with some open set $U' \in \operatorname{Open}(M)$ such that $m^0 \in U' \subseteq U$ in place of U. Hence we may assume without loss of generality that $U = \mu_{i_0}^{-1}(U_{i_0})$. Let $g_a : V_a \to \mathbb{R}$ be defined in coordinates by

$$g_a(p_1,\ldots,p_d,\ldots,p_e) \underset{\text{def}}{=} f_i(p_1,\ldots,p_d)$$

where this is well-defined since every point in V_a is the preimage of a some point in M_j. Since $F(\mu_i^{-1}(U_i))$ is open in S with $n^0 \in F(\mu_i^{-1}(U_i)) = F(U)$ we can pick an index $b \geq a$ and an open set $V_b \subseteq \nu_{ab}^{-1}(V_a)$ such that $n^0 \in S \cap \nu_b^{-1}(V_b) \subseteq F(U)$. Since $W_a \subseteq \nu_{ab}(V_b)$ we can define $g_b \underset{\text{def}}{=} g_a \circ \nu_{ab}\big|_{V_b} : V_b \to \mathbb{R}$.

Let $n \in S \cap \nu_b^{-1}(V_b)$ and note that if we can show that $f \circ F^{-1}(n) = (g_b \circ \nu_b)(n)$ then (as described above) the proof will be complete. Let $m \underset{\text{def}}{=} F^{-1}(n)$ so that in particular $m \in U = \mu_i^{-1}(U_i)$. Since $\mu_i(m) \in U_i$ we can express $\mu_i(m)$ in terms of coordinates, say $\mu_i(m)$ is $(p_1,\ldots,p_{\dim M_i})$, which implies that $\mu_i(m)$ has $(p_1,\ldots,p_d)$ as its coordinate representation. Pick $j \geq i$ such that $\mu_j(m) \in \operatorname{ODom}_j F^b$ and observe that

$$\nu_a(n) = \nu_{ab}(\nu_b(n)) = \nu_{ab}(F(m)) = \nu_{ab}(F_j^b(\mu_j(m))) = F_j^a(\mu_j(m)) = F_i^a(\mu_i(m))$$

This together with the coordinate representation of F_i^a implies that $\nu_a(n) = F_i^a(\mu_i(m))$ has

$(p_1, \dots, p_r, 0, \dots, 0)$ as its coordinate representation. Now,

$$\begin{aligned}
(g_b \circ \nu_b)(n) &= (g_a \circ \nu_{ab} \circ \nu_b)(n) \text{ by definition of } g_b \\
&= g_a(\nu_a(n)) \\
&= f_i(p_1, \dots, p_d) \quad \text{ since } \nu_a(n)\text{'s coordinates are } (p_1, \dots, p_d, \dots, p_r, 0, \dots, 0) \\
&= f_i(\mu_i(m)) \quad \quad \text{ since } \mu_i(m)\text{'s coordinates are } (p_1, \dots, p_d) \\
&= f(m) = (f \circ F^{-1})(n)
\end{aligned}$$

which was to be shown. ∎

Corollary 11.6.2. If $F : M \to N$ is a smooth $\nu_\bullet$-regular immersion then F is a smooth embedding if and only if it is a topological embedding.

Theorem 11.6.3. If $F : M \to N$ is a smooth pointwise isomersion then F is a smooth embedding if and only if it is a topological embedding.

Proof. Every smooth embedding is by definition a topological embedding. Conversely, since F is a pointwise isomersion then each F_i^a is a smooth submersion for all indices i and a so that in particular it has $\nu_\bullet$-constant rank at every point of M which allows us to apply theorem 11.6.1. ∎

Remark 11.6.4. Theorem 11.6.3 implies that if $F : M \to N$ is an injective smooth pointwise isomersion then *only* topological reasons can cause F to fail to be a smooth embedding. Consequently, in the search for an inverse function theorem for promanifolds we must find conditions that will make a smooth Pointwise isomersive map both locally injective at a point as well as an open map onto its image.

The Whitney Embedding Theorem for Promanifolds

Theorem 11.6.5 (Whitney Embedding Theorem for Promanifolds)**.** Every promanifold can be properly smoothly embedded into $\mathbb{R}^{\mathbb{N}}$.

Proof. For each index i let $c_i = 2\dim M_i + 1$, $\epsilon_i : M_i \to \mathbb{R}^{c_i}$ be a proper smooth embedding, $e_i \underset{\text{def}}{=} \sum_{l=1}^{i} c_i$, $F_i \underset{\text{def}}{=} \prod_{l=1}^{i} \epsilon_i : \prod_{l=1}^{i} M_l \to \prod_{l=1}^{i} \mathbb{R}^{c_l}$. Observe that $(F_\bullet, \mathrm{Id}_{\mathbb{N}})$ forms a smooth inverse system morphism between the canonical profinite systems (where all connecting maps are the canonical projections) and that $F \underset{\text{def}}{=} \varprojlim F_\bullet : \prod_{l=1}^{\infty} M_l \to \mathbb{R}^{\mathbb{N}}$ is a smooth $\mathrm{Pr}_{\leq\bullet}$-regular immersion that is also a proper topological embedding. It follows from theorem 11.6.1 that $F : \prod_{l=1}^{\infty} M_l \to \mathbb{R}^{\mathbb{N}}$ is a smooth embedding. The conclusion follows by recalling that lemma 5.4.2 showed that (M, C_M^∞) can be smoothly embedded into $\prod_{l=1}^{\infty} M_l$ as a closed subset. ■

Pointwise Submersions

Example 11.7.1. Each projection $\mu_i : M \to M_i$ is a pointwise submersion.

Theorem 11.7.2. Let $F : (M, m) \to (N, n)$ be a smooth map between promanifolds. Then $F : M \to N$ is a pointwise submersion at m if and only if $\nu_a \circ F : M \to N_a$ is a pointwise submersion at m for each index a.

Proof. ($\Longrightarrow$) is immediate. The converse follows by observing that $\mathrm{Im}\, \mathrm{T}_m F$ is dense in $\mathrm{T}_n N$ by proposition 2.2.1(3) and closed in $\mathrm{T}_n N$ by proposition 7.4.1. ■

Remark 11.7.3. If $F_\bullet : \mathrm{Sys}_M \to \mathrm{Sys}_N$ is an inverse system morphism with every F_a a smooth surjective submersion then we although know that the image of $F = \varprojlim F_\bullet$ is dense in N, example 3.4.5 show that it need not be surjective. However, theorem 11.7.2 guarantees that at least the map $\mathrm{T}\, F : \mathrm{T}\, M \to \mathrm{T}(\mathrm{Im}\, F)$ will be surjective.

Lemma 11.7.4. If $F : M \to N$ be a smooth pointwise submersion and N is a manifold then $F : M \to N$ is open.

Proof. Let $W \in \mathrm{Open}(M)$ be an open set and let $m \in W$. Pick an index i and an open set $U_i \in \mathrm{Open}(M_i)$ such that $m \in U \underset{\text{def}}{=} \mu_i^{-1}(U_i) \subseteq W$ and $F = F_i \circ \mu_i$ on U. Since $F_i : U_i \to N$ is a smooth map between manifolds and since F_i has full rank it follows that F_i is a smooth submersion and thus an open map. Since $\mu_i : M \to M_i$ is an open map and $F = F_i \circ \mu_i$ on $U = \mu_i^{-1}(U_i)$ it follows that $F(U)$ is open in N. Thus $F : M \to N$ is an open map. ∎

Corollary 11.7.5. Assume that $F : M \to N$ is a smooth pointwise submersion. If N is a manifold then for any compact $K \subseteq F(M)$ there exists some index i such that $F = F_i \circ \mu_i$ on $U = \mu_i^{-1}(\operatorname{ODom}_i F)$ with $K \subseteq F(U)$ and $F_i : U_i \to N_a$ a smooth submersion.

Proof. For each index i let $U_i \underset{\text{def}}{=} \operatorname{ODom}_i F$ so that $F = F_i \circ \mu_i$ on $U^i \underset{\text{def}}{=} \mu_i^{-1}(U_i)$ and $U^i \subseteq U^{i+1}$. Observe that $F_i(U_i) = F_i(\mu_i(\mu_i^{-1}(U_i))) = F(U_i)$ so that $F_i(U_i) \subseteq F_{i+1}(U_{i+1})$. Since each $F_i : U_i \to N$ is a smooth submersion, the sets $(F_i(U_i))_{i=1}^{\infty}$ form an increasing open cover of K so that by compactness there exists some index i such that $K \subseteq F_i(U_i)$. ∎

Remark 11.7.6. Observe that for this proof it was only necessary to know that F was open, locally cylindrical, and that K was compact.

Example 11.7.7 (A non-open $\nu_\bullet$-surjective smooth pointwise submersion with a dense image that contains no $\nu_\bullet$-fiber and whose interior in its codomain is empty)**.** Let Sys_M, Sys_P, $(F_\bullet, \iota) : \operatorname{Sys}_M \to \operatorname{Sys}_P$, and $F : M \to M_1^{\mathbb{N}}$ be as in example 3.4.5 so that $M_i = \prod_{l=1}^{i} \mathbb{R}^{\geq 0}$ and $\operatorname{Sys}_P \underset{\text{def}}{=} (M_i, \rho_{a,a+1}, \mathbb{N})$ with limit $(M_1^{\mathbb{N}}, \nu_\bullet)$ where $F_a \underset{\text{def}}{=} \rho_{a,a+1} : M_{a+1} \to M_a$ and $\nu_a \underset{\text{def}}{=} \rho_a : M_1^{\mathbb{N}} \to M_a$ are the canonical projections for each $a \in \mathbb{N}$. Observe that F is a smooth pointwise submersion since the same is true of each F_a and that F is $\nu_\bullet$-surjective since each F_a is surjective, which implies that $\operatorname{Im} F$ is dense in $M_1^{\mathbb{N}}$. Recall from example 3.4.5 that for any $r_\bullet = (r_1, r_2, \dots) \in M_1^{\mathbb{N}}$, $r_\bullet \in \operatorname{Im} F \iff \sum_{a=1}^{\infty} r_a$ converges. Suppose that there existed some index a and some $(r_1, \dots, r_a) \in M_1^a$ such that $\nu_a^{-1}(r_1, \dots, r_a) \subseteq \operatorname{Im} F$ Define $s_\bullet = (r_1, \dots, r_a, 1, 1, \dots)$ and note that $\sum_{a=1}^{\infty} s_a = \infty$ implies that $s_\bullet \notin \operatorname{Im} F$, but this contradicts the fact that $s_\bullet \in \nu_a^{-1}(V_a) \subseteq \operatorname{Im} F$ consists solely of convergent series. Thus $\operatorname{Im} F$ contains no $\nu_\bullet$ fiber which makes it impossible for $\operatorname{Im} F$ to contain a non-empty basic open subset of $M_1^{\mathbb{N}}$ so that $\operatorname{Im} F$ has empty interior in $M_1^{\mathbb{N}}$, which in turn implies that $M_1^{\mathbb{N}}$ $F : M \to M_1^{\mathbb{N}}$ is has no points of openness and so cannot be an open map. ∎

Proposition 11.7.8. Let $F : M \to N$ be a smooth pointwise isomersion such that $F : M \to \operatorname{Im} F$ is open. Then for any map $G : \operatorname{Im} F \to P$ into any promanifold P, $G \circ F : M \to P$ is smooth $\iff$ $G : \operatorname{Im} F \to P$ is smooth.

Proof. Assume that $G \circ F : M \to P$ is smooth and note that $G : \operatorname{Im} F \to P$ is continuous. Let $\alpha \in \mathbb{N}$, $W_\alpha \in \operatorname{Open}(P_\alpha)$, $h_\alpha \in C^\infty_{P_\alpha}(W_\alpha)$, $W \underset{\text{def}}{=} \pi_\alpha^{-1}(W_\alpha)$, $V \underset{\text{def}}{=} G^{-1}(W)$, $U \underset{\text{def}}{=} F^{-1}(V)$, and $h \underset{\text{def}}{=} h_\alpha \circ \pi_\alpha\big|_W$. To show that G is smooth, it suffices to show that $h \circ G : W \cap \operatorname{Im} F \to \mathbb{R}$ is smooth so let $n \in W \cap \operatorname{Im} F$ and $n_\bullet = \nu_\bullet(n)$. Pick any $m \in F^{-1}(n)$ and let $m_\bullet = \mu_\bullet(m)$. Since $G \circ F$ is smooth so is $f \underset{\text{def}}{=} h \circ G \circ F : U \to \mathbb{R}$ so there exists some $i \in \mathbb{N}$, $U_i \in \operatorname{Open}(M_i)$, and $f_i \in C^\infty_{M_i}(U_i)$ such that $m \in \mu_i^{-1}(U_i) \subseteq U$ and $f_i \circ \mu_i = h \circ G \circ F$ on $\mu_i^{-1}(U_i)$. Using proposition 11.5.1 and shrinking U_i if necessary, pick indices a and $j \geq i$ and smooth coordinate boxes (U_i, φ_i), (U_j, φ_j), and (V_a, ψ_a) centered respectively at m_i, m_j, and n_a such that $n \in \nu_a^{-1}(V_a) \subseteq V$, $F_j^a(U_j) = V_a$, and $\mu_j^{-1}(U_j) \subseteq (F \circ \nu_a)^{-1}(V_a) \cap \mu_i^{-1}(U_i)$, and the coordinate representations of μ_{ij} and F_i^a are

$$\begin{tikzcd}
(p_1, \ldots, p_d, \ldots, p_{d_j}) \arrow[r, "\widehat{F_j^a}"] \arrow[d, "\widehat{\mu_{ij}}"'] & (p_1, \ldots, p_d, \ldots, p_{r_a}) \\
(p_1, \ldots, p_d) &
\end{tikzcd}$$

where $r_a = \dim_{n_a} N_a$, $d = \dim_{m_i} M_i$, and $d_j = \dim_{m_j} M_j$. Define $g : F\left(\mu_j^{-1}(U_j)\right) \to \mathbb{R}$ by $g(\hat{n}) = (f_i \circ \varphi_i)(p_1, \ldots, p_d)$ where $(\psi_a \circ \nu_a)(\hat{n}) = (p_1, \ldots, p_d, \ldots, p_{r_a})$. The above diagram makes it is easy to see that g is well-defined, smooth, and that $g = h \circ G$ on the open subset $F\left(\mu_j^{-1}(U_j)\right)$ of $\operatorname{Im} F$. ∎

Chapter 12

Sard's Theorem

Subsets of Measure 0

Recall that a subset of Euclidean space, with a compatible metric d, is said to be *a set of measure* 0 if for all $\epsilon > 0$ there exists a countable collection of open balls of radius r_n whose union contains this set such that $\sum_{n \in \mathbb{N}} r_n < \epsilon$, It is clear if d' is a metric equivalent to d then it would determine the same sets of measure 0 so that measure 0 subsets are independent of the choice of metric. Furthermore, measure 0 subsets of Euclidean space are diffeomorphism-invariants, which allows us to define the *subsets of measure* 0 of a manifold as being those subsets E such that for every chart (U, φ) on M the set $\varphi(E)$ has measure 0 in its codomain.

Similarly, we will now define what it means for a subset of a promanifold to be a set of measure 0 without defining any measure on it. The following definitions 12.1.1(2) - (4) may be viewed as analogs of "measure 0" subsets of a manifold that are contained within a single chart in the sense that both may be seen as being the "basic" measure 0 subsets' that are subsequently used to define all remaining measure 0 subsets. It also shows that there are several different competing immediate generalizations of this notion to promanifolds.

Definition 12.1.1. Let E be any subset of M.

(1) For any index i, say that μ_i *measures* E *as* 0 if $\mu_i(E)$ has measure 0 in M_i.

(2) Say that E *cofinally has measure* 0 *(in* Sys_M*)* if $\mu_l(E)$ has measure 0 in M_l for cofinally many l.

(3) By a $\mu_\bullet$*-basic measure* 0 *set* we mean a set of the form $\mu_i^{-1}(E_i)$ that cofinally has measure 0 in Sys_M.

(4) If i is an index such that $\mu_l(E)$ has measure 0 in M_l for all $l \geq i$ then we will say that E *eventually has measure* 0 *(after i) (in* Sys_M*)*.

Lemma 12.1.2. Let $E \subseteq M$.

(1) If $E = \mu_i^{-1}(E_i)$ is a $\mu_\bullet$-basic measure 0 subset of M and with E_i a measure 0 subset of M_i then $\mu_j(E) = \mu_{ij}^{-1}(E_i)$ has measure 0 for all $j \geq i$.

(2) If $i \in \mathbb{N}$ and $(E_i^l)_{l=1}^{\infty}$ are subsets of M_i such that $\mu_i^{-1}(E_i^l)$ is a $\mu_\bullet$-basic measure 0 for all $l \in \mathbb{N}$, then the same is true of $\underset{l\in\mathbb{N}}{\cup} E_i^l$ and $\underset{l\in\mathbb{N}}{\cap} E_i^l$.

(3) If Sys_M and $\mathrm{Sys}_{\widehat{M}}$ are two smoothly equivalent profinite systems then E is a $\mu_\bullet$-basic measure 0 set if and only if it is a $\mathrm{Sys}_{\widehat{M}}$-basic measure 0 set.

(4) Consider the statements:

   (a) E is contained in a $\mu_\bullet$-basic measure 0 subset of M.

   (b) E eventually has measure 0 in Sys_M.

   (c) E cofinally has measure 0 in Sys_M.

   (d) There exists an index i such that $\mu_i(E)$ has measure 0 in M_i.

   then $(a) \implies (b) \implies (c) \implies (d)$ and if each component of each M_i has positive dimension then also $(d) \implies (a)$.

(5) If there exists a point in the interior of E at which M has non-zero dimension then E is not a $\mu_\bullet$-basic measure 0 set.

Proof. (1), (2), (3): For each index i, let Z_i denote the connected components of M_i that have dimension 0. Suppose that $E = \mu_i^{-1}(E_i)$ is a $\mu_\bullet$-basic measure 0 set in M and let $j \geq i$. Pick $k \geq j$ such that $\mu_k(E) = \mu_{ik}^{-1}(E_i)$ has measure 0 in M_k. Since $E_i \cap Z_i$ is open in M_i, $\mu_{ik}^{-1}(E_i \cap Z_i)$ is an open measure 0 in M_k, which is only possible if $\mu_{ik}^{-1}(E_i \cap Z_i) \subseteq Z_k$ and since μ_{jk} is a smooth submersion this implies that $\mu_{jk}\big(\mu_{ik}^{-1}(E_i \cap Z_i)\big) = \mu_{ij}^{-1}(E_i \cap Z_i) \subseteq Z_j$. That $\mu_{ij}^{-1}(E_i \cap (M_i \setminus Z_i))$ has measure 0 follows immediately from lemma C.5.3. Thus $\mu_j(E) = \mu_{ij}^{-1}(E_i)$ has measure 0 in M_j, which proves (1) and (2) follows immediately. Furthermore, it is clear that if $\mu_{ij} = F \circ G$ is the composition of two smooth submersions, where $F : N \to M_i$ and $G : M_j \to N$ for some manifold N, then this same argument proves that $G^{-1}(E_i)$ has measure 0 in N. Thus if Sys_M is smoothly equivalent to $\operatorname{Sys}_{\widehat{M}}$ then a subset of M is a $\mu_\bullet$-basic measure 0 set in M if and only if it is a $\operatorname{Sys}_{\widehat{M}}$-basic measure 0 set in M. This proves (3).

(4): The $(a) \implies (b) \implies (c) \implies (d)$ follows from (1) so assume that each component of each M_i has positive dimension and that $\mu_i(E)$ has measure 0 in M_i for some index i. Since M_i has dimension ≥ 0 for each $j \geq i$ the set $\mu_{ij}^{-1}(\mu_i(E))$ has measure 0 in M_j by lemma C.5.3 so that $\mu_i^{-1}(\mu_i(E))$ is a $\mu_\bullet$-basic set of measure 0 containing E.

(5): Suppose that $m \in \operatorname{Int}(E)$ and pick an i and a connected non-empty $U_i \in \operatorname{Open}(M_i)$ such that $\dim_{\mu_i(m)} M_i \geq 1$ and $m \in \mu_i^{-1}(U_i) \subseteq \operatorname{Int} E$. For all $j \geq i$ we have $\mu_{ij}^{-1}(U_i) \subseteq \operatorname{Int} E \subseteq \mu_j(E)$ where since $\mu_{ij}^{-1}(U_i)$ is open, $\mu_j(E)$ is not of measure 0 in M_j. ∎

Lemma 12.1.2(4) shows, in particular, that under the following (mild) assumption, the various definitions of a "basic measure 0 subset of M" all coincide.

Assumption 12.1.3. We will henceforth only consider profinite systems where each component of each manifold has positive dimension.

Definition 12.1.4. If $E \subseteq M$ then we will say that E is a

(1) *cylindrical set of measure* 0 if it is a $\mu_\bullet$-basic set of measure 0.

(2) *locally cylindrical set of measure* 0 if for every $m \in E$ there exists a neighborhood $m \in O \in \mathrm{Open}(M)$ such that $E \cap O$ is a cylindrical set of measure 0.

(3) *trivial set of measure* 0 if any of the equivalent condition in lemma 12.1.2(4) are satisfied.

(4) *set of measure* 0 *(in M)* or *locally a set of measure* 0 *(in M)* if E is contained in a countable union of trivial sets of measure 0.

Furthermore, we will say that two subsets of M *differ by a set of measure* 0 if their symmetric difference is a set of measure 0.

Remark 12.1.5. Our assumption 12.1.3 and lemma 12.1.2(3) justify our omission of Sys_M in our terminology when referring to a set of measure 0 in M. Since M is separable, it is clear that any locally cylindrical set of measure 0 can always be written as a countable union of cylindrical sets of measure 0 and it is also clear that every finite union of cylindrical (resp. trivial) sets of measure 0 is again a cylindrical (resp. trivial) set of measure 0. Hence, our assumption implies that a set has measure 0 in M if and only if it is contained in a locally cylindrical set of measure 0.

Sard's Theorem for Promanifolds

Lemma 12.2.1. Let $F : M \to N$ be a smooth map between promanifolds. Let C (resp. D) denote the set of all critical points (resp. values) of F. For all indices a, let C^a (resp. D_a) denote the set of all critical points (resp. values) of $F^a \underset{\text{def}}{=} \nu_a \circ F : M \to N_a$. Then

(1) $C = \bigcup\limits_{a \in \mathbb{N}} C^a$ is an F_σ set and $D = \mathrm{Im}\, F \cap \left(\bigcup\limits_{a \in \mathbb{N}} \nu_a^{-1}(D_a) \right)$.

(2) If N is a manifold then C is closed in M, D is a meager F_σ set of measure 0 in N, and D consists of the union of all critical values of all F_i's, where $F = F_i \circ \mu_i$ on $\mathrm{ODom}_i\, F$.

Proof. Let $m \in M$, $n \underset{\text{def}}{=} F(m)$ and for all indices a let $n_a = \nu_a(n)$. By theorem 11.7.2 we have that $\mathrm{T}_m F$ is surjective if and only if $\mathrm{T}_m F^a : \mathrm{T}_m M \to \mathrm{T}_m N_a$ is surjective for all a. Thus m is a regular point of $\mathrm{T}_m F$ if and only if it is a regular value of all $\mathrm{T}_m F^a$ and so $C = \underset{a \in \mathbb{N}}{\cup} C^a$. Let $n \in N$. Suppose that $n \in \operatorname{Im} F \cap \nu_a^{-1}(D_a)$. Since $\nu_a(n) \in D_a$, there exists some $m \in (F^a)^{-1}(n)$ such that m is a critical point of $F^a = \nu_a \circ F$ so that $\mathrm{T}_m F$ can not be surjective. Hence $n = F(m)$ is a critical value of F and so $\operatorname{Im} F \cap \left(\underset{a \in \mathbb{N}}{\cup} \nu_a^{-1}(D_a) \right) \subseteq D$. The reverse inclusion follows from theorem 11.7.2 as before so that (1) is proved.

Suppose now that N is a manifold so that F is locally cylindrical. For all indices i, let $U_i = \mathrm{ODom}_i F$ so that $F = F_i \circ \mu_i$ on $U^i \underset{\text{def}}{=} \mu_i^{-1}(U_i)$ with $M = \underset{i \in \mathbb{N}}{\cup} U^i$ and $U^i \subseteq U^{i+1}$. Since $(U^i)_{i=1}^{\infty}$ is an open cover of M, to show that C is closed in M it suffices to show that $C \cap U^i$ is closed in U^i for all $i \in \mathbb{N}$ so fix an index i. Since $F = F_i \circ \mu_i$ on U^i (and Sys_N is the trivial system), it is clear that $m \in C \cap U^i$ if and only if $\mu_i(m)$ is a critical point of F_i so if C_i denotes the set of all critical points of F_i (in U_i) then C_i is closed in U_i and $C \cap U^i = \mu_i^{-1}(C_i)$ from which it follows that $C \cap U^i$ is closed in U^i, as desired. It also follows n is a critical value of F if and only if n is a critical value of F_i for some index i so that $D = \underset{i \in \mathbb{N}}{\cup} F_i(C_i)$. By Sard's theorem, for each index i the set of critical values of F_i, that is $F_i(C_i)$, has measure 0 in N. Since this measure 0 set is the continuous image of a σ-compact set it follows that $F_i(C_i)$ is a meager F_σ set in N. Thus $D = \underset{i \in \mathbb{N}}{\cup} F_i(C_i)$ is a meager F_σ measure 0 in N, so (2) holds. ∎

Theorem 12.2.2 (Sard's Theorem for Promanifolds)**.** Let $F : M \to N$ is a smooth map between promanifolds (of non-0 dimension), let $F^\bullet = \mu_\bullet \circ F$, and let D_a denote the set of critical values of $F^a : M \to N_a$ for all $a \in \mathbb{N}$. Then $\underset{a \in \mathbb{N}}{\cup} \nu_a^{-1}(D_a)$ is a meager F_σ locally cylindrical set of measure 0 in N that contains F's critical values. In particular, the set of critical values of F is a meager set of measure 0 in N.

Proof. Let D denote the set of critical values of F. By lemma 12.2.1, we have $D = \operatorname{Im} F \cap \left(\underset{a \in \mathbb{N}}{\cup} \nu_a^{-1}(D_a) \right)$ so that it suffices to prove that each of the sets $\operatorname{Im} F \cap \nu_a^{-1}(D_a)$ has measure 0 in N. By this same lemma, the set of critical values of F^a is a meager F_σ set of measure 0 in N_a and since, by assumption, $\dim N_a \geq 1$ it follows that $\nu_a^{-1}(D_a)$ is also a meager F_σ

set of measure 0 in N and hence that $\bigcup_{a\in\mathbb{N}} \nu_a^{-1}(D_a)$ is a meager F_σ locally cylindrical set of measure 0 in N. ∎

Theorem 12.2.3. Let $F : M \to N$ is a smooth map between promanifolds, let $E \subseteq M$ be a set of measure 0 in M, and assume that $1 \le \dim_m M \le \dim_{F(m)} N$ for all $m \in E$. If for each $m \in E$ either $\dim_m M < \infty$ or otherwise F is a pointwise immersion at m then $F(E)$ has measure 0 in N. In particular, diffeomorphisms preserve sets of measure 0.

Proof. Observe that if suffices to assume that E is a subset of a trivially cylindrical basic measure 0 subset of M and to prove that $F(E)$ has measure 0 in N. By Sard's theorem 12.2.2, we need only to consider the case E is a subset of the set of regular values of F. Suppose that $E = \mu_i^{-1}(E_i)$ where E_i has measure 0 in M_i and for all $j \ge i$ let $E_j = \mu_{ij}^{-1}(E_i)$. Observe that it suffices to show that for every $m \in E$ there exists an index j and a neighborhood $\mu_j(m) \in U_j \in \operatorname{Open}(M_j)$ such that $F\big(E \cap \mu_j^{-1}(U_j)\big)$ has measure 0 in N so fix $m^0 \in E$, let $n^0 = F(m^0)$ and for all $a, j \in \mathbb{N}$ let $m_j^0 = \mu_i j(m^0)$ and $n_a^0 = \nu_a(n^0)$.

If $d \underset{\text{def}}{=} \dim_{m^0} M < \infty$ then since $1 \le \dim_m M \le \dim_{F(m)} N$ we may pick indices a and $j \ge i$ such that $\dim_{n_a^0} N_a \ge d$, $\dim_{m_j^0} M_j = d$, and m_j^0 belongs to the connected component of $\operatorname{ODom}_j F^a$, call it O_j. But then $S_a \underset{\text{def}}{=} F^a\big(\mu_j^{-1}(O_j) \cap E\big) = F_j^a(O_j \cap E_j)$ is a set of measure 0 in N_a, where $\dim_{n_a^0} N_a \ge 1$ implies that $\nu_a^{-1}(S_a)$ is a set of measure 0 in N. Thus, assume that $\dim_m M = \infty$ and that F is a pointwise immersion for all $m \in E$, which, since each $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_{F(m)} N$ is surjective, implies that F is in fact a ($\nu_\bullet$-regular) pointwise isomersion for all $m \in E$.

Fix an index a such that $\dim_{n_a^0} N_a > \dim_{m_i^0} M_i$. Let $j \ge i$ be such that $\dim_{m_j^0} M_j > \dim_{n_a^0} N_a$ and m_j^0 belongs to the connected component of $\operatorname{ODom}_j F^a$, call it O_j. Pick any chart (U_j, φ_j) around m_j^0 such that U_j is a subset of O_j. By proposition 11.5.1 we can find indices $k > j$ and $b > a$ and coordinates (U_k, φ_k) and (V_b, ψ_b) around m_k^0 and n_b^0, respectively, such that the following diagram commutes:

$$\begin{tikzcd}
(p_1, \ldots, p_{d_j}, \ldots, p_{d_k}) \arrow[r, "\widehat{F_k^b}"] \arrow[d, "\widehat{\mu_{jk}}"'] & (p_1, \ldots, p_{d_j}, \ldots, p_r) \\
(p_1, \ldots, p_{d_j}) &
\end{tikzcd}$$

where $d_k = \dim_{m_k^0} M_k$, $r = \dim_{n_b^0} N_b$ is the rank of F_k^b. Note that $F^b\big(E \cap \mu_k^{-1}(U_k)\big) = F_k^b(E_k \cap U_k)$ is, in V_b's coordinates, a subset of the set

$$V_b \cap \big((E_j \cap \mu_{jk}(U_k)) \times \mathbb{R}^{d_r - d_j}\big)$$

where this is a set of measure 0 in N_b since $E_j \cap \mu_{jk}(U_k)$ is a set of measure 0 in M_j, which is a manifold of dimension ≥ 1. ∎

Chapter 13

The Inverse Function Theorem

We recall one of the statements of the inverse function theorem for Banach spaces.

Theorem 13.0.1 (The Inverse Function Theorem for Banach Spaces)**.** Let X and Y be Banach spaces, $U_0 \in \operatorname{Open}(X)$, and let $F : U_0 \to Y$ be a C^2-map. Suppose that $x_0 \in U_0$ is such that $D_{x_0} F : X \to Y$ is a TVS-isomorphism. Then there exists $x_0 \in U \in \operatorname{Open}(U_0)$ such that $F|_U : U \to F(U)$ is a diffeomorphism onto an open subset of Y.

One may consequently hope to prove a generalization of the inverse function theorem to promanifolds that is similar to the following (false) conjecture 13.0.2.

Conjecture 13.0.2 (An Ideal Inverse Function Theorem for Promanifolds)**.** A smooth isomersion between monotone promanifolds is a local diffeomorphism (def. 4.1.3).

Counterexamples

In addition to being a counterexample to both the inverse function theorem for promanifolds (i.e. conjecture 13.0.2) and invariance of domain for promanifolds, the following example 13.1.1 also suggests a potential substitute to the inverse function theorem for promanifolds that would ultimately lead one to theorem 13.2.3. Indeed, this example shows that conjecture

13.0.2 fails even for very well-behaved smooth maps between very well-behaved promanifolds.

Example 13.1.1. The natural inclusion of $]-1,1[^{\mathbb{N}}$ into $\mathbb{R}^{\mathbb{N}}$ disproves both the inverse function theorem for promanifolds (i.e. conjecture 13.0.2) and the direct generalization to promanifolds of the statement of invariance of domain.

Example 13.1.2. A smooth embedding and pointwise isomersion whose image has empty interior: Let $M_i =]0,1[^i$, $N_a = \mathbb{R}^a$, $M =]0,1[^{\mathbb{N}}$, $N = \mathbb{R}^{\mathbb{N}}$, $\mu_i : M \to M_i$, $\nu_a : N \to N_a$, μ_{ij}, and ν_{ab} be the canonical projections so that $(M, \mu_\bullet) = \varprojlim(M_\bullet, \mu_{ij})$ and $(N, \nu_a) = \varprojlim(N_a, \nu_{ab})$. Let $F_i = \mathrm{In}_{M_i}^{N_i} : M_i \to N_i$ and $F : M \to N$ be the natural inclusion maps so that $F = \varprojlim F_\bullet$. Each F_i is a smooth embedding of M_i onto an open subset of N_i so that $F : M \to N$ is smooth and a topological embedding into N. Since each $\mathrm{T}_{m_i} F_i : \mathrm{T}_{m_i} F_i \to \mathrm{T}_{F_i(m_i)} N_i$ is a vector space isomorphism for all $m_i \in M_i$, the same is true of $\mathrm{T}_m F : \mathrm{T}_m M \to \mathrm{T}_{F(m)} N$ for all $m \in M$. However, the (topological) interior of $\mathrm{Im}\, F$ is empty: Suppose that $m^0 \in \mathrm{Int}_N(\mathrm{Im}\, F)$ and let $m^0 \in \nu_a^{-1}(V_a) \in \mathrm{Open}\,(\mathrm{Int}_N(\mathrm{Im}\, F))$ for some $a \in \mathbb{N}$ and some $V_a \in \mathrm{Open}\,(N_a)$. Let $n = (m_1^0, \ldots, m_a^0, 1, 1, \ldots)$ and observe that $\nu_a(n) = (m_1^0, \ldots, m_a^0) = \nu_a(m^0) \in V_a$ so that $n \in \nu_a^{-1}(V_a) \subset \mathrm{Int}_N(\mathrm{Im}\, F) \subseteq \mathrm{Im}\, F = (0,1)^{\mathbb{N}}$ and since $F : M \to N$ is the natural inclusion, this clearly gives a contradiction and hence $\mathrm{Int}_N(\mathrm{Im}\, F) = \varnothing$. In fact, $\mathrm{Im}\, F$ is even nowhere dense in N. The closure of $\mathrm{Im}\, F$ in N is $[0,1]^{\mathbb{N}}$ and we can similarly show that its interior in N is also empty. Furthermore, observe that if we had defined $N_a =]0,1]^a$ then the same conclusion would have held but in addition we'd have that the $\mathrm{Im}\, F$ is a dense subset of $N =]0,1]^{\mathbb{N}}$ with empty interior.

Example 13.1.3. A smooth surjective pointwise isomersion that is nowhere locally injective: Suppose that $g : M_1 \to N_1$ is a surjective local diffeomorphism between manifolds. For all indices i and a let $M_i = M_1^i$, $M = M_1^{\mathbb{N}}$, $N_a = N_1^a$, $N = N_1^{\mathbb{N}}$ and let $\mu_{i,i+1}$, $\nu_{a,a+1}$, μ_i, and ν_a all be the canonical projections so that $(M, \mu_\bullet) = \varprojlim \mathrm{Sys}_M$ and $(N, \nu_\bullet) = \varprojlim \mathrm{Sys}_N$. For each

$i \in \mathbb{N}$ let

$$\begin{aligned} F_i : M_a &\longrightarrow N_a \\ (p_1, \ldots, p_a) &\longmapsto (g(p_1), \ldots, g(p_a)) \end{aligned}$$

so that the limit is

$$\begin{aligned} F \underset{\text{def}}{=} \varprojlim F_i : M &\longrightarrow N \\ (p_1, p_2, \ldots) &\longmapsto (g(p_1), g(p_2), \ldots) \end{aligned}$$

Observe that F is smooth a surjective pointwise isomersion, let $n = (q_1, q_2, \ldots) \in N$, and note that $F^{-1}(n) = g^{-1}(q_1) \times g^{-1}(q_2) \times \cdots$ where each fiber has cardinality at most $\mathbb{N}^{\mathbb{N}}$ (since g is a local diffeomorphism between second countable manifolds). Since every basic open subset of M is of the form $\prod_{h \leq i} U_h \times \prod_{j \geq i} M_1$, it is easy to see that for any $U \in \operatorname{Open}(M)$, if $g^{-1}(q_a)$ fails to be singleton for infinitely many indices a then either $U \cap F^{-1}(n) = \varnothing$ or otherwise $F\big|_U^{-1}(n)$ will have the cardinality of continuum so that there can be no open subset of M on which F is injective and $U \cap F^{-1}(n) \neq \varnothing$. In particular, if g is a map such that each of its fibers contains at least two points, for instance,

$$\begin{aligned} g : \mathbb{C} \setminus \{0\} &\longrightarrow \mathbb{C} \setminus \{0\} \\ z &\longmapsto z^2 \end{aligned} \qquad \text{or} \qquad \begin{aligned} g : \mathbb{C} &\longrightarrow \mathbb{C} \setminus \{0\} \\ z &\longmapsto e^z \end{aligned}$$

then it follows that there is no non-empty open subset U of M on which $F\big|_U : U \to N$ is injective i.e. F is nowhere locally injective.

Example 13.1.4. A nowhere locally injective pointwise isomersion with nowhere dense image: Let $M_1 = \mathbb{R}^{\neq 0} \underset{\text{def}}{=} \mathbb{R} \setminus \{0\}$, $N_1 =]0, \infty[$, $M_i = M_1^i = \prod_{l=1}^{i} M_1$, and $N_a = N_1^a$ for all $i, a \in \mathbb{N}$. Let the canonical projections be the bonding maps for Sys_M and Sys_N and observe that $M_1^{\mathbb{N}}$ (resp. $N_1^{\mathbb{N}}$) together with the canonical projections are the limits of these systems. Let $F_a : M_a \to N_a$ be defined by $F_a(m_1, \ldots, m_a) = (m_1^2, \ldots, m_a^2)$ and let $F \underset{\text{def}}{=} \varprojlim (F_\bullet, \operatorname{Id}_{\mathbb{N}}) : M \to N$

is given by $F(m) = (m_1^2, m_2^2, m_3^2, \ldots)$ for $m = (m_1, m_2, \ldots) \in M$ so that F is nowhere locally injective by example 13.1.3. Observe also that had we instead used $N_1 = \mathbb{R}$ then F would still be everywhere a smooth pointwise isomersion that is nowhere locally injective but in addition its image would now also have empty interior in $N = \mathbb{R}^{\mathbb{N}}$ (since otherwise $\operatorname{Im} F$ would necessarily contain some $n = (n_1, n_2, \ldots)$ with some $n_a < 0$). In fact, the image of F would even be nowhere dense in N.

Example 13.1.5. Fix an interval $J \subseteq \mathbb{R}$ and considering the unit circle S^1 as a subspace of $\mathbb{C}$, let $(S^1)^{\mathbb{N}}$ denote the infinite torus. Clearly,

$$\begin{aligned} F : J^{\mathbb{N}} &\longrightarrow (S^1)^{\mathbb{N}} \\ (r_1, r_2, \ldots) &\longmapsto \left(e^{ir_1}, e^{ir_2}, \ldots\right) \end{aligned}$$

defines a smooth pointwise isomersion. If $J = \mathbb{R}$ or $J = [0, 2\pi]$ then $F : \mathbb{R}^{\mathbb{N}} \to (S^1)^{\mathbb{N}}$ is a surjective quotient map that is at no point locally injective. If $J = [0, 2\pi[$ then $F : J^{\mathbb{N}} \to (S^1)^{\mathbb{N}}$ is bijective but not a quotient map. If $J =]0, 2\pi[$ then $F : J^{\mathbb{N}} \to (S^1)^{\mathbb{N}}$ is a non-surjective topological embedding (and thus a smooth embedding) whose image $\left(S^1 \setminus \{(0,1)\}\right)^{\mathbb{N}}$ is dense in $(S^1)^{\mathbb{N}}$ but has empty interior in $(S^1)^{\mathbb{N}}$.

The next examples gives a smooth surjective pointwise isomersion that fails to be injective on any non-empty open subset of its domain.

Example 13.1.6. A surjective pointwise isomersion with fibers of cardinality $\mathfrak{c}$ (the cardinality of continuum) on each open subset: Let $R = \mathbb{R} \setminus \{0\}$, $S = \mathbb{R}^{>0}$, $M = R^{\mathbb{N}}$, and $N = S^{\mathbb{N}}$. For all $i, a \in \mathbb{N}$ let $M_i = R^i = \prod_{l=1}^{i} R$, $N_a = S^a$, and let $\mu_{i,i+1}$, μ_i, $\nu_{a,a+1}$, and ν_a be the canonical projections. Define

$$\begin{aligned} F : M &\longrightarrow N \\ (r_1, r_2, \ldots) &\longmapsto (r_1^2, r_2^2, \ldots) \end{aligned}$$

Clearly, F is a smooth surjective pointwise isomersion and for each $n = (s_1, s_2, \ldots) \in N = S^{\mathbb{N}}$

the fiber over F over n is $F^{-1}(n) = \prod_{l=1}^{\infty}\{\pm\sqrt{s_l}\}$. Now suppose that $U = O_1 \times \cdots \times O_i \times \prod_{l=i+1}^{\infty} R$ is a non-empty basic open subset of M, let $m = (r_1, r_2, \ldots) \in U$, and let $n = F(m^0)$. Observe that $F(U)$ is open and the fiber of $F\big|_U$ over n has cardinality $\mathfrak{c}$ since it contains at least the set $\{r_1\} \times \cdots \times \{r_i\} \times \prod_{l=i+1}^{\infty}\{\pm r_l\}$ (for we can define a surjective map $\theta : \prod_{l=i+1}^{\infty}\{\pm r_l\} \to [0,1]$ by $\theta(s_{i+1}, s_{i+2}, \ldots) = 0.b_1 b_2 \ldots$ where $b_l = 1$ if $s_{i+l} > 0$, $b_l = 0$ if $s_{i+l} < 0$, and where we view $0.b_1 b_2 \ldots$ as a real number's binary representation) and so it should now be easy to see that this is in fact true for all non-empty open subsets of M. Observe that F is open and it is easy to see that it is even a compact covering. If we define for each $a \in \mathbb{N}$ the map

$$\begin{aligned} F_a : M_a &\longrightarrow N_a \\ (r_1, \ldots, r_a) &\longmapsto (r_1^2, \ldots, r_a^2) \end{aligned}$$

then $F = \varprojlim F_\bullet$ so that F fails to be locally injective despite how (relatively) well-behaved each F_a, each bonding map, and each M_i and N_a is. ■

Sub-Promanifold Inverse Function Theorem for Fibrated Promanifolds

Before beginning the proof of the "sub-promanifold inverse function theorem," we remind the reader of the following proposition C.1.5 that is proved in the appendix.

Proposition 13.2.1. Suppose M and N are smooth manifolds of dimensions $c \geq b$, respectively, and that the following diagram of smooth maps commutes where μ and F are smooth surjective submersions and $\sigma : \mathbb{R}^a \to M$ a smooth section of $\pi \circ \mu : M \to \mathbb{R}^a$ such that $\eta \underset{\text{def}}{=} F \circ \sigma : \mathbb{R}^a \to N$ is a smooth embedding.

$$\begin{tikzcd} M \arrow[d, "\mu"] \arrow[dr, "F"] & \\ \mathbb{R}^a \arrow[u, bend left, "\sigma"] & N \end{tikzcd}$$

There exist smooth surjective charts $\varphi : U \to \mathbb{R}^c$ and $\psi : V \to \mathbb{R}^b$ on M and N, respectively, such that

(1) $\operatorname{Im}\sigma \subseteq U$ and $(\varphi \circ \sigma)(t_1, \ldots, t_a) = (t_1, \ldots, t_a, 0, \ldots, 0)$ for all $(t_1, \ldots, t_a) \in \mathbb{R}^a$.

(2) $\mu\big|_U : U \to \mathbb{R}^a$ and $F\big|_U : U \to V$ are both surjective.

(3) The coordinate representations of μ and F are the canonical projection.

If in the above proposition we had used $\theta \circ \mu$ in place of μ and $\sigma \circ \theta$ in place of σ where θ is some smooth chart $\theta : W \to \mathbb{R}^a$ on some smooth manifold Q, then writing the conclusion of the above proposition in terms of a commutative diagram would give the following corollary, which is better suited for proving the sub-promanifold inverse function theorem.

Corollary 13.2.2. Let Q, N, and M be, respectively, a, d, and e dimensional manifolds, let $\mu : M \to Q$ and $F : M \to N$ be smooth submersions. Suppose that $\sigma : W \to M$ is a smooth local section of $\mu : M \to Q$ such that $F \circ \sigma : W \to N$ is a smooth embedding and that $\theta : W \to \mathbb{R}^a$ is a smooth surjective chart. Then there exist smooth charts (U, φ) and (V, ψ) on M and N, respectively, and a smooth local section $\rho : V \to U$ of $F : M \to N$ such that $\operatorname{Im}\rho \subseteq U$, $\mu(U) = W$, $F(U) = V$, both $\varphi : U \mapsto \mathbb{R}^e$ and $\psi : V \mapsto \mathbb{R}^d$ are surjective, and the following diagram commutes:

$$
\begin{tikzcd}
\mathbb{R}^e \arrow[d, bend right, "\mathrm{Pr}"'] & & U \arrow[ll, dashed, "\varphi"'] \arrow[dl, bend right, "F|_U"'] \arrow[dd, "\mu|_U"] \\
\mathbb{R}^d \arrow[u, bend right, "\mathrm{In}"'] \arrow[d, bend right, "\mathrm{Pr}"'] & V \arrow[l, dashed, "\psi"'] \arrow[ur, dashed, bend right, "\rho"'] & \\
\mathbb{R}^a \arrow[u, bend right, "\mathrm{In}"'] & & W \arrow[ll, "\theta"] \arrow[uu, bend right, "\sigma"']
\end{tikzcd}
$$

where $\mathrm{In} : \mathbb{R}^a \to \mathbb{R}^d$ is *the canonical inclusion* $(t_1, \ldots, t_a) \mapsto (t_1, \ldots, t_a, 0, \ldots, 0)$. The properties expressed by the above commutative diagram can equivalently be described by the following list of properties:

(1) the coordinate representations of both F and μ are the canonical projections, that is, for each $(t_1, \ldots, t_e) \in \mathbb{R}^e$, the following diagram commutes:

$$\begin{tikzcd}
(t_1, \ldots, t_a, \ldots, t_d, \ldots t_e) \arrow[r, "\psi \circ F \circ \varphi^{-1}"] \arrow[d, "\theta \circ \mu \circ \varphi^{-1}"'] & (t_1, \ldots, t_a, \ldots, t_d) \\
(t_1, \ldots, t_a) &
\end{tikzcd}$$

(2) ρ's (resp. σ's) coordinate representation $\varphi \circ \rho \circ \psi^{-1} : \mathbb{R}^d \to \mathbb{R}^e$ (resp. $\varphi \circ \sigma \circ \theta^{-1} : \mathbb{R}^a \to \mathbb{R}^e$) is the canonical inclusion. e.g. $(t_1, \ldots, t_d) \mapsto (t_1, \ldots, t_d, 0, \ldots, 0)$

In particular, for all $m, p \in U$ and all $(x, y) \in \mathbb{R}^a \times \mathbb{R}^{d-a}$

(a) if we write $\widehat{\mu} = \theta \circ \mu\big|_U \circ \varphi^{-1}$ and $\widehat{F} = \psi \circ F\big|_U \circ \varphi^{-1}$ we have $\widehat{F}\left(\widehat{\mu}^{-1}(x)\right) = \{x\} \times \mathbb{R}^{e-a}$ and $\widehat{F}{-1}(x, y) = \{(x, y)\} \times \mathbb{R}^{e-d}$,

(b) $F(m) = F(u) \iff \rho(F(m)) = \rho(F(p))$, in which case $\mu(m) = \mu(\rho(F(m))) = \mu(\rho(F(p))) = \mu(p)$,

(c) $\mu(m) = \mu(p) \iff \sigma(\mu(m)) = \sigma(\mu(p))$.

In addition to being smooth submersions, we will require all of Sys_M's the bonding maps in the following theorem to be Serre fibrations for then whenever we are given any $d > c$, smooth $\Lambda^c : [-1,1]^c \to M$, and smooth $\lambda_i : [-1,1]^d \to M_i$ that extends $\mu_i \circ \Lambda^c$, we will be able to lift λ_i to a smooth map $\Lambda^d : [-1,1]^d \to M$ satisfying $\mu_i \circ \Lambda^d = \lambda_i$ that also extends $\Lambda^c : [-1,1]^c \to M$.

Theorem 13.2.3 (Sub-promanifold Inverse Function Theorem)**.** Suppose that $F : (M, m^0) \to (N, n^0)$ is a smooth isomersion between two promanifolds (i.e. the tangent map is a TVS-isomorphism at each point), $\dim_{m^0} M = \infty$, and that all $\mu_{ij} : M_j \to M_i$ are Serre fibrations. Let i_0 be any index and let $m^0_\bullet = \mu_\bullet(m^0)$, $n^0_\bullet = \nu_\bullet(n^0)$, $F^\bullet = \nu_\bullet \circ F$. There exist

- $\alpha : \mathbb{N} \to \mathbb{N}$ and $\iota : \mathbb{Z}^{\geq 0} \to \mathbb{N}$ both increasing with $\iota(0) = i_0$,

- a $\mu_\bullet$-open sub-promanifold $U \subseteq M$ containing m^0 such that $V = F(U)$ is a $\nu_\bullet$-open sub-promanifold of N, and

- smooth charts $\left(\mu_{\iota(\bullet)}(U), \varphi_\bullet\right) = \left(\mu_{\iota(l)}(U), \varphi_l\right)_{l=0}^{\infty}$ and $\left(\nu_{\alpha(\bullet)}(V), \psi_\bullet\right) = \left(\nu_{\alpha(l)}(V), \psi_l\right)_{l=1}^{\infty}$,

such that, upon letting $U_\bullet := \mu_\bullet(U)$, $V_\bullet := \nu_\bullet(V)$, $e_\bullet := \dim_{n^0_{\alpha(\bullet)}} N_{\alpha(\bullet)}$, and $d_\bullet := \dim_{m^0_{\iota(\bullet)}} M_{\iota(\bullet)}$, the following will hold:

(1) $U_{\iota(l)} \subseteq \operatorname{ODom}_{\iota(l)} F^{\alpha(l)}$, meaning that $F^{\alpha(l)}_{\iota(l)} \circ \mu_{\iota(l)} = F^{\alpha(l)}$ on $\mu^{-1}\left(U_{\iota(l)}\right)$, and $F^{\alpha(l)}_{\iota(l)} : U_{\iota(l)} \to V_{\alpha(l)}$ is a smooth surjective submersion for all $l > 0$,

(2) $d_0 < e_1 < d_1 < e_2 < d_2 < \cdots$,

(3) $\varphi_l : \left(U_{\iota(l)}, m^0_{\iota(l)}\right) \to (\mathbb{R}^{d_l}, \mathbf{0})$ and $\psi_l : \left(V_{\alpha(l)}, n^0_{\alpha(l)}\right) \to (\mathbb{R}^{e_l}, \mathbf{0})$ is surjective for all l,

(4) the coordinate representations of all $F^{\alpha(l)}_{\iota(l)}$, $\mu_{\iota(l),\iota(l+1)}$, and $\nu_{\alpha(l),\alpha(l+1)}$ with respect to these charts are the canonical projections,

(5) $\varphi_\bullet : U_{\iota(\bullet)} \to \mathbb{R}^{d_\bullet}$ and $\psi_\bullet : V_{\alpha(\bullet)} \to \mathbb{R}^{e_\bullet}$ form inverse system morphisms from the inverse system of subsets $U_\bullet$ and $V_{\alpha(\bullet)}$ whose limits $\varphi := \varprojlim \varphi_\bullet : U \to \mathbb{R}^{\mathbb{N}}$ and $\psi := \varprojlim \psi_\bullet : V \to \mathbb{R}^{\mathbb{N}}$ are diffeomorphisms,

(6) $F\big|_U : U \to V$ is a diffeomorphism and the following diagram commutes for all $l \in \mathbb{N}$:

$$
\begin{tikzcd}
& U \arrow[rr, "F|_U"] \arrow[dl, "\varphi"'] \arrow[dd, "\mu_j|_U"] & & V \arrow[dl, "\psi"'] \arrow[dd, "\nu_b|_V"] \\
\mathbb{R}^{\mathbb{N}} \arrow[rr, "\mathrm{Id}"] \arrow[dd, "\mathrm{Pr}"'] & & \mathbb{R}^{\mathbb{N}} \arrow[dd, "\mathrm{Pr}"] & \\
& U_j \arrow[rr, "F_j^b|_{U_j}"] \arrow[dl, "\varphi_{l+1}"'] \arrow[dd, "\mu_{ij}|_{U_j}"] & & V_b \arrow[dl, "\psi_{l+1}"'] \arrow[dd, "\nu_{ab}|_{V_b}"] \\
\mathbb{R}^{d_j} \arrow[rr, "\mathrm{Pr}"] \arrow[dd, "\mathrm{Pr}"'] & & \mathbb{R}^{e_b} \arrow[ll, bend left, "\mathrm{In}_b"] \arrow[dd, "\mathrm{Pr}"] & \\
& U_i \arrow[rr, "F_i^a|_{U_i}"] \arrow[dl, "\varphi_l"'] \arrow[dd, "\mu_{ij}|_{U_j}"] & & V_a \arrow[dl, "\psi_l"'] \\
\mathbb{R}^{d_i} \arrow[rr, "\mathrm{Pr}"] \arrow[dd, "\mathrm{Pr}"'] & & \mathbb{R}^{e_a} \arrow[ll, bend left, "\mathrm{In}_a"] & \\
& U_{i_0} \arrow[dl, "\varphi_0"'] & & \\
\mathbb{R}^{d_0} & & &
\end{tikzcd}
$$

where we let $i = \iota(l)$, $j = \iota(l+1)$, $a = \alpha(l)$, $b = \alpha(l+1)$, and where Pr denote the canonical projection while In denotes the canonical inclusion into the first coordinates (i.e $\mathrm{In}_a(t_1, \ldots, t_{e_a}) = (t_1, \ldots, t_{e_a}, 0, \ldots, 0)$).

(7) Furthermore, if for each $l \in \mathbb{N}$ we define $\rho_l : V_{\alpha(l)} \to U_{\iota(l)}$ by

$$
V_{\alpha(l)} \xrightarrow{\psi_l} \mathbb{R}^{e_l} \xrightarrow{\mathrm{In}_{\alpha(l)}} \mathbb{R}^{d_l} \xrightarrow{\varphi_l^{-1}} U_{\iota(l)}
$$

then ρ_l is a smooth section of $F_{\iota(l)}^{\alpha(l)}\big|_{U_{\iota(l)}} : U_{\iota(l)} \to V_{\alpha(l)}$ and the maps $G_{\alpha(l)}^{\iota(l)} \coloneqq \mu_{\iota(l-1),\iota(l)} \circ \rho_l : V_{\alpha(l)} \to U_{\iota(l-1)}$ define a smooth inverse system morphism from $\mathrm{Sys}_V \coloneqq \mathrm{Sys}_N\big|_V$ into $\mathrm{Sys}_U \coloneqq \mathrm{Sys}_M\big|_U$ making the $\mathbb{N}$-indexed families $F_{\iota(\bullet)}^{\alpha(\bullet)}$ and $G_{\alpha(\bullet)}^{\iota(\bullet)}$ into a smooth equivalence transformation of profinite systems.

Proof. Since $]-1,1[$ is diffeomorphism to $\mathbb{R}$, it suffices to prove the theorem by obtaining smooth charts which, except for having their images be products of $]-1,1[$ instead of products of $\mathbb{R}$, otherwise satisfy all of the claims made above. Also observe that if we can construct charts such that the above diagram commutes, then all of the other claims of this theorem will follow from known results. To simplify the proof, refine the notation $d_\bullet$ and $e_\bullet$ to mean $d_\bullet = \dim_{m^0_\bullet} M_\bullet$ and $e_\bullet = \dim_{n^0_\bullet} N_\bullet$ and by replacing Sys_M and Sys_N with subsystems if necessary, we assume without loss of generality that $i_0 = 1$ and that $d_\bullet$ and $e_\bullet$ are strictly increasing.

Step $l = 0$: Let $\iota(0) = 1$ and let $\phi_0 : U'_1 \to \mathbb{R}^{d_0}$ be any smooth chart on M_1 centered at m^0_1. Since all bonding maps in Sys_M are Serre fibrations, we may inductively construct a smooth μ_1-lift $\Lambda^0 : \left(\mathbb{R}^{d_0}, \{0\}^{d_0}\right) \to (\mu_1^{-1}(U_1), m^0_1)$ of $\phi_0^{-1} : \mathbb{R}^{d_0} \to U'_1$.

Remark 13.2.4. Even if we could somehow always conclude that for every embedding $\lambda : \mathbb{R}^d \to M$, there is some index a such that $\nu_a \circ F \circ \lambda$ was a smooth embedding, which we in general can't, it may still not be possible to able obtain the conclusions described above since even under this assumption, there is there no guarantee that this is an index i such that the image of $\mu_i \circ \lambda$ is contained in $\operatorname{Dom}_i F^a$. This proof's construction uses compactness to overcomes these issues, where it is often these same issues that in general prevent us from finding open subsets $U \in \operatorname{Open}(M)$ and $V \in \operatorname{Open}(N)$ for which $F\big|_U : U \to V$ is a diffeomorphism.

Since $F : M \to N$ is a pointwise immersion at m^0, so too is $F \circ \lambda^0$ at $\{0\}^{d_0}$ so there exists some $a_1 = \alpha(1) > 1 = \iota(0)$ such that $e_{a_1} > d_0$ and for which neighborhood $D_0 \underset{\text{def}}{=} [c_0, c_0]^{d_0}$ of $\{0\}^{d_0}$ in $\mathbb{R}^{d_0}$ such that $F^{a_1} \circ \lambda^0\big|_{D_0} : D_0 \to N_{a_1}$ is a smooth embedding. Since $\Lambda^0(D_0)$ is compact, there exists some $i_1 = \iota(1) > \alpha(1)$ such that $d_{i_1} > e_{a_1}$ and $\mu_{i_1}(\Lambda^0(D_0)) \subseteq \operatorname{ODom}_{i_1} F^{a_1}$. Letting $\varrho :]-2,2[^{d_0} \to]-c_0, c_0[^{d_1}$ be any diffeomorphism mapping the origin to itself, we may replace ϕ_0 with $\varrho^{-1} \circ \phi_0\big|_{\phi_0^{-1}(]-\delta_0,\delta_0[^{d_1})}$ and replace Λ^0 with $\Lambda^0 \circ \varrho$ and thus assume without loss of generality that c_0 had been 2 and that the domain (resp. range) of Λ^0 (resp.

ϕ_0) had been $]-2,2[^{d_0}$. Let $(c_l)_{l=1}^{\infty} \subseteq]1,2[$ be any strictly decreasing sequence converging to 1 and then pick $(b_l)_{l=1}^{\infty}$ such that $c_0 > b_1 > c_1 > \cdots c_l > b_{l+1} > c_{l+1}$. Let $I_0 =]-c_0, c_0[$ and for all $l > 0$ let $I_l =]-c_l, c_l[$ and $J_l = [-b_l, b_l]$.

Step $l = 1$:

Since $\mu_{i_1} \circ \lambda^1$ is valued in $\operatorname{ODom}_{i_1} F^{a_1}$, the composition $F_{i_1}^{a_1} \circ (\mu_{i_1} \circ \lambda^1) = F^{a_1} \circ \lambda^0 : I_1^{d_{i_0}} \to N_{a_1}$ is defined, where recall that this map is a smooth embedding so that we can consequently apply corollary C.1.6 by substituting each item in the top row for the respective item in the bottom row:

$$\begin{array}{c|c|c|c|c|c|c|c}
M_{i_0} & \operatorname{ODom}_{i_1} F^{a_1} & U_{i_0}^1 & N_{a_1} & F_{i_1}^{a_1} & \mu_{i_1} \circ \lambda^0 \circ \varphi_0^0 & \mu_{i_0,i_1}\big|_{\operatorname{ODom}_{i_1} F^{a_1}} & \varphi_0^0 \\
 & & & & & & & \\
Q & M & W & N & F & \sigma & \mu & \theta
\end{array}$$

so as to obtain smooth charts $\Phi_1 : \left(U'_{i_1}, m_{i_1}^0\right) \to \left(I_0^{d_{i_1}}, \mathbf{0}\right)$ and $\Psi_1 : \left(V'_{a_1}, n_{a_1}^0\right) \to \left(I_0^{e_{a_1}}, \mathbf{0}\right)$, and a smooth section $\Gamma_1 : V'_{a_1} \to U'_{i_1}$ of the surjective submersion $F_{i_1}^{a_1}\big|_{U'_{i_1}} : U'_{i_1} \to V'_{a_0}$, such that $U'_{i_1} \subseteq \operatorname{ODom}_{i_1} F^{a_1}$, $V'_{a_1} = F_{i_1}^{a_1}\left(U'_{i_1}\right)$, $\mu_{i_0,i_1}\left(U'_{i_1}\right) = U_{i_0}^0$, and such that the following diagram commutes:

$$\begin{tikzcd}
I_0^{d_{i_1}} \arrow[d, bend right, "\mathrm{Pr}"'] & & U'_{i_1} \arrow[ll, dashed, "\Phi_1"'] \arrow[dl, bend right, "F_{i_1}^{a_1}|_{U'_{i_1}}"'] \arrow[d, "\mu_{i_0,i_1}|_{U'_{i_1}}"] \\
I_0^{e_{a_1}} \arrow[u, bend right, "\mathrm{In}"'] \arrow[d, bend right, "\mathrm{Pr}"'] & V'_{a_1} \arrow[l, dashed, "\Psi_1"'] \arrow[ur, dashed, bend right, "\Gamma_1"'] & \\
I_0^{d_{i_0}} \arrow[u, bend right, "\mathrm{In}"'] & & U_{i_0}^1 \arrow[ll, "\varphi_0^0"] \arrow[uu, bend right, "\mu_{i_1} \circ \lambda^1 \circ \varphi_0^0"']
\end{tikzcd}$$

where this diagram's meaning in terms of coordinates is that the following diagram commutes for each $\left(t_1, \ldots, t_{d_{i_0}}\right) \in I_0^{d_{i_1}}$:

$$\begin{tikzcd}
\left(t_1, \ldots, t_{d_{i_0}}, \ldots, t_{e_{a_0}}, \ldots t_{d_{i_1}}\right) \arrow[r, "\Psi_1 \circ F_{i_1}^{a_1} \circ (\Phi_1)^{-1}"] \arrow[d, "\varphi_0^0 \circ \mu_{i_0,i_1} \circ (\Phi_1)^{-1}"'] & \left(t_1, \ldots, t_{d_{i_0}}, \ldots, t_{e_{a_0}}\right) \\
\left(t_1, \ldots, t_{d_{i_0}}\right) &
\end{tikzcd}$$

where $\Phi_{l+1}\Gamma_1$'s and $\Phi_{l+1} \circ \mu_{i_1} \circ \lambda^1 \circ \varphi_0^0$'s coordinate representations are both just the canonical inclusions.

In particular, the following diagram commutes:

$$\begin{tikzcd}
& U'_{i_1} \arrow[rr, "F^{a_1}_{i_1}"] \arrow[dl, "\Phi_1"'] \arrow[dd, "\mu_{i_0,i_1}"] & & V'_{a_1} \arrow[ll, bend left, "\Gamma_1"] \arrow[dl, "\Psi_1"] \\
I_0^{d_{i_1}} \arrow[rr, "\mathrm{Pr}"] \arrow[dd, "\mathrm{Pr}"'] & & I_0^{e_{a_1}} \arrow[ll, bend left, "\mathrm{In}"] & \\
& U'_{i_0} \arrow[dl, "\varphi_0^0"] & & \\
I_0^{d_{i_0}} & & &
\end{tikzcd}$$

Recall that $J_1 = [-b_1, b_1] \subseteq I_0$ and observe that since Sys_M's bonding maps consist of Serre fibrations and $(\Phi_1)^{-1} : I_0^{d_{i_1}} \to U'_{i_1}$'s restriction to $J_1^{d_{i_0}} \times \{0\}^{d_{i_1} - d_{i_0}}$ agree with $\mu_{i_1} \circ \lambda^1 : I_0^{d_{i_0}} \to U'_{i_1}$'s restriction to $J_1^{d_{i_0}}$, we may use induction to find a smooth μ_{i_1}-lift $\Lambda^1 : J_1^{d_{i_1}} \to M$ of $(\Phi_1)^{-1}\big|_{J_1}^{d_{i_1}} : J_1^{d_{i_1}} \to U'_{i_1}$ extending $\lambda^1\big|_{J_1^{d_{i_1}}} : J_1^{d_{i_1}} \to M$.

The compactness of $J_1^{d_{i_1}}$ allows us to find $a_2 > a_1$ such that $e_{a_2} > d_{i_1}$ and $F^{a_2} \circ \Lambda^1$ is a smooth local embedding around every point of $J_1^{d_{i_1}}$. And since $\Lambda^1\left(J_1^{d_{i_1}}\right)$ is a compact subset of M, we may find some $i_2 > a_2$ such that $\mu_{i_2}\left(\Lambda^1\left(J_1^{d_{i_1}}\right)\right) \subseteq \mathrm{ODom}_{i_2} F^{a_2}$ and $d_{i_2} > e_{i_2}$. Now, although $F^{a_2} \circ \Lambda^1$ need not be an embedding, the fact that Γ_1 is a section of $F^{a_1}_{i_1}\big|_{U'_{i_1}} : U'_{i_1} \to V'_{a_1}$ whose coordinate representation is the canonical inclusion allows us to conclude that $F^{a_1} \circ (\Lambda^1 \circ \Gamma_1)$ is the identity map on $\Psi_1^{-1}\left(J_1^{e_{a_1}}\right)$ so that the equality

$$F^{a_1} \circ \Lambda^1 \circ \Gamma_1 = \nu_{a_1,a_2} \circ F^{a_2}_{i_1} \circ \mu_{i_1} \circ \Lambda^1 \circ \Gamma_1$$

implies that $F^{a_2}_{i_1} \circ \mu_{i_1} \circ \Lambda^1 \circ \Gamma_1$ is also an embedding. Thus $F^{a_2} \circ \Lambda^1$'s restriction to $J_1^{e_{a_1}} \times \{0\}^{d_{i_1} - e_{a_1}}$ is an embedding where since it's also local embedding, we can therefore find some neighborhood O of $J_1^{e_{a_1}} \times \{0\}^{d_{i_1} - e_{a_1}}$ in $J_2^{d_{i_1}}$ on which $F^{a_2} \circ \Lambda^1$ restricts to an embedding. By the tube lemma, we may find some $\epsilon > 0$ such that $J_1^{e_{a_1}} \times [-\epsilon, \epsilon]^{d_{i_1} - e_{a_1}}$ is contained O. Let $T \underset{\text{def}}{=} I_1^{d_{i_1}} \times]-\epsilon, \epsilon[^{d_{i_1} - e_{a_1}}$ and $U_{i_1}^1 = \Phi_1^{-1}(T)$ where recall that $I_1 =]-c_1, c_1[\subseteq J_1$ so that T is in Φ_1's range. Let $\varphi_1^1 : U_{i_1}^1 \to I_1^{d_{i_1}}$ (resp. $\lambda^1 : I_1^{d_{i_1}} \to M$) denote the map that results from restricting Φ_1 to $U_{i_1}^1$ (resp. restricting Λ^1 to T) and then uniformly scaling each of its last $d_{i_1} - e_{a_1}$ coordinates by c_2/ϵ (resp. uniformly scaling each of the last $d_{i_1} - e_{a_1}$ coordinates in Λ^1 domain by ϵ/c_2) and observe that λ^1 is a μ_{i_1}-lift of $(\varphi_1^1)^{-1}$. Let $V_{a_1}^1 = F_{i_1}^{a_1}\left(U_{i_1}^1\right)$ and let ψ_1^1 denote Ψ_1's restriction to this set.

Note that since all we did to obtain φ_1^1 was to restrict Φ_1's domain and scale only the large $d_{i_1} - e_{i_1}$, all of the above diagrams continue to commute with if we use I_1, φ_1^1, ψ_1^1, and ϕ_0^0's restriction to $\phi_0^1\left(I_1^{d_{i_0}}\right)$ in place of I_0, Φ_1, Ψ_1, and ϕ_0^0, respectively.

Let $\psi_1^2 = \psi_1^1\big|_{I_2^{e_{a_1}}}$, $\rho_1^2 = \rho_1^1\big|_{I_2^{e_{a_1}}}$, let $\lambda^2 = \Lambda^1\big|_{I_2^{d_{i_1}}}$, and for $h = 0, 1$, let $\varphi_h^2 = \varphi_h^1\big|_{I_2^{d_{i_h}}}$.

Inductive case $l > 1$:

Remark 13.2.5. The construction of all φ_l^1, ψ_l^1, λ^l, i_l and a_i for all $l > 1$ is essentially the same as the construction that was done above. The only real differences being that now we must state the long and complicated inductive hypotheses and occasionally observe that nothing goes wrong by having ψ's on the lower portions of our diagrams. Also, observe that λ^1 (and all subsequent λ^l's) implicitly hold all of the information needed to reconstruct the charts; however, although constructing only these λ^l's would simplify this construction, attempting to verify that the charts induced by theses λ^l's satisfy all of this theorem's conclusions becomes significantly more difficult.

Suppose that $l \geq 1$, φ_0^0 and φ_1^1 are the maps defined above, and that we've found $i_{l+1} > a_{l+1} > \cdots > i_0$ and constructed smooth charts $\varphi_0^0, \ldots, \varphi_l^l$ whose images are $I_0^{d_{i_0}}, \ldots, I_l^{d_{i_l}}$ and whose domains are contained in $\operatorname{ODom}_{i_h} F^{a_h}$ (for $h = 1, \ldots, l$) and charts $\psi_1^1, \ldots, \psi_l^l$ whose images are $I_a^{e_{i_a}}, \ldots, I_l^{e_{i_l}}$ and a smooth μ_{i_l}-lift λ^l of $\left(\varphi_l^l\right)^{-1} : I_l^{d_{i_l}} \to U_{i_l}^1$ such that if we define $U_{i_h}^l = \varphi_h^{-1}\left(I_1^{d_{i_h}}\right)$ and $V_{a_h}^l = \varphi_h^{-1}\left(I_l^{e_{a_h}}\right)$ and if we denote these charts respective restrictions to these sets by $\varphi_0^l, \ldots, \varphi_l^l$ and $\psi_1^l, \ldots, \psi_l^l$ then defining $O_{i_{l+1}} = \operatorname{ODom}_{i_l+1} F^{a_{l+1}} \cap \mu_{i_l, i_{l+1}}^{-1}\left(U_{i_l}^l\right)$ and $O = \mu_{i_{l_1}}^{-1}\left(O_{i_{l+1}}\right)$ will make the following diagrams commute:

$$
\begin{tikzcd}
 & U_{i_l}^l \arrow[rr, "F_{i_l}^{a_l}"] \arrow[dl, "\varphi_l^l"'] \arrow[dd, "\mu_{i_h,i_l}"] & & V_{a_l}^l \arrow[dl, "\psi_l^l"'] \arrow[dd, "\nu_{a_h,a_l}"] \\
I_l^{d_{i_l}} \arrow[rr, "\mathrm{Pr}"] \arrow[dd, "\mathrm{Pr}"'] & & I_l^{e_b} \arrow[ll, bend left, "\mathrm{In}_{a_l}"] \arrow[dd, "\mathrm{Pr}"] & \\
 & U_{i_h}^l \arrow[rr, "F_{i_h}^{a_h}"] \arrow[dl, "\varphi_h^l"'] \arrow[dd, "\mu_{i_h,i_l}"] & & V_{a_h}^l \arrow[dl, "\psi_h^l"'] \\
I_l^{d_{i_h}} \arrow[rr, "\mathrm{Pr}"] \arrow[dd, "\mathrm{Pr}"'] & & I_l^{e_{a_h}} \arrow[ll, bend left, "\mathrm{In}_{a_h}"] & \\
 & U_{i_0}^l \arrow[dl, "\varphi_0^l"'] & & \\
I_l^{d_0} & & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & O \arrow[dd, "\mu_{i_{l+1}}"] \arrow[ddrr, "F^{a_{i_{l+1}}}"] & & \\
\mathbb{R}^{\mathbb{N}} & & & \\
 & O_{i_{l+1}} \arrow[rr, "F_{i_{l+1}}^{a_{l+1}}"] \arrow[d, "\mu_{i_l,i_{l+1}}"] & & F^{a_{l+1}}(O) \arrow[d, "\nu_{a_l,a_{l+1}}"] \\
 & U_{i_l}^l \arrow[rr, "F_{i_l}^{a_l}"] \arrow[dl, "\varphi_l^l"'] & & V_{a_l}^l \arrow[dl, "\psi_l^l"'] \\
I_l^{d_{i_l}} \arrow[rr, "\mathrm{Pr}"] \arrow[uuuur, "\lambda^l"] & & I_l^{e_{a_l}} \arrow[ll, bend left, "\mathrm{In}_{a_l}"] &
\end{tikzcd}
$$

(where all bonding maps and $F_{u_\bullet}^{a_\bullet}$ are restricted to the domains shown above) and such that

$F^{a_{l+1}} \circ \lambda^1$ is a smooth embedding with λ^1 valued in O.

Observe that we are now in almost the exact same situation as we were in step $l = 1$ and by the same reasoning as described above, we may find so as to obtain smooth charts $\Phi_{l+1} : \left(U'_{i_{l+1}}, m^0_{i_{l+1}}\right) \to \left(I_0^{d_{i_{l+1}}}, \mathbf{0}\right)$ and $\Psi_{l+1} : \left(V'_{a_{l+1}}, n^0_{a_{l+1}}\right) \to \left(I_0^{e_{a_{l+1}}}, \mathbf{0}\right)$, and a smooth section $\Gamma_1 : V'_{a_{l+1}} \to U'_{i_{l+1}}$ of the surjective submersion $F^{a_{l+1}}_{i_{l+1}}\big|_{U'_{i_{l+1}}} : U'_{i_{l+1}} \to V'_{a_{l+1}}$, such that $U'_{i_{l+1}} \subseteq O_{i_{l+1}}$, $V'_{a_{l+1}} = F^{a_{l+1}}_{i_{l+1}}\left(U'_{i_{l+1}}\right)$, $\mu_{i_l, i_{l+1}}\left(U'_{i_{l+1}}\right) = U^l_{i_l}$, and such that the following diagram commutes:

$$\begin{tikzcd}
I_0^{d_{i_{l+1}}} \arrow[d, bend right, "\mathrm{Pr}"'] & & U'_{i_{l+1}} \arrow[ll, dashed, "\Phi_{l+1}"'] \arrow[dl, "F^{a_{l+1}}_{i_{l+1}}\big|_{U'_{i_{l+1}}}"'] \arrow[dd, "\mu_{i_l, i_{l+1}}\big|_{U'_{i_{l+1}}}"'] \\
I_0^{e_{a_{l+1}}} \arrow[u, bend right, "\mathrm{In}"'] \arrow[d, bend right, "\mathrm{Pr}"'] & V'_{a_{l+1}} \arrow[l, dashed, "\Psi_{l+1}"'] \arrow[ur, dashed, bend right, "\Gamma_{l+1}"'] & \\
I_0^{d_{i_l}} \arrow[u, bend right, "\mathrm{In}"'] & & U^1_{i_l} \arrow[ll, "\varphi^l_l"] \arrow[uu, bend right, "\mu_{i_{l+1}} \circ \lambda^l \circ \varphi^l_l"']
\end{tikzcd}$$

where this diagram's meaning in terms of coordinates is that the following diagram commutes for each $\left(t_1, \ldots, t_{d_{i_{l+1}}}\right) \in I_0^{d_{i_{l+1}}}$:

$$\begin{tikzcd}
\left(t_1, \ldots, t_{d_{i_l}}, \ldots, t_{e_{a_{l+1}}}, \ldots t_{d_{i_{l+1}}}\right) \arrow[r, "\Psi_{l+1} \circ F^{a_{l+1}}_{i_{l+1}} \circ (\Phi_{l+1})^{-1}"] \arrow[d, "\varphi^l_l \circ \mu_{i_l, i_{l+1}} \circ (\Phi_{l+1})^{-1}"'] & \left(t_1, \ldots, t_{d_{i_l}}, \ldots, t_{e_{a_{l+1}}}\right) \\
\left(t_1, \ldots, t_{d_{i_l}}\right) &
\end{tikzcd}$$

where $\Phi_{l+1} \circ \Gamma_{l+1}$'s and $\Phi_{l+1} \circ \mu_{i_{l+1}} \circ \lambda^l \circ \varphi^l_l$'s coordinate representations are both just the canonical inclusions.

In particular, observe that these properties imply, the following diagram commutes :

$$
\begin{tikzcd}
 & \mu_{i_{l+1}}^{-1}\left(U'_{i_{l+1}}\right) \arrow[d, "\mu_{i_{l+1}}"] \arrow[drr, "F^{a_{i_{l+1}}}"] & & \\
 & U'_{i_{l+1}} \arrow[rr, "F^{a_{l+1}}_{i_{l+1}}"] \arrow[dl, "\Phi_{l+1}"'] \arrow[dd, "\mu_{i_l,i_{l+1}}"] & & V'_{a_{l+1}} \arrow[ll, bend left, "\Gamma_{l+1}"] \arrow[dl, "\Psi_{l+1}"] \arrow[dd, "\nu_{a_l,a_{l+1}}"] \\
I_l^{d_{i_{l+1}}} \arrow[rr, "\mathrm{Pr}"] \arrow[dd, "\mathrm{Pr}"] & & I_l^{e_{a_{l+1}}} \arrow[ll, bend left, "\mathrm{In}_{a_l}"] \arrow[dd, "\mathrm{Pr}"] & \\
 & U^l_{i_l} \arrow[rr, "F^{a_l}_{i_l}"] \arrow[dl, "\varphi^l_l"'] & & V^l_{a_l} \arrow[dl, "\psi^l_l"] \\
I_l^{d_{i_l}} \arrow[rr, "\mathrm{Pr}"] \arrow[uuuur, bend left, "\lambda^l"] & & I_l^{e_{a_l}} \arrow[ll, bend left, "\mathrm{In}_{a_l}"] &
\end{tikzcd}
$$

Recall that $J_{l+1} = [-b_{l+1}, b_{l+1}] \subseteq I_l$ and observe $(\Phi_{l+1})^{-1} : I_0^{d_{i_1}} \to U'_{i_1}$'s restriction to $J_{l+1}^{d_{i_l}} \times \{0\}^{d_{i_{l+1}} - d_{i_l}}$ agree with $\mu_{i_{l+1}} \circ \lambda^l : I_l^{d_{i_l}} \to U'_{i_{l+1}}$'s restriction to $J_{l+1}^{d_{i_l}}$, we may use induction to find a smooth $\mu_{i_{l+1}}$-lift $\Lambda^{l+1} : J_{l+1}^{d_{i_{l+1}}} \to M$ of $(\Phi_{l+1})^{-1}\big|_{J_1}^{d_{i_1}} : J_1^{d_{i_1}} \to U'_{i_1}$ extending $\lambda^1\big|_{J_{l+1}^{d_{i_{l+1}}}} : J_{l+1}^{d_{i_{l+1}}} \to M$.

The compactness of $J_{l+1}^{d_{i_{l+1}}}$ allows us to find $a_{l+2} > a_{l+1}$ such that $e_{a_{l+2}} > d_{i_{l+1}}$ and $F^{a_{l+2}} \circ \Lambda^{l+1}$ is a smooth local embedding around every point of $J_{l+1}^{d_{i_{l+1}}}$. And since $\Lambda^{l+1}\left(J_{l+1}^{d_{i_{l+1}}}\right)$ is a compact subset of M, we may find some $i_{l+2} > a_{l+2}$ such that $\mu_{i_{l+2}}\left(\Lambda^{l+1}\left(J_{l+1}^{d_{i_{l+1}}}\right)\right) \subseteq \mathrm{ODom}_{i_{l+2}} F^{a_{l+2}}$ and $d_{i_{l+2}} > e_{i_{l+2}}$.

By exactly the same reasoning as in step $l = 1$, we may find an $\epsilon > 0$ and such that $F^{a_{l+2}} \circ \Lambda^{l+1}$ restricts to a smooth embedding on $T = I_{l+1}^{d_{i_{l+1}}} \times]-\epsilon, \epsilon[^{d_{i_{l+1}} - e_{a_{l+1}}}$ and then by defining $U_{i_{l+1}}^{l+1} = Phi_{l+1}(T)$ and scaling only the last $d_{i_{l+1}} - e_{a_{l+1}}$ coordinates we may obtain the desired $\varphi_{l+1}^{l+1} : U_{i_{l+1}}^1 \to I_{l+1}^{d_{i_{l+1}}}$ and $\lambda^{l+1} : I_{l+1}^{d_{i_{l+1}}} \to M$ from Φ_{l+1} and Λ^{l+1}. Let $V_{a_{l+1}}^{l+1} = F_{i_{l+1}}^{a_{l+1}}\left(U_{i_{l+1}}^{l+1}\right)$ and let ψ_{l+1}^{l+1} denote Ψ_{l+1}'s restriction to this set. As in step $l = 1$, all of the above diagrams continue to commute with if we respectively use I_{l+1}, φ_{l+1}^{l+1}, and ψ_{l+1}^{l+1}, in place of I_l, Φ_{l+1}, Ψ_{l+1}, and also use, for all indices $h \leq l$, ϕ_h^l's restriction to $\phi_h^l\left(I_{l+1}^{d_{i_h}}\right)$ and ψ_h^l's restriction to $\psi_h^l\left(I_{l+1}^{e_{a_h}}\right)$. These maps just described together with λ^{l+1} now allow to continue the inductive construction.

It should now be clear that the desired domains are $U_{i_l} := \left(\varphi_l^l\right)^{-1}\left(]-1,1[^{d_{i_l}}\right)$, $V_{i_l} := \left(\psi_l^l\right)^{-1}\left(]-1,1[^{e_{a_l}}\right)$, that the desired $\varphi_\bullet$'s and \$$psi_\bullet$ are the restrictions of $\varphi_\bullet^\bullet$ and $\psi_\bullet^\bullet$ to these domains, that $U := \varprojlim U_{i_\bullet}$, $V := \varprojlim V_{a_\bullet}$ are the desired sub-promanifolds making $F\big|_U : U \to V$ into a diffeomorphism. ■

Local Injectivity and Vector Field Germ Submersions and Immersions

The following definition will provide a sufficient condition for a smooth map to be locally injective at a point.

Definition 13.3.1. Let $F : (M, m) \to (N, n)$ be smooth and let $\mathcal{G}$ (resp. $\mathcal{H}$) denote collections of rough vector fields defined on neighborhoods of m in M (resp. n in N). For $\Phi \in [\mathcal{G}]_m$ and $\Psi \in [\mathcal{H}]_n$, we will write $F(\Phi) = \Psi$ and say that F *pushes (forward)* Φ *to* Ψ if there exist $X \in \Phi$ and $Y \in \Psi$ such that

(1) $F(\operatorname{Dom} X) \subseteq \operatorname{Dom} Y$, and

(2) $(\mathrm{T}\, F)(X_{\widehat{m}}) = Y_{F(\widehat{m})}$ for every $\widehat{m} \in \operatorname{Dom} X$.

Remark 13.3.2. Although it may appear that we now have ambiguous notation for the pushforward of a germ of maps due to the fact that this definition's notation is identical to notation 1.1.26, this is not the case since with notation 1.1.26 we are pushing forward germs of maps valued in $M = \operatorname{Dom} F$ while in the above notation we are pushing forward germs of maps valued in $\mathrm{T}\, M$.

If we write $F(\Phi) \in [\mathcal{H}]_n$ for some $\Phi \in [\mathcal{G}]_m$ then we mean that there exists some $\Psi \in [\mathcal{H}]_n$ such that $F(\Phi) = \Psi$, in which case we will say that F *pushes forward* (or that F *can push forward*) Φ *to (a germ of)* $\mathcal{H}$. If $\Phi \in \mathcal{H}_m$ then we will say that $F(\Phi)$ *is (roughly) definable* *(resp. C^k-definable)* if $F(\Phi) \in \left[\mathscr{X}_{loc}^{rough}(N,n)\right]_n$ (resp. $F(\Phi) \in \left[\mathscr{X}_{loc}^{k}(N,n)\right]_n$), where recall that $\mathscr{X}_{loc}^{rough}(N,n)$ (resp. $\mathscr{X}_{loc}^{k}(N,n)$) denotes the set of all rough (resp. C^k) vector fields defined on neighborhoods of n in N. If we write

$$F : [\mathcal{G}]_m \to [\mathcal{H}]_n$$

or $F([\mathcal{G}]_m) \subseteq [\mathcal{H}]_n$ then we mean that $F(\Phi) \in [\mathcal{H}]_n$ for all $\Phi \in [\mathcal{G}]_m$ and in this case we will say that $F : M \to N$ *maps germs of $\mathcal{G}$ (at m) to germs of $\mathcal{H}$ (at n).*

We will say that $F : [\mathcal{G}]_m \to [\mathcal{H}]_n$ *is a vector field germ*

(1) *submersion at m* if for all $\Psi \in [\mathcal{H}]_n$ there exists some $\Phi \in [\mathcal{G}]_m$ such that $F(\Phi) = \Psi$.

(2) *immersion at m* if for all $\Phi, \widehat{\Phi} \in [\mathcal{G}]_m$, $F(\Phi) = F\left(\widehat{\Phi}\right)$ implies $\Phi = \widehat{\Phi}$.

(3) *bijection at m* if it is both a vector field germ submersion at m and a vector field germ immersion at m.

If we say that *$F : M \to N$ is a C^k-vector field germ submersion (resp. immersion, bijection) at m* then we mean that this is true of

$$F : \left[\mathscr{X}_{loc}^{k}(M,m)\right]_m \nwarrow \left[\mathscr{X}_{loc}^{k}(N,n)\right]_n \quad \text{(resp.} \quad F : \left[\mathscr{X}_{loc}^{k}(M,m)\right]_m \to \left[\mathscr{X}_{loc}^{k}(N,n)\right]_n\text{)}$$

■

Although theorem A.3.2 provides a simple sufficient condition for a smooth map $F : (M,m) \to (N,n)$ to be locally injective on some dense open subset of M, it may be desirable to know that F is locally injective at m in particular and so it is for these ends that we provide the following result.

Proposition 13.3.3. Let $F : (M,m) \to (N,n)$ be a smooth pointwise immersion and suppose that F can pushforward germs of smooth global vector fields at m to germs of rough local vector fields at n (i.e. $F : [\mathscr{X}^{\infty}(M)]_m \to \left[\mathscr{X}_{loc}^{rough}(N,n)\right]_n$). If $\dim_m M = \infty$ then there exists some $m \in U \in \operatorname{Open}(M)$ on which F is injective.

Proof. If m is an accumulation point of $F^{-1}(n) \setminus \{m\}$ then we may apply lemma 8.1.13 to the appropriately selected sequences to obtain a contradiction. Thus there exists a neighborhood W of m in M such that $\{m\} = F^{-1}(n) \cap W$. Now suppose that there is no such U. Then we may again obtain a contradiction by applying lemma 8.1.13 to appropriately selected sequences, where it is obvious how this sequence should be selected so as to obtain a contradiction. ∎

Substitute Inverse Function Theorems

Most of the rest of this paper is dedicated to characterizing the topologies of monotone promanifolds and it is recommended that the reader return to this section after becoming familiar with the contents of this paper's subsequent chapters.

We will see that theorem 16.5.3 provides sufficient condition for a smooth isomersion into a monotone promanifold to be open in terms of smooth almost arcs, which are smooth topological embeddings whose derivatives vanish at exactly one point. But as we have seen in proposition 13.3.3, a map being a vector field germ immersion that can push forward germs of vector fields provides a sufficient condition for local injectivity at a point so if we combine these two results then it follows from theorem 11.6.1 that the map is a local diffeomorphism at a point. We summarize this in theorem 13.4.2.

Theorem 13.4.1 (Local Diffeomorphism Characterization)**.** Let $F : (M, m^0) \to (N, n^0)$ be a smooth isomersion from any promanifold M into a monotone promanifold N and suppose that $\dim_{m^0} M = \infty$. Then $F : M \to N$ a local diffeomorphism at m^0 if and only if

(1) there is some open neighborhood U of m^0 such that for all smooth almost arcs η in N, each non-empty connected component of $U \cap F^{-1}(\operatorname{Im} \eta)$ contains at least two distinct point, and

(2) $F : [\mathscr{X}^{\infty}(M)]_{m^0} \to \left[\mathscr{X}_{loc}^{rough}(N, n^0)\right]_{n^0}$ is a vector field germ-immersion at m^0.

Proof. By theorem 16.5.2, the first condition guarantees that $F\big|_U : U \to N$ is an open map while the second condition and proposition 13.3.3 guarantees local injectivity at m^0. Thus there must exist some open neighborhood of m^0 on which F is an open embedding so that theorem 11.6.1 then allows us to conclude that $F : M \to N$ is a diffeomorphism on this neighborhood. ∎

Theorem 13.4.2 (Local Diffeomorphism Characterization)**.** Let $F : (M, m^0) \to (N, n^0)$ be a smooth isomersion from any promanifold M into a monotone promanifold N and suppose that $\dim_{m^0} M = \infty$. Then $F : M \to N$ a local diffeomorphism at m^0 if and only if

(1) F is a germ submersion on some neighborhood of m^0 in M from smooth almost arcs in N to C^0-paths in M, and

(2) $F : \left[\mathscr{X}^{\infty}_{loc}(M, m^0)\right]_{m^0} \to \left[\mathscr{X}^{\infty}(N)\right]_{n^0}$ is a vector field germ-bijection at m^0.

Proof. This is an immediate consequence of theorem 13.4.1. ∎

Corollary 13.4.3. Let $F : (M, m^0) \to (N, n^0)$ be a smooth isomersion into a monotone promanifold N suppose that $\dim_{m^0} M = \infty$. Then $F : M \to N$ is a local diffeomorphism at m^0 if and only if $F : \left[\mathscr{X}^{\infty}_{loc}(M, m^0)\right]_{m^0} \to \left[\mathscr{X}^{\infty}_{loc}(N, n^0)\right]_{n^0}$ is a vector field germ bijection at m^0 and a germ submersion on some neighborhood of m^0 in M from smooth almost arcs in N to C^0-paths in M.

Remark 13.4.4. Observe that the two ingredients in the above characterize are vector fields and smooth curves, which are of course related to each other through the concept of integral curves. Now integral curves may fail to exist for smooth vector fields on promanifolds and if they do exist then they need not be unique, *but* if an integral curve does exist at a point and if it has non-zero derivative at that point then lemma 8.2.3 shows that it will necessarily be locally unique. In particular, smooth almost arcs have non-vanishing derivatives everywhere exist for at one point. The author suspects that an inverse function theorem similar to the traditional inverse functional theorem (i.e. theorem 13.0.1) could exist for monotone promanifolds.

Conjecture 13.4.5. Let $F:(M,m^0)\to(N,n^0)$ be a smooth isomersion monotone promanifolds and suppose that $\dim_{m^0} M=\infty$. Then $F:M\to N$ is a local diffeomorphism at m^0 if and only if $F:\left[\mathscr{X}^{\infty}_{loc}(M,m^0)\right]_{m^0}\to\left[\mathscr{X}^{\infty}_{loc}(N,n^0)\right]_{n^0}$ is a vector field germ-bijection at m^0.

Chapter 14

Coherence with C^p-Paths and C^p-Embeddings of Intervals

Before continuing, the reader be familiar with the Topology appendix, especially the sections dealing with sequential spaces and coherence of topologies with sets of continuous maps.

Lemma 14.0.1. Let $\mathcal{P}$ be a set of continuous paths into M such that for all $\gamma \in \mathcal{P}$, whenever $h : [0,1] \to J$ is an affine linear bijection onto a non-degenerate closed subinterval J of $\operatorname{Dom} \gamma$ then $\gamma \circ h \in \mathcal{P}$. Then M is coherent with $\mathcal{P} \iff$ for all $m^0 \in M$ and all sequences $(m^l)_{l=1}^{\infty}$ converging to $m^0 \in M$ that contains infinitely many distinct elements there exists

(1) a strictly increasing sequence of integers $(l_k)_{k=1}^{\infty}$ such that for all $j, k \in \mathbb{N}$, $j \neq k \implies m^{l_j} \neq m^{l_k}$ and $m^{l_k} \neq m^0$,

(2) a strictly decreasing sequence $(t^{l_k})_{k=1}^{\infty} \subseteq]0,1[$ converging to 0, and

(3) a map $\gamma : ([0,1], 0) \to (M, m^0)$ belonging to $\mathcal{P}$

such that $\gamma(t^{l_k}) = m^{l_k}$ for all $k \in \mathbb{N}$. Furthermore,

(a) if M is instead a Hausdorff TVS then ($\implies$) remains true while the converse is true if M is also sequential.

(b) this equivalence remains true if M is any Hausdorff sequential topological space and $\mathcal{P}$ is the set of all continuous paths in M.

Proof. Apply lemma A.5.9. ∎

Corollary 14.0.2. Let M be a manifold or a Hausdorff TVSs and let $p \in \{0, 1, \ldots, \infty\}$. For any interval I, let $\mathcal{A}_I$ denote the set of all C^p-embeddings $I \to M$ that are smooth everywhere except for possibly at a single point. Let $\mathcal{A}^*_{[0,1]}$ denote those maps in $\mathcal{A}_{[0,1]}$ that are smooth on $]0,1]$. If M is coherent with $\mathcal{A}^*_{[0,1]}$ or if there is any interval I such that M is coherent with $\mathcal{A}_I$ then M is coherent with $\mathcal{A}^*_{[0,1]}$ and with $\mathcal{A}_J$ and for all intervals J.

Proof. This follows easily from lemmata A.5.9 and A.5.15 and the fact that every C^p-embedding of $[0,1]$ into M has an extension to an open interval containing $[0,1]$. ∎

Example 14.0.3. If $p \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$ and $F : [a,b] \to M$ is a C^p-path (resp. C^p pointwise immersion) into a promanifold M then $\operatorname{Im} F$ is coherent with C^p-paths (resp. C^p-arcs).

Coherence with C^p-arcs ($p \geq 1$)

The material in this section is not needed for the study of promanifolds.

Proposition 14.1.1. Let X be a Hausdorff TVS, let Y be a closed vector subspace of X whose continuous dual space Y' separates points on Y, and let $p \in \{1, 2, \ldots, \infty\}$. If X is coherent with a set $\mathcal{C}$ of C^p-arcs in X and if there exists a continuous projection $\rho : X \to Y$ onto Y, then Y is coherent with its C^p-arcs

Proof. If $\dim Y = 0$ or 1 then Y is trivially coherent with its C^p-arcs so assume that $\dim M > 1$. Let $S \subseteq Y$ be a subset such that for all C^p-arcs γ in Y, $\gamma^{-1}(S)$ is closed in γ's domain. Let $\gamma \in \mathcal{C}$ and suppose for the sake of contradiction that $\gamma^{-1}(S)$ is not closed in γ's domain. Pick $t_0 \in \overline{\gamma^{-1}(S)} \setminus \gamma^{-1}(S)$ and let $(t_l)_{l=1}^{\infty}$ be a sequence in $\gamma^{-1}(S)$ converging to t_0. By continuity of γ, $\gamma(t_0) \in \overline{Y \cap \operatorname{Im}\gamma} \subseteq \overline{Y} = Y$, which implies that $\gamma'(t_0) = \lim\limits_{l \to \infty} \dfrac{\gamma(t_l) - \gamma(t_0)}{t_l - t_0}$ also belongs to the closed space Y. In particular, this show that $(\rho \circ \gamma)'(t_0) = \rho(\gamma'(t_0)) = \gamma'(t_0)$ does not vanish. Since Y's continuous dual space separates points on Y, we may pick an interval $I \subseteq J$ containing t_0 and some $N \in \mathbb{N}$ such that $\eta = \rho \circ \gamma\big|_I : I \to Y$ is a C^p-arc and $(t_{l+N})_{l=1}^{\infty} \subseteq I$. By assumption, $\eta^{-1}(S)$ is closed in η's domain so that $t_0 \in \eta^{-1}(S)$, which implies that $t_0 \in I \cap \gamma^{-1}(S) \subseteq \gamma^{-1}(S)$, a contradiction. Hence, $\gamma^{-1}(S)$ is closed and since γ was an arbitrary C^p-arc in X, it follows that S is closed in X and thus closed in Y, as desired. ∎

By applying proposition 14.1.1 we obtain the following corollary.

Corollary 14.1.2. If X is a TVS with a continuous dual space that separates points on X and if $p \geq 1$ then X is coherent its C^p-arcs if and only if the same is true of every closed complement (def. B.1.5) vector subspace of X.

Lemma 14.1.3. Let $(b_l)_{l=1}^{\infty}$ be a bounded sequence in a Hausdorff TVS X, $(c_l)_{l=1}^{\infty}$ non-zero reals such that $\lim\limits_{l\to\infty} |c_l| = \infty$, $\gamma : (J, t_0) \to (X, \mathbf{0})$ a C^1-curve with non-zero derivative at t_0, and $(t_l)_{l=1}^{\infty} \subseteq J$ a sequence converging to t_0 such that $\gamma(t_l) = \frac{b_l}{c_l}$ for all $l \in \mathbb{N}$. Then $((t_l - t_0)\, c_l)_{l=1}^{\infty}$is bounded. Furthermore, if $(l_k)_{k=1}^{\infty}$ is increasing and $(c_{l_k}(t_{l_k} - t_0))_{k=1}^{\infty}$ is convergent then $\lim\limits_{k\to\infty} b_{l_k} = \gamma'(t_0) \lim\limits_{k\to\infty} c_{l_k}(t_{l_k} - t_0)$ exists.

Proof. Suppose not and pick an increasing sequence $(l_i)_{i=1}^{\infty}$ of positive integers such that $(c_{l_i})_{i=1}^{\infty}$ and $(t_{l_i})_{i=1}^{\infty}$ are monotone and $((t_{l_i} - t_0)\, c_{l_i})_{i=1}^{\infty}$ is monotone and divergent to either ∞ or $-\infty$. Let U and V be balanced neighborhoods of $\mathbf{0}$ in X such that $V + V \subseteq U$ and let $v = \gamma'(t_0)$. Let $N_0 \in \mathbb{N}$ be such that $i \geq N_0$ implies $\frac{\gamma(t_{l_i}) - \gamma(t_0)}{t_{l_i} - t_0} - v = \frac{b_{l_i}}{c_{l_i}(t_{l_i} - t_0)} - v \in V$ and observe that since V is balanced, we have $v \in \frac{b_{l_i}}{c_{l_i}(t_{l_i} - t_0)} + V$. Since $(b_l)_{l=1}^{\infty}$ is bounded and $((t_{l_i} - t_0)\, c_{l_i})_{i=1}^{\infty}$ is divergent to $\pm\infty$, the sequence $\left(\frac{b_{l_i}}{(t_{l_i} - t_0) c_{l_i}}\right)_{i=1}^{\infty}$ converges to $\mathbf{0}$ so pick an integer $N \geq N_0$ such that $i \geq N$ implies $\frac{1}{(t_{l_i} - t_0) c_{l_i}} b_{l_i} \in V$. In particular, $v \in \frac{b_{l_N}}{c_{l_N}(t_{l_N} - t_0)} + V \subseteq V + V \subseteq U$ and since U was an arbitrary neighborhood of $\mathbf{0}$, this gives us the contradiction $v = \mathbf{0}$.

Now suppose that $(l_k)_{k=1}^{\infty}$ is increasing and $(c_{l_k}(t_{l_k} - t_0))_{k=1}^{\infty}$ is convergent. Since $\gamma'(t_0) :=$

$\lim_{k \to \infty} \dfrac{b_{l_k}}{c_{l_k}\left(t_{l_k} - t_0\right)}$ exists, the continuity of scalar multiplication implies that

$$\lim_{k \to \infty} b_{l_k} = \lim_{k \to \infty} \left[c_{l_k}\left(t_{l_k} - t_0\right) \cdot \frac{b_{l_k}}{c_{l_k}\left(t_{l_k} - t_0\right)} \right] = \left[\lim_{k \to \infty} c_{l_k}\left(t_{l_k} - t_0\right) \right] \cdot \left[\lim_{k \to \infty} \frac{b_{l_k}}{c_{l_k}\left(t_{l_k} - t_0\right)} \right]$$

also exists, as desired. ∎

Theorem 14.1.4. Let $p \in \{1, 2, \ldots, \infty\}$, X be a Hausdorff TVS, and $\mathcal{C}$ be a collection of C^p-curves in X whose derivatives never vanish. If X is coherent with $\mathcal{C}$ then every closed and bounded subset of X is sequentially compact.

Proof. Let B be a closed and bounded subset of X and suppose that $(b_l)_{l=1}^{\infty} \subseteq B$ is a sequence. Since B is bounded we have $\lim_{l \to \infty} \dfrac{b_l}{l} = \mathbf{0}$ so by lemma A.5.9 there exists a C^1-arc γ, an increasing sequence $(l_k)_{k=1}^{\infty} \subseteq \mathbb{N}$, and a sequence $(t_k)_{k=0}^{\infty} \subseteq \operatorname{Dom} \gamma$ with $(t_k)_{k=1}^{\infty}$ monotone converging to t_0 such that $\gamma(t_0) = \mathbf{0}$ and $\gamma(t_k) = \frac{b_{l_k}}{l_k}$ for all $k \in \mathbb{N}$. Lemma 14.1.3 now implies that $b_\bullet$ has a convergent subsequence. ∎

Theorem 14.1.5. A normable TVS is finite-dimensional if and only if it's coherent with its C^1-arcs, in which case it is coherent with the set of all its weak C^1-almost arcs.

Proof. This follows immediately from theorems 14.1.4 and 14.5.4 and the fact that a Hausdorff TVS is finite-dimensional if and only if it contains a non-empty compact neighborhood. ∎

Corollary 14.1.6. A Hausdorff TVS that contains a closed complemented infinite-dimensional normable subspace is not coherent with any set C^1-curves with non-vanishing first derivatives.

Proof. Apply theorem 14.1.5 and proposition 14.1.1. ∎

Corollary 14.1.7. Let $p \in \{1, 2, \ldots, \infty\}$. In the category of C^p-manifolds with boundary modeled on normable TVSs, C^p-manifolds are exactly those objects that are coherent with their C^1-embeddings of $\mathbb{R}$.

Proof. This follows immediately from corollary A.5.13 and theorems 14.1.5 and 14.5.4. ■

Since there always exists a (potentially non-linear) homeomorphism between any two infinite-dimensional separable Fréchet spaces [3], one is naturally led to conjecture 14.1.8.

Conjecture 14.1.8. A Fréchet space is finite-dimensional if and only if it's coherent with its C^1-arcs.

Remark 14.1.9. It's straightforward to verify that this conjecture holds if and only if it holds for all separable Fréchet spaces.

Non-Coherence with C^p-Embeddings $(p > 1)$ of Intervals

In this section, we prove theorem 14.2.3, which show that the only Hausdorff LCTVSs that are coherent with a set of C^p-embeddings for $p \geq 2$ are $\{0\}$ and the field $\mathbb{R}$. Consequently, we will subsequently be primarily interested in studying spaces that are coherent with their C^1-arcs.

The next lemma follows immediately from the inverse function theorem.

Lemma 14.2.1. Let $\gamma = (x, y) : J \to \mathbb{R}^2$ be a C^p-curve ($p \in \{1, 2, \ldots, \infty\}$) where J contains 0. If $x'(0) \neq 0$ then there exists some $\epsilon > 0$ such that $x\big|_{[-\epsilon,\epsilon]} : J \cap [-\epsilon, \epsilon] \to \mathbb{R}$ is a C^p-isomorphism onto its image and the map $y \circ \left(x\big|_{[-\epsilon,\epsilon]}^{-1}\right) : \operatorname{Im}\left(x\big|_{[-\epsilon,\epsilon]}\right) \to \mathbb{R}$ is C^p.

Example 14.2.2. For all $l \in \mathbb{N}$, let $x_l = \frac{1}{l}$ and let $y_l = x_l^{3/2}$. Then there does *not* exist any C^2-curve $\gamma : (J, 0) \to \left(\mathbb{R}^2, \{0\}^2\right)$ with $\gamma'(0) \neq \{0\}^2$ such that $\gamma(t_k) = (x_{l_k}, y_{l_k})$ for some monotone sequence $(t_k)_{k=1}^{\infty}$ in J converging to 0 and some increasing sequence $(l_k)_{k=1}^{\infty} \subseteq \mathbb{N}$.

Proof. Suppose for the sake of contradiction that a such a curve $\gamma = (x, y) : (J, 0) \to \left(\mathbb{R}^2, \{0\}^2\right)$ and sequences $(t_k)_{k=1}^{\infty}$ and $(l_k)_{k=1}^{\infty}$ did exist. If necessary, we may replace γ, J, and $(t_k)_{k=1}^{\infty}$ with, respectively, $t \mapsto \gamma(-t)$, $-J$, and $(-t_k)_{k=1}^{\infty}$ so as to assume without loss of generality that $(t_k)_{k=1}^{\infty}$ is decreasing. Since $\gamma'(0)$ does not vanish, we may by lemma 14.2.1 pick

$\epsilon > 0$ such that $\epsilon \in J$ and at least one coordinate of $\gamma\big|_{[0,\epsilon]} : [0,\epsilon] \to \mathbb{R}^2$ is a C^1-isomorphism onto its image in $\mathbb{R}$. Pick $k_0 \in \mathbb{N}$ be such that $k \geq k_0$ implies $t_k \in [0,\epsilon[$. Replacing γ, $(t_k)_{k=1}^{\infty}$, and $(l_k)_{k=1}^{\infty}$ with $\gamma\big|_{[0,\epsilon]}$, $(t_{k_0+k})_{k=1}^{\infty}$, and $(l_{k_0+k})_{k=1}^{\infty}$, respectively, we may assume without loss of generality that $\operatorname{Dom}\gamma = [0,\epsilon]$ and that at least one of γ's coordinates is a C^1-isomorphism onto its image.

If y was a C^1-isomorphism onto its image so that the map $G : \operatorname{Im} y \to \mathbb{R}$ defined by $G(r) = x(y^{-1}(r))$ is C^1 and $G(y_{l_k}) = x_{l_k}$ for all $k \in \mathbb{N}$. But then

$$G'(0) = \lim_{k\to\infty} \frac{G(y_{l_k})}{y_{l_k}} = \lim_{k\to\infty} \frac{x_{l_k}}{y_{l_k}} = \lim_{k\to\infty} l_k^{1/2} = \infty$$

gives a contradiction. Thus y is a not a C^1-isomorphism onto its image, which implies that x is a C^1-isomorphism onto its image.

So define $F : \operatorname{Im} x \to \mathbb{R}$ by $F(r) = y(x^{-1}(r))$ and observe that since $F(x_{l_k}) = y_{l_k}$ for all $k \in \mathbb{N}$ we have

$$F'(0) = \lim_{k\to\infty} \frac{y_{l_k}}{x_{l_k}} = \lim_{k\to\infty} x_{l_k}^{1/2} = 0 \quad \text{ and } \quad \lim_{k\to\infty} \frac{y_{l_k}}{x_{l_k}^2} = \lim_{k\to\infty} l_k^{1/2} = \infty$$

Since x^{-1} and y are C^2, F is twice differentiable at 0 so Taylor's theorem implies that $F^{(2)}(0) = 2 \lim\limits_{k\to\infty} \frac{y_{l_k}}{x_{l_k}^2}$ exists, which gives a contradiction. ■

In contrast to coherence with C^1-arcs, we now show that the only Hausdorff LCTVSs that are coherent with a set of C^p-embeddings of intervals for some $p > 1$, are $\mathbb{R}^1$ and $\mathbb{R}^0$.

Theorem 14.2.3. Let X be a Hausdorff TVS and $p \in \{2, 3, \ldots, \infty\}$. If X's continuous dual space has dimension 2 or more, then X is not coherent with any set C^p-embeddings of intervals.

Proof. Clearly, that $\dim X' \geq 2$ implies that there exists a closed 2-dimensional vector subspace Y of X and a continuous projection map $\rho : X \to Y$ onto Y. Now if X were coherent with a set of C^p-embeddings then by proposition 14.1.1, Y would be coherent with its C^p-arcs but since Y is linearly isomorphic to $\mathbb{R}^2$ this would imply that $\mathbb{R}^2$ is coherent with its C^p-arcs, which would contradict example 14.2.2. ■

The following corollaries follow from theorem 14.2.3, lemma A.5.9, and corollary A.5.14.

Corollary 14.2.4. If $p \in \{2, 3, \ldots, \infty\}$ and k is a set of cardinality strictly greater than 1, then $\mathbb{R}^k$ is not coherent with any set of C^p-embeddings of intervals into $\mathbb{R}^k$.

Corollary 14.2.5. If $p \in \{2, 3, \ldots, \infty\}$ then no manifold modeled on a Hausdorff LCTVS of dimension 2 or more can be coherent with any set of C^p-embeddings of intervals into it.

Corollary 14.2.6. If X is a Hausdorff TVS whose continuous dual space separates points on X and if $S \subseteq X$ is convex then S is contained in a 1-dimensional affine subspace of X if and only if S is coherent with a set of S-valued C^p-curves in X for some/all $p \in \{2, 3, \ldots, \infty\}$.

Non-Coherence of $\mathbb{R}^d$ (d infinite) with C^p-Arcs ($p \geq 1$)

Theorem 14.3.1. If $p \in \{1, 2, \ldots, \infty\}$ and d is an infinite set then the topology of $\mathbb{R}^d$ is *not* coherent with any set of C^p-embeddings of intervals.

Proof. Since every C^p-embedding is a C^1-embedding, by observation A.5.4 it suffices to prove that $\mathbb{R}^d$ is not coherent with the set of all C^1-embeddings of intervals. So suppose for the sake of contradiction that $\mathbb{R}^d$ was coherent with a set of C^1-embeddings of intervals. Let $\mathbb{N}$ denote an arbitrary countable subset of d and let $c = d \setminus \mathbb{N}$ so that we may write $\mathbb{R}^d = \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^c$. For all $l \in \mathbb{N}$, let $m^l = (m^l_i)_{i \in d}$ be the indicator function:

$$m^l_i = \begin{cases} 1 & \text{if } i = l \\ 0 & \text{otherwise} \end{cases}$$

By lemma A.5.9 there exists an increasing sequence of integers $(l_k)_{k=1}^{\infty}$, a C^1-embedding $\gamma = (\gamma_i)_{i \in d} : J \to \mathbb{R}^d$ of an interval J, and a sequence $(t_k)_{k=0}^{\infty} \subseteq J$ such that $(t_k)_{k=1}^{\infty}$ monotone converging to t_0, $\gamma(t_0) = \{0\}^d$, and $\gamma(t_k) = m^{l_k}$ for all $k \in \mathbb{N}$. For any $i \in d$ and $k \in \mathbb{N}$, if $i \in c$ then $\gamma_i(t_k) = m_i^{l_k} = 0$ while if $i \in \mathbb{N}$ then $\gamma_i(t_k) = m_i^{l_k} = 0$ for any $k \in \mathbb{N}$ such that $k > i$; either way we have

$$\gamma_i'(t_0) = \lim_{k \to \infty} \frac{0}{t_{l_k} - t_0} = 0$$

But then $\gamma'(t_0) = (\gamma_i'(t_0))_{i \in d} = \{0\}^d$, contradicting the fact that γ is a C^1-embedding. ■

Coherence of $\mathbb{R}^d$ ($d < \infty$) with Smooth Almost Arcs

Definition 14.4.1. By a *smooth almost arc (on I) (in M)* we mean a smooth topological embedding $\gamma : I \to M$ of a non-degenerate interval I whose derivative vanishes at no more than one point and where if there exists such a point $c \in I$, then all of γ's derivatives at c also vanish (i.e. $\gamma^{(p)}(c) = \mathbf{0}$ for all $p \in \mathbb{N}$). If $c \in I$ and we say that γ is a *smooth almost arc (on I) (in M) at c* then we mean that γ is smooth almost arc and if γ has a point at which its derivative vanishes then this point is c. If we say that γ is a *smooth almost arc (on I) (in M) vanishing at c* then we mean that γ is smooth almost arc and γ' vanishes at c. We may also replace the words "on I in M" with "$I \to M$".

Lemma 14.4.2. Let $\alpha : \mathbb{R} \to [0,1]$ be a smooth non-decreasing function such that $\alpha^{-1}(0) =]-\infty, 0]$, $\alpha^{-1}(1) = [1, \infty[$, and $\alpha' > 0$ on $]0,1[$. Let $\beta = \alpha'$ and for all $n \in \mathbb{N}$, let $M_n = \sup |\operatorname{Im} \alpha^{(n)}|$, $S_n = \max\{1, M_1, \ldots, M_n\}$, and $\beta_n : \left[\frac{1}{n+1}, \frac{1}{n}\right] \to \mathbb{R}$ be $\beta_n(t) = n(n+1)\beta\left(n(n+1)\left(t - \frac{1}{n+1}\right)\right)$. Let $x_\bullet$ be a sequence in $\mathbb{R}$ and define $\gamma :]0,1] \to \mathbb{R}$ on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ by $\gamma(t) = (x_n - x_{n+1})\beta_n(t)$. Then the map $f :]0,1] \to \mathbb{R}$ defined by $f(t) = x_1 - \int_t^1 \gamma(t)dt$ is a smooth map such that $f\left(\frac{1}{n}\right) = x_n$ for all $n \in \mathbb{N}$. If in addition for all $n \in \mathbb{N}$, $|x_n - x_{n+1}| \le \epsilon_n := \frac{1}{n(n(n+1))^{n+1} S_n}$ then the map $]-\infty, 1] \to \mathbb{R}$ that extends f and is identically 0 on $]-\infty, 0]$, is smooth.

Proof. Note that for all $n \in \mathbb{N}$, $\int_{1/(n+1)}^{1/n} \gamma(t)dt = x_n - x_{n+1}$ so that $f(1) = x_1$ implies that

$f\left(\frac{1}{n}\right) = x_n$ for all $n \in \mathbb{N}$. For positive integers $n \geq k$ and $t \in \left[\frac{1}{n+1}, \frac{1}{n}\right]$, observe that

$$\left|\gamma^{(k)}(t)\right| = |x_n - x_{n+1}| \left(n(n+1)\right)^{k+1} \beta^{(k)}\left(n(n+1)\left(t - \frac{1}{n+1}\right)\right) \leq |x_n - x_{n+1}| \left(n(n+1)\right)^{n+1} S_n$$

so that $|x_n - x_{n+1}| \leq \epsilon_n$ implies that $|\gamma^{(k)}(t)| \leq \frac{1}{n}$, which proves the γ's, and thus f's, constantly 0 extension to $]-\infty, 1]$ is smooth. ∎

Lemma 14.4.3. Let $\alpha : \mathbb{R} \to [0,1]$, $\beta = \alpha'$, $\beta_\bullet$, and $\epsilon_\bullet$ be as in lemma 14.4.2. If $x_\bullet \subseteq \mathbb{R}$ is a strictly decreasing sequence such that $x_n \leq \epsilon_n$ and $|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n|$ for all $n \in \mathbb{N}$, then there exists a smooth almost arc $f : ([0,1], 0) \to (\mathbb{R}, 0)$ vanishing at 0 such that $f\left(\frac{1}{n}\right) = x_n$ for all $n \in \mathbb{N}$.

Proof. Using lemma 14.4.2 twice (with the same α) gives us smooth non-decreasing functions $F, G :]-\infty, 1] \to \mathbb{R}$ that are both 0 on $]-\infty, 0]$ and that satisfy $F\left(\frac{1}{n}\right) = x_n$ and $G\left(\frac{1}{n}\right) = x_{n+1}$ for all $n \in \mathbb{N}$ where since we're using the same α, it becomes easy to see that $G < F$ and $(F - G)' \geq 0$ on $]0,1]$. Let $\varphi :]-\infty, 1] \to \mathbb{R}$ be any smooth function such that $\varphi^{-1}(0) =]-\infty, 0]$, $\varphi' > 0$ on $]0,1]$, and $0 < \varphi < 1$ on $]0,1]$. Note that since $F - G$ and φ are positive on $]0,1]$, the same is true of their product $\delta := \varphi \cdot (F - G) :]-\infty, 1] \to \mathbb{R}$ while $0 < \varphi < 1$ on $]0,1]$ implies that $0 < \delta < G$ on $]0,1]$. Since $(F - G)' \geq 0$ while $G, \varphi, \varphi' > 0$ on $]0,1]$, the product rule gives us $\delta' > 0$ on $]0,1$. For all $n \in \mathbb{N}$, let $y_n = x_n - (G + \delta)\left(\frac{1}{n}\right) = \left(1 - \varphi\left(\frac{1}{n}\right)\right)(x_n - x_{n+1})$ where observe that $x_{n+1} < y_n < x_n$ gives us $|y_{n+1} - y_n| \leq 2\epsilon_n$, which allows us to apply lemma 14.4.2 to obtain a smooth non-decreasing function $H :]-\infty, 1] \to \mathbb{R}$ such that $H = 0$ on $]-\infty, 0]$ and $H\left(\frac{1}{n}\right) = y_n$ for all $n \in \mathbb{N}$. The desired smooth map f is $H + G + \delta$ where $f' > 0$ on $]0,1]$ since $(H + G)' \geq 0$ and $\delta' > 0$ on $]0,1]$. ∎

Proposition 14.4.4. Let $d \in \mathbb{N}$ and let $x^\bullet = \left(x_1^\bullet, \ldots, x_d^\bullet\right) \subseteq \mathbb{R}^d$ be an infinite sequence converging to x in $\mathbb{R}^d$. There exists some subsequence $x^{l_\bullet}$ of $x^\bullet$ and some smooth almost arc $\gamma : ([0,1], 0) \to (\mathbb{R}^d, x)$ vanishing at 0 such that $x^{l_\bullet} \to x$ is injective in $\mathbb{R}^d$ and $\gamma\left(\frac{1}{n}\right) = x^{l_n}$ for all $n \in \mathbb{N}$.

Proof. Assume without loss of generality $x = \{0\}^d$, $x_1^\bullet$ strictly decreasing, and for all $h = 2, \ldots, d$ either $x_h^\bullet$ is constantly 0 or otherwise it is strictly decreasing. For all $h \in \{1, \ldots, d\}$ such that $x_h^\bullet$ is constantly 0, let $\gamma_h : [0,1] \to \mathbb{R}$ be the constant 0 function. Let $\alpha : \mathbb{R} \to [0,1]$, $\beta = \alpha'$, $\beta_\bullet$, and $\epsilon_\bullet$ be as in lemma 14.4.2. For each $h \in \{1, \ldots, d\}$ such that $x_h^\bullet$ is not constant, find an increasing sequence $(n_l)_{l=1}^\infty \subseteq \mathbb{N}$ such that $x^{n_l} \le \epsilon_l$ and $x_h^{n_{l+1}} \le x_h^{n_l}/2$ for all $l \in \mathbb{N}$ and then replace $x^\bullet$ with $x^{l_\bullet}$. This allows us to assume that for all $h \in \{1, \ldots, d\}$ such that $x_h^\bullet$ is not constant, $x_h^\bullet$ satisfies the hypotheses of lemma 14.4.3 which gives us a smooth almost arc $\gamma_h : [0,1] \to \mathbb{R}$ vanishing at 0 such that $\gamma_h\left(\frac{1}{n}\right) = x_h^n$ for all $n \in \mathbb{N}$. We have thus constructed the desired smooth almost arc $\gamma = (\gamma_1, \ldots, \gamma_d) : ([0,1], 0) \to \left(\mathbb{R}^d, \{0\}^d\right)$. ∎

Theorem 14.4.5. Smooth manifolds with corners are coherent with their smooth almost arcs.

Proof. Lemma A.5.9 shows that coherence is a local property so we may assume without loss of generality that the manifold is $\mathbb{R}^d$ for some $d \in \mathbb{Z}^{\ge 0}$. If $d = 0$ then the result follows vacuously so we may assume that $d > 0$. The conclusion now follows by applying lemma A.5.9 with proposition 14.4.4. ∎

Coherence of $\mathbb{R}^d$ ($d < \infty$) with C^1-Embeddings of Intervals

Lemma 14.5.1. Suppose that $(x_l)_{l=1}^\infty$, $(y_l)_{l=1}^\infty$, and $(\epsilon_l)_{l=1}^\infty$ are sequences of reals converging to 0 with $(\epsilon_l)_{l=1}^\infty$ positive, $(x_l)_{l=1}^\infty$ positive and decreasing, and $|y_{l+1} - y_l| < \epsilon_l |x_{l+1} - x_l|$ for all $l \in \mathbb{N}$. Then there exists a C^1-function $f : ([0, x_1], 0) \to (\mathbb{R}, 0)$ such that $f'(0) = 0$, f is smooth on $]0, x_1]$, and $f(x_l) = y_l$ for all $l \in \mathbb{N}$. Furthermore, if $(y_l)_{l=1}^\infty$ is decreasing (resp. increasing, non-increasing, non-decreasing) then f is increasing (resp. decreasing, non-decreasing, non-increasing).

Proof. For each $l \in \mathbb{N}$, since $|y_{l+1} - y_l| < \epsilon_l |x_{l+1} - x_l|$ we may find a smooth function $\beta_l : [0, x_1] \to \mathbb{R}$ such that $\beta_l^{-1}(\mathbb{R} \setminus \{0\}) =]x_{l+1}, x_l[$ and $y_l - y_{l+1} = \int_{x_{l+1}}^{x_l} \beta_l(t)dt$, where if $y_l - y_{l+1} \ge 0$

(resp. $y_l - y_{l+1} \leq 0$) then $\beta_l \geq 0$ (resp. $\beta_l \leq 0$). Define $\beta : [0, x_1] \to \mathbb{R}$ by $\beta(0) = 0$ and $\beta\big|_{[x_{l+1}, x_l]} = \beta_l\big|_{[x_{l+1}, x_l]}$ for all $l \in \mathbb{N}$, where β is clearly well-defined and also smooth on $]0, x_1]$. Observe that if $(y_l)_{l=1}^{\infty}$ is monotone then β is always either non-negative or non-positive where if $(y_l)_{l=1}^{\infty}$ is strictly monotone then all of β's zeros in $]0, x_1]$ are also isolated. Note that for all $l \in \mathbb{N}$, $\sup\limits_{x \in [0, x_l]} |\beta(x)| \leq \sup\limits_{k \geq l} \epsilon_k$ so that the assumption that $\lim\limits_{l \to \infty} \epsilon_l = 0$ implies that β is continuous at 0 and hence continuous everywhere.

Let $f : [0, x_1] \to \mathbb{R}$ be the C^1 function defined by $f(x) = y_1 + \int_{x_1}^{x} \beta(t) dt$. Observe that $f(x_1) = y_1$ and for any $k \geq 2$,

$$f(x_k) = y_1 - \int_{x_2}^{x_1} \beta_1(t) dt - \cdots - \int_{x_{k-1}}^{x_k} \beta_l(t) dt = y_1 - (y_1 - y_2) - \cdots - (y_{k-1} - y_k) = y_k$$

Since f is continuous, $f(0) = \lim\limits_{l \to \infty} f(x_l) = \lim\limits_{l \to \infty} y_l = 0$. If $(y_l)_{l=1}^{\infty}$ is decreasing (resp. increasing) then since $\beta \geq 0$ (resp. $\beta \leq 0$) and β has isolated zeros in $]0, 1]$, $\int_{x_1}^{x} \beta(t) dt$ is an increasing (resp. decreasing) function of x so that the same is true of f. Similarly, if $(y_l)_{l=1}^{\infty}$ is non-increasing (resp. non-decreasing) then $\beta \geq 0$ (resp. $\beta \leq 0$) so that f is non-decreasing (resp. non-increasing). ∎

Lemma 14.5.2. Suppose that $x_\bullet = (x_l)_{l=1}^{\infty}$, $y_\bullet = (y_l)_{l=1}^{\infty}$, and $\epsilon_\bullet = (\epsilon_l)_{l=1}^{\infty}$ are sequences of reals converging to 0 with $(\epsilon_l)_{l=1}^{\infty}$ positive, $(x_l)_{l=1}^{\infty}$ never zero, and $\left(\frac{y_l}{x_l}\right)_{l=1}^{\infty}$ converging to 0. Then there exists some increasing $\iota : \mathbb{N} \to \mathbb{N}$ such that $\left|y_{\iota(k)} - y_{\iota(l)}\right| < \epsilon_k \left|x_{\iota(k)} - x_{\iota(l)}\right|$ for all $k < l$ in $\mathbb{N}$, $\left(x_{\iota(l)}\right)_{l=1}^{\infty}$ is strictly monotone, and $\left(y_{\iota(l)}\right)_{l=1}^{\infty}$ is either constantly 0 or otherwise strictly monotone.

Proof. Observe that if $(\hat{\epsilon}_l)_{l=1}^{\infty}$ is a sequence of positive reals such that $\hat{\epsilon}_l \leq \epsilon_l$ for all $l \in \mathbb{N}$ then the desired conclusion follows if we prove this lemma with $(\hat{\epsilon}_l)_{l=1}^{\infty}$ in place of $\epsilon_\bullet$, so we may assume without loss of generality that $\epsilon_1 < 1$ and $\sum\limits_{k=l+1}^{\infty} \epsilon_k < \epsilon_l / 2$ for all $l \in \mathbb{N}$. If there is an infinite subsequence of $y_\bullet$ consisting entirely of zeros then we're done, so assume otherwise and find increasing $(i_l)_{l=1}^{\infty}$ such that $(y_{i_l})_{l=1}^{\infty}$ is strictly monotone. By replacing $x_\bullet$, $y_\bullet$, and $\epsilon_\bullet$ with $(x_{i_l})_{l=1}^{\infty}$, $(y_{i_l})_{l=1}^{\infty}$, and $(\epsilon_{i_l})_{l=1}^{\infty}$ we may henceforth assume without loss of generality that $y_\bullet$ is strictly monotone. Similarly, we may assume that $x_\bullet$ is strictly monotone. Since the inequality $|y_i - y_j| < \epsilon_l |x_i - x_j|$ holds if and only if $|y_i - y_j| < \epsilon_l |(-x_i) - (-x_j)|$ holds, by replacing $x_\bullet$ with $(-x_l)_{l=1}^{\infty}$ if necessary, we may assume without loss of generality that all x_l are positive.

Pick $\iota(1) \in \mathbb{N}$ such that $l \geq \iota(1)$ implies $\left|\frac{y_l}{x_l}\right| < \epsilon_3/2$. Suppose we've picked increasing integers $0 < \iota(1), \ldots, \iota(n)$, where $n \geq 1$, such that

(1) for all $k = 1, \ldots, n$, if $l \geq \iota(k)$ then $\left|\frac{y_l}{x_l}\right| < \epsilon_{k+2}/2$,

(2) for all $k \in \mathbb{Z}$ and l, if $1 \leq k < l \leq n$ then $\left|\frac{y_{\iota(k)} - y_{\iota(l)}}{x_{\iota(k)} - x_{\iota(l)}}\right| < (\epsilon_l + \cdots + \epsilon_{k+1})/2$.

Pick $\iota(n+1) > \iota(n)$ such that for all $l \geq \iota(n+1)$, both of $\left|\frac{y_l}{x_l}\right|$ and $\left|\frac{y_{\iota(n)} - y_l}{x_{\iota(n)} - x_l} - \frac{y_{\iota(n)}}{x_{\iota(n)}}\right|$ are strictly less than $\epsilon_{n+3}/2$. Note that

$$\left|\frac{y_{\iota(n+1)} - y_{\iota(n)}}{x_{\iota(n+1)} - x_{\iota(n)}}\right| \leq \left|\frac{y_{\iota(n+1)} - y_{\iota(n)}}{x_{\iota(n+1)} - x_{\iota(n)}} - \frac{y_{\iota(n)}}{x_{\iota(n)}}\right| + \left|\frac{y_{\iota(n)}}{x_{\iota(n)}}\right| < \epsilon_{n+3}/2 + \epsilon_{n+2}/2 < \epsilon_{n+1}/2$$

which is (2) with $(n, n+1, n+1)$ in place of (k, l, n). If $n = 1$ then this completes the inductive step so that we may henceforth assume that $n > 1$.

If $1 \leq k \leq l < n+1$ with $k < n-1$ then $\left|y_{\iota(l+1)} - y_{\iota(l)}\right| < (\epsilon_{l+1}/2)\left|x_{\iota(l+1)} - x_{\iota(l)}\right| < (\epsilon_{n+1}/2)\left|x_{\iota(n+1)} - x_{\iota(k)}\right|$ so that $\left|y_{\iota(n+1)} - y_{\iota(k)}\right| \leq \left|y_{\iota(n+1)} - y_{\iota(n)}\right| + \cdots + \left|y_{\iota(k+1)} - y_{\iota(k)}\right| < (\epsilon_{n+1}/2 + \cdots + \epsilon_{k+1}/2)\left|x_{\iota(n+1)} - x_{\iota(k)}\right|$, which proves (2) and completes the inductive construction. Observe that (2) together with the fact that $\sum\limits_{l=k+2}^{\infty} \epsilon_l < \epsilon_{k+1}/2$ for all $k \in \mathbb{N}$, implies that for all $1 \leq k < l$, $\left|\frac{y_{\iota(k)} - y_{\iota(l)}}{x_{\iota(k)} - x_{\iota(l)}}\right|$ is bounded above by $(\epsilon_l + \cdots + \epsilon_{k+1})/2 < \epsilon_{k+1} < \epsilon_k$. ■

Lemma 2 of the preprint [37] appears to be false, since it would allow one to conclude that through any sequence $(x_l)_{l=1}^{\infty}$ of non-zero points in $\mathbb{R}^n$ that converges to zero and for which there exists a $v \in \mathbb{R}^n$ such that

$$\lim_{l \to \infty} \frac{d(x_l, \mathbb{R}^{\geq 0} v)}{\|x_l\|_2} = 0$$

there exists a smooth curve with nowhere vanishing derivative, whose range contains infinitely many points of $(x_l)_{l=1}^{\infty}$. But as we have seen in example 14.2.2, such a curve, even if it's merely required to be C^2 rather than smooth, does may fail to exist if $n > 1$. However, we will now show in theorem 14.5.4 that if we reduce the requirement of smoothness to C^1 then such a curve will necessarily exist.

Despite lemma 2 of [37] being false for dimensions greater than 1, elements of this lemma's attempted proof were headed in the right direction and although the author proved the below statements independently, there are nevertheless several commonalities between the attempted proofs in [37] and the author's proof of theorem 14.5.4 below. Since the author cannot guarantee that these commonalities are not the result of having read [37] prior to attempting the independent proof of theorem 14.5.4, for full disclosure and honesty, the author has encapsulated all ideas that are common to both [37] and the proof of theorem 14.5.4 below in the following lemma 14.5.3, which is in fact actually a generalization of lemma 2 of [37] with a proof that includes details omitted form [37].

Lemma 14.5.3. For all $d \in \mathbb{Z}^{\geq 0}$ and all $p \in \{0, 1, \ldots, \infty\}$ let $\star(p, d)$ denote the following statement:

$\star(p, d)$: Whenever $m_\bullet = (m^l)_{l=1}^{\infty}$ is an infinite-ranged sequence in $\mathbb{R}^d$ converging to m^0 then there exists a C^p-embedding $\gamma : ([0, \epsilon], 0) \to (\mathbb{R}^d, m^0)$, where $\epsilon > 0$, that can lift some subsequence of $m_\bullet$ to a monotone injective sequence in $[0, \epsilon]$.

Then for all p, both $\star(p, 0)$ and $\star(p, 1)$ are true, if $\star(p, 2)$ is true then $\star(p, d)$ is true for all $d \geq 2$, and to prove $\star(p, 2)$ it suffices to prove it for those sequences $(m^l)_{l=0}^{\infty} = (x^l, y^l)_{l=0}^{\infty}$ in $\mathbb{R}^2$ for which $x^0 = 0 = y^0$, both $(x^l)_{l=1}^{\infty}$ and $\left(\frac{y^l}{x^l}\right)_{l=1}^{\infty}$ are increasing to 0, and $(y^l)_{l=1}^{\infty}$ is decreasing to 0. Furthermore, this remains true if we add to statement $\star(p, d)$ the condition:

(S): γ is a smooth embedding on $]0, \epsilon]$.

Proof. If $d = 0$ then this is vacuously true while $d = 1$ is obvious. So assume that $\star(p, 2)$ is true, let $d > 2$, and for each $l \in \mathbb{N}$ write $m^l = \left(m_1^l, \ldots, m_d^l\right)$ and let $m_{\geq 2}^l = \left(m_2^l, \ldots, m_d^l\right)$. Observe that if $h : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism and $(m^{l_k})_{k=1}^{\infty}$ is any infinite-ranged subsequence of $(m^l)_{l=0}^{\infty}$ then it suffices to prove the theorem with $(h(m^{l_k}))_{k=1}^{\infty}$ in place of $(m^l)_{l=0}^{\infty}$ so that in particular, we may assume without loss of generality that (1) $m^0 = \{0\}^d$, (2) all $(m_i^l)_{l=1}^{\infty}$ are non-negative and non-increasing, (3) $(m_1^l)_{l=1}^{\infty}$ is decreasing, and (4) for all $i \in \mathbb{N}$ such that $2 \le i \le d$, either $(m_1^l)_{l=1}^{\infty}$ is constant or else it's decreasing. If $\left(m_{\ge 2}^l\right)_{l=1}^{\infty}$ contains an infinite constant subsequence $\left(m_{\ge 2}^{l_k}\right)_{l=1}^{\infty}$ then each $m_{\ge 2}^{l_k} = \{0\}^{d-1}$ so let $t_k = m_1^{l_k}$ for all $k \in \mathbb{N}$ and let $\gamma := \left(\mathrm{Id}_{\mathbb{R}}, \{0\}^{d-1}\right) : \mathbb{R} \to \mathbb{R}^d$. Thus we may assume that $\left(m_{\ge 2}^l\right)_{l=1}^{\infty}$ is injective.

Proceeding by induction, suppose that the theorem has been proved for all dimensions less than d. Pick $\delta > 0$, a C^p-embedding $\beta : ([0, \delta], 0) \to \left(\mathbb{R}^{d-1}, \{0\}^{d-1}\right)$, a sequence $(s_p)_{p=1}^{\infty} \subseteq]0, \delta[$ decreasing to 0, and an increasing sequence $(q_p)_{p=1}^{\infty} \subseteq \mathbb{N}$ such that $\beta(s_p) = m_{\ge 2}^{q_p}$ for all $p \in \mathbb{N}$, where if (S) had been assumed then we also assume that β satisfies this additional condition. Now pick $\epsilon_0 > 0$, a C^p-embedding $\alpha = (\alpha_1, \alpha_2) : ([0, \epsilon_0], 0) \to \left(\mathbb{R} \times D_\delta, \{0\}^2\right)$, a decreasing sequence $(t_k)_{k=1}^{\infty}$ in $]0, \epsilon_0[$ converging to 0, and an increasing sequence of integers $(p_k)_{k=1}^{\infty}$ such that $\alpha(t_k) = \left(m_1^{q_{p_k}}, s_{p_k}\right)$ for all $k \in \mathbb{N}$, where if (S) had been assumed then we also assume that α satisfies this additional condition. Define $\eta = (\alpha_1, \beta \circ \alpha_2) : [0, \epsilon_0] \to \mathbb{R}^d$ and observe that η is a C^p-map and an injection whose derivative (if $p \ge 1$) vanishes nowhere and that will satisfies (S) if (S) had been assumed.

Now suppose $d = 2$ and that we want to prove $\star(p, 2)$. For all $l \in \mathbb{Z}^{\ge 0}$ we may now write $m^l = (x_l, y_l)$. Observe that both $(x_l)_{l=1}^{\infty}$ and $(y_l)_{l=1}^{\infty}$ are decreasing. If $\left(\frac{y_l}{x_l}\right)_{l=1}^{\infty}$ does not contain a convergent subsequence then it contains a subsequence that diverges to infinity so switch all x_l's and y_l's and assume without loss of generality that $\left(\frac{y_l}{x_l}\right)_{l=1}^{\infty}$ monotone converges to a non-negative s. If there exists a subsequence $(x_{l_k}, y_{l_k})_{k=1}^{\infty}$ and a polynomial $p(x)$ such that $y_{l_k} = p(x_{l_k})$ for all $k \in \mathbb{N}$ then let $\gamma(x) = (x, p(x))$ and we're done, so assume without loss of generality that no such subsequence and polynomial exist. In particular, this assumption implies that $\left(\frac{y_l}{x_l}\right)_{l=1}^{\infty}$ does not contain any infinite constant subsequence so we may further assume without loss of generality that $\left(\frac{y_l}{x_l}\right)_{l=1}^{\infty}$ is strictly monotone which, in particular, implies that no (x_l, y_l) lies on the line $x \to (x, sx)$. If after rotating the points (x_l, y_l) by

$-\arctan(s)$, selecting strictly monotone subsequences, reflecting them across the x-axis if necessary (to have infinitely many positive y_l's), and then replacing $(x_l, y_l)_{l=1}^{\infty}$ with this new sequence, we may assume without loss of generality $\left(\frac{y_l}{x_l}\right)_{l=1}^{\infty}$ is strictly monotone decreasing to $s = 0$. Since $(x, y) \mapsto (-x, y)$ is a diffeomorphism we may assume without loss of generality that both $(x^l)_{l=1}^{\infty}$ and $\left(\frac{y^l}{x^l}\right)_{l=1}^{\infty}$ are increasing to 0. ∎

Theorem 14.5.4. Let $d \in \mathbb{Z}^{\geq 0}$, let $\mathcal{A}_I$ and $\mathcal{A}^*_{[0,1]}$ be defined as in corollary 14.0.2 for $p = 1$ and $M = \mathbb{R}^d$. Then $\mathbb{R}^d$ is coherent with $\mathcal{A}^*_{[0,1]}$ and with $\mathcal{A}_I$ for all intervals I. If $d > 0$ and if we instead had $M = \mathbb{R}^{d-1} \times \mathbb{R}^{\geq 0}$ then M would be coherent with $\mathcal{A}^*_{[0,1]}$, $\mathcal{A}_{[0,1]}$, $\mathcal{A}_{[0,1[}$, but not with $\mathcal{A}_{]0,1[}$.

Proof. Corollary 14.0.2 shows that it suffices to show that $\mathbb{R}^d$ is coherent with $\mathcal{A}^*_{[0,1]}$ while lemma 14.5.3 shows that it suffices to prove this statement for $d = 2$. So let $\{m^l = (x^l, y^l) : l \in \mathbb{N}\}$ be an infinite-ranged sequence in $\mathbb{R}^2$. By lemma A.5.9, it suffices to find

(1) a C^1-embedding $\gamma : ([0,1], 0) \to \left(M, \{0\}^d\right)$ that is smooth on $]0,1]$,

(2) a decreasing sequence $(t_k)_{k=1}^{\infty}$ in $]0,1]$ converging to 0,

(3) an increasing sequence of integers $(l_k)_{k=1}^{\infty}$

such that $\gamma(t_k) = m^k$ for all $k \in \mathbb{N}$ and $\gamma\big|_{D \setminus \{0\}}$ is a smooth embedding. By lemma 14.5.3, it suffices to prove this statement under the assumption that $m^0 = (0,0)$, both $(x^l)_{l=1}^{\infty}$ and $\left(\frac{y^l}{x^l}\right)_{l=1}^{\infty}$ are decreasing to 0, and $(y^l)_{l=1}^{\infty}$ is decreasing to 0 while lemma 14.5.2, we may also assume without loss of generality that $|y_k - y_l| < \frac{1}{l}|x_k - x_l|$ for all $k, l \in \mathbb{N}$ with $k > l$. We obtain the desired curve by reparameterizing the curve constructed in lemma A.5.9 by a linear transformation.

If $d > 0$ and we instead had $M = \mathbb{R}^{d-1} \times \mathbb{R}^{\geq 0}$ then the above proof goes through unchanged, except that the non-empty manifold boundary of M would, as shown in corollary A.5.13, now prevent us from concluding that M is coherent with $\mathcal{A}_{]0,1[}$. The proof fails to generalize since we may now no longer extend any $\gamma : ([0,1], 0) \to \left(M, \{0\}^d\right)$ satisfying the above properties (or any of its reparameterizations) to an open interval containing its domain. ∎

Theorem 14.5.5 follows immediately.

Theorem 14.5.5. Smooth manifolds with corners are coherent with their C^1-arcs.

Corollary 14.5.6. For any set d,

(1) d is finite if and only if $\mathbb{R}^d$ is coherent with its C^1-arcs, and

(2) d is empty or a singleton set if and only if $\mathbb{R}^d$ is coherent with its C^p-arcs for some/all $p \in \{2, 3, \ldots, \infty\}$.

Proof. This follows immediately from theorems 14.5.4 and 14.3.1. ■

Theorem 14.5.7. A C^k-manifold with corners ($k = 1, 2, \ldots, \infty$) is a C^k-manifold if and only if it's coherent with its C^1-embedding of open intervals.

Proof. This follows immediately from corollary A.5.13 and theorem 14.5.4. ■

If M is a metrizable TVS and $m^\bullet$ is a sequence in M converging to m^0 then it is well known (see [27, p. 17-18]) that one may find an increasing sequence $(l_k)_{k=1}^{\infty}$ and a smooth path $\gamma : (\mathbb{R}, 0) \to (M, m^0)$ such that $\gamma\left(\frac{1}{k}\right) = m^{l_k}$, where γ's derivative at 0 will necessarily vanish. Consequently, M's topology is coherent with the set of all C^p-paths in $\mathbb{R}^d$ for all $p \in \mathbb{Z}^{\geq 0} \cup \{\mathbb{N}\}$, which in particular gives us the following corollary.

Corollary 14.5.8. $\mathbb{R}^{\mathbb{N}}$ and all manifolds are coherent with their set of smooth paths.

Characterization of Local Path-Connectedness

The following definition will allow us to simultaneously apply the subsequent lemma 14.6.2 to various notions of path-connectedness such as C^p-path connectedness ($p \in \{0, 1, \ldots, \infty\}$), piecewise C^p-path connectedness, C^p-arc connectedness, etc.

Definition 14.6.1. Let X be a space and let $\mathcal{C}$ be a collection of maps in X. If U is a subset of X, $x, y \in U$ and $n \in \mathbb{N}$ then by a *simple $\mathcal{C}$-chain of length n in U from x to y* we mean a sequence $\gamma_1, \ldots, \gamma_n$ of U-valued maps in $\mathcal{C}$ where $x \in \operatorname{Im}\gamma_1$, $y \in \operatorname{Im}\gamma_n$, and whenever $i, j \in \{1, \ldots, n\}$ are such that $|i - j| = 1$ then $\operatorname{Im}\gamma_i \cap \operatorname{Im}\gamma_j \neq \varnothing$. By a *simple $\mathcal{C}$-chain in U from x to y* we mean a simple $\mathcal{C}$-chain of any length in U from x to y.

We will say that *U is piecewise $\mathcal{C}$-connected (resp. U is $\mathcal{C}$-connected)* if for all distinct $x, y \in U$ there exists some simple $\mathcal{C}$-chain in U from x to y (resp. that has length 1). If either (1) x is isolated, or else (2) x is non-isolated and every neighborhood of x in X contains some neighborhood of x that is piecewise $\mathcal{C}$-connected (resp. is $\mathcal{C}$-connected) then we'll say that X is *neighborhood locally piecewise $\mathcal{C}$-connected* (resp. is *neighborhood locally $\mathcal{C}$-connected) at x* where if this neighborhood can always be chosen to be open in X then we'll remove the word "neighborhood." If we don't mention a point in the above definition then we mean that it's true at every point of X. ■

Surprisingly, despite the generality of definition 14.6.1, lemma 14.6.2's mild requirements on X will nevertheless always allow us to conclude that the space in question is locally piecewise $\mathcal{C}$-connected and neighborhood locally $\mathcal{C}$-connected.

Lemma 14.6.2. Let X be a Hausdorff Fréchet-Urysohn space and let $\mathcal{C}$ be a collection of continuous maps in X such that every non-isolated $x \in X$ is contained in the image of some $\gamma \in \mathcal{C}$, $\mathcal{C}$ satisfies condition $(\star)$ in lemma A.5.9, and the following presheaf-like condition is satisfied:

$(\star\star)$: for all $\gamma \in \mathcal{C}$, $t \in \operatorname{Dom}\gamma$, and neighborhoods N of t in $\operatorname{Dom}\gamma$, there exists some neighborhood W of t contained in N such that $\gamma\big|_W \in \mathcal{C}$.

For all $n \in \mathbb{N}$, $x \in X$, and neighborhoods U of x in X, let $C_n(x, U)$ denote the set of all $y \in U$ for which there exists a simple $\mathcal{C}$-chain of length n in U from x to y and let $C(x, U) = \bigcup_{n=1}^{\infty} C_n(x, U)$. Then for all non-isolated $x \in X$ and neighborhoods U of x in X,

(1) $C_n(x, U)$ is a neighborhood of x in X for all $n \in \mathbb{N}$,

(2) if U is open in X then $C(x,U)$ is an open and closed subset of U containing x, which shows, in particular, that X is locally piecewise $\mathcal{C}$-connected at x,

(3) if $C_1(x,U) = C_2(x,U)$ then $C(x,U) = C_n(x,U)$ for all $n \in \mathbb{N}$,

(4) if all U-valued $\gamma \in \mathcal{C}$ have a connected domain then all $C_n(x,U)$'s are connected and $C(x,U)$ is the connected component of U containing x,

(5) $C(x,U) = C(y,U)$ for any $y \in C(x,U)$,

(6) $C_n(x,U) \subseteq C_{n+1}(x,U)$ for all $n \in \mathbb{N}$,

(7) if $y \in C_n(x,U)$ and $z \in C_k(y,U)$ then $x \in C_n(y,U)$ and $z \in C_{n+k}(x,U)$, and

(8) if U is open in X and for all $m \in U$ and $y, z \in C_1(m,U)$ there exists some U-valued $\gamma \in \mathcal{C}$ such that $m, y, z \in \operatorname{Im}\gamma$ then $C_1(x,U) = C(x,U)$ is open and closed in U.

Proof. Let $x \in X$ and let U be a neighborhood of x in X.

(6) - (7): Note that if $(\gamma_1, \ldots, \gamma_n)$ is a simple $\mathcal{C}$-chain in U from x to some point $y \in U$, then $(\gamma_1, \ldots, \gamma_n, \gamma_n)$ is trivially a simple $\mathcal{C}$-chain of length $n+1$, which shows that $C_n(x,U) \subseteq C_{n+1}(x,U)$ for all $n \in \mathcal{N}$. And if there is a simple $\mathcal{C}$-chain $(\eta_1, \ldots, \eta_k)$ of length k in U from y to some $z \in U$ then $(\gamma_1, \ldots, \gamma_n, \gamma_n, \eta_1, \ldots, \eta_k)$ simple $\mathcal{C}$-chain $(\eta_1, \ldots, \eta_k)$ of length $k+n$ in U from y so that $z \in C_{n+k}(x,U)$.

(1): Note that our assumptions imply that $C_1(x,U)$ contains x. Suppose that $C_1(x,U)$ was not a neighborhood of x in X. Then $x \in \operatorname{Cl}_U(U \setminus C_1(x,U))$ where since U is Fréchet-Urysohn, there exists some sequence $(x^l)_{l=1}^{\infty}$ of distinct points in $U \setminus C_1(x,U)$ that converge to x. By condition $(\star)$, there exists some X-valued $\gamma \in \mathcal{C}$ and some γ-liftable subsequence of $x^\bullet$ so that by replacing $x^\bullet$ with this subsequence, we may assume without loss of generality that $x^\bullet$ is γ-liftable. If $(t_i)_{i=1}^{\infty} \to t$ is a γ-lift of $(x^i)_{i=1}^{\infty} \to x$ then by $(\star\star)$ we may pick some neighborhood W of t contained in $\gamma^{-1}(U)$ such that $\gamma\big|_W \in \mathcal{C}$. Pick $N \in \mathbb{N}$ such that $t_N \in W$ and observe that $\gamma\big|_W(t_N) = x^N$, which contradicts the fact that $x^N \in U \setminus C_1(x,U)$. Thus $C_1(x,U)$ is a neighborhood of x in X that is contained in U and now (1) follows from (6).

(2) and (5): Clearly, for any $n \in \mathbb{N}$, if $y \in C_n(x,U)$ and $z \in C_1(y,U)$ then $z \in C_{n+1}(x,U)$ so that $y \in C_1(y,U) \subseteq C_{n+1}(x,U) \subseteq C(x,U)$, where $C_n(y,U)$ is a neighborhood of y in X. This shows that $C(x,U)$ is a neighborhood in X of each of its points and that $C(y,U) \subseteq C(x,U)$ for any $y \in C(x,U)$. With n and y as above, note that if $z \in C_k(x,U)$ for some $k \in \mathbb{N}$ then since $x \in C_n(y,U)$ we have that $z \in C_{k+n}(y,U) \subseteq C(y,U)$, which gives the reverse inclusion $C(x,U) \subseteq C(y,U)$ and proves (5).

Now suppose that $y \in U$ belongs to the closure of $C(x,U)$. Since X is Fréchet-Urysohn, we may pick a sequence $(x^l)_{l=1}^{\infty}$ in $C(x,U)$ in converging to y. By condition $(\star)$, there exists some X-valued $\gamma \in \mathcal{C}$ and some γ-liftable subsequence of $(x^l)_{l=1}^{\infty}$ where as before, we may assume without loss of generality that $(x^l)_{l=1}^{\infty}$ is γ-liftable. Let $(t_i)_{i\in\mathbb{N}} \to t$ is an γ-lift of $(x^i)_{i\in\mathbb{N}} \to y$ and using $(\star\star)$, pick some neighborhood W of t contained in $\gamma^{-1}(U)$. Pick $k \in \mathcal{P}$ such that $t_k \in W$ and let $n \in \mathbb{N}$ such that $\gamma\big|_W((t_k) = x^k \in C_n(x,U)$ where note that this, together with the fact that both x and x^k are in the image of $\gamma\big|_W$, implies that $x \in C_{n+1}(x,U)$. Thus $x \in C_{n+1}(x,U) \subseteq C(x,U)$, as desired.

(3) is proved by a straightforward induction argument.

(4): Note that $C_1(x,U)$ is just the union of all images of all U-valued maps in $\mathcal{C}$ whose images contain x so that if all U valued maps in $\mathcal{C}$ had a connected domain then $C_1(x,U)$ would be connected and from this observation, it is easy to see that one may inductively prove that all $C_n(x,U)$, and thus $C(x,U)$, are be connected. That $C(x,U)$ is the connected component of U containing x now follows from (2).

(8): If $m \in \operatorname{Cl}_U(C_1(x,U))$ then since $C_1(m,U)$ is a neighborhood of m in X there is some $z \in C_1(x,U) \cap C_1(m,U)$ so our assumption gives us a U-valued $\gamma \in \mathcal{C}$ such that $x, m, z \in \operatorname{Im}\gamma$, which implies that $m \in C_1(x,U)$. Our assumption clearly implies that $C_1(x,U) = C_2(x,U)$ so we may apply (2) and (3) to obtain the rest of (8). ■

Corollary 14.6.3. Let M be a C^p-manifold modeled on Hausdorff TVSs where $0 \leq p \leq \infty$, let $S \subseteq M$ be a Fréchet-Urysohn subspace, let $0 \leq k \leq p$, and denote the set of all C^k-paths into S by $\mathcal{P}_k$. If S is coherent with $\mathcal{P}_k$ then S is locally path-connected, neighborhood locally

$\mathcal{P}_k$-connected, and locally piecewise $\mathcal{P}_k$-connected

Warning 14.6.4. Observe that corollary 14.6.3 does *not* claim that S is locally $\mathcal{P}_k$-connected.

Theorem 14.6.5. A first-countable Hausdorff space is locally path-connected if and only if it's coherent with it continuous paths.

Proof. Let X be a first-countable Hausdorff space. If X is coherent with its paths then it's locally path-connected by lemma 14.6.2 so suppose that X is locally-path-connected and let $S \subseteq X$ be such that for all paths γ in X, $\gamma^{-1}(S)$ is closed in γ's domain. Let $x \in \overline{S}$ and let $(U_l)_{l=1}^{\infty}$ be a countable decreasing neighborhood basis of x consisting of path-connected open sets and $(x_l)_{l=1}^{\infty}$ be a sequence of points in S such that for all $k, l \in \mathbb{N}$, if $l \geq k$ then $x_l \in U_k$. For all $l \in \mathbb{N}$, define γ on $\left[\frac{1}{l+1}, \frac{1}{-l}\right]$ to be a path in U_l from $\gamma(1/(l+1)) = x_{l+1}$ to $\gamma(1/l) = x_l$ and then let $\gamma(0) = x$. Clearly, $\gamma : [0,1] \to X$ is continuous on $]0,1]$ and since $[0, 1/l] \subseteq \gamma^{-1}(U_l)$ for all $l \in \mathbb{N}$, γ is also continuous at 0. Since $\gamma^{-1}(S)$ is closed in $[0,1]$ and $\{1, 1/2, \ldots\} \subseteq \gamma^{-1}(S)$, it follows that $0 \in \gamma^{-1}(S)$ so that $x = \gamma(0) \in S$, as desired. ∎

Lemma 14.6.6. Let $S \subseteq M$, $p \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$, and suppose that both

$$\gamma_{-1} : ([-1,0], -1, 0) \to (S, m^{-1}, m) \quad \text{and} \quad \gamma_1 : ([0,1], 0, 1) \to (S, m, m^1)$$

are C^p-paths. Then the map $(\gamma_{-1} \cup \gamma_1) \circ \beta : ([-1,1], -1, 1) \to (S, m^{-1}, m^1)$ is C^p where $\beta : [-1,1] \to [-1,1]$ is some increasing homeomorphism and smooth almost arc vanishing at 0 (which exists by example B.0.1).

Definition 14.6.7. Let $\mathcal{P}$ is a set of continuous paths in a promanifold M, $S \subseteq M$, and $p \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$. For $m^0, m^1 \in S$, we'll write $m^0 \sim_{\mathcal{P}} m^1$ and say that *m^0 and m^1 are $\mathcal{P}$ path-connected in S* if $m^0 \neq m^1$ implies that there is a simple $\mathcal{P}$-chain of length 1 in S from m^0 and m^1, where if $\mathcal{P}$ consists of all C^p-paths in M then we'll write $m^0 \sim_p m^1$. If $\mathcal{P}$ consists of all C^p-paths (resp. C^p-arcs) in M then we'll say that C^p *path-connected* (resp. C^p

arc(wise)-connected) if S is $\mathcal{P}$-connected, where since lemma 14.6.6 shows that $\sim_p$ forms an equivalence relation on S, we will call each equivalence class a *C^p path-component of S*. For any $m \in M$, M is *locally C^p path-connected (resp. arc(wise)-connected) at m* if there exists a neighborhood basis at m consisting of open C^p path-connected (resp. C^p arcwise-connected) sets. A promanifold M is called *locally C^p path (resp. arc(wise)) connected* if it is locally C^p path-connected (resp. C^p arcwise-connected) at each $m \in M$. ■

Remark 14.6.8. Observe that each singleton set is vacuously C^p arcwise connected.

Coherence and 0-Dimensionality

The following lemma explores some consequences of a promanifold being coherent with its smooth paths for points at which the promanifold is 0-dimensional.

Lemma 14.7.1. Let $p \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$. If M is coherent with its C^p paths then

(1) For any $m \in M$, $\dim_m M = 0 \iff \{m\} \in \operatorname{Open}(M)$.

(2) $Z \underset{\text{def}}{=} \{m \in M \mid \dim_m M = 0\}$ is a discrete open and closed submanifold of M that equals $\{m \in M \mid \{m\} \in \operatorname{Open}(M)\}$ and each $\mu_i(Z)$ is a discrete closed and open zero-dimensional submanifold of M_i.

(3) $\left(M \setminus Z, \mu_\bullet\big|_{M \setminus Z}\right)$ is the limit of the profinite system

$$\operatorname{Sys}_{M \setminus Z} \underset{\text{def}}{=} \left(\mu_\bullet(M \setminus Z), \mu_{ij}\big|_{\mu_j(M \setminus Z)}, \mathbb{N}\right)$$

where $\mu_{ij}\big|_{\mu_j(M \setminus Z)} : \mu_j(M \setminus Z) \to \mu_i(M \setminus Z)$ is a smooth surjective submersion between open submanifolds for each $i \leq j$. Furthermore, $M \setminus Z$ is a promanifold that is coherent with its C^p paths and if M is coherent with its C^p arcs then so is $M \setminus Z$.

- In particular, this will often allow us to assume without loss of generality that M has non-zero dimension at each of its points.

(4) Z is exactly the set of points through which no non-constant C^p path passes (i.e. none of these points is not in the image of any non-constant C^p path).

- In particular, if M is coherent with its C^p-arcs then Z is exactly the set of points through which no C^p-arc passes.

Proof. (1): ($\Longleftarrow$) is always true and is proved in lemma 7.2.2 so to prove ($\Longrightarrow$) suppose that $\dim_m M = 0$ but that $\{m\}$ is not open in M. Pick a sequence $(m^l)_{l=1}^{\infty}$ converging to m in M such that $k \neq l \implies m^l \neq m^k$ and $m^l \neq m$ for all $l \in \mathbb{N}$. Pick $(l^k)_{k=1}^{\infty}$, $(t^{l_k})_{k=1}^{\infty} \subseteq]0,1[$, and a C^p path $\gamma : ([0,1],0) \to (M,m)$ as in lemma 14.0.1. For each index i, $0 = \dim_{\mu_i(m)} M_i$ so $\{\mu_i(m)\}$ is a connected component of M_i so that it contains the image of $\gamma_i \underset{\text{def}}{=} \mu_i \circ \gamma : ([0,1],m) \to (M_i, \mu_i(m))$. In particular, γ_i is constant for each index i so that γ is constantly m, which gives a contradiction.

(2): That Z is a discrete open subset of M is immediate from (1). If $m \in M \smallsetminus Z$ belongs to the closure of Z in M then every open neighborhood of m must contain infinitely many distinct elements of Z so we may pick a C^p path $\gamma : ([0,1],0) \to (M,m)$ whose image contains infinitely many of these elements of Z, which is a contradiction. Thus Z is closed in M. If $m \in Z$ then let $U_j \underset{\text{def}}{=} \{\mu_j(m)\}$ for each index j and pick an index i such that $\{m\} = \mu_i^{-1}(U_i)$, which exists since $U \underset{\text{def}}{=} \{m\}$ is open in M. Observe that $\mu_{ij}^{-1}(U_i) = U_j$ for all $j \geq i$ so that $\mu_{ij}\big|_{U_j} : U_j \to U_i$ is a diffeomorphism between open subsets, which implies that the same is true of $\mu_i\big|_U : U \to U_i$. This implies that Z is a zero-dimensional open submanifold of M.

(3): This is immediately verified.

(4): If $m \in Z$ then since $\{m\} \in \operatorname{Open}(M)$ and $[0,1]$ is connected, there can be no non-constant C^p path that passes through m. Now suppose that $m \in M$ is such that there exists no non-constant C^p path that passes through m (i.e. that contains m). Let $C = M \smallsetminus \{m\}$ and $\gamma : [0,1] \to M$ be any C^p path. If $\operatorname{Im}\gamma$ does not intersect C then $\operatorname{Im}\gamma \subseteq C = \varnothing$ is closed in $\operatorname{Im}\gamma$ while if it does intersect C then our assumption implies that $\operatorname{Im}\gamma \subseteq C$ so that we again have that $C \cap \operatorname{Im}\gamma = \operatorname{Im}\gamma$ is closed in $\operatorname{Im}\gamma$. Since γ was an arbitrary C^p path it follows that C is closed in M so that $\{m\} = M \setminus C$ is open in M. Hence, (2) implies that $m \in Z$.

Now assume that M is coherent with its C^p arcs and $m \in M$ is such that there exists no C^p-arc that passes through m. Let $C = M \setminus \{m\}$ and $\gamma : [0,1] \to M$ be any C^p arc. Since $\gamma : [0,1] \to M$ is injective, $\operatorname{Im}\gamma$ necessarily intersects C and conclusion that $m \in Z$ follows by repeating the remainder of the last paragraph but with "'C^p path" replaced with 'C^p arc." ■

Coherence with C^p-Arcs ($p > 0$)

Proposition 14.8.1. For each index i, let D^i denote the set of all $m \in M$ such that $\mu_i^{-1}(\mu_i(m)) = \{m\}$ and let O^i denote the interior (in M) of D^i. If $p \in \mathbb{N} \cup \{\infty\}$ and M is coherent with its C^p-arcs then

(1) M is coherent with its C^p paths. In particular, M is locally C^p path-connected.

(2) For all $m \in M$ there exists some $i \in \mathbb{N}$ such that $\mu_i^{-1}(\mu_i(m)) = \{m\}$ in which case $\dim_m M = \dim_{\mu_j(m)} M_j$ for all $j \geq i$. In particular, $\dim_m M < \infty$ for all $m \in M$.

(3) At each $m^0 \in M$, M is locally finite-dimensional of dimension $\dim_{m^0} M$. That is, there exists some $m^0 \in U \in \operatorname{Open}(M)$ such that $\dim_{m^0} M = \dim_m M$ for all $m \in U$.

(4) For each index i, $D_i \underset{\text{def}}{=} \mu_i(D^i)$ is closed in M_i, $D^i = \mu_i^{-1}(D^i)$ is closed in M, $i \leq j$ implies $D^i \subseteq D^j$, $\mu_i\big|_{D^i} : D^i \to D_i$ is a diffeomorphism, and there exists some non-empty O^j.

- So in particular, $\mu_i\big|_{O^i} : O^i \to \mu_i(O^i)$ is a diffeomorphism from an open subset of M onto an open submanifold of M_i.

(5) $\underset{i \in I}{\cup} O^i$ is the unique maximal open O such that M such that for all $m \in O$ there exists some index i and some $\mu_i(m) \in U_i \in \operatorname{Open}(M_i)$ and such that $\mu_i\big|_{\mu_i^{-1}(U_i)} : \mu_i^{-1}(U_i) \to U_i$ is a diffeomorphism onto U_i. Furthermore, $\underset{i \in I}{\cup} O^i$ is dense in M and it is exactly the set of all $m \in M$ for which there exists a diffeomorphism $G : U \to N$ from some $m \in U \in \operatorname{Open}(M)$ onto a $(\dim_m M)$-dimensional smooth manifold N.

- In particular, $\underset{i\in I}{\cup} O^i$ is a manifold (possibly of inhomogeneous dimension) that is dense and open in M and contains the zero-dimensional manifold Z from lemma 14.7.1.

(6) Either $p = 1$ or else M is 0 or 1 dimensional at every point in its domain.

Proof. (1): Since every C^p arc is also a C^p path it is immediate that M is coherent with its smooth paths so that lemma 14.6.2 applies.

(2): Suppose there did not exist such an index i and let $m_i \underset{\text{def}}{=} \mu_i(m)$ for each $i \in \mathbb{N}$. Let $i_1 = 1$ and pick $m^1 \in \mu_1^{-1}(m_1)$ distinct from m. Suppose we've picked $i_l > \cdots > i_1$ and $m^l \in \mu_{i_l}^{-1}(m_{i_1})$ such that all $m, m^1, \ldots, m^l$ are distinct. Pick $i_{l+1} > i_l$ such that $m_{i_{l+1}}, \mu_{i_{l+1}}(m^1), \ldots, \mu_{i_{l+1}}(m^l)$ are all distinct and pick any $m^{l+1} \in \mu_{i_{l+1}}^{-1}(m_{i_{l+1}})$ distinct from m, where note that m^{l+1} is also necessarily distinct from $m^1, \ldots, m^l$. Observe that for each $i \in \mathbb{N}$, $\mu_i^{-1}(m_i)$ contains all but finitely many elements of $\{m^l \,|\, l \in \mathbb{N}\}$.

Pick $(l_k)_{k=1}^{\infty}$, $(t^{l_k})_{k=1}^{\infty}$, and a C^p arc $\gamma : ([0,1],0) \to (M,m)$ as in lemma 14.0.1 where by replacing $(m^l)_{l=1}^{\infty}$ by $(m^{l_k})_{k=1}^{\infty}$ we may assume that $l_k = k$ for all $k \in \mathbb{N}$. Pick $l \in \mathbb{N}$ such that $(\mu_{i_l} \circ \gamma)'(0) \neq \mathbf{0}$, let $i = i_l$, and let $\gamma_i = \mu_i \circ \gamma$. Pick $0 < \delta < 1$ sufficiently small so that $\gamma_i\big|_{[0,\delta]} : [0,\delta] \to M_i$ is a C^p embedding and then pick $L \in \mathbb{N}$ greater than l such that $\mu_i^{-1}(m_i)$ contains $\{m^k \,|\, k \geq L\}$ and $k \geq L \implies t^k \in [0,\delta]$. Observe that this implies $\{m^k \,|\, k \geq L\} \subseteq \mu_i^{-1}(m_i)$ that $\gamma\big|_{[0,\delta]} : [0,\delta] \to M$ passes through each point of $\{m^k \,|\, k \geq L\}$ so that $\{m^k \,|\, k \geq L\} \subseteq \mu_i^{-1}(m_i) \cap \gamma([0,\delta])$. Since $j \neq k \implies m^j \neq m^k$ and $\mu_i(m^k) = m_i$ for all $k \geq L$, this contradicts the injectivity of $\gamma_i\big|_{[0,\delta]} : [0,\delta] \to M_i$. If $\dim_m M \neq \dim_{m_i} M_i$ then since all μ_{ij} are surjective submersions there would be some $j > i$ such for which $\mu_{jk}^{-1}(\mu_j(m))$ is infinite, contradicting what was just shown.

(3): Recall promanifolds have constant dimension on connected subsets so this follows from (1).

(4): That each D_i is closed in $M_i = \operatorname{Im} \mu_i$ and D^i is closed in M is an application of lemma A.3.1. That $i \leq j$ implies $D^i \subseteq D^j$ follows from the fact that $\mu_i^{-1}(\mu_i(m)) = \mu_j^{-1}\big(\mu_{ij}^{-1}(\mu_i(m))\big)$ for all $m \in M$. Observe that (2) implies that $M = \underset{i\in\mathbb{N}}{\cup} D^i$ so by the Baire category theorem there must exist some D^i with non-empty interior. Fix an index i, let $h \underset{\text{def}}{=} \left(\mu_i\big|_{D^i}\right)^{-1} : D_i \to D^i$, and for all $j \geq i$ let $h_j \underset{\text{def}}{=} \mu_j \circ F$ so that $h_j = \left(\mu_{ij}\big|_{\mu_{ij}^{-1}(D_i)}\right)^{-1} : D_i \to \mu_{ij}^{-1}(D_i)$. For any $j \geq i$ and any $m_i \in D_i$, since $\mu_{ij} : M_j \to M_i$ is a smooth submersion at $h_j(m_i) = \mu_{ij}^{-1}(m_i)$ it follows that there exists an open neighborhood of $h_j(m_i)$ in M_j on which μ_{ij} is a diffeomorphism between open sets so that it follows immediately that h_j is smooth at m_i. Since h_j is smooth with smooth inverse $\mu_{ij}\big|_{\mu_{ij}^{-1}(D_i)} : \mu_{ij}^{-1}(D_i) \to D_i$, it is a diffeomorphism. Since $h = \varprojlim h_\bullet$ is the limit of diffeomorphisms it follows $h : D_i \to D^i = \mu_i^{-1}(D_i)$ is also a diffeomorphism. Since μ_i is an open map, this implies that the open subset O^i of M is diffeomorphic to the finite-dimensional smooth open submanifold $\mu_i(O^i)$ of M_i.

(5): That $O \underset{\text{def}}{=} \bigcup_{i \in \mathbb{N}} O^i$ is the unique maximal open subset of M with the claimed property is easily seen. Suppose that there existed some index i and some non-empty $U_i \in \operatorname{Open}(M_i)$ such that $\mu_i^{-1}(U_i) \subseteq M \smallsetminus \overline{O}$. For each $j \geq i$ let $U_j \underset{\text{def}}{=} \mu_{ij}^{-1}(U_i)$, let $U = \mu_i^{-1}(U_i)$, and let $\operatorname{Sys}_U \underset{\text{def}}{=} \left(U_j, \mu_{jk}\big|_{U_k}, \mathbb{N}^{\geq i}\right)$ where $U = \varprojlim \operatorname{Sys}_U$. Clearly, U is coherent with the collection of all image of all smooth arcs in U so applying part (4) of this theorem to U and Sys_U in place of M and Sys_M gives us an index $j \geq i$ and a non-empty $W_j \in \operatorname{Open}(U_j)$ such that $\mu_j\big|_{\mu_j^{-1}(W_j)} : \mu_j^{-1}(W_j) \to U_j$ is injective. Hence $\mu_j^{-1}(W_j) \subseteq D^j$ and since $\mu_j^{-1}(W_j)$ is open in M it follows that $\mu_j^{-1}(W_j) \subseteq O^j \subseteq O$. But this is a contradiction since $\mu_j^{-1}(W_j) \subseteq U \subseteq M \smallsetminus \overline{O}$. Thus O is dense in M.

Now suppose that $m \in M$ is a point for which there exists $m \in U \in \operatorname{Open}(M)$ and a diffeomorphism $G : U \to N$ onto a $d = \dim_m M$-dimensional smooth manifold N. Let h be an index such that $\dim_{\mu_h(m)} M = d$. Since $\mu_h \circ G^{-1} : N \to \mu_h(U)$ is a smooth surjective submersion between d-dimensional manifolds there exists some $G^{-1}(m) \in \widetilde{W} \in \operatorname{Open}(N)$ such that $\mu_h \circ G^{-1}\big|_{\widetilde{W}} : \widetilde{W} \to \mu_h(G(\widetilde{W}))$ is a diffeomorphism, which implies that $\mu_h\big|_W : W \to \mu_h(W)$ is a diffeomorphism from $W \underset{\text{def}}{=} G(\widetilde{W})$, an open subset of U, onto $\mu_h(W)$. Pick $i \geq h$ and $\mu_i(m) \in U_i \in \operatorname{Open}(M_i)$ such that $\mu_i^{-1}(U_i) \subseteq W$ and observe that $\mu_i\big|_{\mu_i^{-1}(U_i)} : \mu_i^{-1}(U_i) \to U_i$ is a diffeomorphism between open sets since the same is true of $\mu_h\big|_{\mu_i^{-1}(U_i)} = \mu_{hi} \circ \mu_i\big|_{\mu_i^{-1}(U_i)} : \mu_i^{-1}(U_i) \to \mu_{hi}(U_i)$. In particular, this implies that $\mu_i^{-1}(U_i) \subseteq O^i$.

(6): Follows from (5) and corollary 14.5.6. ■

Lemma 14.8.2. Let $p \in \mathbb{N} \cup \{\infty\}$ and suppose that M is coherent with its C^p arcs. If a smooth map $F : (M, m) \to (N, n)$ is a pointwise immersion at m then $\{m\}$ is open in $F^{-1}(n)$. In particular, if $F : M \to N$ is a smooth pointwise immersion then each fiber of F is discrete.

Proof. Suppose that $\{m\}$ was not open in $F^{-1}(n)$. Since $F^{-1}(n)$ is sequential, this implies that there exists a sequence $(m^l)_{l=1}^{\infty} \subseteq F^{-1}(n)$ such that $m^{\bullet} \to m$ is injective in M. Pick $(l^k)_{k=1}^{\infty}$, $(t^{l_k})_{k=1}^{\infty} \subseteq]0,1[$, and a C^p arc $\gamma : ([0,1], 0) \to (M, m)$ as in lemma 14.0.1. Since F is a pointwise immersion at m and $\gamma'(0) \neq \mathbf{0}$, we have $(F \circ \gamma)'(0) \neq \mathbf{0}$ so that there exists some $0 < \epsilon \leq 1$ such that $F \circ \gamma\big|_{[0,\epsilon]} : [0, \epsilon] \to N$ is injective. But this is impossible since $(F \circ \gamma)(t^{l_k}) = n$ for all $k \in \mathbb{N}$ and t^{l_k} converges to 0. ■

QUESTIONS:

(1) If C is a connected component of $M \smallsetminus O$ and $m \in C$, then is there a cofinal subset J of $\mathbb{N}$ such that $\mu_j(C)$ is a finite-dimensional submanifold of M_j?

(2) The ultimate goal of the above question would be to answer the following question: under the above circumstances, is M necessarily (homeomorphic or diffeomorphic to) a stratification of manifolds?

- The reason for this idea stems from the fact that prop. 14.8.1 shows that a connected M is "almost" a manifold of top-dimension, except for on a closed nowhere dense subset of M where this closed set could (maybe after the introduction of additional hypotheses) possibly be identified as the lower-dimensional manifolds of a stratified manifold (ex: it could possibly be the boundary of the unique top-dimension submanifold of M).

Chapter 15

Constructions of Curves into Monotone Promanifolds

Most of results in this section will be used as lemmas in the proof of theorem 15.4.11, from which the more important theorem 16.1.7 follows almost immediately. Although the some of the lemmas statements' appear somewhat lengthy and technical, they are all written in a way so that in practice, it is usually easy to find a subsystem and sequence(s) that satisfy their hypotheses.

Smooth Almost Arcs

Smooth almost arcs are defined in definition 14.4.1. Note that the equality "$\eta_a^{-1}(\eta_a([t,1])) = [t,1]$" in the following lemma would not necessarily have held had $\eta : [0,1] \to N$ be an arbitrary smooth topological embedding instead of a smooth almost arc at 0.

Lemma 15.1.1. Suppose that $\eta : [0,1] \to N$ is a smooth almost arc at 0 and let $\eta_\bullet = \nu_\bullet \circ \eta$. Then for all $t \in]0,1]$, there exists an $a \in \mathbb{N}$ such that $\eta_a\big|_{[t,1]} : [t,1] \to N_a$ is a smooth embedding and $\eta_a^{-1}(\eta_a([t,1])) = [t,1]$, in which case $V_a \underset{\text{def}}{=} N_a \setminus \eta_a([0,t])$ is an open subset of N_a such that $\eta_a\big|_{\eta_a^{-1}(V_a)} : \eta_a^{-1}(V_a) \to V_a$ is a smooth embedding whose image is $\eta_a\left(]t,1]\right)$.

Proof. Let $a_0 \in \mathbb{N}$ be such that $\eta_{a_0}\big|_{[t/2,1]} : [t/2,1] \to N_{a_0}$ is a smooth embedding, which implies that for all $a \geq a_0$, $\eta_a^{-1}(\eta_a([t,1])) \cap [t/2,t[= \varnothing$. For all $a \geq a_0$, let $I^a = [0,t/2] \cap \eta_a^{-1}(\eta_a([t,1]))$ and observe that I_a is compact and that if $a \leq b$ then $I^b \subseteq I^a$. Suppose that there was no $a \in \mathbb{N}$ such that $\eta_a^{-1}(\eta_a([t,1])) \subseteq [t,1]$. Then all $I^\bullet$ are non-empty so that there would exist some $t \in \cap I^\bullet$. Since $\eta_{a_0}\big|_{[t/2,1]} : [t/2,1] \to N_{a_0}$ is injective and $\eta_{a_0}(t) \in \eta_{a_0}([t,1])$, we can define $s = \eta_{a_0}^{-1}(\eta_{a_0}(t))$ For all $a \geq a_0$, our assumption implies that $J^a \underset{\text{def}}{=} [t,1] \cap \eta_a^{-1}(\eta_a(t))$ is a non empty compact set where since $a \leq b$ implies $J^b \subseteq J^a$, we have that $\cap J^\bullet$ contains some element s. Since $\eta_a(s) = \eta_a(t)$ for all $a \geq a_0$, it follows that $\eta(s) = \eta(t)$ so that η's injectivity gives us the contradiction $s = t$. ∎

Fast Converging Sequences

We now extend the notion fast converging sequences found in [27] to promanifolds.

Definition 15.2.1. If $(M, \mu_\bullet) = \varprojlim \operatorname{Sys}_M$, $m^0 \in M$, and $m^\bullet = (m^l)_{l=1}^{\infty} \subseteq M$ then say that $m^\bullet$ *converges fast to* m^0 if $\mu_i(m^\bullet) = (\mu_i(m^l))_{l=1}^{\infty}$ converges fast to $\mu_i(m^0)$ for every index i.

Remark 15.2.2. Recall that smooth maps between manifolds preserve the fast convergent sequences so that in particular, whether or not a sequence of points in a smooth manifold converges fast to some given point is invariant under diffeomorphisms. This observation and the fact that smooth maps between promanifolds are locally cylindrical makes it easy to see that smooth maps between promanifolds send fast convergent sequences to fast convergent sequences. Consequently, fast convergence in a promanifold is a diffeomorphism invariant of the promanifold which implies, in particular, that the above definition 15.2.1 of fast convergence is actually independent of the choice of the smooth projective representation of M that the definition uses (i.e. definition 15.2.1 is independent of the choice of system Sys_M and projections $\mu_\bullet$).

Finally, observe that every convergent sequence has a fast converging subsequence: let $m^\bullet = (m^l)_{l=1}^{\infty}$ be a sequence in M converging to m^0 and pick a subsequence $m^{\bullet,1}$ such that

$\mu_1(m^{\bullet,1})$ converges fast to $\mu_1(m^0)$. Now inductively pick subsequences $m^{\bullet,i+1}$ of $m^{\bullet,i}$ such that $\mu_{i+1}(m^{\bullet,i+1})$ converges fast to $\mu_{i+1}(m^0)$ and note that the diagonal $m^{\bullet,\bullet} = (m^{l,l})_{l=1}^{\infty}$ is a fast converging subsequence of $m^{\bullet}$. ■

Curves Through Sequences

Through Possibly Non-Convergent Sequences

Lemma 15.3.1. Suppose Sys_M is monotone and $(U_i)_{i=1}^{\infty}$ are open subsets of $M_{\bullet}$ such that for all i, $\mu_{i,i+1}(U_{i+1}) \subseteq U_i$ and $\mu_{i,i+1}\big|_{U_{i+1}} : U_{i+1} \to \mu_{i,i+1}(U_{i+1})$ is monotone. Let $m^{\bullet} = (m^l)_{l=1}^{\infty}$ be a sequence in M such that if $l \geq i$ then $m^l \in \mu_i^{-1}(U_i)$. Let γ_1 be a smooth curve into M_1 and let $(t_i)_{i=1}^{\infty} \subseteq \operatorname{Dom}\gamma$ be a sequence decreasing to $a \in \mathbb{R}$ such that for all $i \in \mathbb{N}$, $\gamma_1(t_i) = \mu_1(m^i)$ and if $l \geq i$ then γ_1 maps the interval containing t_i and t_l into $\mu_{1i}(U_i)$. Let $D = \{t \in \operatorname{Dom}\gamma_1 \,|\, t > a\}$. Then there exists a smooth μ_1-lift $\gamma : D \to M$ of $\gamma_1\big|_D$ such that for all $l \in \mathbb{N}$, $\gamma(t_l) = m^l$ and $(\mu_l \circ \gamma)(]a, t_l]) \subseteq U_l$. Furthermore, if $U_{\bullet}$ is a $\mu_{\bullet}$-neighborhood base of some point m^0 and $a > -\infty$ then γ can be extended to a continuous map on $\{a\} \cup D$ by sending a to m^0.

Proof. For all $l \in \mathbb{N}$ let $m_{\bullet}^l = \mu_{\bullet}(m^l)$, $D_{\geq t_l} = [t_l, \infty[\cap \operatorname{Dom}(\gamma_1)$, and $D_{\leq t_l} =]a, t_l] \cap \operatorname{Dom}(\gamma_1)$. Suppose we've constructed smooth maps $\gamma_1 : D \to M_1, \ldots, \gamma_k : D \to M_k$ such that for all $i = 2, \ldots, k$, $\mu_{i-1,i} \circ \gamma_k = \gamma_k\big|_D$ and for all $l \in \mathbb{N}$ and all $i = 1, \ldots, k$, $\gamma_i(t_l) = m_i^l$ and $\gamma_i(D_{\leq t_i}) \subseteq U_i$. Let $\gamma_{k+1}^R : (D_{\geq t_{k+1}}, t_{k+1}, \ldots t_1) \to \left(M_{k+1}, m_{k+1}^{k+1}, \ldots, m_{k+1}^1\right)$ be a smooth μ_{12}-lift of $\gamma_1\big|_{D_{\geq t_{k+1}}}$. By corollary C.3.4 there exists a smooth $\mu_{k,k+1}$-lift $\gamma_{k+1}^L : (D_{\leq t_{k+1}}, t_{k+1}) \to (U_{k+1}, m_{k+1}^{k+1})$ of $\gamma_k\big|_{D_{\leq t_{k+1}}}$ such that $\gamma_{k+1}^L(t_l) = m_{k+1}^l$ for all $l \geq k+1$, where it is clear that we may arrange to have γ_{k+1}^L be such that the map $\gamma_{k+1} \underset{\text{def}}{=} \gamma_{k+1}^L \cup \gamma_{k+1}^R : D \to M_2$ is a smooth extension of γ_{k+1}^R. This completes the inductive step and it should be clear that the limit map $\gamma \underset{\text{def}}{=} \varprojlim \gamma_{\bullet} : D \to M$ is the desired map. The final remark of this lemma is immediate. ■

Arcs Through Convergent Sequences

Lemma 15.4.1 (Replacing $\gamma\big|_{]0,b]}$ with a smooth arc through a sequence in $M \setminus \operatorname{Im}\gamma$ that is eventually in each $\mu_i^{-1}(\operatorname{Im}\gamma_i)$)**.** For all $i \in \mathbb{N}$, let $d_i = \dim_{\mu_i(m^0)} M_i$ and assume that $d_{i+1} > d_i + 1$. Let $\gamma : ([a,b], 0) \to (M, m^0)$ be a smooth curve, $\gamma_\bullet \underset{\text{def}}{=} \mu_\bullet \circ \gamma$, $\widehat{m}^\bullet \subseteq M \setminus \operatorname{Im}\gamma$ be an $\mathbb{N}$-indexed injective sequence converging to m^0, and let $t_\bullet \subseteq \operatorname{Dom}\gamma$ be a sequence decreasing to 0 such that for all $i \leq l$ in $\mathbb{N}$, $\mu_i(\widehat{m}^l) = \gamma_i(t_l)$ and $\mu_{i+1}(\widehat{m}^i) \notin \operatorname{Im}\gamma_{i+1}$. There exists a smooth $\widehat{\gamma} : ([a,b], 0) \to (M, m^0)$ such that for all $i \in \mathbb{N}$, $\widehat{\gamma}(t_i) = \widehat{m}^i$, $\mu_i \circ \widehat{\gamma} = \gamma_i$ on $[a, t_i]$, and if $i > 1$ then $\mu_i \circ \widehat{\gamma}_i\big|_{]t_i,b]} :]t_i, b] \to M_i$ is a smooth embedding whose image is disjoint from $\operatorname{Im}\gamma_i$.

Remarks 15.4.2. Observe that these properties imply that:

- while both γ and $\widehat{\gamma}$ are μ_1-lifts of γ_1 and smooth extensions of $\gamma\big|_{[a,0]} : [a,0] \to M$, only $\widehat{\gamma}$'s restriction to $[0,b]$ is guaranteed to be a topological embedding that is also a smooth embedding on $]0,b]$.

- if γ is a topological embedding, $a = 0$, all of γ's derivatives vanish at 0, and γ' does not vanish on $]0,b]$ then we obtain a smooth topological embedding $\xi : ([-b,b],0) \to (M, m^0)$ extending γ that is defined on $[-b,0]$ by $\xi(t) = \widehat{\gamma}(-t)$. Since $\mu_i(\xi(t)) = \mu_i(\xi(-t))$ on $[-t_i, t_i]$, the germ at 0 of each $\mu_i \circ \xi$ can be thought of as an "even function about 0" so that in this sense, despite ξ being injective, it can still be though of as "a topological embedding extending γ whose $\mu_\bullet$-germ at 0 is even."

Proof. Let $m^0_\bullet \underset{\text{def}}{=} \mu_\bullet(m^0)$ and for all $l \in \mathbb{N}$, let $\widehat{m}^l_\bullet = \mu_\bullet(\widehat{m}^l)$, $m^l = \gamma(t_l)$, $m^l_\bullet = \mu_\bullet(m^l)$, and $\widehat{\gamma}_1 = \gamma_1$. Observe that our assumptions imply that for all $l, i \in \mathbb{N}$, $l < i \iff \widehat{m}^l_i \notin \operatorname{Im}\gamma_i$. Suppose we've constructed $\widehat{\gamma}_1, \ldots, \widehat{\gamma}_i$ such that for all $h = 1, \ldots, i$, (1) $\widehat{\gamma}_h = \mu_{hi} \circ \widehat{\gamma}_i$, (2) $\widehat{\gamma}_h = \gamma_h$ on $[a, t_h]$, (3) $\widehat{\gamma}_i(t_h) = \widehat{m}^h_0$. (4) if $h > 1$ then $\widehat{\gamma}_h\big|_{]t_h,b]}$ is a smooth embedding whose image is disjoint from $\operatorname{Im}\gamma_h$, Since $d_{i+1} > d_i$ and $\mu_{i,i+1}$ is a monotone smooth submersion, it is clear that we may construct a smooth $\mu_{i,i+1}$-lift $\widehat{\gamma}_{i+1}$ of $\widehat{\gamma}_i$ satisfying the above inductive hypotheses. It is now easy to see that $\widehat{\gamma} \underset{\text{def}}{=} \varprojlim \widehat{\gamma}_\bullet$ is the desired smooth map. ∎

If the Sequence is Eventually in Every $\mu_\bullet$-Fiber of m^0

Lemma 15.4.3 (Almost arc through a sequence that is eventually in each fiber and projects onto arcs)**.** Suppose Sys_M is monotone, $m^0 \in M$ and let $m^0_\bullet = \mu_\bullet(m^0)$ and $d_\bullet = \dim_{m^0_\bullet} M_\bullet$. Let $m^\bullet \subseteq M$ be a sequence converging to m^0 that is contained in the component of M containing m^0. Suppose there exists some $i_\bullet \subseteq \mathbb{N}$ increasing such that for all $l \in \mathbb{N}$, $d_{i_{l+1}} > d_{i_l}$, $m^{i_l} \in \mu_{i_l}^{-1}(m^0_{i_l})$, and $\mu_{i_l}(m^{i_1}), \ldots, \mu_{i_l}(m^{i_l}) = m^0_{i_l}$ are all distinct. Then for any strictly monotone sequence $t_\bullet \subseteq \mathbb{R}$ converging to $t_0 \in \mathbb{R}$ there exists a smooth almost arc $\gamma : (J, t_0) \to (M, m^0)$ vanishing at t_0, where $J \underset{\text{def}}{=} \mathrm{co}(t_0, t_1)$ is the closed interval containing t_0 and t_1, such that for all $l \in \mathbb{N}$,

(1) $\gamma(t_l) = m^{i_l}$,

(2) $\mu_{i_1} \circ \gamma \equiv m^0_{i_1}$,

(3) $\mu_{i_l} \circ \gamma \equiv m^0_{i_l}$ on $\mathrm{co}(t_0, t_l)$, the interval between t_0 and t_l,

(4) if $l > 1$ then $\mu_l \circ \gamma$'s restriction to $\mathrm{co}(t_1, t_l)$ is a smooth almost vanishing at t_l, and

(5) if $l > 1$ then there exists a smooth embedding $\xi_{i_l} : \mathrm{co}(t_1, t_l) \to M_{i_l}$ and a smooth increasing homeomorphism $\rho^l : \mathrm{co}(t_1, t_l) \to \mathrm{co}(t_1, t_l)$ with non-vanishin derivative on $\mathrm{co}(t_1, t_l) \smallsetminus \{t_l\}$ such that $\mu_{i_l} \circ \gamma = \xi_{i_l} \circ \rho^l$ on $\mathrm{co}(t_1, t_l)$ and ρ^l is the identity map on $[t_1, (t_{l-1} + t_l)/2]$.

Furthermore, if $C \subseteq M \smallsetminus \{m^0\}$ is such that for all sufficiently large $l \in \mathbb{N}$, $C^*_{i_l} \underset{\text{def}}{=} \overline{\mu_{i_l}(C)} \smallsetminus \{m^0_{i_l}\}$ does not contain any of $\mu_{i_l}(m^{i_1}), \ldots, \mu_{i_l}(m^{i_{l-1}})$ and $\mu_{i_l, i_{l+1}}\big|_{M_{i_{l+1}} \smallsetminus C^*_{l_i}} : M_{i_{l+1}} \smallsetminus C^*_{l_i} \to M_{i_l}$ is a monotone surjection, then we may pick γ so that $\operatorname{Im} \gamma \cap \overline{C} = \varnothing$.

Remarks 15.4.4.

- If $t_0 < t_1$ and $l > 1$ then part (5) implies that even though $\mu_{i_l} \circ \gamma$ is identically $m^0_{i_l}$ on $[t_0, t_l]$, the image of $\mu_{i_l} \circ \gamma$ is still a smooth embedded submanifold of M_{i_l} that is diffeomorphic to the closed unit interval while part (4) then further gives us that

$\mu_{i_l} \circ \gamma\big|_{]t_l,t_1]} :]t_l, t_1] \to \operatorname{Im}(\mu_{i_l} \circ \gamma) \smallsetminus \{m^0_{i_l}\}$ is a diffeomorphism. Thus $(\mu_{i_l} \circ \gamma)^{-1}\left(m^0_{i_l}\right) = [t_0, t_l]$ and $(\mu_{i_l} \circ \gamma)^{-1}\left(\operatorname{Im}(\mu_{i_l} \circ \gamma) \smallsetminus \{m^0_{i_l}\}\right) =]t_l, t_1]$.

- If $d_{i_{l+1}} \geq d_{i_l} + 2$ for all $l \in \mathbb{N}$, $\widehat{J}$ is an interval such that $J \cap \widehat{J} = \{t_0\}$, and if $\widehat{\gamma} : (\widehat{J}, t_0) \to (M, m^0)$ is a smooth topological embedding such that all $\operatorname{Im}(\mu_{i_l} \circ \widehat{\gamma}) \smallsetminus \{m^0_{i_l}\}$ are smoothly embedded intervals whose images do not contain any of $\mu_{i_l}(m^{i_1}), \ldots, \mu_{i_l}(m^{i_l}) = m^0_{i_l}$ then applying this lemma with $C \underset{\text{def}}{=} \widehat{\gamma}\left(\widehat{J} \smallsetminus \{t_0\}\right)$ will allow us to pick γ so that $\widehat{\gamma} \cup \gamma : (\widehat{J} \cup J, t_0) \to (M, m^0)$ is injective. This is improved upon in corollary 15.4.6.

- It can be easily seen that ρ^h being the identity map on $[t_1, (t_{h-1}+t_h)/2]$ implies that the smooth embeddings $\xi_l\big|_{\operatorname{co}(t_1,(t_{l-1}+t_l)/2)}$ (for $l \geq 2$) form a downward complete generalized cone whose limit map is $\gamma\big|_{J \smallsetminus \{t_0\}}$.

- If $\widehat{m}^\bullet \subseteq M$ was a sequence converging to m^0 such that for all $i \in \mathbb{N}$, $\mu_i^{-1}(m_i^0)$ contains infinitely many distinct $\widehat{m}^\bullet$ then it is easy to see that one can always inductively construct some subsequence $\widehat{m}^{l_\bullet}$ of $\widehat{m}^\bullet$ and some increasing sequence $i_\bullet \subseteq \mathbb{N}$ satisfying the hypotheses of this lemma.

- Our assumptions imply that whenever $l \leq L$ then $\mu_{i_l}(m^L) = m^0_{i_l}$.

Proof. Replacing Sys_M with its restriction to $\{i_l \,|\, l \in \mathbb{N}\}$ and then relabeling we may assume without loss of generality that $i_l = l$ for all $l \in \mathbb{N}$. For all $l \in \mathbb{Z}^{\geq 0}$, let $m^l_\bullet = \mu_\bullet(m^l)$. Assume without loss of generality that $t_\bullet$ is increasing and that $t_0 = 1$. Let $\gamma_1 : [t_1, t_0] \to M_1$ be identically m^0_1 and let $\xi_2 : [t_1, t_2] \to \mu^{-1}_{12}(m^0_1)$ any smooth arc from $h_2(t_1) = m^1_2$ to $h_2(t_2) = m^2_2 = m^0_2$. Let $\rho^2 : [t_1, t_2] \to [t_1, t_2]$ be a smooth increasing homeomorphism whose derivative does not vanish on $[t_1, t_2[$, all of whose derivatives vanish at t_2, and that is the identity map on $[t_1, (t_1 + t_2))/2]$. Let $\alpha_2 \underset{\text{def}}{=} \xi_2 \circ \rho^2 : [t_1, t_2] \to M_2$ and let $\gamma_2 : [t_1, t_0] \to M_2$ be the smooth map defined to be α_2 on $[t_1, t_2]$ and identically m^0_2 on $[t_2, t_0]$.

Suppose that for $j \geq 2$ we've constructed smooth paths $\gamma_1, \gamma_2, \ldots, \gamma_j$ on $[t_1, t_0]$ such that for all $h = 1, \ldots, j$, γ_h is M_h-valued and

(1) $\mu_{hi} \circ \gamma_i = \gamma_h$ for all $i = h, \ldots, j$,

(2) γ_h is a smooth embedding on $[t_1, t_h[$,

(3) γ_h is identically $m^0_{i_h}$ on $[t_h, t_0]$,

(4) $\gamma_h(t_l) = m^l_h$ for any $l = 1, \ldots, h$, and

(5) if $h > 1$ then there exists a smooth embedding $\xi_h : [t_1, t_h] \to M_{i_l}$ and a smooth increasing homeomorphism $\rho^h : [t_1, t_h] \to [t_1, t_h]$ such that $\gamma_h = \xi_h \circ \rho^h$ on $[t_1, t_h]$, $\rho^h = \mathrm{Id}$ on $[t_1, (t_{h-1} + t_h)/2]$.

Let $\lambda_{j+1} : [t_1, t_j] \to M_{j+1}$ be a smooth $\mu_{j,j+1}$-lift of $\xi_j : [t_1, t_j] \to M_j$ such that $\lambda_{j+1}(t_1) = m^1_{j+1}, \ldots, \lambda_{j+1}(t_j) = m^j_{j+1}$. Since ξ_j is a smooth embedding so is λ_{j+1} and since $\lambda_{j+1}(t_j) = m^j_{j+1} \neq m^0_{j+1}$, λ_{j+1} does not pass through m^0_{j+1}. Find smooth coordinate boxes (U_j, φ_j) and (U_{j+1}, φ_{j+1}) centered at m^0_j and m^j_{j+1} in M_j and M_{j+1}, respectively, such that $m^0_{j+1} \notin U_{j+1}$, $\mu_{j,j+1}(U_{j+1}) = U_j$, $\varphi_j \circ \mu_{ij} \circ \varphi^{-1}_{j+1}$ is the canonical projection, $\xi^{-1}_j(U_j) =]s, t_j]$ for some $s \in]t_{j-1} - t_j[$, and $\varphi_{j+1}(\lambda_{j+1}(t)) = (t, 0, \ldots, 0)$ for all $t \in]s, t_j]$. Note that $\mu_{j,j+1} \circ (\lambda_{j+1} \circ \rho^j) = \xi_j \circ \rho^j = \gamma_j$, from which it follows that all derivatives of $(\lambda_{j+1} \circ \rho^j)'$ vanish at t_j so that we may, by appropriately altering $\lambda_{j+1} \circ \rho^j$ only on $]s, t_j]$, construct a smooth embedding $\alpha_{j+1} : ([t_1, t_j], t_j) \to (M_{j+1}, m^j_{j+1})$ such that $\alpha_{j+1} = \lambda_{j+1}$ on $[t_1, s]$, $\alpha'_{j+1}(t_j)$ is $\mu_{j,j+1}$-vertical, and α_{j+1} is a $\mu_{j,j+1}$-lift of $\gamma_j\big|_{[t_1,t_j]}$.

Since $\mu^{-1}_{j,j+1}(m^0_j)$ is a smooth connected manifold, $\operatorname{Im} \alpha_{j+1}$ does not contain m^0_{j+1}, and $\alpha'_{j+1}(t_j)$ is $\mu_{j,j+1}$-vertical, we can extend α_{j+1} to a smooth embedding $\xi_{j+1} : ([t_1, t_{j+1}], t_{j+1}) \to (M_{j+1}, m^0_{j+1}$ such that $\xi_{j+1}([t_j, t_{j+1}]) \subseteq \mu^{-1}_{j,j+1}(m^0_j)$. Pick any smooth increasing homeomorphism $\rho^{j+1} : [t_1, t_{j+1}] \to [t_1, t_{j+1}]$ such that $\rho^{j+1} = \mathrm{Id}$ on $[t_1, (t_j + t_{j+1})/2]$, ρ^{j+1}'s first derivative does not vanish on $[t_1, t_{j+1}[$, and all of its derivatives vanish at t_{j+1}. Let $\gamma_{j+1} : [t_1, t_0] \to M_{j+1}$ be the map defined to be $\xi_{j+1} \circ \rho^{j+1}$ on $[t_1, t_{j+1}]$ and constantly m^0_{j+1} on $[t_{j+1}, t_0]$. Clearly, γ_{j+1} is a smooth map whose restriction to $[t_1, t_{j+1}[$ is a smooth embedding. Since ρ^{j+1} is the identity on $[t_1, (t_j + t_{j+1})/2]$, $\gamma_{j+1}(t_h) = m^h_{j+1}$ for all $h = 1, \ldots, j$ and $\mu_{j,j+1} \circ \gamma_{j+1} = \gamma_j$ on $[t_1, t_j]$.

Since γ_j is identically m_j^0 on $[t_j, t_0]$ and $\gamma_{j+1}([t_j, t_0])$ is contained in $\mu_{j,j+1}^{-1}(m_j^0)$, it follows that γ_{j+1} is a $\mu_{j,j+1}$-lift of γ_j. Clearly, $\gamma \underset{\text{def}}{=} \varprojlim \gamma_\bullet$ is the desired map. If C is as described above then it is clear how to alter this construction so that the image of each γ_i does not intersect $\overline{\mu_i(C)} \setminus \{m_i^0\}$, which implies that $\operatorname{Im}\gamma \cap C = \varnothing$. ∎

Corollary 15.4.5. Suppose Sys_M is monotone, $m^0 \in M$ is such that $\dim_{m^0} M = \infty$, and let $m_\bullet^0 = \mu_\bullet(m^0)$. Let $m^\bullet \subseteq M$ be a sequence converging to m^0 such that for all $i \in \mathbb{N}$, $\mu_i^{-1}(m_i^0)$ contains infinitely many distinct $m^\bullet$. Then there exists some $i_\bullet \subseteq \mathbb{N}$ satisfying the hypotheses of lemma 15.4.3.

Corollary 15.4.6 (Arc through sequences that are eventually in each fiber and projects onto arcs)**.** Suppose Sys_M is monotone, $m^0 \in M$ and let $m_\bullet^0 = \mu_\bullet(m^0)$ and $d_\bullet = \dim_{m_\bullet^0} M_\bullet$. Let $m^\bullet, \widehat{m^\bullet} \subseteq M$ be a two sequences converging to m^0 that are contained in the component of M containing m^0. Suppose there exists some $i_\bullet \subseteq \mathbb{N}$ increasing such that for all $l \in \mathbb{N}$, $m^{i_l}, \widehat{m^{i_l}} \in \mu_{i_l}^{-1}(m_{i_l}^0)$, $d_{i_{l+1}} \geq d_{i_l} + 2$, and for $l \geq 2$, $\mu_{i_l}(\widehat{m}^{i_1}), \ldots, \mu_{i_l}(\widehat{m}^{i_{l-1}})$, $\mu_{i_l}(\widehat{m}^{i_l}) = m_{i_l}^0 = \mu_{i_l}(m^{i_l})$, $\mu_{i_l}(m^{i_{l-1}}), \ldots, \mu_{i_l}(m^{i_1})$ are all distinct. Then for any strictly monotone sequences $\widehat{t}_\bullet, t_\bullet \subseteq \mathbb{R}$ converging to $t_0 \in \mathbb{R}$ with $\widehat{t}_\bullet$ increasing and $t_\bullet$ decreasing, there exists a smooth almost arc $\gamma : \left([\widehat{t}_1, t_1], t_0\right) \to (M, m^0)$ vanishing at t_0 such that for all $l \in \mathbb{N}$,

(1) $\gamma(\widehat{t}_l) = \widehat{m}^{i_l}$ and $\gamma(t_l) = m^{i_l}$,

(2) $\mu_{i_l} \circ \gamma \equiv m_{i_l}^0$ on $[\widehat{t}_l, t_l]$, and

(3) $\mu_{i_1} \circ \gamma \equiv m_{i_1}^0$ and if $l > 1$ then $\operatorname{Im}(\mu_{i_l} \circ \gamma)$ is a smooth embedded submanifold of M_{i_l} diffeomorphic to $[0,1]$.

Furthermore, if $C \subseteq M \setminus \{m^0\}$ is such that for all sufficiently large $l \in \mathbb{N}$, $C_{i_l}^* \underset{\text{def}}{=} \overline{\mu_{i_l}(C)} \setminus \{m_{i_l}^0\}$ does not contain any of $\mu_{i_l}(\widehat{m}^{i_1}), \ldots, \mu_{i_l}(\widehat{m}^{i_{l-1}}), \mu_{i_l}(m^{i_{l-1}}), \ldots, \mu_{i_l}(m^{i_1})$ and $\mu_{i_l, i_{l+1}}\big|_{M_{i_{l+1}} \setminus C_{l_i}^*} : M_{i_{l+1}} \setminus C_{l_i}^* \to M_{i_l}$ is a monotone surjection, then we may pick γ so that $\operatorname{Im}\gamma \cap \overline{C} = \varnothing$.

Proof. Assume without loss of generality that $i_l = l$ for all $l \in \mathbb{N}$. We can repeat the construction in the proof of lemma 15.4.3 once for $\widehat{m}^\bullet$ and $\widehat{t}_\bullet$ to construct $\widehat{\gamma}_1, \widehat{\gamma}_2, \ldots$ on $[\widehat{t}_1, t_0]$

satisfying the inductive hypotheses (1) - (5) while simultaneously repeating the construction with $m^\bullet$ and $t_\bullet$ to construct $\gamma_1, \gamma_2, \ldots$ on $[t_0, t_1]$ also satisfying the inductive hypotheses (1) - (5) but this time, add to the list of inductive hypotheses:

(6) $\operatorname{Im}\gamma_j \cap \operatorname{Im}\widehat{\gamma}_j = \{m_j^0\}$

(7) if $j \geq 2$ then $\operatorname{Im}(\mu_{i_l} \circ \gamma)$ is a smooth embedded submanifold diffeomorphic to the closed unit interval.

Note that $\widehat{\gamma}_1 = \gamma_1$ are both the constantly m_1^0 maps. Observe that at each step in the respective constructions of γ_j and $\widehat{\gamma}_j$ from γ_{j-1} and $\widehat{\gamma}_{j-1}$ (for $j > 1$), this corollary's additional hypotheses allow us to construct γ_j and $\widehat{\gamma}_j$ so that these additional inductive hypotheses (6) and (7) are satisfied: the assumption $d_{j+1} \geq d_j + 2$ guarantees that the fiber $\mu_{j,j+1}^{-1}(m_j^0)$ will have dimension ≥ 2, which allows us construct $\widehat{\gamma}_j$ and γ_j so that the smooth arcs $\operatorname{Im}\gamma_j$ and $\operatorname{Im}\widehat{\gamma}_j$ intersect only at $\{m_j^0\}$ with their union, which equals the image of $\widehat{\gamma}_j \cup \gamma_j : [\widehat{t_1}, t_1] \to M_j$, also forming a smooth submanifold of M_j diffeomorphic to a the unit interval. The desired curve will then by $\varprojlim (\widehat{\gamma}_i \cup \gamma_i)_{i=1}^{\infty} : [\widehat{t_1}, t_1] \to M$. ■

Corollary 15.4.7. Suppose Sys_M is monotone and that $m^\bullet = (m^l)_{l=1}^{\infty}$ is a sequence in M containing infinitely many distinct points that converges to $m^0 \in M$. Suppose that $\dim_{m^0} M = \infty$ and that for each indices i, $\{\mu_i(m^l) \,|\, l \in \mathbb{N}\}$ is finite. Then there exists an increasing sequence $(l_k)_{k=1}^{\infty}$ of integers and a smooth topological embedding $\eta : ([-1,1], 0) \to (M, m^0)$ such that $\eta\big|_{[-1,0[\,\cup\,]0,1]} : [-1,0[\,\cup\,]0,1] \to M$ is a smooth embedding, $\eta^{(p)}(0)$ vanishes for all $p \in \mathbb{N}$, $\eta\left(\frac{1}{k}\right) = m^{l_k}$, $\mu_1 \circ \eta$ is identically $\mu_1(m^0)$. Furthermore, upon replacing Sys_M with a certain subsystem we can also arrange so that $\mu_k \circ \eta\big|_{\left[\frac{1}{k+1}, 1\right]} : \left[\frac{1}{k+1}, 1\right] \to M_k$ is a smooth arc and $\mu_k \circ \eta\big|_{[0, c_k]}$ is identically $\mu_k(m^0)$ for all $k \in \mathbb{N}$, where $c_k \underset{\text{def}}{=} \frac{1}{2}\left(\frac{1}{k+2} + \frac{1}{k+1}\right)$.

If No Subsequence is Eventually in Every $\mu_\bullet$-Fiber of m^0

Lemma 15.4.8. Suppose Sys_M is monotone and $m^\bullet = (m^l)_{l=1}^{\infty} \subseteq M \setminus \{m^0\}$ is an injective sequence fast converging to $m^0 \in M$ in M such that all $\mu_1(m^0), \mu_1(m^1), \mu_1(m^2), \ldots$ are distinct. Suppose that $\dim_{m^0} M = \infty$. Then there exists an increasing sequence $(l_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ and a smooth almost arc $\eta : ([-1,1],0) \to (M, m^0)$ vanishing at 0 such that $\eta\big(\frac{1}{k}\big) = m^{l_k}$ for all $k \in \mathbb{N}$, and there is some index i such that $\mu_i \circ \gamma : [-1,1] \to M_i$ is a smooth almost arc at 0.

Furthermore, if $C \subseteq M \setminus \{m^0\}$ is such that for all sufficiently large $l \in \mathbb{N}$, $C^*_{i_l} \underset{\text{def}}{=} \overline{\mu_{i_l}(C)} \setminus \{m^0_{i_l}\}$ does not contain any of $\mu_{i_l}(\widehat{m}^{i_1}), \ldots, \mu_{i_l}(\widehat{m}^{i_{l-1}}), \mu_{i_l}(m^{i_{l-1}}), \ldots, \mu_{i_l}(m^{i_1})$ and $\mu_{i_l, i_{l+1}}\big|_{M_{i_{l+1}} \setminus C^*_{l_i}} : M_{i_{l+1}} \setminus C^*_{l_i} \to M_{i_l}$ is a monotone surjection, then we may pick γ so that $\operatorname{Im} \gamma \cap \overline{C} = \varnothing$.

Proof. Since there exists some $i_0 \in \mathbb{N}$ such that infinitely many $\mu_{i_0}(m^1), \mu_{i_0}(m^2), \ldots$ are distinct then replacing Sys_M by $\operatorname{Sys}_M\big|_{\mathbb{N}^{\geq i_0}}$ we may assume without loss of generality that $i_0 = 1$. Let $(U_i)_{i=1}^{\infty}$ a fast descending $\mu_\bullet$-neighborhood basis of m^0 with smooth surjective charts $\varphi_i : U_i \to]-r_i, r_i[^{d_i}$ centered at $\mu_i(m^0)$, where $d_i \underset{\text{def}}{=} \dim_{\mu_i(m^0)} M_i$. Replacing $m^\bullet$ with a subsequence, we may assume without loss of generality that all $m^\bullet$ are distinct, no m^l is equal to m^0, $m_1^\bullet$ is fast convergent to m_1^0, and that for all $i, l \in \mathbb{N}$, whenever $l \geq i-1$ then $m^i \in \mu_i^{-1}(U_i)$. Replacing $m^\bullet$ with a subsequence, we may assume without loss of generality that there is some coordinate j such that $\varphi_1^j(\mu_1(m^1)), \varphi_1^j(\mu_1(m^2)), \ldots$ is strictly monotone. For all $l \in \mathbb{Z}^{\geq 0}$, let $m^l_\bullet \underset{\text{def}}{=} \mu_\bullet(m^l)$ and for all $l \neq 0$ let $d_l \underset{\text{def}}{=} \dim_{m^0_l} M_l$.

For all $l \in \mathbb{N}$ let $n_1^l = \varphi_1^{-1}(-\varphi_1(m^l))$ and pick $n^l \in \mu_1^{-1}(n_1^l)$ such that $\mu_l(n^l) \in U_l$, which exists since $n_1^l \in \mu_{1l}(U_l)$. Since $\varphi_1^j(\mu_1(m^1)), \varphi_1^j(\mu_1(m^2)), \ldots$ is strictly monotone, $\mu_1(m^1), \mu_1(m^2), \ldots$ belongs to W_1^R where if $\varphi_1^j(\mu_1(m^1)) > 0$ then $W_1^R \underset{\text{def}}{=} \big(\varphi_i^j\big)^{-1}(\mathbb{R}^{>0})$ and where $W_1^R \underset{\text{def}}{=} \big(\varphi_i^j\big)^{-1}(\mathbb{R}^{<0})$ otherwise. If $W_1^R = \big(\varphi_i^j\big)^{-1}(\mathbb{R}^{>0})$ (resp. if $W_1^R = \big(\varphi_i^j\big)^{-1}(\mathbb{R}^{<0})$) then let $W_1^L \underset{\text{def}}{=} \big(\varphi_i^j\big)^{-1}(\mathbb{R}^{<0})$ (resp. $W_1^L \underset{\text{def}}{=} \big(\varphi_i^j\big)^{-1}(\mathbb{R}^{>0})$). Let $W^R = \mu_1^{-1}(W_1^R)$, $W^L = \mu_1^{-1}(W_1^L)$, and for all $l \in \mathbb{N}$ let $m_1^{l,j} = \varphi_1^j(\mu_1(m_1^l))$. Since $m_1^\bullet$ is converges fast to m_1^0, there is a smooth maps $\gamma_1^R : ([0,1], 0, 1) \to (U_1, \mathbf{0}, m_1^1)$ such that if $\gamma_1^L : ([-1,0], 0, -1) \to (U_1^R, \mathbf{0}, n_1^1)$ is defined by $\varphi_1 \circ \gamma_1^L(t) \underset{\text{def}}{=} -\varphi_1 \circ \gamma_1^R(-t)$ then these maps are topological embeddings that satisfy, respectively,

(1) $(\varphi_1 \circ \gamma_1^R)\big(\big]0, \frac{1}{l}\big]\big) \subseteq \mu_{1l}(U_l) \cap W_1^R$ (resp. $(\varphi_1 \circ \gamma_1^R)\big(\big[-\frac{1}{l}, 0\big[\big) \subseteq \mu_{1l}(U_l) \cap W_1^L$).

(2) $\gamma_1^R\left(\frac{1}{l}\right) = m_1^l$ (resp. $\gamma_1^L\left(-\frac{1}{l}\right) = n_1^l$) for all $l \in \mathbb{N}$.

(3) $(\gamma_1^R)'(t) \neq \mathbf{0}$ for $t \in]0,1]$ (resp. $(\gamma_1^L)'(t) \neq \mathbf{0}$ for $t \in [-1,0[$).

Observe that (1) - (3) imply that for all $l \in \mathbb{N}$, $\gamma_1^R\left(\left[0,\frac{1}{l}\right]\right) \subseteq \mu_{1l}(U_l)$ so that using lemma 15.3.1 with $t_l = \frac{1}{l}$ and $U_l \cap \mu_{1l}^{-1}(W_1^R)$ for all $l \in \mathbb{N}$, there exists a continuous μ_1-lift $\gamma^R : [0,1] \to M$ of γ_1^L that is smooth on $]0,1]$ and such that for all $l \in \mathbb{N}$, $\gamma^R\left(\frac{1}{l}\right) = m^l$ and $\mu_l \circ \gamma^R\left(\left]0,\frac{1}{l}\right]\right) \subseteq U_l \cap \mu_{1l}^{-1}(W_1^R)$. Since $\mu_l \circ \gamma^R([0,1/l]) \subseteq U_l$ for all $l \geq 2$, we have by lemma 5.3.9 that γ^L is smooth and that all of γ^R's derivatives exist and vanish at 0. From $\mu_1 \circ \gamma^R = \gamma_1^R$'s properties it follows that γ^R is a topological embedding whose derivative does not vanish on $]0,1]$. Similarly, there exists a smooth μ_1-lift $\gamma^L : [-1,0] \to M$ of γ_1^L such that for all $l \in \mathbb{N}$, $\gamma^L\left(-\frac{1}{l}\right) = n^l$ and $\gamma^L\left(\left[-1,-\frac{1}{l}\right]\right) \subseteq U_l \cap \mu_{1l}^{-1}(W_1^L)$. Let $\gamma \underset{\text{def}}{=} \gamma^L \cup \gamma^R : [-1,1] \to M$ and observe that γ is smooth at 0 since all derivatives of γ^L and γ^R vanish at 0. Since $\gamma([-1,0[)$ and $\gamma(]0,1])$ belong, respectively, to the disjoint sets W^L and W^R it follows that γ is injective and hence a topological embedding. ∎

Smooth Topological Embeddings of $\mathbb{R}$ Through Sequences

Proposition 15.4.9. Let $\eta : ([0,1],0) \to (M,m^0)$ be a smooth curve, let $m^\bullet$ be an $\mathbb{N}$-indexed injective sequence in $M \setminus \operatorname{Im}\eta$ converging to m^0, and assume that Sys_M is monotone with $\dim_{m^0} M = \infty$. There exists $\lambda : \mathbb{N} \to \mathbb{N}$ increasing and a smooth almost arc $\gamma : ([-1,0],0) \to (M,m^0)$ vanishing at 0 such that $\gamma([-1,0[) \cap \operatorname{Im}\eta = \varnothing$ and $\gamma\left(-\frac{1}{k}\right) = m^{\lambda(k)}$ for all $k \in \mathbb{N}$. In particular, if η is a topological embedding all of whose derivatives vanish at 0 then $\gamma \cup \eta : ([-1,1],0) \to (M,m^0)$ is smooth topological embedding.

Proof. Let $C = \operatorname{Im}\eta$, $t_\bullet = -\frac{1}{\bullet}$, $\eta_\bullet = \mu_\bullet \circ \eta$, for all $l \in \mathbb{Z}^\bullet$ let $m_\bullet^l = \mu_\bullet(m^l)$, and let $d_\bullet = \dim_{m_\bullet^0} M_\bullet$. By replacing Sys_M with a subsystem and replacing $m^\bullet$ with a subsequence, we may assume without loss of generality that for all $i \in \mathbb{N}$, $d_{i+1} > d_i + 1 > 3$, $m^\bullet$ converges fast to m^0, all $m^\bullet$ belong to the same connected component of M, and that all $m_i^0, \ldots, m_i^{i-1}$ are distinct. If we may find subsequences of $m^\bullet$ that satisfies corollary 15.4.6 then this corollary essentially gives us our desired curve so assume that there is no subsequence of $m^\bullet$ that satisfies this corollary. Then there is some $i \in \mathbb{N}$ such that $\{\mu_i(m^l) \,|\, l \in \mathbb{N}\}$ is infinite so by replacing Sys_M with a subsystem and replacing $m^\bullet$ with subsequences, we may assume without loss of generality that $\mu_1(m^\bullet)$ is injective and that if $i > 1$ then $m_i^1, \ldots, m_i^{i-1}$ distinct and not contained in $\operatorname{Im} \eta_i$. If we may find some infinite subset $J \subseteq \mathbb{N}$ and subsequence $m^{l_\bullet}$ of $m^\bullet$ to which we the hypotheses of lemma 15.4.1 apply, then we may using the arc resulting from this lemma to construct our desired γ. So assume that not such subset $J \subseteq \mathbb{N}$ and subsequence exist, which is only possible if there is some infinite $J \subseteq \mathbb{N}$ and subsequence $m^{l_\bullet}$ of $m^\bullet$ to which we may apply lemma 15.4.3 to once again obtain our desired curve. This is only possible if we ∎

Corollary 15.4.10. Suppose Sys_M is monotone and that $m^\bullet = (m^l)_{l=1}^\infty$ is a sequence in M containing infinitely many distinct points that converges to $m^0 \in M$. If $\dim_{m^0} M = \infty$ then there exists a strictly increasing sequence of integers $(l_k)_{k=1}^\infty$ and a smooth almost arc $\eta : ([-1,1], 0) \to (M, m^0)$ vanishing at 0 such that $\eta\left(\frac{1}{k}\right) = m^{l_k}$ for all $k \in \mathbb{N}$ and where if in addition there exists some subsequence $(m^{p_j})_{j=1}^\infty$ of $m^\bullet$ containing infinitely many distinct points such that for all indices i, $\{\mu_i(m^{p_j}) \,|\, j \in \mathbb{N}\}$ is finite then upon replacing Sys_M with a certain subsystem, we can pick $(l_k)_{k=1}^\infty$ and η so that we also have:

(5) $\mu_k \circ \eta\big|_{\left[\frac{1}{k+1}, 1\right]} : \left[\frac{1}{k+1}, 1\right] \to M_k$ is a smooth arc, and

(6) $\mu_k \circ \eta\big|_{[0,1/k]}$ is identically $\mu_k(m^0)$ for all $k \in \mathbb{N}$.

We extend proposition 15.4.9 so as to be able to find $l_\bullet \subseteq \mathbb{N}$ increasing and a smooth topological embedding going through two subsequences $m^{l_\bullet}$ and $\widehat{m}^{l_\bullet}$ of two given disjoint infinite sequences $m^\bullet$ and $\widehat{m}^\bullet$ in M, where observe in particular that the same sequence of integers $l_\bullet$ is used for both of these subsequences.

Theorem 15.4.11. Let $m^0 \in M$ be such that $\dim_{m^0} M = \infty$ and let $m^\bullet$ and $\widehat{m}^\bullet$ be $\mathbb{N}$-indexed injective sequences in M converging to m^0 such that $m^\bullet \cap \widehat{m}^\bullet = \varnothing$. If Sys_M is monotone then there exists $\lambda : \mathbb{N} \to \mathbb{N}$ increasing and a smooth almost arc $\gamma : ([-1,1],0) \to (M,m^0)$ vanishing at 0 such that $\gamma\left(\frac{1}{k}\right) = m^{\lambda(k)}$ and $\gamma\left(-\frac{1}{k}\right) = \widehat{m}^{\lambda(k)}$ for all $k \in \mathbb{N}$.

Proof. Let $m^0_\bullet = \mu_\bullet(m^0)$ and for all $i \in \mathbb{N}$, let $t_i = \frac{1}{i}$, $\widehat{t_i} = -t_i$, $d_i = \dim_{\mu_i(m^0)} M_i$ and assume without loss of generality that $d_{i+1} > d_i + 2 > 3$. Suppose first that for all $i \in \mathbb{N}$, both $\mu_i(m^\bullet)$ and $\mu_i(\widehat{m}^\bullet)$ are finite. Let $i_1 = 1$ and pick $L_1 \in \mathbb{N}$ such that for all $l \geq L_i$, $\mu_1(m^l) = m^0_1 = \mu_1(\widehat{m}^l)$. Let $l_1 = L_1$. Clearly, there is some $i_{k+1} > i_k$ for which there exists some $l_{k+1} > L_k$ such that $\mu_{i_{k+1}}(m^{l_{k+1}}) \notin \mu_{i_{k+1}}(\{m^{l_1},\ldots,m^{l_k}\})$ and $\mu_{i_{k+1}}(\widehat{m}^{l_{k+1}}) \notin \mu_{i_{k+1}}(\{\widehat{m}^{l_1},\ldots,\widehat{m}^{l_k}\})$. Pick $L_{k+1} \geq l_{k+1}$ such that for all $l \geq L_i$, $\mu_{i_{k+1}}(m^l) = m^0_{i_{k+1}} = \mu_{i_{k+1}}(\widehat{m}^l)$. Observe that $i_\bullet$ and $l_\bullet$ are increasing sequences such that for all $k > 1$, $\mu_{i_k}(\widehat{m}^{l_1}),\ldots,\mu_{i_k}(\widehat{m}^{l_{k-1}})$, $\mu_{i_k}(\widehat{m}^{l_k}) = m^0_{i_k} = \mu_{i_k}(m^{l_k})$, $\mu_{i_k}(m^{i_{k-1}}),\ldots,\mu_{i_k}(m^{i_1})$ are all distinct and if $p \geq k$ then $m^{l_p},\widehat{m}^{l_p} \in \mu^{-1}_{i_k}(m^0_k)$. So by replacing Sys_M with $\mathrm{Sys}_M\big|_{\{i_k \,|\, k\in\mathbb{N}\}}$ and replacing both $m^\bullet$ and $\widehat{m}^\bullet$ with $m^{l_\bullet}$ and $\widehat{m}^{l_\bullet}$, we may assume without loss of generality that for all $i \in \mathbb{N}$, if $i > 1$ then $\mu_i(\widehat{m}^i),\ldots,\mu_i(\widehat{m}^{i-1})$, $\mu_i(\widehat{m}^i) = m^0_i = \mu_i(m^i)$, $\mu_i(m^{i-1}),\ldots,\mu_i(m^1)$ are all distinct and that for all $l \geq i$, $m^l,\widehat{m}^l \in \mu^{-1}_i(m^0_i)$. We may now apply corollary 15.4.6 to obtain the desired curve.

Thus we may henceforth assume that there is some $i \in \mathbb{N}$ such that at least one of $\mu_i(m^\bullet)$ and $\mu_i(\widehat{m}^\bullet)$ is infinite. Clearly, it suffices to find λ and γ such that $\gamma\left(\frac{1}{k}\right) = \widehat{m}^{\lambda(k)}$ and $\gamma\left(-\frac{1}{k}\right) = m^{\lambda(k)}$ so that we may assume without loss of generality that $\mu_i(m^\bullet)$ is infinite. Pick $(l_i)_{i=1}^\infty \subseteq \mathbb{N}$ increasing such that $\mu_i(m^{l_\bullet}) \to \mu_i(m^0)$ is injective and then replace Sys_M with $\mathrm{Sys}_M\big|_{\mathbb{N}^{\geq i}}$ and replace both $m^\bullet$ and $\widehat{m}^\bullet$ with $m^{l_\bullet}$ and $\widehat{m}^{l_\bullet}$, respectively so that in this case we may assume without loss of generality that $\mu_1(m^\bullet) \to \mu_1(m^0)$ is injective.

Suppose that $\mu_i(\widehat{m}^\bullet)$ is finite for all $i \in \mathbb{N}$. Then we may find $J \subseteq \mathbb{N}$ infinite and $l_\bullet \subseteq \mathbb{N}$ increasing such that upon replacing Sys_M with a $\mathrm{Sys}_M\big|_J$ and replacing both $m^\bullet$ and $\widehat{m}^\bullet$ with $m^{l_\bullet}$ and $\widehat{m}^{l_\bullet}$, we will be able to assume without loss of generality that if $i > 1$ then $\mu_i(\widehat{m}^i),\ldots,\mu_i(\widehat{m}^{i-1})$, $\mu_i(\widehat{m}^i) = m^0_i$ are all distinct and that for all $l \geq i$, $\widehat{m}^l \in \mu^{-1}_i(m^0_i)$. We may thus apply lemma 15.4.8 to obtain a smooth topological embedding $\eta : ([-1,1],0) \to (M,m^0)$ such that $\eta\big|_{[-1,0[\,\cup\,]0,1]} : [-1,0[\,\cup\,]0,1] \to M$ is a smooth embedding, $\eta^{(p)}(0)$ vanishes for all $p \in \mathbb{N}$, $\eta\left(\frac{1}{k}\right) = m^{l_k}$ for all $k \in \mathbb{N}$ where it is easy to see from this lemma's proof that can pick η so that $C \underset{\text{def}}{=} \eta([0,1])$ does not contain any point $\widehat{m}^{\bullet}$. Now define $\gamma^R \underset{\text{def}}{=} \eta\big|_{[0,1]}$: $([0,1],0) \to (M, m^0)$ and use lemma 15.4.3 to construct a smooth topological embedding $\gamma^L : ([-1,0],0) \to (M, m^0)$ such that all of γ^L's derivatives vanish at 0, γ^L's first derivative does not vanish on $[-1,0[$, $\gamma^L\left(\widehat{t_l}\right) = \widehat{m}^l$ for all $l \in \mathbb{N}$, and $\gamma^L([-1,0[) \cap \operatorname{Im}\gamma^R = \varnothing$. The map $\gamma \underset{\text{def}}{=} \gamma^L \cup \gamma^R : ([-1,1],0) \to (M, m^0)$ is clearly our desired smooth topological embedding.

Thus we may henceforth assume that there is some $i \in \mathbb{N}$ such that $\mu_i(\widehat{m}^{\bullet})$ is infinite where as before, we may assume without loss of generality that $\mu_1(m^{\bullet}) \to m_1^0$ is injective. Define $\gamma^R : ([0,1],0) \to (M \setminus \{\widehat{m}^l \,|\, l \in \mathbb{N}\}, m^0)$ as before and then using either lemma 15.4.8 or lemma 15.4.1 to obtain $l_{\bullet} \subseteq \mathbb{N}$ a smooth topological embedding $\gamma^L : ([-1,0],0) \to (M, m^0)$ such that all of γ^L's derivatives vanish at 0, γ^L's first derivative does not vanish on $[-1,0[$, $\gamma^L\left(\widehat{t_k}\right) = \widehat{m}^{l_k}$ for all $k \in \mathbb{N}$, and $\gamma^L([-1,0[) \cap \operatorname{Im}\gamma^R = \varnothing$. Let $\rho : [0,1] \to [0,1]$ be a smooth increasing homomorphism such that $\rho(1/k) = \frac{1}{l_k}$ for all $k \in \mathbb{N}$ and ρ' does not vanish on $]0,1]$. Then $\gamma \underset{\text{def}}{=} \gamma^L \cup (\gamma^R \circ \rho) : ([-1,1],0) \to (M, m^0)$ is our desired smooth topological embedding. ■

Chapter 16

Monotone Promanifolds

Definition 16.0.1. Call a promanifold *monotone* if it is the limit of some monotone profinite system (def. 2.1.56).

Remark 16.0.2. Almost all of the results of this subsection can be extended to promanifolds that are limits of a profinite system that is locally cofinally (def. 2.1.60) a monotone system.

Lemma 16.0.3. Suppose that U is a non-empty path-connected open subset of M and that for each index i, U_i is an open subset of M_i such that $\cup \mu_\bullet^{-1}(U_\bullet) = U$ with $\mu_{i,i+1}^{-1}(U_i) \subseteq U_{i+1}$. Let $m^0 \in U$ and for each index i, if $\mu_i(m^0) \in U_i$ then let O_i denote the connected component of U_i containing $\mu_i(m^0)$ and let $O_i = \varnothing$ otherwise. Then for any index i_0, $U = \underset{i \geq i_0}{\cup} \mu_i^{-1}(O_i)$ and if Sys_M is monotone then $\mu_{i,i+1}^{-1}(O_i) \subseteq O_{i+1}$ for all indices i. Furthermore, if $\dim_{m^0} M \geq 2$ and Sys_M is monotone then $U \setminus \{m^0\}$ is path-connected.

Proof. Let $O = \underset{i \geq i_0}{\cup} \mu_i^{-1}(O_i)$, $m \in U$, and pick a continuous path $\gamma : ([0,1], 0, 1) \to (U, m^0, m)$. Observe that since $\mu_i^{-1}(U_i) \subseteq \mu_{i+1}^{-1}(U_{i+1})$ for all indices $i \geq i_0$ and since these sets form an open cover of $\operatorname{Im} \gamma$ we can pick an index $i \geq i_0$ such that $\operatorname{Im} \gamma \subseteq \mu_i^{-1}(U_i)$. This implies that the connected set $\operatorname{Im}(\mu_i \circ \gamma)$ is contained in U_i and thus contained in O_i so that $m \in \operatorname{Im} \gamma \subseteq \mu_i^{-1}(O_i) \subseteq O$, as desired. If Sys_M is monotone then since for all i, we have $\mu_{i,i+1}^{-1}(O_i) \subseteq O_i$ since $\mu_{i,i+1}^{-1}(O_i)$ is either $\varnothing$ or otherwise it is a connected subset of U_{i+1} containing $\mu_{i+1}(m^0)$. ∎

Coherence with Smooth Almost Arcs

Theorem 16.1.1. Let $\mathcal{E}$ denote the set of all smooth almost arcs $\gamma : [0,1] \to M$ vanishing at 0 for which there exists some index i and some $U_i \in \operatorname{Open}(M_i)$ such that $\mu_i \circ \gamma : [0,1] \to M_i$ is a U_i-valued smooth almost arc vanishing at 0 and U_i is the domain of a smooth chart on M_i. Let $\mathcal{V}$ denote the set of all smooth almost arcs $\gamma : [0,1] \to M$ vanishing at 0 for which there exists some increasing $\iota : \mathbb{N} \to \mathbb{N}$ such that for all $l \in \mathbb{N}$, $\gamma_{\iota(l)}$ is constant on $[0, 1/l]$ and if $l > 1$ then

(1) $\operatorname{Im} \gamma_{\iota(l)}$ is a smooth submanifold of $M_{\iota(l)}$ diffeomorphic to $[0,1]$, and

(2) $\gamma_{\iota(l)}\big|_{[1/l,1]} : [1/l, 1] \to M_{\iota(l)}$ is a smooth almost arc vanishing $1/l$.

where $\gamma_\bullet \underset{\text{def}}{=} \mu_\bullet \circ \gamma$. If Sys_M is monotone then M is coherent with $\mathcal{V} \cup \mathcal{E}$.

Remark 16.1.2. If M is a manifold then $\mathcal{V}$ is clearly empty. If $\gamma \in \mathcal{V}$ then for all $l \in \mathbb{N}$, $\gamma_{\iota(l)}\big|_{]1/l,1]} :]1/l, 1] \to \operatorname{Im} \gamma_{\iota(l)} \smallsetminus \{\gamma_{\iota(l)}(0)\}$ is a diffeomorphism and $\gamma_{\iota(l)}^{-1}\left(\{\gamma_{\iota(l)}(0)\}\right) = [0, 1/l]$ while if $\gamma \in \mathcal{E}$ then $\gamma_i\big|_{]0,1]} :]0,1] \to M_i$ is a smooth embedding for all sufficiently large i.

Proof. Let $m^0 \in M$ and let $m^\bullet = (m^l)_{l=1}^{\infty} \subseteq M$ be an infinite-ranged sequence converging to m^0 in M. By lemma A.5.9, it suffices to show that there exists some $\gamma \in \mathcal{V} \cup \mathcal{E}$ and some γ-liftable injective subsequence $m^{l_\bullet}$ of $m^\bullet$. If $\dim_{m^0} M < \infty$ then by lemma 16.2.1, M is a smooth manifold on an open neighborhood of m^0 in M so that the desired $\gamma \in \mathcal{E}$ can clearly be found. So assume henceforth that $\dim_{m^0} M = \infty$ and let $m^l_\bullet = \mu_\bullet(m^l)$ and $d_\bullet = \dim_{m^0_\bullet} M_\bullet$ for all $l \in \mathbb{Z}^{\geq 0}$. Assume without loss of generality that $m^\bullet \to m^0$ is injective in M and that $d_{i+1} > d_i$ for all $i \in \mathbb{N}$. Suppose first that $m^\bullet$ contains an infinite subsequence $m^{l_\bullet}$ such that for all $i \in \mathbb{N}$, $\{\mu_i(m^{l_k}) \,|\, k \in \mathbb{N}\}$ is finite and replace $m^\bullet$ with this subsequence. Inductively construct a sequence $(i_l)_{l=1}^{\infty}$ such that for all $l \in \mathbb{N}$, $m^{i_l} \in \mu_{i_l}^{-1}(m^0_{i_l})$ and $\mu_{i_l}(m^{i_1}), \ldots, \mu_{i_l}(m^{i_l}) = m^0_{i_l}$ are all distinct and then apply lemma 15.4.3 to obtain the desired $\gamma \in \mathcal{V}$. Now suppose that $m^\bullet$ does not contain any infinite subsequence $m^{l_\bullet}$ such that for all $i \in \mathbb{N}$, $\left\{m_i^{l_k} \,\middle|\, k \in \mathbb{N}\right\}$ is finite so that there exists some index i such that $\left\{m_i^{l_k} \,\middle|\, k \in \mathbb{N}\right\}$ is infinite. Replace $m^\bullet$ with a subsequence $m^{l_\bullet}$ such that $m_i^{l_\bullet} \to m_i^0$ is injective in M_i and $m^{l_\bullet}$ converges fast to m^0 in M. Restrict to $[0,1]$ the map given by lemma 15.4.8 to obtain the desired $\gamma \in \mathcal{E}$. ∎

Corollary 16.1.3. Let $\mathcal{A}$ denote the set of all smooth almost arcs $[0,1] \to M$ that vanish at 0. If M is a monotone promanifold then M is coherent with $\mathcal{A}$.

Corollary 16.1.4. Monotone promanifolds are coherent with their smooth paths and (consequently) are smoothly locally path-connected.

Corollary 16.1.5. Assume that Sys_M is monotone, let $S \subseteq M$, and let $m^0 \in M$. Let $\mathcal{E}$ and $\mathcal{V}$ denote the sets of smooth almost arcs defined in theorem 16.1.1. Then the following are equivalent:

(1) S is an open (resp. closed) subset of M.

(2) $\gamma^{-1}(S)$ is an open (resp. closed) subset of $[0,1]$ for all $\gamma \in \mathcal{E} \cup \mathcal{V}$.

If $m^0 \in S$ then the following are equivalent:

(1) S is a neighborhood of m^0 in M.

(2) for all $\gamma \in \mathcal{E} \cup \mathcal{V}$ such that $\gamma(0) = m^0$, $\gamma^{-1}(S)$ is a neighborhood of 0 in $[0,1]$.

Proof. The characterization of when S is an open or closed subset of M is from the definition of M being coherent with $\mathcal{E} \cup \mathcal{V}$. One direction of the second characterization is immediate and for the other direct suppose that S is not a neighborhood of m^0 in M. Since M is Fréchet-Urysohn, there exists an injective sequence $m^\bullet = (m^l)_{l=1}^\infty \subseteq M \setminus S$ converging to m^0 in M. Just as was done in the proof of theorem 16.1.1, we can obtain some $\gamma \in \mathcal{E} \cup \mathcal{V}$ and some γ-liftable subsequence $m^{l_\bullet}$ of $m^\bullet$ such that $\gamma(0) = m^0$. Observe that $\gamma^{-1}(S)$ is not a neighborhood of 0 in $[0,1]$ since $\gamma^{-1}\left(m^{l_\bullet}\right)$ is a sequence in $[0,1] \setminus \gamma^{-1}(S)$ that converges to 0 in $[0,1]$. ∎

Remark 16.1.6. The statement of theorem 16.1.1 is analogous to the fact that a smooth manifold with or without corners are coherent with their set of C^1-arcs while the statement of theorem 16.1.7 below is analogous to how smooth manifolds are coherent with C^1 embeddings of $\mathbb{R}$. However, recall that a smooth manifold with corners is a smooth manifold if and only if it is coherent set of C^1 embeddings of $\mathbb{R}$. Similarly, theorem 16.1.1 does not by itself imply theorem 16.1.7.

The following theorem is proved in a manner similar to the proof of theorem 16.1.1, where the desired curve can be obtained from corollary 15.4.10.

Theorem 16.1.7. Let $\mathcal{P}$ denote the set of all restrictions to $]-1,1[$ of smooth almost arcs $[-1,1] \to M$ that vanish at 0. If M is a monotone promanifold then M is coherent with $\mathcal{P}$.

Finite Dimensional Monotone Promanifolds

The following proposition shows that finite-dimensional promanifolds that arise as limits of monotone systems are necessarily manifolds.

Proposition 16.2.1. Suppose Sys_M is monotone, $m^0 \in M$, and $d_0 \underset{\text{def}}{=} \dim_{m^0} M$ is finite. For all $i \in \mathbb{N}$, let O_i denote the connected component of M_i containing $\mu_i(m^0)$. Then for any index i such that $\dim_{\mu_i(m^0)} M_i = d_0$ and any $j \geq i$, both

$$\mu_i\big|_{\mu_i^{-1}(O_i)} : \mu_i^{-1}(O_i) \to O_i \quad \text{and} \quad \mu_{ij}\big|_{O_j} : O_j \to O_i$$

are diffeomorphisms, $O_j = \mu_{ij}^{-1}(O_i)$, and $\mu_i^{-1}(O_i)$ is an open connected component of M.

Proof. Let $m_\bullet^0 \underset{\text{def}}{=} \mu_\bullet(m^0)$. Since O_j is connected and μ_{ij} is continuous we have $\mu_{ij}(O_j) \subseteq O_i$ so $O_j \subseteq \mu_{ij}^{-1}(O_i)$. But μ_{ij} is a monotone continuous open map so the connectedness of O_i implies that $\mu_{ij}^{-1}(O_i)$ is connected. That fact that $m_j^0 \in \mu_{ij}^{-1}(O_i)$ implies that $\mu_{ij}^{-1}(O_i)$ is contained in the connected component of M_j containing m_j^0 and thus $O_j = \mu_{ij}^{-1}(O_i)$.

The map $\mu_{ij}\big|_{\mu_{ij}^{-1}(O_i)} : \mu_{ij}^{-1}(O_i) \to O_i$ is a local diffeomorphism since it is a smooth surjective submersion between manifolds of equal dimensions, which forces its fibers to be discrete and so the monotonicity of μ_{ij} implies that this map is injective and hence a diffeomorphism. Applying what we've just proved to all $j, k \in \mathbb{N}^{\geq i}$ with $j \leq k$ allows us to conclude that all $\mu_{jk}\big|_{O_k} : O_k \to O_j$ are diffeomorphisms and since $\mu_i\big|_{\mu_i^{-1}(O_i)} : \mu_i^{-1}(O_i) \to O_i$ is the limit of the cone $\left(\mu_i^{-1}(O_i), \mu_{i\bullet}\big|_{O_\bullet}\right)$ into $(O_k, \mu_{kl}, \mathbb{N}^{\geq i})$, where $\mu_{i\bullet}\big|_{O_\bullet} \underset{\text{def}}{=} \left(\mu_{ik}\big|_{O_k}\right)_{k=i}^{\infty}$, it follows that $\mu_i\big|_{\mu_i^{-1}(O_i)}$ is an isomorphism in the category of commutative locally $\mathbb{R}$-ringed spaces (i.e. a diffeomorphism). Finally, observe that if $m \in \mathrm{Cl}_M(\mu_i^{-1}(O_i)) \smallsetminus \mu_i^{-1}(O_i)$ then since $\mu_i(m) \notin O_i$, $\mu_i(m)$ belongs to the open set $M_i \smallsetminus O_i$ so that m belongs to the open set $\mu_i^{-1}(M_i \smallsetminus O_i)$, which gives us a contradiction since this set is disjoint from $\mu_i^{-1}(O_i)$. Thus $\mathrm{Cl}_M(\mu_i^{-1}(O_i)) = \mu_i^{-1}(O_i)$, where the connectedness of $\mu_i^{-1}(O_i)$ implies that this set is a connected component of M. ∎

Connectedness of Monotone Promanifolds

Proposition 16.3.1. Suppose Sys_M is monotone and $p \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$.

(1) If i is an index and $C_i \subseteq M_i$ then C_i is C^p-path-connected if and only if the same is true of $\mu_i^{-1}(C_i)$. If C_i is C^p-arcwise-connected then so is $\mu_i^{-1}(C_i)$.

(2) Each connected component of each open subset of M is open in M and smoothly arcwise-connected.

(3) If U is a connected open subset of M, $m^0 \in U$, and $\dim_{m^0} M \geq 2$, then $U \smallsetminus \{m^0\}$ is connected.

(4) Every tangent vector is a kinematic tangent vector.

In particular, M is locally smoothly arcwise connected.

Proof. (1): If $\mu_i^{-1}(C_i)$ is C^p-path-connected then it is clear that the same is true of $C_i = \mu_i(\mu_i^{-1}(C_i))$ so suppose that C_i is C^p-path-connected (resp. C^p-arcwise-connected) and let

$m^0, m^1 \in \mu_i^{-1}(C_i)$. Let $\gamma_i : ([0,1], 0, 1) \to (M_i, \mu_i(m^0), \mu_i(m^1))$ be a C^p-path (resp. C^p-arc) contained in C_i. Inductively construct smooth maps $\gamma_j : ([0,1], 0, 1) \to \left(\mu_{ij}^{-1}(U_i), m_j^0, m_j^1\right)$ such that $\gamma_{j-1} = \mu_{j-1,j} \circ \gamma_j$ for all $j > i$. Then $\gamma \underset{\text{def}}{=} \varprojlim \gamma_\bullet$ is a smooth path in $\mu_i^{-1}(C_i)$ from $m^0 = \gamma(0)$ to $m^1 = \gamma(1)$ that is a C^p-arc whenever γ_i is.

(2): M is locally path-connected by proposition 2.5.12 so it consequently suffices to show that each connected open set is smoothly arc-wise connected. Let U be non-empty connected open subset of M. If U is singleton then U is vacuously smoothly arcwise connected so suppose that $m^0, m^1 \in U$ are two distinct points and let $m_\bullet^l \underset{\text{def}}{=} \mu_\bullet(m^l)$ $(l = 0, 1)$. Since U contains a continuous path from m^0 to m^1, we may pick an index i and a connected open subset $U_i \in \operatorname{Open}(M_i)$ such that $m^0, m^1 \in \mu_i^{-1}(U_i) \subseteq U$ with $m_i^0 \neq m_i^1$. By (1), $\mu_i^{-1}(U_i) \subseteq U$ is smoothly arc-wise connected so that the conclusion follows.

(3): Let $m_\bullet^0 = \mu_\bullet(m^0)$ and for each index i let U_i denote the largest open subset of M_i such that $\mu_i^{-1}(U_i) \subseteq U$ and if $\mu_i(m^0) \in U_i$ then let O_i denote the connected component of U_i containing $\mu_i(m^0)$ and let $O_i = \varnothing$ otherwise. If $m_i^0 \in U_j$ then let O_j denote the connected component of U_i containing this point and let $O_i = \varnothing$ otherwise. Since $U = \cup \mu_\bullet^{-1}(O_\bullet)$ by lemma 16.0.3 we may pick an index i such that $\dim_{m_i^0} M_i \geq 2$ and $m^0 \in \mu_i^{-1}(O_i)$. For all $j \geq i$ let $W_j \underset{\text{def}}{=} O_j \setminus \{m_j^0\}$ and observe that W_j is connected. Since $U = \underset{j \geq i}{\cup} \mu_j^{-1}(O_j)$ it is clear that $U \setminus \{m^0\} = \underset{j \geq i}{\cup} \mu_j^{-1}(W_j)$, which is an increasing union of connected sets, which is thus connected.

(4): That every tangent vector is kinematic follows immediately from lemma 7.5.4. ■

A Characterization of $\operatorname{Dom}_i F^a$

The following lemma, which will not be used in this paper, shows that if $F : M \to N$ is any smooth map from a monotone promanifold then each set $\operatorname{Dom}_i F^a$ is completely determined by $\operatorname{T} F$ and the sets of μ_i-vertical and ν_a-vertical tangent vectors. Furthermore, this lemma immediately produces a necessary and sufficient condition for the equality $\operatorname{Dom}_i F^a = M_i$ to hold.

Lemma 16.4.1. Suppose that Sys_M is monotone and that $F : M \to N$ is a smooth map. Pick indices i and a and let $m_i^0 \in M_i$. Then $m_i^0 \in \operatorname{Dom}_i F^a$ if and only if $\mathrm{T}\, F : \mathrm{T}\, M \to \mathrm{T}\, N$ maps every μ_i-vertical tangent vector over m_i^0 to a ν_a-vertical tangent vector (i.e. for every $\mathbf{v} \in (\mathrm{T}\, \mu_i)^{-1}\big(\mathbf{0}_{m_i^0}\big)$, $\mathrm{T}\, F\mathbf{v}$ belongs to $(\mathrm{T}\, \nu_a)^{-1}(\mathbf{0}_{n_a})$ where $n_a = \nu_a(F(\mathrm{T}_M\, \mathbf{v}))$).

Proof. If $m_i \in \operatorname{Dom}_i F^a$ then since every μ_i-vertical vector arises as the derivative of smooth curve contained in $\mu_i^{-1}(m_i^0)$, it follows that any such curve's image under $\nu_a \circ F$ will be the singleton set $(\mu_a \circ F)(\mu_i^{-1}(m_i^0)) = \{F_i^a(m_i^0)\}$ so that the right hand side now follows immediately. For the converse, fix two points $m^0, m^1 \in \mu_i^{-1}(m_i^0)$ and let $\gamma : [0,1] \to M$ be a smooth path from m^0 to m^1 whose image is contained in $\mu_i^{-1}(m_i^0)$. Observe that for all $t \in [0,1]$, $\gamma'(t)$ is a μ_i-vertical tangent vector of m_i^0 so that by assumption $\mathrm{T}\, F(\gamma'(t))$ is a ν_a-vertical tangent vector, which implies that $(\nu_a \circ F \circ \gamma)'(t) = 0$. This implies that $\nu_a \circ F \circ \gamma : [0,1] \to N_a$ is constant so that

$$\nu_a\left(F\left(m^0\right)\right) = \nu_a\left(F(\gamma(0))\right) = \nu_a\left(F(\gamma(1))\right) = \nu_a\left(F\left(m^1\right)\right)$$

Since $m^1 \in \mu_i^{-1}(m_i^0)$ was arbitrary it follows that $(\nu_a \circ F)(\mu_i^{-1}(m_i^0)) = \{\nu_a(F(m^0))\}$ so that m_i^0 belongs to $\operatorname{Dom}_i F^a$ by definition of $\operatorname{Dom}_i F^a$. ∎

Sufficient Conditions for Openness at a Point

Theorem 16.5.1. Assume that Sys_N is monotone and let $\mathcal{E}$ and $\mathcal{V}$ denote the sets of smooth almost arcs defined in theorem 16.1.1. Let $F : (M, m^0) \to (N, n^0)$ be a smooth pointwise isomersion. If for all $\eta \in \mathcal{E} \cup \mathcal{V}$ whose image contains n^0, the path component of $F^{-1}(\operatorname{Im}\eta)$ containing m^0 contains at least two distinct points then m^0 is a point of openness of $F : M \to N$. Thus, if for all $\eta \in \mathcal{E} \cup \mathcal{V}$, each path component of $F^{-1}(\operatorname{Im}\eta)$ contains at least two distinct points then $F : M \to N$ is an open map.

Proof. Let $m_\bullet^0 = \mu_\bullet(m^0)$, $n_\bullet^0 = \mu_\bullet(n^0)$, $F^\bullet = \nu_\bullet \circ F$, let $\eta \in \mathcal{E} \cup \mathcal{V}$ be a smooth almost arc through n^0 vanishing at 0, and let $(n^l)\,l = 1^\infty$ be any sequence in $\operatorname{Im}\eta$ converging to n^0. Let C be a path-component of $F^{-1}(\operatorname{Im}\eta)$ containing m^0 and note that since η is a smooth almost arc and C contains at least two distinct elements, $F(C)$ contains at least two distinct elements. So let $\alpha : ([0,b],0) \to (C,m^0)$ be a C^0-arc such that $\operatorname{Im}(F \circ \alpha)$ consists of more than one point and $F(\alpha(b)) = \eta(b)$. Let $\alpha_\bullet = \mu_\bullet \circ \alpha$. Assume without loss of generality that $m_i^0 \in \operatorname{ODom}_i F^i$ for all $i \in \mathbb{N}$ and that $\nu_1(F(C))$ consists of at least two distinct elements. Let $U_\bullet$ be a $\mu_\bullet$-neighborhood basis of M at m^0 such that $m_i^0 \in U_i \subseteq \operatorname{ODom}_i F^i$ for all $i \in \mathbb{N}$ and observe that $(F_i^i(U_i))_{i=1}^\infty$ forms a $\nu_\bullet$-neighborhood basis of N at n^0. Inductively pick $b_1 > b_2 > \cdots$ converging to 0 such that $\alpha_i([0,b_i]) \subseteq U_i$ contains more than one element, which implies that $F(\alpha([0,b_i]))$ consists of at least two elements. We may thus pick $(l_k)_{k=1}^\infty$ increasing such that $n^{l_k} \in F(\alpha([0,b_{l_k}])\,)$, which allows us to pick $m^k \in \alpha([0,b_{l_k}]) \cap F^{-1}(n^{l_k})$ for all $k \in \mathbb{N}$. Since $b_\bullet \to 0$, the sequence $(m^k)_{k=1}^\infty$ necessarily converges in M to m^0. Thus, every injective sequence in N that converges to n^0 has an F-liftable subsequence that converges to m^0 so that lemmata A.5.9 and A.2.2 allows us to conclude that m^0 is a point of openness of F. ∎

Theorem 16.5.2. Assume that Sys_N is monotone and let $\mathcal{E}$ and $\mathcal{V}$ denote the sets of smooth almost arcs defined in theorem 16.1.1. Let $F : (M,m^0) \to (N,n^0)$ be a smooth pointwise isomersion. If for all $\eta \in \mathcal{E} \cup \mathcal{V}$ whose image contains n^0, the connected component of $F^{-1}(\operatorname{Im}\eta)$ containing m^0 contains at least two distinct points then m^0 is a point of openness of $F : M \to N$. In particular, if for all $\eta \in \mathcal{E} \cup \mathcal{V}$, each non-empty connected component of $F^{-1}(\operatorname{Im}\eta)$ contains at least two distinct point then $F : M \to N$ is an open map.

Proof. Let $m_\bullet^0 = \mu_\bullet(m^0)$, $n_\bullet^0 = \mu_\bullet(n^0)$, $F^\bullet = \nu_\bullet \circ F$, and let $(n^l)\,l = 1^\infty \subseteq N$ be a sequence converging to n^0 in N such that $n^\bullet \to n^0$ is injective. By lemma A.2.2, to conclude that m^0 is a point of openness of $F : M \to N$ it suffices to find some increasing $(l_k)_{k=1}^\infty$ and some sequence $(m^k)_{k=1}^\infty \subseteq M$ such that $m^\bullet \to m^0$ is an F-lift of $(n^{l_k})_{k=1}^\infty$. By lemma A.5.9, we may assume without loss of generality that there is some $\eta \in \mathcal{E} \cup \mathcal{V}$ be a smooth almost arc through

n^0 vanishing at 0 such that $n^\bullet \subseteq \operatorname{Im} \eta$. Let $\eta_\bullet = \nu_\bullet \circ \eta$ and let C be a connected component of $F^{-1}(\operatorname{Im} \eta)$ containing m^0.

Note that since η is a smooth almost arc and C contains at least two distinct elements, $F(C)$ contains at least two distinct elements. Assume without loss of generality that $m_i^0 \in \operatorname{ODom}_i F^i$ for all $i \in \mathbb{N}$ and that $\nu_1(F(C))$ consists of at least two distinct elements. Let $U_\bullet$ be a $\mu_\bullet$-neighborhood basis of M at m^0 such that $m_i^0 \in U_i \subseteq \operatorname{ODom}_i F^i$ and U_i is connected and the domain of a smooth chart centered at m_i^0 for all $i \in \mathbb{N}$. Observe that $(F_i^i(U_i))_{i=1}^\infty$ forms a $\nu_\bullet$-neighborhood basis of N at n^0. Let $U^\bullet = \mu_\bullet^{-1}(U_\bullet)$ and note that since C is connected and contains at least two distinct elements, every $C \cap U^i$ must also contain at least two distinct elements and since the fiber $F^{-1}(n^0)$ is totally disconnected by corollary 11.5.7, this implies that every $F(C \cap U^i)$ contains at least two distinct elements.

Let D^i be the connected component of $U^i \cap C$ containing m^0. By assumption, D^i contains at least two distinct elements so that proposition 11.5.1 implies that the same is true of $F(D^i)$ so that the connectedness of D^i implies that $F(D^i)$ is a neighborhood of n^0 in $\operatorname{Im} \eta$. Let $l_1 \in \mathbb{N}$ and $m^1 \in D^1$ be such that $F(m^1) = n^{l_1}$. Having picked increasing integers $l_1, \ldots, l_k$ and $m^1, \ldots, m^k$ such that $m^h \in D^h$ and $F(m^h) = n^{l_h}$ for all $h = 1, \ldots, k$, pick $l_{k+1} > l_k$ and $m^{k+1} \in D^h$ such that $F(m^{k+1}) = n^{l_{k+1}}$. Observe that $m^\bullet \to m^0$ is an F-lift of $(n^{l_k})_{k=1}^\infty$, as desired. ∎

In addition to characterizing when some given point is a point of openness of a smooth isomersion, the following theorem provides a sufficient condition for the inverse function theorem to hold "topologically almost everywhere."

Theorem 16.5.3. Let $F : (M, m) \to (N, n)$ be a smooth pointwise isomersion on M and suppose that N is a monotone promanifold. Then m is a point of openness of $F : M \to N$ (def. A.0.6(d)) if any one of the following conditions holds, where each condition begins with "F is a germ submersion at m (def. 1.1.26(1)) from "

(1) smooth almost arcs in N to C^0-paths in M.

(2) smooth almost arcs in N to smooth almost arcs in M.

(3) smooth paths in N to C^0-paths in M.

(4) from smooth paths in N to smooth paths in M.

Furthermore, if F has discrete fibers and at least one of the above conditions holds at every $m \in M$, then there exists a dense (in M) open subset $U \in \operatorname{Open}(M)$ such that $F\big|_U : U \to N$ is an open map that is a local diffeomorphism around each of its points.

Proof. It follows from theorem 16.5.2 that any one of these conditions is sufficient for m to be a point of openness of F. If F has discrete fibers and the above conditions are satisfied at every $m \in M$, then the existence of the dense open subset U of M on which F is a local diffeomorphism follows from theorems A.3.2 and 11.6.1. ∎

If N is not a monotone promanifold then it may still be possible to use the following proposition to deduce that some given point is a point of openness (def. A.0.6(d)) of a smooth pointwise isomersion.

Proposition 16.5.4. Let $F : (M, m) \to (N, n)$ be a smooth pointwise isomersion on M and suppose that N is coherent with its smooth paths at n (def. A.5.1). Then m is a point of openness of $F : M \to N$ if either of the following two equivalent conditions hold:

(1) F is a germ submersion at m from smooth paths in N to C^0-paths in M (def. 1.1.26(1)).

(2) F is a germ submersion at m from smooth paths in N to smooth paths in M.

Proof. The equivalence of (1) and (2) follows from corollary 11.5.3. That these conditions are sufficient for m to be a point of openness of F follows from theorem 16.5.2 and the definition A.5.1. ∎

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Appendix A

Topology

In this appendix we (1) explicitly define some basic topological terminology whose definition may vary between authors or that are not frequently used by specialists in differential geometry, (2) include some rarer topological definitions, (3) prove some results in point-set topology that (to the author's knowledge) have not appeared anywhere else, (4) provide proofs of some results for which no citation could be found despite their relatively frequent implicit use in mathematical literature.

Definition A.0.1 (Filters and filter bases)**.** Let X be a non-empty set, let $\mathcal{F}$ be a non-empty collection $\mathcal{F}$ of subsets of X, and consider the following statements:

(1) Directed downwards: if $F_1, F_2 \in \mathcal{F}$ there exists some $F_3 \in \mathcal{F}$ such that $F_3 \subseteq F_1 \cap F_2$.

(2) Proper: $\varnothing \notin \mathcal{F}$.

(3) Closed upwards: if $F \in \mathcal{F}$ and $F \subseteq S \subseteq X$ then $S \in \mathcal{F}$.

We call $\mathcal{F}$ a *filter base* (resp. *filter*) *on* X if it satisfies (1) and (2) (resp. and (3)). Every filter base $\mathcal{F}$ is contained in a unique smallest filter on X called *the filter (on X) generated by $\mathcal{F}$*.

If $(x_i)_{i \in I}$ is a net in X then for every $i_0 \in I$ the *tail of* $(x_i)_{i \in I}$ *(after i_0)* is the set $\{x_i \,|\, i \in I,\, i \geq i_0\}$. The set of all tails of $x_\bullet = (x_i)_{i \in I}$, denoted by $\operatorname{Tails}(x_\bullet)$, is called *the filter* *base (of tails) generated by* $(x_i)_{i \in I}$ and the filter on X that it generates is called *the filter (on* X*) generated by* $(x_i)_{i \in I}$. ■

Definition A.0.2 (Neighborhood filters and convergent filter bases)**.** If S is a subset of a topological space (X, τ_X) then let

$$\tau_X(S) \underset{\text{def}}{=} \{U \in \tau_X \mid S \subseteq U\}$$

and call its elements *open neighborhoods of* S *(in* X*)*. Observe that $\tau_X(S)$ is a filter base on X for any $S \neq \varnothing$ and denote the filter that it generates in X by $\mathrm{Nhd}_{\tau_X}(S)$, $\mathrm{Nhd}_X(S)$, or simply $\mathrm{Nhd}(S)$ and call its elements *neighborhoods of* S *(in* X*)*. If $S = \{x\}$ is a singleton set then we may omit writing braces with the above notation and refer to (resp. open) neighborhoods of $\{x\}$ as *(resp. open) neighborhoods of* x.

If $\mathcal{F}$ is a filter base in X and $S \subseteq X$ then $\mathcal{F}$ *converges to* S *(in* X*)* if $\mathcal{F}$ is finer than $\mathrm{Nhd}_X(S)$. A net $(x_i)_{i \in I}$ *converges to* $x \in X$ *(in* X*)* if its filter base of tails converges to x. More generally, if I is directed, S_i is a subset of X for each $i \in I$, and $S \subseteq X$ then we may say that the net of sets $(S_i)_{i \in I}$ *converges to* S *(in* X*)* if its *filter base of tails* $\left\{ \bigcup\limits_{\substack{i \in I \\ i \geq i_0}} S_i \,\middle|\, i_0 \in I \right\}$ converges to S in X. ■

Observation A.0.3 below show that the notion of filters together with the well-known Stone topology on some given set allow us to topologize the set of topologies with a simple, but non-trivial, topology defined entirely in terms of the underlying set. That this is even possible appears to have gone unnoticed even though it allows one to, for instance, sensibly and rigorously speak of "a continuous path of topologies" and "the homotopy class of a topology." Since we will not be needing this notion for the theory of promanifolds, we relegate it to the following observation and state without proof assertions that the reader may readily verify.

Observation A.0.3 (Topologizing the set of topologies)**.** Let X be any non-empty set, let

Fltr_X denotes the set of all filters on X, and for any $S \subseteq X$ let $\operatorname{Fltr}_X(S) \underset{\text{def}}{=} \{\mathfrak{f} \in \operatorname{Fltr}_X \,|\, S \in \mathfrak{f}\}$. The sets in $\{\operatorname{Fltr}_X(S) \,|\, S \subseteq X\}$ constitute the basic open sets of a topology on Fltr_X called the *Stone topology*, where although this topology is usually only defined on the set of all ultrafilters on X, one may readily verify that $\operatorname{Fltr}_X(S) \cap \operatorname{Fltr}_X(S') = \operatorname{Fltr}_X(S \cap S')$ for all $S, S' \subseteq X$ so that these sets still forms a basis for a topology on Fltr_X. Note that every topology τ on X induces a canonical map of neighborhoods $\operatorname{Nhd}_\tau : X \to \operatorname{Fltr}_X$ that sends $x \in X$ to its neighborhood filter $\operatorname{Nhd}_\tau(x)$ where one may show that $\operatorname{Nhd}_\tau : (X, \tau) \to \operatorname{Im}(\operatorname{Nhd}_\tau)$ is necessarily a continuous, open, and closed map. Indeed, if $f : X \to \operatorname{Fltr}_X$ is any map such that for all $x \in X$, $x \in \cap f(x)$ then for any $x \in X$ and $N \in f(x)$, $f(N)$ is a neighborhood of $f(x)$ in $\operatorname{Im} f$ (to prove these assertions, it helps to first observe that for any $\mathcal{S} \subseteq \operatorname{Fltr}_X$ and $\mathfrak{f} \in \operatorname{Fltr}_X$, $\mathfrak{f} \in \operatorname{Cl}_{\operatorname{Fltr}_X}(\mathcal{S}) \iff \mathfrak{f} \subseteq \cup \mathcal{S}$ and that $\mathcal{S}$ is a neighborhood of $\mathfrak{f}$ in $\operatorname{Fltr}_X \iff$ there exists some $R \in \mathfrak{f}$ such that for all $\mathfrak{g} \in \operatorname{Fltr}_X$, $R \in \mathfrak{g} \implies \mathfrak{g} \in \mathcal{S}$). Now place the topology of pointwise convergence (or any other topology of uniform convergence) on the set of maps $\{\operatorname{Nhd}_\tau \,|\, \tau \text{ is a topology on } X\}$ and observe that since this set is in 1-to-1 correspondence with the set of all topologies on X via the canonical map $\tau \mapsto \operatorname{Nhd}_\tau$, we have thus also topologized the set of all topologies on X which, in particular, now allows us to talk sensibly about things such as "a continuous path of topologies on X."

Definition A.0.4. Let X be a topological space and let $x \in X$. Say that

(a) x is *isolated (in X)* if $\{x\}$ is open in X.

(b) X *is locally compact at* x if there exists a compact neighborhood of x.

Say that a subset S of X is

(c) *comeager (in X)* if $X \setminus S$ is meager in X.

(d) G_δ (resp. F_σ) if S is the intersection (resp. union) of countably many open (resp. closed) subsets of X.

(e) *meager (in X)* or *of the first category (in X)* if S is a countable union of nowhere dense (in X) subsets of X.

> **Remark:** A meager set is a topological generalization of "a subset of measure 0" so Baire spaces are spaces with no non-empty open subsets "of topological measure 0." This interpretation lends itself particularly well to the conclusion of Sard's theorem.

(f) *nonmeager (in X)* or *of the second category (in X)* if S is not meager in X.

(g) *nowhere dense (in X)* if any of the following equivalent conditions hold:

  (i) S is contained in the boundary of an open set.

  (ii) $\operatorname{Int}(\operatorname{Cl}_X(S)) = \varnothing$.

  (iii) The exterior of S is dense in X ([11]).

  (iv) There is no non-empty open set U such that $S \cap U$ is dense in U.

(h) *perfect* or *a perfect subset (of X)* if it is closed in X and has no isolated points.

(i) a *regular closed* subset of X if $\overline{\operatorname{Int}(S)} = S$ or equivalently, if $\operatorname{Bd}(\operatorname{Int}(S)) = \operatorname{Bd}(S)$.

(j) a *regular open* subset of X if $\operatorname{Int}\left(\overline{S}\right) = S$ or equivalently, if $\operatorname{Bd}\left(\overline{S}\right) = \operatorname{Bd}(S)$.

(k) *relatively compact (in X)* if $\operatorname{Cl}_X(S)$ is compact. ∎

Definition A.0.5. A topological space (X, τ_X) is said to be

(a) *Baire* if no open subset of X is meager in X or equivalently, if whenever a countable union of closed subsets of X has non-empty interior in X then the same is true of at least of these closed sets.

(b) *discrete* if each point of X is isolated in X.

(c) *first countable* if for every $x \in X$ there exists a countable neighborhood basis of x in X.

(d) *Polish* if it is a separable completely metrizable topological space.

(e) *second countable* if X has a countable basis.

(f) *semiregular* if it has a basis consisting of regular open sets.

(g) *strongly Baire* each of its closed subsets is a Baire space.

(h) *totally disconnected* if its only connected subsets are singleton sets and the empty set. ■

Definition A.0.6. Let $f:(X,\tau_X)\to(Y,\tau_Y)$ be a map between topological spaces. Then we will say that f has/is/is a(n)

(a) *monotone* if each fiber of f is connected.

(b) *light* if every fiber of f is totally disconnected.

(c) *pseudo-open* if for each $y \in Y$ whenever $U \in \operatorname{Open}(X)$ contains $f^{-1}(y)$ then $y \in \operatorname{Int}_Y(f(U))$.

(d) *a point of openness at* $x \in X$ and call x *is a point of openness for* f if for every $x \in U \in \operatorname{Open}(X)$, $f(U)$ is a neighborhood of $f(x)$ in Y.

(e) *almost open* if f is surjective and for each $y \in Y$, there exists some $x \in f^{-1}(y)$ such that x is a point of openness for f.

(f) *open* or *strongly open* (resp. *closed* or *strongly closed*) if for all $R \subseteq X$, whenever R is open (resp. closed) in X then $f(R)$ is *open* (resp. *closed*) in Y.

(g) *open map (resp. closed map, homeomorphism, etc.) onto its image* if the natural induced map $f:X\to\operatorname{Im} f$ is an open map (resp. closed map, homeomorphism, etc.) when $\operatorname{Im} f$ is given the subspace topology induced by Y.

(h) *locally injective* if for every $x \in X$ there exists some $x \in U \in \operatorname{Open}(X)$ such that $f\big|_U : U \to Y$ is injective.

(i) *proper* if the preimage of every compact subset of Y is compact in X.

(j) *universally closed* if it is continuous and for every space Z, $f \times \mathrm{Id}_Z : X \times Z \to Y \times Z$ is closed.

(k) *compact-covering* or *k-covering* (resp. *countable-compact-covering* or *s-covering*) if for every compact (resp. countable and compact) subset $K \subseteq Y$ there exists a compact set $C \subseteq X$ such that $f(C) = K$.

(l) *hereditarily quotient* if for every $S \subseteq Y$ the map $f\big|_{f^{-1}(S)} : f^{-1}(S) \to S$ is a quotient map.

(m) *sequentially quotient* if every convergent sequence in Y has a subsequence that f can lift.

(n) *sequence-covering* if f can lift every convergent sequence in Y.

(o) 1-*sequence-covering* if for all $y \in Y$ there exists some $x \in f^{-1}(y)$ such that every sequence in Y that converges to y has an f-lift that converges to x.

(p) 2-*sequence-covering* if f is surjective and for all $y \in Y$ and $x \in f^{-1}(y)$, every sequence in Y that converges to y has an f-lift that converges to x.

Furthermore,

- a continuous map $H : X \times [0,1] \to Y$ *is a homotopy for f* ([34]) if $H(\cdot, 0) = f$.
- if H and G are homotopies for f then we will say that they *have the same germ* ([34]) if there is some open set $X \times \{0\} \subseteq W \in \operatorname{Open}(X \times [0,1])$ for which $H|_W = G|_W$.
- a *polytope* is a finite simplicial complex ([34]).

If X and Y are metric spaces with metrics d_X and d_Y, respectively, then say that f is

- *contractive* (resp. *non-expansive*) if there exists a positive real number L such that $L < 1$ (resp. $L \leq 1$) and whenever $x_1, x_2 \in X$ then $d_Y(f(x_1), f(x_2)) \leq L\, d_X(x_1, x_2)$. ■

Definition A.0.7. Let $f : X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a map and let $x_0 \in X$. Then f is *lower (resp. upper) semicontinuous at* x if for all $r \in \mathbb{R}$ such that $r < f(x)$ (resp. $r > f(x)$) there exists $x \in U \in \operatorname{Open}(X)$ such that $r < f(u)$ (resp. $r > f(u)$) for all $u \in U$.

Remark A.0.8. It is straightforward to show that if a continuous surjection $f : (X, \tau_X) \to (Y, \tau_Y)$ is sequentially quotient then $f : (X, \operatorname{SeqOpen}(X, \tau_X)) \to (Y, \operatorname{SeqOpen}(Y, \tau_Y))$ is a quotient map (where $\operatorname{SeqOpen}(X, \tau_X)$ is defined in def. A.1.1) and that the converse is true if (Y, τ_Y) is Hausdorff.

The following example and its definitions are from [31].

Example and Definition A.0.9. The *rank (resp. nullity)* of a linear map is defined to be the dimension of its range (resp. kernel). If $x_0 \in X$ and $d \in \mathbb{Z}^{\geq 0}$ then we will say that x_0 is a *constant rank (resp. nullity) point of* Λ *(of rank (resp. nullity)* d*)* or that Λ *has constant rank (resp. nullity)* d *around* x_0 if there exists a neighborhood U of x_0 in X such that $\operatorname{rank}(\Lambda(x)) = d$ for all $x \in U$ and otherwise, we will call x_0 a *non-constant rank point* or a *rank singular point of* Λ.

If $\Lambda : X \to L(\mathbb{R}^m \to \mathbb{R}^n)$ is a continuous map from a topological space X into $L(\mathbb{R}^m \to \mathbb{R}^n)$, the TVS of all linear maps from $\mathbb{R}^m$ into $\mathbb{R}^n$, then the rank and nullity maps

$$\begin{aligned} \operatorname{rank}\Lambda : X &\longrightarrow \mathbb{Z}^{\geq 0} \\ x &\longmapsto \operatorname{rank}(\Lambda(x)) \end{aligned} \qquad \text{and} \qquad \begin{aligned} \operatorname{nullity}\Lambda : X &\longrightarrow \mathbb{Z}^{\geq 0} \\ x &\longmapsto \operatorname{nullity}(\Lambda(x)) \end{aligned}$$

are lower-semicontinuous and upper-semicontinuous, respectively, and the set of all rank regular points is a dense open subset of X. ■

The following proposition is applicable to all promanifolds.

Proposition A.0.10. Let $f : X \to Y$ be a continuous surjection between first-countable Hausdorff spaces. Then

(a) f is almost open $\iff$ f is 1-sequence-covering.

(b) f is open $\iff$ f is 2-sequence covering.

(c) f a compact covering $\implies$ f is a quotient map.

Furthermore, the following are equivalent: f is

(1) a quotient map.

(2) sequentially quotient.

(3) pseudo-open.

If in addition X and Y are separable metric spaces then we may add to this list:

(5) hereditarily quotient.

Proof. This is a combination of the results found in [44], [18], and [32]. ■

Example A.0.11 (Both g and $g \circ f$ are monotone while f isn't)**.** Let $P = \{\star\}$ denote a topological space with one point, let X and Y denote connected space, let $g : Y \to P$ be the constant map, and let $f : X \to Y$ be arbitrary. Observe that g and $g \circ f : X \to P$ are both monotone but that f may fail to be monotone.

Example A.0.12 (Both f and $g \circ f$ are monotone while g isn't)**.** Let X be any topological space and let X' be a disjoint copy of X where for each $x \in X$ we denote the element in X' corresponding to x by x'. Let $f : X \to X \sqcup X'$ denote the natural inclusion and let $g : X \sqcup X' \to X$ denote the map that restricts to the identity on X and sends $x' \in X'$ to x. Then both $g \circ f = \mathrm{Id}_X$ and f are monotone but g fails to be monotone.

Sequential Spaces

Definition A.1.1 (([5, p. 446], [14], [44])). For any subset S of a space (X, τ_X), *the sequential closure of S in X* is

$$\operatorname{SeqCl}_X(S) = \{x \in X : \text{ there exists some sequence } (s_l)_{l=1}^{\infty} \subseteq S \text{ such that } s_\bullet \to x \text{ in } X\}$$

which is a subset of $\operatorname{Cl}_X(S)$. The set S is said to be *sequentially closed in* X if $S = \operatorname{SeqCl}_X(S)$ and *sequentially open in* X if whenever $s \in S$ and $(x^l)_{l=1}^{\infty}$ is a sequence in X converging to s then there exists some integer L such that $l \geq L \implies x^l \in S$. A subset S of X is sequentially open (resp. sequentially closed) in X if and only if $X \setminus S$ is sequentially closed (resp. sequentially open) in X and the set $\operatorname{SeqOpen}(X, \tau_X)$ of all sequentially open subsets of X forms a topology on X that is finer than τ_X. Although every open (resp. closed) subset of X is necessarily sequentially open (resp. sequentially closed), the converse is not necessarily true so we call those spaces for which the converse does hold *sequential spaces*. ■

Sequential spaces are characterized by the following universal property.

Universal Property A.1.2. A space (X, τ_X) is sequential if and only if for all spaces Y and maps $f : X \to Y$, f is continuous if and only if whenever $x_\bullet = (x_l)_{l=1}^{\infty} \to x$ in X then $f(x_\bullet) \to f(x)$ in Y.

Even if a space X is a sequential space it is possible that there are subsets $S \subseteq X$ such that $\operatorname{Cl}_X(S) \neq \operatorname{SeqCl}_X(S)$ where if X is a sequential space then this inequality holds for a subset $S \subseteq X$ if and only if $\operatorname{SeqCl}_X(\operatorname{SeqCl}_X(S)) \neq \operatorname{SeqCl}_X(S)$.

Definition A.1.3. A space X is a *Fréchet-Urysohn space* if $\operatorname{Cl}_X(S) = \operatorname{SeqCl}_X(S)$ for all subsets $S \subseteq X$, which can easily be shown to be equivalent to all subspaces of X being sequential spaces.

The following lemma is essentially an observation.

Lemma A.1.4. If X is sequential Hausdorff and $S \subseteq X$ then (1) - (3) are equivalent:

(1) S is sequentially open X,

(2) whenever $(x^l)_{l=1}^{\infty} \subseteq X$ converges in X to some point in S then $S \cap \{x^l \mid l \in \mathbb{N}\} \neq \varnothing$,

(3) whenever $x \in S$ and $(x^l)_{l=1}^{\infty} \subseteq X$ is such that $x^\bullet \to x$ is injective in X then $S \cap \{x^l \mid l \in \mathbb{N}\} \neq \varnothing$.

If $s \in S$ and X is in addition a Fréchet-Urysohn space then the following are equivalent:

(4) S is a neighborhood of s in X,

(5) whenever $(x^l)_{l=1}^{\infty} \subseteq X$ converges to s in X then $S \cap \{x^l \mid l \in \mathbb{N}\} \neq \varnothing$,

(6) whenever $(x^l)_{l=1}^{\infty} \subseteq X$ is such that $x^\bullet \to s$ is injective in X then $S \cap \{x^l \mid l \in \mathbb{N}\} \neq \varnothing$.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) is obvious and we now prove (3) $\Longrightarrow$ (1). Suppose that S is not sequentially open in X. Then there exists $(x^l)_{l=1}^{\infty} \subseteq X$ converging in X to $s \in S$ such that for all $k \in \mathbb{N}$, there exists some $l \geq k$ such that $x^l \notin S$. Thus, we may pick a subsequence $(x^{l_k})_{k=1}^{\infty}$ converging to s in X such that $x^{l_k} \notin S$. In particular, $x^{l_k} \neq s$ for all k and by replacing $x^\bullet$ with $(x^{l_k})_{k=1}^{\infty}$, we may assume without loss of generality that $S \cap \{x^l \mid l \in \mathbb{N}\} = \varnothing$. If any subsequence $(x^{l_k})_{k=1}^{\infty}$ of $x^\bullet$ was constant then since it converges to s, we'd necessarily have $x^{l_k} = s$ for all $k \in \mathbb{N}$ (since X is Hausdorff) so that $x^{l_k} \in S$, giving us a contradiction. This implies that each x^l appears in the sequence at most finitely many times which allows us to pick a subsequence $(x^{l_k})_{k=1}^{\infty}$ of $(x^l)_{l=1}^{\infty}$ such that $n > l$ implies $x^l \neq x^n$. Thus $S \cap \{x^{l_k} \mid k \in \mathbb{N}\} = \varnothing$, which contradicts our assumption.

Assume now that X is a Hausdorff Fréchet-Urysohn space and let $s \in S$. (4) $\Longrightarrow$ (5) $\Longrightarrow$ (6) is always true so assume (6) holds but that (4) fails. Then $s \in \operatorname{Cl}_X(X \setminus S) = \operatorname{SeqCl}_X(X \setminus S)$ so there exists a sequence $x^\bullet \subseteq X \setminus S$ that converges to s in X. Since X is

Hausdorff, no subsequence of $x^\bullet$ is constant so we may pick an injective subsequence $(x^{l_k})_{k=1}^\infty$ of $x^\bullet$ so that $x^{l_\bullet} \to s$ is injective in X where $x^{l_\bullet} \subseteq X \setminus S$, which is a contradiction. ■

Open Mapping Sufficient Conditions

The following corollary of lemma A.1.4 is essentially an observation.

Corollary A.2.1. If Y is a sequential Hausdorff space, $f : X \to Y$ is a surjective map, and $\mathcal{B}$ is a basis for X, then the following are equivalent:

(1) $f : X \to Y$ is open.

(2) For all $x \in X$ and $B \in \mathcal{B}$, whenever $(y_l)_{l=1}^\infty \to f(x)$ in Y then $f(B) \cap y_\bullet \neq \varnothing$.

(3) For all $x \in X$ and $B \in \mathcal{B}$, whenever $(y_l)_{l=1}^\infty \subseteq Y$ is a sequence such that $y_\bullet \to f(x)$ is injective, then $f(B) \cap y_\bullet \neq \varnothing$.

Proof. (1) $\implies$ (2) $\implies$ (3) is obvious and we prove (3) $\implies$ (1). Suppose $B \in \mathcal{B}$ is such that $f(B)$ is not open in Y. Then as shown by the characterization in lemma A.1.4, there exists a sequence $(y_l)_{l=1}^\infty \subseteq Y$ and some $y \in f(B)$ such that $y_\bullet \to y$ is injective in Y and $f(B) \cap \{y_l : l \in \mathbb{N}\} = \varnothing$. Now pick $x \in B$ such that $y = f(x)$ and observe that this contradicts our assumption. ■

Lemma A.2.2. Suppose that Y is a Fréchet-Urysohn Hausdorff space and $f : (X, x^0) \to (Y, y^0)$ is a map whenever $(y^l)_{l=1}^\infty \subseteq Y$ is a sequence such that $y^\bullet \to y^0$ is injective in Y then there exists some increasing $(l_k)_{k=1}^\infty$ and some sequence $(x^k)_{k=1}^\infty \subseteq X$ such that $x^\bullet \to x$ is an f-lift of $(y^{l_k})_{k=1}^\infty$. Then x^0 is a point of openness of $f : X \to Y$.

Proof. Suppose that $U \in \operatorname{Open}(X)$ is such that $f(U)$ is not a neighborhood of y^0 Then there exists some sequence $(y^l)_{l=1}^\infty \subseteq Y \setminus f(U)$ such that $y^\bullet \to f(x)$ is injective in Y. By assumption, exists some increasing $(l_k)_{k=1}^\infty$ and some sequence $(x^k)_{k=1}^\infty \subseteq X$ such that $x^\bullet \to x$ is an f-lift of $(y^{l_k})_{k=1}^\infty$. Since $x \in U \in \operatorname{Open}(X)$, there exists some $k \in \mathbb{N}$ such that $x^k \in U$, which gives us the contradiction $y^{l_k} = f\left(x^k\right) \in f(U)$. ■

Proposition A.2.3. Suppose that $F : M \to N$ is a continuous surjection onto a Hausdorff space. If M is regular and $F\big|_U : U \to F(U)$ is a sequentially quotient for all $U \in \operatorname{Open}(M)$ then $F : M \to N$ maps open sets to sequentially open subsets of N.

Proof. Suppose first that $F\big|_U : U \to F(U)$ is a sequentially quotient for all $U \in \operatorname{Open}(M)$ but that $O \in \operatorname{Open}(M)$ is such that $F(O)$ is not sequentially open in N. Then there exists $n \in F(O)$ and an injective sequence $(n_l)_{l=1}^{\infty} \subseteq N \setminus F(O)$ that converges to n in N. Since $F : M \to N$ is sequentially quotient there exists some subsequence $(n_{l_k})_{k=1}^{\infty}$ and some F-lift $(m_{l_k})_{k=1}^{\infty} \to m$ of $n_{l_\bullet} \to n$, where observe that this necessitates that $m_{l_k} \notin O$ for all l and $m \notin O$. By replacing $n_\bullet$ with $n_{l_\bullet}$ we may assume without loss of generality that $l_k = k$ for all $k \in \mathbb{N}$.

By M's regularity, we may pick $U_0 \in \operatorname{Open}(M)$ such that $\operatorname{Cl}_M(U_0) \subseteq O$ and $n \in F(U_0)$. For all $l \in \mathbb{N}$ pick $n_l \in V_l \in \operatorname{Open}(N)$ such that $n \notin V_l$ and then pick $m_l \in U_l \in \operatorname{Open}(F^{-1}(V_l))$ such that $U_l \cap \operatorname{Cl}_M(U_0) = \varnothing$. Let $U = \bigcup\limits_{l=0}^{\infty} U_l$ and observe that n and all $n_l = F(m_l)$ all belong to $F(U)$. Since $F\big|_U : U \to F(U)$ is sequentially quotient there exists some subsequence $(n_{l_k})_{k=1}^{\infty}$ and some $F\big|_U$-lift $(\hat{m}_{l_k})_{k=1}^{\infty} \to \hat{m}$ of $n_{l_\bullet} \to n$. Since $F(U_{l_k}) \subseteq V_{l_k}$ and $n = F(\hat{m}) \notin V_{l_k}$ for all $k \in \mathbb{N}$ we have that $\hat{m} \notin U_{l_k}$ for all $l \in \mathbb{N}$, where this together with the fact that $\hat{m} \in U = \bigcup\limits_{l=0}^{\infty} U_l$ implies that $\hat{m} \in U_0$. But since $\hat{m}_{l_\bullet} \to \hat{m}$ there exist some $k \in \mathbb{N}$ such that $\hat{m}_{l_k} \in U_0$, which implies that $n_{l_k} = F(\hat{m}_{l_k}) \in F(U_0) \subseteq F(O)$, a contradiction. ∎

The next proposition follows almost immediately from proposition A.2.3 and remark A.0.8.

Proposition A.2.4. A continuous surjection $F : M \to N$ from a regular sequential space onto a Hausdorff sequential space is an open map if and only if $F\big|_U : U \to F(U)$ is sequentially quotient for all $U \in \operatorname{Open}(M)$.

Theorem A.2.5. Let M be a sequential Hausdorff space, N a first countable Hausdorff space, $F : M \to N$ a sequence covering, and $\widehat{n} \in N$. If $F^{-1}(\widehat{n})$ is countable then there exists some $m \in F^{-1}(\widehat{n})$ such that m is a point of openness for F (i.e. for all $U \in \operatorname{Open}(X)$ containing m, $f(U)$ is a neighborhood of $\widehat{n}$ in N).

Proof. Enumerate $F^{-1}(\widehat{n}) = (\widehat{m}_l)_{l=1}^{\infty}$ and let $(V^l)_{l=1}^{\infty}$ be a neighborhood basis of $\widehat{n}$ consisting of open subsets of N such that $V^{l+1} \subseteq V^l$. Suppose that the conclusions is false. Then for all $l \in \mathbb{N}$ there exists an open neighborhood U_l of $\widehat{m}_l$ such that $F(U_l)$ is not a neighborhood of $\widehat{n}$ in N. For each $l \in \mathbb{N}$, since $F(U_l)$ is not a neighborhood of $\widehat{n}$ and since N is sequential we can pick a sequence $\left(n_l^p\right)_{p=1}^{\infty} \subseteq V_l$ converging to $\widehat{n}$ such that $n_l^p \notin F(U_l)$ for all $p \in \mathbb{N}$ where by replacing this sequence with a subsequence we may assume that $n_l^p \in V_p$ for all $p \in \mathbb{N}$.

Observe that for all $l, p, q \in \mathbb{N}$, if $\max\{p, l\} \geq q$ then $n_l^p \in V_q$: If $l \geq q$ then $V_l \subseteq V_q$ so this follows from $n_l^p \in V_l$ while if $p \geq q$ then $V_p \subseteq V_q$ so this follows from $n_l^p \in V_p$. In particular, this shows that for all $q \in \mathbb{N}$ there are at most finitely many $(l, p) \in \mathbb{N} \times \mathbb{N}$ such that n_l^p is not contained in V_q. Thus by picking any bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$ we can view $\left(n_l^p\right)_{(l,p) \in \mathbb{N} \times \mathbb{N}}$ as a convergent sequence in N, whose limit is $\widehat{n}$.

Since F is sequence covering there exists a convergent sequence $\left(m_l^p\right)_{(l,p) \in \mathbb{N} \times \mathbb{N}}$ in M such that $F(m_l^p) = n_l^p$ for each $(l, p) \in \mathbb{N} \times \mathbb{N}$. Since $F\left(\lim\limits_{(l,p) \in \mathbb{N} \times \mathbb{N}} m_l^p\right) = \widehat{n}$ there is some $L \in \mathbb{N}$ such that $\left(n_l^p\right)_{(l,p) \in \mathbb{N} \times \mathbb{N}}$ converges to $\widehat{m}_L$. Since U_L is a neighborhood of $\widehat{m}_L$ we can pick $P \in \mathbb{N}$ such that $p \geq P$ implies $m_L^p \in U_L$. But then $n_L^P = F(m_L^P) \in F(U_L)$, which contradicts the choice of $(n_L^p)_{p=1}^{\infty}$. ∎

Corollary A.2.6. Let $F : M \to N$ be a quotient map between first-countable Hausdorff spaces. If each fiber of F is countable then F is an almost open map and (consequently) a 1-sequence covering.

Proposition A.2.7. If $F : M \to N$ is an almost open map then $F : M \to N$ is open $\iff$ whenever $m, \widehat{m} \in M$ belong to the same fiber of F then for any $m \in U \in \operatorname{Open}(M)$ there exists some $\widehat{m} \in \widehat{U} \in \operatorname{Open}(M)$ such that $F\left(\widehat{U}\right) \subseteq F(U)$.

Remark A.2.8. Observe that while the LHS of the equivalence explicitly depends on knowledge of both M and N's topologies, the RHS does ***not*** require *any* knowledge of N's topology (only of M's topology). The condition on the RHS of the equivalence makes rigorous the idea that around any two points of the same fiber of F, the result of F's "infinitesimal deformation of regions around one of these points" (i.e. images of neighborhoods of this point) does not differ much from its "infinitesimal deformation of regions around the other point."

Proof. Let $U \in \operatorname{Open}(M)$, $m^0 \in U$, and $n^0 = F(m^0)$. Let $m^1 \in F^{-1}(n^0)$ be a point of openness of F so our assumption gives us some $m^1 \in U^1 \in \operatorname{Open}(M)$ such that $F(U^1) \subseteq F(U)$. But then $F(U^1)$ is a neighborhood in N of n^0 that is contained in $F(U)$, as desired. ∎

Corollary A.2.9. A map $F : M \to N$ is open if and only if (1) it is almost open and (2) whenever $m, \widehat{m} \in M$ belong to the same fiber of F then for any $m \in U \in \operatorname{Open}(M)$ there exists some $\widehat{m} \in \widehat{U} \in \operatorname{Open}(M)$ such that $F\left(\widehat{U}\right) \subseteq F(U)$.

In light of corollary A.2.6, we obtain the following corollary to proposition A.2.7.

Corollary A.2.10. Let $F : M \to N$ be a quotient map between first-countable Hausdorff spaces such that each fiber of F is countable. Then $F : M \to N$ is open $\iff$ whenever $m, \widehat{m} \in M$ belong to the same fiber of F then for any $m \in U \in \operatorname{Open}(M)$ there exists some $\widehat{m} \in \widehat{U} \in \operatorname{Open}(M)$ such that $F\left(\widehat{U}\right) \subseteq F(U)$.

Local Homeomorphism Conditions

Lemma A.3.1. Let $F : M \to N$ be a continuous map between Hausdorff sequential spaces such that $F : M \to \operatorname{Im} F$ is open. Let $R = \{m \in M \mid F^{-1}(F(m)) = \{m\}\}$ and let $S = F(R)$. Then S is closed in $\operatorname{Im} F$, R is closed in M, $R = F^{-1}(S)$, and $F\big|_R : R \to N$ is injective.

Proof. It is clear that if $m \in F^{-1}(S)$ then there exists some $m^0 \in R$ such that $F(m^0) = F(m)$ but since $F^{-1}(F(m^0)) = \{m^0\}$ it follows that $m^0 = m$. Thus $m \in R$ so we've shown that $R = F^{-1}(S)$ and from here it is clear that $F\big|_R : R \to N$ is injective. Now let $n^0 \in \operatorname{Cl}_{\operatorname{Im} F}(S)$ and let $(n^l)_{l=1}^{\infty} \subseteq S$ be a sequence converging to n^0. Since $F\big|_{F^{-1}(S)} : F^{-1}(S) \to N$ is injective we can define $m^l \underset{\text{def}}{=} F^{-1}(n^l)$ for each $l \in \mathbb{N}$. Suppose that $m, \widehat{m} \in F^{-1}(n^0)$ are distinct points. Pick $m \in U \in \operatorname{Open}(M)$ and $\widehat{m} \in \widehat{U} \in \operatorname{Open}(M)$ such that U and $\widehat{U}$ are disjoint. Since $n^0 = F(m) \in F(U) \in \operatorname{Open}(\operatorname{Im} F)$ and $n^0 = F(\widehat{m}) \in F(\widehat{U}) \in \operatorname{Open}(\operatorname{Im} F)$ there exists some $l_0 \in \mathbb{N}$ such that $n^l \in F(U) \cap F(V)$ for all $l \geq l_0$. In particular, there exists $p \in U$ and $\widehat{p} \in \widehat{U}$ such that $F(p) = n^{l_0} = F(\widehat{p})$ so that $\{p, \widehat{p}\} \subseteq F^{-1}(n^{l_0}) = \{m^{l_0}\}$, which implies $p = m^{l_0} = \widehat{p}$. But this contracts the fact that $U \cap \widehat{U} = \varnothing$. ∎

Theorem A.3.2. Let $F : M \to N$ be a continuous map between Hausdorff second countable spaces where M is a Baire space and $\operatorname{Im} F$ is normal. Let O be the (unique) largest open subset of M on which F restricts to a locally injective map (i.e. for every $m \in O$ there exists some $m \in U \in \operatorname{Open}(M)$ such that $F\big|_U : U \to N$ is injective). Suppose that $F : M \to \operatorname{Im} F$ is an open map and that each fiber of F is a discrete subspace of M. Then O is dense in M and furthermore, this O is also the largest open subset of M on which $F\big|_O : O \to \operatorname{Im} F$ is a local homeomorphism.

Proof. Since $F : M \to \operatorname{Im} F$ is a continuous open surjection from a Baire space, $\operatorname{Im} F$ is also a Baire space. Let $B^\bullet = (B^l)_{l=1}^\infty$ be a countable basis for M. Since each fiber of F is discrete, for every $m \in M$ we may pick $\lambda(m) \in \mathbb{N}$ such that $\{m\} = F^{-1}(F(m)) \cap B^{\lambda(m)}$. For all $l \in \mathbb{N}$, let $R^l \underset{\text{def}}{=} \{m \in B^l \,|\, F^{-1}(F(m)) = \{m\}\}$ and let $S^l \underset{\text{def}}{=} F(R^l)$ and observe that lemma A.3.1 implies that R^l is closed in B^l and S^l is closed in $F(B^l)$. By definition of $\lambda(m)$ we have $m \in R^{\lambda(m)}$ so in particular, $M = \bigcup\limits_{l \in \mathbb{N}} R^l$ and $\operatorname{Im} F = \bigcup\limits_{l \in \mathbb{N}} S^l$.

Since $\operatorname{Im} F$ is normal and since for each $m \in M$, $F(B^{\lambda(m)})$ is an open (in $\operatorname{Im} F$) neighborhood of $F(m)$ such that $F(R^{\lambda(m)})$ is closed in $F(B^{\lambda(m)})$ we can pick some $\widehat{\lambda}(m) \in \mathbb{N}$ such that $m \in B^{\widehat{\lambda}(m)}$ and $\operatorname{Cl}_{\operatorname{Im} F}\left(F\left(B^{\widehat{\lambda}(m)}\right)\right) \subseteq F(B^{\lambda(m)})$. For each $m \in M$ let $E^m = S^{\lambda(m)} \cap \operatorname{Cl}_{\operatorname{Im} F}\left(F\left(B^{\widehat{\lambda}(m)}\right)\right)$ and note that since $S^{\lambda(m)}$ is closed in the open (in $\operatorname{Im} F$) set $F(B^{\lambda(m)})$ while $\operatorname{Cl}_{\operatorname{Im} F}\left(F\left(B^{\widehat{\lambda}(m)}\right)\right)$ is closed in $\operatorname{Im} F$, the set E^m is closed in $\operatorname{Im} F$. Observe that there are only countably map E^m's since each E^m is of the form $S^i \cap \operatorname{Cl}_{\operatorname{Im} F}(F(B^j))$ for some $i, j \in \mathbb{N}$ and that $E^m \subseteq S^{\lambda(m)} \cap F(B^{\lambda(m)})$ since $\operatorname{Cl}_{\operatorname{Im} F}\left(F\left(B^{\widehat{\lambda}(m)}\right)\right) \subseteq F(B^{\lambda(m)})$.

Since $\{E^m, m \in M\}$ is a countable closed cover of $\operatorname{Im} F$, which is a Baire space, there exists some $m \in M$ such that $V \underset{\text{def}}{=} \operatorname{Int}_{\operatorname{Im} F}(E^m)$ is non-empty. Since $V \subseteq E^m \subseteq S^{\lambda(m)} \cap F(B^{\lambda(m)}) \subseteq F(B^{\lambda(m)})$ and $V \neq \varnothing$ it follows that $U \underset{\text{def}}{=} U^m \cap F^{-1}(V)$ is non-empty (note that whether or not $m \in U$ will be unimportant). Since $U \subseteq U^{\lambda(m)}$ and

$$F(U) \subseteq V \subseteq S^{\lambda(m)} = F\left(R^{\lambda(m)}\right) = \left\{n \in \operatorname{Im} F \,\middle|\, F^{-1}(n) \cap B^{\lambda(m)} \text{ is singleton }\right\}$$

it follows that for all $u \in U$, $F^{-1}(F(u)) \cap B^{\lambda(m)} = \{u\}$. Thus $F\big|_U : U \to N$ is injective and since $F\big|_U : U \to F(U)$ is a continuous open bijection it is a homeomorphism onto $F(U)$, where $F(U) \in \operatorname{Open}(\operatorname{Im} F)$.

Let $\mathcal{O}$ denote all those $U \in \operatorname{Open}(M)$ such that $F\big|_U : U \to N$ is injective and let $O \underset{\text{def}}{=} \bigcup_{O \in \mathcal{O}} O$. Suppose that $U \underset{\text{def}}{=} M \setminus \operatorname{Cl}_M(O)$ is not empty and let $G \underset{\text{def}}{=} F\big|_U : U \to N$. Observe that $G : U \to G(U)$ is an open map since $U \in \operatorname{Open}(M)$ and $F : M \to \operatorname{Im} F$ is open. Since the fibers of G are discrete we may apply the first part of this theorem to conclude that there exists some non-empty $V \in \operatorname{Open}(U)$ such that $G\big|_V : V \to \operatorname{Im} F$ is injective. But since $V \in \operatorname{Open}(U)$ it follows that $V \in \mathcal{O}$, which gives us a contradiction. Thus $\operatorname{Cl}_M(O) = M$. ■

Properties of Continuous Open Maps

The conclusions of the following lemma A.4.1 are straightforward to prove and while the results that relate preimages with closures, boundaries, and interiors are well-know, the results relating closures of images with regular closed/open sets appear to be new observations.

Lemma A.4.1. Suppose $f : X \to Y$ is a continuous open map, $S \subseteq Y$, and $A \subseteq X$. Then $f^{-1}(\operatorname{Bd}(S)) = \operatorname{Bd}(f^{-1}(S))$, $f^{-1}\left(\overline{S}\right) = \overline{f^{-1}(S)}$, and whenever $\overline{A} = \overline{\operatorname{Int}(A)}$ then $\overline{\operatorname{Int}(f(A))} = \overline{f(A)} = \overline{f(\operatorname{Int}(A))} = \overline{f\left(\overline{\operatorname{Int}(A)}\right)}$ is a regular closed set. In particular, if A is a regular closed (resp. open) set then so is $\overline{f(A)}$ (resp. $\overline{f(X \setminus A)}$). If in addition f is surjective then $\operatorname{Int}(f^{-1}(S)) = f^{-1}(\operatorname{Int}(S))$ and S is a regular open (resp. regular closed) subset of Y if and only if $f^{-1}(S)$ is a regular open (resp. regular closed) subset of X.

Proof. That $f^{-1}(\overline{S}) = \overline{f^{-1}(S)}$ is well-known and $f^{-1}(\mathrm{Bd}(S)) = \mathrm{Bd}(f^{-1}(S))$ follows immediately from this. Recall that from the continuity of f we have $\overline{f(\overline{A})} = \overline{f(A)}$ so that $\overline{A} = \overline{\mathrm{Int}(A)}$ implies $\overline{f(A)} = \overline{f(\overline{A})} = \overline{f\left(\overline{\mathrm{Int}(A)}\right)} = \overline{f(\mathrm{Int}(A))} \subseteq \overline{\mathrm{Int}(f(A))} \subseteq \overline{f(A)}$ (since $\mathrm{Int}(f(A)) \subseteq f(A)$) from where equality follows. Since the closure of any open set is a regular closed set so that in particular $\overline{f(A)} = \overline{\mathrm{Int}(f(A))}$ is a regular closed set.Suppose that f is surjective. Then $f(\mathrm{Int}(f^{-1}(S))) \subseteq \mathrm{Int}(f(f^{-1}(S))) = \mathrm{Int}(S)$ so that $\mathrm{Int}(f^{-1}(S)) \subseteq f^{-1}(\mathrm{Int}(S))$. By continuity, $f^{-1}(\mathrm{Int}(S)) \subseteq \mathrm{Int}(f^{-1}(S))$ and so equality holds. It is not hard to show that S is a regular open (resp. closed) set if and only if $\mathrm{Bd}(S) = \mathrm{Bd}(\overline{S})$ (resp. $\mathrm{Bd}(\mathrm{Int}(S)) = \mathrm{Bd}(S)$). That S is regular open (resp. closed) if and only if the same is true of $f^{-1}(S)$ is now immediate. ∎

Lemma A.4.2. Suppose that $\mu : M \to S$ is a surjective open continuous map where M is first countable and S is Hausdorff. Let $D \subseteq S$, $s \in \overline{D}$, $m \in \mu^{-1}(s)$, and $(m^k)_{k=1}^{\infty}$ be a sequence of points in $\mu^{-1}(D)$ converging to m (which necessarily exists). Then for any $n \in \mu^{-1}(s)$ there exists a sequence $(n^l)_{l=1}^{\infty}$ in $\mu^{-1}(D)$ converging to n and there exists a subsequence $(m^{k_l})_{l=1}^{\infty}$ such that $\mu(n^l) = \mu(m^{k_l})$ for all $l \in \mathbb{N}$.

Proof. Since $m \in \mu^{-1}(s) \subseteq \mu^{-1}(\overline{D}) = \overline{\mu^{-1}(D)}$ and M is first countable the sequence $(m^k)_{n \in \mathbb{N}}$ in $\mu^{-1}(D)$ with m^k converging to m is guaranteed to exist. If $s \in D$ the result is obvious so assume that $s \notin D$ and so for all $k \in \mathbb{N}$, $\mu(m^k) \neq s$. So replacing $(m^k)_{k=1}^{\infty}$ with a subsequence we may assume (since S is Hausdorff) that all m^k are distinct. Let $C = \{\mu(m^k) \mid k \in \mathbb{N}\} \subseteq \mu(\mu^{-1}(D)) = D$ and note that $\mu(n) = m \in \overline{C}$ so that $n \in \mu^{-1}(\overline{C}) = \overline{\mu^{-1}(C)}$ so there exists a sequence $(n^i)_{i=1}^{\infty}$ in $\mu^{-1}(C)$ with n^i converging to n so that $\mu(n^i) \in C$ converges to $\mu(n) = m$. Since $\lim_{i\to\infty} \mu(n^i) = \mu\left(\lim_{i\to\infty} n^i\right) = \mu(n) = s$ converges and since $(\mu(n^i))_{i=1}^{\infty}$ is a subsequence of $(m^k)_{k=1}^{\infty}$ we may replace $(m^k)_{k=1}^{\infty}$ with this subsequence which will guarantee that for all $k \in \mathbb{N}$, there exists some i such that $\mu(n^i) = \mu(m^k)$ where in addition this i is necessarily $i \geq k$. Let $k_1 = 1$ and pick i_1 be such that $\mu(n^{i_1}) = \mu(m^{k_1})$, where we have that $i_1 \geq k_1$. Having picked k_{l-1} and i_{l-1} let $k_l = 1 + \max\{k_{l-1}, i_{l-1}\}$ and pick i_l to be such that $\mu(n^{i_l}) = \mu(m^{k_l})$, where we necessarily have that $i_l \geq k_l$ so that $i_l > i_{l-1}$. Thus $(n^{i_l})_{l=1}^{\infty}$ and $(m^{k_l})_{l=1}^{\infty}$ are the desired sequences. ■

Lemma A.4.3. Suppose that $F : M \to N$ and $\mu : M \to S$ is are maps where μ is surjective. Then there exists a unique largest subset $D \subseteq S$ on which it is possible to define a map $f : D \to N$ such that $F = f \circ \mu$ on $\mu^{-1}(D)$, where this map $f : D \to N$ will then necessarily be unique. If $\mu : M \to S$ is also a quotient map then for any $R \subseteq D$, $F|_{\mu^{-1}(R)} : \mu^{-1}(R) \to N$ is continuous if and only if $f|_R$ is continuous. If in addition $\mu : M \to S$ is continuous and open, $F : M \to N$ is continuous, M is first countable, and S and N are Hausdorff then

(1) D is closed in S,

(2) $f : D \to N$ continuous,

(3) $U \underset{\text{def}}{=} \operatorname{Int}(D)$ is the unique maximal open subset of S and $f|_U : U \to N$ is the unique continuous map such that $F|_{\mu^{-1}(U)} = f \circ \mu|_{\mu^{-1}(U)}$ in the sense that if $\tilde{U} \in \operatorname{Open}(M)$ is any set for which such a representation exists then $\tilde{U} \subseteq U$,

(4) U is necessarily a regular open set.

Proof. The uniqueness, existence, and maximality of the (possibly empty) set D is clear. Assume that μ is a quotient map. Since $\mu : M \to S$ is a quotient map so is $\mu|_{\mu^{-1}(R)} : \mu^{-1}(R) \to R$ where $R \subseteq D$. Since $F|_{\mu^{-1}(R)} = f|_R \circ \mu|_{\mu^{-1}(R)}$ is continuous it follows from the universal property of quotient maps that $F|_{\mu^{-1}(R)}$ is continuous if and only if $f|_R$ is continuous.

Now assume that $\mu : M \to S$ is continuous and open, $F : M \to N$ is continuous, M is first countable, and S and N are Hausdorff. Let $C = \mu^{-1}(D)$. Since $\mu : M \to S$ is a quotient map the continuity of $F = f \circ \mu$ on C implies that f is continuous. Let $s \in \overline{D}$ and let $m, n \in \mu^{-1}(S)$. Note that if we can show that $F(m) = F(n)$ then since m and n were arbitrary we'll have shown that $F(\mu^{-1}(s))$ is a singleton set so that we can define $\hat{f} : D \cup \{s\} \to N$ by $\hat{f}(s) = F(\mu_i^{-1}(s))$ and $\hat{f}(d) = f(d)$ for $d \in D$ so that from the maximality of D it will follow that $s \in D$. By the previous lemma, there exist sequences $(m^l)_{l=1}^{\infty}$ and $(n^l)_{l=1}^{\infty}$ in $\mu^{-1}(D) = C$ converging in M to m and n, respectively, such that $\mu(m^l) = \mu(n^l)$ for all $l \in \mathbb{N}$. Since F is continuous the following limit

$$F(m) = F(\lim_l m^l) = \lim_l F(m^k) = \lim_l f(\mu(m^l))$$

exists and likewise

$$F(n) = \lim_l f(\mu_i(n^l)) = \lim_l f(\mu_i(m^l))$$

exists so that $F(m) = F(n)$, as desired.

Let U consist of all those $s \in S$ for which there exist $s \in U_s \in \operatorname{Open}(S)$ and continuous $f_s : U_s \to N$ such that $F|_{\mu^{-1}(U_s)} = f_s \circ \mu|_{\mu^{-1}(U_s)}$. Note that if $s \in U$ then $U_s \subseteq U$ so that $U = \bigcup_{s \in U} U_s$ is open in S. By definition of D we have $U \subseteq D$ so that $U \subseteq \operatorname{Int}(D)$. By lemma 6.1.4 $F|_{\mu^{-1}(\operatorname{Int}(D))} = f \circ \mu|_{\mu^{-1}(\operatorname{Int}(D))}$ and since $\operatorname{Int}(D)$ is open it follows from lemma 6.1.12 and the definition of U that $\operatorname{Int}(D) \subseteq U$ so that $\operatorname{Int}(D) = U$. The maximality and uniqueness of U is clear from its definition. Since D_i is closed in M_i and U_i is the interior of this closed set we have that U_i is a regular open set. ∎

Coherence of Topologies with Collections of Subsets

Definition A.5.1. Suppose X is a set and $\mathcal{F}$ is a collection of X-valued maps where for each $f \in \mathcal{F}$, f's domain, denoted by $\operatorname{Dom} f$, has a topology τ_f. Then *the final topology $\tau_{\mathcal{F}}$ on X induced by $\mathcal{F}$* is the finest topology on X making all $f : (\operatorname{Dom} f, \tau_f) \to (X, \tau_{\mathcal{F}})$ continuous. This topology's open (resp. closed) sets are characterized by the following property: a subset S of X is open (resp. closed) in $(X, \tau_{\mathcal{F}})$ if and only if for all $f \in \mathcal{F}$, $f^{-1}(S)$ is open (resp. closed) in $(\operatorname{Dom} f, \tau_f)$. If τ_X is a topology on X then let us say that τ_X (or (X, τ_X) or simply X if τ_X is understood) *is coherent with $\mathcal{F}$* if $\tau_X = \tau_{\mathcal{F}}$, where in place of τ_X we may write (X, τ_X) or even simply X if τ_X is understood. If $x \in X$ then say that τ_X *is coherent with $\mathcal{F}$ at x* if the following condition holds: for all $S \subseteq X$ containing x, S is a neighborhood of x in (X, τ_X) $\iff$ for all $f \in \mathcal{F}$ whose image contains x, $f^{-1}(S)$ is a neighborhood of $f^{-1}(x)$.

Example and Definition A.5.2. Let (X, τ_X) be a space and let $\operatorname{Cmpt}(X, \tau_X)$ denote the set of all compact subspaces of (X, τ_X) with their subspace topologies. We call (X, τ_X) a *k-space* and say that it is *compactly generated* if τ_X is coherent with the set of all inclusion maps $\operatorname{In}_K^X : K \to X$ as K varies over $\operatorname{Cmpt}(X, \tau_X)$.

Remark A.5.3. Every space that is first-countable or locally compact is a k-space but it is well-known that there are k-spaces whose product is not a k-space.

Observations A.5.4. Let (X, τ_X) and (Y, τ_Y) be spaces, let $\mathcal{F}$ and $\mathcal{G}$ be two collections of continuous maps into (X, τ_X), and suppose that τ_X is coherent with $\mathcal{F}$.

- If $\mathcal{F} \subseteq \mathcal{G}$ then τ_X is coherent with $\mathcal{G}$.
- A surjection $\pi : X \to Y$ is a quotient map if and only if τ_Y is coherent with $\pi \circ \mathcal{F} := \{\pi \circ f : f \in \mathcal{F}\}$.
- For every $f \in \mathcal{F}$ let $\mathcal{E}_f$ be a collection of continuous maps into $(\operatorname{Dom} f, \tau_{\operatorname{Dom} f})$. If $\tau_{\operatorname{Dom} f}$ is coherent with $\mathcal{E}_f$ for each $f \in \mathcal{F}$ then τ_X is coherent with

$$\bigcup_{f \in \mathcal{F}} \{f \circ e : e \in \mathcal{E}_f\}$$

The following lemma is readily verified.

Lemma A.5.5. If (X, τ_X) is a space and each $f \in \mathcal{F}$ is a quotient map onto its image (i.e. $f : (\operatorname{Dom} f, \tau_f) \to \left(\operatorname{Im} f, \tau_X\big|_{\operatorname{Im} f}\right)$ is a quotient map) then the following are equivalent:

(1) τ_X is coherent with $\mathcal{F}$.

(2) A subset $S \subseteq X$ is open in $(X, \tau_X) \iff$ for all $f \in \mathcal{F}$, $S \cap \operatorname{Im} f$ is open in $\left(\operatorname{Im} f, \tau_X\big|_{\operatorname{Im} f}\right)$.

(3) A subset $S \subseteq X$ is closed in $(X, \tau_X) \iff$ for all $f \in \mathcal{F}$, $S \cap \operatorname{Im} f$ is closed in $\left(\operatorname{Im} f, \tau_X\big|_{\operatorname{Im} f}\right)$.

Observe in particular that if τ_X is Hausdorff and every $f \in \mathcal{F}$ is a continuous map with a compact domain then the hypotheses of lemma A.5.5 are satisfied since any such map is necessarily a closed map onto its image.

Definition A.5.6 ([49, pp. 68-69])**.** If $\mathcal{A}$ is a collection of subsets of X then say that a topology τ_X on X is *coherent with* $\mathcal{A}$ and call this topology *the weak topology induced by* $\mathcal{A}$ if when each $A \in \mathcal{A}$ is given the subspace topology from (X, τ_X), then τ_X is coherent with set of inclusion maps $\left\{\operatorname{In}_A^X : A \in \mathcal{A}\right\}$.

Remark A.5.7. In this case, we may apply lemma A.5.5 to conclude that a subset S of X is open (resp. closed) in (X, τ_X) if and only if for all $A \in \mathcal{A}$, $S \cap A$ is open (resp. closed) in $\left(A, \tau_X\big|_A\right)$, which shows that these definitions of coherence, weak topology, and k-space are equivalent to their usual definitions (e.g. as found in [12] or [49]). Observe that if τ_X and $\mathcal{F}$ are as in lemma A.5.5 then this terminology allows us to restate that lemma's conclusion as: τ_X is coherent with (the set of all maps in) $\mathcal{F}$ if and only if τ_X is coherent with the set of all images of maps in $\mathcal{F}$.

Proposition A.5.8. Let $\mathcal{F}$ be a collection of continuous maps into (X, τ_X) whose domains are Fréchet-Urysohn spaces. If τ_X is coherent with $\mathcal{F}$ then X is a sequential space.

Proof. Let $S \subseteq X$ be sequentially closed, let $f : D \to X$ be in $\mathcal{F}$, and let $d \in \operatorname{Cl}_D(f^{-1}(S))$. Let $(d_l)_{l=1}^{\infty} \subseteq f^{-1}(S)$ converge to d. Since f is continuous, $f(d_\bullet) \to f(d)$ in X so that $f(d) \in \operatorname{SeqCl}_X(S) = S$, which shows that $\operatorname{Cl}_D(f^{-1}(S)) = f^{-1}(S)$ is closed in D. ■

The following original lemma will be an important tool for this paper.

Lemma A.5.9. Let (X, τ_X) be Hausdorff, let $\mathcal{C}$ be a collection of continuous maps in X, and let $(\star)$ denote the following statement:

($\star$): whenever $x^\bullet = (x^l)_{l=1}^{\infty} \subseteq X$ is an infinite-ranged sequence converging to x in X then there exists some $\gamma \in \mathcal{C}$ and some γ-liftable subsequence $(x^{l_k})_{k=1}^{\infty}$ of $x^\bullet$ such that $(x^{l_k})_{k=1}^{\infty} \to x$ is injective in X.

If X is Fréchet-Urysohn and ($\star$) holds then τ_X is coherent with $\mathcal{C}$. If τ_X is coherent with $\mathcal{C}$ and if every map in $\mathcal{C}$ is that is not sequentially quotient onto its image has a Fréchet-Urysohn domain, then ($\star$) holds.

Remark A.5.10. It will be clear from the proof that we may replace ($\star$) with: "whenever $x^\bullet = (x^l)_{l=1}^\infty \to x$ is injective in X then there exists some $\gamma \in \mathcal{C}$ and some γ-liftable subsequence of $x^\bullet$."

Convention A.5.11. If $\gamma \in \mathcal{C}$ is a curve, then we will pick the γ-liftable subsequence $(x^{l_k})_{k=1}^\infty$ in ($\star$) so that it has a monotone convergent γ-lift.

Proof. Assume first that τ_X is coherent with $\mathcal{C}$ and that whenever a map in $\mathcal{C}$ is not sequentially quotient onto its image then it has a Fréchet-Urysohn domain. Let x and $x^\bullet = (x^l)_{l=1}^\infty \subseteq X$ be as in ($\star$) and observe that it suffices to prove ($\star$)'s conclusion under the additional assumption that $x^\bullet \to x$ is injective in X. Let $S = \{x^l : l \in \mathbb{N}\}$ and note that $x \notin S$. We will assume that such a γ and subsequence does not exist and obtain a contradiction by concluding that S is closed in X. Let $\gamma \in \mathcal{C}$ and note that since τ_X is coherent with $\mathcal{C}$ it's enough to show that $\gamma^{-1}(S)$ is closed. If $S \cap \operatorname{Im}\gamma$ is finite then $S \cap \operatorname{Im}\gamma$ is compact so $\gamma^{-1}(S)$ is closed. So assume that $S \cap \operatorname{Im}\gamma$ is infinite and consists of the subsequence $(x^{n_k})_{k=1}^\infty$.

Suppose first that $\gamma : \operatorname{Dom}\gamma \to \operatorname{Im}\gamma$ is sequentially quotient. If $x \in \operatorname{Im}\gamma$ then since $(x^{n_k})_{k=1}^\infty \to x$ in $\operatorname{Im}\gamma$ and $\gamma : \operatorname{Dom}\gamma \to \operatorname{Im}\gamma$ is sequentially quotient, we can pick a subsequences of $(x^{n_k})_{k=1}^\infty$ that we had been assumed not to exist. Thus $x \notin \operatorname{Im}\gamma$ so that since $S \cup \{x\}$ is closed in X, $\gamma^{-1}(S) = \gamma^{-1}(S \cup \{x\})$ is closed, as desired. So we may henceforth assume that γ's domain is Fréchet-Urysohn.

Suppose that $\gamma^{-1}(S)$ is not closed and let $t_0 \in \overline{\gamma^{-1}(S)} \setminus \gamma^{-1}(S)$. Since $\operatorname{Dom}\gamma$ is Fréchet-Urysohn and $t_0 \in \overline{\gamma^{-1}(S)}$, there exists some sequence $(t_j)_{j=1}^\infty$ in $\gamma^{-1}(S)$ converging to t_0. Let $j_1 = 1$ and let l_1 be the unique integer such that $\gamma(t_{j_1}) = x^{l_1}$. Having picked $j_1 < \cdots < j_k$ and $l_1 < \cdots < l_k$ such that $\gamma(t_{j_1}) = x^{l_1}, \ldots, \gamma(t_{j_k}) = x^{l_k}$, let $j_{k+1} > j_k$ be such that for all $j \geq j_{k+1}$, t_j belongs to the open neighborhood $\gamma^{-1}(X \setminus \{x^l : 1 \leq l \leq l_k\})$ of t_0 and then let

l_{k+1} be the unique index such that $\gamma(t_{j_{k+1}}) = x^{l_{k+1}}$. Since $(t_{j_k})_{k=1}^{\infty} \to t_0$ and X is Hausdorff, $\gamma(t_0) = \lim_{k\to\infty} \gamma(t_{j_k}) = \lim_{k\to\infty} x^{l_k} = x$, a contradiction that finishes the proof.

Now assume that X is Fréchet-Urysohn and that $(\star)$ holds. Let $S \subseteq X$ be such that $\gamma^{-1}(S)$ is closed for all $\gamma \in \mathcal{C}$. Let $x \in \mathrm{Cl}_X(S)$ and suppose for the sake of contradiction that $x \notin S$. Since X is Fréchet-Urysohn we may pick a sequence $(x^l)_{l=1}^{\infty} \subseteq S$ converging to x where since $x \notin S$, this sequence has infinite range. By assumption, there is some $\gamma \in \mathcal{C}$ and some γ-liftable subsequence $(x^{l_k})_{k=1}^{\infty}$ of $(x^l)_{l=1}^{\infty}$. Let $(t_k)_{k=1}^{\infty} \to t_0$ be a γ-lift of $(x^{l_k})_{k=1}^{\infty} \to x$. Since $\gamma(t_k) = x^{l_k} \in S$ for all $k \in \mathbb{N}$, this implies that $t_0 \in \overline{\gamma^{-1}(S)}$, where this set is just $\gamma^{-1}(S)$ so that $\gamma(t_0) \in S$. Since X is Hausdorff it follows that $x = \gamma(t_0) \in S$. ■

Corollary A.5.12. Suppose (X, τ_X) is a Hausdorff Fréchet-Urysohn space and $\mathcal{C}$ is a collection of continuous maps in X where each map either has a Fréchet-Urysohn domain or is otherwise sequentially quotient onto its image. Then τ_X is coherent with $\mathcal{C}$ if and only if $(\star)$ from lemma A.5.9 holds.

Corollary A.5.13. If $X = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x^i \geq 0, \ldots, x^n \geq 0\}$ (where $1 \leq i \leq n$) then X is *not* coherent with any set of C^1-embeddings into X whose domains are all open intervals.

Corollary A.5.14. Let J be an interval and $\mathcal{A}$ be a cover of J by intervals such that for all $x \in J$, if x does not belong to the interior of an interval in $\mathcal{A}$ then there exist intervals L (resp. R) in $\mathcal{A}$ containing x as a left (resp. right) endpoint. Then J is coherent with $\mathcal{A}$. In particular, J is coherent with $\{[a,b] : a, b \in J, a < b\}$.

The following lemma follows easily from corollary A.5.14. Its last conclusion essentially states that if a space is coherent with a set of curves $\mathcal{C}$ then it will also be coherent with the set of curves that results if one replaces each non-path curve in $\mathcal{C}$ with a set of restrictions of this curve to compact intervals that cover its domain.

Lemma A.5.15. Let (X, τ_X) be a space, $\mathcal{C}$ be a set continuous maps in X, and for every

$\gamma \in \mathcal{C}$ let $\mathcal{A}_\gamma$ denote a collection of subsets of $\operatorname{Dom} \gamma$. Let

$$\mathcal{P} = \bigcup_{\gamma \in \mathcal{C}} \left\{ \gamma\big|_A : A \in \mathcal{A}_\gamma \right\}.$$

If τ_X is coherent with $\mathcal{P}$ then τ_X is coherent with $\mathcal{C}$. If τ_X is coherent with $\mathcal{C}$ and the domain of every $\gamma \in \mathcal{C}$ is coherent with $\mathcal{A}_\gamma$ then τ_X is coherent with $\mathcal{P}$.

In particular, if $\mathcal{C}$ is a set of curves and if for all $\gamma \in \mathcal{C}$ we let $\mathcal{A}_\gamma = \{\operatorname{Dom} \gamma\}$ whenever γ is a path and let $\mathcal{A}_\gamma = \{[a, b] : a, b \in \operatorname{Dom} \gamma, a < b\}$ otherwise, then τ_X being coherent with $\mathcal{C}$ implies that it is coherent with $\mathcal{P}$.

Remark A.5.16. Theorem 14.5.5 and corollary A.5.13 show that the last conclusion could fail if we were to restrict paths to open intervals instead of restricting non-path curves to compact intervals.

Corollary A.5.17. Let $k \in \{0, 1, \ldots, \infty\}$ and $0 \leq p \leq k$. If S is a subset of a C^k-manifold with corners that is coherent with the set of all S-valued C^p-curves (resp. C^p-embeddings) whose domains are open intervals then S is coherent with the set of all S-valued C^p-paths (resp. C^p-embeddings) with domain $[0, 1]$.

Corollary A.5.18. If a space (X, τ_X) is coherent with a set $\mathcal{P}$ of continuous maps into X then τ_X is coherent with any set of continuous extensions into X. In particular, if (X, τ_X) is coherent with a set of paths and if each path has a continuous extension to an open interval, then τ_X is coherent with these extensions.

Characterization of Points in a Map's Image

Definition A.6.1. Let X is a topological space, $x \in X$, $S \subseteq X$, and let $S_\bullet = (S_i)_{i \in I}$ be an I-indexed collection of subsets of X. Say that

(a) $S_\bullet$ is *locally finite (resp. index-locally finite) at x (in X)* if there exists some $x \in U \in \operatorname{Open}(X)$ such that $\{S_i \,|\, S_i \cap U \neq \varnothing, i \in I\}$ (resp. $\{i \in I \,|\, S_i \cap U \neq \varnothing\}$) is finite.

(b) $S_\bullet$ is *locally finite (resp. index-locally finite) on S (in X)* if it is locally finite (resp. index-locally finite) in X at every point of S where if $S = X$ then we may instead say that $S_\bullet$ is *locally finite (resp. index-locally finite) (in X)*.

(c) a collection of points $x_\bullet = (x_i)_{i \in I}$ in X is *locally finite (resp. index-locally finite) (on S) (in X)* if this is true of $\{x_i\}_{i \in I}$.

Example A.6.2. If $x \in X$ and we let $x^l \underset{\text{def}}{=} x$ for all $l \in \mathbb{N}$ then although $\{\{x^l\} \mid l \in \mathbb{N}\} = \{\{x\}\}$ is locally finite at x in X, $x^\bullet = (x^l)_{l=1}^\infty$ is not index-locally finite at x in X.

The most important part of the following lemma is the implication (4) $\implies$ (1), which reduces the question of whether or not $y \in \operatorname{Im} f$ down to a question about the fibers of f. This result may be used with continuous maps between promanifolds to show, for instance, that a particular point in the map's codomain lies in the map's image.

Lemma A.6.3. Let $f : X \to Y$ be a continuous map between first-countable Hausdorff spaces and let $y \in Y$. Consider the following statements:

(1) $y \in \operatorname{Im} f$.

(2) There exists a sequence $y^\bullet = (y^l)_{l=1}^\infty \subseteq Y$ converging to y in Y such that $f^{-1}(y^\bullet) \underset{\text{def}}{=} (f^{-1}(y^l))_{l=1}^\infty$ is *not* index-locally finite in X (def. A.6.1(c)).

(3) There exists a countable filter base $B^\bullet = (B^l)_{l=1}^\infty$ on Y converging to y such that $f^{-1}(B^\bullet)$ is *not* index-locally finite in X.

(4) Statement (2) with "index-locally finite" replaced with "locally finite."

(5) Statement (3) with "index-locally finite" replaced with "locally finite."

Then (5) $\iff$ (4) $\implies$ (3) $\iff$ (2) $\iff$ (1) and if there is no open subset of X on which f is constant then also (3) $\implies$ (4).

Proof. (1) $\implies$ (2): If $y \in \operatorname{Im} f$ then let $y^l \underset{\text{def}}{=} y$ for all $l \in \mathbb{N}$ and observe that $f^{-1}(y^\bullet)$ is not locally finite.

(2) $\implies$ (3) and (4) $\implies$ (5): If $y^\bullet$ is as in (2) (resp. (4)) then for each $l \in \mathbb{N}$, let $B^l \underset{\text{def}}{=} \{y^k \,|\, k \geq l\}$ be the tail of $y^\bullet$ after l. Then $B^\bullet$ satisfies (3) (resp. (5)).

(3) $\implies$ (1): Let $B^\bullet$ is as in (3). Since $f^{-1}(B^\bullet)$ is not index-locally finite in X there exists some $x \in X$ such that $f^{-1}(B^\bullet)$ is not index-locally finite at x. Let $V^\bullet = (V^k)_{k=1}^\infty$ be a decreasing neighborhood basis of y in Y. Let $B^{l_\bullet} = (B^{l_k})_{k=1}^\infty$ be such that $B^{l_k} \subseteq V^k$ for all $k \in \mathbb{N}$ and observe that for each $k_0 \in \mathbb{N}$, $(f^{-1}(B^{l_k}))_{k=k_0+1}^\infty$ is not index-locally finite at x (since otherwise $(f^{-1}(B^l))_{l=1}^\infty$ would be index-locally finite at x).

Let $U^\bullet = (U^l)_{l=1}^\infty$ be a decreasing neighborhood basis of x in X. Since $f^{-1}(B^\bullet)$ is not locally finite at x, there exists some $l_1 \in \mathbb{N}$ for which there is some $x^1 \in U^1 \cap f^{-1}(B^{l_1}) \neq \varnothing$. Suppose we've picked $l_1, \ldots, l_k$ and $x^1 \in U^1 \cap f^{-1}(B^{l_1}), \ldots, x^k \in U^k \cap f^{-1}(B^{l_k})$ such that $l_i < l_j$ for all $1 \leq i < j \leq k$. Since $(f^{-1}(y^l))_{l=l_k+1}^\infty$ is not index-locally finite in at x there exists some $l_{k+1} \leq l_k + 1$ such that $U^{k+1} \cap f^{-1}(B^{l_{k+1}}) \neq \varnothing$ and now pick any $x^{k+1} \in U^{k+1} \cap f^{-1}(B^{l_{k+1}})$. For each $k \in \mathbb{N}$, $f(x^k) \in B^{l_k} \subseteq V^{l_k}$ and $x^k \in U^k$ where since $U^\bullet$ and $V^{l_\bullet}$ are decreasing we have $\lim\limits_{k\to\infty} f(x^k) = y$ and $\lim\limits_{k\to\infty} x^k = x$. The continuity of f implies that $f(x) = \lim\limits_{k\to\infty} f(x^k) = y$.

(4) $\implies$ (2) is immediate.

(5) $\implies$ (4): Let $B^\bullet$ be as in (5) and let $x \in X$ be such that $f^{-1}(B^\bullet)$ is not locally-finite at x. Let $U^\bullet = (U^l)_{l=1}^\infty$ be a decreasing neighborhood basis of x in X. Observe that for all $k \in \mathbb{N}$, $(f^{-1}(B^l))_{l=k}^\infty$ is not locally finite at x so we may construct $(l_k)_{k=1}^\infty$ increasing and a sequence $x^k \in f^{-1}(B^{l_k}) \cap U^k$ such that $f(x^k) \neq y$ for all $k \in \mathbb{N}$.

Since $x^\bullet$ converges to x, $f(x^\bullet)$ converges to y so that the fact that N is Hausdorff implies that $f(x^\bullet)$ has a subsequence $(f(x^{k_i}))_{i=1}^\infty$ such that $i \neq j$ implies $f(x^{k_i}) \neq f(x^{k_j})$. For all $i \in \mathbb{N}$ let $y^i \underset{\text{def}}{=} f(x^{k_i})$ and observe that $f^{-1}(y^\bullet)$ is not locally finite at x since $x^{k_\bullet}$ is a sequence of points converging to x with each x^{k_i} contained in a distinct fiber of f.

(3) $\implies$ (4): Suppose that is no open subset of X on which f is constant and let $x \in f^{-1}(y)$. Let $U^\bullet = (U^l)_{l=1}^\infty$ be a decreasing neighborhood basis of x in X. For each $l \in \mathbb{N}$

pick $x^l \in U^l$ such that $f(x^l) \neq y$. The rest of this proof is the same as the second paragraph of (5) $\implies$ (4). ■

Miscellaneous Lemmata

Lemma A.7.1. Let $F : M \to N$ and $G : N \to P$ be continuous maps. Then

(1) $G \circ F : M \to P$ is injective (resp. a homeomorphism, an embedding) if and only if the same is true of both $F : M \to N$ and $G|_{\operatorname{Im} F} : \operatorname{Im} F \to P$.

(2) If N is Hausdorff Fréchet-Urysohn, P is Hausdorff, and $G \circ F : M \to P$ is injective and sequentially quotient then $\operatorname{Im} F$ is closed in $G^{-1}(\operatorname{Im}(G \circ F))$.

(3) If $\sigma : W \to M$ is a continuous local section of $G \circ F : M \to P$ then $F \circ \sigma : W \to N$ is a continuous local section of $G : N \to P$ and $F|_{\operatorname{Im} \sigma} : \operatorname{Im} \sigma \to P$ is an embedding.

Proof. (1): The statement for injectivity is immediate and note that the statement regarding embeddings follows from the statement regarding homomorphisms. So assume that $G \circ F : M \to P$ is a homeomorphism. Let $V \in \operatorname{Open}(N)$, $U \underset{\text{def}}{=} F^{-1}(V)$, and $W \underset{\text{def}}{=} G(V)$. Since F is continuous U is open and hence so is $(G \circ F)(U) = (G \circ F)(F^{-1}(V)) = G(\operatorname{Im} F \cap V)$ so that $G|_{\operatorname{Im} F} : \operatorname{Im} F \to P$ is a continuous open bijection. Since $F = \left(G|_{\operatorname{Im} F}\right)^{-1} \circ (G \circ F)$ is a composition of homeomorphisms it follows that F is a homeomorphism.

(2): Let $n^\bullet \subseteq \operatorname{Im} F$ be a sequence converging to $n \in G^{-1}(\operatorname{Im}(G \circ F))$. Let $p^\bullet = G(n^\bullet)$, which converges to $p \underset{\text{def}}{=} G(n)$ in $\operatorname{Im}(G \circ F)$. Since $G \circ F$ is sequentially quotient there exists an increasing sequence $(l_k)_{k=1}^\infty \subseteq \mathbb{N}$ and a sequence $(m^{l_k})_{k=1}^\infty$ converging in M to some $m \in (G \circ F)^{-1}(p)$ such that $m^{l_\bullet}$ is an $(G \circ F)$-lift of $p^{l_\bullet}$. Since $G \circ F$ is injective and since it was known that $n^{l_\bullet} \subseteq \operatorname{Im} F$, we have that $F(m^{l_\bullet}) = n^{l_\bullet}$ so that $m^{l_\bullet} \to m$ in M implies that $n^{l_\bullet} = F(m^{l_\bullet}) \to F(m)$, which implies that $F(m) = n$ and thus that $n \in \operatorname{Im} F$. If $p^\bullet \to p$ in $\operatorname{Im}(G \circ F)$ then picking $m, (l_k)_{k=1}^\infty, m^{l_\bullet}$ as before shows that $\lim F(m^{l_\bullet}) \to F(m)$ is a G-lift of $p^{l_\bullet} \to p$, which proves that the bijection $F\big|_{\operatorname{Im} \gamma} : \operatorname{Im} \gamma \to \operatorname{Im} \eta$ is sequentially quotient.

(3) is immediate from $G \circ (F \circ \sigma) = \mathrm{Id}_V$. ∎

Lemma A.7.2. Let $E : L \to M$ and $F : M \to N$ be maps with L compact, L and M Hausdorff, F continuous, and $F \circ E : L \to N$ a topological embedding. Then $F\big|_{\operatorname{Im} E} : \operatorname{Im} E \to N$ is a continuous bijection and the following are equivalent:

(1) $\operatorname{Im} E$ is compact.

(2) $F\big|_{\operatorname{Im} E} : \operatorname{Im} E \to N$ is a topological embedding.

(3) $E : L \to M$ is a topological embedding.

(4) $E : L \to M$ is continuous.

Proof. (2) $\implies$ (3) follows from the equality $E = \left(F\big|_{\operatorname{Im} E}^{-1} \right) \circ (F \circ E)$ and the rest are immediate. ∎

Lemma A.7.3. Suppose that $\mu : X \to Y$ is a surjective continuous open map between σ-compact spaces and let $\mathcal{V} = (V^n)_{n=1}^{\infty}$ be a sequence of open subsets of Y whose closures form an exhaustion of Y by compact sets. Then there exists a sequence of open subsets of X, $\mathcal{U} = (U^n)_{n=1}^{\infty}$, whose closures form an exhaustion of X by compact sets and such that $\overline{V^n} \subseteq \mu(U^n)$ for each $n \in \mathbb{N}$.

Proof. Let $(W^n)_{n=1}^{\infty}$ be any exhaustion of X by relatively compact open sets. Since $\mu : X \to Y$ is a surjective open continuous map, the sets $(\mu(W^n))_{n=1}^{\infty}$ form an open cover of Y where each $\mu(W^n)$ is a relatively compact open subset of Y. From the compactness of $\overline{V^1}$ there exists some integer $N(1)$ such that $\overline{V^1} \subseteq \bigcup_{n=1}^{N(1)} \mu(W^n)$. Having picked $N(l) > N(l-1)$ such that $\overline{V^l} \subseteq \bigcup_{n=1}^{N(l)} \mu(W^n)$, pick N such that $N > N(l)$ and $\overline{V^{l+1}} \subseteq \mu\left(\bigcup_{n=1}^{N} W^n \right)$ and then let $N(l+1) \underset{\text{def}}{=} N$. For all $n \in \mathbb{N}$, let $U^n = \bigcup_{l=1}^{N(l)} W^l$ and note that $\mathcal{U} \underset{\text{def}}{=} (U^n)_{n=1}^{\infty}$ satisfies $\overline{U^n} \subseteq U^{n+1}$ and $\overline{V^n} \subseteq \mu(U^n)$. ∎

Lemma A.7.4. Let $(K_i)_{i=1}^{\infty}$ be a locally finite collection of compact subsets of a locally compact Hausdorff normal space M. Let $(W_i)_{i=1}^{\infty}$ be a collection of open subsets of M such that $K_i \subseteq W_i$ for all $i \in \mathbb{N}$. Then there exists a sequence $(V_i)_{i=1}^{\infty}$ of relatively compact open subsets of M such that for each $i \in \mathbb{N}$, $K_i \subseteq V_i \subseteq \overline{V_i} \subseteq W_i$ and $\overline{V_i}$ intersects at most only finitely many other $\overline{V_l}$'s.

Proof. Note that we may assume without loss of generality that $(W_i)_{i=1}^{\infty}$ covers M. Observe that it suffices to show that there exists a sequence $(V_i)_{i=1}^{\infty}$ of relatively compact open subsets of M such that for each $i \in \mathbb{N}$, $K_i \subseteq V_i$ and $\overline{V_i}$ intersects at most only finitely many other V_l's. To see this pick, for each $i \in \mathbb{N}$, an open set $\widehat{V_i}$ such that $K_i \subseteq \widehat{V_i} \subseteq \overline{\widehat{V_i}} \subseteq W_i \cap V_i$ so that $\left(\widehat{V_i}\right)_{i=1}^{\infty}$ are relatively compact open set that satisfy the conclusion of the lemma. Note that for each i there exists a relatively compact open set $O_i \in \operatorname{Open}(M)$ containing K_i such that O_i intersects at most finitely many K_h's. To see this, fix an index i. For all $p \in K_i$ there exists some relatively compact open set $p \in O_p \in \operatorname{Open}(M)$ such that O_p intersections only finitely many K_h's. Since K_i is compact there exist finitely many of these p's, say $p_1^i, p_2^i, \ldots, p_{N(i)}^i$ such that $O_{p_1^i}, \ldots, O_{p_{N(i)}^i}$ cover K_i. Now $O_i \underset{\text{def}}{=} O_{p_1^i} \cup \cdots \cup O_{p_{N(i)}^i}$ is the desired set.

Let O_1 be any relatively compact open subset of M containing K_1 that intersects only finitely many K_h's and let V_1 be an open subset of O_1 containing K_1 such that $\overline{V_1} \subseteq O_1$. Continuing inductively, suppose we've picked a relatively compact open subset O_{i-1} of M such that $\overline{O_i}$ intersects at most finitely many K_h's and have also some open set V_i containing K_i such that $\overline{V_i} \subseteq O_i$ and if for some $h = 1, \ldots, i-2$ we have $V_{i-1} \cap V_h \neq \varnothing$ then $O_h \cap K_{i-1} \neq \varnothing$. Pick a relatively compact open set $U_i \subseteq M$ containing K_i such that $\overline{U_i}$ intersects only finitely many K_h's. Let $i_1, \ldots, i_q$ denote all the indices from among $\{1, \ldots, i-1\}$ such that $\overline{O_{i_l}} \cap K_i = \varnothing$ and let $j_1, \ldots, j_r$ denote all the indices from among $\{1, \ldots, i-1\}$ such that $\overline{V_{j_l}} \cap K_i = \varnothing$. Let

$$O_i \underset{\text{def}}{=} U_i \setminus \left(\overline{O_{i_1}} \cup \cdots \cup \overline{O_{i_q}} \cup \overline{V_{j_1}} \cup \cdots \cup \overline{V_{j_r}}\right)$$

and observe that O_i is a relatively compact open set containing K_i. Pick an open set V_i containing K_i such that $\overline{V_i} \subseteq O_i$. Suppose that $h \in \{1, \ldots, i-1\}$ is such that $V_i \cap V_h \neq \varnothing$. If $O_h \cap K_i = \varnothing$ then $\overline{V_h} \cap K_i = \varnothing$ so that h is among $j_1, \ldots, j_r$. By definition of O_i we have

$O_i \cap \overline{V_h} = \varnothing$ and hence $\overline{V_i} \cap \overline{V_h} = \varnothing$ (since $\overline{V_i} \subseteq O_i$), giving a contradiction. Thus $O_h \cap K_i \neq \varnothing$, and so the inductive step is complete.

Assume for the sake of contradiction that $i_0 \in \mathbb{N}$ is such that V_{i_0} intersects infinitely many of the sets $V_{i_0+1}, V_{i_0+2}, \ldots$. Since O_{i_0} intersects only finitely many K_h's there exists some $j_0 > i_0$ such that for all $j \geq j_0$, $O_{i_0} \cap K_j = \varnothing$. By assumption, there exists some $j > j_0$ such that $V_{i_0} \cap V_j \neq \varnothing$. By the inductive hypothesis in the case of $i := j$ and $h := i_0$ we have $O_{i_0} \cap K_j \neq \varnothing$ (since $V_{i_0} \cap V_j \neq \varnothing$), which contradicts $O_{i_0} \cap K_j = \varnothing$. ∎

Appendix B

Analysis

Example B.0.1. Using the well-known properties of $e^{-1/t}$ it is easy to verify that the map

$$\begin{aligned} \beta : [-1,1] &\longrightarrow [-1,1] \\ t &\longmapsto \begin{cases} -2e^{(\ln 2)/t} & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ 2e^{-(\ln 2)/t} & \text{if } t > 0 \end{cases} \end{aligned}$$

is a smooth strictly increasing homeomorphism such that $\beta^{(n)}(0) = 0$ for all $n \in \mathbb{Z}^{\geq 0}$ and $\beta'(t) = 0 \iff t = 0$.

Functional Analysis

For readers who are primarily familiar with Banach space theory, we will now give brief overview of some of the more general functional analytic terminology that will be relevant. For a more detailed exposition the author recommends [47], [40], and especially [42].

Definition and Notation B.1.1. Let X be a vector space over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, endowed with a topology τ_X and give the field its usual topology. Then we call (X, τ_X) a *topological vector space* (abbreviated *TVS*) and say that τ_X *is compatible with the vector space structure of* X or that τ_X *is a TVS-topology (on X)* if the operations of addition and scalar multiplication are each continuous with respect to τ_X. By X' we mean the TVS's *(continuous) dual space*, which is the set of all continuous linear functions and we will let $\langle \cdot, \cdot \rangle : X' \times X \to \mathbb{F}$ denote composition: $\langle x', x \rangle \underset{\text{def}}{=} x'(x)$. For any $x, y \in X$ let $[x, y] \underset{\text{def}}{=} \{tx + (1-t)y \,|\, 0 \leq t \leq 1\}$ and if $A, R \subseteq X$ then say that A *absorbs* R if there exists some $r > 0$ such that $R \subseteq rA$. A subset $S \subseteq X$ is

- *absorbing* or *radial in* X if it absorbs every point of X.
- *(von-Neumann) bounded* if every $0 \in U \in \operatorname{Open}(X)$ absorbs S.
- *balanced* or *circled* if $\alpha S \subseteq S$ for every scalar $|\alpha| \leq 1$.
- *convex* for all $s_1, s_2 \in S$, the line segment between s_1 and s_2 is contained in S.
- a *barrel* if it is closed, convex, balanced, and absorbing.

Call a continuous linear map $\Lambda : X \to Y$ between two Hausdorff LCTVSs *nuclear* if there exists a convex, balanced, and bounded subset $B \subseteq Y$ and sequences $\alpha_\bullet = (\alpha_1, \alpha_2, \ldots) \in \ell^1(\mathbb{N} \to \mathbb{R})$, $x'_\bullet \subseteq X'$, and $y_\bullet \subseteq B$ such that $\Lambda(\cdot) = \sum_{i=1}^{\infty} \alpha_i \langle x'_i, \cdot \rangle y_i$ and the normed space $(\operatorname{span}(B), \mu_B)$ is complete, where $\mu_B(x) \underset{\text{def}}{=} \inf\{t > 0 \,|\, x \in tB\}$ is called *the gauge (*or *Minkowski) functional of B*. A *TVS-isomorphism* is a continuous linear bijection with a continuous inverse. A continuous linear map $\Lambda : X \to Y$ between two TVSs is a *TVS-homomorphism* if the induced map $\tilde{\Lambda} : X / \ker \Lambda \to \operatorname{Im} \Lambda$ is a TVS-isomorphism onto its image.

A TVS X is

- said to *have the Heine-Borel property* if every closed and bounded subset is compact.
- *barreled* if every barrel in X is a neighborhood of the origin.
- a *locally convex TVS* (abbreviated *LCTVS*) if $\mathbf{0}$ has a neighborhood basis consisting of convex open sets.

- *Fréchet* if it is a complete metrizable LCTVS.

- *Montel* if it is a barreled Hausdorff LCTVS with the Heine-Borel property.

- *nuclear* if it is a Hausdorff LCTVS such that whenever $\Lambda : X \to Y$ is a continuous linear operator into a Banach space Y then Λ is necessarily nuclear.

The topology (on X') of uniform convergence on singleton (resp. compact, bounded) subsets of X will be denoted by $\sigma(X', X)$ (resp. $c(X', X)$, $b(X', X)$) and when X' is endowed with this topology then it will be denoted by X'_σ (resp. X'_c, X'_b). Unless indicated otherwise we will henceforth assume that X' is endowed with $b(X', X)$ so that in particular, when we write X'' then we will mean $\left(X'_b\right)'_b$. For each $x \in X$ let ev_x denote the evaluation map at x defined on X'. If X is a Locally convex TVS then the evaluation map is called *the canonical injection* of X into X'', where if the map $\mathrm{ev} : X \to X''$ that sends x to ev_x is a vector space-isomorphism (resp. a TVS-isomorphism) then X is called *semi-reflexive* (resp. *reflexive*).

If Y is another TVS then $L(X \to Y)$ will denote the vector space of all continuous linear maps from X to Y. When the vector space $L(X \to Y)$ is endowed with the topology of uniform convergence on singleton (resp. compact, bounded) subsets of X then it will be denoted by $L_\sigma(X \to Y)$ (resp. $L_c(X \to Y)$, $L_b(X \to Y)$). ■

Remark B.1.2. If a TVS X has the Heine-Borel property then for any TVS Y, the topology (on $L(X \to Y)$) of compact convergence is identical to the topology of bounded convergence. If X is Montel then recall ([47]) that it is reflexive, its strong dual X'_b is also Montel, and on any bounded subset of X'_b the weak and strong topologies coincide, which in particular implies that any weakly convergent sequence of continuous linear functionals is strongly convergent.

Example and Definition B.1.3 ($\mathbb{R}^{\mathbb{N}} \underset{\text{def}}{=} \prod_{i=1}^{\infty} \mathbb{R}$, the space of real sequences)**.** It is well-known ([47]) that $\mathbb{R}^{\mathbb{N}}$ is an infinite-dimensional nuclear Fréchet Montel space. Furthermore, every continuous linear functional $x' : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ is of the form $x'(x_1, x_2, \ldots) = c_{i_1} x_{i_1} + \cdots + c_{i_n} x_{i_n}$ for some finite collection of indices $i_1, \ldots, i_n$ and constants $c_{i_1}, \ldots, c_{i_n}$. Equivalently, every continuous linear functional x' on $\mathbb{R}^{\mathbb{N}}$ is of the form $x' = x'_i \circ \operatorname{Pr}_i$ for some index i and some continuous linear functional $x'_i : \mathbb{R}^i \to \mathbb{R}$. It should now be clear that the continuous dual space of $\mathbb{R}^{\mathbb{N}}$ is TVS-isomorphic to $\mathbb{R}^\infty \underset{\text{def}}{=} \bigoplus\limits_{i \in \mathbb{N}} \mathbb{R}$ with it's usual inductive limit topology, which is itself a non-metrizable locally convex nuclear Souslin Montel space. Recall ([47, p. 201]) also that every linear functional on $\mathbb{R}^\infty$ is necessarily continuous and that the weak and strong dual topologies on $\mathbb{R}^{\mathbb{N}}$ induced by $\mathbb{R}^\infty$ (i.e. $\sigma(\mathbb{R}^{\mathbb{N}}, \mathbb{R}^\infty)$ and $b(\mathbb{R}^{\mathbb{N}}, \mathbb{R}^\infty)$) are identical. Furthermore, if Y is a Fréchet space then every separately continuous bilinear form on $(\mathbb{R}^{\mathbb{N}})'_b \times Y'_b$ is continuous and if Y is Banach then every continuous linear map $\Lambda : \mathbb{R}^{\mathbb{N}} \to Y$ is nuclear. ∎

Remark B.1.4. Recall ([39, prop. 4.2.2]) that for any locally convex nuclear space and for any given collection of elements $(x^\alpha)_{\alpha \in A}$ the notions of weakly summable (i.e. $\sum\limits_\alpha |\langle \lambda, x^\alpha \rangle| < \infty$ for all continuous linear functionals λ), summable (i.e. the net of finite partial sums indexed by inclusion converges), and absolutely summable (i.e. $\sum\limits_\alpha p(x^\alpha) < \infty$ for all continuous seminorms p) are all equivalent. In particular, since $\mathbb{R}^{\mathbb{N}}$ is a Fréchet space, any such summable family will necessarily have only countably many non-zero terms.

Definition B.1.5 ([42], [29])**.** If X is a TVS and Y is a vector subspace of X then recall ([42, p. 22]) that Y is said to be *complemented in* X and that Y *splits in* X if there exists a vector subspace Z of X such that the continuous linear map $\Lambda : Y \times Z \to X$ defined by $\Lambda(y, z) = y + z$ is an isomorphism of TVSs, in which case we'll say that Y and Z are *complements in* X and that Y *is complemented in* X *by* Z. In this case, X will be a direct sum of Y and Z (in the category of TVSs) and $\Lambda^{-1} = (\pi_Y, \pi_Z) : X \to Y \times Z$ will be continuous, where π_Y and π_Z are projections onto Y and Z, respectively. In particular, both projections will be continuous and if X is Hausdorff then both $Y = \ker \pi_Z$ and $Z = \ker \pi_Y$ will be closed in X. A continuous linear injection between TVSs is said to *split* if its image is complemented in its codomain.

Remark B.1.6. If the map $\Lambda : Y \times Z \to X$ from above is a TVS-isomorphism then X will be a direct sum of Y and Z (in the category of TVSs) and $\Lambda^{-1} = (\pi_Y, \pi_Z) : X \to Y \times Z$ will be continuous, where π_Y and π_Z are projections onto Y and Z, respectively. In particular, both projections will be continuous and if X is Hausdorff then both $Y = \ker \pi_Z$ and $Z = \ker \pi_Y$ will be closed in X.

Remarks B.1.7.

- If the map $A : Y \times Z \to X$ from def. B.1.5 is bijective with inverse $(\operatorname{Pr}_Y, \operatorname{Pr}_Z) : X \to Y \times Z$ then it is straightforward to verify that both Pr_Y and Pr_Z are projections (i.e. $\operatorname{Pr}_Y \circ \operatorname{Pr}_Y = \operatorname{Pr}_Y$), $Z = \ker \operatorname{Pr}_Y$, $Y = \ker \operatorname{Pr}_Z$, $Y \cap Z = \{\mathbf{0}\}$, and $\operatorname{Id}_X = \operatorname{Pr}_Y + \operatorname{Pr}_Z$. It follows ([42]) that $X = Y \oplus Z$ in the category of TVSs if and only if at least one of Pr_Y and Pr_Z is continuous, in which case both are continuous so that their kernels Y and Z will be closed in X if X is Hausdorff.

- It's easily seen that Y is complemented in X if and only if Y is the image of some continuous projection $\operatorname{Pr}_Y : X \to Y$ for if given such a map then set $Z := \ker \operatorname{Pr}_Y$ and $\operatorname{Pr}_Z := \operatorname{Id}_X - \operatorname{Pr}_Y$.

- Many authors (e.g. [40]) define $X = Y \oplus Z$ to mean that X is the algebraic direct sum of Y and Z with both Y and Z closed in X. If X is Hausdorff then our definition B.1.5 implies this definition while if X is complete and metrizable then they are equivalent ([40]).

Lemma B.1.8. Let X be a Hausdorff LCTVS with a vector subspace Y that is TVS-isomorphic to $\mathbb{R}^I$ for some set I. Then Y is closed in X and Y splits in X.

Proof. Let $\lambda = (\lambda_i)_{i \in I} : Y \to \mathbb{R}^I$ be a TVS-isomorphism. Y is closed in X since it is a complete subspace of a Hausdorff space. For all $i \in I$ let $\Lambda_i : X \to \mathbb{R}$ be a continuous linear functional extending $\lambda_i : Y \to \mathbb{R}$. Then $\Lambda := (\Lambda_i)_{i \in I} : X \to \mathbb{R}^I$ is a continuous linear map extending $\lambda = (\lambda_i)_{i \in I}$. Since $\Lambda\big|_Y = \lambda$ it follows that $\pi := \lambda^{-1} \circ \Lambda : X \to Y$ is a continuous projection onto Y. Thus $X = Y \oplus \ker \pi$ in the category of TVSs. ∎

Differentiation in Topological Vector Spaces

We will now describe two of the most common definitions of continuous differentiability, which we will observe (see B.2.3) to be equivalent in the spaces that are most relevant to promanifolds.

Definition B.2.1. Let X and Y be Hausdorff LCTVSs, $U \in \operatorname{Open}(X)$, and $F : U \to Y$ be any map. If $m \in U$ then $F : U \to Y$ is *Gâteaux differentiable at* m if $\mathrm{d}_m F \mathbf{v} \underset{\text{def}}{=} \lim_{\tau \to 0} \frac{F(m+\tau\mathbf{v}) - F(m)}{\tau}$, which is called *the directional derivative of* F *(at* m*) in the direction* $\mathbf{v}$ and also denoted by $\mathrm{d}F(m)\mathbf{v}$, exists in all directions $\mathbf{v} \in X$. If F is Gâteaux differentiable at all points of U then we will let $\mathrm{d}F$ denote the map

$$\begin{aligned} \mathrm{d}F : U \times X &\longrightarrow Y \\ (m, \mathbf{v}) &\longmapsto \mathrm{d}_m F \mathbf{v} \end{aligned}$$

and we will denote its associate $U \to Y^X$ by

$$\begin{aligned} \mathrm{D}F : U &\longrightarrow Y^X \\ m &\longmapsto \mathrm{d}_m F \end{aligned}$$

where if each $\mathrm{d}_m F : X \to Y$ is linear and continuous then we will instead consider $\mathrm{D}F$ as a map into $L_b(X \to Y)$. We say that F is *Gâteaux* C^1 (resp. *Fréchet* C^1) on U if it is Gâteaux differentiable at all points of U and if the map $\mathrm{d}F : U \times X \to Y$ (resp. $\mathrm{D}F : U \to L_b(X \to Y)$) is continuous. Proceeding by induction on $k > 1$, we define F to be *Gâteaux* C^k (resp. *Fréchet* C^k) if F is Gâteaux C^1 (resp. Fréchet C^1) and $\mathrm{d}F : U \times X \to Y$ (resp. $\mathrm{D}F : U \to L_b(X \to Y)$) is Gâteaux C^{k-1} (resp. Fréchet C^{k-1}). ∎

Remark B.2.2. It can be shown ([21]) that if $F : U \to Y$ is Gâteaux C^1 then each $\mathrm{d}F(m) : X \to Y$ is necessarily linear and continuous so $\mathrm{D}F$'s prototype will indeed be $\mathrm{D}F : U \to L_b(X \to Y)$.

The following observation, which appears to have gone unnoticed elsewhere, shows that Montel spaces are well-suited for analysis.

Observation B.2.3 (Conditions for equivalence of Gâteaux C^1 with Fréchet C^1)**.** Suppose that X is a Hausdorff LCTVS with the Heine-Borel property, $U \in \operatorname{Open}(X)$, Y is a TVS such that $U \times Y$ is a k-space, and $F : U \to Y$ is a map that is Gâteaux differentiable at each point of U. Since $U \times Y$ is a k-space,

$$\mathrm{d}F : U \times X \to Y$$

is continuous if and only if its associate $\mathrm{d}F : U \to Y^X$ is continuous when Y^X is given the compact open topology. So if each map $\mathrm{d}_m F : X \to Y$ is linear and continuous it follows that the map

$$\begin{aligned} \mathrm{D}F : U &\longrightarrow L_c(X \to Y) = L_b(X \to Y) \\ m &\longmapsto \mathrm{d}_m F \end{aligned}$$

is continuous if and only if $\mathrm{d}F : U \times X \to Y$ is continuous, where the equality $L_c(X \to Y) = L_b(X \to Y)$ was described in remark B.1.2. Since continuity of $\mathrm{d}F : U \times X \to Y$ implies the continuity and linearity of all $\mathrm{d}_m F : X \to Y$ it follows that the map $F : U \to Y$ is Gâteaux C^1 if and only if it is Fréchet C^1. In particular, if $X = \mathbb{R}^k$ for some $k \in \mathbb{Z}^{\geq 0} \cup \{\mathbb{N}\}$ then a map $F : U \to Y$ from an open subset U of $\mathbb{R}^k$ into a TVS Y is Gâteaux C^1 if and only if it is Fréchet C^1. ■

Fréchet-Urysohn Hausdorff LCTVSs are Coherent with Arcs

Lemma B.3.1. Let X be a Hausdorff LCTVS and suppose $(x^l)_{l=1}^{\infty} \to x^0$ in X has infinite range. Then there exists an increasing sequence $(l_k)_{k=1}^{\infty} \subseteq \mathbb{N}$ and an arc $\gamma : ([-1,1], 0) \to$

(X, x^0) such that $\gamma\left(\frac{1}{k}\right) = x^{l_k}$ for all $k \in \mathbb{N}$.

Remark B.3.2. The idea of the construction of γ is similar to how it would be constructed in a $\mathbb{R}^d$ ($d \in \mathbb{N}$) except that the necessity of guaranteeing that γ be both injective and pass through infinitely many $x^\bullet$'s complicates the construction. It's due to these requirements that to construct γ, it may not be enough to simply construct a continuous curve passing through these points and then evoke the fact the image of a path is arc-wise connected. Indeed, it is not even clear that such a curve even necessarily exists without local convexity.

Proof. Assume without loss of generality that $x^0 = \mathbf{0}$ and that $x^\bullet \to \mathbf{0}$ is injective in X. If there exists a subsequence of $(x^l)_{l=0}^\infty$ that is contained in a finite-dimensional affine subspace then the conclusion is obvious so assume that no such subsequence exists. For any $S \subseteq X$ let $\operatorname{Aff}(S)$ (resp. $\operatorname{co}(S)$) denote the affine span (resp. convex hull) of S in X.

Let $l_1 = 1$ and pick $l_2 > l_1$ be such that $\mathbf{0} \notin \operatorname{Aff}(x^{l_1}, x^{l_2})$. Suppose we have increasing integers $l_1, \ldots, l_k$ such that $\mathbf{0} \notin \operatorname{Aff}(x^{l_1}, \ldots x^{l_h})$ and for all $h = 2, \ldots, k$, $x^h \notin \operatorname{Aff}(x^{l_1}, \ldots, x^{l_{h-1}})$. Observe that if $x \in X$ is such that $\mathbf{0} \in \operatorname{Aff}(x, x^{l_1}, \ldots, x^{l_k})$ then the fact that $\mathbf{0} \notin \operatorname{Aff}(x^{l_1}, \ldots, x^{l_k})$ implies that x belongs to $\operatorname{Span}(x^{l_1}, \ldots, x^{l_k})$. Since no infinite subsequence of $(x^l)_{l=0}^\infty$ is contained in any finite-dimensional affine subspace, this implies that there exists some $L > l_k$ such that $l \leq L \implies \mathbf{0} \notin \operatorname{Aff}(x^{l_1}, \ldots, x^{l_k}, x^l)$. Pick a non-empty balanced open set U_{k+1} such that $U_{k+1} \cap \operatorname{Aff}(x^{l_1}, \ldots, x^{l_k}) = \varnothing$ and let $l_{k+1} \geq L$ be such that $l \geq l_{k+1} \implies x^l \notin U_{k+1}$. Observe that $l \geq l_{k+1} \implies x^l \notin \operatorname{Aff}(x^{l_1}, \ldots, x^{l_k}, x^l)$, which completes the construction.

Define $\gamma_k : \left[\frac{1}{k+1}, \frac{1}{k}\right] \to X$ by $\gamma_k(t) = x^{l_{k+1}} + \left[\frac{t - \frac{1}{k+1}}{\frac{1}{k} - \frac{1}{k+1}}\right](x^{l_{k+1}} - x^{l_k})$. Note that for all $k \in \mathbb{N}$, $\operatorname{co}(x^{l_{k+1}}, x^{l_k}) \cap \operatorname{Aff}(x^{l_1}, \ldots, x^{l_k}) = \{x^{l_k}\}$ and $\mathbf{0} \notin \operatorname{co}(x^{l_{k+1}}, x^{l_k})$ so that $\operatorname{Im}\gamma_k \cap \operatorname{Im}\gamma_{k-1} = \{x^{l_k}\}$, and $\operatorname{Im}\gamma_k \cap \operatorname{Im}\gamma_{k+1} = \{x^{l_{k+1}}\}$, and for any $h \in \mathbb{N}$, if $|h - k| > 2$ then γ_h and γ_k have disjoint images that do not contain $\mathbf{0}$. Define $\gamma : [0,1] \to X$ by $\gamma(0) = \mathbf{0}$ and $\gamma = \gamma_k$ on $\left[\frac{1}{k+1}, \frac{1}{k}\right]$ and observe that γ is injective and that $\gamma\big|_{]0,1]}$ is continuous. To see that γ is continuous at $\mathbf{0}$, let U be any convex neighborhood of $\mathbf{0}$ in X and pick $L \in \mathbb{N}$ such that $k \geq L$ implies $x^{l_k} \in U$. Observe that for any $k \geq l$, $\gamma\left(\left[\frac{1}{k+1}, \frac{1}{k}\right]\right) = \operatorname{Im}\gamma_k = \operatorname{co}(x^{l_{k+1}}, x^{l_k}) \subseteq U$ so that $\gamma\left(\left]0, \frac{1}{k}\right]\right) \subseteq U$. Now since $\operatorname{co}(\operatorname{Im}\gamma)$ is not all of X (since $\dim X > 3$) we may pick $x \neq \mathbf{0}$ in X such that

$\{rx : r \geq 0\} \cap \operatorname{Im}\gamma = \varnothing$ and now extend γ to $[-1,1]$ by defining $\gamma(t) = -tx$ for $t \in [-1,0]$, where this extension is necessarily injective and continuous and thus a topological embedding. ∎

Proposition A.5.8 implies that Hausdorff TVSs coherent with C^0-arcs are necessarily sequential. We now prove theorem B.3.3, which together with proposition A.5.8 shows that in the category of Hausdorff LCTVSs, the class of spaces coherent with their arcs lies in-between the class of Fréchet-Urysohn spaces and the class of sequential spaces.

Theorem B.3.3. If a Hausdorff LCTVS is Fréchet-Urysohn then it is coherent with its arcs.

Proof. Let $S \subseteq X$ be a non-empty subset of the Fréchet-Urysohn Hausdorff LCTVS X such that for all arcs γ in X, $S \cap \operatorname{Im}\gamma$ is closed in $\operatorname{Im}\gamma$. Let $x \in \operatorname{Cl}_X(S)$ be a non-isolated point and pick a sequence $(x^l)_{l=1}^{\infty}$ in S converging to x. Let $(l_k)_{k=1}^{\infty}$ and $\gamma : ([0,1],0) \to (X,x)$ be as in lemma B.3.1 and note that since each $\gamma\left(\frac{1}{k}\right)$ belongs to $S \cap \operatorname{Im}\gamma$, $x = \gamma(0)$ belongs to $\operatorname{Cl}_{\operatorname{Im}\gamma}(S \cap \operatorname{Im}\gamma)$. But since $S \cap \operatorname{Im}\gamma$ is by assumption a closed subset of $\operatorname{Im}\gamma$, it follows that $x \in S \cap \operatorname{Im}\gamma \subseteq S$. ∎

Corollary B.3.4. Every metrizable LCTVS is coherent with its C^0-arcs.

Appendix C

Differential Geometry

Definition C.0.1. Following [2], a manifold M is said to be *regular* if for any compact $K \subseteq M$ and any open neighborhood $U \subseteq M$ of K there exists a neighborhood S of K with S a compact submanifold-with-boundary of U.

Definition C.0.2. Call a chart (U, φ) on a d-dimensional manifold a *coordinate box* if $\operatorname{Im}\varphi$ is a product of d open intervals. i.e. $\operatorname{Im}\varphi = \,]a_1, b_1[\, \times \cdots \times \,]a_d, b_d[$ for some $a_1, \ldots, a_d, b_1, \ldots, b_d \in \mathbb{R}$ with $a_i < b_i$ for all $i = 1, \ldots, d$.

Remark C.0.3. Canonical identification of manifolds as commutative locally $\mathbb{R}$-ringed spaces: Let M be a set and recall that for any d-dimensional smooth atlas $\mathcal{A}$ on M induces a unique topology on M (locally dependent only on d) and a sheaf, $C^\infty_{(M,\mathcal{A})}$, of smooth $\mathbb{R}$-valued functions on the open subsets of $(M, \mathcal{A})$. It is easiest to see that distinct (maximal) smooth atlases on M induced distinct sheaves of smooth ($\mathbb{R}$-valued) functions if, for each smooth atlas $\mathcal{A}$ on M, we denote the set of all coordinates of all charts in $\mathcal{A}$ by $\operatorname{Pr}_{\mathbb{R}} \circ \mathcal{A}$.

Suppose that $\mathcal{A}$ and $\widehat{\mathcal{A}}$ are any maximal smooth atlases on M and note that for the induced sheaves to be equal, these atlases must induce the same topology, which we will henceforth assume. By considering the coordinates of smooth charts, it is easy to see that

$$\mathcal{A} = \widehat{\mathcal{A}} \quad \Longleftrightarrow \quad \operatorname{Pr}_{\mathbb{R}} \circ \mathcal{A} = \operatorname{Pr}_{\mathbb{R}} \circ \widehat{\mathcal{A}} \quad \Longleftrightarrow \quad C^\infty_{(M,\mathcal{A})} = C^\infty_{(M,\widehat{\mathcal{A}})}$$

which allows us to identify each smooth manifold $(M, \mathcal{A})$ with a unique commutative locally $\mathbb{R}$-ringed space $\left(M, C^\infty_{(M,\mathcal{A})}\right)$. It is now straightforward to see that for any map $F : M \to N$ between two manifolds $(M, \mathcal{A})$ and $(N, \mathcal{B})$, $F : (M, \mathcal{A}) \to (N, \mathcal{B})$ is smooth if and only if $(F, F^*) : \left(M, C^\infty_{(M,\mathcal{A})}\right) \to \left(N, C^\infty_{(N,\mathcal{B})}\right)$ is a morphism of commutative locally $\mathbb{R}$-ringed spaces, where as usual $F^*(g) \underset{\text{def}}{=} g \circ F\big|_{F^{-1}(\operatorname{Dom} g)}$. We have thus defined a functor from the category Man of smooth manifolds into the category of commutative locally $\mathbb{R}$-ringed spaces that is injective on objects and arrows. ∎

Tubular Neighborhood Constructions

Definition C.1.1 ([25])**.** Let M be a manifold, S be a smoothly embedded submanifold with boundary in M, let T be an open neighborhood of S in M. We will say that T *is a tubular neighborhood of S in M with projection* $\pi : T \to S$ or that $\pi : T \to S$ *is a tubular neighborhood (of S) in M* if there exists a smooth locally trivial vector bundle structure making $\pi : T \to S$ into a smooth locally trivial vector bundle such that the natural inclusion $\operatorname{In} : S \to T$ is the 0-section of π.

Remarks C.1.2.

- If S is a closed smoothly embedded submanifold with boundary in M then S has a tubular neighborhood in M ([25, Thm. 2.2]).
- Recall that any smooth fiber bundle over a contractible manifold is a globally trivial bundle so that, in particular, if $\pi : T \to S$ is a tubular neighborhood of a contractible submanifold S in M, then $\pi : T \to S$ has a global trivialization $\tau : T \to S \times \mathbb{R}^{\dim M - \dim S}$.
- S is a smoothly embedded submanifold without boundary in M then since S is locally closed in M, we may find an open neighborhood W of S in M such that $W \cap S$ is closed in W. Since any tubular neighborhood of S in W is also a tubular neighborhood of

S in M, it follows from [25, Thm. 2.2] that tubular neighborhoods exist around all smoothly embedded submanifolds of a manifold.

Lemma C.1.3. Let $\mu : M \to \mathbb{R}^d$ be a smooth submersion from an e-dimensional manifold M such that $\operatorname{Im}\mu$ contains $S \underset{\text{def}}{=} \mathbb{R}^a \times \{0\}^{d-a}$ for some $a \in \mathbb{Z}^{\mathbb{N}}$ and let $\sigma : S \to M$ be a smooth map such that $\mu \circ \sigma = \operatorname{Id}_S$. There exists a smooth chart (T, ϕ) on M with T containing $\operatorname{Im}\sigma$ and $\phi : T \to \mu(T) \times \mathbb{R}^{e-d}$ surjective such that

(1) $\mu \circ \phi^{-1} : \mu(T) \times \mathbb{R}^{e-d} \to \mu(T)$ is the canonical projection,

(2) $\mu(T)$ is an open tubular neighborhood of S that is diffeomorphic to $\mathbb{R}^d$, and

(3) $\phi \circ \sigma : S \to \mu(T) \times \mathbb{R}^{e-d}$ is the canonical inclusion $p \mapsto \left(p, \{0\}^{e-d}\right)$,

In particular, this implies that the map $\mu(T) \cong \mu(T) \times \{0\}^{e-d} \xrightarrow{\phi^{-1}} T$ is a smooth section of $\mu\big|_T : T \to \mu(T)$ extending σ.

Proof. Let $b = e-d$, $c = d-a$, and $\mu = (\mu_a, \mu_c) : M \to \mathbb{R}^a \times \mathbb{R}^c$ so that $\mu_a\big|_{\operatorname{Im}\sigma}$ is a smooth chart on $\operatorname{Im}\sigma$. Observe that $\operatorname{Im}\sigma$ is a closed submanifold of $\mu^{-1}(S)$, which is itself a closed submanifold of M. Let $\rho : R \to \operatorname{Im}\sigma$ be an open tubular neighborhood of $\operatorname{Im}\sigma$ in $\mu^{-1}(S)$ say with a smooth global trivialization $\nu = (\rho, \nu_2) : R \to \operatorname{Im}\sigma \times \mathbb{R}^b$. Since R is an embedded submanifold of M, there exists an open tubular neighborhood $\xi : W \to R$ of R in M where since R is contractible there also exists a smooth global trivialization $\Theta = (\xi, \Theta_2) : W \to R \times \mathbb{R}^c$. Clearly, $\rho \circ \xi : W \to \operatorname{Im}\sigma$ is a smooth vector bundle with $(\nu \circ \xi, \Theta_2) = (\rho \circ \xi, \nu_2 \circ \xi, \Theta_2) : W \to \operatorname{Im}\sigma \times \mathbb{R}^b \times \mathbb{R}^c$ as a global trivialization. Let $\varphi \underset{\text{def}}{=} (\mu_a \circ \rho \circ \xi, \nu_2 \circ \xi, \Theta_2) : W \to \mathbb{R}^d \times \mathbb{R}^b \times \mathbb{R}^c$ and observe that for all $(s, x, y) \in \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$,

(1) $\varphi \circ \sigma\,(x, \{0\}^c) = \left(x, \{0\}^b, \{0\}^c\right)$,

(2) φ is a slice chart for R both and $\operatorname{Im}\sigma$ where $\varphi\big|_R = (\mu_a \circ \rho, \nu_2, \{0\}^c)$ and $\varphi\big|_{\operatorname{Im}\sigma} = \left(\mu_a, \{0\}^b, \{0\}^c\right)$ have images $\varphi(R) = \mathbb{R}^a \times \mathbb{R}^b \times \{0\}^c$ and $\varphi\,(\operatorname{Im}\sigma) = \mathbb{R}^a \times \{0\}^b \times \{0\}^c$,

(3) both $\varphi \circ \rho \circ \varphi\big|_R^{-1} : \mathbb{R}^a \times \mathbb{R}^b \times \{0\}^c \to \mathbb{R}^e$ and $\varphi\big|_R \circ \xi \circ \varphi^{-1} : \mathbb{R}^e \to \mathbb{R}^a \times \mathbb{R}^b \times \{0\}^c$ are the canonical projections $(s, x, \{0\}^c) \mapsto \left(s, \{0\}^b, \{0\}^c\right)$ and $(s, x, y) \mapsto (s, x, \{0\}^c)$, respectively,

(4) $\varphi\big|_{\operatorname{Im}\sigma} \circ \Theta \circ \varphi^{-1}(s, x, y) = (s, x, y)$ and $\varphi\big|_{\operatorname{Im}\sigma} \circ \nu \circ \varphi\big|_R^{-1} (s, x, \{0\}^c) = (s, x, \{0\}^c)$.

By replacing M with $\operatorname{Im}\varphi$, μ with $\mu \circ \varphi^{-1}$, and ρ, ξ, ν, Θ, and σ with their coordinate representations, it is clear that suffices to prove his lemma under the additional assumption that $M = \mathbb{R}^e$, $\operatorname{Im}\sigma = \mathbb{R}^a \times \{0\}^{b+c}$ with $\sigma = \operatorname{Id}_S \times \{0\}^b$, $R = \mathbb{R}^a \times \mathbb{R}^b \times \{0\}^c$, all three of $\xi : \mathbb{R}^e \to \mathbb{R}^{a+b} \times \{0\}^c$, $\rho : \mathbb{R}^{a+b} \times \{0\}^c \to \mathbb{R}^a \times \{0\}^{b+c}$, and $\mu_a\big|_R : R = \mathbb{R}^a$ are the canonical projections, and Θ and ν are the identity maps on their domains.

Let $I = \mathbb{R}^a \times \{0\}^b \times \mathbb{R}^c$, which is a smooth manifold containing $\operatorname{Im}\sigma$ and observe that for all $m = \left(s, \{0\}^{b+c}\right) \in \operatorname{Im}\sigma = \mathbb{R}^a \times \{0\}^{b+c}$, the map $\mu\big|_I : I \to \mathbb{R}^a \times \mathbb{R}^c$ has full rank at m so there exists some $m \in U_m \in \operatorname{Open}(I)$ such that $\mu_{W_m} : W_m \to \mathbb{R}^d$ is an open embedding, which also implies that the map $\Psi : \mathbb{R}^e \, to \mathbb{R}^e$ that is defined by $\Psi(s, x, y) = (\pi_a(s, x, y), \pi_b(s, x, y), x)$ (note that the position of x has moved) has full rank at m so we may pick some $m \in U_m \in \operatorname{Open}(\mathbb{R}^e)$ with $U_m \cap I \subseteq W_m$ such that $\Psi_{W_m} : W_m \to \mathbb{R}^e$ is an open embedding. Note that $U \underset{\text{def}}{=} \bigcup_{m \in \operatorname{Im}\sigma} U_m$ is an open subset $\mathbb{R}^e$ and that $\Psi\big|_U : U \to \mathbb{R}^e$ is a smooth local diffeomorphism whose restriction to the closed subset $\operatorname{Im}\sigma$ is a the natural inclusion (and thus a topological embedding). Since $\Psi : U \to Q$ is a local homeomorphism between paracompact spaces whose restriction to the closed set $\operatorname{Im}\sigma$ is an embedding, by [25, Lemma 7.2] there exists some open neighborhood O of $\operatorname{Im}\sigma$ in U on which Ψ's restriction becomes an open topological embedding. Observe that $\psi \underset{\text{def}}{=} \mu \circ \Psi\big|_O^{-1} : \Psi(O) \to \mathbb{R}^d$ is the canonical projection, that $\mu_{I \cap O} : I \cap O \to \mu(I \cap O)$ is a diffeomorphism onto an open neighborhood of S, and that every point of $\mathbb{R}^a \times \{0\}^{b+c}$ is a fixed point of Ψ so that the map $\mu(O) \to \Psi(W)$ defined by $(s, y) \mapsto \left(s, y, \{0\}^b\right)$ is a smooth extension of $\Psi^{-1} \circ \sigma$.

Pick a tubular neighborhood T_0 of S contained in $\mu(I \cap O)$ and replace O with $O \cap \mu^{-1}(T_0)$, which allows us to assume that $\mu(O)$ is diffeomorphic to $\mathbb{R}^e$ where since $\mu_{I \cap O}$ is an open embedding, the same is true of $I \cap O$. Pick an open tubular neighborhood $\beta : V \to \mu(O) \times \{0\}^b$ of the contractible manifold $\mu(O) \times \{0\}^{b+c}$ in $\Psi^{-1}(O)$ and observe that $\mu(O) = \mu(\Psi(V))$ so that by replacing O with $\Psi(V)$ we may now assume that O is also diffeomorphic to $\mathbb{R}^e$. Let $\omega : V \to \mu(O) \times \mathbb{R}^b$ be a global trivialization of β, where we identified $\mu(O) \times \{0\}^b$ with $\mu(O)$, and observe that we are now done since by letting $\tau \underset{\text{def}}{=} \Psi \circ \omega^{-1} : \mu(O) \times \times \mathbb{R}^b \to \psi(V)$, we have that $\mu \circ \tau : \mu(O) \times \mathbb{R}^b \to \mu(O)$ is the canonical projection, the canonical injection of S into $\mu(O) \times \mathbb{R}^b$ is a $(\mu \circ \tau)$-lift of σ, and the canonical injection of $\mu(O) \to \mu(O) \times \mathbb{R}^b$ of a smooth extension of the above lift of σ. ■

Corollary C.1.4. Let M (resp. N) have dimension e (resp. d), let $F : M \to N$ be a smooth submersion, and let $\sigma : \mathbb{R}^a \to M$ be a smooth map such that $F \circ \sigma : \mathbb{R}^a \to N$ is a smooth embedding. There exist a smooth charts (T, φ) on M and $(F(T), \psi)$ on N with T containing $\operatorname{Im} \sigma$ and both $\varphi : T \to \mathbb{R}^e$ and $\psi : F(T) \to \mathbb{R}^d$ surjective such that

(1) $\psi \circ F \circ \varphi^{-1} : \mathbb{R}^e \to \mathbb{R}^d$ is the canonical projection, and

(2) $\varphi \circ \sigma : \mathbb{R}^a \to \mathbb{R}^e$ is the canonical inclusion $p \mapsto \left(p, \{0\}^{e-a}\right)$,

Proof. Since $\operatorname{Im}(F \circ \sigma)$ is contractible smooth submanifold of N, any open tubular neighborhood V of it has a global trivialization, say $\chi : V \to \operatorname{Im}(F \circ \sigma) \times \mathbb{R}^{d-a}$ so that by replacing N, M, and F with $\operatorname{Im} \chi$, $F^{-1}(\operatorname{Im} \chi)$, and $\chi \circ F$, respectively, we may assume without loss of generality that $N = \operatorname{Im}(F \circ \sigma) \times \mathbb{R}^{d-a}$. Since the map $h : \operatorname{Im}(F \circ \sigma) \times \mathbb{R}^{d-a} \to \mathbb{R}^a \times \mathbb{R}^{d-a}$ defined by $(s, x) \mapsto ((F \circ \sigma)^{-1}(s), x)$ is a diffeomorphism, we may further assume that $N = \mathbb{R}^d$ and that $F(\sigma(s)) = \left(s, \{0\}^{d-a}\right)$ for all $s \in \mathbb{R}^a$. Let (T, ϕ) be as in lemma C.1.3 where by assumption, $\operatorname{Im} \sigma \subseteq T$, $V \underset{\text{def}}{=} F(T)$ is an open tubular neighborhoodof $S \underset{\text{def}}{=} \operatorname{Im}(F \circ \sigma)$, $F \circ \phi^{-1}$ is the canonical projection, and $\phi \circ \sigma$ is the canonical inclusion $p \mapsto \left(p, \{0\}^{e-d}\right)$. Since V is a tubular neighborhood of the contractible submanifold S, it has a smooth global trivialization $\tau : V \to S \times \mathbb{R}^{d-a}$. Identifying $S = \mathbb{R}^a \times \{0\}^{d-a}$ with $\mathbb{R}^a$, τ becomes a diffeomorphism $\psi : V \to \mathbb{R}^a \times \mathbb{R}^{d-a}$ as does the map $\varphi : T \to \mathbb{R}^e$ defined by $\varphi \circ \phi^{-1}(s, x, y) \mapsto (\psi(F(\phi^{-1}(s, x, y))), y)$. Since $F \circ \phi^{-1}$ is the canonical projection, the same is clearly true of $\psi \circ F \circ \varphi^{-1}$. That $\varphi \circ \sigma$ is the canonical inclusion is immediately seen. ■

Proposition C.1.5. Suppose M and N are smooth manifolds of dimensions $c \geq b$, respectively, and that the following diagram of smooth maps commutes where μ and F are smooth surjective submersions and $\sigma : \mathbb{R}^a \to M$ a smooth section of $\pi \circ \mu : M \to \mathbb{R}^a$ such that $\eta \underset{\text{def}}{=} F \circ \sigma : \mathbb{R}^a \to N$ is a smooth embedding.

$$\begin{tikzcd} M \arrow[d, "\mu"] \arrow[dr, "F"] & \\ \mathbb{R}^a \arrow[u, bend left, "\sigma"] & N \end{tikzcd}$$

There exist smooth surjective charts $\varphi : U \to \mathbb{R}^c$ and $\psi : V \to \mathbb{R}^b$ on M and N, respectively, such that

(1) $\operatorname{Im}\sigma \subseteq U$ and $(\varphi \circ \sigma)(t_1, \ldots, t_a) = (t_1, \ldots, t_a, 0, \ldots, 0)$ for all $(t_1, \ldots, t_a) \in \mathbb{R}^a$.

(2) $\mu\big|_U : U \to \mathbb{R}^a$ and $F\big|_U : U \to V$ are both surjective.

(3) The coordinate representations of μ and F are the canonical projection.

Proof. Let $\eta = F \circ \sigma : \mathbb{R}^a \to N$ and $\sigma = \mu \circ \sigma : \mathbb{R}^a \to M$. Observe that to prove this theorem, it suffices to find non-surjective chart (U, φ) and (V, ψ) such that upon replace "surjective maps" with "well-defined maps" in (2), all of (1) - (1) are satisfied, for after appropriately restricting these chart's domains so that they become open disk bundles and so that all of μ and F restrictions become surjective, we may then compose these charts with bundle morphisms to obtain surjective charts.

Thus, by lemma C.1.4, we may assume without loss of generality that $N = \mathbb{R}^b$, $\eta(t_1, \ldots, t) = \left(t_1, \ldots, t, \{0\}^{b-a}\right)$, and that there is a smooth section $\xi : N \to M$ of $F : M \to N$ such that $\sigma(t_1, \ldots, t_a) = \xi(t_1, \ldots, t_a, 0, \ldots, 0)$ for all $(t_1, \ldots, t_a) \in \mathbb{R}^a$. Observe that $\pi \circ \xi : N \to \mathbb{R}^a$ has full rank at every point of $\operatorname{Im}\sigma$ so there exists some open neighborhood O of $\operatorname{Im}\eta = \mathbb{R}^a \times \{0\}^{b-a}$ in N such that $\pi \circ \xi\big|_O : O \to \mathbb{R}^a$ has full rank everywhere on O. Since $\pi \circ \xi\big|_O : O \to \mathbb{R}^a$ is a smooth submersion, we may apply lemma C.1.3 and obtain a smooth chart $\psi : V \to \mathbb{R}^b$ on O such that $\pi \circ \xi \circ \psi^{-1} : \mathbb{R}^a \times \mathbb{R}^{b-a} \to \mathbb{R}^a$ is the canonical projection and $\psi^{-1} \circ \eta : \mathbb{R}^a \to \mathbb{R}^b$ is the canonical section. By replacing N with $\operatorname{Im}\psi$ and making the necessary changes discussed above, we may assume without loss of generality that $O = N = \mathbb{R}^b$ and that $\pi \circ \xi : \mathbb{R}^b \to \mathbb{R}^a$ is the canonical projection with $\eta : \mathbb{R}^a \to \mathbb{R}^b$ as its canonical zero section. Let D be the integrable subbundle of the $\mathrm{T}\,M$ defined by $D_m = \ker \mathrm{T}_m F$ for all $m \in M$ and observe that for all $x \in \mathbb{R}^a$, $\xi\left(\{x\} \times \mathbb{R}^{b-a}\right)$ is a smooth codimension $c-b$ submanifold of the fiber $\pi^{-1}(x)$. The construction of (U, φ) can now be accomplished by using the $D\big|_U$'s foliation to construct an open tubular neighborhood $\pi : U \to \operatorname{Im} \xi$ in M of the closed smooth submanifold $\operatorname{Im} \xi$ such that π's fibers are the integral manifolds of D, so that contractability of $\operatorname{Im} \xi \cong \mathbb{R}^b$ guarantees a smooth global trivialization $\tau : U \to \operatorname{Im} \xi \times \mathbb{R}^{c-b}$ of π from which the φ's definition follows. We now give the details.

For so that for all $(x, y) \in \mathbb{R}^a \times \mathbb{R}^{b-a}$, $F\big|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times \mathbb{R}^{b-a}$ being a smooth submersion at $\xi(x, y)$ makes it clear that the integral manifold of D at $\xi(x, y)$ is locally contained in the fiber $\mu^{-1}(x)$. So let U be an open neighborhood of $\operatorname{Im} \xi$ in M such that $F^{-1}(x, y) \cap U$ is contained in $\mu^{-1}(x)$ for each $(x, y) \in \mathbb{R}^a \times \mathbb{R}^{b-a}$. Give $\mathbb{R}^a$ and $N = \mathbb{R}^b$ the standard Riemannian embedding and by shrinking M if necessary, we give M a Riemannian metric making $\xi : N \to M$ into a Riemannian submanifold and making $F : U \to \mathbb{R}^b$ into a Riemannian submersion with geodesically complete fibers (i.e. a geodesic tangent to a fiber $\pi\big|_U^{-1}(n)$ stays within this fiber). Let exp denote the exponential map for M, identify $\operatorname{Im} \xi$ with the zero-section of M's tangent bundle, and recall [25, thm. 2.2] that there exists an open neighborhood $O \subseteq \operatorname{Dom}(\exp)$ of $\operatorname{Im} \xi$ such that $O \cap \mathrm{T}_M^{-1}(m)$ is a connected open subset of $\mathrm{T}_M^{-1}(m)$ for all $m \in \operatorname{Im} \xi$, thereby making O into an open disc bundle, and $\exp\big|_O : O \to U$ is an open smooth embedding. Since the subspace $\operatorname{Im} \xi$ is contractible and the open disc bundle $\exp\big|_O : O \to U$ has a smooth global trivialization $\tau = (\tau_1, \tau_2) : U \to \operatorname{Im} \xi \times \mathbb{R}^{c-b}$ so that our desired smooth chart on M is then clearly the map $\varphi := \left(\tau_1 \circ F\big|_U, \tau_2\right) : U \to \mathbb{R}^b \times \mathbb{R}^{c-b}$. ■

Corollary C.1.6. Let Q, N, and M be, respectively, a, d, and e dimensional manifolds, let $\mu : M \to Q$ and $F : M \to N$ be smooth submersions. Suppose that $\sigma : W \to M$ is a smooth local section of $\mu : M \to Q$ such that $F \circ \sigma : W \to N$ is a smooth embedding and that $\theta : W \to \mathbb{R}^a$ is a smooth surjective chart. Then there exist smooth charts (U, φ) and (V, ψ)

on M and N, respectively, and a smooth local section $\rho : V \to U$ of $F : M \to N$ such that $\operatorname{Im}\rho \subseteq U$, $\mu(U) = W$, $F(U) = V$, both $\varphi : U \mapsto \mathbb{R}^e$ and $\psi : V \mapsto \mathbb{R}^d$ are surjective, and the following diagram commutes:

$$
\begin{tikzcd}
\mathbb{R}^e \arrow[d, bend right, "\mathrm{Pr}"'] & & U \arrow[ll, dashed, "\varphi"'] \arrow[dl, bend right, "F|_U"'] \arrow[dd, "\mu|_U"] \\
\mathbb{R}^d \arrow[u, bend right, "\mathrm{In}"'] \arrow[d, bend right, "\mathrm{Pr}"'] & V \arrow[l, dashed, "\psi"'] \arrow[ur, dashed, bend right, "\rho"'] & \\
\mathbb{R}^a \arrow[u, bend right, "\mathrm{In}"'] & & W \arrow[ll, "\theta"] \arrow[uu, bend right, "\sigma"']
\end{tikzcd}
$$

where $\mathrm{In} : \mathbb{R}^a \to \mathbb{R}^d$ is *the canonical inclusion* $(t_1, \ldots, t_a) \mapsto (t_1, \ldots, t_a, 0, \ldots, 0)$. The properties expressed by the above commutative diagram can equivalently be described by the following list of properties:

(1) the coordinate representations of both F and μ are the canonical projections, that is, for each $(t_1, \ldots, t_e) \in \mathbb{R}^e$, the following diagram commutes:

$$
\begin{tikzcd}
(t_1, \ldots, t_a, \ldots, t_d, \ldots t_e) \arrow[r, "\psi \circ F \circ \varphi^{-1}"] \arrow[d, "\theta \circ \mu \circ \varphi^{-1}"'] & (t_1, \ldots, t_a, \ldots, t_d) \\
(t_1, \ldots, t_a) &
\end{tikzcd}
$$

(2) ρ's (resp. σ's) coordinate representation $\varphi \circ \rho \circ \psi^{-1} : \mathbb{R}^d \to \mathbb{R}^e$ (resp. $\varphi \circ \sigma \circ \theta^{-1} : \mathbb{R}^a \to \mathbb{R}^e$) is the canonical inclusion. e.g. $(t_1, \ldots, t_d) \mapsto (t_1, \ldots, t_d, 0, \ldots, 0)$

In particular, for all $m, p \in U$ and all $(x, y) \in \mathbb{R}^a \times \mathbb{R}^{d-a}$

(a) if we write $\widehat{\mu} = \theta \circ \mu\big|_U \circ \varphi^{-1}$ and $\widehat{F} = \psi \circ F\big|_U \circ \varphi^{-1}$ we have $\widehat{F}\left(\widehat{\mu}^{-1}(x)\right) = \{x\} \times \mathbb{R}^{e-a}$ and $\widehat{F}{-1}(x, y) = \{(x, y)\} \times \mathbb{R}^{e-d}$,

(b) $F(m) = F(u) \iff \rho(F(m)) = \rho(F(p))$, in which case $\mu(m) = \mu(\rho(F(m))) = \mu(\rho(F(p))) = \mu(p)$,

(c) $\mu(m) = \mu(p) \iff \sigma(\mu(m)) = \sigma(\mu(p))$.

Proof. This corollary's statement is just detailing the properties of the charts that result from applying proposition C.1.5 with $O \underset{\text{def}}{=} \mu^{-1}(W) \cap F^{-1}(N)$, $F(O)$, $\theta \circ \mu\big|_O$, $\sigma \circ \theta^{-1}$, and $F\big|_O$, substituting for M, N, μ, σ, and F, respectively in a format that will be more helpful for proving the inverse function theorem for promanifolds (theorem 13.2.3) and proposition C.2.1. ∎

Lemma C.1.7. Let $\gamma : \mathbb{R}^a \to M$ and $F : M \to N$ be smooth and suppose that $F \circ \gamma : \mathbb{R}^a \to N$ is a smooth embedding. Suppose that O_0 is an open neighborhood of $\operatorname{Im}\gamma$ in M on which F has constant rank r. Then there exists some connected neighborhood $O \in \operatorname{Open}(O_0)$ of $\operatorname{Im}\gamma$ such that $F(O)$ is a smooth embedded submanifold of N and $F\big|_O : O \to F(O)$ is a smooth submersion.

Proof. Let $S = \operatorname{Im}\sigma$. Pick countable locally finite connected open covers $V_1, V_2, \ldots$ of $F(S)$ in N and $U_1, U_2, \ldots$ of S in O_0 such that for all $l \in \mathbb{N}$, there exists a smooth slice charts $\psi_l : V_l \to \mathbb{R}^d$ of $F(S)$ in N and $\varphi_i : U_i \to \mathbb{R}^e$ of S in M such that $F(U_l) \subseteq V_l$ and $\psi_l \circ F \circ \varphi_l^{-1}\big|_{\operatorname{Im}\varphi} : \operatorname{Im}\varphi_l \to \operatorname{Im}\psi_l$ is the canonical map $(t_1, \ldots, t_r, \ldots, t_e) \mapsto (t_1, \ldots, t_r, 0, \ldots, 0)$ and $\psi_l(F(U_l)) = \left]-s_l^1, s_l^1\right[\times \cdots \times \left]-s_l^r, s_l^r\right[\times \{0\}^{d-r}$. Furthermore, since $F \circ \sigma : \mathbb{R}^a \to N$ is a smooth embedding, if necessary we may, for each $l \in \mathbb{N}$, decrease $s_l^{a+1}, \ldots, s_l^r$ so that if $k \in \mathbb{N}$ is such that $F(U_l) \cap F(U_k) \neq \varnothing$ then $\sigma^{-1}(U_l) \cap \sigma^{-1}(U_k) \neq \varnothing$. For all $s \in S$, let U_s denote the intersection of all U_l's such that $s \in U_l$ and let $R_s = F(U_s)$, which is clearly an r-dimensional smooth submanifold of N. Using the facts that for all $s \in S$ and $l, k \in \mathbb{N}$, if $U_s \subseteq U_l$ then R_s is an open subset of the r-dimensional manifold $F(U_l)$ and that if $F(U_l) \cap F(U_k) \neq \varnothing$ then $\sigma^{-1}(U_l) \cap \sigma^{-1}(U_k) \neq \varnothing$, it is readily seen that $R_s \cap R_t$ is an r-dimensional manifold for all $s, t \in S$. It follows that $R := \bigcup_{s \in S} R_s$ is the desired smooth r-dimensional submanifold of N and that $U := \bigcup_{s \in S} U_s$ is the desired open neighborhood of S in M such that $F\big|_U : U \to R$ is a smooth submersion. ∎

From lemma C.1.7 and proposition C.1.5, it is easy to see how one may prove the following generalization of proposition C.1.5, where we weaken the requirement that $F : M \to N$ be a submersion to instead require that F have constant rank r on some neighborhood of $\operatorname{Im}\sigma$.

Proposition C.1.8. Suppose M and N are smooth manifolds of dimensions $c \geq b$, respectively, and that the following diagram of smooth maps commutes, where μ are smooth surjective submersions, $\sigma : \mathbb{R}^a \to M$ a smooth section of $\pi \circ \mu : M \to \mathbb{R}^a$ such that $\eta \underset{\text{def}}{=} F \circ \sigma : \mathbb{R}^a \to N$ is a smooth embedding, and $F : M \to N$ is a smooth map with constant rank r on some neighborhood O_0 of $\operatorname{Im}\sigma$.

$$\begin{tikzcd} M \arrow[d, "\mu"] \arrow[dr, "F"] & \\ \mathbb{R}^a \arrow[u, bend left, "\sigma"] & N \end{tikzcd}$$

There exist smooth surjective charts $\varphi : U \to \mathbb{R}^c$ and $\psi : V \to \mathbb{R}^b$ on M and N, respectively, such that $F(U) \subseteq V$ and

(1) $\operatorname{Im}\sigma \subseteq U$ and $(\varphi \circ \sigma)(t_1, \ldots, t_a) = (t_1, \ldots, t_a, 0, \ldots, 0)$ for all $(t_1, \ldots, t_a) \in \mathbb{R}^a$.

(2) $\mu\big|_U : U \to \mathbb{R}^a$ is a smooth surjective submersion, $\mu \circ \varphi^{-1}$ is the canonical projection, and $\operatorname{Im}\varphi = \mathbb{R}^c$.

(3) F has rank r on U, and $\psi \circ F \circ \varphi^{-1} : \mathbb{R}^c \to \mathbb{R}^b$ is the canonical map $(t_1, \ldots, t_r, \ldots, t_c) \mapsto (t_1, \ldots, t_r, 0, \ldots, 0)$.

Corollary C.1.9. Let Q, N, and M be, respectively, a, d, and e dimensional manifolds, let $\mu : M \to Q$ and $F : M \to N$ be smooth maps, and let O_0 be a neighborhood of $\operatorname{Im}\sigma$ on which F has constant rank $r \geq b$. Suppose that $\mu : M \to Q$ is a smooth submersion with a smooth local section $\sigma : W \to O_0$ such that $F \circ \sigma : W \to Q$ is a smooth embedding and that $\theta : W \to \mathbb{R}^a$ is a smooth surjective chart. Then there exist smooth charts (U, φ) and (V, ψ) on M and N, respectively, making $R \underset{\text{def}}{=} F(U)$ into a smooth r-dimensional submanifold of V for which there is a smooth local section $\rho : R \to U$ of $F\big|_U : U \to R$ such that

(1) $\operatorname{Im}\rho \subseteq U \subseteq O_0$, $\mu(U) = W$, and $F(U) \subseteq V$,

(2) both $\varphi : U \mapsto \mathbb{R}^e$ and $\psi : V \mapsto \mathbb{R}^d$ are surjective, and $\psi \circ F \circ \varphi^{-1}$ is the canonical map $(t_1, \ldots, t_r, \ldots t_e) \mapsto (t_1, \ldots, t_r, 0, \ldots, 0)$

(3) ρ's coordinate representation is the canonical inclusion $(t_1, \ldots, t_r) \mapsto (t_1, \ldots, t_r, 0, \ldots, 0)$,

(4) if we use the canonical inclusion to identify $\mathbb{R}^a$ with the submanifold $\mathbb{R}^a \times \{0\}^{r-a}$ of $\mathbb{R}^r$, then φ's first r coordinates are $(\mu \circ \rho)^{-1} \circ \mu\big|_U$

(5) if we use $(\mu \circ \Sigma)^{-1} : \operatorname{Im}(\mu \circ \rho) \to \mathbb{R}^b$ as a smooth chart on Q, then μ's coordinate representation $(\mu \circ \rho)^{-1} \circ \mu \circ \varphi^{-1} : \mathbb{R}^e \mapsto \mathbb{R}^r$ is the canonical projection.

(6) $\psi \circ F \circ \varphi^{-1}$ is the canonical projection, or equivalently, $\varphi^{\leq d} = \psi \circ F\big|_U$,

This implies that for each $(t_1, \ldots, t_e) \in \mathbb{R}^e$, the following diagram commutes:

$$\begin{tikzcd}
(t_1, \ldots, t_b, \ldots, t_r, \ldots p_e) \arrow[r, "\psi \circ F \circ \varphi^{-1}"] \arrow[d, "(\mu \circ \rho)^{-1} \circ \mu \circ \varphi^{-1}"'] & (t_1, \ldots, t_b, \ldots, t_r, 0, \ldots, 0) \\
(t_1, \ldots, t_b) &
\end{tikzcd}$$

In the particular case that $F : M \to N$ is a smooth submersion and $R = V$ the following diagram commute:

$$\begin{tikzcd}
\mathbb{R}^e \arrow[d, bend right, "\mathrm{Pr}"'] & & U \arrow[ll, dashed, "\varphi"'] \arrow[dl, bend right, "F|_U"'] \arrow[dd, "\mu|_U"] \\
\mathbb{R}^d \arrow[u, bend right, "\mathrm{In}"'] \arrow[d, bend right, "\mathrm{Pr}"'] & V \arrow[l, dashed, "\psi"'] \arrow[ur, bend right, "\rho"'] & \\
\mathbb{R}^a \arrow[u, bend right, "\mathrm{In}"'] & & W \arrow[ll, "\theta"] \arrow[uu, bend right, "\sigma"']
\end{tikzcd}$$

where $\mathrm{In} : \mathbb{R}^a \to \mathbb{R}^r$ is *the canonical inclusion* $(t_1, \ldots, t_a) \mapsto (t_1, \ldots, t_a, 0, \ldots, 0)$.

Canonical Form

In a typical application of the following proposition C.2.1, we will have the maps shown in the left diagram together with some chart ρ on Q and desire charts on M and N such that, without changing ρ, when we pass into coordinates then the left diagram becomes the right diagram.

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\mu"'] & N \\ Q & \end{tikzcd} \qquad\qquad \begin{tikzcd} (p_1, \ldots, p_m) \arrow[r] \arrow[d] & (p_1, \ldots, p_q, \ldots, p_r, 0, \ldots, 0) \\ (p_1, \ldots, p_q) & \end{tikzcd}$$

Proposition C.2.1. Let $\mu : (M, m^0) \to (Q, q^0)$ be a smooth submersion and let $F : (M, m^0) \to (N, n^0)$ be a smooth map with constant rank r on some neighborhood of m^0, where M, N, and Q are manifolds of dimensions d_M, d_N, and d, respectively. Let (W_0, ρ) be a smooth chart on Q centered at q^0 and let $X \leq \mathrm{T}_{m^0} M$ be a vector subspace such that $\mathrm{T}_{m^0} \mu\big|_X : X \to \mathrm{T}_{q^0} Q$ is bijective and $\mathrm{T}_{m^0} F\big|_X : X \to \mathrm{T}_{n^0} N$ is injective. There exist smooth charts (U, φ) and (V, ψ) centered at m^0 and n^0, respectively, with $\mu(U) \subseteq W_0$, $F(U) \subseteq V$, and such that

$$\psi \circ F\big|_U = \left(\rho \circ \mu\big|_U, \psi^{d+1} \circ F\big|_U, \ldots, \psi^r \circ F\big|_U, 0, \ldots, 0\right)$$

and

$$\begin{aligned} \varphi &= \left(\mathrm{Pr}_{\leq r} \circ \psi \circ F\big|_U,\ \varphi^r, \ldots, \varphi^{d_M}\right) \\ &= \left(\rho \circ \mu\big|_U,\ \psi^{d+1} \circ F\big|_U, \ldots, \psi^r \circ F\big|_U,\ \varphi^r, \ldots, \varphi^{d_M}\right) \end{aligned}$$

i.e μ and F have, respectively, the coordinate representations

$$(p_1, \ldots, p_{d_M}) \mapsto (p_1, \ldots, p_d) \quad \text{and} \quad (p_1, \ldots, p_{d_M}) \mapsto (p_1, \ldots, p_d, \ldots, p_r, 0, \ldots, 0)$$

and where in addition both (U, φ) and (V, ψ) can be chosen to be coordinate boxes with the property whenever a point $(p_1, \ldots, p_r, \ldots, p_{\dim N})$ belongs to V then the point $(p_1, \ldots, p_d, \ldots, p_r, 0, \ldots, 0)$ is the preimage under F of some point in U. Furthermore, if with $W_\sigma \in \mathrm{Open}\,(Q)$ and $\sigma : W_\sigma \to M$ is some smooth local section of μ through m^0 such that $\mathrm{Im}\, \mathrm{T}_{q^0} \sigma = X$ then we can also arrange it so $\mu(U) \subseteq W_\sigma$, $\sigma(\mu(U)) \subseteq U$, and so that in coordinates, $F \circ \sigma\big|_{\pi(U)} : \pi(U) \to V$

is the canonical zero section of (V, ψ).

Proof. Our assumptions allow us to construct a smooth local section σ with the above stated properties if one was not given. The result now follows from corollary C.1.9 where, if necessary, we may assume without loss of generality that F is a submersion at m^0 by applying lemma C.1.7. ■

Lifts of Curves and Monotonicity

The following lemma's primary purpose is to prove proposition C.3.3.

Lemma C.3.1. Let $c \in [0,1]$, $k \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$, $\mu : M \to N$ be a smooth surjective submersion between smooth manifolds, and let $\eta : [0,1] \to N$, $\gamma^L : [0,c] \to M$, and $\gamma^R : [c,1] \to M$ all be C^k-maps with γ^L (resp. γ^R) a π-lift of $\eta\big|_{[0,c]}$ (resp. $\eta\big|_{[c,1]}$). Let $n^c = \eta(c)$, $m^L = \gamma^L(c)$, $m^R = \gamma^R(c)$, and suppose $m^L \neq m^R$ and that $\xi : ([0,1], 0, 1) \to (\mu^{-1}(n^c), m^L, m^R)$ is a smooth arc. Then for any $\epsilon_0 > 0$ there exists $0 < \epsilon < \epsilon_0$ and a C^k-path $\gamma : ([0,1], 0, 1) \to (M, \gamma^L(0), \gamma^R(1))$ that is a μ-lift of η such that $\gamma = \gamma^L$ on $[0, c-\epsilon]$ and $\gamma = \gamma^R$ on $[c+\epsilon, 1]$ (where $[s,r] \underset{\text{def}}{=} \varnothing$ for $r < s$).

Proof. Assume without loss of generality that $\epsilon_0 < \frac{1}{4}$ and let (V, ψ) be a smooth chart on N centered at n^c. For each $t \in [0,1]$, pick a smooth chart (U_t, φ_t) on M centered at $\xi(t)$ such that $\overline{\mu(U_t)} \subseteq V$ and $\mu\big|_{U_t}$'s representation in the charts (U_t, φ_t) and (V, ψ) is the canonical projection where by shrinking U_t we may also assume that $\operatorname{Im} \varphi_t$ is a cube and (since ξ is an arc) that $\xi^{-1}(U_t)$ is a connected subset of $[0,1]$.

Pick $R \in \mathbb{Z}^{\geq 0}$ and $0 \leq p_0 < \cdots < p_R \leq 1$ such that $U_{p_0}, \ldots, U_{p_R}$ is a cover $\operatorname{Im} \xi$. If any U_{p_j} contains $\operatorname{Im} \xi$ then the result is immediate so assume otherwise, which forces $R \geq 1$. Since $R \in \mathbb{N}$, $\{\xi^{-1}(U_{p_0}), \ldots, \xi^{-1}(U_{p_R})\}$ covers $[0,1]$, and each $\xi^{-1}\big(U_{p_j}\big)$ is a connected sub-interval of $[0,1]$ we can pick $L \in \mathbb{N}$ and $p_{k_1}, \ldots, p_{k_L} \in \{p_0, \ldots, p_R\}$ so that $U_{p_{k_1}}, \ldots, U_{p_{k_L}}$ forms a simple chain from 0 to 1. For all $i = 0, \ldots, L$, let us now denote $U_{p_{k_i}}$ (resp. $\varphi_{p_{k_i}}$) by U_i (resp. φ_i) so that $U_0, \ldots, U_L$ forms a simple chain from 0 to 1 (recall this means that $0 \in U_0$, $1 \in U_L$, and $U_i \cap U_j \neq \varnothing \iff |i-j| \leq 1$) and let $O_i \in \operatorname{Open}(\mathbb{R}^{\dim N})$ and $W_i \in \operatorname{Open}(\mathbb{R}^{\dim M - \dim N})$ denote the sets such that $\operatorname{Im} \varphi_i = O_i \times W_i$. Pick any $0 < \epsilon < \frac{1}{4} \min\{\epsilon_0, 1-\epsilon_0, |c-\epsilon_0|\}$ such that after we let $a \underset{\text{def}}{=} \max\{0, c-\epsilon_0\}$ and $b \underset{\text{def}}{=} \min\{1, c+\epsilon_0\}$, we'll have $\eta([a,b]) \subseteq \bigcap\limits_{i=0}^{L} \mu(U_i)$, $\gamma^L([a,c]) \subseteq U_0$, and $\gamma^R([c,b]) \subseteq U_L$.

Write $\varphi_0 \circ \gamma^L = (\gamma_0^L, \gamma_1^L)$ with γ_1^L valued in W_0 and observe that we may assume without losing generality that all k derivatives of γ_1^L at a are $\mathbf{0}$: let $a \in J \subseteq \operatorname{Dom} \gamma_1^L$ be a closed interval and let $\beta : (J, a) \to (\mathbb{R}, 1)$ be a smooth function all of whose derivatives (of order ≥ 1) vanish at a such that $\operatorname{Im}(\beta \cdot \gamma_1^L) \subseteq W_i$ and it is 1 outside of a neighborhood of a, chosen sufficiently small so that upon replacing $\gamma^L\big|_J$ by $\varphi_0^{-1} \circ (\gamma_0^L, \beta \cdot \gamma_1^L)$, the result is a C^k map. Similarly, if we write $\varphi_L \circ \gamma^R = (\gamma_0^R, \gamma_1^R)$ with γ_1^R valued in W_L then we may assume that all k derivatives of γ_1^R at b are $\mathbf{0}$. Let $s_0 = 0, s_{L+1} = 1$, and pick $0 < s_1 < \cdots < s_L < 1$ such that for all $i = 1, \ldots, L$, $\xi(s_i) \in U_{i-1} \cap U_i$. Let $\sigma : [a,b] \to [0,1]$ be a smooth increasing homeomorphism such that $\sigma^{(p)}(r_i) = 0$ for all $p \in \mathbb{N}$ and all $i = 0, \ldots, L+1$, where $r_i \underset{\text{def}}{=} \sigma^{-1}(s_i)$ (where note that $r_0 = a$ and $r_{L+1} = b$). Observe that the connectivity of $\xi^{-1}(U_i)$ implies that $\xi([s_i, s_{i+1}]) \subseteq U_i$ for all $i = 0, \ldots, L$ so that we may write $\varphi_i \circ \xi\big|_{[s_i, s_{i+1}]} = \left(\mathbf{0}, \widehat{\xi_i}\right)$ for some smooth $\widehat{\xi_i} : [s_i, s_{i+1}] \to W_i$.

For each $i = 0, \ldots, L$, let $\gamma_i : [r_i, r_{i+1}] \to U_i$ be defined by $\varphi_i \circ \gamma_i = \left(\psi \circ \eta\big|_{[r_i, r_{i+1}]}, \ \widehat{\xi_i} \circ \sigma\big|_{[r_i, r_{i+1}]}\right)$. Observe that $\gamma_0(a) = \gamma^L(a)$, $\gamma_{L+1}(b) = \gamma^R(b)$, and for all $i = 0, \ldots, L$, $\gamma_i(r_{i+1}) = \gamma_{i+1}(r_i)$ so that we may define the continuous path $\gamma : [0,1] \to M$ by $\gamma \underset{\text{def}}{=} \gamma^L\big|_{[0,a]} \cup \gamma_0 \cup \cdots \cup \gamma_L \cup \gamma^R\big|_{[b,1]}$. For each $i = 0, \ldots, L$, write $\varphi_i \circ \gamma = (\gamma_i^<, \gamma_i^>)$ with $\gamma_i^>$ valued in W_i and observe that $\gamma_i^< = \psi \circ \eta\big|_{[r_i, r_{i+1}]}$ and that all k of $\gamma_i^>$'s derivatives exist and are equal to $\mathbf{0}$ at both r_i and r_{i+1}. Thus γ is the desired a C^k μ-lift of η. ∎

Corollary C.3.2. Let $\mu : M \to N$ be a smooth submersion between smooth manifolds, $k \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$, $\eta : [0,1] \to N$ be a C^k path, and $m^i \in \mu^{-1}(\eta(i))$ $(i = 0,1)$. Then there exists a C^k μ-lift $\gamma : ([0,1], 0, 1) \to (M, m^0, m^1)$ of $\eta \iff$ there exists some $c \in [0,1]$ and C^k maps $\gamma^L : ([0,c], 0) \to (M, m^0)$ and $\gamma^R : ([c,1], 1) \to (M, m^1)$ with γ^L (resp. γ^R) a π-lift of $\eta\big|_{[0,c]}$ (resp. $\eta\big|_{[c,1]}$) such that $\gamma^L(c)$ and $\gamma^R(c)$ belong to the same connected component of

$\mu^{-1}(\eta(c))$.

The following result, for which the author could not find a reference (but that is likely already known due to the simplicity of its statement), is a strengthening of the well-known fact that monotone (def. A.0.6) smooth surjective submersions are 1-fibrations and it can be easily proved by inductively applying lemma C.3.1.

Proposition C.3.3. Let $\mu : M \to N$ be a smooth surjective submersion between manifolds and let $k \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$. Then μ is monotone $\iff$ μ has the extension lifting property from $\{0,1\}$ to $[0,1]$ (def. 1.1.13) for C^k-paths (i.e. for any C^k-path $\eta : [0,1] \to N$ and any $m^i \in \mu^{-1}(\eta(i))$ $(i = 0,1)$ there exists a C^k μ-lift $\gamma : ([0,1], 0, 1) \to (M, m^0, m^1)$ of η).

Corollary C.3.4. Let $\mu : M \to N$ be a monotone smooth surjective submersion between manifolds, $k \in \mathbb{Z}^{\geq 0} \cup \{\infty\}$, $\eta :]a,b[\to N$ a C^k-curve, $(t_l)_{l=1}^{\infty}$ a strictly decreasing sequence in $]a,b[$ converging to a, and for all $l \in \mathbb{N}$ let $m^l \in \mu^{-1}(\eta(t_l))$. Then there exists a C^k μ-lift $\gamma :]a,b[\to M$ of η such that $\gamma(t_l) = m^l$ for all $l \in \mathbb{N}$.

Proof. Inductively apply proposition C.3.3. ∎

Partial Replacement of Lifts

Definition C.4.1. Let M and N be manifolds, $C \subseteq M$ be any subset, and let $F : C \to N$ be a map. Recall that F is said to be *smooth at* $c \in C$ if there exists a *smooth local extension of* F *around* c, that is, a smooth map $F_c : U_c \to N$ such that U_c is a neighborhood of c in M and $F_c = F$ on $U_c \cap C$.

The following theorem is almost certainly already known and is stated here for later reference. It can be proven by first considering the case where M is compact and then using the fact that μ is a smooth submersion to construct the smooth μ-lift in the obvious way. The general case follows by using an exhaustion of M by relatively compact neighborhoods and induction.

Theorem C.4.2. Let M, N, and Q be smooth manifolds, let $\mu : N \to Q$ be a smooth submersion, and let $f : M \to Q$ be a smooth map. Let $D \subseteq M$ and suppose that $F : M \to N$ is a continuous μ-lift of f such that $F\big|_D : D \to N$ is smooth. Then f has a smooth μ-lift $G : M \to N$ of f extending $F\big|_D$.

Corollary C.4.3. Let M, N, and Q be smooth manifolds and let $\mu : N \to Q$ be a smooth submersion. If a smooth map $f : M \to Q$ has a continuous μ-lift then it has a smooth μ-lift.

Miscellaneous Lemmata

Lemma C.5.1. Let M, N, and Q be smooth manifolds and let F, f, and μ be smooth maps making the following diagram commute:

$$\begin{tikzcd} & N \arrow[d, "\mu"] \\ M \arrow[ur, "F"] \arrow[r, "f"'] & Q \end{tikzcd}$$

where in addition μ is a smooth submersion. If f is a smooth embedding then

(1) $F : M \to N$ is a smooth embedding,

(2) $\mu\big|_{\operatorname{Im} F} : \operatorname{Im} F \to Q$ is a smooth embedding onto $\operatorname{Im} F$,

(3) if f is also proper then so are $F : M \to N$, $\mu\big|_{\operatorname{Im} F} : \operatorname{Im} F \to Q$, and $\mu\big|_{\operatorname{Im} F} : \operatorname{Im} F \to \operatorname{Im} f$.

Proof. Let $S = \operatorname{Im} F$, $R = \operatorname{Im} f$, and $e = \mu\big|_S : S \to Q$. If $F(m) = F(\hat{m})$ then $f(m) = \mu(F(m)) = \mu(F(\hat{m})) = f(\hat{m})$ so that $m = \hat{m}$. Similarly e is injective. If $U \in \operatorname{Open}(M)$ then $F(U) = S \cap \mu^{-1}(f(U))$ so the continuity of $\mu\big|_S$ implies that $F : M \to S$ is an open map and thus a homeomorphism. If $V \in \operatorname{Open}(N)$ then $e(V) = \mu(S \cap V)$, which equals $f(F^{-1}(S \cap V))$ so that $e(V)$ is open in $\operatorname{Im} e = R$ and thus $e : S \to R$ is a homeomorphism. Since $\mu \circ F = f$ we have for all $m \in M$ that $\mathrm{T}_s \mu \circ \mathrm{T}_m F = \mathrm{T}_m f$, where $s = F(m)$, so that the injectivity of $\mathrm{T}_m f$ implies that $\mathrm{T}_m F$ is injective and thus $S = \operatorname{Im} F$ is an embedded submanifold of N.

Since $\operatorname{Im}(\mathrm{T}_m F) \subseteq \mathrm{T}_{F(m)} S$ and $\mathrm{T}_{F(m)} \mu \circ \mathrm{T}_m F = \mathrm{T}_m f$ it is clear that $\mathrm{T}_s : \mathrm{T}_s S \to \mathrm{T}_{f(m)} R$ has full rank and thus $e : S \to R$ is a diffeomorphism. Suppose that f is a proper map so that in particular, $R = \operatorname{Im} f$ is closed in Q. Let $K \subseteq N$ be compact so that $\mu(K)$, and hence $\mu(K) \cap R$, is compact. So $F^{-1}(K) = f^{-1}(\mu(K) \cap R)$ is compact and thus F is a proper map, which implies that $\operatorname{Im} F = S$ is closed in N. That $e = \mu\big|_S : S \to Q$ and $\mu\big|_S : S \to Q$ are proper maps is now apparent. ■

The following lemma is included for completeness.

Lemma C.5.2. Let $d \leq e$ be non-negative integers and let M and N be the closed unit balls centered at the origin in $\mathbb{R}^d$ and $\mathbb{R}^e$, respectively and let $\mathrm{Pr} : N \to M$ be the canonical projection of the first d coordinates of N. Suppose that P is a manifold of dimension $d = \dim M$ and that both $F : N \to P$ and $G : P \to M$ are smooth surjective submersion. Then $G : P \to M$ is a diffeomorphism and $F : N \to P$ maps the interior of N onto the interior of P.

Proof. Since N is compact and $F : N \to P$ is surjective, P is compact. By Ehresmann's fibration theorem it follows that both $\mathrm{Pr} : N \to M$ and $G : P \to M$ are locally trivial fibre bundles and since M is contractible they are globally trivial. Let $E = G^{-1}(\mathbf{0})$ and $D = \mathrm{Pr}^{-1}(\mathbf{0})$ be the typical fibers of G and Pr over the origin, where D is diffeomorphic to the closed unit ball centered at the origin in $\mathbb{R}^{d-e}$. By using global trivializations we may, respectively, replace N and P with $N \cong M \times D$ and $P \cong M \times E$ to get the commutative diagram

$$\begin{tikzcd} N \cong M \times D \arrow[r, "F"] \arrow[dr, "\mathrm{Pr}"'] & M \times E \cong P \arrow[d, "G"] \\ & M \end{tikzcd}$$

where if $F = (F_1, F_2)$ then H, Pr, and $F_1 : M \times D \to M$ are the canonical projections onto the first coordinate.

Since $G : M \times E \to M$ is a smooth submersion between manifolds of the same dimension it is a local diffeomorphism and since it is also proper it is a smooth covering map. This implies that, for all $m \in M$, the fiber $G^{-1}(m) = \{m\} \times E \cong E$ is compact and discrete and hence finite. Since $D \cong \mathrm{Pr}^{-1}(m) = F^{-1}(G^{-1}(m)) = F^{-1}(\{m\} \times E)$ and since points in E are open and closed we have that D has as many connected components as E has points, which must be 1 since D is connected. Thus for all $m \in M$, $G^{-1}(m) = \{m\} \times E$ is a singleton set which says exactly that G is injective, from which it follows that G is a diffeomorphism. Since $\mathrm{Pr} : N \to M$ maps the interior of N onto the interior of M and since interiors of manifolds are diffeomorphism invariant it follows that F maps the interior of N onto the interior of P. ∎

Lemma C.5.3. Let $\mu : M \to N$ be a smooth submersion and assume that either $\dim M = \dim N$ or otherwise that $\dim N \geq 1$. If $S \subseteq N$ is a set of measure 0 in N then so is $\mu^{-1}(S)$.

Proof. Let $R = \mu^{-1}(S)$ and note that may assume without loss of generality that μ is surjective. Let $d_M = \dim M$ and let $d_N = \dim N$. Let $(U^l)_{l=1}^{\infty}$ be a locally finite open cover of R by relatively compact open sets where for each $l \in \mathbb{N}$ there exist coordinates containing $\overline{U^l}$ such that the coordinate representation of $\mu\big|_{\overline{U^l}}$ is the canonical projection from a compact cube onto a compact cube. Since $R = \bigcup_{l \in \mathbb{N}} (R \cap U^l)$ to show that R has measure 0 it suffices to show that each $U^l \cap R$ has measure 0 in M. Since being a set of measure 0 in a manifold is diffeomorphism invariant we may assume without loss of generality that R is contained in a compact ball in $\mathbb{R}^{d_M}$, S is contained in a compact coordinate ball in $\mathbb{R}^{d_N}$ and that μ is the canonical projection. If $d_M = d_N$ then we're done so assume that $d_N \geq 1$ and define $c = d_M - d_N$. Writing $M = N \times C$ where $C \subseteq \mathbb{R}^c$ we have by definition $R = S \times C$ but then since $\dim N \geq 1$ we have that R has measure 0 by definition of the product metric on $N \times C$. ∎

Definition C.5.4 ([27, p. 17])**.** Let X be a LCTVS, $x \in X$, and $x^{\bullet}$ be a sequence in X. Say that $x^{\bullet}$ *converge fast to* x if for all $k \in \mathbb{N}$, $(n^k(x^n - x))_{n=1}^{\infty}$ is (von-Neumann) bounded in X.

Remark C.5.5. Observe that if X is metrizable, $|\cdot| : X \to \mathbb{R}$ is a pseudonorm on X such that $d(x, y) \underset{\text{def}}{=} |x - y|$ forms a metric that is compatible with X's topology then the above condition is equivalent to requiring that for each $k \in \mathbb{N}$ there exist some $L \in \mathbb{N}$ such that $n \geq L \implies d(x^n, x) < \frac{1}{n^k}$. It is straightforward to see that smooth maps send fast converging sequences to fast convergent sequences so that in particular, whether or not a given sequence of points converges fast to some given point is diffeomorphism invariant. Observe that if $X^\bullet$ is a sequence of points in $X = \mathbb{R}^d$ converging to x then $x^\bullet$ converges fast to $x \iff x_i^\bullet$ converge fast to x_i for each $i = 1, \ldots, d$, where $x = (x_1, \ldots, x_d)$ and each $x^l = (x_1^l, \ldots, x_d^l)$. Finally, by a standard argument involving taking the diagonal of an appropriately constructed sequence of sequences, it can shown that every convergent sequence in a metrizable TVS has a fast converging subsequence.

The following example is taken from [27, p. 18] and is repeated here for the sake of making the additional observations found in remark C.5.7.

Example C.5.6. Let X be a Hausdorff LCTVS, $x \in X$, $(x^n)_{n=1}^\infty$ be a sequence in X fast converging to x, and let $\varphi : \mathbb{R} \to [0,1]$ be a smooth map such that $\varphi = 0$ on $]-\infty, 0]$ and $\varphi = 1$ on $[1, \infty[$. Then

$$\begin{aligned} \gamma : \mathbb{R} &\longrightarrow X \\ t &\longmapsto \begin{cases} x & \text{if } t \leq 0 \\ x^{n+1} + \varphi\left(\frac{t - \frac{1}{n+1}}{\frac{1}{n} - \frac{1}{n+1}}\right)\left(x^n - x^{n+1}\right) & \text{if } \frac{1}{n+1} \leq t \leq \frac{1}{n} \\ x^1 & \text{if } t \geq 1 \end{cases} \end{aligned}$$

is a smooth map such that $\gamma(0) = x$ and for all $k, n \in \mathbb{N}$, $\gamma^{(k)}\left(\frac{1}{n}\right) = \mathbf{0} = \gamma^{(k)}(0)$ and $\gamma\left(\frac{1}{n}\right) = x^n$.

Remark C.5.7. Suppose that the above construction of [27] we had picked φ to be such that $\varphi^{-1}(0) =]-\infty, 0]$, $\varphi^{-1}(1) = [1, \infty[$, $\varphi' \neq 0$ on $]0, 1[$. Then whenever $x^n \neq x^{n+1}$, γ would be a topological embedding on $\left[\frac{1}{n+1}, \frac{1}{n}\right]$ with non-vanishing derivative on $\left]\frac{1}{n+1}, \frac{1}{n}\right[$. In particular, if $X = \mathbb{R}^n$ $(n \neq 0)$ and if any coordinate of $(x^n)_{n=1}^\infty$ were to be strictly monotone then $\gamma\big|_{[0,1]}$ is a homeomorphism such that $\gamma(0) = x$, $\gamma\left(\frac{1}{n}\right) = x^n$ for all $n \in \mathbb{N}$, and for all $t \in [0,1]$ the following are equivalent:

(1) $\gamma'(t) = \mathbf{0}$,

(2) $\gamma^{(k)}(t) = \mathbf{0}$ for all $k \in \mathbb{N}$,

(3) $t = 0, 1$ or $t = \frac{1}{n}$ for some $n \in \mathbb{N}$.

So if $X = \mathbb{R}$, $x^\bullet$ is monotone convergent to x, $\lambda : \mathbb{N} \to \mathbb{N}$ is a strictly increasing map such that $x^{\lambda(\bullet)} = \left(x^{\lambda(n)}\right)_{n=1}^{\infty}$ converges rapidly to x, and if we let J denote the closed interval formed by x and $x^{\lambda(1)}$, then we have thus constructed a smooth homeomorphism $\gamma : [0, 1] \to J$ such that $\gamma\left(\frac{1}{n}\right) = x^{\lambda(n)}$ and the above equivalence holds when $x^\bullet$ is replaced by $x^{\lambda(\bullet)}$.