CALCULUS AND COMMUTATIVITY:

AN INVESTIGATION OF STUDENT THINKING REGARDING THE SEQUENCING OF MATHEMATICAL PROCESSES IN CALCULUS

by

LAKEN MICHELLE TOP

Associate of Arts, Science Option
Yakima Valley Community College, 2007

Bachelor of Science, Chemistry and Mathematics
University of Idaho, 2009

Master of Science, Chemistry
Massachusetts Institute of Technology, 2012

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written by Laken Michelle Top

has been approved for the Department of Mathematics

____

Dr. Eric Stade
Professor of Mathematics
Thesis Supervisor

____

Dr. David Grant
Professor of Mathematics

____

Dr. Katherine Stange
Assistant Professor of Mathematics

Date_______________

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

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ABSTRACT

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Calculus and Commutativity: An Investigation of Student Thinking Regarding the Sequencing of Mathematical Processes in Calculus

Thesis directed by Professor Eric Stade

Previous research has demonstrated that students enrolled in calculus courses continue to struggle with old algebra ideas along with new calculus-specific concepts. In this study we seek to investigate student thinking around the sequencing of mathematical processes (SMP) by examining student work in a one-semester university calculus course. Our results indicate that students enrolled at the calculus level continue to struggle with algebraic order of operations in addition to making new mistakes with respect to the commutativity of differentiation and other operations; these errors are often evidenced in student failure to apply the product, quotient, or chain rules. We examine these SMP mistakes through the theoretical lenses of APOS theory, conceptual and procedural thinking, and structural and operational thinking. In addition, we look at two student sense-making strategies - using demarcating symbols and naming differentiation rules - to explore whether these are correlated with student propensity to make SMP-type errors.

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CHAPTER 1

INTRODUCTION

"Does it matter which step I do first?" This is a student question that could be heard in a college mathematics classroom with nearly the same frequency as in an elementary mathematics classroom. The order in which mathematical operations are performed is truly a consideration that spans all levels of mathematics. In middle school, this concern most often surfaces in problems where students must decide whether to prioritize working inside parentheses, multiplying, dividing, adding, or subtracting. For this reason, the order of operations in mathematics is often associated with the PEMDAS mnemonic which is intended to help students remember the correct order for simplifying an expression: Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction. The importance of sequencing mathematical processes extends far beyond this short list of operations however. For high school and undergraduate students, the situation becomes even more complicated since performing geometric transformations, composing and otherwise manipulating functions, finding limits, derivatives, and integrals, etc. all require some understanding of whether the processes involved commute with one another. Specifically, in calculus, the non-commutativity of differentiation with other operations such as multiplication and division may cause issues for students. In addition to learning how to sequence these new processes that they are encountering, students at higher levels also need to retain their understanding of the order of basic operations as well since mathematics is a subject that continues to build on itself.

Because of their omnipresence in every level of mathematics, sequencing issues deserve more attention in classrooms and in educational research. Since order of operations is so frequently tied to PEMDAS, we introduce a new term 'sequencing of mathematical processes' or

SMP to refer to all situations where the order in which mathematical operations are performed may affect the result. We hypothesize that student difficulties with the sequencing of mathematical processes in earlier mathematics courses will continue to present themselves as students advance through the math curriculum. We also hypothesize that more advanced students may have a better understanding of the mathematical procedures involving multiple operations, but may still struggle when asked more abstract, conceptual questions about the commutativity of such processes. Additionally, it may be the case that errors resulting from reasoning misconceptions for younger students may persist as procedural errors for more advanced students. These conjectures led us to construct this study with the purpose of examining student thinking about SMP for students enrolled in a one-semester calculus course.

Research Questions

Specifically, in this study we seek to investigate the following research questions in the context of a one-semester undergraduate calculus course:

- How do problems with the sequencing of mathematical processes manifest themselves across levels of mathematics? What is the relationship between typical student issues with the order of mathematical operations at lower levels of math (e.g. Pre-Algebra) and those that appear in more advanced classes (e.g. Calculus)?
- Are there any apparent differences between students' conceptual and procedural understandings of the sequencing of mathematical processes?
- How often are student errors in calculus related to sequencing mathematical processes and using the order of operations?
- Which student errors stem from misconceptions in reasoning, and which stem from carelessness or lack of procedural knowledge?

Limitations of Study

Due to the lack of prior research on the broader category of SMP, this study is intended to be an initial foray into the field. As such, there are several limitations of which to be cautious and aware.

First, the data set used in this study is limited. The primary sources of data analyzed come from student work submitted as part of a one-semester calculus course. Most of the assessments analyzed were not specifically designed for this study, but were simply intended to evaluate student understanding of the course topics in general. Thus, it must be acknowledged that some of the questions could have been better worded in order to elicit student thinking that would better illuminate their understanding of SMP. Furthermore, the data was collected across two semesters and the types, timing, and wording of assessments changed over that period. Thus, results are not directly comparable across the two semesters. However, since the intention of this study was to provide an overview of students' conceptions and use of SMP, this comparison is not necessary at this stage.

Second, the data analysis reported in this thesis was conducted by a single researcher. Certainly, cross-checking with other members of the research team would benefit the reliability of the data analysis. The consistency of results found by triangulating the multiple data sources analyzed, however, is also reassuring of the validity of this work.

Third, this study took place in a calculus course that was intentionally designed for students majoring in the life sciences, with specific emphasis on material applicable to their chosen career trajectories and a de-emphasis on some other traditional calculus topics. Thus the results may not reflect the work of students enrolled in a more typical multi-semester calculus course.

Students were allowed to self-select into the study for both semesters of the course in which student work was examined. As a result, there may be some selection bias in terms of participants. It seems reasonable to assume, however, that the population consists of a mix of high-achieving, ambitious students, and struggling students in need of the extra credit offered, thus averaging out to typical classroom representation.

Finally, it must be emphasized that this work is preliminary and naturally incomplete. Certainly, more research is needed to further investigate where and how SMP appears in calculus contexts and how it intersects with student learning of calculus concepts. Our hope in this study is simply to establish a warrant for research in this area and provide motivation for further investigation.

Organization of Thesis

This thesis is organized into six chapters. This first chapter is intended to introduce the language and definition of SMP. Chapter 2 provides a brief literature review of prior educational research on order of operations as well as student learning in calculus. Additionally, the second chapter presents three theories of mathematical learning that will be used as a conceptual framework for subsequent analysis. Chapter 3 describes our methodology and the data used for this study. Chapter 4 presents the results and findings while Chapter 5 provides a discussion intended to consider the findings in more depth and link them back to the theoretical perspectives described in the literature review. Finally, Chapter 6 is a conclusion and offers suggestions for future research.

CHAPTER 2

LITERATURE REVIEW

Because SMP is an extension of the more commonly used term "order of operations" we review research conducted in that field in this chapter. Additionally, since our focus is on order of operations in calculus, we take this occasion to offer a brief review of research concerning student learning and misconceptions in calculus. Third, we summarize three theoretical perspectives on mathematical learning that have been used to understand student learning with respect to the order of operations, with the goal of using these perspectives as a lens through which to examine our own findings in future chapters. Finally, we provide a more in-depth look at some studies that are closely related to our own, with the intent of distinguishing our study and illustrating how it will contribute to the existing research in the field. While we acknowledge that this literature review is not exhaustive, we do feel that it provides a good overview of the types of prior studies and findings that have been published.

Order of Operations Research

Throughout the literature exists a general acknowledgement that the order of operations is important at all levels of math and a better understanding of these concepts is important for student success throughout their mathematics careers (Schrock & Morrow, 1993). Furthermore, beyond its importance, students also struggle with order of operations at all levels (Stephens, 2016). Despite these claims, relatively little research has been done on the order of operations in math education, particularly within higher levels of mathematics. The types of studies that do exist primarily fall into two categories: student misconceptions about order of operations or more practitioner-focused methods for teaching order of operations.

Student Learning and Misconceptions

Research on student learning with respect to the order of operations often focuses on middle to high school students enrolled in pre-algebra or algebra classes. This is likely due to the emphasis of this branch of research on the PEMDAS acronym. PEMDAS is frequently used to introduce students to the correct order of elementary math operations and stands for Parentheses, Exponents, Multiplication, Division, Addition, and Subtraction. Students are often taught to remember this mnemonic by reciting the phrase, "Please Excuse My Dear Aunt Sally" or some other variation.

Many of the studies on student learning emphasize that PEMDAS is often treated literally in instruction, leading to student errors; interpreting PEMDAS literally causes students to think that multiplication always precedes division or similarly that addition always comes before subtraction (Dupree, 2016). In addition, students (and teachers) routinely struggle with interpreting parentheses, failing to see them as a symbol of grouping rather than an operation that must always be performed first (Dupree, 2016). PEMDAS has also been shown to encourage left-to-right thinking, that is, that operations must be performed from left to right in an expression (Ameis, 2011; Dupree, 2016).

Beyond issues with PEMDAS, research has demonstrated that students also struggle with interpretation of negatives in the context of the order of operations (Booth et al., 2014). Authors have suggested that this may have to do with children's difficulties in comprehending and conceptualizing a number less than zero (perhaps due to their lack of exposure with negative integers in real life situations) or that students may find it challenging to understand the concept of an additive inverse (Fuadiah, Suryadi, and Turmudi, 2017). Furthermore, students demonstrate difficulty in dealing with the combination of exponents and negative signs, often struggling to understand when the negative sign is “sticky” or included in the exponent operation (as in $(-2)^2$ but not in -2^2) or when the exponent itself is negative (Pitta-Pantazi, Christou, and Zachariades, 2007; Cangelosi et al., 2013). Similar findings have been produced in a developmental math study in which student difficulties with negative signs, exponents, fractions, and parentheses were highlighted (Titus, 2010).

Teaching Methods

Order of operations studies directed toward a practitioner audience also highlight PEMDAS, but primarily discuss the pros and cons of the method or propose alternatives to PEMDAS. Lee and Messner (2000) highlight interpretation difficulties and a lack of curricular resources dealing with concatenations in expressions; they point out that calculators are not even standardized in their simplification of certain expressions without grouping symbols and also suggest that adding steps to PEMDAS may increase the potential of clarity. They are not alone in their concern about these types of problems as others have also called for further support for difficulties with the lack of grouping symbols amongst pre-service teachers (Glidden, 2008). Glidden (2008) also points out that the literal interpretation of PEMDAS is not simply a student misconception, but that pre-service teachers even have issues with order (e.g. multiplication before division becomes prioritized over left-to-right simplification). These teacher difficulties are especially important because teacher understanding (or misunderstanding) can be taken up by students in ways that are very hard to correct (Papadopoulos, 2015).

A few solutions to these challenges have been proposed. For example, Schrock and Morrow (1993) suggested that using calculators, mnemonics (such as PEMDAS) and increased awareness among students and teachers may improve student execution of the order of operations. Others have suggested that an emphasis on teaching division and subtraction as multiplicative and additive inverses, respectively, may improve comprehension (Dupree, 2016). Another alternative approach to PEMDAS is a "hierarchy of operators" triangle in which visuals are used to indicate which operators are prioritized thus helping to dismantle the "rule" of left-to-right thinking (Ameis, 2011). More recently, it has been suggested that technology could be useful in order of operations instruction; Stephens (2016) proposes that equation editors may help with the awareness of the importance of the order of operations since most editors require slow and careful input of terms.

Misconceptions in Calculus

In general, as illustrated in the previous section, order of operations research has not expanded to include a broader definition of SMP and still focuses on PEMDAS-type operations in algebra. Research exists that examines misconceptions in higher levels of math such as calculus, but these studies tend focus on other areas of student understanding. In a brief literature review of calculus-specific education research, Sabella and Redish (2003) note that many students lack a conceptual understanding of calculus topics, and are rarely asked to engage in deeper thinking and more challenging problems, prompting the 'calculus reform' movement. The authors provide a concise overview of research on student conceptual difficulties with four broad calculus topics, namely functions and variables, limits and continuity, derivatives, and integrals (Sabella and Redish, 2003). More recently, Rasmussen, Marrongelle, and Borba (2014) provided an update on the state of calculus education research noting that studies have concentrated on four main objectives: identifying students' conceptual difficulties, investigating learning processes, studying classrooms (specifically classroom interventions intended to improve student learning), and understanding teachers' ideas and methods. It is not our goal in this section to recreate these prior literature reviews; rather, we present a brief overview of studies that seem to most closely align with our goals in studying how SMP manifests itself in calculus.

Any review of the calculus education research literature would be remiss to neglect the two seminal papers published by Orton. One article (Orton, 1983a) was among the first to investigate student learning with respect to differentiation, and the second study (Orton, 1983b) remains among the few research articles to discuss student understanding of integration. In these papers, Orton analyzes student work on a set of problems covering a wide variety of differentiation and integration topics and subsequently categorizes student errors as *structural*, *executive*, and *arbitrary*. These error categories were based on the work of Donaldson (1963) and can briefly be described as students' failures to follow correct procedures (structural), students' inabilities to understand the rule or concept necessary to solve the problem (executive) and students' tendencies to ignore part of the problem (arbitrary). In both his integration and differentiation studies, Orton (1983a, 1983b) found that the majority of student difficulties were structural, though a significant number of executive mistakes were made as well. Later authors have attempted similar error classification studies, often providing a broad overview of student errors and not focusing in on any specific topic (e.g. Muzangwa and Chifamba, 2012). Other studies narrow in on student learning with respect to a particular calculus concept such as limits (Williams, 1991), Riemann sums (Sealey, 2014), or the chain rule (Kabael, 2010) to name a few.

Only a few studies seem to directly address SMP in calculus either as a focus concept itself or as a lens through which to analyze student understanding of other concepts. One such study that does specifically allude to the order of operations as it presents itself in calculus is Musgrave, Hatfield, and Thompson's (2015) paper on how students interpret differences in calculus. They found that students had a broad range of definitions for the word 'difference' and also identified differences in a variety of ways when asked to label them in a given expression. The authors point out that how students related definitions, symbols, and structure are directly related to the order of operations and leave this as an area for future research (Musgrave, Hatfield, and Thompson, 2015). Another study investigated student difficulties with respect to negative signs and exponents, analyzing work from students enrolled in college algebra, pre-calculus, first-semester calculus, and second-semester calculus (Cangelosi et al., 2013). The authors claim that one of their main findings was that students struggle with the idea of inverses (both additive and multiplicative) across all levels of mathematics, and they call for further research in this area. Note that both Cangelosi et al. (2013) and Musgrave, Hatfield, and Thompson (2015) restrict their definition of the order of operations to mean the operations named in PEMDAS; research that expands on that definition is surprisingly difficult to find.

Theoretical Perspectives on Mathematical Learning

A number of theoretical perspectives have grown in popularity due to their usefulness in making sense about how students develop in mathematical understanding. Many of these perspectives are evolutions and more modern interpretations of Donaldson's (1963) error classifications used in Orton's (1983a, 1983b) widely cited calculus studies. Three perspectives in particular have been used prominently in order of operations analyses, namely APOS theory, conceptual and procedural thinking, and structural and operational thinking. We describe these three theories in more detail below.

APOS Theory

The first perspective that we will consider classifies student sense-making as taking place through actions, processes, objects, and schemas, and thus is aptly named APOS theory (Dubinsky & McDonald, 2001). The origins of this theory started with Piaget who studied children’s learning development as they aged. Piaget claimed that children learned mathematics by beginning with concrete actions (for example, moving pebbles) and gradually interiorizing those actions, constantly building on previous understanding as they progressed (Piaget, 1975). He coined the term “reflective abstraction” to describe his theory, which he claimed was a “reorganization of mental activity, as it reconstructs at a higher level everything that was drawn from the coordinates of actions” (Piaget, 1975, p. 7). Others have built on Piaget’s ideas to develop APOS theory in which students participate in sense-making through reflecting on and interiorizing actions, processes, objects, and schemas.

Czarnocha et al. (1999) describe actions as student reactions to external instructions, that is, when students are able to carry out specific steps that were detailed for them. Process, on the other hand, requires students to reflect on this action and to begin to internalize and take control of it. Object represents a further stage of development in which a student has begun to recognize a construct as an entity which can be acted upon by transformations. Finally, students begin to build schemas which link processes, objects, and actions in a coherent and structured way (Czarnocha et al., 1999). The authors describe these levels of development in the context of cosets. An action level understanding of cosets might involve students understanding how to construct a coset given a starting element and a process for constructing other elements such as “begin with 2 and add 4” (Czarnocha et al., 1999, p. 99). At the process level students may begin to think about constructing a coset by using a single element to operate on other elements. Students can make another conceptual leap by beginning to think of cosets as objects (e.g. a left coset) that were formed through specific processes and have particular properties (e.g. cardinality). Finally, thinking about operations between sets or developing an organization of algebraic structures in which groups, rings etc. are located indicates that students are at a schema level of sense-making.

APOS theory has been used as a tool to understand student's sense-making in a number of calculus contexts including graphical interpretations of derivatives (Asiala et al., 1997), related rates (Tziritas, 2011), the chain rule (Jojo, Maharaj, and Brijlall, 2013) and integration (Maharaj, 2014).

Conceptual and Procedural Thinking

A second perspective that has been employed to understand student misconceptions at all levels of mathematics is that of *conceptual* and *procedural* thinking. This framework is laid out by Tall et al. (2001) where they describe mathematics as a land of procedures and concepts. Symbols that we use often stand for both; for example, addition and sum can both be denoted with a $+$ sign and differentiation and derivative are often represented by $\frac{dy}{dx}$ or other symbolic variations. Although often identified by the same symbol, concepts and procedures also remain distinct. Counting is definitively a procedure, while number is a concept. In some ways, the concept is the result of performing the process (addition leads to a sum, counting results in a number) and in fact, Gray and Tall (1994) formed the portmanteau 'procept' to describe this duality that symbols can represent. Within this perspective, students begin at the procedure level where they are simply doing specific steps. Increasing sophistication and development are indicated by moving on to process, where students are beginning to grow in flexibility and efficiency when performing procedures, and finally to procept, where students demonstrate symbolic understanding (Tall et al., 2001).

Structural and Operational Thinking

The third and final perspective that we consider here is structural and operational thinking, which shares many similarities with the conceptual and procedural lens described above. In this perspective, abstract concepts can be thought of *structurally* (as objects) and *operationally* (as processes) (Sfard, 1991). Typically, operational thinking comes first and development happens through interiorization, condensation, and reification. According to Sfard (1991), interiorization is the stage in which students begin to develop the ability to mentally envision what would happen when they performed a process without actually having to carry out the action. At the same time, the student is becoming more proficient at performing the process as well. Condensation happens when students become more comfortable combining and comparing processes, while in reification students begin to think of the result of the process as an object in and of itself. The main difficulty for students in moving through this process is due to the fact that concepts often have multiple representations, but there is no way to actually visualize the concepts concretely. Structural-level understanding that develops in the reification stage requires students to view mathematical things as real objects that exist which is extremely challenging. To illustrate this, Sfard (1991) considers the example of a function. Structurally, one might think of a function as a set of ordered pairs, while operationally a function is a way of transforming (or mapping) an input to an output. Functions are represented in multiple ways, however, such as through algebraic expressions, graphs, or algorithms. This creates confusion when students are trying to actually comprehend what a function is and trying to see it as an object because each representation captures a piece of the structural understanding, but none fully encompass the concept.

This perspective shares many similarities with the conceptual and procedural thinking framework laid out by Tall et al. (2001). Indeed, Sfard (1991) uses the same counting versus number example that Tall et al. (2001) describe as procedure versus concept to demonstrate differences in young children's thinking and the progression from operational to structural. Sfard (1991) however, emphasizes that the seemingly disparate ways of understanding mathematical concepts as operational and structural actually complement one another and are inseparable; she highlights this as a distinguishing factor in operational/structural theory, as opposed to the conceptual/procedural theories described previously. Sfard (1991) also strives to combine epistemology and ontology in developing the theory and calls attention to the ways that structure and operation interact with math representations, psychology and concept development, and cognitive processes.

Similar Studies

In the final section of this literature review, we note that there are two studies that closely model our research questions, and thus deserve some careful explication. Both studies that we highlight in this section are trying to understand the order of operations in the context of higher education; the first study bears methodological similarities to our own study and the second shares content and theoretical framing aspects.

Order of Operations and Business Students

In their study, Pappanastos, Hall, and Honan (2002) ask if business students understand the order of operations. The researchers distributed a survey instrument consisting of pre-algebra and algebra-level problems. They then compare student responses across the number of years the student had been in college (e.g. freshmen versus sophomores versus juniors). The authors found that college-level business students struggled with the same things as in prior middle and high school studies, namely exponents, parentheses, negatives, and left-to-right order of operations. This study is methodologically similar to our own, since we also distributed assessments and analyzed student responses for order of operations issues (to be described in detail in Chapter 3). We viewed Pappanastos, Hall, and Honan's (2002) study as a foundation to build upon as it exemplified that students in higher education also struggle with order of operations and provided guidelines for the types of errors that we may see arise with our own population of interest. One of the main distinguishing factors that sets our study apart from this one is that we did not ask specific algebra questions, but rather looked at where these issues naturally arose in the context of calculus problems (see our methodology described in Chapter 3).

The Chain Rule and Natural Science Students

The second study that we highlight here used an APOS perspective to look at students majoring in the natural sciences and their understanding of derivatives, particularly the chain rule (Maharaj, 2013). The author suggests some initial benchmarks for examples of how student understanding of derivatives might appear at different APOS levels. For example, Maharaj (2013) proposes that a student with an action level understanding may be able to perform a simple power rule derivative when given an exact expression for the original function, whereas a student with a process orientation might be able to combine a number of steps, such as first simplifying the original expression and then applying a differentiation rule. In the object phase, students may be able to recognize a composition of two functions, enabling them to identify the two functions making up the expression in order to apply the chain rule to the problem. According to Maharaj (2013) the schema phase involves understanding multiple aspects or characteristics of the function and knowing how to find them; he suggests finding maxima and minima of a function which requires making connections between differentiation rules, critical points, and where the derivative is positive or negative. After establishing this framework, Maharaj (2013) analyzes six questions that were distributed to students as an assessment. One major finding that he reports is the difficulty that students have with applying the chain rule for differentiation, and he implies that greater instructional emphasis on the object conception of function composition may be helpful in remedying this struggle, but he points to the need for further research on this subject. Our study is similar to Maharaj's (2013) study in many ways. First, our populations are similar, since many of the students participating in our study are also life (or natural) science majors. Second, our assessments are also designed to be a set of standard calculus problems. Finally, we also hope to analyze student responses through the APOS perspective. The main factor that distinguishes our study from that of Maharaj (2013) is the SMP lens that we also hope to apply to our analysis; it is possible that some of the errors that were presented in his research could also be attributed to difficulties understanding SMP.

We feel that our study design begins to fill a void in the world of mathematics education research as it seeks to merge calculus studies with order of operations studies. In particular, we seek to expand our definition of the order of operations to the sequencing of mathematical processes in order to ask how order and commutativity manifest themselves in the context of "new-to-students" calculus operations. As a result, our current study has a slightly broader scope than most prior order of operations studies. We also find that our focus is different than typical calculus misconceptions studies because we are not looking at student understanding of a particular calculus concept, but rather we seek to understand how SMP appears in calculus and where students struggle with those concepts including and beyond the typical order of operations. In other words, our study is less about exploring student understanding of what a derivative is and more about investigation student understanding of how other operations commute with differentiation.

CHAPTER 3

METHODOLOGY

Data Set

The data used in this study consisted of exams, homework assignments, and quizzes collected from students enrolled in MATH 1310 at the University of Colorado Boulder (CU Boulder) during the Fall 2017 and Spring 2018 semesters. MATH 1310, officially titled *Calculus, Systems, and Modeling*, is a one-semester 5-credit hour calculus course that is aimed at students intending to major in the life sciences. As such, it addresses conventional calculus topics such as differentiation and integration methods, but primarily develops these concepts in the context of biological applications. A full syllabus is included in the Appendix for the curious reader. There were three sections of the course taught in each of the Fall 2017 and Spring 2018 semesters, with relatively equal enrollment across the sections. Each section had a different instructor, with the exception that one instructor taught a section in both semesters. The course coordinator remained the same across the two semesters. Additionally, all sections took common exams and were assigned the same homework, quizzes, and tutorials.

The initial semester, Fall 2017, was primarily intended as a pilot investigation, and only one exam was collected for analysis that term. The data for that semester was requested retroactively per Institutional Review Board standards and blinded before research commenced; in total, 96 students enrolled in MATH 1310 in Fall 2017 elected to participate in the study. The exam analyzed was the second of three midterms that were given over the course of the semester and assessed concepts such as basic differentiation rules (constant multiple, sum, product, chain, and quotient rules), maxima, minima, and inflection points, local linearity and the Microscope Equation, and application problems involving related rates and exponential growth. The full exam is included in the Appendix. This exam was chosen for analysis since it was administered about halfway through the Fall 2017 semester, and thus provided the opportunity to investigate what types of problems students were struggling with, what strategies they were using, and how SMP-related ideas were being exhibited as students progressed through the course.

Table 1. Summary of data analyzed.

Assessment Name	**Semester Given**	**Description**	**N**
Second Exam	Fall 2017	1.5 - hour second midterm assessing: • Basic differentiation rules • Extrema, concavity and inflection points • Differentiability • Related rates • Exponential growth	96
Pop Quiz	Spring 2018	Quiz given during class assessing: • Combining differentiation rules to find a given function's derivative with or without the requirement to name the rules used*	46
Homework	Spring 2018	Take-home assignment assessing: • Ability to write and evaluate correctness of differentiation rules using formal mathematical notation • Ability to write and evaluate correctness of differentiation rules using mathematical language	62
Third Exam	Spring 2018	1.5 - hour third midterm assessing: • Derivatives involving $arctan$ • Exponential decay • Definite integrals • Riemann sums • Distance and velocity applications • Fundamental Theorem of Calculus	52
Final Exam	Spring 2018	2.5 - hour final assessing: • Differentiation rules • Definite and indefinite integrals • Hypothesis testing and confidence intervals • Riemann sums • Tangent lines	56

* Two versions of the Spring 2018 pop quiz were created as described in the body of the text.

The remaining data was collected over the course of the Spring 2018 semester. Students were asked during the first month of the course if they would be willing to allow analysis of their class assignments. In total, 63 of the 92 students enrolled in the course agreed to participate. Note that the number of papers analyzed for a given assignment may not equal 63 since not all students submitted every assignment. The exact number of students whose work was analyzed for each individual assignment is indicated in Table 1. The assignments examined from the Spring 2018 semester consisted of one homework assignment specifically designed to assess SMP-related issues, one midterm exam, one "pop quiz" developed on the basis of preliminary results, and the final exam. These assignments were collected at various points throughout the semester, starting in Week 6. Table 1 describes each of the assignments in more detail, but the full assessments can also be seen in the Appendix. Notably, two versions of the pop quiz were created. The first version was given to students in two sections of the course, and simply asked them to find the derivative of the given function. The second version was distributed to the other section and asked students to find the same derivative, but included additional instructions asking them to name the differentiation rules they used and how many times they used each of them in the process. The rationale for this choice is described in more detail in the analysis section below. Both versions of the quiz are included in the Appendix.

Data Analysis

Phase 1

The data analysis occurred in two phases. First, the midterm collected during the Fall 2017 semester was analyzed with the goal of uncovering preliminary results and informing the wording and types of questions asked on Spring 2018 assessments. Three questions (one multi-part) from Fall 2017 were selected for analysis, namely Question 1 (parts a – d), Question 2, and

Question 4; these are listed in Table 2 in Chapter 4 for reference. These questions were chosen because of their potential to highlight students' understanding of SMP, and all three problems required students to find derivatives using the sum rule, constant multiple rule, chain rule, product rule, quotient rule, or a combination of them. Other questions on the Fall 2017 midterm were left out of the study because they presented problems for a clear analysis of students' abilities to correctly sequence mathematical processes. Some of the questions were word problems which required students to translate the language into mathematical symbols to set up the problem; we felt that difficulties in this process may have overshadowed the most relevant process-sequencing components to the problem since not all students may have started with the same mathematical expression. Other questions involved graphing or fill-in-the-blank responses and we felt that it was too difficult to discern student thought processes in their solutions to these problems without having a think aloud session. Thus we restricted our analysis to the questions on the exam that asked students to find the derivative of a given function expressed in standard mathematical notation.

In analyzing student responses to this midterm, we carefully combed through each of the problems selected, and recorded every type of mistake that students made or noted whether students correctly completed the entire problem. After listing each mistake, we grouped them into larger categories which we called *SMP*, *miscopy*, *forgotten rule or step*, and *misinterpretation of math symbol* respectively. Every mistake found fell under one of these categories and all of the categories emerged inductively from the data, with the exception of the *SMP* category. Under the *SMP* heading we gathered any student mistake that resulted from students performing mathematical processes out of order, or using notation that implied a different sequencing than the correct response. For example, students often placed parentheses in the wrong places in their mathematical expressions or left parentheses out altogether but performed the subsequent steps correctly – we categorized these mistakes as students having an incorrect understanding of SMP. As the name implies, *miscopy* was coded whenever a student rewrote the problem incorrectly or did not carry or incorrectly copied a term from a previous line in their work. The third category, *forgotten rule or step*, was coded whenever students did not perform a necessary operation in the problem (e.g. forgot to take the derivative of a term) or when they incorrectly remembered something generally regarded as a fact (e.g. wrote that the derivative of $\cos(x)$ was $\sin(x)$ instead of $-\sin(x)$). Finally, the category *misinterpretation of math symbol* was reserved for student mistakes that indicated that they did not understand mathematical notation; for example, students often interpreted exponentials incorrectly assuming that expressions like $\sqrt{x^5}$ could be rewritten as $x^{1/5}$. In addition to noting student mistakes on each problem, we also made note of some student strategies used to make sense of the problem; examples include *demarcation* and *rule identification* where students drew on their pages to help them group and label components of the problem or explicitly named the relevant rule or formula that could be used to solve the problem. These mistakes and strategies are described in more detail in the findings in Chapter 4.

Phase 2

Given our interest in SMP, we used the information we collected from the Fall 2017 data to construct problems on a variety of Spring 2018 assessments that allowed us to further investigate student understanding of SMP in calculus. In particular, we formulated problems that required some students to name the differentiation rules that they applied when taking the derivative of a given function, while some students were not required to list the rules by name. This allowed us to analyze if rule-naming allowed students to better connect the conceptual and procedural elements of the problem and ultimately helped them derive the correct result. In addition, we asked questions about the differentiation rules in words and using formal mathematical notation to see if students were making conceptual and procedural connections when thinking about the formulas they had been given. We also designed exams to include differentiation and integration problems that were conducive to further SMP analysis.

After collecting the new set of assessments from students, we again analyzed them and coded the mistakes that students made. In this second phase, however, we primarily used deductive codes that were found to be sub-categories of our *SMP* code in Phase 1. In particular, we focused on students' propensity to assume commutativity of differentiation with other operations such as multiplication, division, and function composition. For example, we looked at how frequently students tended to equate the derivative of the product of two functions with the product of the derivatives of the two functions; we named this code *product/differentiation commutativity*. Similarly, we noted when students made this error with the quotient rule and the chain rule and called these codes *quotient/differentiation commutativity* and *chain/differentiation commutativity*. We also continued to look at the strategies that students employed when solving differentiation or integration problems which were noticed in the first phase: *demarcation* and *rule identification*. Our analysis also involved cross-tabulation between strategies and these *SMP* errors to see if certain strategies seemed to lead to fewer student mistakes. Finally, we made notes of any new SMP-related errors that students made when solving the Phase 2 problems.

CHAPTER 4

FINDINGS

Preliminary Phase 1 Results

In our Phase 1 analysis, we looked only at the second exam given to all MATH 1310 sections in Fall 2017. As indicated in the methodology, we looked only at student responses to Question 1 (parts a – d), Question 2, and Question 4 from this exam. These problems are shown in Table 2 for reference.

Table 2. Problems from the Fall 2017 Second Exam used for the Phase 1 analysis.

Number	Question
1a	$$\frac{d}{dx}[x^2 - 3\cos(x) + 2]$$
1b	$$\frac{d}{dx}\left[\sqrt[3]{x} + \frac{1}{x^3} + \frac{\sqrt{x^5}}{x}\right]$$
1c	$\frac{d}{dx}[e^{\ln(3x^2-\cos(x)+5)} \cdot \ln(e^{\tan(x)})$ Hint: simplify first, *then* differentiate.
1d	$$\frac{d}{dx}\left[\frac{x^2 + 3^{\ln(x)}}{\sin(x)}\right]$$
2	Suppose that $$f(x) = (g(x) - 1)^3 + 2$$ Find $f'(3)$, given that $g(3) = 2$ and $g'(3) = -1$.
4	Find $$\frac{d}{dx}[3\cos(\arctan(x)\, e^{\sin(x)})].$$ At the end, please list *all* the rules you used, and *how many times you used each.* (The possible choices for rules are: Constant Multiple, Sum, Chain, Product, or Quotient.)

We listed each of the mistakes made by students on these questions and found that they could be grouped into four major categories: *SMP*, *miscopy*, *forgotten rule or step*, and *misinterpretation of math symbol*. We also coded student responses if they were *completely correct* for a reference point. The breakdown of student mistakes by category for each of the questions is shown in Table 3. It is important to note that the frequency of the codes does not necessarily represent the number of students who made a mistake in a given category. This is because each student was coded once for each unique sub-category mistake that they made, which could result in a student being coded multiple times for each major category. For example, if a student left out necessary parentheses when solving a problem, and switched the order of terms in the numerator when using the quotient rule, the student would have been coded once for each of these mistakes, resulting in the student being coded twice for *SMP* since each of those errors was a sub-category of *SMP*. The *completely correct* code count does indicate the number of students that correctly solved the problem however, since that category had no sub-codes (either students answered the problem perfectly or not).

Table 3. Frequency of major category codes for student responses to Fall 2017 second exam questions. Student N = 96.

	Exam Questions					
	1a	1b	1c	1d	2	4
Completely correct	79	29	43	22	41	30
Miscopy	4	3	3	1	17	10
Forgotten rule or step	7	23	4	33	11	35
Misinterpretation of math symbol	--	33	3	10	1	--
SMP	6	23	49	43	16	34

As indicated in Table 3, there was a definite disparity in the relative difficulty of each of the exam problems. As expected, Problem 1a was relatively straightforward, as it only required students to remember the sum rule, the power rule, the constant multiple rule, and the derivative of $\cos x$; in our experience, these rules are not difficult for students to remember. As a result, 79 out of 96 or 82% of students answered Problem 1a perfectly. The remaining problems presented more of a challenge, however, as less than half of the students answered each of the remaining problems correctly.

Miscopy

The *miscopy* code also had no sub-codes since it was used only when students rewrote the problem incorrectly or did not carry a term or sign from one line to the next when working through the problem. These errors happened relatively infrequently, though student responses to Question 2 did seem to demonstrate more issues in this regard. Most of these mistakes on Question 2 were due to the use of $f(x)$ and $f'(x)$ notation; students seemed to forget to write the prime symbol to indicate the derivative when moving from line to line. Additionally, students forgot to carry through the coefficient that resulted from taking the derivative even though they initially took the derivative correctly. Overall, errors resulting from miscopying work seem minimal and they do not present much opportunity for intervention from the instructor. Certainly, students can be encouraged to double-check and keep their work tidy to make sure that they do not write things down incorrectly, but these errors are not a huge cause for concern from our perspective since they do not indicate any deep misunderstanding of the material.

Forgotten Rule or Step

Most of the *forgotten rule or step* errors resulted from students forgetting to take the derivative of at least one term in the problem, or forgetting how to take the derivative. On Question 1a, all of the student errors in this category were from students who forgot that the derivative of $\cos x$ is $-\sin x$, not $+\sin x$. On Question 1b, students often made mistakes with the power rule (likely due to the presence of fractional exponents), or spent so much time simplifying a term in the expression that they ultimately forgot to take the derivative of the term. Forgetting to take the derivative of a term also occurred frequently on Question 1c and 1d.

Question 1d also indicated that students frequently forget to include the denominator when performing the quotient rule, and that students often do not remember the rule for derivatives of the form a^x where a is a number and x is a variable. Finally, on Questions 2 and 4, students often forgot the chain rule, taking only the derivative of the outside function and neglecting to multiply by the derivative of the inside. Question 4 also asked students to list which differentiation rules they applied to the problem and how many times they used each of them; this also led to a number of *forgotten rule or step* errors as students often did not list the constant multiple rule or incorrectly counted the number of times they used the simpler rules.

Misinterpretation of Math Symbols

This preliminary analysis also indicated that students struggle with making sense of certain mathematical symbols, particularly exponents. Students made no mistakes in interpretation on Questions 1a or 4, but Question 1b proved especially challenging for students in this regard. This was overwhelmingly due to the combination of exponents and roots found in the problem. One-third of the students struggled with interpreting $\sqrt{x^5}$ and $\frac{1}{x^3}$ correctly, often converting them to $x^{1/5}$ and $x^{-1/3}$ respectively. Interestingly, students did not struggle as much with $\sqrt[3]{x}$, as most correctly rewrote this as $x^{1/3}$. Nonetheless, it is clear that students' comfort levels are low when converting between root symbols and exponents, particularly when dealing with fractional exponents. Student misinterpretations of symbols on Questions 1c, 1d, and 2 were more scarce and scattered, with the largest issues involving the $3^{\ln(x)}$ term in Question 1d. We did notice that most of these interpretation struggles did not involve symbols introduced in calculus; rather, the types of expressions that students had difficulties with (e.g. exponents) should have been discussed more deeply in prior classes. This is not surprising, as we have often heard it said that students' greatest struggle in calculus is algebra. This data simply confirms the truth in this phrase.

Sequencing of Mathematical Processes

The main category of interest for us in this study is *SMP*. As indicated in Table 3, a large proportion of the errors that students made on differentiation problems were SMP related. The vast majority of these *SMP* errors fell into two sub-categories: *parentheses* or *operation/differentiation commutativity*. All sub-categories and the distribution of student errors on the problems analyzed are indicated in Table 4. Note that in Table 4, since the numbers represent sub-codes of *SMP*, students were only coded at most once for each error. That is, even if a student misused parentheses multiple times in their solution to a problem, they were only coded once for *parentheses*. Thus, the numbers in Table 4 represent the number of students who made the mistake on each problem.

Table 4. Frequency of *SMP* sub-codes for student responses to Fall 2017 second exam questions. Student N = 96.

	Exam Questions					
	1a	1b	1c	1d	2	4
Operation/differentiation commutativity	1	14	20	8	8	18
Parentheses	3	3	29	24	8	16
Cancellation	--	3	--	3	--	--
Distributing or factoring	2	--	--	3	--	--
Addition or subtraction	--	3	--	5	--	--

Table 4 clearly indicates the prominence of incorrect commutativity assumptions between differentiation and other operations, namely, multiplication, division, and composition of functions. The table also shows the difficulties that students had in working with parentheses. We analyze these sub-codes in more detail in the following sections.

Other *SMP* sub-codes included *cancellation*, *distributing or factoring*, and *addition or subtraction* as displayed in the table. Errors in student responses were coded as *cancellation* if students incorrectly assumed that terms cancelled; this usually occurred in quotient settings. For example, several students equated expressions of the form $\frac{a-b}{a^2}$ to $\frac{1-b}{a}$. We viewed this as an *SMP* error because students divided the a in the numerator by the a^2 in the denominator before subtracting the b in the numerator. *Distributing or factoring* errors occurred in multiplication settings, such as factoring expressions like $(x + ay)$ and rewriting them as $a(x + y)$ or distributing the denominator of a single term to multiple terms in an expression (without appropriately modifying the numerators). This was also a sub-code of *SMP* because students assumed that multiplication by y by a and then adding x was the same as adding x and then multiplying by a. Finally, *addition or subtraction* errors were coded when students reversed the order of terms being subtracted (as if they followed addition rules). The most common example of this error was students reversing the order of the terms in the numerator after differentiation using the quotient rule. These last three categories only occurred on a couple of problems and were relatively infrequent; at most 5% of students made such an error on any given problem. Thus we concentrate our attention on the first two *SMP* sub-codes.

Parentheses and SMP. Student work demonstrated that students often understood the correct order in which to perform operations, but they did not recognize the importance of parentheses in signaling that order. In essence, they were following PEMDAS with assumed parentheses. Students would incorrectly write an expression in the form $a + b \cdot x$ when it should have been written as $(a + b) \cdot x$. However, in subsequent lines of work, students would treat the expression as if it had been written in the latter form. An example of this type of student mistake is shown in Figure 1. Instructor marks can be seen in red in Figure 1, and denote places where the student neglected to use the appropriate parentheses. Despite the fact that the student left the parentheses out of their third line of work, their subsequent derivative expression indicates that they operated as if the parentheses were there. This type of mistake was extremely common among students, especially on Questions 1c, 1d, and 4 (see Table 4). Parenthetical errors also occurred when multiplying single terms with negative signs; on Question 1a, three students wrote $3 - \sin x$ instead of $3(-\sin x)$ but proceeded with the problem as if they had used the latter notation.

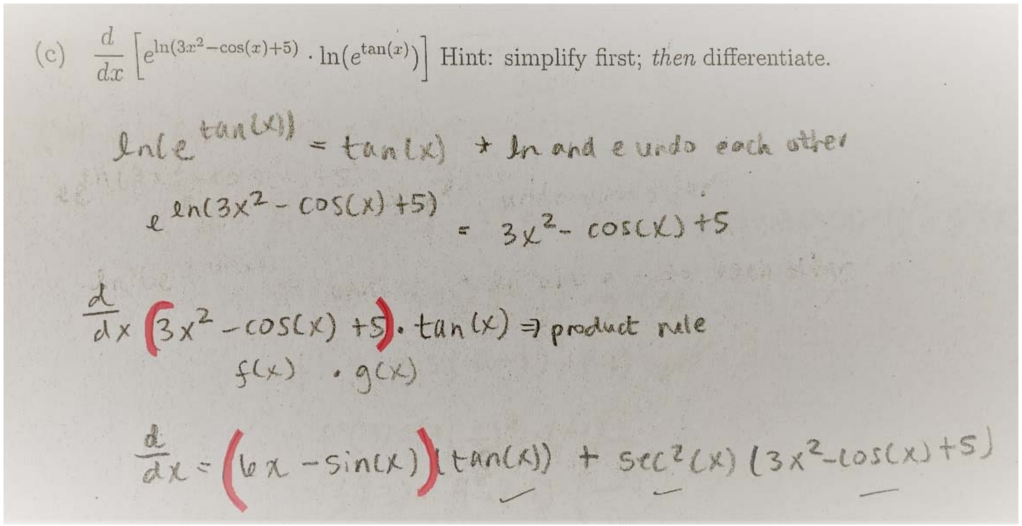

Figure 1. Example of student work on Fall 2017 Second Exam Question 1c demonstrating forgotten parentheses. Instructor marks are noted in red.

This error seems to indicate that students do understand something about the order of operations, but do not understand the significance of proper notation in signaling the correct sequencing.

Operation/differentiation commutativity. The other widespread *SMP* error discovered through our Phase 1 analysis was that of students assuming that differentiation could commute with other operations. There were three main examples of this that we uncovered; *product/differentiation commutativity*, *quotient/differentiation commutativity*, *chain/differentiation commutativity*. We named the codes as such because these errors corresponded to students' lack of use of the product, quotient, and chain differentiation rules respectively. As might be expected, *product/differentiation commutativity* codes indicated that students used multiplication and differentiation as commuting operations, that is, they assumed that the derivative of a product of functions was equivalent to the product of the derivatives of the functions. Similarly, *quotient/differentiation commutativity* was coded where students indicated that the derivative of a quotient of functions was the quotient of the derivatives of the functions, and *chain/differentiation commutativity* was coded where students equated the derivative of composed functions as the composition of the derivatives of the functions. As shown in Table 4, these errors were made in all problems analyzed, but were particularly prevalent in Questions 1b, 1c, and 4. Given proper simplification of terms, Question 1b did not actually require the use of the product, chain, or quotient rule, but most students were unable to (or chose not to) simplify the $\frac{\sqrt{x^5}}{x}$ term, instead opting to apply the product or quotient rule. This resulted in four students making a *product/differentiation commutativity* error, and 10 students making a *quotient/differentiation commutativity* error. On Question 1c, 19 out of the 20 *operation/differentiation commutativity* errors were made when using the product rule, while one of the errors was made when using the chain rule (note that the problem involved using both the product rule and the chain rule). Question 4 was a significantly more complicated problem, requiring the use of multiple differentiation rules. On this problem, students made 12 *product/differentiation commutativity* errors, and six *chain/differentiation commutativity errors*. The number of chain rule errors may have been larger relative to Question 1c given that Question 4 required two uses of the chain rule and most students missed one of them, likely due to the large and complicated expression that the solution entailed. We did notice however, that only 12 students (13%) made mistakes with the product rule on Question 4 compared to 19 students (20%) on Question 1c despite the fact that Question 4 was a significantly more challenging derivative to compute. One hypothesis that we developed to explain this was that requiring students to name the rules they used (as was done on Question 4) may help students in making conceptual and procedural connections between the rule formulas they used and applying them to actual functions. We explored this hypothesis and all three *operation/differentiation commutativity* sub-codes further in Phase 2.

Phase 2 Results

In Phase 2, we chose to narrow our focus to just the *operation/differentiation commutativity* sub-category of our main *SMP* code. Our primary reason for this restriction of attention was the calculus-specificity of these errors. Although parenthetical errors were also a major component of the Phase 1 analysis that we conducted, most student mistakes involving parentheses are an extension of past mistakes and rules learned in previous classes; rarely, if at all, were the parentheses errors from Phase 1 unique to calculus. The commutativity (or lack thereof) of differentiation with other operations, however, is a concept that students do not encounter until they reach calculus. Since one of our goals in conducting this study was to propose ideas for interventions for students struggling in calculus, it made sense to devote our attention to these distinct calculus misconceptions.

As noted in Chapter 3, we used four different assessments from the Spring 2018 semester of MATH 1310 at CU Boulder for this Phase 2 analysis. We approached the study of operation and differentiation commutativity from five different angles. First, we looked at students' propensities to make these types of SMP errors when differentiating products of functions, quotients of functions, or compositions of functions, to see if students were more inclined to make the mistake when encountering one type of problem over another. Then we looked at two strategies that students employed to make sense of those types of problems - rule-naming and demarcation – to study how effective these tactics were in improving students' ability to correctly sequence mathematical processes. We also looked at students' relative tendencies to make commutativity errors when differentiating versus integrating, i.e. we tried to identify whether operation/integration commutativity errors were also common. Finally, we studied students' likelihood of assuming commutativity between differentiation and other operations when presented with the differentiation rules in words or symbols instead of in the context of a problem. We present the findings from each of these five approaches to analysis in the subsequent sections.

SMP Errors: Products, Quotients, and Chains

For our detailed comparison of problems whose correct solutions involved the product rule, the quotient rule, and the chain rule, we used one problem from the Spring 2018 third exam (Question 6 parts b, c, and d) and one problem from the Spring 2018 final exam (Question 1 parts b, c, and d); these problems are listed in Table 5.

Table 5. Problems from the Spring 2018 Third and Final Exams used for the Phase 2 analysis.

Number	Question
Spring 2018 Third Exam	Find the following derivatives involving the arctangent function.
6b	$\frac{d}{dx}[\ln(x)\arctan(8x)]$
6c	$\frac{d}{dx}[e^{\arctan(x)}]$
6d	$\frac{d}{dx}\left[\frac{\arctan(x)}{x}\right]$
Spring 2018 Final Exam	Find the derivative of each of the following functions. You do not need to simplify your answers.
1b	$h(x) = \frac{7x^6}{\sqrt[3]{x}+2}$
1c	$p(x) = (4x^3+5)\tan(3x)$
1d	$e^{3\ln(5x)}$

Our analysis presented problems, however, since the problems assigned to students on these assessments proved to be relatively easy for them; they seemed to be less challenging problems than the corresponding problems given on the Fall 2017 second exam (Question 1 parts b, c, and d shown earlier in Table 2). In fact, over half of the students correctly solved each of the problems on the Spring 2018 assessments. The most challenging problem proved to be Question 1d on the Spring 2018 final which 55% of the students answered incorrectly; the percentage of students with completely correct answers was higher on the other five problems analyzed from the Spring 2018 exams. This presented problems for our analysis, since the number of students making mistakes was so small to begin with that the number making SMP-related errors was even smaller. Thus, we also included the Fall 2017 second exam data in this portion of the study for comparison. The results are presented in Table 6. Note that these findings are presented as percentages since the number of students in the study differed significantly across the three assessments.

Table 6. Percentage of students who made operation/differentiation commutativity errors on three assessments.

	Product	Quotient	Chain
Fall 2017 Second Exam (N=96)	24%	14%	6%
Spring 2018 Third Exam (N=52)	--	2%	2%
Spring 2018 Final Exam (N=56)	2%	2%	5%

The Fall 2017 second exam data clearly indicate that students tend to have greater difficulty recognizing the need to use the product rule rather than the quotient or chain rule. It must be acknowledged that the Spring 2018 data do not support this claim, but we see a lower number of operation/differentiation commutativity errors on those exams in general. This may be due to the fact that the Spring 2018 exams are assessments given later in the semester, that is, we are comparing a second exam with a third and final exam. It is possible that students enrolled in the Spring 2018 semester made the same number of SMP errors on their second exam, but had simply learned from their mistakes by the time they took the third and final exams. Unfortunately, since we are comparing across different semesters, it is not possible to confirm this hypothesis with this data. An additional problem with the Spring 2018 analysis is that the percentage of students making mistakes on each of the exams that semester are much lower, thus skewing the data slightly. In fact, one out of 10 students who answered the Spring 2018 final exam questions incorrectly, made a *product/differentiation commutativity* error; this number corresponds to roughly 10% of mistakes made on those problems. When controlling for the number of students making mistakes, the quotient rule and chain rule for the same exam accounted for 12% and 6% respectively. This does seem to indicate that the product and quotient rule are more challenging to remember for students, which agrees with the Fall 2017 exam data.

One hypothesis for students' inclination to simply take the product or the quotient of the function derivatives rather than applying the appropriate rule is that these two scenarios present the extremes of delineating the two (or more) functions that are being dealt with in the problem. Two functions are often adjoined with no symbols in between them to denote a product, e.g. $x^2 \tan(x)$. Even if a symbol is used, it is often a subtle " $\cdot$ " sign as in $e^{\ln(3x^2-\cos(x)+5)} \cdot \ln(e^{\tan(x)})$ which was found on the Fall 2017 second exam. It may be that these types of notations are not sending a clear signal to students that a product is present. A third way to represent a product is through the use of parentheses, such as $(4x^3 + 5)\tan(3x)$, but our preliminary data analysis from Phase 1 indicated that students often do not see parentheses as necessary when multiplying and thus they may not be cued to think of the product rule through this type of notation either. On the other hand, the notation typically used for the quotient rule (e.g. $[\frac{x^2+3^{\ln(x)}}{\sin(x)}]$ from Fall 2017 second exam) may provide too much separation between the functions, causing students to see differentiating them almost as separate problems. We will explore this idea of demarcation in greater depth in the next section.

Student Use of Demarcation Symbols

In addition to analyzing student errors across problems, we also looked at student strategies for making sense of the problems they were given. In particular, we found two strategies that emerged inductively from the data, demarcation and rule-naming. For the former, we found that students used a variety of symbols or labels to separate out components of the problem. For example, students would specifically write $f(x)$ and $g(x)$ on their papers to denote the presence of two functions in a product, quotient, or chain rule situation. Students also tended to make use of parentheses, square brackets, or underlining to clarify the different pieces. We analyzed student use of these two methods by examining two questions from the Spring 2018 third and final exams (Question 6 parts b, c, and d, and Question 1 parts b, c, and d, respectively – shown in Table 5). Combining student responses from both exams, we found that when students specifically labelled the individual functions involved in the problem, they were generally more likely to get the problem completely correct than other students. On product rule problems, 80% of students who labeled functions $f(x)$ and $g(x)$ on their papers answered the problem correctly, while only 70% of students who did not label the functions got the solution right. Similarly, for quotient rule problems, 90% of students who labeled the functions correctly found the derivative, while only 64% of students who did not label ultimately answered the problem correctly. This pattern did break down for chain rule problems, however; only 60% of students who labelled solved the full problem correctly, while 69% of students who did not label ended with the right answer. It must be noted, however, that only 5 students labelled chains of functions across both exams, while 20 and 29 students labelled functions on product or quotient problems respectively. The small sample size of students who actually used this sense-making strategy on chain rule problems may skew the data. This strategy did seem effective for the nearly 20% and 30% of students who used it on product or quotient rule problems and this seems to support the hypothesis from the previous section that students who are able to recognize a product or a quotient are more often able to apply the rule correctly. In other words, student difficulties with SMP do not seem to be the result of them forgetting the rule or misapplying the rule, but rather with their ability to connect the need for the rule with the given problem. Labeling the individual functions that make up the problem may be one strategy for helping students make this link.

Rule-Naming Strategy Effectiveness

Another strategy that students used frequently was rule-naming. This strategy was actually required on one of the problems on the Fall 2017 second exam (Question 4, shown in Table 2), and we noticed that students were less likely to make SMP errors on Question 4 than on Question 1c (shown in Table 2) which similarly required the application of the product and chain rules. On Question 4, students were required to list each of the differentiation rules that they had applied and count the number of times they had used each, while on Question 1c no additional instructions were given aside from differentiating the function. Despite Question 4 being more challenging from a mathematical point of view, only 34 students made SMP errors on that problem, while 49 students made SMP errors on Question 1c (Table 3). Out of these SMP errors, 12 students (13%) made the product/differentiation commutativity error on Question 4 while 19 students (20%) made the same error on Question 1c. This led us to postulate that rule-naming was a strategy for helping students make sense of the connections between the conceptual rules and the procedural aspect of taking a derivative.

To test this hypothesis we developed another assessment. We wanted to make sure that the difference in the functions given was not a confounding factor in our previous results. To that end, we gave the Spring 2018 students an in-class quiz and asked two of the sections to simply differentiate the given function while the other section was asked to name the differentiation rules they used and the number of times they used them. The given function was $f(x) = \sec^2(x^4)\ln(x^2 + 1)$; both versions of the quiz can be found in the Appendix. We found that 24% (seven out of 29) of students who were not given the additional instructions made the product/differentiation commutativity error, while only 6% (one out of 17) students who were required to name the rules made the same error. This was yet another indicator to us that rule-naming could be an effective strategy in helping students correctly make conceptual and procedural connections.

SMP Errors: Derivatives v. Integrals

In addition to understanding the differences in students' SMP errors related to product, chain, and quotient derivative problems, we also wanted to see if students had a greater propensity to make SMP errors when dealing with integrals versus derivatives. For this analysis we focused on comparing derivative problems involving products to integral problems involving products, and derivatives of quotients to integrals of quotients. Due to the lack of an appropriate integral "chain" problem, we left function composition out of this examination. The problems analyzed are shown in Table 7.

When coding student responses to these questions for *operation/differentiation commutativity* or *operation/integration commutativity* errors, we found that no students made this type of SMP error on the integrals of products problems (Questions 7a and 7c). These mistakes were not prevalent in the derivative of a product analog either though; only one student out of 56 made a *product/differentiation commutativity* error. The quotient results were similarly uneventful. Only one out of 56 students made commutativity errors on each of the derivative and integral problems (Questions 1b and 7b).

Table 7. Problems analyzed from the Spring 2018 Final Exam for comparison of SMP errors between integrals and derivatives.

Number		Question
Products:		
	1c	Find the derivative of each of the following functions. You do not need to simplify your answers. $$p(x) = (4x^3 + 5)\tan(3x)$$
	7a	Calculate the following definite and indefinite integrals using substitution. Please show your work. In particular, clearly specify what you are calling u, and what is du. $$\int 6t^4(6t^5 + 2)^{11}\,dt$$
	7c	Calculate the following definite and indefinite integrals using substitution. Please show your work. In particular, clearly specify what you are calling u, and what is du. $$\int_{\ln(\frac{\pi}{2})}^{\ln(\pi)} e^x \sin(e^x)\,dx$$
Quotients:		
	1b	Find the derivative of each of the following functions. You do not need to simplify your answers. $$h(x) = \frac{7x^6}{\sqrt[3]{x} + 2}$$
	7b	Calculate the following definite and indefinite integrals using substitution. Please show your work. In particular, clearly specify what you are calling u, and what is du. $$\int \frac{q^2}{q^3 + \ln(3)}\,dq$$

Part of this SMP-error absence may be due to the high number of students who answered these problems correctly, Questions 1b and 1c, as well as Questions 7a and 7c had correct responses rates of 70% or higher. Less than half (approximately 48%) of the students answered Question 7b correctly, but the errors were primarily due to forgetting absolute value signs or forgetting a constant when changing variables; none of these errors were SMP-related. Another reason that the mistakes on the integral problems were limited may have been the specificity of the instructions given. Students were told in the problem to use the substitution method to compute the integrals and our previously reported findings indicate that rule names may be critical in helping students make sense of a problem. The comparison of integrals and derivatives certainly warrants further exploration with better designed problems that would allow us to see student mistakes more clearly.

Students' Conceptual and Procedural Understanding of Differentiation Rules

A final approach that we used to learn about students' understanding of SMP and differentiation was to look at their ability to assess SMP errors in mathematical statements of the differentiation rules. We gave students a take-home worksheet on which they were asked to identify errors in statements of the common differentiation rules when they were written in words or in mathematical notation. Students were first given examples, such as the one seen below:

> "State whether each of the given statements is true or false. If the statement is *true*, identify it by name. If it's *false*, make it *true* by replacing everything that comes after "is equal to" with appropriate verbiage and identify the corrected statement by name.
>
> (i) The derivative of the sum of two functions is equal to the sum of the derivatives of those functions.
>
> **Answer:** True (sum rule).
>
> (ii) The derivative of the product of two functions is equal to the product of the derivatives of those functions.

Answer: False. The derivative of the product of two functions is equal to the first function times the derivative of the second, plus the second function times the derivative of the first (product rule)."

Similarly, students were asked to identify mistakes in the differentiation rules when written using symbols:

"Listed below are eight mathematical statements: **not all of them are true!** Identify each statement as being either a **correct** or an **incorrect** version of a known differentiation rule.

Examples:

(i) $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$ **Answer**: product rule, incorrect version

(ii) $\frac{d}{dx}[cf(x)] = cf'(x)$ **Answer**: constant multiple rule, correct version"

The full worksheet can be seen in the Appendix.

When analyzing student responses to this worksheet, about 50% of the students scored perfectly, correctly identifying all of the correct and incorrect versions of the differentiation rules and modifying them appropriately if necessary. There were a number of inconsistencies in the remaining students' responses however. Many students answered the statements written using mathematical symbols correctly, but incorrectly assessed their counterparts written using words. Three students thought that the statement for the constant multiple rule expressed in words was incorrect because they thought the derivative of a constant multiplied by a function was zero. In effect, this is a product/differentiation commutativity error because students were multiplying the derivative of the constant by the derivative of the function to find the derivative of the constant multiplied by the function. Many students also agreed with the statement that "the derivative of a chain of two functions is equal to the chain of the derivatives of those functions", though they failed to offer any explanation for this belief. Students also agreed with the incorrect version of the quotient rule written in words which reversed the order of the terms in the numerator of the derivative. There seemed to be much confusion around the quotient rule in general, since students often thought that both versions of the quotient rule written in symbols were correct, or that both versions were incorrect when in fact one was expressed correctly and the other expression reversed the order of the terms in the numerator. This indicated that students struggle to recognize the importance of SMP when dealing with the subtraction in the numerator of the quotient rule. Finally, many students interpreted the version of the product rule written as $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$ as incorrect. This was mystifying to us for a while until we noticed one student's response to this question which stated " product rule, correct. (Usually written as $f'(x)g(x) + g'(x)f(x)$, the example given is written differently but produces the same answer)." This student's explanation may be a clue as to why other students struggled with this problem; since they were used to seeing the derivative of the first function being written first in the expression, they automatically assumed that this version of the product rule was incorrect. This is interesting, however, given that students often assumed that the order of the terms in the quotient rule did not matter. It is difficult to determine with our data whether students thought that the product rule expression did not commute because of the addition or because of the multiplication order (both are changed on the worksheet from the student's suggested expression) but it is striking that students struggled with accepting commutativity in the product rule even though they were willing to overlook the term-switching in the quotient rule expressions. These results certainly merit further investigation, perhaps by conducting think-aloud interviews with students so that they are able to fully explain their thought processes. In any case, it is clear that students struggle with translating between different ways of writing the differentiation rules which may be one reason that students also made certain operation/differentiation commutativity errors when asked to apply the rules to find the derivative of given functions.

CHAPTER 5

DISCUSSION

Many of our findings can be connected back to prior foundational research and theory, but analyzing student work through the lens of SMP also allows us to add to the body of knowledge about students' ideas and struggles in calculus. In particular, we found that our findings confirmed many of the errors presented in previous studies, especially those related to misinterpretation of exponents and grouping symbols. We also found that naming rules may help students make procedural and conceptual links. Our findings also demonstrated that students frequently lack flexibility in notation and expression which may be explained through learning theory. Finally, we noticed connections between students' APOS levels with respect to a given concept and their ability to correctly manipulate and sequence mathematical processes when working with that concept. We use this section to elaborate on each of these points and link our own results to prior research.

Confirmation of Errors

Perhaps not surprisingly, our findings confirmed many of the previously published errors and misconceptions that students have when solving procedural math problems. Of these typical errors, two that stood out as exceptionally common were misinterpretation of exponents and misuse or lack of grouping symbols. While our research agrees with previous studies, it is important to note that we found that these mistakes are made by students enrolled at the calculus level and in the context of calculus problems. This differentiates our findings to some extent from prior research which primarily locates these mistakes in the context of algebra problems often directly intended to assess student understanding and manipulation of exponents and parentheses. Though these results are not obviously related to our SMP-related research questions, errors involving exponents and grouping symbols were so prevalent that we felt they merited a brief discussion here.

Misinterpretation errors with exponents

Our results demonstrated that undergraduate students misinterpret exponents in many of the same ways that high school students have been shown to do so. Previous research illustrates that students have particular difficulty in interpreting negative exponents. For many students, their initial understanding of an exponential expression of the form x^n is tied to the procedure of multiplying x by itself n times; this process begins to break down and becomes harder to conceptualize when n is not a natural number (Pitta-Pantazi, Christou, and Zachariades, 2007). The students in our study also struggled with the interpretation of negative exponents. Previous research has suggested that this may be due to lack of understanding about inverses (Cangelosi et al., 2013). We also hypothesize that many of students' misconceptions about exponents may be related to difficulties in combining concepts since they are still working at an operational level of thinking instead of a structural level (as per Sfard, 1991). For example, when exponents and root symbols are combined, as in $\sqrt{x^3}$ or $\sqrt[3]{x}$, students often forget which expression can be rewritten as $x^{1/3}$. In our experience, students view these as rules to remember, and have not yet reached the stage of thinking about square roots or exponents as objects in and of themselves that can be operated on and combined. Students instead think of the square root as requiring them to complete the process of raising something to a fractional exponent, and determining what the exponent might be is a matter of rote memorization.

Importance of grouping symbols

In addition to agreeing with previous findings on exponents, our study also confirmed prior research related to student use of grouping symbols. Cangelosi et al. (2013) found that students often deemed expressions with and without parentheses as equal even if they were not (e.g. $(-2)^2$ and -2^2), and we found that our students also viewed parentheses as non-essential when writing out their solutions. We also discovered that students often completed the correct steps in the right order despite misusing parentheses or leaving them out altogether. Again this is consistent with previous research which found that students often use *personal shorthand notation*, leaving out crucial components of the expression, but somehow remembering and operating as if they were there (Cangelosi et al., 2013). Lee & Messner (2000) point to middle and high school textbooks as lacking explanations about when and why grouping symbols make a difference in evaluating expressions involving negatives and exponents. If this is the case, then it should come as no surprise that students enrolled in calculus courses still struggle to understand the importance and meaning of parentheses.

What's in a Name? Connecting Procedures and Concepts

Another major finding that we presented in Chapter 4 related to students' use of rule names to facilitate finding the derivative correctly. We found that students who were required to (or opted to) label the differentiation rule that they were applying at each step in the problem were less likely to make SMP-related errors. Students who did not name the rules they were using, on the other hand, were more likely to make *operation/differentiation commutativity* errors, assuming for example that the derivative of a product of functions was the product of the derivatives of those functions. We hypothesize that this is because students were not connecting procedures and concepts, but that rule-naming helps them to do so. According to Tall et al. (2001), multiplication is a process while a product is a concept. It may be that students saw the process of two functions being multiplied, but were not making the connection to the concept of a product of functions. As suggested by the theory of conceptual and procedural learning (Tall et al., 2001) students may still have a procedural level of understanding about functions, not yet recognizing them as entities that can be combined to form products, quotients, etc. Since the differentiation rules are usually named after the concept instead of the process (e.g. product rule not multiplication rule), a lack of procedural and conceptual connection could mean that students are not triggered to use the product rule when looking at function multiplication. The same hypothesis could also explain students' *operation/differentiation commutativity* errors when working with other differentiation problems since both the chain rule and quotient rule are named after concepts instead of the processes of function composition and division. We hypothesize that having students explicitly name the rules that they are applying forces them to consider the connections between the concepts and the processes. Rather than seeing two functions multiplied together and simply taking the derivative of each, they must think through their repertoire of rules and realize that a product is the result of two functions being multiplied, and thus this is the correct occasion to apply the product rule. We propose that renaming the differentiation rules to reflect the processes instead (e.g calling the 'product rule' the 'multiplication rule') might make it even easier for students to correctly find the derivative since the name would more closely match their current procedural level of understanding of functions. A consequence of this choice, however, may be that it leaves students "stuck" in this procedural way of thinking and does not encourage further development toward structural understanding. Further research is certainly warranted to determine the full benefits and disadvantages of this approach.

Flexibility in Notation

In our study we also found that students did not always choose the most efficient path to reach an answer, nor did they recognize the equivalence of the same expression written in multiple ways. One prime example of this was on an exam where students were asked to differentiate the term $\frac{\sqrt{x^5}}{x}$ (as part of a larger expression). The vast majority of students identified this as an opportunity to apply the quotient rule; we postulate that this is because the division bar is a prominent symbol that cues students to think of a quotient. This is clearly not an incorrect approach, but it is arguably the least efficient way to take the derivative of the term. The best approach is probably to simplify the expression to $x^{3/2}$ first and then simply apply the power rule, and the second best option may be to recognize that the expression can be rewritten as the product $x^{5/2} \cdot x^{-1}$ and then use the product rule. Students often did not choose either of these more expedient options however. Certainly, students' struggles with interpreting exponents may have played a role in this decision, but we hypothesize that an explanation may go deeper than this. Gray and Tall (1994) discuss that lack of flexibility in notation often indicates that students may not have reached the 'procept' level of understanding yet. This could certainly be the case with the students in our study. As students are still hovering between the procedure and process stages of understanding differentiation rules, they may still be applying a prescription for which rule to use based on what the initial expression looks like rather than thinking about how they could rearrange the expression to use a different approach. This may in part be due to the lack of emphasis on different approaches in their classroom experiences.

Another example of students' lack of flexibility in mathematical expression was shown in our findings on students' identification of correct and incorrect differentiation rule statements. As described in Chapter 4, many students interpreted the version of the product rule written as $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$ as incorrect. One student's explanation next to their correct identification of the rule provided some insight into this misconception since the student pointed out that the product rule was "usually written as $f'(x)g(x) + g'(x)f(x)$" while "the example given is written differently but produces the same answer". While this particular student demonstrated flexibility in their ability to identify equivalent mathematical expressions, their answer may provide a clue as to why other students were unable to make the same connection. Students may be trained to think that in the product rule, the derivative of the first function comes first, and may lack the advanced understanding required to recognize that in this case the order does not matter. This is certainly an SMP-related issue and may explain why even more advanced students continue to struggle with SMP errors. Interpreting mathematical expressions in a flexible way is definitely a prerequisite to understanding when and where order matters in mathematical operations. Gray and Tall's (1994) 'procept' idea offers an explanation for this difficulty, much as it justified students' automatic use of the differentiation rule when seeing a quotient as described above. Sfard's (1991) distinction between structural and operational thinking also provides a framework for understanding this student difficulty, however. The student who correctly identified the product rule and defended their decision illustrated more of a structural level of understanding of functions, recognizing that they are objects that can be combined using various operations. Moreover, these operations sometimes result in objects that are mathematically equivalent, even though they look slightly different. Students who struggle with flexibility in notation and recognizing equivalent expressions may have only an operational understanding of working with functions, viewing the expression as a sequence of operations that must be performed in a certain order, rather than as an object that can be manipulated without changing its outcome.

APOS and SMP

APOS theory as described by Dubinsky and McDonald (2001) also provides a strong justification for student difficulties with SMP. Although the 'P' in APOS stands for process, correct sequencing of mathematical processes often requires students to be at a more advanced level of understanding. It is reasonable to assume that students who are able to think about functions, products, quotients, derivatives, etc. as objects are more likely to understand commutativity (or the lack thereof) between such constructs. Furthermore, understanding the connectivity between combinations of functions, derivatives, and differentiation rules requires students to begin constructing a schema to organize all of these concepts. Maharaj (2013) describes requisite components for students' function and derivative schemas in order for students to truly understand derivative problems encountered in calculus. His theory is based on an initial description of constructing a schema for the chain rule by Clark (1997). Through interviews with students, Clark discovered that for students to understand the chain rule, they needed to construct schemas linking function composition and differentiation. Maharaj generalized this by looking at functions and derivatives in general. He notes that not only do students need to have an action and process understanding of these concepts, but they also need to be able to assess the properties of a function as an object that allow a particular differentiation rule to be applied (such as function composition requiring the chain rule). Beyond that, building the arsenal of differentiation tools and recognizing various function types that cue when and how to use the differentiation rules is really part of generating schema for derivatives and functions and making connections between them (Maharaj, 2013).

Based on this prior research, it should come as no surprise that the students participating in our study struggled to make all of the connections between functions, operations on functions, and differentiation. Students who made *operation/differentiation* commutativity errors often demonstrated action and process level understanding of functions and of derivatives through correct algebraic manipulations and through applications of the more basic differentiation rules (i.e. the power rule, the sum rule, and the constant multiple rule). Where students began to struggle with SMP errors was in contexts where multiple concepts were rolled in to one, such as function composition, powers, and the chain rule, or multiplication, trigonometric functions, and the product rule. As Clark (1997) and Maharaj (2013) point out, these problems require students to combine ideas about each of the differentiation rules, functions, derivatives, and operations on functions, as well as the correct sequencing of processes to truly understand the solution to the problem. The struggles that students had with correctly linking each of these components can be viewed as an indicator that they have not yet progressed to the object and schema stages of APOS. Furthermore, these findings demonstrate a need for more focused research and classroom instruction on the role of SMP in calculus. Although SMP may seem simple like a simple set of rules to remember, looking through the lens of APOS it becomes apparent that its ubiquity as well as its intersections with many other concepts requires students to engage in higher-order thinking in order to understand the sequencing of mathematical processes.

CHAPTER 6

CONCLUSION

This study demonstrated that SMP transcends levels of mathematics. Our analysis of student work in a one-semester calculus course indicated that not only do students at the undergraduate level still struggle with PEMDAS-type algebra errors but they also encounter new sequencing problems as they are introduced to operations specific to calculus such as differentiation and integration. Specifically, we found that students often make *operation/differentiation commutativity* mistakes, particularly with respect to the product, chain, and quotient rules. That is, students frequently assume that the derivative of the product of two functions is the product of the derivatives of those two functions. Likewise, students equate the derivative of a quotient of two functions with the quotient of the derivatives of those functions or the derivative of composed functions as the composition of the derivatives of the functions. Up to a quarter of the students in our study made these SMP types of errors on given assessments, demonstrating that it is an issue worth further discussion.

Better pre-algebra and algebra instruction is not sufficient to strengthen students' development of SMP. As students are introduced to new operations and new commutativity rules in calculus, focus on SMP still needs to be strong at that level of mathematics. Since combining concepts requires the development of schemas (as per APOS theory), it is not enough to assume that students will extend their understanding of order of operations in middle school to SMP in calculus. Furthermore, interchangeability of order becomes less of a given as students move upward in mathematics; differentiation and integration often do not commute with basic mathematical operations, and once linear algebra or abstract algebra are reached, students begin encountering new algebraic structures that do not necessarily obey the commutative law. Thus it cannot be assumed that instructors do not need to directly address SMP past algebra.

Theories about student learning in mathematics provide clues about why correctly sequencing operations in calculus may be so difficult. Often, understanding the correct order of operations requires students to be flexible with how mathematical notation is presented, which in turn requires students to have an advanced ‘procept’ level understanding of the concepts involved. Understanding the connections between functions, derivatives, and other operations requires students to begin to develop a structural understanding of each of the notions individually, and beyond that to begin to build schemas that organize and link the concepts with one another.

Our study did indicate that students might find certain strategies useful for making procedural and conceptual connections, namely the use of demarcating symbols and rule-naming. Whether self-imposed or required in the problem, we found some evidence to indicate that students who identified differentiation rules at each step as they attempted to find a derivative were less likely to make SMP-type errors. We also discovered that students used parentheses, brackets, or labelled individual functions in the problem as a way of making sense about the type of differentiation problem at hand.

Implications for Future Research

Certainly many questions remain to be answered about the sequencing of mathematical processes in calculus. Our study primarily investigated the occurrences of SMP-type errors in product, chain, and quotient rule problems. There are undoubtedly other areas of calculus (and courses beyond calculus) in which SMP plays a large role. Understanding where SMP issues are most prominent could help us better dissect student learning in this area and in turn assist us in developing better instructional strategies and interventions that target these SMP errors.

Further studies about student strategies such as rule-naming and demarcation symbols and how they affect students' propensities to make SMP-errors are also warranted. In particular, conducting talk aloud interviews with students as they apply these strategies may help us understand how they interface with students' sense-making processes. Related to this, investigating how the specific names of rules cue students' thoughts would also be a worthwhile venture. Would naming rules after processes instead of concepts help students make stronger connections between functions, derivatives, and differentiation rules?

Symbols have been shown to play an important role in students' understanding of mathematical processes. It is not entirely clear however, how symbol choice might impact students' sense-making about SMP or student development in creating conceptual and procedural links. For example, does using parentheses versus using a · symbol make a difference in students' abilities to make connections between the process of multiplication and the concept of product? Does the choice of symbol affect students' likelihood of choosing the correct differentiation rule to apply to the problem? These are also questions worthy of further investigation.

Finally, this study primarily focused on SMP in the context of differentiation problems. Our preliminary investigation into the differences in SMP errors in integration compared to differentiation was inconclusive, largely due to the lack of suitable problems for analysis. Designing an assessment that allows for direct comparison of operation/differentiation commutativity errors with operation/integration commutativity errors would provide much more insight into how students combine and sequence mathematical processes. Overall, SMP is an important and relatively unexplored area of mathematics education research, and we hope that this study encourages others to dive in and investigate further.

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APPENDIX

A: Spring 2018 MATH 1310 Syllabus

Math 1310–001: Calculus, Systems, and Modeling (CSM) **Course Syllabus** Spring 2018

Class Meetings: Monday through Friday 9:00–9:50 AM, in BESC 1B81.

Instructor: Albany Thompson (email: albany.thompson@colorado.edu)

Office Hours: Tuesdays and Thursdays 10–11 AM in Math 216, and Fridays 11 AM–12 PM in MATH 175.

Teaching Assistant: Natalie Coston (email: natalie.coston@colorado.edu)

Learning Assistant: Abigail Tubman (email: abigail.tubman@colorado.edu)

Prerequisites. Requires prerequisite course of APPM 1235 or MATH 1021 or MATH 1150 or MATH 1160 (minimum grade C-) or an ALEKS math exam taken in 2016 or earlier, or placement into calculus based on your admissions data and/or CU Boulder coursework.

Course website. Please see

`http://math.colorado.edu/~stade/CSM/CSM_Spring2018.html`

for homework assignments and other stuff relevant to the course.

Text. *Calculus in Context, Revised*, by Stade, Callahan, et al., is *free*, and can be found online at

`http://math.colorado.edu/~stade/CSM/textbook.html`

(The text is still under development; this is not the final version.) There is also a link to this text on our above course page.

Technology:

- **Sage.** In this course, we will make regular use of a FREE mathematical software package called Sage, to model, program, graph, etc. To this end, you will NEED to set up a free account on the CU Sage server. Instructions on how to do this, and further information on the Sage package and on Sage requirements for this course, can be found by clicking on the link entitled "The Sage Page" under the "General Information" header of our above course page.

  For this course, you will be *required* to bring to class, on selected Tuesdays and Thursdays (and sometimes Fridays), a device – laptop or tablet – with wifi and a web browser. You will be using this device to create and run Sage code in your tutorials (see item **(b)** on the next page). Your instructor will provide more information on this requirement.

- **Calculator.** For exams in this course, you will **need** to own an *approved* calculator that has keys for basic operations ($+, -, \times$, and $\div$), and for basic transcendental (trigonometric and exponential/logarithmic) functions, but does NOT have programming or graphing features. Permissible calculators (all of which are in the same price range) are:

  - Sharp EL-500W Electronic Calculator – the CU Bookstore should have tons of these; see `http://www.cubookstore.com/p-68896-sharp-electronic-calculator.aspx`
  - Texas Instruments TI-30Xa Scientific Calculator
  - Texas Instruments TI-30XIIS Scientific Calculator
  - Hewlett Packard HP6S Scientific Calculator
  - Casio FX-260 Solar Scientific Calculator

If you have a calculator that you would like to use for exams, and it is NOT one of the above, you MUST get it approved by your instructor BEFORE the day of the exam.

Mathematics Academic Resource Center, also known as MARC. You may seek assistance with your math questions at the MARC, which will be open (excluding holidays) Monday–Thursday 9 AM–9 PM and Friday 9 AM–4 PM, in Math 175.

Please see our course web page for the schedule of MARC tutors who are most familiar with, and will be most able to help you with, the material for this course.

Requirements and grades. Your grade in this course will be computed on the basis of:

(a) Exams (68% of your final grade). You will have three evening exams, each worth **16%** of your final grade. These exams will be take place **from 6:15 until 7:45 PM on the following Wednesdays:**

February 7, March 7, and April 11, 2018 (location to be determined).

If you miss an evening exam and *do not* have a valid excuse, you will receive a zero for that exam grade. If you *do* have a valid excuse, we will compute your course grade from the rest of the data we have; that is, you will not be penalized for missing the exam. (A note from a doctor will be considered a valid excuse, as well as a note from the Office of the Dean of Students. No excuse will be considered valid unless it is *documented.*)

We will *review* for each evening exam in class on the day before, and the day of, the exam itself.

You will also have a **final exam**, worth **20%** of your final grade, on

Monday, May 7, 10:30 AM–1:00 PM (location to be determined).

(b) Tutorials (a.k.a. "worksheets") (10% of your final grade). There will be group assignments to be completed in class on *Tuesdays and Thursdays.* (On certain weeks, e.g. the week before an evening exam, these tutorials may instead be on Thursdays and Fridays.)

These tutorials will be distributed in class, and you will work on them with your classmates in **groups of four or so**. Your instructor (on Tuesdays), or your Teaching Assistant (TA) and Learning Assistant (LA) (on Thursdays), will be present during tutorials to facilitate your work, but the goal is for you (and your groupmates) to *work through, and complete, these worksheets on your own* as much as possible.

To get full credit for a tutorial, you must attend class on that day, and *participate* in your group's explorations and discoveries. Another really good reason to take part in tutorials is: material covered in tutorials **WILL** be on your exams.

Missed tutorials cannot be made up; if you miss a tutorial, you will receive a zero for that tutorial grade. *If you are more than five minutes late for any tutorial, you will get at most half credit for that day.* However, we will drop your lowest *five* tutorial scores.

(c) Homework (22% of your final grade). Homework for this class will come in three flavors: **(i)** Individual written assignments (worth 10% of your final grade); **(ii)** "Mini Project" assignments (worth 6% of your final grade); and a Term Project, which will be due in three stages (collectively worth 6% of your final grade).

Mini Projects and the Term Project are to be completed collaboratively, in the same groups that you work with for tutorials. We will discuss formation of groups during the first week of classes.

All assignments will be posted on our course web page. Missed homeworks *cannot be made up for any reason*; if you miss a homework, you will receive a zero for that homework grade. We will drop your lowest *two* individual homework scores.

Note: Scores for your Mini Projects and Term Project may *NOT* be dropped. More details on the Mini Projects and Term Projects will be supplied in class.

Please see the "Homework Assignment Guidelines" link under the "General Information" header of our course page for important instructions on completing Individual, Mini Project, and Term Project assignments!!!

Other important course information. Please see our course web page for important policy information regarding disabilities, religious holidays, classroom behavior, discrimination and harassment, and the CU Honor Code.

B: Spring 2018 MATH 1310 Daily Schedule

	Topic(s) covered and relevant readings	HW due and additional information
WEEK 1: JANUARY 15–JANUARY 19		
MONDAY	MLK Day: no classes	
TUESDAY	Introduction and tutorial: CSM and SIR	
WEDNESDAY	Section 1.2: The spread of disease: the *SIR* model	
THURSDAY	**Tutorial:** SIR	
FRIDAY	Section 1.3: Prediction using *SIR*	
WEEK 2: JANUARY 22–JANUARY 26		
MONDAY	Section 1.3: Prediction using *SIR* (continued)	First individual assignment due
TUESDAY	**Tutorial:** Graphing with Sage, part I	
WEDNESDAY	Section 1.4: Functions and graphs	
THURSDAY	**Tutorial:** Graphing with Sage, part II	
FRIDAY	Section 1.5: Linear functions	Second individual assignment due
WEEK 3: JANUARY 29–FEBRUARY 2		
MONDAY	Section 2.1: Rates of change	
TUESDAY	Section 2.2: Local linearity (differentiability)	First mini project due
WEDNESDAY	Section 2.3: The microscope equation	
THURSDAY	**Tutorial:** Secant and tangent lines; rates of change	
FRIDAY	**Tutorial:** SIR and Euler's method (review)	Third individual assignment due
WEEK 4: FEBRUARY 5–FEBRUARY 9		
MONDAY	Section 2.4: A global view	
TUESDAY	Review for Exam 1	
WEDNESDAY	Review for Exam 1	Exam 1, 6:15--7:45 PM, in HUMN 1B50
THURSDAY	**Tutorial:** Functions and derivatives	
FRIDAY	Section 2.5: The chain rule	
WEEK 5: FEBRUARY 12–FEBRUARY 16		
MONDAY	Section 2.5: The chain rule (continued)	
TUESDAY	**Tutorial:** The chain rule	
WEDNESDAY	Section 2.6: More differentiation rules	
THURSDAY	**Tutorial:** More differentiation; the microscope equation	
FRIDAY	Section 2.6: More differentiation rules (continued)	Fourth individual assignment due
WEEK 6: FEBRUARY 19–FEBRUARY 23		
MONDAY	Section 3.1: The exponential function	
TUESDAY	**Tutorial:** Population growth with Sage	
WEDNESDAY	Section 3.2: Modeling with differential equations	Second mini project due
THURSDAY	**Tutorial:** Monomers, Dimers, and Trimers	
FRIDAY	Section 3.3: Modeling populations	Fifth individual assignment due

	Topic(s) covered and relevant readings	HW due and additional information
WEEK 7: FEBRUARY 26–MARCH 2		
MONDAY	Section 3.4: Modeling other phenomena	
TUESDAY	Section 3.5: The logarithm function	
WEDNESDAY	Section 3.5: The logarithm function (continued)	
THURSDAY	**Tutorial:** Derivatives of the logarithm function	
FRIDAY	**Tutorial:** More modeling with differential equations	Sixth individual assignment due
WEEK 8: MARCH 5–MARCH 9		
MONDAY	Section 3.5: The logarithm function (continued)	
TUESDAY	Review for Exam 2	
WEDNESDAY	Review for Exam 2	Exam 2, 6:15--7:45 PM, in HUMN 1B50
THURSDAY	**Tutorial:** Exponential growth and decay	
FRIDAY	Section 3.6: Inverse functions and the arctangent function	
WEEK 9: MARCH 12–MARCH 16		
MONDAY	Section 4.1 Power and energy	
TUESDAY	**Tutorial:** Arctangent derivatives	
WEDNESDAY	Section 4.2: Accumulation functions	
THURSDAY	**Tutorial:** The second derivative	
FRIDAY	Section 4.3: Riemann sums	Seventh individual assignment due
WEEK 10: MARCH 19–MARCH 23		
MONDAY	Section 4.4: The definite integral	Term project part 1 (proposal) due
TUESDAY	**Tutorial:** Riemann Sums	
WEDNESDAY	Section 4.5: The Fundamental Theorem of Calculus	
THURSDAY	**Tutorial:** Polyhedra	
FRIDAY	**Tutorial:** Spirographs	Eighth individual assignment due
WEEK 11: MARCH 26–MARCH 30		
MONDAY	SPRING BREAK—NO CLASSES	
TUESDAY	SPRING BREAK—NO CLASSES	
WEDNESDAY	SPRING BREAK—NO CLASSES	
THURSDAY	SPRING BREAK—NO CLASSES	
FRIDAY	SPRING BREAK—NO CLASSES	
WEEK 12: APRIL 2–APRIL 6		
MONDAY	Section 5.1: Antiderivatives	
TUESDAY	Section 5.1: Antiderivatives (continued)	Third mini project due
WEDNESDAY	**Tutorial:** Riemann sums and Ebola	
THURSDAY	**Tutorial:** The Boulder Flood	
FRIDAY	**Tutorial:** Basic Integration	Ninth individual assignment due

	Topic(s) covered and relevant readings	HW due and additional information
WEEK 13: APRIL 9–APRIL 13		
MONDAY	**Tutorial:** Integration and accumulation	
TUESDAY	Review for Exam 3	
WEDNESDAY	Review for Exam 3	Exam 3, 6:15--7:45 PM, in HUMN 1B50
THURSDAY	**Tutorial:** Coin flipping and histograms	
FRIDAY	Section 5.2: Integration by substitution	
WEEK 14: APRIL 16–APRIL 20		
MONDAY	Section 5.2: Integration by substitution (continued)	
TUESDAY	**Tutorial:** Integration by substitution	
WEDNESDAY	Section 5.3: Separation of variables	Term project part 2 (first draft) due
THURSDAY	**Tutorial: Separation of Variables**	
FRIDAY	Section 6.1: Relative frequency density	Tenth individual assignment due
WEEK 15: APRIL 23–APRIL 27		
MONDAY	Section 6.1: Relative frequency density (continued)	
TUESDAY	Section 6.2	
WEDNESDAY	Section 6.2 (continued)	
THURSDAY	**Tutorial: Probability and Statistics**	
FRIDAY	Section 6.2 (continued)	Eleventh individual assignment due
WEEK 16: APRIL 30–MAY 4		
MONDAY	Section 6.2 (continued)	
TUESDAY	Section 6.2 (continued)	Term project part 3 (final version) due
WEDNESDAY	Final Exam Review	
THURSDAY	Final Exam Review	Twelfth individual assignment due
FRIDAY	**Reading Day: no classes**	
	FINAL EXAM MONDAY, MAY 7, 10:30 AM -- 1 PM,	
	IN HUMN 150 (NOT HUMN 1B50!!!)	

C: Fall 2017 MATH 1310 Second Exam

MATH 1310: CSM

October 18, 2017

SECOND EXAM

I have neither given nor received unauthorized assistance on this exam.

Name:____

Signature:____

Section: ◯ **001** Eric Stade .... (11AM)
◯ **002** Natalie Coston .... (1PM)
◯ **003** Albany Thompson .... (2PM)

You must **show your work** on every problem of this exam, and provider **units** with your answers wherever appropriate.
Please supply **at least five decimal places** for all numerical answers (but leave out trailing zeroes; e.g. you don't need to write 0.58000 if the exact answer is 0.58).

GOOD LUCK!!

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	16 pts	
2	6 pts	
3	15 pts	
4	9 pts	
5	12 pts	
6	15 pts	
7	12 pts	
8	15 pts	
TOTAL	100 pts	

1. Find the following derivatives. Do not simplify your answer.

(a) $\dfrac{d}{dx}\left[x^2 - 3\cos(x) + 2\right]$

(b) $\dfrac{d}{dx}\left[\sqrt[3]{x} + \dfrac{1}{x^3} + \dfrac{\sqrt{x^5}}{x}\right]$

(c) $\dfrac{d}{dx}\left[e^{\ln(3x^2-\cos(x)+5)} \cdot \ln(e^{\tan(x)})\right]$ Hint: simplify first; *then* differentiate.

(d) $\dfrac{d}{dx}\left[\dfrac{x^2+3^{\ln(x)}}{\sin(x)}\right]$

2. Suppose that

$$f(x) = (g(x) - 1)^3 + 2.$$

Find $f'(3)$, given that $g(3) = 2$ and $g'(3) = -1$.

3. You have one of those fancy ice spheres, and you place it in a glass of water. When you first take it out of the freezer, the radius of the sphere is 2 inches, and you observe that the radius of the sphere is melting at a rate of 0.5 inches per hour. Recall that the volume of a sphere is $V = \frac{4}{3}\pi r^3$.

(a) What is the radius of the ice sphere after 2 hours? Include units.

(b) What is the volume of the ice sphere after 2 hours? Include units.

(c) How fast is the volume of the ice sphere shrinking after 2 hours? include units.

4. Find

$$\frac{d}{dx}\left[3\cos(\arctan(x)e^{\sin(x)})\right].$$

At the end, please list *all* the rules you used, and *how many times you used each.* (The possible choices for rules are: Constant Multiple, Sum, Chain, Product, or Quotient.)

5. Let $f(x) = \sqrt{3 + x^3}$.

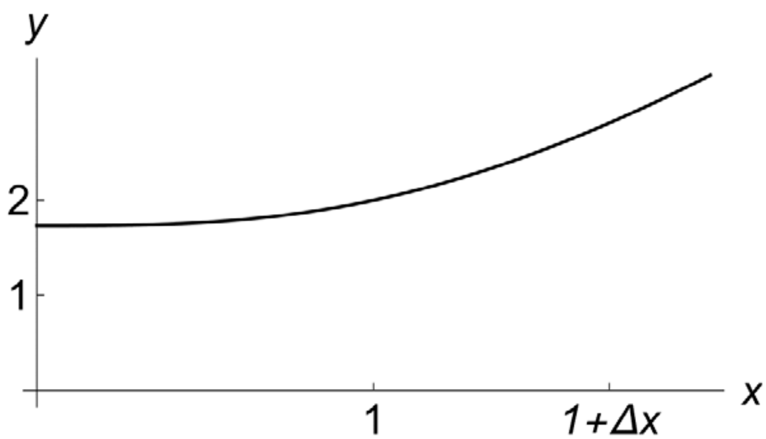
(a) Write down the Microscope Equation for $f(x)$ at $x = 1$.

(b) To which of the following quantities does the microscope equation, from part (a) of this problem, give a better estimate (circle the correct answer):

(i) $\sqrt{3 + (1.05)^3}$ (ii) $\sqrt{3 + (.7)^3}$

Please explain briefly, *without* using a calculator. (Though you may check your answer with a calculator if you would like.)

(c) A graph of $f(x) = \sqrt{3 + x^3}$ appears below.



i. On the above graph, draw the line tangent to $y = f(x)$ at $x = 1$.

ii. Let Δx be a small number. Does the microscope equation, from part (a) of this problem, give an overestimate or underestimate to $\sqrt{3 + (1 + \Delta x)^3}$? Circle the correct answer:

(A) underestimate (B) overestimate

Please explain briefly, by referring to the graph above.

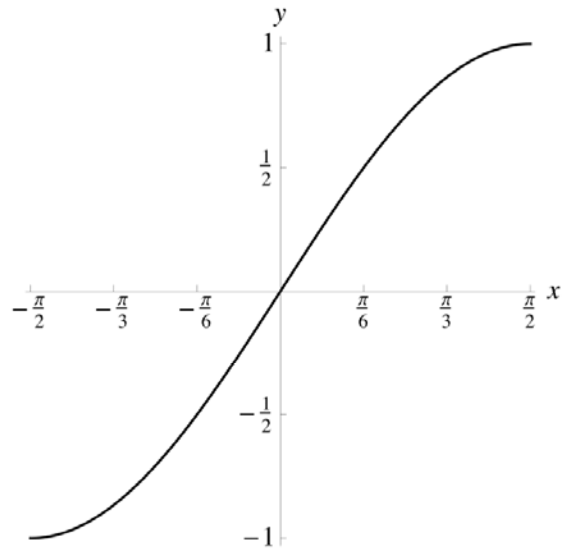
iii. On the above graph, clearly mark (with dots) the three points with these coordinates:

(A) $(1, f(1))$ (B) $(1 + \Delta x, f(1 + \Delta x))$ (C) $(1 + \Delta x, f(1) + f'(1)\Delta x)$.

Put the correct letter ((A), (B), or (C)) next to each dot.

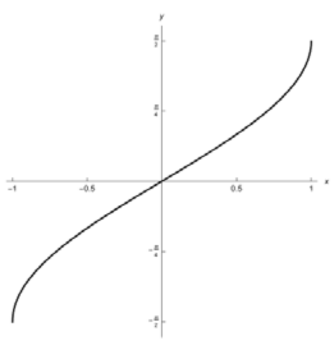
(d) Approximate $\sqrt{4.331}$. Hint: $4.331 = 3 + (1.1)^3$.

6. On the axes below is a graph of the function $y = \sin(x)$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

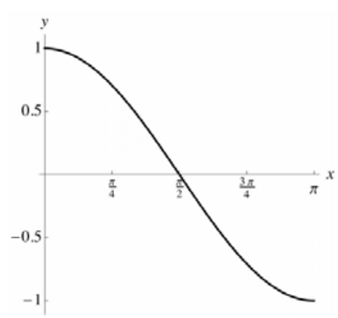


(a) Explain why reflecting $y = \sin(x)$, on this domain, about the line $y = x$ gives a new function, which we'll call $y = \arcsin(x)$.

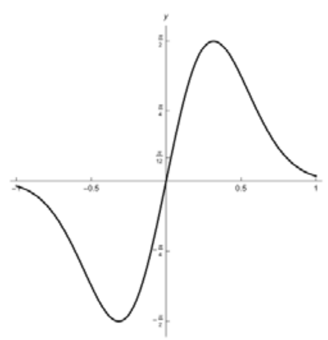
(b) Which of the following gives the graph of $y = \arcsin(x)$? Circle the letter ((A), (B), or (C)) below the correct graph.



(A)



(B)



(C)

(c) Find $\dfrac{d}{dx}[\arcsin(x)]$, as follows (fill in the blanks; there are six of them).

Since the function $y = \arcsin(x)$ takes an input x to an output $\arcsin(x)$, we know that the reflection $y = \sin(x)$ must take an input $\arcsin(x)$ to an ____________ x. That is,

$$\sin(\arcsin(x)) = \underline{\qquad\qquad}. \tag{1}$$

We differentiate both sides of this equation to get

$$\frac{d}{dx}[\sin(\arcsin(x))] = 1$$

or, using the chain rule on the left,

$$\cos(\underline{\qquad\qquad\qquad}) \cdot \frac{d}{dx}[\underline{\qquad\qquad\qquad}] = 1.$$

Dividing by $\cos(\arcsin(x))$ then gives

$$\frac{d}{dx}[\arcsin(x)] = \frac{1}{\cos(\arcsin(x))}. \tag{2}$$

Now for any real number θ, we have $\cos(\theta) = \sqrt{1 - (\sin(\theta))^2}$. But then

$$\cos(\arcsin(x)) = \sqrt{1 - (\sin(\arcsin(x)))^2} = \sqrt{1 - (\underline{\qquad\qquad})^2},$$

the last step by equation (1). Putting this back into equation (2) gives

$$\frac{d}{dx}[\arcsin(x)] = \underline{\qquad\qquad\qquad},$$

and we're done.

(d) What is the *slope* of the line tangent to the graph of $y = \arcsin(x)$ at $x = 0$? Use the previous part of this problem to answer. Please express your answer as a *whole number.*

7. Allison is an eccentric cat lover. In 1993 she owned only 3 cats. Assume that the number of cats that Allison owns grows exponentially.

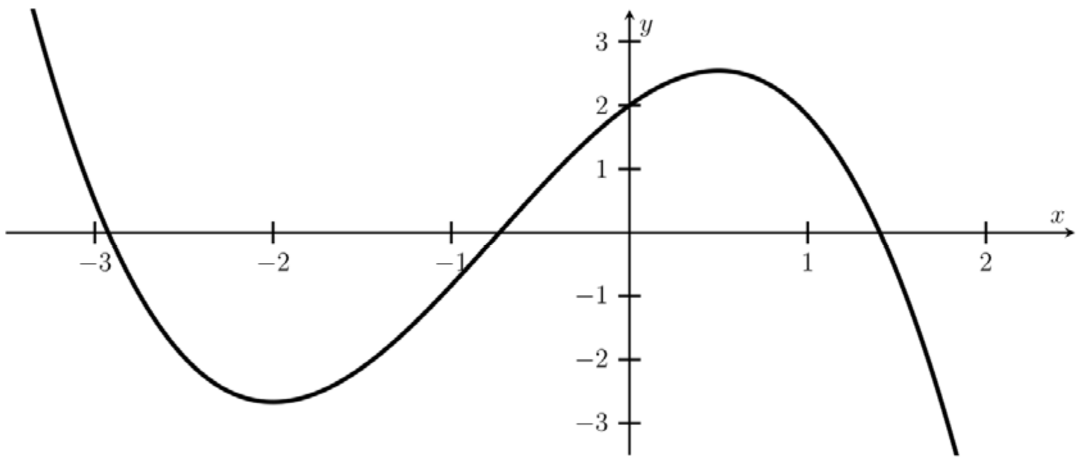
(a) Write an initial value problem to model the growth of the number of cats that Allison owns $C(t)$.

(b) In 1996, Allison owned 6 cats. Given this additional information, write down a formula $C(t)$ to model the number of cats that Allison will own after t years (measured since 1993).

(c) In what year does Allison own 100 cats?

(d) According to your model, how many cats does Allison own in 2017?

8. Consider the function $f(x) = -\frac{2}{3}x^3 - \frac{3}{2}x^2 + 2x + 2$ between $x = -3$ and $x = 2$. The graph of $f(x)$ is shown below.



(a) Find $f'(x)$.

(b) Find $f''(x)$.

(c) On the graph, indicate the points where the function $f(x)$ changes from increasing to decreasing, or from decreasing to increasing. Use your answer to part (a) to find the exact values of x where this occurs. Hint: $-2x^2 - 3x + 2 = -(x + 2)(2x - 1)$.

(d) On the graph, indicate the point where *the derivative*, $f'(x)$, changes from increasing to decreasing. Use your answer to part (b) to find the exact value of x where this occurs.

(e) What is the slope of $f(x)$ at the x-value that you found in part (c)?

D: Spring 2018 MATH 1310 Homework

Math 1310: CSM **Spring 2018**

Individual Homework #8: Due in class Friday, March 23

Part A: Integrals and Accumulation. Important note: the Sage program RIEMANN.sws, which you will need for parts of this assignment, may be found on our course page, under the link that says "The Sage Page."

Please **read** Sections 4.1–4.4. Also please **do:**

(a) **Section 4.1** (pages 197–199): Exercises 2a, 4.

(b) **Section 4.2, Part 1: Work as force × distance** (page 206): Exercise 2abcd.

(c) **Section 4.3, Part 1: Riemann sums "by hand"** (pages 214–215): Exercises 1, 3.

(d) **Section 4.3, Part 2: Using RIEMANN.sws** (page 215): Exercises 4, 5, 6, 8.

(e) **Section 4.4, Part 1: Evaluating integrals geometrically** (pages 225–226): Exercises 1abcd, 4.

(f) **Section 4.4, Part 2: Integrals and Riemann sums** (pages 226–227): Exercise 7ac.

Part B: Differentiation review. The **GOAL** of the following exercises is explore differentiation rules and formulas through an "order of operations" perspective.

(a) State whether each of the given statements is true or false. If the statement is *true*, identify it by name. If it's *false*, make it *true* by replacing everything that comes after "is equal to" with appropriate verbiage, and identify the *corrected* statement by name.

Examples:

(i) The derivative of the sum of two functions is equal to the sum of the derivatives of those functions.

Answer: True (sum rule).

(ii) The derivative of the product of two functions is equal to the product of the derivatives of those functions.

Answer: False. The derivative of the product of two functions is equal to the first function times the derivative of the second, plus the second function times the derivative of the first (product rule).

OK, these ones are for you. (Please provide an **Answer** according to the instructions above, and as illustrated in the above examples.)

(iii) The derivative of a constant times a function is equal to the constant times the derivative of that function.

(iv) The derivative of a chain of two functions is equal to the chain of the derivatives of those functions.

(v) The derivative of the quotient of two functions is equal to the top function times the derivative of the bottom, minus the bottom function times the derivative of the top, all divided by the bottom function squared.

(b) Listed below are eight mathematical statements: **not all of them are true!** Identify each statement as being either a **correct** or an **incorrect** version of a known differentiation rule.

Examples:

(i) $\frac{d}{dx}[f(x)g(x)] = f'(x)g'(x)$ **Answer:** product rule, incorrect version

(ii) $\frac{d}{dx}[cf(x)] = cf'(x)$ **Answer:** constant multiple rule, correct version

OK, here are the rest. (Please provide an **Answer** according to the instructions above, and as illustrated in the above examples.)

(iii) $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$

(iv) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f(x)g'(x) - g(x)f'(x)}{(g(x))^2}$

(v) $\frac{d}{dx}[f(g(x))] = f'(g'(x))$

(vi) $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

(vii) $\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

(viii) $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$

E: Spring 2018 MATH 1310 Quiz Version A

Math 1310: CSM Derivative review quiz **Name:**

Find $f'(x)$ if

$$f(x) = \sec^2(x^4)\ln(x^2+1).$$

F: Spring 2018 MATH 1310 Quiz Version B

Math 1310: CSM Derivative review quiz **Name:**

Find $f'(x)$ if

$$f(x) = \sec^2(x^4)\ln(x^2+1).$$

Please also list **all of the rules** you used and **how many times** you used each. (The possible choices for rules are: Constant Multiple, Sum, Chain, Product, or Quotient.)

G: Spring 2018 MATH 1310 Third Exam

MATH 1310: CSM

APRIL 11, 2018

THIRD EXAM

I have neither given nor received unauthorized assistance on this exam.

Name:____

Signature:____

Section: ◯ **001** Albany Thompson ....(9am)
◯ **002** Sarah Arpin .... (11am)
◯ **003** Athena Sparks .... (2pm)

You must **show your work** on every problem of this exam, and provide **units** with your answers wherever appropriate.
Please supply **at least six decimal places** for all numerical answers, **unless otherwise specified** (but leave out trailing zeroes; e.g. you don't need to write 0.580000 if the exact answer is 0.58).
Make sure your calculator is in radian mode!!!
GOOD LUCK!!

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	10 pts	
2	12 pts	
3	12 pts	
4	14 pts	
5	16 pts	
6	12 pts	
7	12 pts	
8	12 pts	
TOTAL	100 pts	

1. Evidence of life has been discovered on the faraway Planet X! There is no life currently on the planet, but fossils have been found which indicate that life once existed on this planet.

   In order to date the fossils, scientists are using a new chemical compound, CSM, which is known to decay exponentially. It is known that the half-life of CSM is 6000 years. Scientists find a Planet X fossil which contains 23% of the CSM that it would have started with. How old is the fossil? Round your final answer to the nearest year, but keep *at least* 6 decimal places of accuracy in your calculations.

2. Given the fact that $\int_{-2}^{4} f(x)dx = 3$ and $\int_{-2}^{4} g(x)dx = -2$, find the values of the following definite integrals:

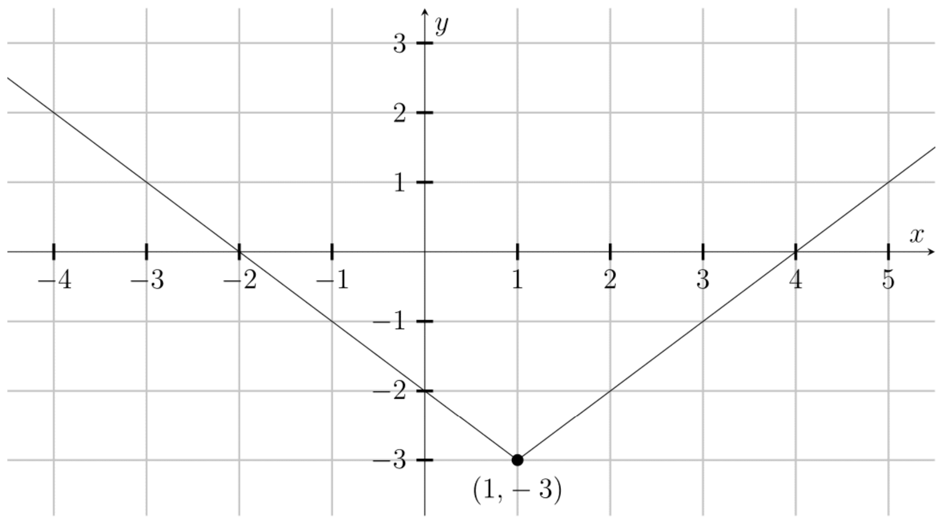
(a) $\displaystyle\int_{4}^{-2} \frac{f(x)}{2}\, dx$

(b) $\displaystyle\int_{-2}^{4} (2f(x) - 3g(x))\, dx$

(c) $\displaystyle\int_{-2}^{4} (g(x) + x)\, dx$

(d) If $\displaystyle\int_{0}^{4} f(x)\, dx = 7$, find $\displaystyle\int_{-2}^{0} f(x)\, dx$.

3. Given the graph of $f(x) = |x - 1| - 3$ below, evaluate the following definite integrals:

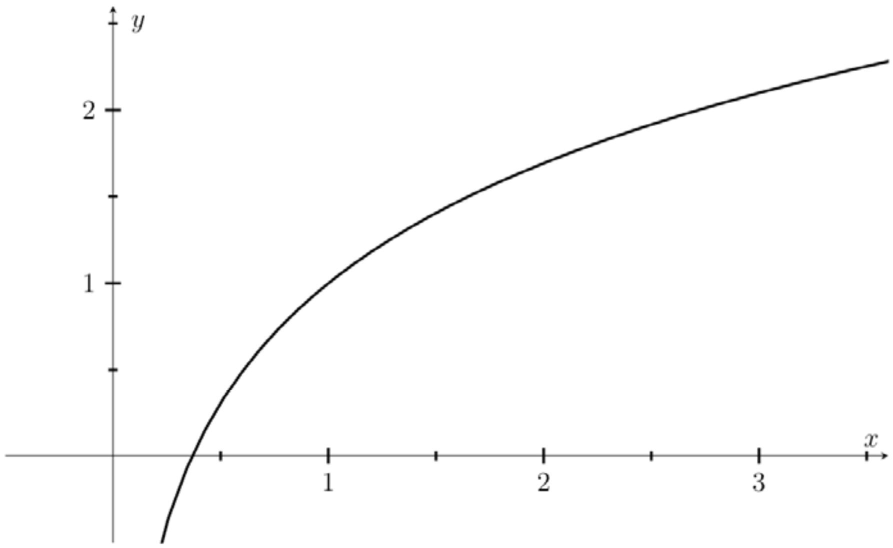


(a) $\displaystyle\int_1^4 f(x)dx$

(b) $\displaystyle\int_4^1 (f(x) - 2)\, dx$

(c) $\displaystyle\int_2^4 (\sqrt{x} - \frac{1}{2} f(x))\, dx$

4. A graph of the function $f(x) = \ln(x) + 1$ is shown below.



(a) On top of the above graph, draw the rectangles corresponding to a left endpoint Riemann sum approximation, with $n = 4$ rectangles of equal baselength, to the area

$$\int_1^3 f(x)\, dx.$$

(b) Use the rectangles in part (a) to provide a Riemann sum approximation to $\displaystyle\int_1^3 f(x)\, dx$. Please supply at least six decimal places.

(c) Is your approximation in part (b) an underestimate or an overestimate? Please explain geometrically, by referring to the graph above.

(d) Given that $F(x) = x\ln(x)$ is an antiderivative of $f(x) = \ln(x) + 1$, calculate the actual area $\int_1^3 f(x)\,dx$. Please provide an answer in terms of the natural logarithm function, and *also* supply a decimal answer with at least six decimal places.

5. For this problem, *please be careful with your units.* (It may help to recall that 1 hour = 60 minutes.)

   A red car, a green car, and a blue car enter a drag race in the desert.

   (a) The red car has a *constant* velocity of 80 mi/hr. (It can instantly jump to 80 mi/hr from start, and maintain that speed.) How far has the red car gone after 1 minute? Include units in your answer.

   (b) The green car has a velocity given by

$$v(t) = \begin{cases} 50 \text{ mi/hr} & \text{for the first 30 seconds,} \\ 90 \text{ mi/hr} & \text{for the next 30 seconds.} \end{cases}$$

   How far has the green car gone after 1 minute? Include units in your answer.

(c) The blue car has a velocity $v(t)$ given by

$$v(t) = 1200\sqrt{t},$$

where $v(t)$ is in mi/hr. How far has the red car gone after 1 minute? Include units in your answer.

(d) If the race was 1.5 miles long, can you tell who won? Hint: compare your answers to parts (a), (b), and (c) of this problem. Please explain your answer.

6. Find the following derivatives involving the arctangent function.

   (a) $\frac{d}{dx}[2\arctan(3x) + 3x]$

   (b) $\frac{d}{dx}[\ln(x)\arctan(8x)]$

   (c) $\frac{d}{dx}[e^{\arctan(x)}]$

   (d) $\frac{d}{dx}[\frac{\arctan(x)}{x}]$

7. Use the Fundamental Theorem of Calculus to find the following indefinite integrals.

(a) $\displaystyle\int \left(x^5 + 5x^3 + \frac{1}{3}x + 1\right)\, dx$

(b) $\displaystyle\int \left(\frac{x^4}{5} + e^{7x} + 2\right)\, dx$

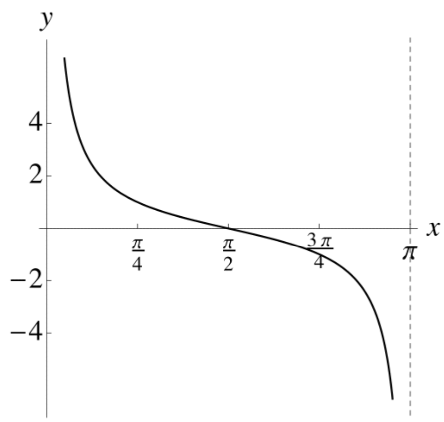
(c) $\displaystyle\int \left(\sqrt[4]{x} + 6\sin(3x) + 9^x\right)\, dx$

(d) $\displaystyle\int \left(\frac{1}{x} + \frac{1}{1+x^2} + \frac{1}{x^5}\right)\, dx$

8. This problem involves the contangent function $\cot(x)$, defined by

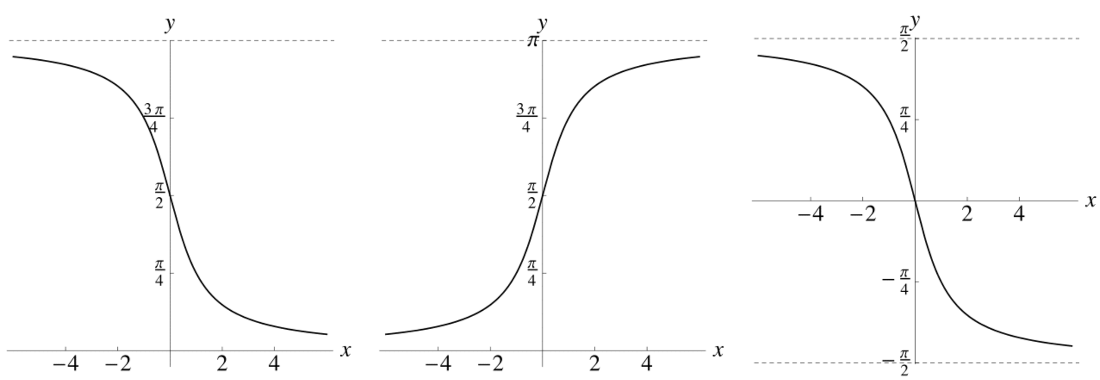
$$\cot(x) = \frac{\cos(x)}{\sin(x)}.$$

On the axes below is a graph of $y = \cot(x)$, for $0 < x < \pi$.



(a) Explain why reflecting $y = \cot(x)$, on this domain, about the line $y = x$ gives a new function, which we'll call $y = \text{arccot}(x)$.

(b) Which of the following gives the graph of $y = \text{arccot}(x)$? Circle the letter ((A), (B), or (C)) below the correct graph.



(A) **(B)** **(C)**

(c) For this part of our problem, we will be using the fact (which is not hard to show) that

$$\frac{d}{dx}[\cot(x)] = -(1+\cot^2(x)).$$

Find $\dfrac{d}{dx}[\operatorname{arccot}(x)]$, as follows (fill in the blanks; there are five of them).

Since the function $y = \operatorname{arccot}(x)$ takes an input x to an output $\operatorname{arccot}(x)$, we know that the reflection $y = \cot(x)$ must take an input $\operatorname{arccot}(x)$ to an output x. That is,

$$\cot(\operatorname{arccot}(x)) = \underline{\hspace{3cm}}. \tag{1}$$

We differentiate both sides of this equation to get

$$\frac{d}{dx}[\cot(\operatorname{arccot}(x))] = 1$$

or, using the chain rule and the fact that $d[\cot(x)]/dx = -(1+\cot^2(x))$ on the left,

$$-(1+\cot^2(\underline{\hspace{5cm}}))\cdot\frac{d}{dx}[\underline{\hspace{5cm}}] = 1. \tag{2}$$

Now again, $\cot(\operatorname{arccot}(x)) = x$ by (1), so equation (2) gives

$$-(1+\underline{\hspace{3cm}})\frac{d}{dx}[\operatorname{arccot}(x)] = 1$$

or, dividing by $-(1+x^2)$,

$$\frac{d}{dx}[\operatorname{arccot}(x)] = \underline{\hspace{5cm}},$$

and we're done.

H: Spring 2018 MATH 1310 Final Exam

MATH 1310: CSM
MAY 7, 2018
FINAL EXAM

I have neither given nor received unauthorized assistance on this exam.

Name:____

Signature:____

Section: ◯ **001** Albany Thompson ....(9am)
◯ **002** Sarah Arpin .... (11am)
◯ **003** Athena Sparks .... (2pm)

You must **show your work** on every problem of this exam, and provide **units** with your answers wherever appropriate.
Please supply **at least six decimal places** for all numerical answers, **unless otherwise specified** (but leave out trailing zeroes; e.g. you don't need to write 0.580000 if the exact answer is 0.58).
Make sure your calculator is in radian mode!!!
GOOD LUCK!!

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	12 pts	
2	7 pts	
3	8 pts	
4	8 pts	
5	12 pts	
6	10 pts	
7	15 pts	
8	12 pts	
9	8 pts	
10	8 pts	
TOTAL	100 pts	

1. Find the derivative of each of the following functions. You do not need to simplify your answers.

   (a) $f(x) = 5^x + \pi^2 x^5 + 5^5$

   (b) $h(x) = \dfrac{7x^6}{\sqrt[3]{x} + 2}$

(this problem is continued on the next page)

(c) $p(x) = (4x^3 + 5)\tan(3x)$

(d) $v(x) = e^{3\ln(5x)}$

2. Solve the initial value problem

$$\frac{dy}{dx} = y^2 + 1, \quad y(0) = 1.$$

Your final answer must be written in the form

$$y = \text{some function of } x.$$

3. The mean weight of a Gala apple is 76 grams. The owner of MacDonald's orchard wants to investigate whether their Gala apples weigh 76 grams, on average.

   The owner collects and weighs a random sample of 40 apples. The mean of this sample is 80.4 grams and the standard deviation is 2.8 grams. Test, at the 95% level, the null hypothesis

$$H_0 \colon \mu = 76 \text{ grams}$$

   against the alternative hypothesis

$$H_A : \mu \neq 76 \text{ grams},$$

   where μ is the mean weight of a Gala apple from MacDonald's orchard. Please show your work, and state the clearly the result of your hypothesis test.

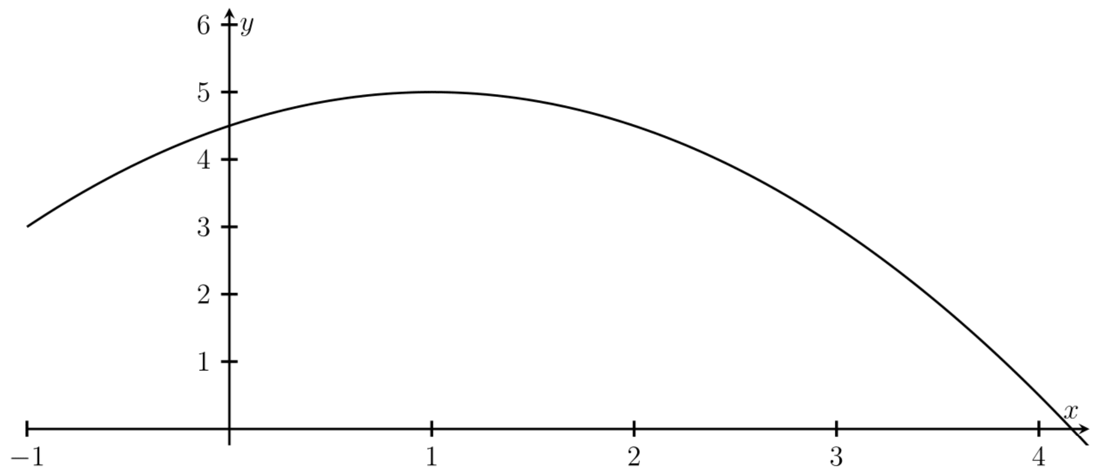
4. You and your research group completed a study, and hired a statistical group to create two confidence intervals for you, based on your sample data. One of the intervals below is a 90% confidence interval, and the other is a 99% confidence interval, created using the same data.

$$(197.1, 202.9)\ ,\ (198.2, 201.8)$$

   (a) Which is the 90% confidence interval, and which is the 99% confidence interval? Explain your answer thoroughly.

   (b) What was the mean of the sample from which these confidence intervals were calculated?

5. Below is the graph of the function $f(x) = -\frac{1}{2}(x-1)^2 + 5$.



(a) On top of the graph, draw the rectangles corresponding to a left endpoint Riemann sum approximation, with $n = 4$ rectangles of equal length, to the area

$$\int_0^4 f(x)dx.$$

(b) Using the rectangles you drew in part (a), together with the equation of the function pictured ($f(x) = -\frac{1}{2}(x-1)^2 + 5$), provide a Riemann sum approximation to $\int_0^4 f(x)dx$.

(this problem is continued on the next page)

(c) On what subinterval of [0,4] is your Riemann sum from part (b) of this problem an overestimate?

(d) On what subinterval of [0,4] is your Riemann sum from part (b) of this problem an underestimate?

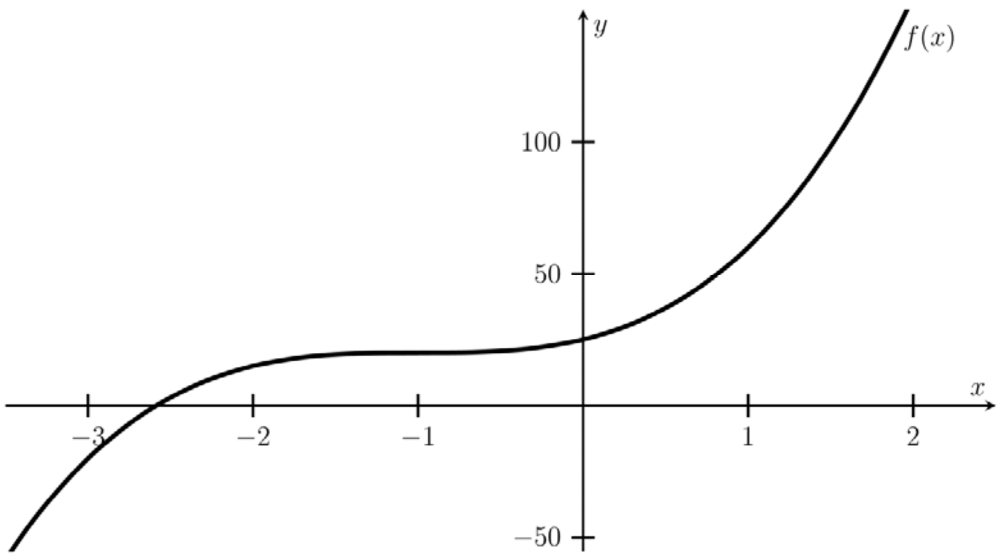
(e) Find the exact value of $\int_0^4 f(x)dx$. Hint: it might help to multiply $f(x)$ out; that is,

$$f(x) = -\frac{1}{2}(x-1)^2 + 5 = -\frac{1}{2}(x^2 - 2x + 1) + 5 = -\frac{1}{2}x^2 + x + \frac{9}{2}.$$

6. Consider the graph of the function

$$f(x) = 5(x+1)^3 + 20$$

below.



(a) Explain how, in general, the derivative of a function at a point is related to the tangent line to the function at that point.

(b) Find the equation of the tangent line to $f(x)$ at $x = -2$. Sketch your answer on the graph above.

7. Calculate the following definite and indefinite integrals using substitution. Please show your work. In particular, clearly specify what you are calling u, and what is du.

   (a) $\displaystyle\int 6t^4(6t^5+2)^{11}\,dt$

   (b) $\displaystyle\int \frac{q^2}{q^3+\ln(3)}\,dq$

(c) $\displaystyle\int_{\ln(\pi/2)}^{\ln(\pi)} e^x \sin(e^x)\, dx$

8. Situated on the border between Xanadu and Cygnus X-1 is a town called La Villa Strangiato, or LVS. LVS has a population of 100,000 demogorgons. A certain mysterious disease (rumored to be contracted from cats) is spreading through this demogorgon population, according to the usual SIR equations:

$$\begin{aligned} S' &= -aSI, \\ I' &= \quad aSI - bI, \\ R' &= \qquad\qquad bI. \end{aligned}$$

Here S, I, and R denote the number of demogorgons susceptible, infected, and recovered, respectively, at any given time t. We agree that t is measured in days, and that S, I, and R are measured in individual demogorgons.

It's known that, for this particular disease, it takes 30 days on average to recover. (That is, on average one stays infected for 30 days.)

It's also observed that, after 10 days, there are 50,858 demogorgons still susceptible and 45,487 demogorgons infected – that is, $S(10) = 50{,}859$ and $I(10) = 45{,}487$. It's also noted that, six hours (one quarter of a day) later, there are 47,388 demogorgons still susceptible – that is, $S(10.25) = 47{,}388$.

Assume throughout this problem that the total number of demogorgons always remains the same.

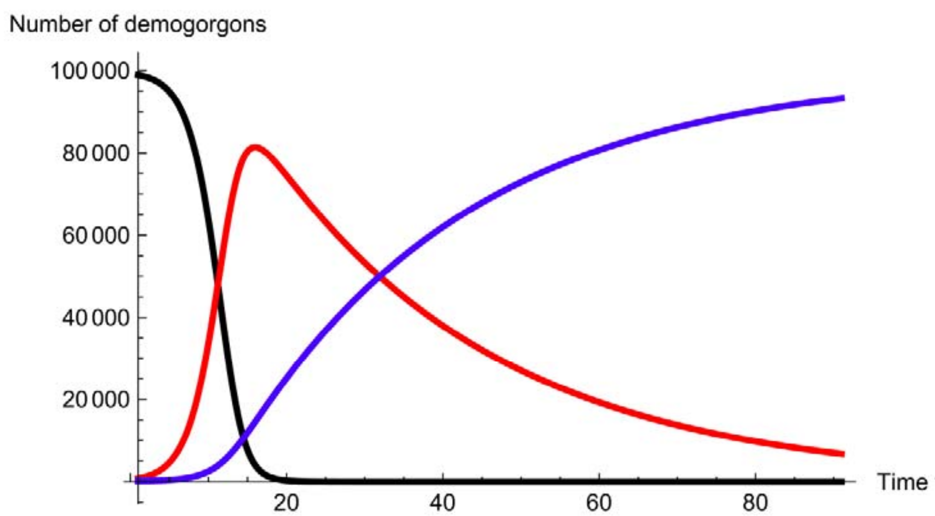
(a) Using information given above, find the average rate of change of the susceptible population from $t = 10$ to $t = 10.25$. Please include units in your answer.

(b) Find the approximate value of the transmission coefficient a. Hint: use your answer to part (a) of this problem, above, as an approximation to $S'(10)$. Then use one of the SIR equations. Please write your answer to *at least six* decimal places, and include units in your answer.

(this problem is continued on the next page)

(c) Using your answer to part (b) above, and additional information given in this problem, find the approximate value of S_T, the threshold value of S, to the nearest whole demogorgon.

(d) Now approximate S_T in a different way, namely: by reading it off of the graph below. (A rough estimate is fine.) Please show your work, by drawing appropriate lines on the graph if necessary, and/or explain your answer. (Note: your answer might be somewhat different from your answer to part (c) above; that's the thing about estimates.) Again, please include units in your answer.



$S_T \approx$ __________________

9. The entire population of CU students is asked to rate *this very* story problem on a scale of 0–4, with 0 denoting "Really?" and 4 meaning "Even better than CSM!" (A rating of 4 is not really possible; this is a *hypothetical* problem.)

CSM instructors Dr. Derivative and P. Probably, PhD are too tired at the end of the semester to review all of the ratings, so they take a random sample of $n = 100$ CU students, and observe the following ratings:

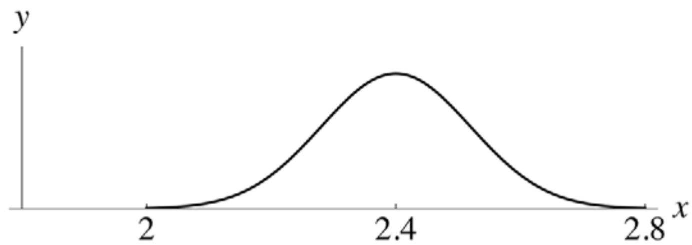
Rating	0	1	2	3	4
Frequency	10	10	30	30	20

(a) Compute the mean $\overline{x}$ and standard deviation s of the above ratings.

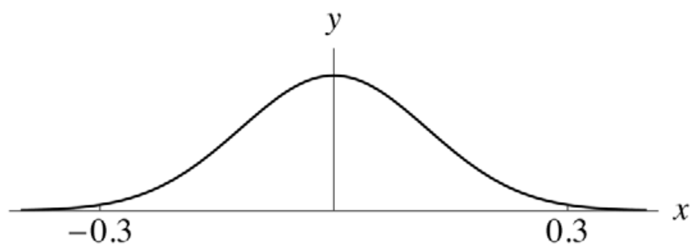
(b) What proportion of the above data is within one standard deviation of the mean? ("Proportion" means the total number of data points satisfying the given criterion, divided by the number of data points in the entire data set.)

(this problem is continued on the next page)

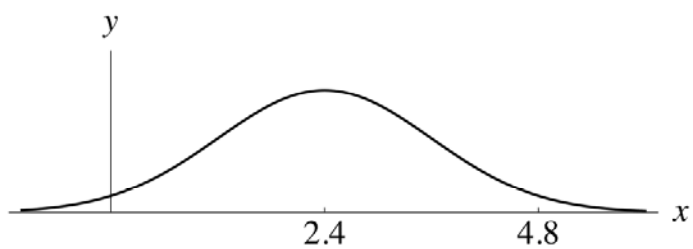
(c) Graduate student Sam Stats computes the mean rating of every single size-100 random sample of CU students. Which of the following graphs – (i), (ii), or (iii) – do you think best reflects the distribution of these sample means? Please explain. (Describe completely how you are eliminating the incorrect choices.)



(i)



(ii)



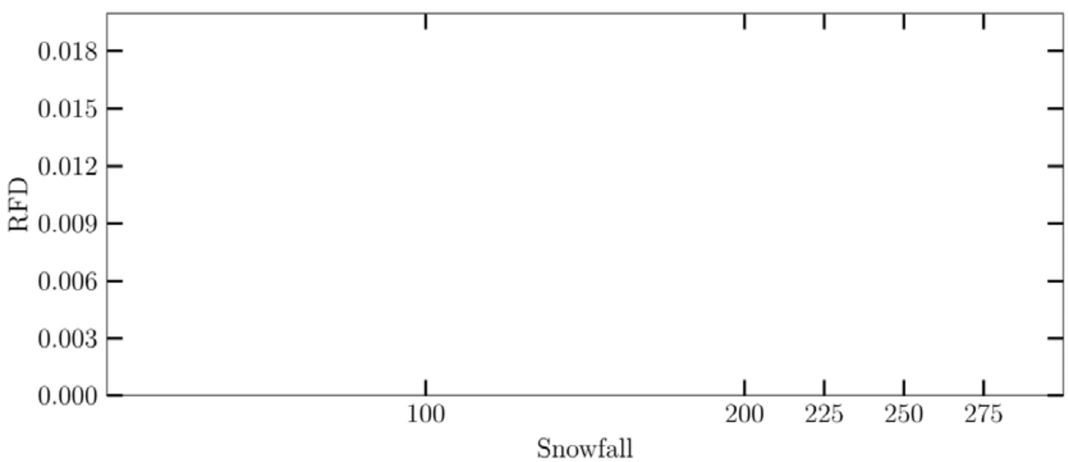
(iii)

10. Each year, the yearly snowfall total in Sapporo, Japan is recorded. Here is a frequency table for this data (in inches) as measured over a period of 75 years:

Snowfall (inches)	Frequency	RFD
[0,100)	4	0.000533
[100,200)	10	
[200,225)	17	
[225,250)	33	0.017600
[250,275)	11	

(a) Complete the third column of the table, by filling in the missing RFD values.

(b) On the axes below, draw an RFD histogram for this data:



(c) Find the area under your RFD histogram on the interval [200,250), and explain the meaning of this area.