

**Fundamental Limitations on the Terminal Behavior of  
Antennas and Nonuniform Transmission Lines**

by

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Fundamental Limitations on the Terminal Behavior of Antennas and Nonuniform Transmission  
Lines

Thesis directed by Prof. Edward F. Kuester

The Authoritative Dictionary of IEEE Standard Terms [1] provides two definitions for the quality factor, or  $Q$ , of a resonant system. These definitions suggest a fundamental relationship between the fractional bandwidth of a resonant system and the ratio of stored energy to dissipated energy within that system. We show this relationship is in general not true. The success of this relationship, however, inspires the research that follows. We seek to find a more general, and fundamental, relationship between the energy within a system and the terminal behavior of that system. We apply our ideas to antennas theory and the theory of nonuniform transmission lines.

## Dedication

*To My God My Strength,  
and to my wife Katie.*

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My greatest delight on earth is my wife Katie. I love her with all of my heart, and without her, this accomplishment has little value. She is the color in all I do. Thank you Love.

Finally, it is only by my God's grace that I finish this thesis. It is His gifts in me, through Christ, that have brought me to where I am. To Him I give all the glory.

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## Chapter 1

### Introduction

$Q$ , or “quality factor,” is a parameter often used to characterize  $RLC$  circuits. Consider, for example, the circuit of Figure 1.1. The transfer function  $V_L(\omega)/V(\omega)$  of the circuit is

$$H(\omega) = \frac{j\frac{\omega}{\omega_0}\frac{1}{Q}}{-\left(\frac{\omega}{\omega_0}\right)^2 + j\frac{\omega}{\omega_0}\frac{1}{Q} + 1} \quad (1.1)$$

where

$$\omega_0 = \frac{1}{\sqrt{LC}} \quad (1.2)$$

is the resonant frequency, and

$$Q = \frac{1}{\omega_0 RC} = \frac{\omega_0 L}{R} \quad (1.3)$$

The 3 dB bandwidth  $\Delta\omega$  is defined as the width of the range of frequencies at which the curve of the transfer function  $|H(\omega)|$  is 3 dB below the peak of the curve. The 3 dB fractional bandwidth  $B_{3dB}$  is

$$B_{3dB} \equiv \frac{\Delta\omega}{\omega_0} \quad (1.4)$$

It can be shown, for the transfer function of Eqn. (1.1), that fractional bandwidth  $B_{3dB}$  is related to  $Q$  by

$$Q = \frac{1}{B_{3dB}} \quad (1.5)$$

$Q$  can also be related to the time-averaged stored energy in this circuit by simple algebraic manipulations

$$Q = \frac{\omega_0 \tilde{E}_{stored}}{P} \quad (1.6)$$

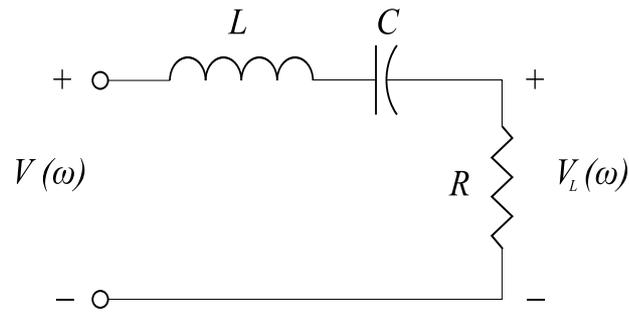


Figure 1.1: RLC circuit.

where  $\tilde{E}_{stored}$  is the time-averaged stored energy within the circuit, and  $P$  is the time-averaged power dissipated in the resistor  $R$ . Clearly,

$$\frac{1}{B_{3dB}} = \frac{\omega_0 \tilde{E}_{stored}}{P} \quad (1.7)$$

The relationship (1.7) is not unique to the circuit of Figure (1.1). A parallel  $RLC$  circuit will also satisfy (1.7). A different system satisfying (1.7) is an electromagnetic mode within a resonant cavity made of lossy metal. The energy  $\tilde{E}_{stored}$  is stored in the field of the mode and the power dissipated  $P$  is lost in the metal walls of the cavity. In fact, there are many resonant systems from engineering and physics where the relationship (1.7) is satisfied (at least approximately). For this reason, it is common to report a value of  $Q$  in association with a resonant system as a figure of merit—large values of  $Q$  indicate that the system has a narrow bandwidth and a large stored energy.

## 1.1 Definitions of $Q$

$Q$  is included in the IEEE dictionary of standards because of its broad applicability to different kinds of resonant systems. But, defining  $Q$  so that (1.6) and (1.5) are true in general is not simple. For the circuit of Figure 1.1, a transfer function was first determined and then parameterized with  $Q$ . It followed that (1.5) and (1.6) could be shown by simple algebra. Because there are so many different kinds of resonant systems with transfer functions that can first be parameterized by  $Q$  and then have the results (1.5) and (1.6) follow, the dictionary of standards simply chooses to define

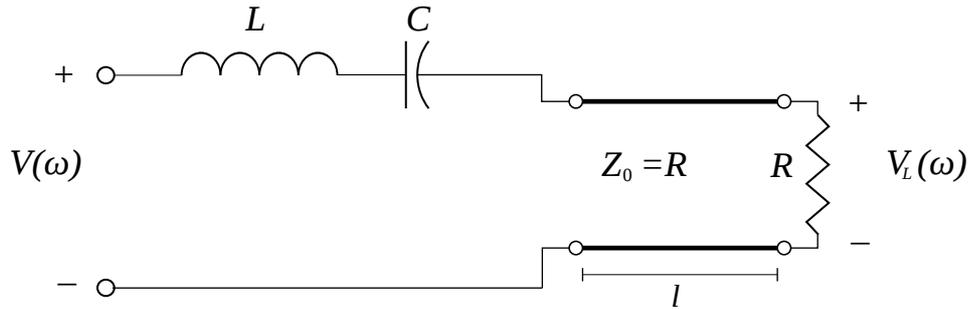


Figure 1.2: RLC circuit, with transmission line.

$Q$  by (1.5) or (1.6). Specifically, in the *The Authoritative Dictionary of IEEE Standard Terms* [1] the definitions are:

*$2\pi$  times the ratio of the maximum stored energy to the energy dissipated per cycle at a given frequency.*

This is equivalent to (1.6). Another definition corresponds to bandwidth:

*An approximate equivalent definition is that the  $Q$  is the ratio of the resonant frequency to the bandwidth between those frequencies on opposite sides of the resonant frequency, where the response of the resonant structure differs by 3 dB from that at resonance.*

This is equivalent to (1.5). There is an obvious problem with these definitions: although there are many systems that do satisfy (1.5) and (1.6), this does not mean that all resonant systems satisfy them.

Figure 1.2 represents a system for which the IEEE definitions are inconsistent. The transmission line in this circuit is terminated in its own characteristic impedance; therefore, the magnitude of the transfer function  $|H(\omega)|$  is identical for both this circuit and the circuit of Figure 1.1. We argue that if the circuit of Figure 1.1 is a resonant system, then the circuit of Figure 1.2 should be one as well. The stored energy of the two circuits, on the other hand, will not be the same (unless  $l = 0$ ). For the circuit of Figure 1.2 the stored energy inside the transmission line can be made arbitrarily large by increasing the length  $l$ . The IEEE definitions suggest that an increase in

the stored energy should lead to a decrease in bandwidth. Clearly, the definitions are inconsistent because the bandwidth of the circuit in Figure 1.2 is the same for any length  $l$ .

## 1.2 Constant Resistance Circuits

There exists a class of circuit known as constant resistance circuits [2]. The transmission line section followed by the resistor  $R$  in Figure 1.2 is in this class. Such circuits are identified by having a constant and real valued input impedance, while having any number of elements within them that can store energy. Multiplexers [3]-[5] are examples of this kind of circuit. We introduce these circuits here because of their importance associated with the example above. Just as the transmission line terminated in its characteristic impedance was used to replace the resistor in the circuit of Figure 1.1, any constant resistance circuit can be used to replace  $R$ . In turn, this means that there are an infinite number of circuits, known as equivalent networks [6, 7], that have the transfer function (1.1). Rather complicated circuits with arbitrary amounts of stored energy will have constant 3 dB bandwidths identical to the circuit of Figure 1.1. The IEEE definitions are inconsistent for all of these.

## 1.3 Purpose of Thesis

Despite these inconsistencies, the definitions of  $Q$  that are in the dictionary of standards suggest an intriguing and fundamental idea. That is, the terminal behavior of a resonant system somehow corresponds to the energy inside that system. This idea inspires the research of this thesis. In the following chapters we seek to find limitations on the terminal behavior of linear systems, and to relate these limitations to the energy inside those systems. These ideas will be used to explore fundamental limitations on antennas, as well as nonuniform transmission lines.

## 1.4 Fundamental Limitations of Antennas

In antenna theory, fundamental limitations are important and have been the topic of much research: [8]-[25] (to cite only a few). Since many antennas are narrow band devices, it would

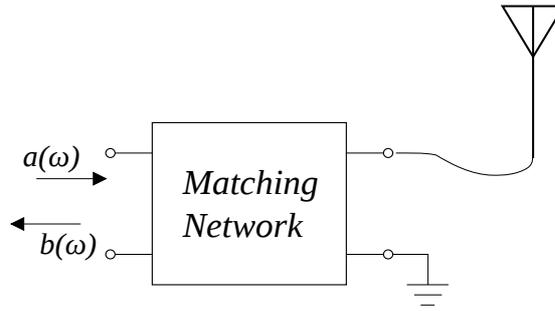


Figure 1.3: One port network. The total time-averaged energy stored in the matching network plus the energy stored in the field of the antenna is infinite.

appear that  $Q$  might apply naturally. However, determining the stored energy of an antenna is problematic by nature.

Consider the one-port network shown in Figure 1.3. The input and output signals are the waves  $a(\omega)$  and  $b(\omega)$ , the transfer function is the reflection coefficient  $\rho(\omega)$ . The energy stored inside the system is contained in the network used to match the input to the antenna, as well as in the field generated by the current of the radiating antenna. The time-harmonic field generated by the antenna occupies all of space, and when the total energy of this field is calculated, the result is infinite. With the current definitions of  $Q$ , an infinite stored energy suggests that the bandwidth at the input terminals must be zero. But, it is well known that all physical antennas have finite, non-zero bandwidths. The definitions of  $Q$ , therefore, are inconsistent.

A parallel can be drawn between the resonant antenna of Figure 1.3 and the circuit of Figure 1.2. If the length  $l$  of the transmission line is allowed to go to infinity, the system will have a finite bandwidth but an infinity stored energy—as was the case for the antenna of Figure 1.3.

Notice that the act of adding the transmission line to the simple  $RLC$  circuit (see Figure 1.2) did not change the terminal behavior of the system. We might conclude, then, that the energy stored in the transmission line does not affect the bandwidth of the system, but only that energy that resides in the capacitor  $C$  and the inductor  $L$ . This leads us to the question: is there a way to separate a portion of the stored energy within a system that makes the definitions (1.5) and (1.6) consistent when that energy is used in place of  $\tilde{E}_{stored}$ ?

We now discuss some of the more important work carried out over the last century, and done for the purpose of understanding fundamental antenna limitations.

Wheeler [8]-[10] and Chu [11] were apparently the first to explore fundamental limitations of antennas. Although Wheeler did not use the symbol  $Q$  directly, a closely related term “power factor” was used and is similar to the definition given by (1.3). Chu on the other hand, based his work on the energy definition of  $Q$ . He too uses circuit models that model the input behavior of the antenna and determines the stored energy within. Harrington [12] followed Chu’s ideas and elaborated on them. It was mentioned above that the total stored energy outside of an antenna at a single frequency is not finite; in 1964 Collin and Rothschild [13] introduced a kind of energy based on subtracting the energy of the radiated field from the total energy, the result is a finite energy, which they used for their definition of  $Q$ . Following Collin and Rothschild, other authors such as Fante [14] and McLean [15] extend Collin and Rothschild’s idea to be applicable more generally. Levis [16], and Rhodes [17, 18] chose to base their definition of  $Q$  on bandwidth. But, as it is pointed out by Yaghjian and Best [19], the reactance doesn’t provide a unique method for determining bandwidth. More recently there have been a number of people working on Bode-Fano like limitations for antennas [20]-[24]. These sort of limitations have not yet been tied to stored energy, but may lead there in future work. Despite all attempts since the late 1940s, fundamental limitations of antennas (especially those based on  $Q$ ) have yet to find a firm footing in physics: all proposed attempts have certain undesired problems. These problems are illustrated below through some of the more important work done in the last century.

Wheeler based his work [8] on modeling electrically small antennas with simple circuit. From the models, he gave expressions for antenna efficiencies based on parameters he called radiation power factors. The power factors are unitless quantities which are closely related to the definition of  $Q$ , given by (1.3). Without resorting to any ideas about stored energy, Wheeler makes claims about the limitations on the bandwidth of small antennas via basic circuit analysis.

Apparently unaware of Wheeler’s work, Chu studied the physical limitations of antennas

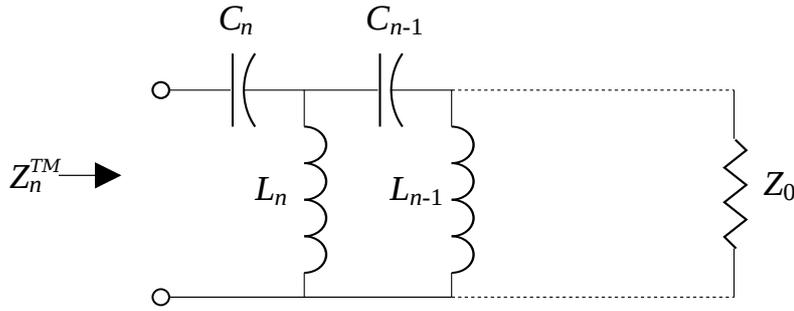


Figure 1.4: Equivalent circuit of TM spherical wave.

using the energy definition of  $Q$  [11] (1948). To obtain his results, Chu was content with calculating only the stored energy outside a sphere that was as small as possible, and yet completely contained the arbitrarily shaped antenna of interest—a so-called minimum sphere. He ignored the energy inside the sphere, arguing that this energy would only increase the total stored energy; since his interest was to find a bound, an energy smaller than the total stored energy was acceptable. He expanded the field outside the minimum sphere into spherical modes and from the field components of each of these modes, he determined an impedance. Focusing on the transverse magnetic modes, Chu determined that each mode had an impedance of

$$Z_n^{TM}(x) = jZ_0 \frac{[xh_n(x)]'}{xh_n(x)} \quad (1.8)$$

where

$$x = kr \quad (1.9)$$

Here  $Z_0$  is the characteristic impedance of free space,  $h_n(x)$  is the spherical Hankel function of the second kind,  $k = 2\pi/\lambda$  where  $\lambda$  is the wavelength, and  $r$  is the radius of the minimum sphere. This impedance by itself has no obvious relationship to the energy stored outside the minimum sphere; however, Chu cleverly recognized that a circuit could be synthesized from the impedance (1.8), and his particular circuit has the form of that in Figure 1.4. The capacitors and inductors can be determined from

$$C_n = \frac{x}{Z_0 n \omega}$$

and

$$L_n = \frac{Z_0 x}{n\omega}$$

From the synthesized circuit, stored energy and dissipated power can be calculated for the circuit; Chu assumed these quantities obtained from the circuit were the correct quantities for the stored energy and dissipated power of the field due to a mode, outside the minimum sphere. Though his method is clever, it is unclear if the circuit of Figure 1.4 represents *the* circuit for the given impedance (1.8). For instance the resistor  $Z_0$  can be replaced by any constant resistance circuit (see Section 1.2) and  $Z_n^{TM}$  will not change. There may exist a different circuit that stores less than the energy of Chu's circuit, and therefore the result Chu obtained would not be a lower bound on energy.

Collin and Rothschild proposed a different method for calculating  $Q$  [13] that is simpler to calculate than the one suggested by Chu. Chu [11] pointed out that after the first couple of modes, calculation of his stored energy becomes unwieldy. Collin and Rothschild base their work on the fact that the time-averaged power  $P$  flowing through a sphere centered on an antenna will be constant. It is certainly true that at distances far enough away from the antenna the fields will radiate at the speed of light  $c$ , and in some region of that distance you may conclude that the energy per unit length along the radial direction is simply

$$U_r = \frac{P}{c} \tag{1.10}$$

From Pointing's theorem, the *total* energy per unit length anywhere outside the minimum sphere containing the antenna is

$$U_t(r) = \frac{1}{4} \int_{\Omega} [\varepsilon |\mathbf{E}|^2 + \mu |\mathbf{H}|^2] r^2 d\Omega \tag{1.11}$$

Here,  $\mathbf{E}$  and  $\mathbf{H}$  are the electric and magnetic fields generated by the antenna,  $\varepsilon$  and  $\mu$  are the permittivity and permeability respectively, and the integral is to be taken over all  $4\pi$  steradians of solid angle  $\Omega$ . Collin and Rothschild's method for calculating stored energy<sup>1</sup> was to assume

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<sup>1</sup> This energy subtraction method was used earlier by Kessenikh in 1939 [26] to determine stored energy around an antenna.

that the energy (1.10) represented the *radiated* energy throughout all space outside of a minimum sphere, so that the total stored energy outside the minimum sphere could be obtained by

$$W_{stored} = \int_a^{\infty} \left[ U_t(r) - \frac{P}{c} \right] dr \quad (1.12)$$

Because of the spherical symmetry, the fields outside of the minimum sphere may be expanded into spherical modes. Collin and Rothschild's results yield a rather simple expression for the  $Q$  of each of these modes, and is much easier to calculate than Chu's method. Although simple in principle, this method has a number of problems. Levis [16] pointed out that the method wrongly assumes the speed of the radiated energy is constant throughout the sphere. This must be postulated and its truth is not obvious. Another defect is that the method requires (1.10) and (1.11) to be subtracted before they are integrated in (1.12), otherwise the stored energy would be

$$W_{stored} = \int_a^{\infty} U_t(r) dr - \int_a^{\infty} \frac{P}{c} dr$$

where each integral by itself is infinite. There are an infinite number of ways to subtract infinity from infinity to get a finite number.

The ambiguity in finding a unique stored energy of an antenna led researchers to pursue limits based on the bandwidth definition of  $Q$  (1.5). Levis [16] and Rhodes [18, 17] claimed that stored energy could not be defined rigorously. Bandwidth, on the other hand, is a value that can be observed by measurement. To make their work rigorous, Rhodes pointed out that the bandwidth definition of  $Q$  approaches asymptotically to

$$Q = \frac{1}{B_{3dB}} \sim \frac{\omega_0 |X'_0|}{2R_0} \quad (1.13)$$

for large  $Q$ .  $R_0$  is the input resistance to the antenna at resonance,  $X_0$  is the input reactance at resonance and the prime denotes differentiation with respect to frequency  $\omega$ . Rhodes further defines what he calls the observable stored energy  $\langle\langle U \rangle\rangle$  via

$$Q = \frac{\omega_0 \langle\langle U \rangle\rangle}{P}$$

where  $Q$  is the asymptotic value (1.13). Since the true stored energy cannot be observed by measurement,  $\langle\langle U \rangle\rangle$  is defined this way assuming that energy and bandwidth are fundamentally related by (1.6) and (1.5). The input resistance and reactance of the antenna can be determined from the fields at the input port of the antenna

$$R(\omega) + jX(\omega) = \frac{-\frac{1}{2} \int_s \mathbf{E} \times \mathbf{H}^* \cdot \hat{\mathbf{n}} da}{\frac{1}{2} |I|^2}$$

where  $I$  is the current fed to the antenna and  $s$  is the surface at the input port. The so-called reactance theorem [17] may be used to find an expression for  $X'(\omega)$  in terms of the fields surrounding the antenna. Provided that the asymptotic formula (1.13) is a reasonable measure of bandwidth, the work done by Rhodes supplies a rigorous definition of antenna  $Q$ .

Yaghjian and Best [19] show that a definition of bandwidth based on the derivative of reactance does not always exist. The problem arises when antennas are tuned in antiresonance frequency ranges. In these ranges, the expression (1.13) does not represent an accurate approximation for bandwidth. Yaghjian and Best introduce the Matched VSWR Bandwidth. Their definition is attractive because it is well defined over all frequency ranges—even in antiresonance regions. Their  $Q$  based on the Matched VSWR Bandwidth is

$$Q_{MVB} \sim \frac{\omega_0 |Z'_0|}{2R_0}$$

where

$$Z_0 = R_0 + jX_0$$

Gustafsson, Sohl and Kristensson [20]-[22] use a different approach for determining antenna limitations based on Bode-Fano limitations. Almost all of the work described above is restricted to antennas that are placed inside a hypothetical sphere. Gustafsson et. al. are not limited by this restriction, and have considered arbitrary shapes. Another advantage of their technique is that the Bode-Fano type limitations can be applied unambiguously to the transfer function of a system as a measure of the terminal behavior, which cannot be said about the measure of  $B$  as it has been introduced in this chapter.

Interesting, and very recent, work done by Yaghjian [25] attempts to approximate the stored energy of an electrically small antenna using the static limit of the fields about the antenna. This work also has the advantage of being independent of having to assume that the antenna resides within a minimum sphere. Results obtained by Yaghjian show very similar results to Gustafsson et. al. who determined their limits only from the properties of the transfer function. It would thus seem that a connection could be made between stored energy and the Bode-Fano type limits. But, this has yet to be determined.

## 1.5 Terminal Behavior

A 3 dB bandwidth is one measure of the terminal behavior of a system, but this form of measurement gives only limited information about how the input and output terminals of a system behave. 3 dB bandwidth is well suited for describing the width of certain kinds of transfer functions, but it is also known that there are other methods for measuring the width of a curve, and that this number alone can be misleading. The standard deviation of a curve, for instance, is another way to measure width, and it may be that this width has advantages over 3 dB bandwidth. There does not exist a universal method for associating a width to a curve, and therefore, other forms of characterizing the transfer function of a system should be considered. For instance, a bandwidth measurement does not (usually) take into consideration the phase of the transfer function. The phase behavior of a system can affect the fidelity with which a signal is transmitted, and it may be that the energy inside of a system not only affects the magnitude of the transfer function, but the phase as well.

## 1.6 Thesis Overview

In this thesis we study limitations on the terminal behavior of linear systems, and seek to relate these limitations to the energy inside those systems. We have shown in this chapter the problems that arise when  $Q$  is used improperly; the following chapters seek to determine if there are definitions similar to the IEEE definitions (see Section 1.1) that are more generally consistent.

This work is important to the study of antenna theory and nonuniform transmission lines. Clear from the discussion above, there is great interest in understanding fundamental limitations of antennas. But, since the definitions of  $Q$  are not always consistent, much of the work done in the past is either in question or wrong. We are also interested in limitations on nonuniform transmission lines used as matching circuits. Nonuniform lines are used to match loads over broad frequency ranges, and understanding fundamental limitations on these devices is useful to designers.

To lay our work on a solid foundation, Chapter 2 introduces basic concepts and notation.

Chapter 3 is a study of the terminal behavior of systems. We introduce methods for measuring both bandwidth and distortion. A couple of these measures require the use of numerical calculation, and we discuss the value of these calculations in seeking fundamental limitations. We will also introduce Bode-Fano limitations. At the end of the chapter a new bound for nonuniform transmission lines is discovered; we discuss its implications and its relationship to the energy within the transmission line.

Recoverable energy from a one-port network will be derived in Chapter 4. This is a new type of energy that can be determined from the terminals of a system alone. We will show that this energy is better suited, in general, than stored energy for the energy definition of  $Q$  (1.6). Chapter 4 will develop the theory of recoverable energy from a one-port network, and several examples will be provided to show the value of this energy. We show, for instance, that recoverable energy is the energy that resides in the capacitor and inductor of Figure 1.2. Therefore, recoverable energy solves the problem of how we separate the portion of stored energy in the system that makes the  $Q$  definitions (see (1.5) and (1.6)) consistent, from the stored energy within the transmission line (see Section 1.4). We will also use recoverable energy to verify Chu's results (see Section 1.4). We will show that his lowest order circuit representing an electric dipole, does indeed store the smallest amount of energy that can be determined from the terminal characteristics alone. We end the chapter by showing that recoverable energy is equal to the stored energy of a minimum phase Darlington circuit. Minimum phase Darlington circuits can be synthesized from an understanding

of the terminal behavior of a system alone. Such circuits have interesting properties which will lead us into a discussion about other energies that would be interesting to study in future research.

Chapter 5 is about transferrable energy. The majority of this thesis is concerned with single-port systems. Transferrable energy, on the other hand, provides insight into two-port systems. We discuss the definition of transferrable energy and show how it can be used for characterizing systems. Closely related to bandwidth is the idea of a bitrate, we define bitrate in this chapter and show how it is related to transferrable energy. An example is considered and the relationship between energy and bandwidth is again discussed.

Chapter 6 summarizes the thesis and provides insight and direction for future research.

## Chapter 2

### Basic Concepts and Notation

In the introduction, we used the word “system” without definition. The intent of this chapter is to lay down a foundation for our work build upon. We assume that the audience has a good understanding of linear systems; but, carefully defining ideas such as “system” or “passivity” will help ensure our discussions in subsequent chapters are made as clear as possible.

We will introduce notation and several definitions below to define the kind of systems we will be discussing in the thesis. We begin by introducing notation for functions of both time and frequency, and we will define the Fourier transform that relates these functions between the two domains. Next we will define what we mean by a system and a network and provide notation that will remain consistent throughout all of the chapters. Finally, we will define passivity and causality for systems. After we have made all the necessary definitions, we clearly state the kind of systems considered in this thesis.

#### 2.1 Functions of Time and Frequency

Functions of time will have a superscribed caret or “hat,”

$$\hat{a}(t) \tag{2.1}$$

In the frequency-domain, the caret is removed and the resulting function is related to the time-domain function by

$$a(\omega) = (\mathcal{F} \hat{a})(\omega) \tag{2.2}$$

where the operator  $\mathcal{F}$  represents the Fourier transform

$$(\mathcal{F} \hat{a})(\omega) = \int_{-\infty}^{\infty} \hat{a}(t) \exp(-j\omega t) dt \quad (2.3)$$

The inverse Fourier transform is

$$(\mathcal{F}^{-1} a)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} a(\omega) \exp(j\omega t) dt \quad (2.4)$$

## 2.2 Systems

For our purposes, a *system* is a physical device that can be modeled mathematically as:

$$\begin{aligned} \hat{b}(t) &= \hat{\rho}(t) * \hat{a}(t) \\ &= \int \hat{\rho}(t - \tau) \hat{a}(\tau) d\tau \end{aligned} \quad (2.5)$$

The function  $\hat{a}(t)$  represents the input signal to the device and the function  $\hat{b}(t)$  represents the output;  $\hat{\rho}(t)$  is the impulse response that characterizes the system. Systems defined with the definition above are generally known as linear and time-invariant systems (see [27]). The representation (2.5) can be derived from an axiomatic treatment [28] whereby a general system is restricted to be linear and time-invariant; nevertheless, the definition given above is sufficient for the proceeding study. In the frequency-domain Eqn. (2.5) becomes simply

$$b(\omega) = \rho(\omega) a(\omega) \quad (2.6)$$

The function  $\rho(\omega)$  will be referred to as the *transfer function* of the system. Systems with two inputs and two outputs will also be considered provided they can be modeled as:

$$\begin{aligned} \hat{b}_1(t) &= \hat{S}_{11}(t) * \hat{a}_1(t) + \hat{S}_{12}(t) * \hat{a}_2(t) \\ \hat{b}_2(t) &= \hat{S}_{21}(t) * \hat{a}_1(t) + \hat{S}_{22}(t) * \hat{a}_2(t) \end{aligned} \quad (2.7)$$

and in the frequency domain this becomes

$$\begin{bmatrix} b_1(\omega) \\ b_2(\omega) \end{bmatrix} = \begin{bmatrix} S_{11}(\omega) & S_{12}(\omega) \\ S_{21}(\omega) & S_{22}(\omega) \end{bmatrix} \begin{bmatrix} a_1(\omega) \\ a_2(\omega) \end{bmatrix} \quad (2.8)$$

### 2.3 Networks

In this thesis, one and two-port networks are considered only. The functions  $a_k(\omega)$  and  $b_k(\omega)$  can be regarded as wave-amplitudes, and  $S_{nm}(\omega)$  of Eqn. (2.8) are the scattering parameters (see Section 4.3 of [29]). For a one-port network  $\rho(\omega)$  (2.6) is the reflection coefficient. It is somewhat unconventional to work with the time-domain wave-amplitudes  $\hat{a}_k(t)$  and  $\hat{b}_k(t)$ ; however, in Chapters 4 and 5 we will see that these signals are quite useful for formulating recoverable and transferrable energy. The wave-amplitudes are normalized such that the power entering port  $k$  is

$$\hat{P}(t) = \hat{a}_k^2(t) - \hat{b}_k^2(t) \quad (2.9)$$

### 2.4 Passivity

Passivity is defined in terms of the wave-amplitudes  $\hat{a}_k(t)$  and  $\hat{b}_k(t)$ . A system that has  $N$  ports is *passive* when

$$\sum_{k=1}^N \int_{-\infty}^{\infty} \left[ \hat{a}_k^2(t) - \hat{b}_k^2(t) \right] dt \geq 0 \quad (2.10)$$

for all input signals  $\hat{a}_k(t)$  with corresponding output signals  $\hat{b}_k(t)$ . Physically, the definition implies that a system itself cannot supply energy to the outside world, it can only store it or dissipate it.

It is of interest to note that there is a more restrictive definition of passivity. In this definition, a system is passive if

$$\sum_{k=1}^N \int_{-\infty}^t \left[ \hat{a}_k^2(\tau) - \hat{b}_k^2(\tau) \right] d\tau \geq 0 \quad (2.11)$$

for all  $t$ ; it implies that there is no instance in time  $t$  for which the system is able to supply energy to the outside world. The definition (2.10) only restricts the total energy over all time to be non-negative; however, (2.10) permits short periods of time for which energy can be supplied from the system—a physical device, with no sources inside, would not permit this. One important consequence of the more restrictive definition (2.11) is that a linear time-invariant and passive system will always be causal (see [30]). For this thesis, we choose to make causality an added

assumption about our systems, and although the definition (2.10) is less physically correct than (2.11), the first definition will be sufficient for the chapters that follow.

## 2.5 Causality

Again, only systems that can be modeled by the convolution (2.5) will be considered in this thesis; for this reason, a system is *causal* if the impulse response  $\hat{\rho}(t)$  of the system satisfies

$$\hat{\rho}(t) = 0 \quad \text{for } t < 0 \tag{2.12}$$

We may now clearly state: all systems in this thesis are modeled by a convolution integral relating inputs to their corresponding outputs, and these systems are both passive and causal.

## 2.6 Summary

In this chapter we introduced notation and definitions for the purpose of clearly stating what we mean by a system. These definitions will remain consistent throughout the thesis.

## Chapter 3

### Terminal Behavior

The transfer function of a system contains the information necessary for characterizing what we call terminal behavior. Any number of operations may be applied to a transfer function to learn something about the properties of a system. We may view these properties as observables—to borrow an idea from quantum mechanics. A transfer function along with the various operations applied to it is what we mean by terminal behavior. We discuss in this chapter certain operations that can be applied to a system's transfer function. One of our goals is to determine whether or not these observed quantities are related to energy within the system.

Fractional bandwidth, introduced in Chapter 1, is one form of measure determined by operations applied to the transfer function. It is not the only possible measure of bandwidth, and others will be discussed in this chapter. It may be that energy within a system is not only affected by the width of the transfer function, but also by its phase. The phase can potentially distort the shape of a signal passed through a system. We will discuss a definition of distortion that may be related to the energy inside the system. Another form of measure applied to the transfer function has the form

$$\int_{-\infty}^{\infty} f(|H(\omega)|) d\omega \quad (3.1)$$

where  $f(\cdot)$  is some known function (see p. 78 of [31]). Such operations are not affected by the phase of the transfer function. Bode-Fano limitations are a measure of this kind, and they pivot on understanding the causality of the system.

The Bode-Fano limitations are intriguing in the study of matching circuits. At the end of this chapter, we find a Bode-Fano limitation for nonuniform transmission lines. This is a new result, and demonstrates the advantage that measures of the form (3.1) have over others. This form of measure can be applied unambiguously to transfer functions, provided the integral in (3.1) exists.

This chapter defines different quantities for measuring terminal behavior and will provide examples for how those quantities can be applied. We will begin by discussing a couple of different measures of bandwidth, including what we call *energy* bandwidth. We will show how energy bandwidth can be calculated and will provide an example of its use. Next, we will define distortion. In this thesis, distortion measures the deviation of the shape of a pulse passed through a system; we will provide an example that calculates the distortion of a pulse passed through an allpass circuit. Finally, we will discuss Bode-Fano limitations and show how they may be used to obtain a bound on the terminal behavior of a nonuniform transmission line.

### 3.1 Bandwidth

The 3 dB bandwidth defined in Chapter 1 is one example of a bandwidth, which could be determined analytically for the circuit in Figure 1.1. For that particular circuit the bandwidth was inversely proportional to the energy stored within, which implies that wide band signals could be transferred through the system more easily when the stored energy of the circuit is low.

The standard deviation of a function is another common measure of its width. For a frequency spectrum, it can be defined as

$$\sigma(|H(\omega)|) = \sqrt{\frac{\int_{-\infty}^{\infty} (\omega - \omega_0)^2 |H(\omega)| d\omega}{\int_{-\infty}^{\infty} |H(\omega)| d\omega}} \quad (3.2)$$

where  $\omega_0$  is the center frequency. Although this measure has been useful in other areas of physics and engineering, it is not well suited here. For example, the RLC circuit of Figure 1.1 in the introduction has a standard deviation of infinity.

### 3.1.1 Energy Bandwidth

An interesting measure of bandwidth, studied by Uffink [32, 33], is one which we will refer to as “energy bandwidth”. Let

$$f(\alpha, \beta) = \int_{\alpha - \Delta\omega_E/2}^{\alpha + \Delta\omega_E/2} \psi(\omega) d\omega \quad (3.3)$$

where

$$\psi(\omega) = \frac{|H(\omega)|^2}{\int_0^\infty |H(\omega)|^2 d\omega}$$

Define the bandwidth as the minimum value of  $\Delta\omega_E$  over all  $\alpha$  such that  $f(\alpha, \Delta\omega_E) = C$ , where  $C$  is a constant between 0 and 1. If this minimum is attained for a unique value of  $\alpha$ ,  $\alpha$  can be thought of as a center frequency. The integral (3.3) can be thought of as the fraction of energy in  $|H(\omega)|$ , if for the moment we regard  $|H(\omega)|$  as a pulse whose square magnitude has units of power, about the frequency  $\alpha$  within the band  $\Delta\omega_E$ . For example, if  $C = 1/2$  we seek to find the smallest band  $\Delta\omega_E$  for some frequency  $\alpha$  over which the energy constitutes half of the total energy. In general, determining this width requires numerical calculations.

To determine the width  $\Delta\omega_E$ , let  $f(\alpha, \Delta\omega_E) = C$  be written as

$$f(\mathbf{r}) = C \quad (3.4)$$

where  $\mathbf{r} = [\alpha, \Delta\omega_E(\alpha)]$ . Let  $\mathbf{r}_0$  be a point that satisfies (3.4) and  $\mathbf{r}_0 + \Delta\mathbf{r}$  be another point, then

$$f(\mathbf{r}_0 + \Delta\mathbf{r}) = C$$

and for small  $\Delta\mathbf{r}$

$$\begin{aligned} f(\mathbf{r}_0 + \Delta\mathbf{r}) &= f(\mathbf{r}_0) + \nabla f(\mathbf{r}_0) \cdot \Delta\mathbf{r} + \mathcal{O}(|\Delta\mathbf{r}|^2) \\ &= C + \nabla f(\mathbf{r}_0) \cdot \Delta\mathbf{r} + \mathcal{O}(|\Delta\mathbf{r}|^2) \end{aligned}$$

thus

$$\nabla f(\mathbf{r}_0) \cdot \Delta\mathbf{r} = \mathcal{O}(|\Delta\mathbf{r}|^2)$$

which may be rewritten as

$$\begin{aligned}\nabla f(\mathbf{r}_0) \cdot \left[ \Delta\alpha, \frac{\partial}{\partial\alpha}\beta(\alpha)\Delta\alpha \right] &= \mathcal{O}\left(|\Delta\mathbf{r}|^2\right) \\ \nabla f(\mathbf{r}_0) \cdot \left[ 1, \frac{\partial}{\partial\alpha}\beta(\alpha) \right] &= \frac{\mathcal{O}\left(|\Delta\mathbf{r}|^2\right)}{\Delta\alpha}\end{aligned}$$

letting  $|\Delta\mathbf{r}| \rightarrow 0$ , we obtain a differential equation for the width  $\Delta\omega_E$  as a function of center frequency  $\alpha$ :

$$\frac{\partial}{\partial\alpha}f(\alpha, \Delta\omega_E) + \frac{\partial}{\partial\Delta\omega_E}f(\alpha, \Delta\omega_E) \frac{d}{d\alpha}\Delta\omega_E(\alpha) = 0$$

A minimum  $\Delta\omega_E(\alpha)$  will occur at  $\alpha$  where  $\Delta\omega'_E(\alpha) = 0$ , and therefore when

$$\frac{\partial}{\partial\alpha}f(\alpha, \Delta\omega_E) = 0 \tag{3.5}$$

From Leibnitz's Rule,

$$\frac{\partial}{\partial\alpha}f(\alpha, \Delta\omega_E) = \psi(\alpha + \Delta\omega_E/2) - \psi(\alpha - \Delta\omega_E/2) = 0 \tag{3.6}$$

The expressions (3.4) and (3.6) represent two equations that need to be solved for the unknowns  $\alpha$  and  $\Delta\omega_E$ .

Note that if

$$\int_{\alpha - \Delta\omega_E/2}^{\alpha + \Delta\omega_E/2} \psi(\omega) d\omega = C \tag{3.7}$$

then

$$\frac{C}{\psi_{\max}} \leq \Delta\omega_E \tag{3.8}$$

and from Chebyshev's inequality, it can be shown that

$$\Delta\omega_E \leq \frac{2}{\sqrt{1-C}} \sigma(|H(\omega)|)$$

This is a result due to Uffink [33].

As an example, consider the Lorentzian

$$\psi(\omega) = \frac{\sigma}{N\pi} \frac{1}{(\omega - \omega_0)^2 + \gamma^2} \tag{3.9}$$

where

$$N = 1 + \frac{2}{\pi} \tan^{-1} \left( \frac{\omega_0}{\gamma} \right) \quad (3.10)$$

Using (3.4) and (3.6) to solve for  $\alpha$  and  $\Delta\omega_E$ , we have

$$\tan^{-1} \left( \frac{\alpha + \Delta\omega_E/2 - \omega_0}{\gamma} \right) - \tan^{-1} \left( \frac{\alpha - \Delta\omega_E/2 - \omega_0}{\gamma} \right) = \frac{\pi CN}{2} \quad (3.11)$$

and

$$\frac{1}{\gamma^2 + (\alpha - \omega_0 + \Delta\omega_E/2)^2} = \frac{1}{\gamma^2 + (\alpha - \omega_0 - \Delta\omega_E/2)^2}$$

From symmetry,  $\alpha = \omega_0$ , and from (3.11)

$$\Delta\omega_E = 2\gamma \tan \frac{\pi CN}{4} \quad (3.12)$$

This may be compared with 3 dB bandwidth  $\Delta\omega_{3dB}$  which is

$$\Delta\omega_{3dB} = 2\gamma \quad (3.13)$$

and if  $C = 1/2$

$$\Delta\omega_E = 2\sigma \tan \left[ \frac{\pi}{8} \left( 1 + \frac{2}{\pi} \tan^{-1} \left( \frac{\omega_0}{\sigma} \right) \right) \right] \quad (3.14)$$

Clearly

$$\Delta\omega_E \leq \Delta\omega_{3dB} \quad (3.15)$$

and for  $\omega_0 \gg \sigma$

$$\Delta\omega_E \approx \Delta\omega_{3dB} \quad (3.16)$$

As another example, consider the transfer function of an electric dipole. Let

$$\psi(\omega) = \frac{\sqrt{N}}{\omega^2 - \omega_0^2 - i\gamma\omega} \quad (3.17)$$

be the transfer function relating the magnitude of the field  $\mathbf{E}$ , of an incident electromagnetic plane wave, to the magnitude of the dipole moment  $\mathbf{p}$  (see p. 309 of [34]).  $N$  is a normalization factor

$$N = \frac{\gamma\omega_0 \sqrt{4\omega_0^2 - \gamma^2}}{\pi\sqrt{2} \operatorname{Re} G} \quad (3.18)$$

and the constant  $G$  is

$$G = \frac{1}{\sqrt{2 - \frac{\gamma^2}{\omega_0^2} + i\sqrt{4\frac{\gamma^2}{\omega_0^2} - \frac{\gamma^4}{\omega_0^4}}}} \quad (3.19)$$

To find  $\beta$  and  $\alpha$ , the two equations and two unknowns are

$$\frac{2}{\pi \operatorname{Re} G} \begin{pmatrix} \operatorname{Re} \left[ G \tan^{-1} \left( \frac{iG\sqrt{2}(\alpha + \Delta\omega_E/2)}{\omega_0} \right) \right] \\ - \operatorname{Re} \left[ G \tan^{-1} \left( \frac{iG\sqrt{2}(\alpha - \Delta\omega_E/2)}{\omega_0} \right) \right] \end{pmatrix} = C$$

$$\frac{N}{\left( (\alpha + \Delta\omega_E/2)^2 - \omega_0^2 \right)^2 + \gamma^2 (\alpha + \Delta\omega_E/2)^2} = \frac{N}{\left( (\alpha - \Delta\omega_E/2)^2 - \omega_0^2 \right)^2 + \gamma^2 (\alpha - \Delta\omega_E/2)^2}$$

then  $\alpha$  is readily identified as

$$\alpha = \omega_0 \sqrt{1 - \frac{2\gamma^2 + \Delta\omega_E^2}{4\omega_0^2}} \quad (3.20)$$

and we find

$$\frac{2}{\pi \operatorname{Re} G} \begin{pmatrix} \operatorname{Re} \left[ G \tan^{-1} \left( iG\sqrt{2} \left( \sqrt{1 - \frac{\gamma^2}{2\omega_0^2} - \frac{\Delta\omega_E^2}{4\omega_0^2} + \frac{\Delta\omega_E}{2\omega_0}} \right) \right) \right] \\ - \operatorname{Re} \left[ G \tan^{-1} \left( iG\sqrt{2} \left( \sqrt{1 - \frac{\gamma^2}{2\omega_0^2} - \frac{\Delta\omega_E^2}{4\omega_0^2} - \frac{\Delta\omega_E}{2\omega_0}} \right) \right) \right] \end{pmatrix} = C$$

For  $\omega_0 \gg \gamma, \Delta\omega_E$  we have  $\alpha \approx \omega_0$ . With  $C = 1/2$

$$\Delta\omega_E \approx \gamma \quad (3.21)$$

Comparing this result with the 3 dB bandwidth for  $\omega_0 \gg \gamma$ , we find that

$$\Delta\omega_{3dB} \approx \Delta\omega_E \quad (3.22)$$

A numerical implementation of the calculation discussed above is easily achieved. A particularly interesting circuit used in the discussions in Chapter 4 (see Figure 4.6) has the reflection coefficient

$$\rho(\omega) = \frac{-LCR_2\omega^2 + j\omega(L + CR_2(R_1 - Z_0)) + (R_1 + R_2 - Z_0)}{-LCR_2\omega^2 + j\omega(L + CR_2(R_1 + Z_0)) + (R_1 + R_2 + Z_0)} \quad (3.23)$$

In the limit as  $R_2$  goes to infinity, the circuit reduces to the simple series RLC circuit found in the introduction (see Figure 1.1). The circuit is resonant at

$$\omega_o = \sqrt{\frac{1}{LC} - \left(\frac{1}{CR_2}\right)^2} \quad (3.24)$$

We choose the normalizations

$$w = \frac{\omega}{\omega_o} \quad (3.25)$$

$$\theta = R_2 \sqrt{\frac{C}{L}} \quad (3.26)$$

and

$$\zeta = \sqrt{\frac{L}{R_1^2 C}} \quad (3.27)$$

so that the reflection coefficient matched at  $\omega_o$  is

$$\rho(\omega) = \frac{-w^2 \left(1 - \frac{1}{\theta^2}\right) + 2jw\sqrt{\theta^2 - 1} \frac{1}{\theta^2} + \left(1 + \frac{1}{\theta^2}\right)}{-w^2 \left(1 - \frac{1}{\theta^2}\right) + 2jw\sqrt{\theta^2 - 1} \left(\frac{1}{\theta^2} + \frac{1}{\theta\zeta}\right) + \left(1 + \frac{2}{\theta\zeta} + \frac{1}{\theta^2}\right)} \quad (3.28)$$

We used Matlab to plot  $1 - |\rho(\omega)|^2$  when  $\theta = 8$  and  $\zeta = 5$  (see Figure 3.1). In this case  $R_2$  is large, therefore the circuit in Figure 1.1 is “nearly” equivalent to a series RLC circuit. The shaded region represents the band of frequencies determined by the energy bandwidth measurement discussed above. Calculation determined that  $\alpha = 1.0096$  (which is close to the resonant frequency  $w = 1$ ),  $\Delta\omega_E = 0.6361$ . In comparison, the 3 dB bandwidth  $\Delta\omega_{3dB}$  of the curve is  $\Delta\omega_{3dB} = 0.6664$ . Figure 3.2 is a plot of  $1 - |\rho(\omega)|^2$  when  $\theta = 2$  and  $\zeta = 5$ . In this case  $R_2$  is smaller and becomes a more significant part of the circuit in Figure 1.1. We determined that  $\alpha = 0.8798$  and  $\Delta\omega_E = 1.1597$ . Notice from the plot that there is no meaningful measure of 3 dB bandwidth  $\Delta\omega_{3dB}$  because  $1 - |\rho(\omega)|^2$  does not fall of to  $1/2$  at two distinct frequencies.

The energy bandwidth is always defined under the assumption that

$$\int_0^{\infty} |H(\omega)|^2 d\omega < \infty \quad (3.29)$$

which means, physically, that the impulse response has finite energy. This makes energy bandwidth a more useful measure than 3 dB bandwidth, which relies on  $|H(\omega)|$  being sharp enough for it to

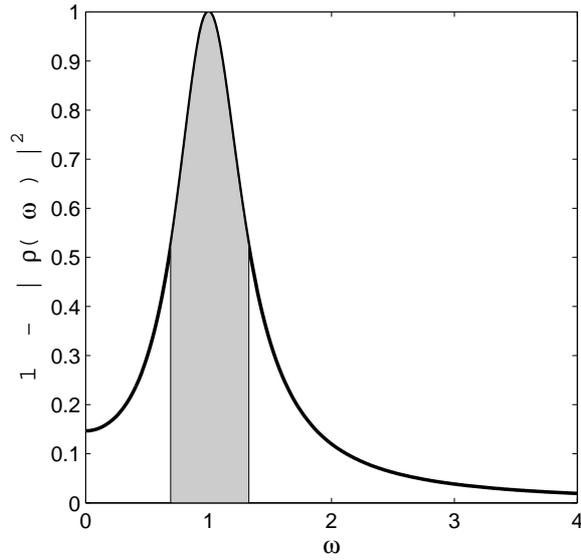


Figure 3.1: Transfer function  $1 - |\rho(\omega)|^2$  for  $\theta = 8$  and  $\zeta = 5$ .

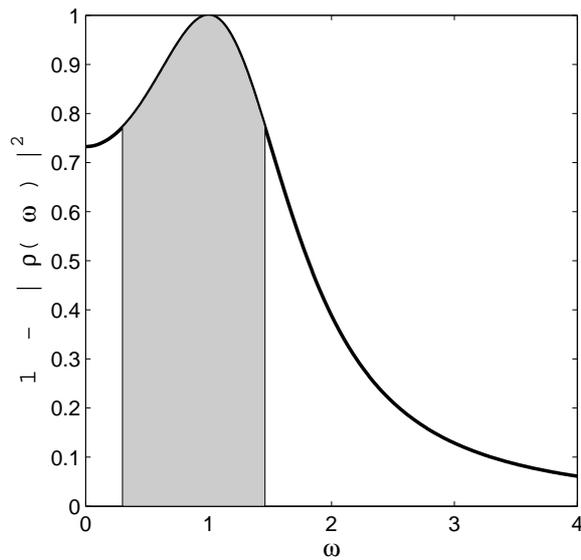


Figure 3.2: Transfer function  $1 - |\rho(\omega)|^2$  for  $\theta = 2$  and  $\zeta = 5$ .

be defined. Using the transfer function of Eqn. (3.28) with  $\zeta = 5$ , Figure 3.3 is a plot of of the bandwidth measured using the 3 dB method and the energy method. Notice that as  $\theta$  increases the 3 dB bandwidth and energy bandwidth converge to the same value. But, the 3 dB measure

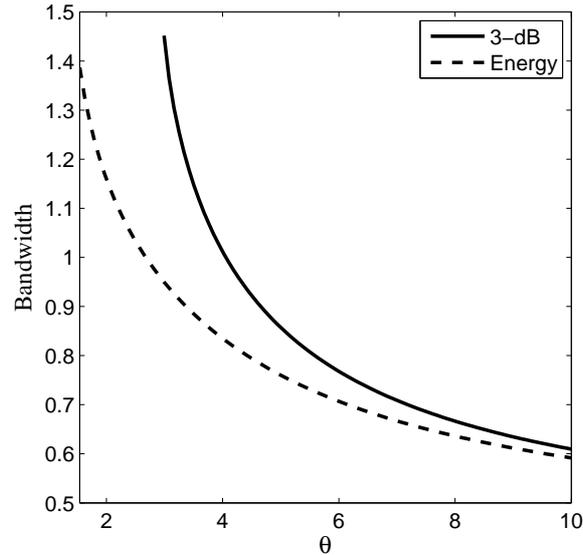


Figure 3.3: 3 dB bandwidth compared to energy bandwidth.

is only defined for  $\theta > 3$ . Not only is the energy bandwidth more generally applicable than 3 dB bandwidth because it can be unambiguously applied in more situations, but—as will see in Chapter 4—it also seems to more closely corresponds to energy within a system in general.

## 3.2 Distortion

The bandwidth definitions of the previous section do not depend on the phase of the transfer function. A signal passed through a system can be distorted from its original shape, even if the magnitude of the transfer function is unity over all frequencies.

Consider, for example, the circuit of Figure 3.4 (a constant resistance circuit studied by P. Nicolas [35]). When  $L = CZ_0^2$ , the transfer function, relating the input wave amplitude  $a(\omega)$  to the output wave amplitude  $b(\omega)$  is

$$H(\omega) = \frac{j + CZ_0\omega}{j - CZ_0\omega} \quad (3.30)$$

where  $Z_0$  is the characteristic impedance of the transmission lines. The magnitude of  $H(\omega)$  is unity for all frequencies  $\omega$ . Although the bandwidth of this circuit is in some sense infinite, the shape of

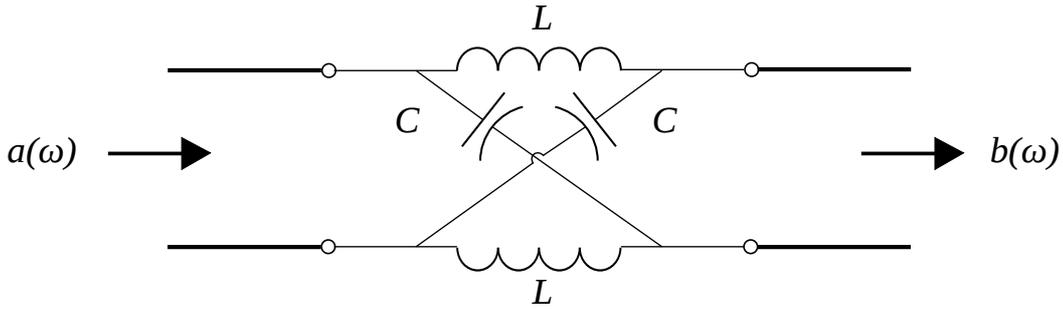


Figure 3.4: All Pass Circuit.

the signal at the input will not be preserved at the output (unless  $L = C = 0$ ). For example, if  $CZ_0 = 1/2$  and the square pulse of Figure 3.5a having a width of 1 unit is passed into the system, the signal of Figure 3.5b will be the resulting output.

In the context of this thesis, distortion is the deviation of a signal from its original shape when it has been passed through a system (This is in the same context as Brillouin [36] understands distortion). If the signal at the output is merely a copy of the input signal which has been translated in time, or vertically scaled, we say that the output has not been distorted. A suitable measure of distortion, therefore, should be invariant to vertical scaling and translations in time.

Determining distortion is useful for analyzing transmitted signals. For example, if a square pulse (Figure 3.5a) is sent through a channel, a measure of distortion can help determine whether the output pulse will be registered by a measuring instrument as a bit.

Quantifying distortion is often done by local approximations of  $H(\omega)$  about a center frequency (see [34]-[37]). However, local approximations do not give a complete picture of the mechanisms that cause distortion.

We desire a measure of distortion that considers the entire transfer function  $H(\omega)$ . Consider the functional

$$W(m, \theta) \equiv \frac{\int_{-\infty}^{\infty} [\hat{g}(t) - m\hat{f}(t - \theta)]^2 dt}{\int_{-\infty}^{\infty} \hat{g}(t)^2 dt} \quad (3.31)$$

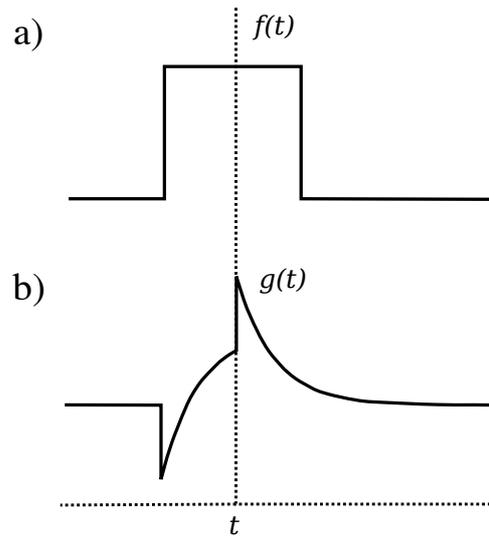


Figure 3.5: All Pass Distortion.

where  $\hat{g}(t)$  is the output of the system:

$$\hat{g}(t) = \int_{-\infty}^{\infty} \hat{h}(t-t') \hat{f}(t') dt' \quad (3.32)$$

Choosing the amplification constant  $m_0$  and the translation constant  $\theta_0$  that minimize  $W(m, \theta)$ , we define distortion as

$$D \equiv W(m_0, \theta_0) \quad (3.33)$$

This definition is similar to one studied by Colombo [38].

To interpret the definition (3.33), consider a square pulse (Figure 3.6a) passed through a linear system. The output of the system  $\hat{g}(t)$  is a time-shifted and attenuated copy of  $\hat{f}(t)$ , which is distorted. To calculate distortion  $D$ , we subtract  $m_0 \hat{f}(t - \theta_0)$  from  $\hat{g}(t)$  (Figure 3.6b) and choose  $m_0$  and  $\theta_0$  so that (3.31) is minimized. Clearly, if  $\hat{g}(t)$  is not distorted, then  $D = 0$ .

$W(m, \theta)$  will have a minimum with respect to  $m$  when

$$\frac{\partial W}{\partial m} = 0 \quad (3.34)$$

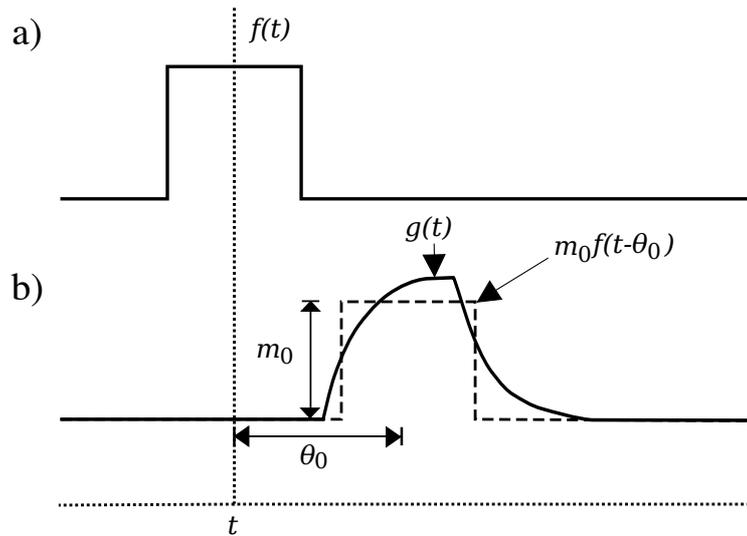


Figure 3.6: Distorted Signal.

The  $m$  that will satisfy (3.34) is

$$m_0(\theta) = \frac{\int_{-\infty}^{\infty} \hat{g}(t) \hat{f}(t - \theta) dt}{\int_{-\infty}^{\infty} \hat{f}(t)^2 dt} \quad (3.35)$$

$W$  may be rewritten

$$\begin{aligned} W(m, \theta) &= \frac{\int_{-\infty}^{\infty} [\hat{g}^2(t) - 2m\hat{g}(t)\hat{f}(t - \theta) + m^2\hat{f}^2(t - \theta)] dt}{\int_{-\infty}^{\infty} \hat{g}(t)^2 dt} \\ &= 1 + \frac{m^2 \int_{-\infty}^{\infty} \hat{f}^2(t) dt - 2m \int_{-\infty}^{\infty} \hat{g}(t)\hat{f}(t - \theta) dt}{\int_{-\infty}^{\infty} \hat{g}(t)^2 dt} \end{aligned}$$

and in terms of  $m_0(\theta)$

$$W(m, \theta) = 1 + m(m - 2m_0(\theta)) \frac{\int_{-\infty}^{\infty} \hat{f}(t)^2 dt}{\int_{-\infty}^{\infty} \hat{g}(t)^2 dt} \quad (3.36)$$

Thus,  $W(m, \theta)$  is minimized with respect to  $\theta$  when  $m_0(\theta)$  is at a maximum.

Distortion (3.33) can therefore be calculated by determining (3.35) and then searching for the  $\theta_0$  that minimizes (3.31). Although there are few examples where this definition of distortion

may be determined analytically, the problem is well suited for numerical computation.

Distortion may also be calculated in the frequency domain. Let  $F(\omega)$ ,  $G(\omega)$  and  $H(\omega)$  be the Fourier transforms of the function  $\hat{f}(t)$ ,  $\hat{g}(t)$  and  $\hat{h}(t)$  respectively. Then

$$m_0(\theta) = \frac{\int_{-\infty}^{\infty} H(\omega) |F(\omega)|^2 \exp(j\omega\theta) d\omega}{\int_{-\infty}^{\infty} |F(\omega)|^2 dt} \quad (3.37)$$

After determining  $\theta_0$ , distortion may then be calculated by

$$D = 1 - m_0^2(\theta_0) \frac{\int_{-\infty}^{\infty} |F(\omega)|^2 dt}{\int_{-\infty}^{\infty} |G(\omega)|^2 dt} \quad (3.38)$$

To handle modulated signals, we use analytic signals [39]. The analytic signal  $\hat{f}_A(t)$  corresponding to the real-valued signal  $\hat{f}(t)$  is a complex function defined by

$$\hat{f}_A(t) \equiv \frac{1}{\pi} \int_0^{\infty} F(\omega) \exp(j\omega t) d\omega \quad (3.39)$$

The signal  $\hat{f}(t)$  can be determined from  $\hat{f}_A(t)$  by

$$\hat{f}(t) = 2 \operatorname{Re} \hat{f}_A(t)$$

In what follows, the subscript  $A$  will be used differentiate an analytic signal from a real-valued signal.

Define the modulated signal as

$$\hat{f}_{Am}(t; \omega_0) = \hat{f}_A(t) \exp(j\omega_0 t) \quad (3.40)$$

where  $\hat{f}_A(t)$  is now a complex envelope, and  $\omega_0$  is the carrier frequency. The modulated output

signal is then

$$\begin{aligned}
\hat{g}_{Am}(t; \omega_0) &= \int_{-\infty}^{\infty} \hat{h}(t') \hat{f}_{Am}(t-t') dt' \\
&= \int_{-\infty}^{\infty} \hat{h}(t') \hat{f}_A(t-t') \exp(j\omega_0(t-t')) dt' \\
&= \left[ \int_{-\infty}^{\infty} \hat{h}(t') \hat{f}_A(t-t') \exp(-j\omega_0 t') dt' \right] \exp(j\omega_0 t)
\end{aligned}$$

We define the output signal envelope as

$$\hat{g}_A(t) = \int_{-\infty}^{\infty} \hat{h}(t') \exp(-j\omega_0 t') \hat{f}_A(t-t') dt' \quad (3.41)$$

and thus

$$\hat{g}_{Am}(t; \omega_0) = \hat{g}_A(t) \exp(j\omega_0 t) \quad (3.42)$$

In terms of the modulated analytic signals

$$\begin{aligned}
W(m, \theta) &= \frac{\int_{-\infty}^{\infty} \left| \hat{g}_{Am}(t; \omega_0) - \mu \hat{f}_A(t-\theta; \omega_0) \right|^2 dt}{\int_{-\infty}^{\infty} |\hat{g}_{Am}(t; \omega_0)|^2 dt} \\
&= \frac{\int_{-\infty}^{\infty} \left| \hat{g}_A(t) - m \hat{f}_A(t-\theta) \right|^2 dt}{\int_{-\infty}^{\infty} |\hat{g}_A(t)|^2 dt}
\end{aligned}$$

where

$$m = \mu \exp(-j\omega_0 \theta) \quad (3.43)$$

To minimize  $W$ ,  $m$  must be split into real and imaginary parts

$$m = x + jy \quad (3.44)$$

then

$$\frac{\partial D(x, y, \theta)}{\partial x} = \frac{2x \int_{-\infty}^{\infty} \left| \hat{f}_A(t) \right|^2 dt - 2 \operatorname{Re} \left[ \int_{-\infty}^{\infty} \hat{g}_A(t) \hat{f}_A^*(t-\theta) dt \right]}{\int_{-\infty}^{\infty} |\hat{g}_A(t)|^2 dt} = 0$$

yields

$$x(\theta) = \frac{\operatorname{Re} \left[ \int_{-\infty}^{\infty} \hat{g}_A(t) \hat{f}_A^*(t - \theta) dt \right]}{\int_{-\infty}^{\infty} |\hat{f}_A(t)|^2 dt}$$

and

$$\frac{\partial D(x, y, \theta)}{\partial y} = \frac{2y \int_{-\infty}^{\infty} |\hat{f}_A(t)|^2 dt - 2 \operatorname{Im} \left[ \int_{-\infty}^{\infty} \hat{g}_A(t) \hat{f}_A^*(t - \theta) dt \right]}{\int_{-\infty}^{\infty} |\hat{g}_A(t)|^2 dt} = 0$$

returns

$$y(\theta) = \frac{\operatorname{Im} \left[ \int_{-\infty}^{\infty} \hat{g}(t) \hat{f}^*(t - \theta) dt \right]}{\int_{-\infty}^{\infty} |\hat{f}(t)|^2 dt}$$

Putting the two results together, we get

$$m_0(\theta) = \frac{\int_{-\infty}^{\infty} \hat{g}_A(t) \hat{f}_A^*(t - \theta) dt}{\int_{-\infty}^{\infty} |\hat{f}_A(t)|^2 dt} \quad (3.45)$$

and finally distortion may be calculated by

$$D = 1 - |m_0(\theta_0)|^2 \frac{\int_{-\infty}^{\infty} |\hat{f}_A(t)|^2 dt}{\int_{-\infty}^{\infty} |\hat{g}_A(t)|^2 dt} \quad (3.46)$$

where  $\theta_0$  is the location where  $m_0(\theta)$  is maximum.

In the frequency domain

$$m_0(\theta) = \frac{\int_{-\infty}^{\infty} G(\omega) F^*(\omega) \exp(j\omega\theta) d\omega}{\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega} \quad (3.47)$$

which is identical to what it was before, except  $m$  is now complex. We have

$$G(\omega) = H_m(\omega) F(\omega) \quad (3.48)$$

$$H_m(\omega) = H(\omega + \omega_0) \quad (3.49)$$

thus

$$m_0(\theta, \omega_0) = \frac{\int_{-\infty}^{\infty} H(\omega + \omega_0) |F(\omega)|^2 \exp(j\omega\theta) d\omega}{\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega} \quad (3.50)$$

Finally, notice that for narrow band pulses

$$m_0(\omega_0) \equiv m(\theta, \omega_0) \approx H(\omega_0) \quad (3.51)$$

Therefore, if the signal  $\hat{f}_A(t)$  is used to transmit pulses that are wide,  $m_0(\omega_0)$  is simply the transfer function  $H(\omega_0)$ .

We used Matlab to implement the procedure above for determining distortion  $D$ . As an example, consider the allpass circuit of Figure 3.4 where  $CZ_0 = 0.1$ , thus

$$H(\omega) = \frac{j + 0.1\omega}{j - 0.1\omega} \quad (3.52)$$

We send through this system a square pulse with unit height and unit width. The distortion  $D$  as a function of carrier frequency  $\omega_0$  is plotted in Figure 3.7. The plot indicates that the distortion is the greatest when  $\omega_0 = 0$ . At  $\omega_0$ , Figure 3.8 shows the signal before it is passed through the system  $\hat{f}(t) = 2 \operatorname{Re} \hat{f}_A(t)$ , the output signal  $\hat{g}(t) = 2 \operatorname{Re} \hat{g}_A(t)$ , and the scaled and shifted function  $m_0 \hat{f}(t - \theta_0)$ . In contrast, the distortion at  $\omega_0 = 200$  is much smaller, which is indicated by Figure 3.9.  $\hat{f}(t)$ ,  $\hat{g}(t)$  and  $m_0 \hat{f}(t - \theta_0)$  are all present in this last plot, but it is difficult to delineate between the three curves because the distortion is small.

Unlike the local approximation methods (see [34]-[37]), our measure provides a complete picture of the mechanisms that cause distortion. The equations (3.46) and (3.50) are in no way approximations. Although, in general, this measure requires numerical computation, routines can be made robust and with little effort. For new research, numerical solutions are useful for experimenting with a large number of examples and doing so very quickly. The measure of distortion we provide here is a good tool for seeing how different kinds of systems distort signals.

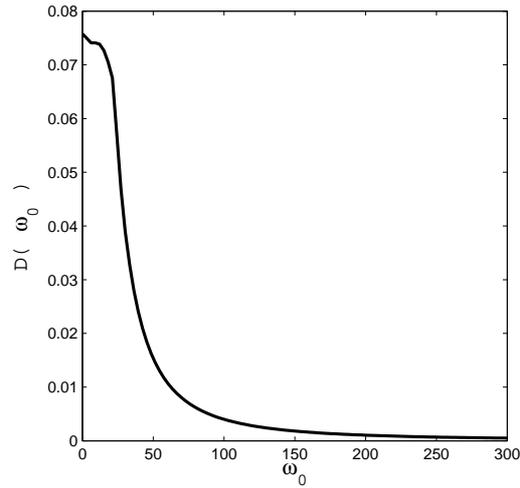


Figure 3.7: Distortion as a function of modulated frequency  $\omega_0$ .

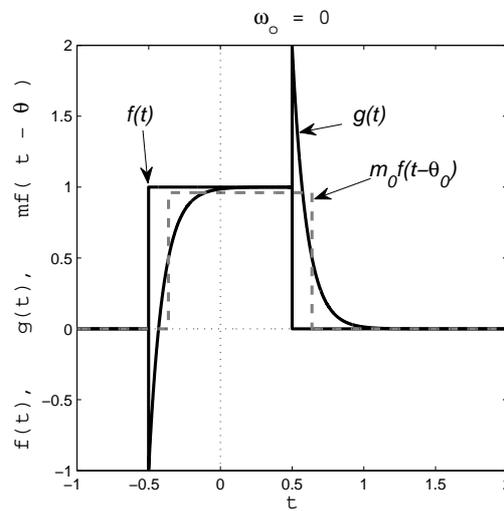


Figure 3.8: Maximum distortion occurs when  $\omega_0 = 0$ .

### 3.3 Bode-Fano Limitations

A class of limitations useful for characterizing transfer functions has the form

$$\int_{-\infty}^{\infty} f(|H(\omega)|) d\omega \leq K \quad (3.53)$$

where  $f(\cdot)$  is a known function and  $K$  is a positive constant. The inequality, known as a gain bandwidth limitation [31], enforces a constraint on the transfer function and is useful for obtaining

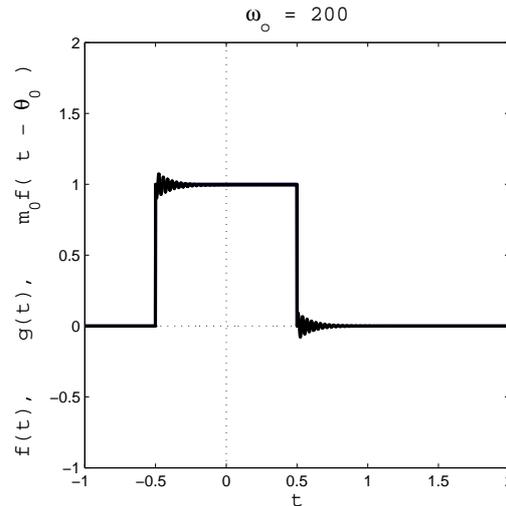


Figure 3.9: Distortion when  $\omega_0 = 200$ .

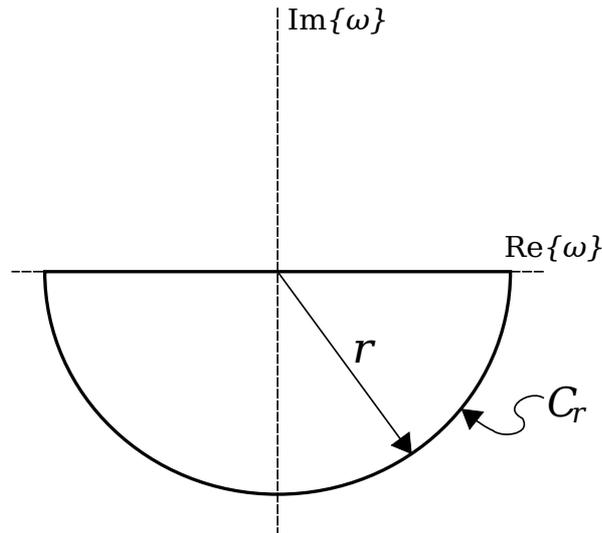
bounds on  $|H(\omega)|$ .

Bode-Fano limitations [40, 41] are a subclass of the gain bandwidth limitations (3.53). The kind of systems considered in this thesis are causal, implying that all transfer functions encountered here are analytic in the lower half of the complex  $\omega$ -plane (so called Herglotz functions). Bode and Fano exploit this behavior as well as certain characteristics of  $H(\omega)$  along the real line to obtain constraints of the form (3.53).

To introduce the Bode-Fano limitations, we consider systems that can be characterized by the reflection coefficient  $\rho(\omega)$ . From causality,  $\rho(\omega)$  is analytic in the lower complex half-plane. Consider the logarithm of the transfer function

$$\ln \rho(\omega) = \ln |\rho(\omega)| + j\phi(\omega). \quad (3.54)$$

where  $\ln |\rho(\omega)|$  and  $\phi(\omega)$  are real functions of real-valued  $\omega$ . The function  $\rho(\omega)$  is the Fourier transform of the real impulse response  $\hat{\rho}(t)$ ; therefore,  $\ln |\rho(\omega)|$  is an even function and  $\phi(\omega)$  is odd. The function (3.54) is not generally analytic in the lower half-plane, because any zeros in  $\rho(\omega)$  will cause (3.54) to be singular. If we know where all the zeros ( $\omega_1, \omega_2, \dots$ ) of  $\rho(\omega)$  are located

Figure 3.10: Integration contour  $C$ .

in the lower complex plane, we can form the function

$$\bar{\rho}(\omega) = \rho(\omega) \frac{(\omega - \omega_1^*)(\omega - \omega_2^*) \dots}{(\omega - \omega_1)(\omega - \omega_2) \dots} \quad (3.55)$$

so that the new function

$$\ln \bar{\rho}(\omega)$$

is analytic in the lower half-plane. With this new function, we form the contour integral

$$I = \oint_C (\ln \bar{\rho}(\omega) - R_\infty) d\omega \quad (3.56)$$

where  $C$  is the combination of a semicircle  $C_r$ , in the lower half-plane (see Figure 3.10) and the real line over the domain  $[-r, r]$ .  $R_\infty$  is the first term in the Laurent expansion taken about the origin

$$\ln \rho(\omega) = R_\infty + j \frac{\phi_\infty}{\omega} + \frac{R_1}{\omega^2} + j \frac{\phi_1}{\omega^3} + \dots \quad (3.57)$$

which is valid provided that  $\ln \rho(\omega)$  is analytic everywhere outside some circle centered on the origin. Clearly,

$$\ln |\rho(\omega)| = R_\infty + \frac{R_1}{\omega^2} + \dots \quad (3.58)$$

and

$$\phi(\omega) = \frac{\phi_\infty}{\omega} + \frac{\phi_1}{\omega^3} + \dots \quad (3.59)$$

From Cauchy's integral formula

$$I = 0 \quad (3.60)$$

therefore, we may break the integral (3.56) into a sum of three parts

$$I_1 + I_2 + I_3 = 0$$

where

$$I_1 = \oint_C (\ln |\rho(\omega)| - R_\infty) d\omega \quad (3.61)$$

$$I_2 = j \oint_C \phi(\omega) d\omega \quad (3.62)$$

and

$$I_3 = \oint_C \ln \left( \frac{(\omega - \omega_1^*)(\omega - \omega_2^*) \dots}{(\omega - \omega_1)(\omega - \omega_2) \dots} \right) d\omega \quad (3.63)$$

As  $r \rightarrow \infty$ ,  $I_1$  becomes

$$I_1 = \int_{-\infty}^{\infty} (\ln |\rho(\omega)| - R_\infty) d\omega \quad (3.64)$$

because the integral along the line  $C_r$  vanishes due to the fact that  $\ln |\rho(\omega)| - R_\infty \sim \frac{R_1}{\omega^2}$  for large  $|\omega|$ . Replacing  $\phi(\omega)$  with the expansion (3.59) in  $I_2$ , it is trivial to show that

$$I_2 = \pi \phi_\infty \quad (3.65)$$

when  $r \rightarrow \infty$ . The magnitude of

$$\frac{(\omega - \omega_1^*)(\omega - \omega_2^*) \dots}{(\omega - \omega_1)(\omega - \omega_2) \dots}$$

is unity for all  $\omega$  (this function is a so-called Blaschke product). Consequently, the logarithm of this product is a pure phase function along the real axis. Therefore, the integral of the logarithm along the real axis is

$$j \int_{-\infty}^{\infty} \sum_{i=1}^N \angle \frac{(\omega - \omega_i^*)}{(\omega - \omega_i)} d\omega \quad (3.66)$$

where  $\angle$  denotes the argument (phase angle) of a complex quantity, and  $N$  is the number of zeros in the lower half-plane. Along the semicircle  $C_r$ , for larger  $|\omega|$ , we have the asymptotic behavior

$$\ln \left( \frac{(\omega - \omega_1^*)(\omega - \omega_2^*) \dots}{(\omega - \omega_1)(\omega - \omega_2) \dots} \right) \sim - \sum_{i=1}^N \frac{2 \operatorname{Im} \omega_i}{j\omega} \quad (3.67)$$

Allowing  $r \rightarrow \infty$ , we may rewrite (3.60) as

$$\int_{-\infty}^{\infty} (\ln |\rho(\omega)| - R_\infty) d\omega + \pi\phi_\infty + j \int_{-\infty}^{\infty} \sum_{i=1}^N \angle \frac{(\omega - \omega_i^*)}{(\omega - \omega_i)} + 2\pi \sum_{i=1}^N \operatorname{Im} \omega_i = 0 \quad (3.68)$$

Taking the real part of the above equation and arranging terms, we may write

$$\int_{-\infty}^{\infty} \left( \ln \frac{1}{|\rho(\omega)|} + R_\infty \right) d\omega - 2\pi \sum_{i=1}^N \operatorname{Im} \omega_i = \pi\phi_\infty$$

Using the fact that the zeros  $\omega_i$  are known to be in the lower complex plane, and the fact that  $\ln |\rho(\omega)|$  is an even function, we obtain the final result

$$\int_0^{\infty} \left( \ln \frac{1}{|\rho(\omega)|} + R_\infty \right) d\omega \leq \frac{\pi\phi_\infty}{2} \quad (3.69)$$

This is one of the results due to Bode and Fano, and is a gain-bandwidth limitation.

As an example of how one might use the limitation (3.69), consider the reflection coefficient of a transmission line connected to a load consisting of a capacitor  $C$  in parallel with a resistor  $R$ :

$$\rho(\omega) = - \frac{Z_0 - R + CRj\omega Z_0}{R + Z_0 + CRj\omega Z_0} \quad (3.70)$$

As the frequency goes to infinity, the reflection coefficient becomes  $-1$ . Therefore, the circuit does not transfer energy to the resistor at higher frequencies. The Bode-Fano limitation characterizes how the limit at infinite frequencies affects  $|\rho(\omega)|$  over all frequencies. We may calculate

$$R_\infty = \lim_{\omega \rightarrow \infty} \ln(\rho(\omega)) = 0 \quad (3.71)$$

and

$$\phi_\infty = \frac{2}{CZ_0} \quad (3.72)$$

The integral becomes

$$\int_0^\infty \ln \frac{1}{|\rho(\omega)|} d\omega \leq \frac{\pi}{CZ_0} \quad (3.73)$$

Suppose now that we are interested in the reflection coefficient in the band  $\omega_1$  to  $\omega_2$ . Since the argument of the integral is always nonnegative, it is true that

$$\int_{\omega_1}^{\omega_2} \ln \frac{1}{\rho_{\max}} d\omega \leq \frac{\pi}{CZ_0} \quad (3.74)$$

where  $\rho_{\max} = \max \{|\rho(\omega)|\}$  within the band  $\omega_1$  to  $\omega_2$ . Then

$$\ln \frac{1}{\rho_{\max}} (\omega_2 - \omega_1) \leq \frac{\pi}{CZ_0} \quad (3.75)$$

and

$$\rho_{\max} \geq \exp \left( -\frac{\pi}{(\omega_2 - \omega_1) CZ_0} \right) \quad (3.76)$$

We have therefore obtained a limitation on the best we can do to minimize a reflection coefficient within a bandwidth of interest.

Another Bode-Fano result, which will be useful to us later, is found when the reflection coefficient of interest has a magnitude of unity at  $\omega = 0$ . Using the same ideas from above, we form the integral

$$I = \operatorname{Re} \oint_C \frac{1}{\omega^2} (\ln \bar{\rho}(\omega) - R_0) d\omega = 0 \quad (3.77)$$

where  $R_0$  is the first term in the Taylor expansion

$$\ln \rho(\omega) = R_0 + j\omega\phi_0 + \omega^2 R_1 + j\omega^3\phi_1 + \dots \quad (3.78)$$

which is taken about the origin of the complex  $\omega$ -plane. We find that

$$\int_0^\infty \frac{1}{\omega^2} \left( \ln \frac{1}{|\rho(\omega)|} + R_0 \right) d\omega \leq -\frac{\pi}{2}\phi_0 \quad (3.79)$$

where  $\phi_0$  is the second term of the Taylor series (3.78).

A transmission line connected to a resistor in series with a capacitor, will serve as an example for the limit given by (3.79). Here the transfer function is

$$\rho(\omega) = \frac{(R - Z_0)Cj\omega + 1}{(R + Z_0)Cj\omega + 1} \quad (3.80)$$

and

$$\begin{aligned} R_0 &= 0 \\ \phi_0 &= -2CZ_0 \end{aligned}$$

Therefore

$$p.v. \int_0^\infty \frac{1}{\omega^2} \ln \frac{1}{|\rho(\omega)|} d\omega \leq \pi Z_0 C \quad (3.81)$$

### 3.3.1 Tapered Transmission Line

A tapered transmission line can be used as a broadband matching system to match a transmission line to a resistive load. In some sense, the nonuniform transmission line is like a broadband antenna, which is used to match a single port system to free space over a large band of frequencies. We introduce a new limitation on nonuniform transmission lines based on Bode-Fano limitations. Energy within the matching system will be considered in a later chapter.

The matching circuit illustrated in Figure 3.11 is used to match a transmission line with real characteristic impedance  $Z_0$  to the resistor  $R_L$ . Carefully tapering the nonuniform transmission line in between the input port and the resistor  $R_L$ , provides an engineer with the capability to design broadband matching devices. This sort of matching device is always limited by its low frequency behavior. At low enough frequencies, the wavelength can be made much larger than the length  $L$  of the nonuniform line. For such frequencies, it is as if the input transmission line with characteristic impedance  $Z_L$  were connected directly to the load  $R_L$ .

Since nonuniform transmission lines are limited by low frequency behavior, we seek a Bode-Fano type limitation of the form

$$p.v. \int_0^\infty \frac{1}{\omega'^2} \ln \left| \frac{\rho(0, 0)}{\rho(0, \omega')} \right| d\omega' \leq -\frac{\pi}{2} \phi_0 \quad (3.82)$$

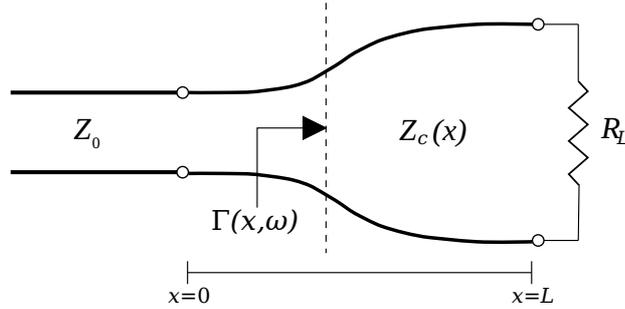


Figure 3.11: Nonuniform transmission line matching circuit.

consistent with the the limitation (3.79) discussed above. To obtain this bound, we must first ensure that the argument of the integral on the left hand side of the inequality is always positive, then we must determine the constant  $\phi_0$ .

For the lossless nonuniform transmission line used as a matching circuit as in Figure 3.11, the reflection coefficient  $\Gamma(x, \omega)$  at any location  $x$  along the line is known to satisfy the nonlinear differential equation<sup>1</sup> [42]

$$\frac{d\Gamma}{dx} - 2j\beta\Gamma + (1 - \Gamma^2) N = 0 \quad (3.83)$$

where

$$N(x) = \frac{1}{2} \frac{d \ln Z_c}{dx} \quad (3.84)$$

$Z_c(x)$  is the characteristic impedance of the nonuniform transmission line, and  $\beta(x)$  is the propagation constant. Following Litvenenko [43], let

$$\Gamma = \rho e^{j\theta} \quad (3.85)$$

where  $\rho(x, \omega)$  and  $\theta(x, \omega)$  are real and continuous functions of location  $x$  and frequency  $\omega$ . Placing (3.85) back into the differential equation (3.83) leads to two real valued differential equations

$$\rho' = -N(1 - \rho^2) \cos \theta \quad (3.86)$$

<sup>1</sup> This is a special form of the Riccati equation.

and

$$\theta' = 2\beta + \left(\rho + \frac{1}{\rho}\right) N \sin \theta \quad (3.87)$$

The first differential equation may be written as

$$\frac{1}{2} \left[ \ln \frac{1+\rho}{1-\rho} \right]' = -N \cos \theta \quad (3.88)$$

If we assume that  $\rho(L, \omega) = 0$ , integrating the previous equation returns

$$\ln \frac{1+\rho(0, \omega)}{1-\rho(0, \omega)} = \int_0^L [\ln Z_c(x)]' \cos \theta(x) dx \quad (3.89)$$

solving this for  $\rho(0, \omega)$  returns

$$\rho(0, \omega) = \tanh \left( \frac{1}{2} \int_0^L [\ln Z_c(x)]' \cos \theta(x) dx \right) \quad (3.90)$$

Now consider

$$\begin{aligned} |\rho(0, \omega)| &= \left| \tanh \left( \frac{1}{2} \int_0^L [\ln Z_c(x)]' \cos \theta(x) dx \right) \right| \\ &= \tanh \left| \frac{1}{2} \int_0^L [\ln Z_c(x)]' \cos \theta(x) dx \right| \\ &\leq \tanh \frac{1}{2} \int_0^L |[\ln Z_c(x)]'| |\cos \theta(x)| dx \end{aligned} \quad (3.91)$$

The function  $|\cos \theta(x)|$  is bounded between 0 and 1 for all  $x$ , therefore

$$|\rho(0, \omega)| \leq \tanh \frac{1}{2} \int_0^L |[\ln Z_c(x)]'| dx \quad (3.92)$$

To proceed, we now restrict the transmission lines to cases where  $[\ln Z_c(x)]'$  is either positive or negative for all  $x$ . This implies that the following analysis will pertain only to those transmission lines where  $Z_c(x)$  is either monotonically growing or decreasing. Consider first the case where  $Z_c(x)$  is monotonically increasing, it follows that

$$|\rho(0, \omega)| \leq \tanh \frac{1}{2} \int_0^L [\ln Z_c(x)]' dx \quad (3.93)$$

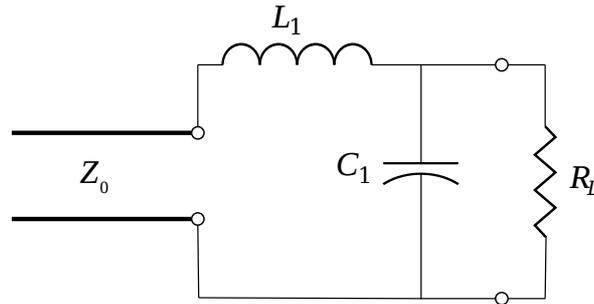


Figure 3.12: Low frequency approximation of nonuniform transmission line.

This integral may be evaluated in closed form

$$\begin{aligned}
 \tanh \frac{1}{2} \int_0^L [\ln Z_c(x)]' dx &= \tanh \left[ \frac{1}{2} \ln \frac{Z_c(L)}{Z_c(0)} \right] \\
 &= \frac{Z_c(L) - Z_c(0)}{Z_c(L) + Z_c(0)} \\
 &= \rho(0,0)
 \end{aligned} \tag{3.94}$$

A similar calculation provides the same result for a monotonically decreasing  $Z_c(x)$ . We have therefore shown that

$$|\rho(0, \omega)| \leq |\rho(0, 0)| \tag{3.95}$$

for all  $\omega$ , provided that  $Z_c(x)$  monotonically increases or decreases. In turn, we have discovered the first desired result, that is that the argument of the integral (3.82) is always positive.

To determine the constant  $\phi_0$ , let

$$Z_c(L) = R_L \tag{3.96}$$

i.e., there is no impedance discontinuity at the load end. We also let

$$Z_c(0) = Z_0 \tag{3.97}$$

i.e., there is no discontinuity at the input end. At low frequencies, the tapered transmission line may be approximated by the lumped circuit in Figure 3.12. If  $l(x)$  and  $c(x)$  are the inductance and capacitance of the transmission line, per unit length, the propagation factor per unit length is

$$\beta(x) = \omega \sqrt{l(x)c(x)} \tag{3.98}$$

and the characteristic impedance per unit length is

$$Z_c(x) = \sqrt{\frac{l(x)}{c(x)}} \quad (3.99)$$

In terms of these functions,  $C_1$  is

$$C_1 = \frac{1}{\omega} \int_0^L \frac{\beta(x)}{Z_c(x)} dx \quad (3.100)$$

and  $L_1$  is

$$L_1 = \frac{1}{\omega} \int_0^L \beta(x) Z_c(x) dx \quad (3.101)$$

Therefore, at low frequencies the input impedance looking into the nonuniform transmission line is

$$Z_{in} = j\omega L_1 + \frac{1}{\frac{1}{R_L} + j\omega C_1} \quad (3.102)$$

From  $Z_{in}$ , the factor  $\phi_0$  is readily determined

$$\phi_0 = -\frac{2Z_0}{R_L^2 - Z_0^2} (C_1 R_L^2 - L_1) \quad (3.103)$$

Using the results above, the Bode-Fano limitation for monotonically increasing or decreasing  $Z_c(x)$  is

$$\begin{aligned} \int_0^\infty \frac{1}{\omega'^2} \ln \left| \frac{\rho(0,0)}{\rho(0,\omega')} \right| d\omega' &\leq \frac{Z_0\pi}{R_L^2 - Z_0^2} (C_1 R_L^2 - L_1) \\ &= \frac{\pi}{4Z_0\omega} \frac{[1 - \rho(0,0)]^2}{\rho(0,0)} \int_0^L \frac{\beta(x)}{Z_c(x)} [R_L^2 - Z_c^2(x)] dx \\ &= \pi \int_0^L \frac{\beta(x)}{\omega} \frac{Z_c(x)}{Z_0} \frac{\rho_s}{\rho(0,0)} \frac{(1 - \rho(0,0))^2}{(1 - \rho_s)^2} dx \end{aligned} \quad (3.104)$$

Further simplifications may be made to determine a simple bound on  $|\rho(0,\omega')|$ . The right side of the above Bode-Fano limitation can be written as

$$\begin{aligned} \frac{Z_0\pi}{R_L^2 - Z_0^2} (C_1 R_L^2 - L_1) &= \frac{\pi}{4Z_0\omega} \frac{[1 - \rho(0,0)]^2}{\rho(0,0)} \int_0^L \frac{\beta(x)}{Z_c(x)} [R_L^2 - Z_c^2(x)] dx \\ &= \pi \int_0^L \frac{\beta(x)}{\omega} \frac{Z_c(x)}{Z_0} \frac{\rho_s(x)}{\rho(0,0)} \frac{(1 - \rho(0,0))^2}{(1 - \rho_s(x))^2} dx \end{aligned} \quad (3.105)$$

where we define

$$\rho_s(x) = \frac{R_L - Z_c(x)}{R_L + Z_c(x)} \quad (3.106)$$

Since  $Z_c(x)$  is either monotonically increasing or decreasing,  $|\rho_s(0)|$  is always the maximum of  $|\rho_s(x)|$ , i.e.,

$$\rho_{s \max} \equiv |\rho_s(0)| = \max \{|\rho_s(x)|\} \quad (3.107)$$

Also note that

$$\frac{\rho_s(x)}{\rho(0,0)}$$

is always positive by our earlier assumption. For  $\beta(x)$  constant, we have

$$\begin{aligned} \frac{Z_0\pi}{R_L^2 - Z_0^2} (C_1 R_L^2 - L_1) &\leq \pi \frac{\beta(1 - \rho^2(0,0))}{\omega \rho(0,0)} \int_0^L \frac{\rho_s(x)}{(1 - \rho_s^2(x))} dx \\ &\leq \pi \frac{\beta L(1 - \rho^2(0,0))}{\omega \rho(0,0)} \max \left\{ \frac{\rho_s(x)}{(1 - \rho_s^2(x))} \right\} \\ &\leq \pi \frac{\beta L}{\omega}. \end{aligned} \quad (3.108)$$

This is a consequence of

$$\frac{(1 - \rho_s^2(0))}{\rho_s(0)} \max \left\{ \frac{\rho_s(x)}{(1 - \rho_s^2(x))} \right\} = \frac{(1 - \rho_s^2(0))}{\rho_s(0)} \frac{\rho_{s \max}}{(1 - \rho_{s \max}^2)} = 1 \quad (3.109)$$

and recognizing that  $\rho_s(0) = \rho(0,0)$ , and that (3.107). Over the range of frequencies

$$\omega_1 \leq \omega \leq \omega_2$$

it is clear that

$$\ln \left| \frac{\rho(0,0)}{\rho_{\max}} \right| \frac{\omega_2 - \omega_1}{\omega_1 \omega_2} \leq \int_0^\infty \frac{1}{\omega'^2} \ln \left| \frac{\rho_{in}(0)}{\rho_{in}(\omega)} \right| d\omega' \quad (3.110)$$

where

$$\rho_{\max} = \max \{\rho(0, \omega)\} \quad (3.111)$$

We have

$$\ln \left| \frac{\rho(0,0)}{\rho_{\max}} \right| \frac{\omega_2 - \omega_1}{\omega_1 \omega_2} \leq \pi \frac{\beta}{\omega} L \quad (3.112)$$

and if we set

$$\begin{aligned} \omega_1 &= \frac{v_p}{\lambda_1} 2\pi \\ \omega_2 &= \frac{v_p}{\lambda_2} 2\pi \end{aligned} \quad (3.113)$$

and note that

$$\beta = \frac{\omega}{v_p} \quad (3.114)$$

where  $v_p$  is the phase velocity along the line, and  $\lambda_1$  and  $\lambda_2$  are the wavelengths corresponding to  $\omega_1$  and  $\omega_2$ , we see that

$$\ln \left| \frac{\rho(0,0)}{\rho_{\max}} \right| \leq \frac{2\pi^2 L}{\lambda_1 - \lambda_2} \quad (3.115)$$

or

$$\rho_{\max} \geq \rho(0,0) \exp\left(-\frac{2\pi^2 L}{\lambda_1 - \lambda_2}\right) \quad (3.116)$$

The Bode-Fano limitation (3.104) provides insight into how well a tapered transmission line performs as a function of  $L$ . We emphasize that the bound is limited to monotonically tapered transmission lines. But, this kind of characterization of the transfer function is attractive because it can be applied unambiguously to  $\rho(0, \omega)$ . The limitation only requires that the integral in (3.104) to exist.

### 3.4 Summary

We provide in this chapter tools useful for quantifying terminal behavior. In particular, energy bandwidth  $\Delta\omega_E$  (see Section 3.1.1) was defined as an alternative to 3 dB bandwidth  $\Delta\omega_{3dB}$ . This kind of width, at least for the examples we discussed, gave similar results to 3 dB bandwidth in certain limits. But, as we have shown, energy bandwidth is more generally applicable than 3 dB bandwidth. From the discussions in Chapter 1, it is not clear that 3 dB bandwidth should always be the choice of measure that corresponds to the energy within a system, we will show in Chapter 4 an example where energy bandwidth is a better choice.

We also in this chapter defined a measure for distortion (see Section 3.2). Bandwidth, at least in the definitions we present here, is only concerned with the magnitude of the transfer function and not the phase. To get a more complete description of terminal behavior, we defined distortion

$D$  (see Eqn. (3.46) and (3.50)). Unlike other measures of distortion, the definition we provide is in no way approximate.

Although measures such as the energy bandwidth and distortion are best suited for numerical computations, they are simple to implement robustly. They can be used with little difficulty to explore many different kinds of examples with little effort. This is useful for experimental purposes and is very helpful when trying to find how such quantities may be related to the energy within the systems to which they are being applied. In an age where computation is inexpensive, tools such as these are valuable assets for research.

The Bode-Fano limitation for nonuniform transmission lines (3.104) is a new result that places a limitation on the reflection coefficient  $\rho(0, \omega)$ . The result shows, clearly, that longer transmission lines will have larger bandwidths (or, equivalently, at higher frequencies the bandwidth is larger). We will show in Chapter 4 a method for determining a portion of the energy within these nonuniform transmission lines at large frequencies, and that one over this energy is consistent with the result found here.

## Chapter 4

### Recoverable Energy

Recoverable energy is a kind of energy that can be calculated *uniquely* from the reflection coefficient  $\rho(\omega)$  of a one-port network. This is in contrast to stored energy, which requires information about the internal structure of a system, e.g., the configuration of inductors and capacitors along with the currents and voltages associated with them. Because of its unique relationship to the terminal behavior, recoverable energy may somehow be uniquely related to the bandwidth of a system in general. This is impossible to do with stored energy, because as we showed in Chapter 1, there are classes of circuits that have different stored energies but the same terminal behavior. We introduce the recoverable energy  $E_{rec}$  of a one-port network in this chapter and show how it can be calculated given the reflection coefficient  $\rho(\omega)$  of a one-port network. We will also define the new parameter

$$Q_{rec} = \frac{\omega_0 E_{rec}}{P}$$

simply by replacing  $U$  with  $E_{rec}$  in the definition of  $Q$  from Chapter 1 (1.6). An example will be studied to see how  $Q_{rec}$  is related to the definitions of bandwidth from Chapter 3.

It was discovered from results leading to a calculation of recoverable energy that Darlington synthesis is closely related to the problem of determining recoverable energy. We show that the recoverable energy from a one-port network is equal to the stored energy of a minimum phase Darlington circuit, which has been synthesized from the reflection coefficient  $\rho(\omega)$ . A Darlington circuit is a circuit constructed from a lossless two-port network terminated in a single resistance, and it will be shown that the resulting lossless two-port is, in general, nonreciprocal.

The first part of this chapter is a derivation of the quantity we call recoverable energy (4.61), and the second part contains four examples revealing how important this energy is for understanding terminal behavior. Deriving recoverable energy is a lengthy process and requires solving a variational problem; the next two sections lead us to a functional (4.12) that will need to be minimized. The section that follows these two, will show that minimizing the functional results in having to solve an integral equation (4.19). We solve this integral equation using the so-called Wiener-Hopf technique and the final result is given by Eqn (4.57). Because we will be interested in time-harmonic recoverable energy only, we provide a method for calculating this quantity and obtain the result given by Eqn. (4.61). Four examples follow that demonstrate how recoverable energy can be calculated. These examples show the importance of this kind of energy, which is determined from the terminal behavior of a system alone.

#### 4.1 Recoverable Energy

Consider the network of Figure 4.1. The real valued signal  $\hat{a}(t)$  represents the wave-amplitude (see Chapter 2) due to some source incident on the one-port network; the real valued signal  $\hat{b}(t)$  represents the wave-amplitude reflected from the network. We assume that the system meets the requirements specified in Chapter 2 that permit  $\hat{b}(t)$  to be expressed as

$$\hat{b}(t) = \hat{\rho}(t) * \hat{a}(t)$$

where  $\hat{\rho}(t)$  is the real valued impulse reflection response of the system. We also assume that the energy inside the network is initially zero. Energy is put into the system by controlling  $\hat{a}(t)$  up to a time  $t_0$ . The energy put into the network is thus

$$E_{in} = \int_{-\infty}^{t_0} [\hat{a}^2(t) - \hat{b}^2(t)] dt \quad (4.1)$$

We require  $\hat{a}(t)$  to be in the set of functions  $L_2$  and that  $E_{in} < \infty$ . After the time  $t_0$ ,  $\hat{a}(t)$  is then used to extract energy back out of the system. The energy extracted from the system is

$$E_{out} = - \int_{t_0}^{\infty} [\hat{a}^2(t) - \hat{b}^2(t)] dt \quad (4.2)$$

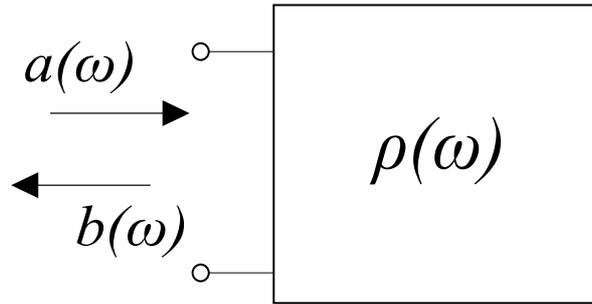


Figure 4.1: Single port network.

We define recoverable energy  $E_{rec}$  as the maximum energy that can be extracted from the system by an optimally chosen  $\hat{a}(t)$  in the time interval  $t \geq t_0$ .

Recoverable energy has been studied before, but in different contexts. In 1963 Breuer and Onat [44, 45] studied what they called “recoverable work” within viscoelastic solids. In this case, Breuer and Onat were interested in determining a fraction of the work done on a solid for  $t < 0$  by straining it mechanically. They determined this fraction of work, which they called recoverable work, by choosing an optimal straining function in a similar manner to what we are doing here. In 1990 Polevoi [46] studied how much electromagnetic energy could be extracted from a dissipative and dispersive medium. He used current densities to transfer energy, in the form of electromagnetic fields into and out of the medium. Using an optimal current density function defined for  $t > 0$  he determined maximum extractable energy. Glasgow, Meilstrup and Peatross et al. [47] as well as Amendola, Fabrizio and Golden [48] also looked into the dissipative and dispersive medium problem that Polevoi had studied.

## 4.2 Calculation of Recoverable Energy

The function  $\hat{a}(t)$  can be split into a sum of two different functions

$$\hat{a}(t) = \hat{a}_p(t) + \hat{a}_f(t) \quad (4.3)$$

The  $p$  in the function  $\hat{a}_p(t)$  stands for the *past* and means  $\hat{a}_p(t)$  is equal to  $\hat{a}(t)$  for  $t < t_0$ . The  $f$  in the function  $\hat{a}_f(t)$  stands for the *future*,  $\hat{a}_f(t)$  is equal to  $\hat{a}(t)$  for  $t \geq t_0$ . Consequently,

$$\hat{a}_p(t) = 0 \quad \text{for } t \geq t_0 \quad (4.4)$$

and

$$\hat{a}_f(t) = 0 \quad \text{for } t < t_0 \quad (4.5)$$

Figure 4.2 illustrates the decomposition of  $\hat{a}(t)$ .

Restating the definition, for a given function  $\hat{a}_p(t) \in L_2$ , recoverable energy is the maximum  $E_{out}$  over all possible  $\hat{a}_f(t)$ . For calculation purposes, it is easier to minimize the energy *lost* to the network

$$E_{lost} = E_{in} - E_{out} \quad (4.6)$$

By definition,  $E_{lost}$  is nonnegative because the system is passive. It is also true that

$$E_{lost} \leq E_{in} < \infty \quad (4.7)$$

which implies that we cannot extract more energy than has been put into the system. From (4.6), (4.1), (4.2), and the definition of passivity in Chapter 2, we have

$$E_{lost} = \int_{-\infty}^{\infty} [\hat{a}^2(t) - \hat{b}^2(t)] dt \geq 0 \quad (4.8)$$

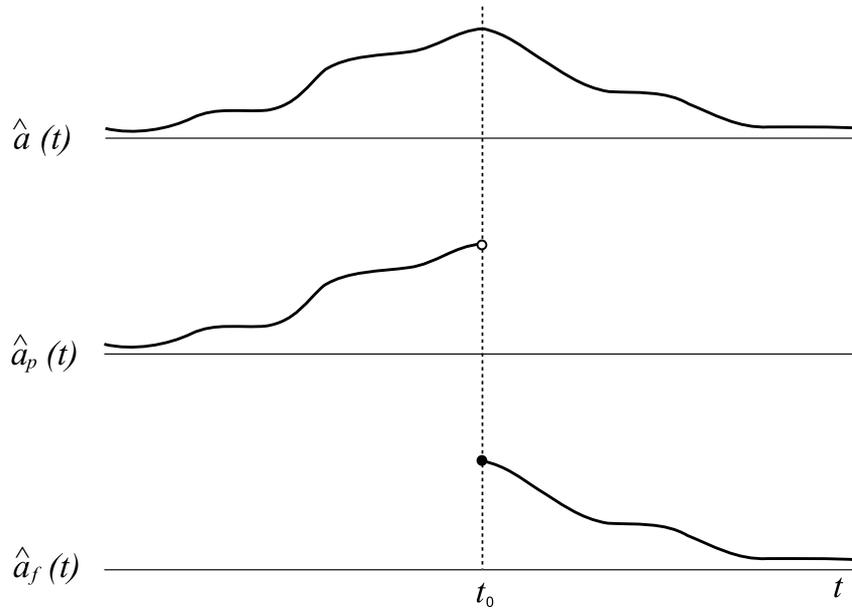
Once a minimum  $E_{lost}$  is found, the maximum  $E_{out}$  (i.e., the recoverable energy) is found via (4.6).

In functional form, energy lost to the system is

$$E_{lost}[\hat{a}_f] = \int_{-\infty}^{\infty} [\hat{a}^2(t) - [\hat{\rho}(t) * \hat{a}(t)]^2] dt \geq 0 \quad (4.9)$$

where  $\hat{a}(t) = \hat{a}_p(t) + \hat{a}_f(t)$ . A clever way to write this last expression that simplifies the minimization calculation later, uses the identity

$$\int_{-\infty}^{\infty} \hat{a}^2(t) dt = \int_{-\infty}^{\infty} \delta(t) [\hat{a}(t) * \hat{a}(-t)] dt \quad (4.10)$$

Figure 4.2: Decomposition of  $\hat{a}(t)$ .

The identity allows us to write the second part of the integral (4.9) as

$$\begin{aligned} \int_{-\infty}^{\infty} [\hat{\rho}(t) * \hat{a}(t)]^2 dt &= \int_{-\infty}^{\infty} \delta(t) [\hat{\rho}(t) * \hat{a}(t) * \hat{\rho}(-t) * \hat{a}(-t)] dt \\ &= \int_{-\infty}^{\infty} [\hat{\rho}(t) * \hat{\rho}(-t)] [\hat{a}(t) * \hat{a}(-t)] dt \end{aligned} \quad (4.11)$$

$E_{lost}[\hat{a}_f]$  can then be transformed to

$$E_{lost}[\hat{a}_f] = \int_{-\infty}^{\infty} [\delta(t) - \hat{\rho}(t) * \hat{\rho}(-t)] [\hat{a}(t) * \hat{a}(-t)] dt \geq 0 \quad (4.12)$$

### 4.3 Minimization Requirements

If  $\hat{a}_f(t)$  minimizes the energy lost to the system (4.12), then any other function  $\hat{a}_f(t) + \hat{\Delta}_f(t)$  will make  $E_{lost}$  larger. The  $\hat{a}_f(t)$  that minimizes  $E_{lost}$  must, therefore, satisfy

$$E_{lost}[\hat{a}_f + \hat{\Delta}_f] - E_{lost}[\hat{a}_f] \geq 0 \quad (4.13)$$

for all  $\hat{\Delta}_f(t)$ , such that

$$\hat{\Delta}_f(t) = 0 \quad \text{for } t < t_0 \quad (4.14)$$

Clearly,

$$\hat{a}_f(t) + \hat{\Delta}_f(t) = 0 \quad \text{for } t < t_0$$

Let

$$\hat{K}(t) = \delta(t) - \hat{\rho}(t) * \hat{\rho}(-t)$$

so that the functional  $E_{lost}[\hat{a}_f + \hat{\Delta}_f]$  can be written as

$$\begin{aligned} E_{lost}[\hat{a}_f + \hat{\Delta}_f] &= \int_{-\infty}^{\infty} \hat{K}(t) \left[ (\hat{a}(t) + \hat{\Delta}_f(t)) * (\hat{a}(-t) + \hat{\Delta}_f(-t)) \right] dt \\ &= E_{lost}[\hat{a}_f] + E_{lost}[\hat{\Delta}_f] \\ &\quad + 2 \int_{-\infty}^{\infty} \hat{K}(t) \left[ \hat{\Delta}_f(t) * \hat{a}(-t) \right] dt \end{aligned} \quad (4.15)$$

It follows that if (4.13) must be true for all  $\hat{\Delta}_f$ , then

$$E_{lost}[\hat{\Delta}_f] + 2 \int_{-\infty}^{\infty} \hat{K}(t) \left[ \hat{\Delta}_f(t) * \hat{a}(-t) \right] dt \geq 0$$

or

$$E_{lost}[\hat{\Delta}_f] + 2 \int_{-\infty}^{\infty} \left[ \hat{K}(t) * \hat{a}(t) \right] \hat{\Delta}_f(t) dt \geq 0 \quad (4.16)$$

must be true for for all  $\hat{\Delta}_f$ . From passivity (see Chapter 2),  $E_{lost}[\hat{\Delta}_f]$  is greater than or equal to zero for all  $\hat{\Delta}_f$ . Thus, ensuring that (4.16) is true for all  $\hat{\Delta}_f$ , requires that

$$\int_{-\infty}^{\infty} \left[ \hat{K}(t) * \hat{a}(t) \right] \hat{\Delta}_f(t) dt \geq 0 \quad (4.17)$$

for all  $\hat{\Delta}_f(t)$ . The function  $\hat{\Delta}_f(t)$  is arbitrary and in the above expression, it can always be replaced by  $-\hat{\Delta}_f(t)$ . Remembering that  $\hat{\Delta}_f(t)$  satisfies (4.14), to ensure that the integral in (4.17) is never negative it is necessary that

$$\hat{K}(t) * \hat{a}(t) = 0 \quad \text{for } t \geq t_0 \quad (4.18)$$

almost everywhere. This last equation can be rewritten as

$$\hat{a}(t) - \int_{-\infty}^{\infty} h(\tau - t) \hat{a}(\tau) d\tau = 0 \quad \text{for } t \geq t_0 \quad (4.19)$$

where

$$h(t) = \hat{\rho}(t) * \hat{\rho}(-t) \quad (4.20)$$

and we remember that

$$\hat{a}(t) = \hat{a}_p(t) + \hat{a}_f(t)$$

Equation (4.19) is also sufficient for (4.13) to be true, for if it is true then

$$E_{lost} [\hat{a}_f + \hat{\Delta}_f] - E_{lost} [\hat{a}_f] = E_{lost} [\hat{\Delta}_f]$$

which is clearly positive because  $E_{lost} [\hat{\Delta}_f]$  is always positive.

#### 4.4 Wiener-Hopf Solution

An analytic solution exists for the integral equation (4.19), subject to the constraint (4.5). The tool we use to determine a solution is the Wiener-Hopf technique (see Chap. 2 of [49]). In this section we use this technique to determine  $\hat{a}_f(t)$  for a particular  $\hat{a}_p(t)$ . Before we do, however, we present a few results from complex variable theory [50] that are needed to carry out the process. We also introduce the projection operators  $\mathcal{P}_{\pm}$  that are useful for simplifying the analysis.

**Cauchy's Integral Formula.** *If the function  $f(z)$  is analytic inside and on a simply connected closed contour  $C$  in the complex plane, then at any point interior to the contour*

$$f(z) = \frac{1}{2\pi j} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta \quad (4.21)$$

where the integration is carried out in the counterclockwise direction

**Jordan's lemma.** *If  $f(z)$  satisfies  $f(z) \rightarrow 0$  uniformly as  $|z| \rightarrow \infty$  in the upper half of the complex  $z$ -plane, as well as on the real axis, then*

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{jzt} dz = 0 \quad \text{for } t > 0 \quad (4.22)$$

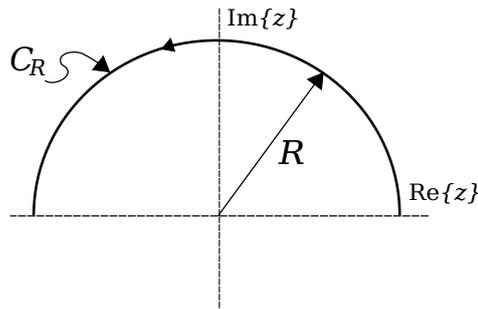


Figure 4.3: The contour  $C_R$  in the complex  $z$ -plane, illustrating Jordan's lemma.

The path  $C_R$  is a semicircle in the upper half-plane with radius  $R$  (see Figure 4.3). If  $f(z) \rightarrow 0$  uniformly in the lower complex plane and on the real axis, the above statement is true when  $C_R$  is a semicircle in the lower complex plane and  $t < 0$ .

**Liouville's theorem.** A function that is analytic and bounded over the whole complex plane must be constant.

From Parseval's theorem,  $E_{lost}$  (4.12) can be written as

$$E_{lost} = \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - |\rho(\omega)|^2) |a(\omega)|^2 d\omega \geq 0 \quad (4.23)$$

(The hats over all functions have been removed to indicate that the functions have been transformed to the frequency domain.) It follows from passivity (see Chapter 2), that if  $E_{lost} \geq 0$  for all  $a(\omega)$ , then

$$1 \geq |\rho(\omega)| \quad (4.24)$$

almost everywhere. For  $E_{loss}$  to remain finite,  $(1 - |\rho(\omega)|^2) |a(\omega)|^2$  must fall off faster than  $1/\omega$  as  $|\omega| \rightarrow \infty$ . For a general reflection coefficient  $\rho(\omega)$  that does not approach unity as  $|\omega| \rightarrow \infty$ , the fall off requirement implies that

$$a(\omega) \sim o\left(\frac{1}{\sqrt{\omega}}\right) \quad \text{for } |\omega| \rightarrow \infty \quad (4.25)$$

where  $o(1/\sqrt{\omega})$  indicates that  $|a(\omega)|$  falls off faster than  $1/\sqrt{\omega}$ . The fall off restriction would be relaxed if  $|\rho(\omega)| \rightarrow 1$  as  $|\omega| \rightarrow \infty$ , however, we do not consider such cases in this thesis.

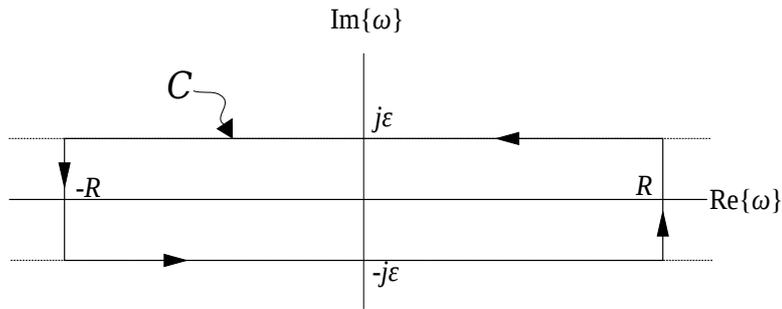


Figure 4.4: Integration path.

If a function  $G(\omega)$  is analytic within a strip containing the real line, i. e., within  $|\text{Im}\omega| \leq \varepsilon$  (see Figure 4.4), and falls off as  $\omega^{-\sigma}$  as  $\text{Re}\omega \rightarrow \pm\infty$  for some  $\sigma > 0$  within that strip, then  $G(\omega)$  can be decomposed into a sum of two functions

$$G(\omega) = G_+(\omega) + G_-(\omega) \quad (4.26)$$

where the subscripts  $+$  and  $-$  indicate that the functions  $G_+(\omega)$  and  $G_-(\omega)$  are analytic in the upper and lower complex  $\omega$ -half-planes respectively. A proof can be found on p. 44 of [49] and is summarized here. Suppose that  $G(\omega)$  is analytic in the strip where  $|\text{Im}\omega| \leq \varepsilon$ . Applying Cauchy's integral theorem, the integral over the contour  $C$  (shown in Figure 4.4) is

$$G(\omega) = \frac{1}{2\pi j} \oint_C \frac{G(\zeta)}{\zeta - \omega} d\zeta$$

Integration over the vertical sections of the contour vanish as  $|R| \rightarrow \infty$  because  $G(\omega)$  decays in the strip as  $\omega^{-\sigma}$ . Letting  $|R|$  to go to infinity, the above integral becomes

$$G(\omega) = \frac{1}{2\pi j} \int_{-\infty - j\varepsilon}^{\infty - j\varepsilon} \frac{G(\zeta)}{\zeta - \omega} d\zeta - \frac{1}{2\pi j} \int_{-\infty + j\varepsilon}^{\infty + j\varepsilon} \frac{G(\zeta)}{\zeta - \omega} d\zeta$$

The first integral is taken along the lower section of the contour, and the second integral is taken along the upper. We define

$$G_+(\omega) \equiv \frac{1}{2\pi j} \int_{-\infty - j\varepsilon}^{\infty - j\varepsilon} \frac{G(\zeta)}{\zeta - \omega} d\zeta \quad (4.27)$$

and

$$G_-(\omega) \equiv \frac{-1}{2\pi j} \int_{-\infty + j\varepsilon}^{\infty + j\varepsilon} \frac{G(\zeta)}{\zeta - \omega} d\zeta \quad (4.28)$$

It is clear by inspection that  $G_+(\omega)$  is a function that is analytic everywhere in the upper half-plane ( $\text{Im } \omega > 0$ ), and  $G_-(\omega)$  is analytic everywhere in the lower half-plane ( $\text{Im } \omega < 0$ ).

The behavior of the functions  $G_+(\omega)$  and  $G_-(\omega)$  at infinity is discussed on p. 44 of [49]. When  $G(\omega)$  falls off as  $\omega^{-\sigma}$ , where  $\sigma > 1$ , then

$$G_{\pm}(\omega) \sim \frac{M_{\pm}}{\omega}$$

where  $M_{\pm}$  are a finite constants. When  $1 > \sigma > 0$  the functions fall off as

$$G_{\pm}(\omega) \sim \frac{M_{\pm}}{\omega^{\sigma}}$$

If  $\sigma = 1$ , then

$$G_{\pm}(\omega) \sim M_{\pm} \frac{\ln \omega}{\omega}$$

To simplify the analysis that follows, it is convenient to introduce the projection operators  $\mathcal{P}_+$  and  $\mathcal{P}_-$ . The purpose of these operators is to project out the functions  $G_+(\omega)$  and  $G_-(\omega)$  from the function  $G(\omega)$  described above. We define the operators by

$$\mathcal{P}_{\pm}\phi(\omega) = \lim_{\varepsilon \rightarrow 0} \pm \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{\phi(\zeta)}{\zeta - (\omega \pm j\varepsilon)} d\zeta \quad (4.29)$$

where  $\varepsilon$  is a small positive real number. Equivalently, the operators can be defined as

$$\mathcal{P}_{\pm}\phi(\omega) = \frac{1}{2}\phi(\omega) \pm \frac{1}{2\pi j} p.v. \int_{-\infty}^{\infty} \frac{\phi(\zeta)}{\zeta - \omega} d\zeta \quad (4.30)$$

where *p.v.* indicates that the integral must be taken as a Cauchy principal value integral. The functions (4.27) and (4.28) can now be written as

$$G_{\pm}(\omega) = \mathcal{P}_{\pm}G(\omega) \quad (4.31)$$

$\mathcal{P}_+$  and  $\mathcal{P}_-$  have several properties needed for later calculations. Let an inner product be defined by

$$\langle \phi, \psi \rangle = \int_{-\infty}^{\infty} \phi(\xi) \psi^*(\xi) d\xi \quad (4.32)$$

The first interesting property of the operators is

$$\langle \mathcal{P}_{\pm} \phi, \psi \rangle = \langle \phi, \mathcal{P}_{\pm} \psi \rangle \quad (4.33)$$

which is to say that the operators  $\mathcal{P}_{\pm}$  are self-adjoint. The proof follows from the definition (4.29)

$$\begin{aligned} \langle \mathcal{P}_{\pm} \phi, \psi \rangle &= \int_{-\infty}^{\infty} \mathcal{P}_{\pm} [\phi(\xi)] \psi^*(\xi) d\xi \\ &= \lim_{\varepsilon \rightarrow 0} \pm \frac{1}{2\pi j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\phi(\zeta) \psi^*(\xi)}{\zeta - (\xi \pm j\varepsilon)} d\zeta d\xi \\ &= \lim_{\varepsilon \rightarrow 0} \mp \frac{1}{2\pi j} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\phi(\zeta) \psi^*(\xi)}{\xi - (\zeta \mp j\varepsilon)} d\zeta d\xi \\ &= \int_{-\infty}^{\infty} \lim_{\varepsilon \rightarrow 0} \mp \frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{\psi^*(\xi) d\xi}{\xi - (\zeta \mp j\varepsilon)} \phi(\zeta) d\zeta \\ &= \int_{-\infty}^{\infty} \mathcal{P}_{\mp} [\psi^*(\zeta)] \phi(\zeta) d\zeta \end{aligned}$$

Since

$$(\mathcal{P}_{\pm} \phi)^* = \mathcal{P}_{\mp} [\phi^*]$$

which follows from (4.30), we have

$$\langle \mathcal{P}_{\pm} \phi, \psi \rangle = \int_{-\infty}^{\infty} \phi(\zeta) (\mathcal{P}_{\pm} [\psi(\zeta)])^* d\zeta$$

and the proof is complete. The properties

$$\mathcal{P}_{\pm} \mathcal{P}_{\pm} \phi = \mathcal{P}_{\pm} \phi \quad (4.34)$$

and

$$\mathcal{P}_{\mp} \mathcal{P}_{\pm} \phi = 0 \quad (4.35)$$

follow trivially from the definitions. The final property

$$\langle \mathcal{P}_{\pm} \phi, \mathcal{P}_{\mp} \phi \rangle = 0$$

follows from (4.33) and (4.35). It states that  $\mathcal{P}_\pm\phi$  and  $\mathcal{P}_\mp\phi$  are orthogonal with respect to the product (4.32).

From the projection operators (4.30), it follows that a function  $\hat{k}(t) \in L_2$  satisfying

$$\hat{k}(t) = 0 \quad \text{for } t < 0 \quad (4.36)$$

has a spectrum analytic in the lower complex  $\omega$ -plane. To demonstrate this, we take the Fourier transform of  $\hat{k}(t)$

$$k(\omega) = \int_0^{\infty} \hat{k}(t) e^{-j\omega t} dt$$

now  $k(\omega)$  falls off like  $\omega^{-1}$  as  $|\omega| \rightarrow \infty$  and, therefore, it can be split into the sum of two functions

$$k(\omega) = \mathcal{P}_+[k(\omega)] + \mathcal{P}_-[k(\omega)] = \mathcal{P}_+ \int_0^{\infty} \hat{k}(t) e^{-j\omega t} dt + \mathcal{P}_- \int_0^{\infty} \hat{k}(t) e^{-j\omega t} dt$$

The operators  $\mathcal{P}_\pm$  can be brought under the integrals to operate on  $e^{-j\omega t}$ , resulting in

$$\mathcal{P}_+ e^{-j\omega t} = 0 \quad \text{for } t \geq 0$$

and

$$\mathcal{P}_- e^{-j\omega t} = e^{-j\omega t} \quad \text{for } t < 0$$

Therefore,

$$\mathcal{P}_+[k(\omega)] = 0$$

and

$$\mathcal{P}_-[k(\omega)] = \int_0^{\infty} \hat{k}(t) e^{-j\omega t} dt = k(\omega)$$

which implies that  $k(\omega)$  is analytic in the lower half-plane. It can similarly be shown that a function  $\hat{k}(t) \in L_2$  satisfying

$$\hat{k}(t) = 0 \quad \text{for } t \geq 0$$

will have a spectrum analytic in the upper complex plane. From (4.4) and (4.5), we can conclude that the spectra  $a_f(\omega)$  and  $a_p(\omega)$  are analytic in the upper and lower complex half-planes respectively. A final tool necessary for the solution of the integral equation (4.19) is the product

factorization of  $K(\omega) \equiv 1 - |\rho(\omega)|^2$ . This function can be factored into the product of two auxiliary functions

$$K(\omega) = \kappa_-(\omega) \kappa_+(\omega) \quad (4.37)$$

where, again,  $+$  and  $-$  indicate that the corresponding functions are analytic in the upper and lower complex planes respectively. The function  $|\rho(\omega)|$  is even, and therefore  $K(\omega)$  is even, thus

$$\kappa_-(\omega) \kappa_+(\omega) = \kappa_-(-\omega) \kappa_+(-\omega) \quad (4.38)$$

If  $\kappa_-(\omega)$  is analytic in the lower complex plane, then  $\kappa_-(-\omega)$  will be analytic in the upper complex plane. The symmetry (4.38) suggests that  $\kappa_-(\omega)$  is equal to  $\kappa_+(-\omega)$  to within a constant factor. So in addition to the factorization (4.37), without loss of generality we require that

$$\kappa_-(\omega) = \kappa_+(-\omega) \quad (4.39)$$

For this thesis we only consider functions  $K(\omega)$  that are both analytic within a strip about the real axis and have no zeros within the same strip. We also require  $K(\omega) \rightarrow K_\infty$  as  $|\omega| \rightarrow \infty$ , where  $0 \leq K_\infty < 1$ . (The restriction means  $|\rho(\omega)|^2 \rightarrow 1$  as  $|\omega| \rightarrow \infty$  is not permitted.) With these demands, a unique factorization (4.37) is possible, and its proof is constructive. Let

$$G(\omega) = \ln \frac{K(\omega)}{K_\infty}$$

where the argument of the logarithm is restricted to  $(-\pi, \pi]$ . From (4.37) and (4.39)

$$G(\omega) = \ln \kappa_-(\omega) + \ln \kappa_+(\omega) - \ln K_\infty$$

Clearly,  $G(\omega)$  is analytic in a strip about the real axis and goes to zero as  $|\omega| \rightarrow \infty$ , hence  $G(\omega)$  itself can be decomposed as

$$G(\omega) = \mathcal{P}_-G(\omega) + \mathcal{P}_+G(\omega)$$

and thus

$$\ln \kappa_-(\omega) + \ln \kappa_+(\omega) - \ln K_\infty = \mathcal{P}_-G(\omega) + \mathcal{P}_+G(\omega)$$

Applying the projections operators  $\mathcal{P}_\pm$  to the left and right side of the above equation, we have

$$\ln \kappa_\pm(\omega) - \frac{1}{2} \ln K_\infty = \mathcal{P}_\pm G(\omega)$$

or

$$\kappa_-(\omega) = \sqrt{K_\infty} \exp(\mathcal{P}_- G(\omega))$$

and

$$\kappa_+(\omega) = \sqrt{K_\infty} \exp(\mathcal{P}_+ G(\omega))$$

From the definition (4.30)

$$\mathcal{P}_\pm G(\omega) = \left[ \frac{1}{2} \ln \frac{K(\omega)}{C^2} \pm \frac{1}{2\pi j} p.v. \int_{-\infty}^{\infty} \frac{\ln K(\zeta) - \ln K_\infty}{\zeta - \omega} d\zeta \right]$$

It is clear that

$$p.v. \int_{-\infty}^{\infty} \frac{\ln K_\infty}{\zeta - \omega} d\zeta = 0$$

therefore, the result for  $\kappa_\pm(\omega)$  becomes

$$\begin{aligned} \kappa_\pm(\omega) &= \sqrt{K_\infty} \exp \left( \left[ \frac{1}{2} \ln \frac{K(\omega)}{K_\infty} \pm \frac{1}{2\pi j} p.v. \int_{-\infty}^{\infty} \frac{\ln K(\zeta)}{\zeta - \omega} d\zeta \right] \right) \\ &= \sqrt{K_\infty} \exp \left( \ln \sqrt{\frac{K(\omega)}{K_\infty}} \right) \exp \left( \pm \frac{1}{2\pi j} p.v. \int_{-\infty}^{\infty} \frac{\ln K(\zeta)}{\zeta - \omega} d\zeta \right) \\ &= \sqrt{K(\omega)} \exp \left( \pm \frac{1}{2\pi j} p.v. \int_{-\infty}^{\infty} \frac{\ln K(\zeta)}{\zeta - \omega} d\zeta \right) \end{aligned}$$

and finally

$$\kappa_\pm(\omega) = \sqrt{K(\omega)} \exp(\mp j \phi_\kappa(\omega)) \quad (4.40)$$

where the phase function  $\phi_\kappa(\omega)$  is

$$\phi_\kappa(\omega) = \frac{1}{2\pi} p.v. \int_{-\infty}^{\infty} \frac{\ln K(\zeta)}{\zeta - \omega} d\zeta$$

It is sometimes convenient to write this as

$$\phi_\kappa(\omega) = \frac{\omega}{\pi} p.v. \int_0^{\infty} \frac{\ln K(\zeta)}{\zeta^2 - \omega^2} d\zeta$$

to emphasize that it is an odd function of  $\omega$ . This completes the proof that  $K(\omega)$  can be uniquely factored.

From (4.40)

$$\kappa_{\pm}(\omega) = \kappa_{\mp}^*(\omega)$$

for real  $\omega$ . For brevity, we will henceforth omit the subscript of  $\kappa_{-}(\omega)$ , and define

$$\kappa(\omega) \equiv \kappa_{-}(\omega)$$

and it follows that the factorization (4.37) can be written as

$$K(\omega) = \kappa(\omega) \kappa^*(\omega) \quad (4.41)$$

or

$$K(\omega) = |\kappa(\omega)|^2$$

Therefore, the function  $\kappa(\omega)$  has the following properties:

$$|\kappa(\omega)|^2 = 1 - |\rho(\omega)|^2 \quad (4.42)$$

and

$$\kappa(\omega) = |\kappa(\omega)| \exp(j\phi_{\kappa}(\omega)) \quad (4.43)$$

where

$$\phi_{\kappa}(\omega) = \frac{\omega}{\pi} p.v. \int_0^{\infty} \frac{\ln |\kappa(\zeta)|^2}{\zeta^2 - \omega^2} d\zeta \quad (4.44)$$

From (4.43), (4.44) and the fact that  $|\kappa(\omega)|$  is an even function,  $\kappa(\omega)$  adheres to the symmetry

$$\kappa(-\omega) = \kappa^*(\omega) \quad (4.45)$$

when  $\omega$  is real.

At this point, it is interesting to identify  $\kappa(\omega)$  as the transmission coefficient of a minimum-phase lossless two-port network. Notice that the magnitude squared of  $\kappa(\omega)$  (4.42) is identical to that of a transmission coefficient associated with a lossless two-port network. The fact that

the phase  $\phi_\kappa(\omega)$  satisfies (4.44) means that the transfer function  $\kappa(\omega)$  is minimum-phase [51]. It is also interesting to note that lossless two-port circuits can be synthesized from  $\rho(\omega)$  such that the transmission coefficient of that circuit is the minimum-phase function  $\kappa(\omega)$  (see [52]). The general synthesis procedure is known as Darlington synthesis [53]. We explore the impact of this observation at the end of this chapter once the recoverable energy has been determined.

We are now ready to solve the integral equation (4.19), with the constraint (4.5). Using the Fourier representations of the functions  $\hat{a}_p(t)$ ,  $\hat{a}_f(t)$  and  $\hat{h}(t)$ , (4.19) and (4.5) can be written as

$$I_1(t) = \int_{-\infty}^{\infty} J(\omega) e^{j\omega t} d\omega = 0 \quad \text{for } t \geq t_0 \quad (4.46)$$

and

$$I_2(t) = \int_{-\infty}^{\infty} a_f(\omega) e^{j\omega t} d\omega = 0 \quad \text{for } t < t_0 \quad (4.47)$$

where  $J(\omega) = |\kappa(\omega)|^2 a(\omega)$ .  $I_1(t)$  is a function that is 0 for  $t \geq t_0$  and  $I_2(t)$  is a function that is 0 for  $t < t_0$ . These functions can be shifted to the left by  $t_0$  so that

$$I_1(t + t_0) = \int_{-\infty}^{\infty} J(\omega) e^{j\omega t_0} e^{j\omega t} d\omega = 0 \quad \text{for } t \geq 0 \quad (4.48)$$

and

$$I_2(t + t_0) = \int_{-\infty}^{\infty} a_f(\omega) e^{j\omega t_0} e^{j\omega t} d\omega = 0 \quad \text{for } t < 0 \quad (4.49)$$

and these conditions imply that we may write

$$\mathcal{P}_- [J(\omega) e^{j\omega t_0}] = 0 \quad (4.50)$$

and

$$\mathcal{P}_+ [a_f(\omega) e^{j\omega t_0}] = 0 \quad (4.51)$$

(see section following (4.36)). We said above that  $E_{loss}$  will be finite, so as a consequence  $J(\omega) e^{j\omega t_0}$  falls off faster than  $\omega^{-1/2}$  as  $|\omega| \rightarrow \infty$ , and it can therefore be decomposed as

$$J(\omega) e^{j\omega t_0} = \mathcal{P}_- [J(\omega) e^{j\omega t_0}] + \mathcal{P}_+ [J(\omega) e^{j\omega t_0}]$$

and (4.50) implies that

$$J(\omega) e^{j\omega t_0} = \mathcal{P}_+ [J(\omega) e^{j\omega t_0}] \quad (4.52)$$

Similarly, since  $a_f(\omega) e^{j\omega t_0}$  falls off as  $\omega^{-1}$  as  $|\omega| \rightarrow \infty$ , (4.51) implies that

$$a_f(\omega) e^{j\omega t_0} = \mathcal{P}_- [a_f(\omega) e^{j\omega t_0}] \quad (4.53)$$

The statements (4.52) and (4.53) together imply that a solution to (4.19) can be found if  $J(\omega) e^{j\omega t_0}$  is analytic in the upper complex plane and falls off faster than  $\omega^{-1/2}$  as  $|\omega| \rightarrow \infty$ , and  $a_f(\omega) e^{j\omega t_0}$  is analytic in the lower complex plane and falls off faster than  $\omega^{-1}$  as  $|\omega| \rightarrow \infty$ . The solution for  $a_f(\omega) e^{j\omega t_0}$  now proceeds as follows: We first split  $a(\omega)$  into the sum  $a_f(\omega)$  and  $a_p(\omega)$

$$J(\omega) e^{j\omega t_0} = \kappa(\omega) \kappa^*(\omega) (a_f(\omega) + a_p(\omega)) e^{j\omega t_0}$$

since  $\kappa^*(\omega)$  is analytic in the upper complex plane, as is  $1/\kappa^*(\omega)$ , we can move it to the left side of the equation

$$\frac{J(\omega) e^{j\omega t_0}}{\kappa^*(\omega)} = \kappa(\omega) a_f(\omega) e^{j\omega t_0} + \kappa(\omega) a_p(\omega) e^{j\omega t_0} \quad (4.54)$$

The left hand side of (4.54) is analytic in the upper half-plane, and the first term in the sum on the right hand side is analytic in the lower half-plane.  $a_p(\omega)$  will fall off as  $\omega^{-1}$ , and we have chosen  $|\kappa(\omega)|^2$  such that  $|\kappa(\omega)|^2 \rightarrow K_\infty$ , where  $0 \leq K_\infty < 1$ , as  $|\omega| \rightarrow \infty$ . We can now split  $\kappa(\omega) a_p(\omega) e^{j\omega t_0}$  into a sum of two functions:

$$\frac{J(\omega) e^{j\omega t_0}}{\kappa^*(\omega)} = \kappa(\omega) a_f(\omega) e^{j\omega t_0} + \mathcal{P}_+ [\kappa(\omega) a_p(\omega) e^{j\omega t_0}] + \mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]$$

and rearranging, we get

$$\frac{J(\omega) e^{j\omega t_0}}{\kappa^*(\omega)} - \mathcal{P}_+ [\kappa(\omega) a_p(\omega) e^{j\omega t_0}] = \mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}] + \kappa(\omega) a_f(\omega) e^{j\omega t_0}$$

Since the left side of the above equation is analytic in the upper half-plane and the right side is analytic in the lower, the only way for the equality to hold is if both sides are equal to the same entire function  $E(\omega)$ ; thus

$$E(\omega) = \mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}] + \kappa(\omega) a_f(\omega) e^{j\omega t_0}$$

We find that

$$a_f(\omega)e^{j\omega t_0} = \frac{E(\omega) - \mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]}{\kappa(\omega)}$$

Since  $a_f(\omega)e^{j\omega t_0} \rightarrow 0$ ,  $\kappa(\omega) \rightarrow \sqrt{K_\infty}$  and  $\mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}] \rightarrow 0$  as  $|\omega| \rightarrow \infty$ , the entire function  $E(\omega)$  must go to zero as  $|\omega| \rightarrow \infty$ . By Liouville's theorem,  $E(\omega)$  must be zero everywhere, and therefore

$$a_f(\omega)e^{j\omega t_0} = -\frac{\mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]}{\kappa(\omega)}$$

The solution  $\hat{a}_f(t)$  is finally

$$\hat{a}_f(t) = \mathcal{F}^{-1} \left[ -\frac{e^{-j\omega t_0} \mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]}{\kappa(\omega)} \right]$$

To calculate the minimum  $E_{loss}$ , notice that

$$\begin{aligned} a(\omega) &= a_p(\omega) + a_f(\omega) \\ &= a_p(\omega) - \frac{e^{-j\omega t_0} \mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]}{\kappa(\omega)} \end{aligned}$$

and that we can split the second term in the sum so that we get simply

$$\begin{aligned} a(\omega)e^{j\omega t_0} &= a_p(\omega) e^{j\omega t_0} - \left[ \frac{\kappa(\omega) a_p(\omega) e^{j\omega t_0}}{\kappa(\omega)} - \frac{\mathcal{P}_+ [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]}{\kappa(\omega)} \right] \\ &= \frac{\mathcal{P}_+ [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]}{\kappa(\omega)} \end{aligned}$$

therefore

$$|a(\omega)|^2 = \frac{|\mathcal{P}_+ [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]|^2}{|\kappa(\omega)|^2}$$

From (4.23), and with the notation  $\min\{E_{loss}\}$  to indicate that the expression is a minimum over all possible  $\hat{a}_f(t)$

$$\begin{aligned} \min\{E_{loss}\} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (1 - |\rho(\omega)|^2) |a(\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\kappa(\omega)|^2 \frac{|\mathcal{P}_+ [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]|^2}{|\kappa(\omega)|^2} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{P}_+ [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]|^2 d\omega \end{aligned}$$

The recoverable energy follows from

$$E_{rec} = E_{in} - \min\{E_{loss}\} \quad (4.55)$$

First, note that by Parsaval's theorem and (4.1)

$$\begin{aligned} E_{in} &= \int_{-\infty}^{\infty} [\hat{a}_p^2(t) - \hat{b}_p^2(t)] dt \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [|a_p(\omega)|^2 - |b_p(\omega)|^2] d\omega \end{aligned}$$

where  $\hat{a}_p(t)$  and  $\hat{b}_p(t)$  are both zero for  $t \geq t_0$ ; consequently, both  $a_p(\omega) e^{j\omega t_0}$  and  $b_p(\omega) e^{j\omega t_0}$  are analytic in the upper half-plane. The function  $b(\omega) e^{j\omega t_0}$  is given by

$$b_f(\omega) e^{j\omega t_0} + b_p(\omega) e^{j\omega t_0} = \rho(\omega) a_p(\omega) e^{j\omega t_0} + \rho(\omega) a_f(\omega) e^{j\omega t_0}$$

Applying the operator  $\mathcal{P}_+$ , we have

$$b_p(\omega) e^{j\omega t_0} = \mathcal{P}_+ [\rho(\omega) a_p(\omega) e^{j\omega t_0}] + \mathcal{P}_+ [\rho(\omega) a_f(\omega) e^{j\omega t_0}] \quad (4.56)$$

The reflection coefficient  $\rho(\omega)$  is the Fourier transform of a (possibly generalized) function  $\hat{\rho}(t)$  that is zero for  $t < 0$  (by causality); it is therefore analytic in the lower half-plane. The function  $a_f(\omega) e^{j\omega t_0}$  is also analytic in the lower half-plane, and this means that  $\mathcal{P}_+ [\rho(\omega) a_f(\omega) e^{j\omega t_0}] = 0$ .

Equation (4.56) is now

$$b_p(\omega) e^{j\omega t_0} = \mathcal{P}_+ [\rho(\omega) a_p(\omega) e^{j\omega t_0}]$$

thus

$$E_{in} = \frac{1}{2\pi} \int_{-\infty}^{\infty} [|a_p(\omega)|^2 - |\mathcal{P}_+ [\rho(\omega) a_p(\omega) e^{j\omega t_0}]|^2] d\omega$$

The recoverable energy (4.55) is now expressible as

$$E_{rec} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ |a_p(\omega)|^2 - |\mathcal{P}_+ [\rho(\omega) a_p(\omega) e^{j\omega t_0}]|^2 - |\mathcal{P}_+ [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]|^2 \right] d\omega$$

From the self-adjoint property (4.33) and the property (4.34)

$$E_{rec} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \begin{bmatrix} |a_p(\omega)|^2 - \rho^*(\omega) a_p^*(\omega) e^{-j\omega t_0} \mathcal{P}_+ [\rho(\omega) a_p(\omega) e^{j\omega t_0}] \\ -\kappa^*(\omega) a_p^*(\omega) e^{-j\omega t_0} \mathcal{P}_+ [\kappa(\omega) a_p(\omega) e^{j\omega t_0}] \end{bmatrix} d\omega$$

We now use  $\mathcal{P}_+[\phi(\omega)] = 1 - \mathcal{P}_-[\phi(\omega)]$  to write

$$E_{rec} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \begin{bmatrix} |a_p(\omega)|^2 \\ -\rho^*(\omega) a_p^*(\omega) e^{-j\omega t_0} [\rho(\omega) a_p(\omega) e^{j\omega t_0} - \mathcal{P}_- [\rho(\omega) a_p(\omega) e^{j\omega t_0}]] \\ -\kappa^*(\omega) a_p^*(\omega) e^{-j\omega t_0} [\kappa(\omega) a_p(\omega) e^{j\omega t_0} - \mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]] \end{bmatrix} d\omega$$

After simplification

$$E_{rec} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \begin{bmatrix} |a_p(\omega)|^2 (1 - |\rho(\omega)|^2 - |\kappa(\omega)|^2) \\ +\rho^*(\omega) a_p^*(\omega) e^{-j\omega t_0} \mathcal{P}_- [\rho(\omega) a_p(\omega) e^{j\omega t_0}] \\ +\kappa^*(\omega) a_p^*(\omega) e^{-j\omega t_0} \mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}] \end{bmatrix} d\omega$$

By definition  $1 - |\rho(\omega)|^2 - |\kappa(\omega)|^2 = 0$ , and from the projection properties of  $\mathcal{P}_+$  and  $\mathcal{P}_-$ :

$$E_{rec} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \begin{bmatrix} \rho^*(\omega) a_p^*(\omega) e^{-j\omega t_0} \mathcal{P}_-^2 [\rho(\omega) a_p(\omega) e^{j\omega t_0}] \\ +\kappa^*(\omega) a_p^*(\omega) e^{-j\omega t_0} \mathcal{P}_-^2 [\kappa(\omega) a_p(\omega) e^{j\omega t_0}] \end{bmatrix} d\omega$$

Using the self-adjoint property one more time

$$E_{rec} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \begin{bmatrix} (\mathcal{P}_- [\rho(\omega) a_p(\omega) e^{j\omega t_0}])^* \mathcal{P}_- [\rho(\omega) a_p(\omega) e^{j\omega t_0}] \\ +(\mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}])^* \mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}] \end{bmatrix} d\omega$$

The final result for recoverable energy, given the functions  $\rho(\omega)$  and  $a_p(\omega)$  is

$$E_{rec} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\mathcal{P}_- [\rho(\omega) a_p(\omega) e^{j\omega t_0}]|^2 + |\mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]|^2 d\omega \quad (4.57)$$

where the function  $\kappa(\omega)$  is determined by (4.43) and (4.44).

#### 4.4.1 Time Harmonic Solution

The solution (4.57) for recoverable energy requires information about the terminal behavior of the one-port network given by the reflection coefficient  $\rho(\omega)$ , and information about the particular

way energy is pumped into the one-port network, which enters the equation by  $a_p(\omega)$ . Of particular interest is the case when the signal used to put energy into the system is time-harmonic. This, after all, would permit a comparison between the known definition of  $Q$  and a modified version

$$Q_{rec} = \frac{\omega_0 E_{rec}}{P_{in}}$$

The difficulty in finding recoverable energy for a sinusoidal input is that an input function like  $\hat{a}_p(t) = \cos(\omega_0 t)$  defined for  $t < t_0$  and zero otherwise, is not  $L_2$ . It would mean that an infinite amount of energy had been put into the system (4.1) before energy began to be extracted. (A unique solution for recoverable energy would then simply be impossible.) To work around this, a technique used by Polevoi [46] will be used. Consider the function

$$\hat{a}_p(t) = \begin{cases} e^{\mu(t-t_0)} A \cos(\omega_0 t) & \text{for } t < t_0 \\ 0 & \text{for } t \geq t_0 \end{cases} \quad (4.58)$$

where  $\mu$  and  $A$  are positive real. This function does belong to  $L_2$ , and therefore a unique solution for recoverable energy can be obtained. After recoverable energy has been calculated we can let the positive number  $\mu$  go to zero, and in doing so, obtain a solution for recoverable energy for a sinusoidal input. The details of this calculation are tedious and deferred to Appendix A. The final result for the time-harmonic recoverable energy is

$$E_{rec}(\omega_0, t_0) = -\frac{A^2}{2} (\text{Im} \{ \rho^*(\omega_0) \rho'(\omega_0) \} + \text{Im} \{ \kappa^*(\omega_0) \kappa'(\omega_0) \}) \\ + \text{Im} \left\{ \frac{A^2}{4} \frac{1 - \rho^2(\omega_0) - \kappa^2(\omega_0)}{\omega_0} e^{2j\omega_0 t_0} \right\}$$

where the prime denotes differentiation with respect to the argument  $\omega_0$ .

Up to this point the parameter  $t_0$  has been left in the derivation of recoverable energy with little explanation for its purpose. Polevoi [46] uses this parameter to determine a time averaged recoverable energy via

$$\tilde{E}_{rec}(\omega_0) = \frac{1}{T} \int_0^T E_{rec}(\omega_0, t_0) dt_0$$

where

$$T = \frac{2\pi}{\omega_0}$$

The time averaged recoverable energy is thus

$$\tilde{E}_{rec}(\omega_0) = -\frac{A^2}{2} (\text{Im} \{ \rho^*(\omega_0) \rho'(\omega_0) \} + \text{Im} \{ \kappa^*(\omega_0) \kappa'(\omega_0) \})$$

and is independent of  $t_0$ . Dropping the subscript on  $\omega$  and noting that the time averaged power sent into the system is  $P_{in} = A^2/2$

$$\tilde{E}_{rec}(\omega) = -P_{in} [\text{Im} \{ \rho^*(\omega) \rho'(\omega) \} + \text{Im} \{ \kappa^*(\omega) \kappa'(\omega) \}] \quad (4.59)$$

From this point on, only the time averaged recoverable energy  $\tilde{E}_{rec}$  will be of any concern to us, and for this reason  $\tilde{E}_{rec}$  will be referred to simply as *recoverable energy*. Alternative, and useful, ways to express recoverable energy are:

$$\tilde{E}_{rec}(\omega) = -P_{in} \left[ |\rho(\omega)|^2 \text{Im} \{ [\ln \rho(\omega)]' \} + |\kappa(\omega)|^2 \text{Im} \{ [\ln \kappa(\omega)]' \} \right] \quad (4.60)$$

or, with

$$\rho(\omega) = |\rho(\omega)| e^{j\phi_\rho(\omega)}$$

and

$$\kappa(\omega) = |\kappa(\omega)| e^{j\phi_\kappa(\omega)}$$

recoverable energy can be expressed as

$$\tilde{E}_{rec}(\omega) = -P_{in} \left[ |\rho(\omega)|^2 \phi'_\rho(\omega) + |\kappa(\omega)|^2 \phi'_\kappa(\omega) \right] \quad (4.61)$$

At this point it is worth pointing out the similarity between the result (4.61) and a result due to Kishi and Nakazawa. In their paper [54], they found that the time-averaged stored energy inside of a lossless two-port network is identical to the recoverable energy given by (4.61), if  $\kappa(\omega)$  is identified with transmission coefficient  $S_{21}$  of the network and  $\rho(\omega)$  with the reflection coefficient  $S_{11}$ . This connection provides an interesting interpretation of the physics behind recoverable energy, which will be discussed further at the end of this chapter.

The equation for recoverable energy (4.59) is similar in form to a result found by W. E. Smith [55], which, however, was obtained with a different goal in mind. Smith knew that determining the stored energy within a circuit from the terminal behavior alone is not possible. A constant resistance circuit (see section 1.2), for example, will look like a pure resistance from the terminals, yet it may store energy. Smith instead determined a minimum energy that must be present inside a circuit given an arbitrary impedance  $Z(\omega)$ . The recoverable energy (4.59) discovered in this thesis is the minimum energy Smith presents. This means that not only must at least this amount of energy be present within the circuit, but we have shown that no more than this amount can be extracted from the circuit, and have explicitly shown how this can be done.

Calculating the recoverable energy of a circuit is simplified by the fact that the reflection coefficient  $\rho(\omega)$  can be expressed as the rational function

$$\rho(\omega) = \rho_0 \frac{\prod_{i=1}^{N_\rho} (\omega - \alpha_i)}{\prod_{i=1}^{M_\rho} (\omega - \beta_i)} \quad (4.62)$$

where  $\alpha_i$  are the zeros of  $\rho(\omega)$ ,  $\beta_i$  are its poles and  $\rho_0$  is a complex constant with the requirement  $|\rho_0| < 1$ <sup>1</sup>. The fact that  $|\rho(\omega)| \leq 1$  puts the following restriction on  $N_\rho$  and  $M_\rho$

$$N_\rho \leq M_\rho$$

For the rational function (4.62) to represent a causal transfer function, all of the poles  $\beta_i$  are required to be in the upper half-plane, which makes  $\rho(\omega)$  analytic in the lower complex plane. The magnitude squared of the function  $\kappa(\omega)$  is

$$|\kappa(\omega)|^2 = \frac{\left| \prod_{i=1}^{M_\rho} (\omega - \beta_i) \right|^2 - |\rho_0|^2 \left| \prod_{i=1}^{N_\rho} (\omega - \alpha_i) \right|^2}{\prod_{i=1}^{M_\rho} (\omega - \beta_i) \left[ \prod_{i=1}^{M_\rho} (\omega - \beta_i) \right]^*}$$

and since  $|\rho_0| < 1$ , the number of zeros that  $|\kappa(\omega)|^2$  has is equal to the number of its poles. The numerator of  $|\kappa(\omega)|^2$  is real, making it necessary for its roots to come in conjugate pairs.

---

<sup>1</sup> Reflection coefficients with  $|\rho_0| = 1$  must be handled with care. It will be shown in the examples that such situations can usually be handled by carefully adding a resistor to a circuit to make sure  $|\rho_0| < 1$  at least initially. That resistor can be removed after the calculation of recoverable energy through some limiting process.

Factoring the numerator as

$$\left| \prod_{i=1}^{M_\rho} (\omega - \beta_i) \right|^2 - |\rho_0|^2 \left| \prod_{i=1}^{N_\rho} (\omega - \alpha_i) \right|^2 = |\kappa_0|^2 \prod_{i=1}^{M_\rho} (\omega - \xi_i) \left[ \prod_{i=1}^{M_\rho} (\omega - \xi_i) \right]^*$$

where  $|\kappa_0|^2 = 1 - |\rho_0|^2$  and  $\xi_i$  are the roots of the numerator of  $|\kappa(\omega)|^2$  that are in the upper complex plane. The factorization

$$|\kappa(\omega)|^2 = \kappa(\omega) \kappa^*(\omega)$$

now amounts to finding the zeros of the numerator of  $|\kappa(\omega)|^2$  and constructing

$$\kappa(\omega) = \kappa_0 \frac{\prod_{i=1}^{M_\rho} (\omega - \xi_i)}{\prod_{i=1}^{M_\rho} (\omega - \beta_i)}$$

where  $\kappa_0 = \sqrt{1 - |\rho_0|^2}$ . To calculate recoverable energy note that

$$\begin{aligned} [\ln \rho(\omega)]' &= \left[ \ln \rho_0 + \sum_{i=1}^{N_\rho} \ln(\omega - \alpha_i) - \sum_{i=1}^{M_\rho} \ln(\omega - \beta_i) \right]' \\ &= \sum_{i=1}^{N_\rho} \frac{1}{\omega - \alpha_i} - \sum_{i=1}^{M_\rho} \frac{1}{\omega - \beta_i} \end{aligned}$$

and

$$[\ln \kappa(\omega)]' = \sum_{i=1}^{M_\rho} \frac{1}{\omega - \xi_i} - \frac{1}{\omega - \beta_i}$$

The formula (4.60) can now be used to find  $\tilde{E}_{rec}$

$$\tilde{E}_{rec}(\omega) = -P_{in} \operatorname{Im} \left[ \sum_{i=1}^{N_\rho} \frac{|\rho(\omega)|^2}{\omega - \alpha_i} + \sum_{i=1}^{M_\rho} \frac{|\kappa(\omega)|^2}{\omega - \xi_i} - \sum_{i=1}^{M_\rho} \frac{1}{\omega - \beta_i} \right] \quad (4.63)$$

We now provide four examples demonstrating how recoverable energy can be calculated, and we show why recoverable energy is important.

#### 4.4.2 Example 1

As an example, consider the circuit in Figure 4.5. The input impedance is

$$Z(\omega) = \frac{j\omega C R_1 R_2 + R_1 + R_2}{j\omega C R_2 + 1}$$

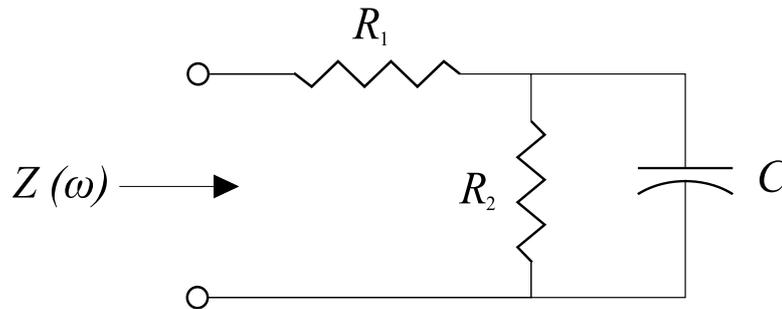


Figure 4.5: Circuit example 1.

The time-averaged stored energy inside of the capacitor is

$$\tilde{E}_{stored} = \frac{1}{4} C |V_C|^2 \quad (4.64)$$

where  $V_C$  is the voltage across the capacitor. We determine this voltage to be

$$V_C = \frac{R_2 \parallel \frac{1}{j\omega C}}{R_1 + R_2 \parallel \frac{1}{j\omega C}} V$$

where  $V$  is the voltage across the circuit terminals. (The notation  $\parallel$  means that the circuit elements are in parallel, i.e.,  $a \parallel b = (1/a + 1/b)^{-1}$ .) For plotting purposes, it is convenient to choose the normalizations

$$r_{1,2} = \frac{R_{1,2}}{Z_0}$$

and

$$\tau = CZ_0$$

The voltage across the capacitor is now

$$V_C = \frac{r_2}{(r_1 + r_2) + j\omega\tau r_1 r_2} V \quad (4.65)$$

Assuming that this one-port network is fed by a transmission line with a characteristic impedance of  $Z_0$ , the reflection coefficient is

$$\begin{aligned} \rho(\omega) &= \frac{Z(\omega) - Z_0}{Z(\omega) + Z_0} \\ &= \frac{j\omega C R_2 (R_1 - Z_0) + (R_1 + R_2) - Z_0}{j\omega C R_2 (R_1 + Z_0) + (R_1 + R_2) + Z_0} \\ &= \frac{j\omega\tau r_2 (r_1 - 1) + (r_1 + r_2) - 1}{j\omega\tau r_2 (r_1 + 1) + (r_1 + r_2) + 1} \end{aligned}$$

At the terminals of the one-port network, the voltage may be written as the sum of a forward traveling voltage  $V_+$  plus the reflected voltage  $V_- = \rho V_+$ ; taking this in conjunction with (4.64) and (4.65), the stored energy inside of this circuit is

$$\begin{aligned}\tilde{E}_{stored}(\omega) &= \frac{1}{4}C \left| \frac{r_2}{(r_1 + r_2) + j\omega\tau r_1 r_2} (1 + \rho(\omega)) V_+ \right|^2 \\ &= 2\tau P_{in} \frac{r_2^2}{\omega^2 \tau^2 r_2^2 (r_1 + 1)^2 + (r_1 + r_2 + 1)^2}\end{aligned}$$

where  $P_{in} = |V_+|^2 / (2Z_0)$  is the average power delivered to the system.

To calculate recoverable energy, notice that  $\rho(\omega)$  has the form

$$\begin{aligned}\rho &= \frac{j\omega a + b}{j\omega c + d} \\ &= \frac{a \left( \omega - j\frac{b}{a} \right)}{c \left( \omega - j\frac{d}{c} \right)}\end{aligned}\tag{4.66}$$

where

$$a = \tau r_2 (r_1 - 1)$$

$$b = r_1 + r_2 - 1$$

$$c = \tau r_2 (r_1 + 1)$$

and

$$d = r_1 + r_2 + 1$$

It follows that

$$\begin{aligned}|\kappa(\omega)|^2 &= 1 - |\rho(\omega)|^2 \\ &= \frac{\omega^2 (c^2 - a^2) + d^2 - b^2}{c^2 \left( \omega - j\frac{d}{c} \right) \left( \omega + j\frac{d}{c} \right)}\end{aligned}$$

or

$$|\kappa(\omega)|^2 = \frac{(c^2 - a^2)}{c^2} \frac{\left( \omega - j\sqrt{\frac{(d^2 - b^2)}{(c^2 - a^2)}} \right) \left( \omega + j\sqrt{\frac{(d^2 - b^2)}{(c^2 - a^2)}} \right)}{\left( \omega - j\frac{d}{c} \right) \left( \omega + j\frac{d}{c} \right)}$$

By inspection we determine

$$\kappa(\omega) = \sqrt{\frac{(c^2 - a^2)}{c^2}} \frac{\left(\omega - j\sqrt{\frac{(d^2 - b^2)}{(c^2 - a^2)}}\right)}{\left(\omega - j\frac{d}{c}\right)} \quad (4.67)$$

which is analytic in the lower complex plane and has no zeros in the lower complex plane. Sticking (4.66) and (4.67) into the expression for  $\tilde{E}_{rec}(\omega)$  (4.63), and after a little algebra we find that

$$\tilde{E}_{rec}(\omega) = 2\tau P_{in} \frac{r_2^2 - 2r_2 \left(\sqrt{r_1(r_1 + r_2)} - r_1\right)}{\omega^2 \tau^2 r_2^2 (r_1 + 1)^2 + (r_1 + r_2 + 1)^2}$$

Comparing  $\tilde{E}_{rec}(\omega)$  to the time-averaged stored energy  $\tilde{E}_{stored}(\omega)$ , we see that

$$\tilde{E}_{rec}(\omega) = \tilde{E}_{stored}(\omega) - 2\tau P_{in} \frac{2r_2 \left(\sqrt{r_1(r_1 + r_2)} - r_1\right)}{\omega^2 \tau^2 r_2^2 (r_1 + 1)^2 + (r_1 + r_2 + 1)^2}$$

The second term to the right of the equal sign is nonnegative for all physical  $r_1$ ,  $r_2$  and  $\tau$ , i.e., when these parameters are nonnegative. We can think of this second term as the part of the stored energy within the circuit that cannot be recovered. It is clear, in this example, that  $\tilde{E}_{rec}(\omega) \leq \tilde{E}_{stored}(\omega)$  for all  $\omega$ . Equality is achieved either if  $r_2 \rightarrow 0$  or if  $r_1 \rightarrow \infty$ .

#### 4.4.3 Example 2

As an important example, consider the circuit of Figure 4.6. The details to find the stored energy and the recoverable energy in this example are messy, the details have been deferred to Appendix B. The input impedance to the circuit is found to be

$$Z(\omega) = \frac{j\omega(LC^2\omega^2 R_2^2 + L - CR_2^2) + \omega^2 C^2 R_2^2 R_1 + R_2 + R_1}{(1 + \omega^2 R_2^2 C^2)}$$

while the time-averaged stored energy is

$$\tilde{E}_{stored}(\omega) = 2CZ_0 P_{in} \frac{1 + CL \left(\omega^2 + \frac{1}{C^2 R_2^2}\right)}{\left(\frac{R_2 + R_1 + Z_0}{R_2} - \omega^2 LC\right)^2 + \omega^2 \left(\frac{L}{R_2} + C(R_1 + Z_0)\right)^2}$$

We have assumed that the one-port network is fed by a transmission line with characteristic impedance  $Z_0$ , and the power delivered to the circuit is  $P_{in}$ . We define the resonant frequency to be

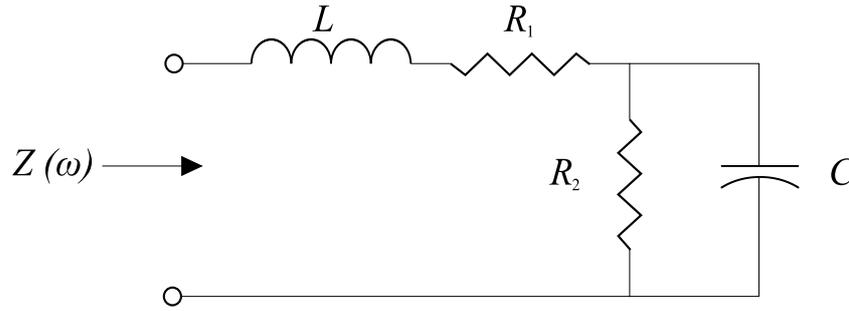


Figure 4.6: Circuit example 2.

$$\omega_0 = \sqrt{\frac{1}{LC} - \left(\frac{1}{CR_2}\right)^2}$$

which is the frequency at which  $Z(\omega)$  is real. At the frequency  $\omega_0$ , we define the resistance  $R_o$  as

$$R_o \equiv Z(\omega_0) = R_1 + \frac{L}{R_2 C}$$

which is the impedance of the circuit at resonance. When  $R_2 \rightarrow \infty$ , the circuit becomes a series  $RLC$  circuit with  $\omega_0 = 1/\sqrt{LC}$  and  $R_o = R_1$ . In this example we only consider a range of  $R_2$  for which

$$R_2 \geq \sqrt{\frac{L}{C}}$$

In this range, the resonant frequency  $\omega_0$  is real and we say that the circuit is resonant.

The reflection coefficient is

$$\rho(\omega) = \frac{-LCR_2\omega^2 + j\omega(L + CR_1R_2 - CR_2Z_0) + (R_1 + R_2 - Z_0)}{-LCR_2\omega^2 + j\omega(L + CR_1R_2 + CR_2Z_0) + (R_1 + R_2 + Z_0)}$$

In Appendix B we find that

$$\tilde{E}_{rec}(\omega) = \tilde{E}_{stored}(\omega) - 2CR_0P_{in} \frac{\frac{2}{R_2} \left( \sqrt{R_1(R_1 + R_2)} - R_1 \right)}{\left( \frac{R_1 + R_2 + R_0}{R_2} - LC\omega^2 \right)^2 + \omega^2 \left( \frac{L}{R_2} + C(R_1 + R_0) \right)^2}$$

if we choose to match the circuit to  $Z_0$  at resonance (i.e., make  $Z_0 = R_0$ ). Notice that the second term in this sum is always positive, therefore

$$\tilde{E}_{rec}(\omega) \leq \tilde{E}_{stored}(\omega) \quad (4.68)$$

for all physical values of  $R_1$ ,  $R_2$ ,  $L$ , and  $C$  (i.e., when  $R_1$ ,  $R_2$ ,  $L$  and  $C$  are nonnegative).

This is an important result, and solves the problem discussed in Section 1.4. When  $R_2 \rightarrow \infty$  the circuit of Figure 4.6 becomes the *RLC* circuit of Figure 1.1, and the inequality (4.68) becomes the equality

$$\tilde{E}_{rec}(\omega) = \tilde{E}_{stored}(\omega)$$

Recoverable energy  $\tilde{E}_{rec}(\omega)$  was determined from the impedance of the circuit alone: no information about the internal structure was required, which was needed to determine  $\tilde{E}_{stored}(\omega)$ . In Section 1.1 the energy definition for quality factor was given by

$$Q_{stored} = \frac{\omega_0 \tilde{E}_{stored}(\omega_0)}{P}$$

where  $\omega_0$  is the resonant frequency and  $P$  is the power dissipated in the circuit. (We put the subscript *stored* on  $Q$  to distinguish it from the other quality factors to be defined below.) For the series *RLC* circuit (see Figure 1.1) we showed that the energy definition of quality factor  $Q_{stored}$  and the fractional bandwidth  $B_{3dB}$  were related by

$$Q_{stored} = \frac{1}{B_{3dB}}$$

If recoverable and stored energy are equal for the circuit of Figure 1.1, and if we define the new quality factor

$$Q_{rec} = \frac{\omega_0 \tilde{E}_{rec}(\omega_0)}{P} \tag{4.69}$$

it must be true that

$$Q_{rec} = \frac{1}{B_{3dB}} \tag{4.70}$$

holds for the circuit of Figure 1.1. The definition (4.69) is invariant to replacing the resistor of the *RLC* circuit by a constant resistance circuit (see Section 1.2). For example, the recoverable energy in the circuit depicted by Figure 1.2 is identical to that of the circuit of Figure 1.1. Both of these circuits also have the same terminal behavior at the input. We conclude that the definition (4.69) is the correct definition for quality factor for all circuits that are equivalent to Figure 1.1.

It is interesting to compare different definitions of quality factor for the circuit of Figure 4.6 when  $R_2$  is allowed to vary. To do this, we choose the normalizations

$$w = \frac{\omega}{\omega_0} \quad \theta = R_2 \sqrt{\frac{C}{L}} \quad \zeta = \sqrt{\frac{L}{R_1^2 C}}$$

and thus

$$|\rho(w)|^2 = \frac{(1-w^2)^2}{\left(\frac{2(\theta/\zeta+1)}{(\theta^2-1)} + 1 - w^2\right)^2 + w^2 \frac{4(\theta/\zeta+1)^2}{(\theta^2-1)}}$$

The quality factors  $Q_{stored}$  (which depends on stored energy (1.6)) and  $Q_{rec}$  (which depends on recoverable energy (4.69)) are

$$Q_{stored} = \frac{\zeta \sqrt{\theta^2 - 1}}{\theta + \zeta}$$

and

$$Q_{rec} = \left[ \left( \frac{1}{\theta\zeta} + 1 \right) - \sqrt{\frac{1}{\theta\zeta} \left( \frac{1}{\theta\zeta} + 1 \right)} \right] Q_{stored} \quad (4.71)$$

We are also interested in the bandwidth definitions of  $Q_{stored}$ . We know from Chapter 1 that the definition (1.5) is exact for the circuit of Figure 1.1. To allow the parameter to be defined for various parameters  $\theta$  and  $\zeta$ , we define

$$Q_{3dB} \equiv \frac{1}{B} = \frac{1}{\Delta\omega_{3dB}}$$

where  $\Delta\omega$  is the 3 dB bandwidth of the transfer function  $|\kappa(w)|^2$ . In Chapter 3 we introduced the energy bandwidth (see Section 3.1.1), we define therefore define

$$Q_{EBW} \equiv \frac{1}{\Delta\omega_E}$$

where  $\Delta\omega_E$  is the width determined by the procedure marked out in that chapter. The constant  $C$  will be chosen as 1/2 in the present example.

Figure 4.7 is a plot of the four quality factors as a function of  $\theta$ . For this case we set  $\zeta = 7$  and let  $\theta$  run over the domain (1, 20) (note that  $\theta \geq 1$  for the circuit to be resonant according to our definition). As  $\theta$  goes to infinity,  $Q_{rec}$  approaches  $Q_{stored}$ , which is clear from Eqn. (4.71). But, for this value of  $\zeta$ ,  $\theta$  needs to be far greater than 20 to see this limit graphically. Notice that

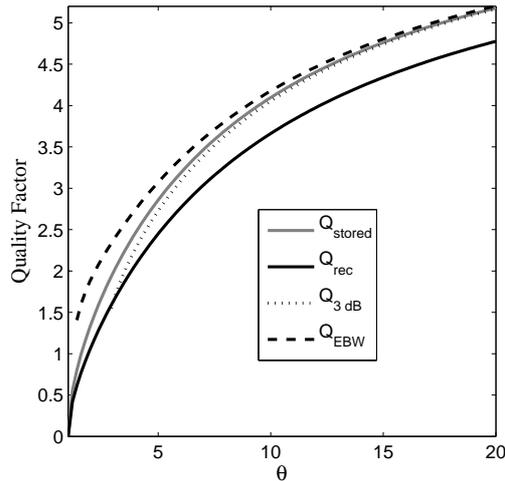


Figure 4.7: Quality factors as functions of  $\theta$ .

while  $Q_{\text{rec}}$  does not directly correspond to  $Q_{3\text{dB}}$  or  $Q_{\text{EBW}}$ , it does present a solution that is of the same order of magnitude.

It is also important to note here the advantages energy bandwidth (see Section 3.1.1) has over 3 dB bandwidth. For small  $\theta$ , the 3 dB bandwidth does not even exist. Also notice how 3 dB bandwidth intersects with  $Q_{\text{rec}}$ . On the other hand,  $Q_{\text{EBW}}$  is always greater than  $Q_{\text{rec}}$ , and the trend of  $Q_{\text{EBW}}$  appears to match that of  $Q_{\text{rec}}$ . In this example, energy bandwidth is the better choice for measuring bandwidth, future studies will show if this is true in general.

It is curious that  $Q_{\text{stored}}$  tracks both  $Q_{3\text{dB}}$  and  $Q_{\text{EBW}}$  better than  $Q_{\text{rec}}$  does; but, we should remember that stored energy is not unique to the terminal behavior. We could replace the resistors in this circuit by constant resistance circuits (see Section 1.2) and arbitrarily increase  $Q$  to whatever we like without changing the terminal behavior. Therefore,  $Q_{\text{rec}}$  is the correct factor to use, in that it is a unique quantity determined from the terminals—a necessary characteristic if energy is to correspond to terminal behavior.

#### 4.4.4 Example 3

We now consider the recoverable energy of a nonuniform transmission line. In Chapter 3 we introduced the differential equation

$$\frac{d\Gamma}{dx} = 2j\beta(x)\Gamma - N(x)(1 - \Gamma^2) \quad (4.72)$$

appropriate for the reflection coefficient  $\Gamma(x, \omega)$  of a nonuniform transmission line. The function  $N(x)$  is related to the characteristic impedance  $Z_c(x)$  of the line by

$$N(x) = \frac{1}{2} \frac{d}{dx} \ln Z_c(x) \quad (4.73)$$

and the propagation constant  $\beta(x)$  is related to the phase velocity  $v(x)$  by

$$\beta(x) = \frac{\omega}{v(x)} \quad (4.74)$$

Following Browning [56], we multiply the above Riccati equation by  $\Gamma^*$  and take the real part

$$\Gamma^* \frac{d}{dx} \Gamma + \Gamma \frac{d}{dx} \Gamma^* = -N(x)(\Gamma + \Gamma^*)(1 - |\Gamma|^2) \quad (4.75)$$

or

$$\frac{d}{dx} (|\Gamma|^2) = -N(x)(\Gamma + \Gamma^*)(1 - |\Gamma|^2)$$

Dividing both sides by  $(1 - |\Gamma|^2)$  we have

$$-\frac{\frac{d}{dx} (1 - |\Gamma|^2)}{(1 - |\Gamma|^2)} = -N(x)(\Gamma + \Gamma^*)$$

or

$$\frac{d}{dx} \ln (1 - |\Gamma|^2) = N(x)(\Gamma + \Gamma^*) \quad (4.76)$$

Integrating both sides over  $x$  from 0 to  $L$  produces

$$\ln (1 - |\Gamma(L, \omega)|^2) - \ln (1 - |\Gamma(0, \omega)|^2) = \int_0^L N(x)(\Gamma + \Gamma^*) dx \quad (4.77)$$

We now suppose that the nonuniform transmission line is matched at the load (see Figure 3.11), i.e.,  $\Gamma(L, \omega) = 0$ . We also may identify the argument of the logarithm as  $|\kappa(\omega)|^2 = 1 - |\Gamma(0, \omega)|^2$ .

Therefore, we have

$$\begin{aligned} \ln \left( |\kappa(\omega)|^2 \right) &= - \int_0^L N(x) (\Gamma + \Gamma^*) dx \\ &= -2 \int_0^L N(x) \operatorname{Re} \Gamma dx \end{aligned} \quad (4.78)$$

From (4.44) we may determine the phase of the function  $\kappa(\omega)$ :

$$\phi_\kappa(\omega) = -p.v. \int_{-\infty}^{\infty} \frac{\int_0^L N(x) (\Gamma + \Gamma^*) dx}{\omega' - \omega} d\omega'$$

From the fact that the reflection coefficient  $\Gamma(x, \omega)$  is causal, and therefore analytic in the lower half-plane, a straight forward calculation will show that

$$\phi_\kappa(\omega) = - \int_0^L N(x) \operatorname{Im} \Gamma dx \quad (4.79)$$

It follows from (4.79) and (4.78) that

$$\begin{aligned} \kappa(\omega) &= |\kappa(\omega)| \exp(j\phi_\kappa(\omega)) \\ &= \exp\left(- \int_0^L N(x) \operatorname{Re} \Gamma dx\right) \exp\left(-j \int_0^L N(x) \operatorname{Im} \Gamma dx\right) \end{aligned}$$

and clearly

$$\kappa(\omega) = \exp\left(- \int_0^L N(x) \Gamma(x, \omega) dx\right) \quad (4.80)$$

Up to this point, the analysis is exact. Provided that the reflection coefficient  $\Gamma(x, \omega)$  is known, it is possible to calculate the recoverable energy using  $\rho(\omega) = \Gamma(0, \omega)$  and the  $\kappa(\omega)$  determined from Eqn. (4.80). In general, these calculations are unwieldy. But, for large  $\omega$ , i.e., when

$$\omega \gg \left| \frac{N(x) v(x)}{2} \right| \quad (4.81)$$

and

$$\omega \gg \left| \frac{v(x)}{2} \frac{d}{dx} \ln [v(x) N(x)] \right| \quad (4.82)$$

the reflection coefficient of a nonuniform transmission line, with the condition  $\Gamma(L, \omega) = 0$ , has the asymptotic behavior  $\Gamma(x, \omega) \sim v(x) N(x) / [2j\omega]$ . The function  $\kappa(\omega)$  becomes

$$\kappa(\omega) = \exp\left(-\frac{1}{2j\omega} \int_0^L v(x) N^2(x) dx\right) \quad (4.83)$$

To calculate recoverable energy (4.59) we find that

$$\kappa^*(\omega) \kappa'(\omega) = -\frac{j}{2\omega^2} \int_0^L v(x) N^2(x) dx$$

and

$$\rho^*(\omega) \rho'(\omega) = -\frac{v(0)^2 N^2(0)}{4\omega^3}$$

Keeping only the term of order  $\omega^{-2}$  in Eqn. (4.59), we find the asymptotic behavior of the recoverable energy is

$$\tilde{E}_{rec}(\omega) \sim \frac{P_{in}}{2\omega^2} \int_0^L v(x) N^2(x) dx \quad (4.84)$$

for when (4.81) and (4.82) are satisfied.

The stored energy within the nonuniform transmission line can be calculated from

$$\tilde{E}_{stored}(\omega) = \frac{1}{4} \int_0^L \left[ c(x) |V(x, \omega)|^2 + \frac{l(x)}{Z_c^2(x)} |V(x, \omega)|^2 \right] dx \quad (4.85)$$

(see [57]). Where  $c(x)$  and  $l(x)$  are the per-unit-length capacitance and inductance of the line. The function  $V(x, \omega)$  represents the voltage along the line. When  $\omega$  satisfies (4.81) and (4.82), then  $|\Gamma(x, \omega)| \ll 1$ ; therefore, we may use the WKB approximation (see [58]) for the voltage along the line

$$|V(x, \omega)| \approx \left| V_+ \sqrt{\frac{Z_c(x)}{Z_c(0)}} \right|$$

where  $V_+$  is the voltage incident to the nonuniform line. In this approximation the stored energy asymptotically approaches

$$\tilde{E}_{stored}(\omega) \sim P_{in} t_d \quad (4.86)$$

where  $t_d$  is the time delay of the line:

$$t_d = \int_0^L \frac{1}{v(x)} dx \quad (4.87)$$

Here we have used

$$v(x) = \frac{1}{\sqrt{l(x)c(x)}} \quad (4.88)$$

$$Z_c(x) = \sqrt{\frac{l(x)}{c(x)}} \quad (4.89)$$

and

$$P_{in} = \frac{|V_+|^2}{2Z_c(0)} \quad (4.90)$$

The two asymptotic limits (4.84) and (4.86) demonstrate a dramatic difference between stored energy and recoverable energy. For example, if we compare the quality factor defined for recoverable energy

$$\begin{aligned} Q_{rec} &= \frac{\omega \tilde{E}_{rec}(\omega)}{P} \\ &= \frac{1}{2\omega} \int_0^L v(x) N^2(x) dx \end{aligned} \quad (4.91)$$

to quality factor defined in the IEEE dictionary of standards

$$\begin{aligned} Q &= \frac{\omega \tilde{E}_{stored}(\omega)}{P} \\ &= \omega t_d \end{aligned} \quad (4.92)$$

we see that for large  $\omega$  that  $Q_{rec}$  goes to zero, while  $Q$  becomes infinite. It is well known that appropriately-smooth nonuniform transmission lines, satisfying  $\Gamma(L, \omega) = 0$ , will be better matched as the frequency  $\omega$  increases; therefore, at high frequencies the bandwidth about  $\omega$  will be broad. This is not what the  $Q$  in Eqn. (4.92) predicts. On the other hand, the inverse dependence of  $Q_{rec}$  on  $\omega$  does appear to properly correspond to the bandwidth.

#### 4.4.5 Example 4

As a final example, we attempt to show that the recoverable energy of the first-order transverse-magnetic Chu circuit (see [11]) is equal to the stored energy within the circuit. This lowest order Chu circuit is representative of an electric dipole antenna and is important to the study of electrically small antennas.

We discussed in Chapter 1 that  $Z_n^{TM}(x)$  of Eqn. (1.8) represents the impedance of each mode on a sphere of radius  $x = kr$ . ( $k$  is the wave number and  $r$  is the radius of the sphere.) From the expression of Eqn. (1.8), Chu observed that he could construct a circuit with the impedance  $Z_n^{TM}(x)$  (see Figure 1.4). He then used this circuit, containing inductors and capacitors, to determine the energy stored within the field outside the sphere of radius  $kr$ . He argued that this energy stored outside the sphere represents a lower bound on the total stored energy of an antenna that just fits within the sphere.

Although the circuit of Figure 1.4 does represent one configuration having the impedance  $Z_n^{TM}(x)$ , constant resistance circuits could always be used to replace the resistor  $Z_0$ . Following the discussion in Chapter 1 (see Section 1.2), an infinite number of configurations can be constructed having the same impedance  $Z_n^{TM}(x)$ . Each of these configurations can potentially have different stored energies. If there are an infinite number of circuits having the same impedance, which of these should be chosen to calculate stored energy?

Chu's intention was to find a lower bound on the stored energy of a radiating antenna. Since there are an infinite number of configurations having the impedance  $Z_n^{TM}(x)$ , it is reasonable then to seek the minimum stored energy that can be determined from the impedance alone. This minimum stored energy, as we have discussed, is the recoverable energy.

Calculating recoverable energy for the circuit of Figure 1.4 is made difficult due to the behavior of the input impedance at  $\omega = 0$ . At this frequency, the circuit looks like an open circuit from the input terminals, i.e.,  $\rho(0) = 1$ . Thus, the product factorization of  $|\kappa(\omega)|^2 = 1 - |\rho(\omega)|^2$  (4.37) cannot be done since  $\ln(|\kappa(\omega)|)$  must be analytic in a strip. The problem can be mitigated by considering the modified Chu circuit of Figure 4.8. We content ourselves with calculating the recoverable energy of the first order Chu circuit only. The resistor  $R$  modifies the original Chu circuit so that there is no longer a zero in the function  $|\kappa(\omega)|$  at zero frequency. Once the recoverable energy has been determined for this modified circuit, we can let  $R \rightarrow \infty$  to remove the resistor. The limiting procedure enables us to determine the recoverable energy of the classic Chu circuit.

We resort to a numerical calculation to find the recoverable energy of the modified Chu

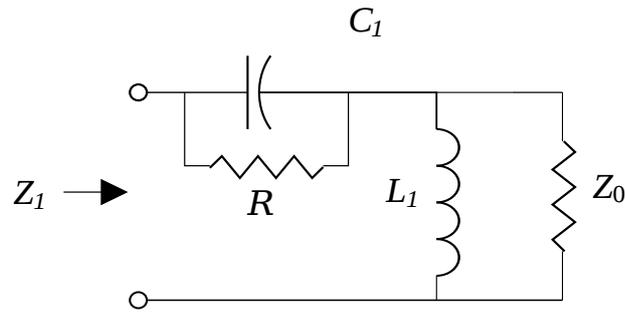


Figure 4.8: Modified Chu circuit.

circuit. It is evident from the algebra in Appendix B that recoverable energy calculations can be unwieldy analytically. The difficulty is, in part, due to factoring second order polynomials and the algebra that follows. We use Matlab to calculate  $\tilde{E}_{rec}(\omega)$  from  $\rho(\omega)$  and (4.63).

We choose the following parameters for the calculation:

$$Z_0 = 1.00 \Omega$$

$$C_1 = 0.01 \mu\text{F}$$

and

$$L_1 = 0.01 \mu\text{H}$$

These parameters correspond to a sphere of radius  $a = 0.47$  meters. Figures 4.9-4.11 are plots of the stored and recoverable energy as a function of  $kr$ . Recoverable energy is plotted for  $R = 10$ ,  $R = 100$ , and  $R = 1000$ . Notice how the plot of recoverable energy converges to that of the stored energy as  $R$  increases. When  $R = 10000$ , the two plots are so close they cannot be distinguished. Although this should be checked analytically, these results suggest that the recoverable energy and the stored energy of the first-order transverse-magnetic Chu circuit are indeed the same.

The result found here is significant. It indicates that the circuit Chu utilized for the impedance  $Z_1^{TM}(x)$  is the circuit associated with the minimum energy that can be determined from the impedance alone. Therefore, the lower bound on stored energy which Chu provides in his paper is justified.

It would be interesting to see if this is true for the rest of the spherical modes, i.e.,  $Z_n^{TM}(x)$

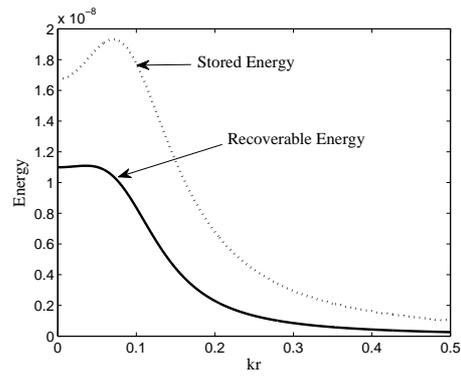


Figure 4.9: Modified Chu circuit with  $R = 10$ .

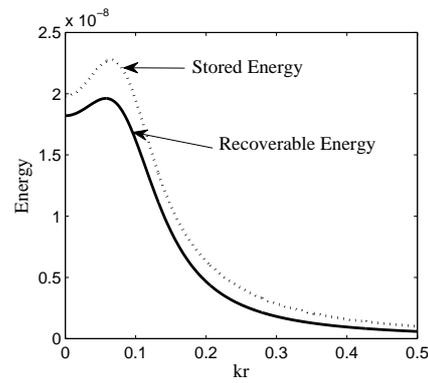


Figure 4.10: Modified Chu circuit with  $R = 100$ .

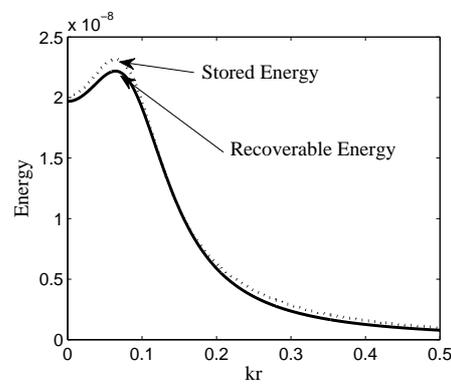


Figure 4.11: Modified Chu circuit with  $R = 1000$ .

where  $n > 1$ . Proving the result found here analytically and determining if stored energy is equal to recoverable energy for higher order modes is left for future work.

## 4.5 Minimum Phase Darlington Circuits

Another result from Smith's paper [55], relevant to the study of recoverable energy, is that a circuit can be synthesized from the impedance  $Z(\omega)$  that will have a stored energy equal to its minimum energy. The synthesis is known as minimum phase Darlington synthesis. Darlington synthesis involves the construction of a circuit having an impedance  $Z(\omega)$ , which is constructed from a lossless two-port network terminated in a single resistance (see Figure 4.13). Conventional Darlington synthesis, discussed by Balabanian [59], will add phase factors to the lossless two-port network. This will make the resulting network both realizable with passive devices and reciprocal. An alternative was found by Hazony [52]. He found that the synthesis could be done without added phase factors, which results in a lossless two-port having minimum phase transmission coefficients. The draw back of this kind of synthesis is that the lossless two-port network will in general be nonreciprocal and the nonstandard circuit element known as a gyrator (see [60]) must be used to realize the circuit. Smith calls Hazony's method of synthesis, minimum phase Darlington synthesis. Since Smith's minimum energy is equal to the recoverable energy, it is in turn true that a circuit can be construct from an impedance  $Z(\omega)$  for which the recoverable energy is identically equal to the stored energy. Minimum phase Darlington synthesis can be used to find this circuit.

As an example, we determine the minimum phase Darlington synthesis of the circuit seen in Figure 4.6. We use the procedure of Hazony's mapped out by Karni in his book: see p. 297-299 of [61]. The resulting circuit is shown in Figure 4.12, where

$$\begin{aligned} C_M &= \frac{CR_2}{R_1 + R_2} \\ R_M &= R_1 + R_2 \end{aligned}$$

and

$$R_G = \sqrt{R_1(R_1 + R_2)}$$

The device at the bottom of the circuit is known as a gyrator, which is a nonreciprocal device. The energy stored in this circuit is in the inductor and the capacitor. Basic circuit analysis may

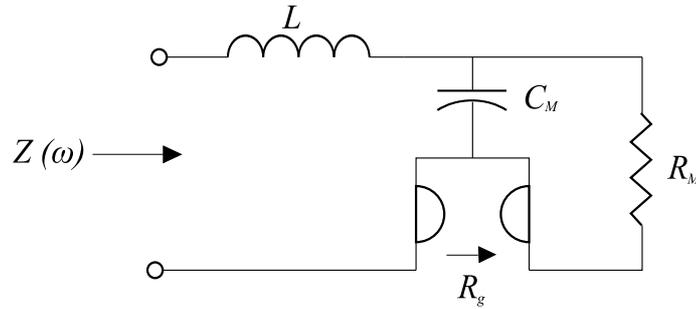


Figure 4.12: Minimum phase shift Darlington synthesis of the circuit of Figure 4.6.

be applied to determine the stored energy within the circuit. The resulting time-averaged stored energy is

$$\tilde{E}_{stored}(\omega) = 2CZ_0P_{in} \frac{\left(1 + LC \left(\frac{1}{C^2R_2^2} + \omega^2\right)\right) - \frac{2}{R_2} \left(\sqrt{R_1(R_1 + R_2)} - R_1\right)}{\left(\frac{R_1 + R_2 + Z_0}{R_2} - LC\omega^2\right)^2 + \omega^2 \left(\frac{L}{R_2} + C(R_1 + Z_0)\right)^2}$$

We see that the stored energy of this circuit is identical to the recoverable energy of the circuit of Figure 4.6.

It is some what curious that the recoverable energy of an arbitrary impedance requires a Darlington synthesis containing a nonreciprocal network. Nonreciprocal passive networks require gyrators, which are not standard devices. (These devices generally require active elements to simulate them.) Smith questioned the possibility of obtaining a minimum phase and *reciprocal* Darlington synthesis. This kind of circuit would have an energy larger than the recoverable energy. It would be of interest to determine if such a synthesis could be done, and if so, does the stored energy of this synthesized circuit produce a quality factor that more accurately tracks bandwidth. Perhaps this quality factor would be more appropriate in Example 2.

## 4.6 Summary

The examples provided in this chapter demonstrate the value of recoverable energy. Eqn. (4.59) provides a simple formula for its calculation, and despite some algebra, calculating  $\tilde{E}_{rec}(\omega)$  for the various examples in this chapter was straightforward.

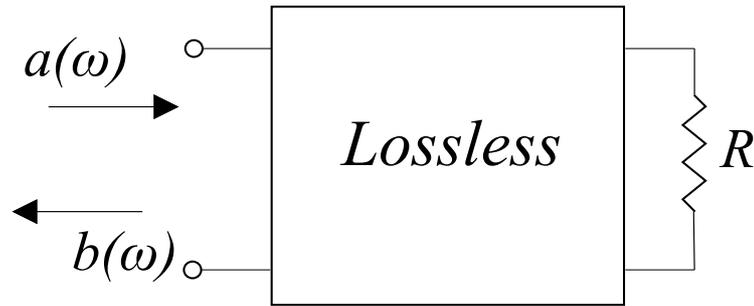


Figure 4.13: Darlington type circuit.

It is clear from Example 2 that recoverable energy solves the problem discussed in Chapter 1 concerning circuits equivalent to the  $RLC$  circuit of Figure 1.1. When the resistor of this circuit was replaced by a transmission line (see Figure 1.2), the stored energy would vary as a function of the length  $l$ . Recoverable energy, on the other hand, returns the correct energy that makes definitions (4.69) and (4.70) consistent for any length  $l$ . In fact, recoverable energy is the right choice for any circuit equivalent to the  $RLC$  circuit. From the discussion on constant resistance circuits (see Section 1.2), there are an infinite number of circuits that are equivalent to the  $RLC$  circuit of Figure 1.1. Clearly, recoverable energy is the right choice for all of these circuits.

Example 3 was a dramatic result showing the difference between stored and recoverable energy of a nonuniform transmission line. Clearly, the quality factor defined using  $\tilde{E}_{rec}(\omega)$  (see Eqn. (4.91)) implied the correct behavior for bandwidth as frequency increased. The quality factor defined with stored energy (see Eqn. 4.92) did not. Also notice that the quality factor  $Q_{rec}$  (4.91) implied the same behavior for bandwidth that the Bode-Fano result of Chapter 3 did.

Example 4 verified Chu's energy bound [11] for electrically small antennas. His choice of circuit (see Figure 1.4) for the lowest order TM mode appears to be the circuit whose stored energy is the smallest energy that can be determined from the mode impedance  $Z_1^{TM}(x)$ . This is the first time this has been verified.

Circuits can be synthesized from the reflection coefficient  $\rho(\omega)$  that have a stored energy equal to their recoverable energy. These circuits, in general, contain nonreciprocal elements. Finding the stored energy of *reciprocal* minimum phase Darlington circuits is left for future research, but the

energy contained in these circuits may provide insight into how energy and terminal behavior are connected.

## Chapter 5

### Transferable Energy

Bandwidth is often associated with the amount of information transferred through a system. For example, when comparing two communication channels, it has become common to say that a channel capable of transferring 10 megabits per second has a lower bandwidth than one capable of transferring 100 megabits per second. The connection to what is usually referred to as bandwidth is obvious. A transfer function with a wide bandwidth can transfer narrow pulses in the time domain with little distortion. More pulses, and therefore more information, can be transferred through such systems.

In this chapter we consider bandwidth as a measure of the rate at which information can be transferred through a system. We use a simple pulse scheme where a pulse defined over a finite period of time is used to represent a bit of information. To connect this idea of bandwidth to energy we assume it is important to transfer energy, localized in a single pulse over a finite period of time  $T$ , from the input to output of a two-port system with as little energy as possible being smeared outside the pulse period  $T$ . We assume that the detector we use to register a bit of information is able to do so by measuring the energy within the time period of interest.

Take, for example, the two-port system of Figure 5.1. The input pulse  $\hat{a}(t)$ , which is equal to zero outside the period  $T$ , is transferred through the two-port system. The resulting output  $\hat{b}(t)$  is translated in time by  $\theta$ , and is distorted such that some of the energy is smeared outside the interval  $[\theta, \theta + T]$ . We would like to select a pulse  $\hat{a}(t)$  defined over the period  $T$ , that will maximize the energy in  $\hat{b}(t)$  within the time interval  $[\theta, \theta + T]$ . The idea here is that if the energy

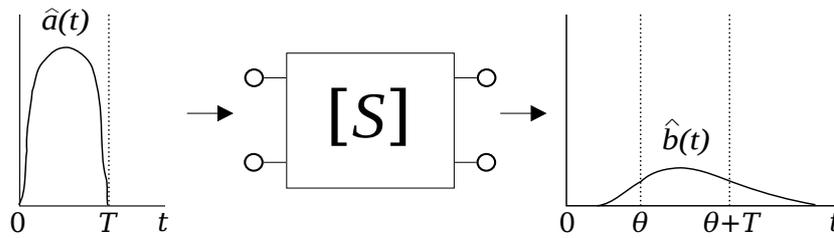


Figure 5.1: Two-port system.

at the output is maximized within  $[\theta, \theta + T]$ , then there is presumably little energy outside this interval, which would indicate that higher bit rates could be achieved.

We have seen that for certain systems, despite having effectively infinite amplitude bandwidths, distortion can make it difficult to identify a pulse at the output. Although these transfer functions pass all of the energy presented at the input, and therefore have a zero recoverable energy, there is a chance that they will distort the pulse and in turn limit the rate at which information can be transferred.

We hope that the transferrable energy, defined below, will be useful as yet another piece of the puzzle for understanding the connection between terminal behavior and energy.

In the following sections we first define transferrable energy, and then provide an example showing the value of this kind of energy. Like recoverable energy, to find transferrable energy we must consider a variational problem. The variational problem leads to the need for solving the integral equation given by Eqn. (5.6). We show that solving this equation will provide a quantity that we will define as the transferrable energy (5.10). We will then provide a definition for bitrate which is determined from transferrable energy. Finally, an example is provided that shows how transferrable energy can be calculated numerically.

## 5.1 Variational Problem

We are interested in finding the input pulse  $\hat{a}(t)$  that has a constant energy  $E_p$ , defined by

$$E_p = \int_0^T \hat{a}^2(t) dt$$

which is zero outside the domain  $[0, T]$ , and that will maximize the energy at the output within the domain  $[\theta, \theta + T]$ . A similar idea has been discussed by [62] and Chalk [63]. We thus require

$$\hat{a}(t) = 0 \text{ for } t \notin [0, T]$$

The interval energy at the output we defined as

$$E_{out}(\theta, T) = \int_{\theta}^{T+\theta} \hat{b}^2(t) dt \quad (5.1)$$

where

$$\hat{b}(t) = \int_0^T \hat{S}(t - \tau) \hat{a}(\tau) d\tau$$

and  $\hat{S}(t) = \hat{S}_{21}(t)$ , with subscripts omitted for brevity, is the impulse response of the two-port system (see Sec. 2.2 of Chapter 2). From causality  $\hat{S}(t) = 0$  for  $t \leq 0$ . The interval energy  $E_{out}(\theta, T)$  may now be rewritten as

$$E_{out}(\theta, T) = \int_0^T \left[ \int_0^T K_{\theta}(\tau_1, \tau_2) \hat{a}(\tau_1) d\tau_1 \right] \hat{a}(\tau_2) d\tau_2$$

where

$$K_{\theta}(\tau_1, \tau_2) = \int_{\theta}^{T+\theta} \hat{S}(t - \tau_1) \hat{S}(t - \tau_2) dt \quad (5.2)$$

The kernel  $K$  is symmetric

$$K_{\theta}(\tau_1, \tau_2) = K_{\theta}(\tau_2, \tau_1) \quad (5.3)$$

In general, the kernel cannot be written as a difference kernel  $K_{\theta}(\tau_1 - \tau_2)$  (see p. 777 of [62]). This would be possible if we were interested in the energy transferred to the output in the domain  $[0, \infty)$  instead of  $[\theta, \theta + T]$ . Chalk [63] was interested in the  $[0, \infty)$  case, and showed that for certain simple examples he could determine the optimal pulse and maximum transferred energy analytically. This was possible because the kernel could be written as  $K_{\theta}(\tau_1 - \tau_2)$  (see [64]-[67]). In general, because we are interested in maximizing energy within the period  $[\theta, \theta + T]$ , we consider numerical solutions.

To find the function  $\hat{a}(t)$  that maximizes the energy at the output within  $[\theta, \theta + T]$ , while the input energy in the pulse is a given constant  $E_p$ , we construct the functional

$$U[\hat{\phi}] = E_{out}(\theta, T)[\hat{\phi}] + \mu \left( E_p - \int_0^T \hat{\phi}^2(t) dt \right) \quad (5.4)$$

where  $\mu$  is a Lagrangian multiplier (see Chp. 2 Sec. 12 of [68]). The stationary point  $\hat{a}(t)$  is the point where the linear principal part of  $U[\hat{a} + \delta\hat{a}] - U[\hat{a}]$  vanishes (see Sec. 7.2.1 of [69]).  $\delta\hat{a}(t)$  can be thought of as an infinitesimal variation from the true stationary point  $\hat{a}(t)$ . We will see below that there are actually an infinite number of stationary points, but the one that returns the largest  $E_{out}(\theta, T)$  is the one we desire. We have thus

$$\begin{aligned} U[\hat{a} + \delta\hat{a}] - U[\hat{a}] &= 2 \int_0^T \left[ \int_0^T K_\theta(\tau_1, \tau_2) \hat{a}(\tau_2) d\tau_2 - \mu \hat{a}(\tau_1) \right] \delta\hat{a}(\tau_1) d\tau_1 \\ &\quad + \left[ \int_0^T \int_0^T K_\theta(\tau_1, \tau_2) \delta\hat{a}(\tau_1) \delta\hat{a}(\tau_2) d\tau_1 d\tau_2 - \mu \int_0^T \delta\hat{a}^2(t) dt \right] \end{aligned}$$

The first integral is identified as the linear principal part, so we require that

$$\int_0^T \left[ \int_0^T K_\theta(t, \tau) \hat{a}(\tau) d\tau - \mu \hat{a}(t) \right] \delta\hat{a}(t) dt = 0 \quad (5.5)$$

The condition

$$\mu \hat{a}(t) = \int_0^T K_\theta(t, \tau) \hat{a}(\tau) d\tau \quad \text{for } 0 \leq t \leq T \quad (5.6)$$

is both necessary and sufficient for (5.5) to be satisfied (see [69]). The function  $\hat{a}(t)$  that maximizes  $E_{out}(\theta, T)$  for a constant  $E_p$  will satisfy this integral equation.

## 5.2 Integral Equation Solution

The integral equation (5.6) is a homogeneous Fredholm integral equation of type two (see Sec. 1.2 of [69]). The operator in the integral equation takes a function  $\hat{a}(t)$  on the interval  $[0, T]$  to  $[0, T]$ , i.e.

$$(K_\theta \hat{a})(t) = \int_0^T K_\theta(t, \tau) \hat{a}(\tau) d\tau \quad \text{for } 0 \leq t \leq T \quad (5.7)$$

It will be compact for the cases we are interested, compactness follows from

$$\int_0^T \int_0^T |K_\theta(t, \tau)|^2 dt d\tau < \infty$$

The operator is also self-adjoint, which follows from the symmetry of the kernel (5.3). As a consequence, the spectrum will consist of 0 along with a finite or countably infinite number of real eigenvalues  $\mu_n$  (see theorem 4.7 p. 101 of [69]). For each of the eigenvalues  $\mu_n$  there will be a corresponding eigenfunction  $\hat{a}_n(t)$ , each of which may be viewed as extrema of the functional (5.4). We, however, are interested in the  $\hat{a}_n(t)$  that returns the largest  $E_{out}$  (5.1).

If the integral equation (5.6) for  $\hat{a}_n(t)$  is multiplied on both sides by  $\hat{a}_n(t)$  and integrated over  $[0, T]$ , we see that

$$\mu_n \int_0^T \hat{a}_n^2(t) dt = \int_0^T \int_0^T \hat{a}_n(t) K_\theta(t, \tau) \hat{a}_n(\tau) d\tau dt \quad (5.8)$$

We choose to normalize the functions  $\hat{a}_n(t)$  such that the input energy of each eigenfunction is one

$$E_p = \int_0^T \hat{a}_n^2(t) dt = 1$$

We recognize that the right side of (5.8) is the output energy of the system for a given  $\hat{a}_n(t)$ , thus

$$\mu_n = E_{out}(\theta, T) [\hat{a}_n]$$

The largest eigenvalue  $\mu_{\max}(\theta, T) \equiv \max\{\mu_n\}$ , therefore, corresponds to the maximum energy, i.e.,

$$\mu_{\max}(\theta, T) = \max\{E_{out}(\theta, T)\} \quad (5.9)$$

The corresponding eigenfunction to the maximum eigenvalue (5.9) we call  $\hat{a}_{\max}(t)$ .

To summarize, we can determine the maximum energy at the output and within the interval  $[\theta, \theta + T]$  by determining the maximum eigenvalue of the equation (5.6). The pulse launched into the system is zero outside the interval  $[0, T]$  and has unit energy. From passivity

$$0 \leq \mu_{\max}(\theta, T) \leq 1$$

### 5.3 Transferrable Energy

A pulse that is transferred through a system will in general be delayed in time at the output. For example, if the system were a lossless transmission line, the pulse at the output of the system would be identical to the input pulse but delayed in time. We determine the delay time  $\theta$  by searching for the maximum of  $\mu_{\max}(\theta, T)$  as a function  $\theta$ . In general this is not something that can be done analytically and requires a numerical search. We define  $\theta_0$  to be the  $\theta$  that corresponds to the largest  $\mu_{\max}(\theta, T)$  for a particular  $T$ . We now define transferrable energy  $E_{tran}(T)$  as

$$E_{tran}(T) \equiv E_p \mu_{\max}(\theta_0, T) \quad (5.10)$$

### 5.4 Bitrate

Once  $E_{tran}(T)$  is found, we can define the bitrate of the system. If the majority of the energy that passes through the system is within  $[\theta, \theta + T]$ , we will say that a bit has been registered; assuming here that we have a device that can detect this event. The ratio of transferrable energy  $E_{tran}(T)$  to the total amount of output energy, or *fractional energy*, is defined as

$$x(T) = \frac{E_{tran}(T)}{\int_0^{\infty} \hat{b}_{\max}^2(t) dt} \quad (5.11)$$

where the output signal  $\hat{b}_{\max}(t)$  corresponds to  $\hat{a}_{\max}(t)$ .  $x(T)$  has a value between zero and one, and as  $T$  goes to infinity  $x(T) \rightarrow 1$ .

As a criterion for registering a bit, we search for the  $T$  that corresponds to 90% of the energy being within the interval  $[\theta, \theta + T]$ . Thus, we seek  $T$  such that

$$x(T) = 0.9$$

For this  $T$ , we define the bitrate  $b_r$  as

$$b_r \equiv \frac{1}{T}$$

Understanding  $b_r$  as a kind of bandwidth, the relationship between transferrable energy  $E_{tran}(T)$

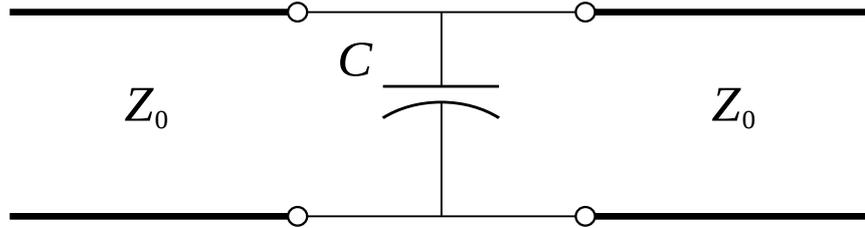


Figure 5.2: Two-port system.

and bitrate  $b_r$  is clear, i.e.,

$$E_{tran}(1/b_r) = 0.9 \int_0^{\infty} \hat{b}_{\max}^2(t) dt$$

### 5.5 Low Pass Filter Example

As an example consider the two-port system shown in Figure 5.2. The impulse response  $\hat{S}(t)$  for this system is

$$\hat{S}(t) = a \exp(-at) u(t)$$

where  $u(t)$  is the unit-step function

$$u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t > 0 \end{cases}$$

and

$$a = \frac{2}{CZ_0}$$

The kernel determined from the impulse response (5.2) is

$$\begin{aligned} K_{\theta}(\tau_1, \tau_2) &= a^2 \int_{\theta}^{T+\theta} \exp(-a(t-\tau_1)) u(t-\tau_1) \exp(-a(t-\tau_2)) u(t-\tau_2) d\tau_1 d\tau_2 \\ &= a^2 \int_{\theta}^{T+\theta} \exp(-2at + a(\tau_1 + \tau_2)) u(t - \max\{\tau_1, \tau_2\}) d\tau_1 d\tau_2 \end{aligned}$$

Let

$$\gamma(t, \tau) = \exp(-2a(T + \theta) + a(t + \tau))$$

it is easy to show that

$$K_{\theta}(t, \tau) = \begin{cases} \frac{a}{2} [\exp(-2a\theta + a(t + \tau)) - \gamma(t, \tau)] & \text{for } 0 \leq t, \tau \leq \theta \\ \frac{a}{2} [\exp(-a|t - \tau|) - \gamma(t, \tau)] & \text{for } \theta < \max\{t, \tau\} < T + \theta \end{cases}$$

We do not expect the system in this case to shift the signal transferred through the system much if  $aT \gg 1$ . In fact, we expect  $\theta$  to be nearly zero. To simplify this example, we set  $\theta$  to zero. For smaller values of  $aT$ , this may not be accurate. The kernel becomes simply

$$K_0(t, \tau) = \frac{a}{2} [\exp(-a|t - \tau|) - \exp(-2aT + a(t + \tau))]$$

To find  $E_{tran}(T)$  we resort to a numerical computation. The integral in the equation (5.6) must be handled with care because of the discontinuous derivative at  $t = \tau$  in  $K_0(t, \tau)$ .

To apply the operator (5.7) numerically, we first split the integral into two parts

$$(K_0 \hat{a})(t) = \int_0^t K_0(t, \tau) \hat{a}(\tau) d\tau + \int_t^T K_0(t, \tau) \hat{a}(\tau) d\tau$$

We then approximate the integrals with Gaussian quadratures

$$(K_0 \hat{a})(t) \approx \sum_{i=1}^N \omega_i^l(t) K_0(t, \tau_i^l(t)) \hat{a}(\tau_i^l(t)) + \sum_{i=1}^N \omega_i^u(t) K_0(t, \tau_i^u(t)) \hat{a}(\tau_i^u(t))$$

where  $N$  is the number of nodes to be used. Here  $\omega_i^{l,u}(t)$  are the weights of the of the quadrature and  $\tau_i^{l,u}(t)$  are the nodes. The  $l$  and  $u$  stand for lower and upper respectively;  $l$  corresponds to the integral that is calculated along  $\tau = [0, t]$ , and  $u$  corresponds to the integral calculated along  $\tau = [t, T]$ . The nodes are given by

$$\begin{aligned} \tau_i^l(t) &= \frac{t}{2} x_i + \frac{t}{2} \\ \tau_i^u(t) &= \frac{T-t}{2} x_i + \frac{T+t}{2} \end{aligned}$$

where  $x_i$  are Legendre nodes (see p. 276 of [70]). The weights are

$$\begin{aligned} \omega_i^l(t) &= \frac{t}{2} \omega_i \\ \omega_i^u(t) &= \frac{T-t}{2} \omega_i \end{aligned}$$



where  $l_i(t)$  are the Lagrange interpolation polynomials. This permits the vector on the right of (5.12) to be written as

$$\begin{bmatrix} \phi(\tau_1^l(t_1)) \\ \phi(\tau_2^l(t_1)) \\ \phi(\tau_1^l(t_2)) \\ \phi(\tau_2^l(t_2)) \\ \phi(\tau_1^u(t_3)) \\ \phi(\tau_2^u(t_3)) \\ \phi(\tau_1^u(t_4)) \\ \phi(\tau_2^u(t_4)) \end{bmatrix} = [\mathbf{L}] \begin{bmatrix} \phi(t_1) \\ \phi(t_2) \\ \phi(t_3) \\ \phi(t_4) \end{bmatrix}$$

where

$$\mathbf{L} = \begin{bmatrix} l_1(\tau_1^l(t_1)) & l_2(\tau_1^l(t_1)) & l_3(\tau_1^l(t_1)) & l_4(\tau_1^l(t_1)) \\ l_1(\tau_2^l(t_1)) & l_2(\tau_2^l(t_1)) & l_3(\tau_2^l(t_1)) & l_4(\tau_2^l(t_1)) \\ l_1(\tau_1^l(t_2)) & l_2(\tau_1^l(t_2)) & l_3(\tau_1^l(t_2)) & l_4(\tau_1^l(t_2)) \\ l_1(\tau_2^l(t_2)) & l_2(\tau_2^l(t_2)) & l_3(\tau_2^l(t_2)) & l_4(\tau_2^l(t_2)) \\ l_1(\tau_1^u(t_3)) & l_2(\tau_1^u(t_3)) & l_3(\tau_1^u(t_3)) & l_4(\tau_1^u(t_3)) \\ l_1(\tau_2^u(t_3)) & l_2(\tau_2^u(t_3)) & l_3(\tau_2^u(t_3)) & l_4(\tau_2^u(t_3)) \\ l_1(\tau_1^u(t_4)) & l_2(\tau_1^u(t_4)) & l_3(\tau_1^u(t_4)) & l_4(\tau_1^u(t_4)) \\ l_1(\tau_2^u(t_4)) & l_2(\tau_2^u(t_4)) & l_3(\tau_2^u(t_4)) & l_4(\tau_2^u(t_4)) \end{bmatrix}$$

The integral equation (5.6) may now be represented numerically by the matrix equation

$$\mu \begin{bmatrix} \phi(t_1) \\ \phi(t_2) \\ \phi(t_3) \\ \phi(t_4) \end{bmatrix} = \begin{bmatrix} \mathbf{K}^l & 0 \\ 0 & \mathbf{K}^u \end{bmatrix}^{4 \times 8} [\mathbf{L}]^{8 \times 4} \begin{bmatrix} \phi(t_1) \\ \phi(t_2) \\ \phi(t_3) \\ \phi(t_4) \end{bmatrix}$$

Once the matrix

$$[\mathbf{A}] = \begin{bmatrix} \mathbf{K}^l & 0 \\ 0 & \mathbf{K}^u \end{bmatrix}^{4 \times 8} [\mathbf{L}]^{8 \times 4}$$

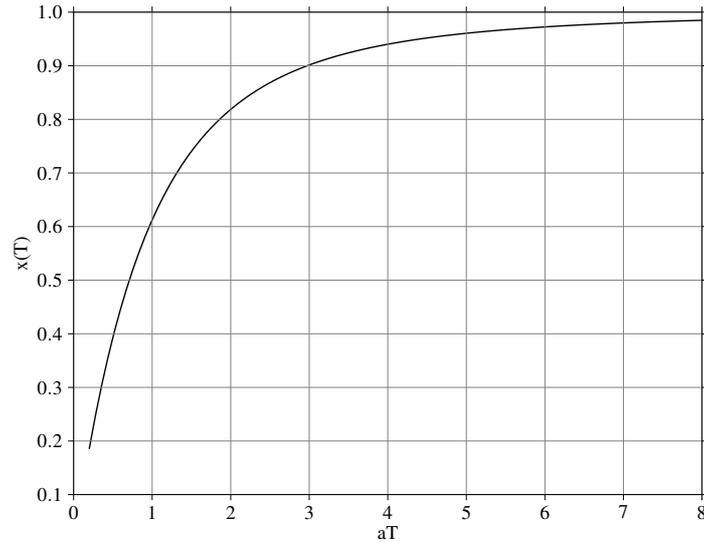


Figure 5.3: The ratio of output energy within  $[0, T]$  to the output energy within  $[0, \infty)$  as a function of  $T$ .

is constructed,  $E_{tran}(T)$  can be determined using a suitable numerical software package.

We implemented the above numerical procedure in Python to calculate the bitrate  $b_r$  (see Section 5.4) for the circuit of Figure 5.2. We choose the following parameters

$$Z_0 = 50 \Omega$$

and

$$C = 20 \mu\text{F}$$

Figure 5.3 is a plot of the fractional energy  $x(T)$  from Eqn. (5.11). Graphically we determine that  $x(T) = 0.9$  when  $aT \approx 3.0$ . So, for this example, the bitrate is

$$b_r \approx 667 \text{ bits per second}$$

In general, we see that for small values of  $aT$  (for which setting  $\theta = 0$  is surely not accurate enough), the fractional energy in the target interval is not very large, and the effect of the system time constant versus signal duration is apparent.

## 5.6 Summary

Transferrable energy (5.9) offers more insight into understanding the relationship between the energy within a system and terminal behavior. Good numerical tools must be written to work with general kernels (5.2), so that we can analyze other (and more complicated) examples.

The transferrable energy defined in this chapter provided us with a first step in understanding how the energy passed through a system is related to bandwidth. Another form of transferrable energy would also be interesting to look into. When we formulated the functional of Eqn. (5.4), we constrained the output energy subject to the condition that the energy within the input signal  $\hat{a}(t)$  must be constant. However, when the signal  $\hat{a}(t)$  arrives at the input of the two-port system, a portion of that signal will be reflected back (unless the system is matched). It would be interesting to constrain the output energy of the system subject to the condition that the energy that enters into the system be held constant. In doing so, we would be able to determine the amount of energy dissipated within the system and compare this to the total energy transferred. Perhaps a factor similar to  $Q$  could then be defined by taking the ratio of energy transferred to energy dissipated. It would be interesting to see the relationship between this factor and bandwidth, but we leave this work to future studies.

## Chapter 6

### Conclusion and Future Work

We have shown that the IEEE standard definitions [1] for quality factor  $Q$  are at best ambiguous (see Chapter 1). The  $RLC$  circuit of Figure 1.2 with an infinitely long transmission line clearly made the definitions (1.5) and (1.6) for  $Q$  inconsistent. We solved this problem with recoverable energy. For nearly a century<sup>1</sup>, quality factor  $Q$  has been used to characterize resonant systems; perhaps the IEEE standard [1] should be amended using recoverable energy if it is to last another.

In Chapter 3, we presented methods for characterizing the terminal behavior of a system. Energy bandwidth (see Section 3.1.1) was defined, and it was shown to have particular advantages over 3 dB bandwidth. For the cases studied, 3 dB bandwidth could not always be defined. Energy bandwidth, however, can always be defined provided that the integral of Eqn. (3.29) exists. We also showed in Chapter 4 that the quality factor defined using energy bandwidth  $Q_{EBW}$  was always greater than the quality factor defined with recoverable energy  $Q_{rec}$ — $Q_{EBW}$  also had the same trend as  $Q_{rec}$ . Future studies may show that this is true in general. We pointed out that bandwidth measures only measured the magnitude of the transfer function; energy within a system may also be related to the phase of a transfer function. To explore this, we introduced distortion and showed how it could be calculated.

The Bode-Fano bounds, in Chapter 3, were used to put an upper bound on the reflection coefficient  $\rho(\omega)$  of a nonuniform transmission line. This is a new result that shows how well a nonuniform line can be used as a matching circuit over a band of frequencies, assuming the line

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<sup>1</sup> See references [71]-[76] for the history of  $Q$ .

is monotonically tapered. It may also be viewed as a first step in finding Bode-Fano limitations for antennas. Antennas, after all, are devices used for matching a single port to free space. These kinds of limitations have potential for making the connection to the energy within a nonuniform line (or antenna). We leave finding this connection to future research.

Time-averaged recoverable energy  $\tilde{E}_{rec}(\omega)$  can be determined from the reflection coefficient  $\rho(\omega)$  of a one-port system alone. This is an important characteristic if energy within the system is to be related to the terminal behavior. Chapter 4 showed how to calculate  $\tilde{E}_{rec}(\omega)$  for a general one-port system (4.59).

Also in Chapter 4, we discussed an example (Example 2) showing that the recoverable energy of the *RLC* circuit (see Figure 1.1) is identical to the stored energy of that circuit. This is an important result meaning that if  $\tilde{E}_{rec}(\omega_0)$  is used to define the quality factor

$$Q_{rec} = \frac{\omega_0 \tilde{E}_{rec}(\omega_0)}{P} \quad (6.1)$$

then the fractional bandwidth  $B_{3dB}$  (define in Chapter 1), is related to  $Q_{rec}$  by

$$Q_{rec} = \frac{1}{B_{3dB}} \quad (6.2)$$

We found that this is also true for the circuit when stored energy was used to define quality factor by (1.6); but, unlike the stored energy definition, Eqn. (6.1) is independent of any equivalent network. For instance, the resistor  $R$  can be replaced with any constant resistance circuit (defined in Chapter 1) and  $Q_{rec}$  will be the same. As long as the transfer function of the *RLC* circuit (1.1) does not change,  $B_{3dB}$  will not change. The definitions (6.1) and (6.2) are, therefore, the correct ones for circuits equivalent to the *RLC* circuit, i.e., having the transfer function (1.1).

The same example was also used to show how well  $Q_{rec}$  could be used to predict bandwidth when the circuit was more complicated than the *RLC* circuit of Figure 1.1. We considered the circuit of Figure 4.6 (which becomes the simple *RLC* circuit when  $R_2 \rightarrow \infty$ ). The quality factor  $Q_{rec}$  tracked the different bandwidth definitions of quality factor (see Figure 4.7) within the same order of magnitude and had the same qualitative behavior. The question naturally arises: is there

another form of energy, determinable from the terminals, that may be smaller than the recoverable energy, but is better suited for defining a quality factor that is related to bandwidth? The answer may result from finding the stored energy of a *reciprocal* minimum-phase Darlington network (see Section 4.5). The construction of such a network was first hypothesized by Smith [55]. Currently, this sort of synthesis has not been discovered, but future work may prove that the stored energy of such a network can be used to define a quality factor that is better suited for making definitions like (6.1) and (6.2) consistent.

Another example in Chapter 4 showed the high frequency asymptotic behavior of recoverable energy from a nonuniform-transmission-line matching circuit. It was shown using this asymptotic formula for  $\tilde{E}_{rec}(\omega)$ , that the quality factor  $Q_{rec}$  went to zero as  $\omega$  became large. This is consistent with what is known about the bandwidth nonuniform transmission lines: as the frequency increases, the transmission line provides a better and better match to the load, i.e.,  $\rho(\omega) \rightarrow 0$ . Thus, the bandwidth becomes broad as  $Q_{rec}$  goes to zero in this limit—consistent with the definitions (6.1) and (6.2). On the other hand, we showed that the quality factor defined using stored energy actually increases as  $\omega$  becomes large. This would suggest, by the definitions (1.6) and (1.5), that the bandwidth should go to zero as  $\omega$  goes to infinity, which is not true. This result may be connected to the Bode-Fano limitation for nonuniform transmission lines in Chapter 3. We leave this for future research.

The final example of Chapter 4 indicates that the stored energy and recoverable energy of the lowest order Chu circuit are identical. This is an important result supporting Chu's work [11]. Chu was in search of a lower bound on the stored energy contained outside of a radiating antenna. In his original work, he merely discovered a particular circuit that could be used to model the impedance of spherical modes. From that circuit he calculated the stored energy and stated that the obtained energy was a lower bound. From our discussion of constant resistance circuits (see Chapter 1), we know there are an infinite number of circuits that can have the same impedance. We showed numerically, however, that Chu did indeed select a circuit whose stored energy was the lowest energy that could be determined from the impedance. We did this by numerically calculating

the recoverable energy of the circuit. In future work we intend to show this result analytically. It is also our intent to determine the recoverable energy of Chu's higher order circuits and see if those too represent the correct circuits for calculating a lower bound on energy.

The recoverable energy (4.59) calculated in Chapter 4 is similar to a result found by Smith [55]. Given an impedance (corresponding to our reflection coefficient  $\rho(\omega)$ ), Smith showed that there must be at *least* the energy given by Eqn. (4.59) present in the circuit. By searching for the maximum amount of energy that can be extracted from the circuit, we have shown that at *most* the energy given by Eqn. (4.59) can be extracted, and we have shown explicitly how this can be done.

There is another interesting result about recoverable energy that follows from Smith's work [55]. The the recoverable energy determined from the reflection coefficient  $\rho(\omega)$ , is equal to the stored energy of a minimum-phase Darlington circuit (see Section 4.5) synthesized from  $\rho(\omega)$ . As a consequence of the methods Smith used to find the minimum energy within a circuit, he showed that minimum-phase Darlington circuits have exactly this much energy stored within them. It follows from the fact that Smith's minimum energy and our recoverable energy are the same, that we find that these Darlington circuits can be synthesized having recoverable energy equal to their stored energy.

Chapter 5 showed how transferrable energy is related to bitrates. This research may yet be another piece of the puzzle in finding out if there exists a relationship between the energy within a system and bandwidth. We defined bandwidth, in this chapter, using the idea of a bitrate. We showed that we could define such a parameter if we maximized the energy within a time interval  $T$  at the output by an optimal choice for an input signal defined over the same time interval  $T$ . The bitrate was defined as:  $b_r = 1/T$ . In future research we would like to find the maximum energy transferred given that the energy that enters the system is constant. This would be more consistent with the current definitions of quality factor.

The work contained in this thesis opened many questions for future research:

- Can recoverable energy be calculated given a reflection coefficient  $\rho(\omega)$  measured over a band of frequencies? In Chapter 4, we derived an expression for recoverable energy that required knowing the magnitude of reflection coefficient  $\rho(\omega)$  at all frequencies; therefore, determining recoverable energy from measured data will have obvious problems. It may be possible to use a priori information to estimate the magnitude of  $\rho(\omega)$  outside the measured band. This estimation along with the measured data might then be used to approximate recoverable energy.
- Is the energy stored in a minimum-phase reciprocal Darlington circuit a more appropriate energy to use in the definition of quality factor? By more appropriate, we mean to ask, to what degree is such a quality factor approximate or exactly equal to  $1/B_{3dB}$ ? We asked this question in Chapter 4 following a discussion on minimum-phase Darlington synthesis. The circuits that result from this kind of synthesis contain stored energy that is equal to recoverable energy. These circuits, however, required the use of non-reciprocal circuit elements (gyrators). Perhaps there is a synthesis procedure that can be performed using only reciprocal elements, and that can be carried out in such a way as to add no unnecessary phase. It may be that the energy stored in such a circuit is better suited for the definition of quality factor, or at least gives a quality factor closer to the widely used value  $Q_{stored}$ .
- Is there a gain bandwidth limitation (see Eqn. (3.53)) that is bounded by recoverable energy? In Section 3.3 we discussed gain bandwidth limitations, of which the Bode-Fano limitation was a common example. This kind of measure of terminal behavior can be applied unambiguously to the magnitude of a transfer function; all that is required is that the relevant integral exists. In Chapter 3, a Bode-Fano limitation was derived for a nonuniform transmission line. We also saw in Chapter 4 how the recoverable energy for a nonuniform transmission line went to zero as the frequency approaches infinity. We discussed at the end of Chapter 4 that there seems to be a relationship between this Bode-Fano limitation and the asymptotic behavior of recoverable energy, but we did not succeed

in finding it. It would be interesting to see if this connection can be made for the nonuniform transmission line, and perhaps for linear systems in general.

- In Chapter 3, the Bode-Fano limitation that was discovered for the nonuniform transmission line assumed that the inductance and capacitance, per unit length, of the line were frequency independent. Is there a Bode-Fano limitation similar to the one in Chapter 3 that incorporates frequency dependence?
- For the simple *RLC* circuit in Chapter 1 (see Figure 1.1), we found in Chapter 4 that the recoverable energy of this circuit is equal to its stored energy. This circuit possesses a single mode of oscillations [77]. For this particular configuration of the inductor, capacitor, and resistor, all the energy stored within it can be recovered from the input port. The circuit of Figure 4.6 has two modes [77]; however, because of the configuration of the inductor, capacitor, and two resistors in this circuit, we have some linear combination of two modes and it appears that we cannot recover all the energy from the single port (perhaps because of this). This may also explain why no energy can be recovered from constant resistance circuits (see Section 1.2): these circuits may contain many modes, but such modes cannot be “seen” from the input port, and so, no energy can be recovered from them. Can an arbitrary circuit be decomposed into a linear combination of modes, and if so, is the energy stored in the individual modes equal to the energy stored in the circuit? Is recoverable energy at a given port generally not equal to stored energy because this port can only partially “see” all these modes in some specific linear combination?
- The last question may also be related to the concepts of controllability and observability from the study of control systems [78]. Perhaps the reason why all the energy stored within a circuit cannot be recovered is because some modes are not observable?

The current IEEE dictionary definitions for  $Q$ , have been shown to have certain shortcomings. But, the great success this parameters has had over the last century suggests that there is an

underlying physics not yet fully understood. It is my hope that the work done in this dissertation will inspire future researchers to explore further the connection between the energy inside a system and that systems terminal behavior.

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## Appendix A

### Time Harmonic Recoverable Energy

The Fourier transform of the function

$$\hat{a}_p(t) = \begin{cases} e^{\mu(t-t_0)} A \cos(\omega_0 t) & \text{for } t < t_0 \\ 0 & \text{for } t \geq t_0 \end{cases} \quad (\text{A.1})$$

where  $\mu$  is a positive constant, is

$$a_p(\omega) = \left( \frac{A}{2} \left[ \frac{e^{j\omega_0 t_0}}{(\mu - j(\omega - \omega_0))} \right] + \frac{A^*}{2} \left[ \frac{e^{-j\omega_0 t_0}}{(\mu - j(\omega + \omega_0))} \right] \right) e^{-j\omega t_0}$$

(Notice that this function is analytic in the upper complex plane.) To simplify the calculation of recoverable energy, let

$$a_p(\omega) = \left( \left[ \frac{C}{(\omega - \lambda_1)} \right] - \left[ \frac{C^*}{(\omega - \lambda_2)} \right] \right) e^{-j\omega t_0}$$

where

$$C = \frac{jAe^{j\omega_0 t_0}}{2}$$

and

$$\lambda_1 = \omega_0 - j\mu \quad \lambda_2 = -\omega_0 - j\mu$$

Calculating recoverable energy requires an evaluation of the integral (4.57). This in turn requires evaluations of both  $\mathcal{P}_- [\rho(\omega) a_p(\omega) e^{j\omega t_0}]$  and  $\mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}]$ . Because of the similarity in calculation, consider

$$\mathcal{P}_- [\gamma(\omega) a_p(\omega) e^{j\omega t_0}]$$

where  $\gamma$  is understood to be a function analytic in the lower half-plane, and can represent either  $\rho$  or  $\kappa$ . From the definition of  $\mathcal{P}_-$  (4.29)

$$\begin{aligned}
& \mathcal{P}_- [\gamma(\omega) a_p(\omega) e^{j\omega t_0}] \\
&= \lim_{\varepsilon \rightarrow 0} -\frac{1}{2\pi j} \int_{-\infty}^{\infty} \frac{\gamma(\zeta) \left( \left[ \frac{C}{(\zeta - \lambda_1)} \right] - \left[ \frac{C^*}{(\zeta - \lambda_2)} \right] \right)}{\zeta - (\omega - j\varepsilon)} d\zeta \\
&= \lim_{\varepsilon \rightarrow 0} -\frac{1}{2\pi j} \int_{-\infty}^{\infty} \left( \left[ \frac{\gamma(\zeta) C}{(\zeta - \lambda_1)(\zeta - (\omega - j\varepsilon))} \right] - \left[ \frac{\gamma(\zeta) C^*}{(\zeta - \lambda_2)(\zeta - (\omega - j\varepsilon))} \right] \right) d\zeta
\end{aligned}$$

The integrand of the above integral decays at least as fast as  $1/\zeta^2$  when  $|\zeta| \rightarrow \infty$ ; consequently, the above integral may be evaluated as the sum of the residues in the lower complex plane, thus

$$\begin{aligned}
& \mathcal{P}_- [\gamma(\omega) a_p(\omega) e^{j\omega t_0}] \\
&= \lim_{\varepsilon \rightarrow 0} \left[ \begin{array}{l} C \left( \frac{\gamma(\omega - j\varepsilon)}{(\omega - j\varepsilon - \lambda_1)} + \frac{\gamma(\lambda_1)}{(\lambda_1 - (\omega - j\varepsilon))} \right) \\ -C^* \left( \frac{\gamma(\omega - j\varepsilon)}{(\omega - j\varepsilon - \lambda_2)} + \frac{\gamma(\lambda_2)}{(\lambda_2 - (\omega - j\varepsilon))} \right) \end{array} \right] \\
&= C \left( \frac{\gamma(\omega)}{\omega - \lambda_1} + \frac{\gamma(\lambda_1)}{\lambda_1 - \omega} \right) - C^* \left( \frac{\gamma(\omega)}{\omega - \lambda_2} + \frac{\gamma(\lambda_2)}{\lambda_2 - \omega} \right)
\end{aligned}$$

and finally:

$$\mathcal{P}_- [\gamma(\omega) a_p(\omega) e^{j\omega t_0}] = C \frac{\gamma(\omega) - \gamma(\lambda_1)}{\omega - \lambda_1} - C^* \frac{\gamma(\omega) - \gamma(\lambda_2)}{\omega - \lambda_2}$$

Using the self-adjoint property of  $\mathcal{P}_{\pm}$  as well as their projection properties, the recoverable energy (4.57) can be written as:

$$E_{rec} = \frac{1}{2\pi} \int_{-\infty}^{\infty} [D_1(\omega) + D_2(\omega)] a_p^*(\omega) d\omega$$

where

$$D_1(\omega) = \mathcal{P}_- [\rho(\omega) a_p(\omega) e^{j\omega t_0}] \rho^*(\omega) e^{-j\omega t_0}$$

and

$$D_2(\omega) = \mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}] \kappa^*(\omega) e^{-j\omega t_0}$$

Notice that

$$D_1(\omega) = C \left( \frac{|\rho(\omega)|^2 - \rho^*(\omega)\rho(\lambda_1)}{\omega - \lambda_1} \right) - C^* \left( \frac{|\rho(\omega)|^2 - \rho^*(\omega)\rho(\lambda_2)}{\omega - \lambda_2} \right) \quad (\text{A.2})$$

and likewise

$$D_2(\omega) = C \left( \frac{\kappa(\omega) - \kappa^*(\omega)\kappa(\lambda_1)}{\omega - \lambda_1} \right) - C^* \left( \frac{|\kappa(\omega)|^2 - \kappa^*(\omega)\kappa(\lambda_2)}{\omega - \lambda_2} \right) \quad (\text{A.3})$$

Summing (A.2) and (A.3) together, we have

$$D_1(\omega) + D_2(\omega) = C \frac{|\rho(\omega)|^2 + |\kappa(\omega)|^2 - \rho^*(\omega)\rho(\lambda_1) - \kappa^*(\omega)\kappa(\lambda_1)}{\omega - \lambda_1} - C^* \frac{|\rho(\omega)|^2 + |\kappa(\omega)|^2 - \rho^*(\omega)\rho(\lambda_2) - \kappa^*(\omega)\kappa(\lambda_2)}{\omega - \lambda_2}$$

and after simplifying:

$$D_1(\omega) + D_2(\omega) = C \frac{1 - \rho^*(\omega)\rho(\lambda_1) - \kappa^*(\omega)\kappa(\lambda_1)}{\omega - \lambda_1} - C^* \frac{1 - \rho^*(\omega)\rho(\lambda_2) - \kappa^*(\omega)\kappa(\lambda_2)}{\omega - \lambda_2}$$

The integral to calculate recoverable energy may now be written as

$$\begin{aligned} & \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \begin{array}{l} \mathcal{P}_- [\rho(\omega) a_p(\omega) e^{j\omega t_0}] \rho^*(\omega) \\ + \mathcal{P}_- [\kappa(\omega) a_p(\omega) e^{j\omega t_0}] \kappa^*(\omega) \end{array} \right] a_p^*(\omega) e^{-j\omega t_0} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \begin{array}{l} C \frac{1 - \rho^*(\omega)\rho(\lambda_1) - \kappa^*(\omega)\kappa(\lambda_1)}{\omega - \lambda_1} \\ - C^* \frac{1 - \rho^*(\omega)\rho(\lambda_2) - \kappa^*(\omega)\kappa(\lambda_2)}{\omega - \lambda_2} \end{array} \right] \left( \left[ \frac{C^*}{(\omega - \lambda_1^*)} \right] - \left[ \frac{C}{(\omega - \lambda_2^*)} \right] \right) d\omega \end{aligned}$$

Note that  $\lambda_1^* = -\lambda_2$ , so that the integrand gives

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \begin{array}{l} |C|^2 \frac{1 - \rho^*(\omega)\rho(\lambda_1) - \kappa^*(\omega)\kappa(\lambda_1)}{(\omega - \lambda_1)(\omega + \lambda_2)} \\ + |C|^2 \frac{1 - \rho^*(\omega)\rho(\lambda_2) - \kappa^*(\omega)\kappa(\lambda_2)}{(\omega - \lambda_2)(\omega + \lambda_1)} \\ - C^* 2 \frac{1 - \rho^*(\omega)\rho(\lambda_2) - \kappa^*(\omega)\kappa(\lambda_2)}{(\omega - \lambda_2)(\omega + \lambda_2)} \\ - C^2 \frac{1 - \rho^*(\omega)\rho(\lambda_1) - \kappa^*(\omega)\kappa(\lambda_1)}{(\omega - \lambda_1)(\omega + \lambda_1)} \end{array} \right] d\omega \quad (\text{A.4})$$

This is another integral whose integrand decays at least as fast as  $1/\omega^2$  as  $|\omega| \rightarrow \infty$ ; thus the integral may be solved by residue calculus. The first half of the integral evaluates to

$$\begin{aligned} F_1 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \begin{array}{l} |C|^2 \frac{1 - \rho^*(\omega)\rho(\lambda_1) - \kappa^*(\omega)\kappa(\lambda_1)}{(\omega - \lambda_1)(\omega + \lambda_2)} \\ + |C|^2 \frac{1 - \rho^*(\omega)\rho(\lambda_2) - \kappa^*(\omega)\kappa(\lambda_2)}{(\omega - \lambda_2)(\omega + \lambda_1)} \end{array} \right] d\omega \\ &= 2\pi j \frac{1}{2\pi} |C|^2 \left( \begin{array}{l} \frac{1 - \rho^*(-\lambda_2)\rho(\lambda_1) - \kappa^*(-\lambda_2)\kappa(\lambda_1)}{(-\lambda_2 - \lambda_1)} \\ + \frac{1 - \rho^*(-\lambda_1)\rho(\lambda_2) - \kappa^*(-\lambda_1)\kappa(\lambda_2)}{(-\lambda_1 - \lambda_2)} \end{array} \right) \\ &= \frac{j|C|^2}{2j\mu} \left( \begin{array}{l} 1 - \rho^*(\omega_0 + j\mu)\rho(\omega_0 - j\mu) - \kappa^*(\omega_0 + j\mu)\kappa(\omega_0 - j\mu) \\ + 1 - \rho^*(-\omega_0 + j\mu)\rho(-\omega_0 - j\mu) - \kappa^*(-\omega_0 + j\mu)\kappa(-\omega_0 - j\mu) \end{array} \right) \end{aligned}$$

We are now free to allow the positive real constant  $\mu$  go to zero since the integral has been evaluated.

Consider the following two expansions for small  $\mu$ :

$$\begin{aligned} \rho^*(\omega_0 + j\mu)\rho(\omega_0 - j\mu) &= |\rho(\omega_o)|^2 + j\rho^{*'}(\omega_o)\rho(\omega_o)\mu - j\rho^*(\omega_o)\rho'(\omega_o)\mu + \mathcal{O}(\mu^2) \\ &= |\rho(\omega_o)|^2 - 2\text{Im}\{\rho^{*'}(\omega_o)\rho(\omega_o)\}\mu + \mathcal{O}(\mu^2) \end{aligned}$$

and

$$\begin{aligned} \kappa^*(\omega_0 + j\mu)\kappa(\omega_0 - j\mu) &= |\kappa(\omega_o)|^2 + j\kappa^{*'}(\omega_o)\kappa(\omega_o)\mu - j\kappa^*(\omega_o)\kappa'(\omega_o)\mu + \mathcal{O}(\mu^2) \\ &= |\kappa(\omega_o)|^2 - 2\text{Im}\{\kappa^{*'}(\omega_o)\kappa(\omega_o)\}\mu + \mathcal{O}(\mu^2) \end{aligned}$$

Summing these together and putting them back into the result for  $F_1$ , we have

$$I_1 = j|C|^2 \left( \begin{array}{c} \left( \begin{array}{c} 1 - |\rho(\omega_o)|^2 - |\kappa(\omega_o)|^2 \\ 2 \operatorname{Im} \{ \rho^{*'}(\omega_o) \rho(\omega_o) \} \mu \\ 2 \operatorname{Im} \{ \kappa^{*'}(\omega_o) \kappa(\omega_o) \} \mu + \mathcal{O}(\mu^2) \end{array} \right) / 2j\mu \\ + \left( \begin{array}{c} 1 - |\rho(\omega_o)|^2 - |\kappa(\omega_o)|^2 \\ 2 \operatorname{Im} \{ \rho^{*'}(-\omega_o) \rho(-\omega_o) \} \mu \\ 2 \operatorname{Im} \{ \kappa^{*'}(-\omega_o) \kappa(-\omega_o) \} \mu + \mathcal{O}(\mu^2) \end{array} \right) / 2j\mu \end{array} \right)$$

Clearly,  $\rho(-\omega) = \rho^*(\omega)$ , which follows from the fact that the impulse response  $\hat{\rho}(t) = \mathcal{F}^{-1}\rho(\omega)$  is real; from (4.45) we have  $\kappa(-\omega) = \kappa^*(\omega)$ . The derivatives of the functions when the arguments are negative give  $\rho'(-\omega_o) = -\rho'(\omega_o)$  and  $\kappa'(-\omega_o) = -\kappa'(\omega_o)$ . Simplifying the above expression and then letting  $\mu$  go to zero gives

$$F_1 = -2|C|^2 (\operatorname{Im} \{ \rho^*(\omega_o) \rho'(\omega_o) \} + \operatorname{Im} \{ \kappa^*(\omega_o) \kappa'(\omega_o) \})$$

The second half of the integral (A.4) evaluates in a similar way to

$$\begin{aligned} F_2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \frac{-C^{*2} \frac{1 - \rho^*(\omega)\rho(\lambda_2) - \kappa^*(\omega)\kappa(\lambda_2)}{(\omega - \lambda_2)(\omega + \lambda_2)}}{-C^2 \frac{1 - \rho^*(\omega)\rho(\lambda_1) - \kappa^*(\omega)\kappa(\lambda_1)}{(\omega - \lambda_1)(\omega + \lambda_1)}} \right] d\omega \\ &= -j \left[ \frac{C^{*2} \frac{1 - \rho^*(-\lambda_2)\rho(\lambda_2) - \kappa^*(-\lambda_2)\kappa(\lambda_2)}{(-\lambda_2 - \lambda_2)}}{+C^2 \frac{1 - \rho^*(-\lambda_1)\rho(\lambda_1) - \kappa^*(-\lambda_1)\kappa(\lambda_1)}{(-\lambda_1 - \lambda_1)}} \right] \end{aligned}$$

Again letting  $\mu$  go to zero, we get

$$\begin{aligned} F_2 &= -j \left[ \frac{C^{*2} \frac{1 - \rho^*(\omega_o)\rho(-\omega_o) - \kappa^*(\omega_o)\kappa(-\omega_o)}{2\omega_o}}{-C^2 \frac{1 - \rho^*(-\omega_o)\rho(\omega_o) - \kappa^*(-\omega_o)\kappa(\omega_o)}{2\omega_o}} \right] \\ &= j \left[ C^2 \frac{1 - \rho^2(\omega_o) - \kappa^2(\omega_o)}{2\omega_o} - C^{*2} \frac{1 - \rho^{*2}(\omega_o) - \kappa^{*2}(\omega_o)}{2\omega_o} \right] \\ &= -\operatorname{Im} \left\{ C^2 \frac{1 - \rho^2(\omega_o) - \kappa^2(\omega_o)}{\omega_o} \right\} \end{aligned}$$

The recoverable energy for the input signal given by (A.1) is the sum of the integrals  $F_1$  and  $F_2$ , thus

$$\begin{aligned}
E_{rec}(\omega_0, t_0) = & -\frac{|A|^2}{2} (\text{Im} \{ \rho^* (\omega_0) \rho' (\omega_0) \} + \text{Im} \{ \kappa^* (\omega_0) \kappa' (\omega_0) \}) \\
& + \text{Im} \left\{ \frac{A^2}{4} \frac{1 - \rho^2(\omega_0) - \kappa^2(\omega_0)}{\omega_0} e^{2j\omega_0 t_0} \right\}
\end{aligned}$$

## Appendix B

### Recoverable Energy Examples

For the circuit of Figure 1.2, we determine the three equations for the three unknowns  $I$ ,  $I_1$  and  $I_2$  from Kirchhoff's circuit laws:

$$I = I_1 + I_2$$

$$V - j\omega LI - IR_1 - I_1 R_2 = 0$$

$$I_1 R_2 - I_2 \frac{1}{j\omega C} = 0$$

In terms of  $I_2$ , the previous equations become

$$I_1 = \frac{I_2}{j\omega R_2 C}$$

$$I = \left( \frac{1}{j\omega R_2 C} + 1 \right) I_2$$

$$V = (j\omega L + R_1) \left( \frac{1}{j\omega R_2 C} + 1 \right) I_2 + \frac{1}{j\omega C} I_2$$

The current through the capacitor is thus

$$I_2 = \frac{V}{(j\omega L + R_1) \left( \frac{1}{j\omega R_2 C} + 1 \right) + \frac{1}{j\omega C}}$$

From the above results, the impedance at the input of the circuit  $Z$  is determined to be

$$\begin{aligned}
Z &= \frac{V}{I} \\
&= \frac{(j\omega L + R_1) \left( \frac{1}{j\omega R_2 C} + 1 \right) + \frac{1}{j\omega C}}{\left( \frac{1}{j\omega R_2 C} + 1 \right)} \\
&= \frac{(R_1 + R_2) - CLR_2\omega^2 + j\omega(L + CR_1R_2)}{(1 + j\omega R_2 C)} \\
&= \frac{j\omega(LC^2\omega^2 R_2^2 + L - CR_2^2) + \omega^2 C^2 R_2^2 R_1 + R_2 + R_1}{(1 + \omega^2 R_2^2 C^2)}
\end{aligned}$$

$Z$  is resonant when its imaginary part becomes zero:

$$(LC^2\omega^2 R_2^2 + L - CR_2^2) = 0$$

At resonance we define the parameter

$$R_o \equiv Z(\omega_o) = \frac{\omega_o^2 C^2 R_2^2 R_1 + R_2 + R_1}{(1 + \omega_o^2 R_2^2 C^2)}$$

where the resonant frequency  $\omega_o$  is

$$\omega_o = \sqrt{\frac{1}{LC} - \left(\frac{1}{CR_2}\right)^2} \quad (\text{B.1})$$

$R_o$  may also be written as

$$R_o = R_1 + \frac{L}{R_2 C}$$

The circuit remains resonant provided that  $\omega_o$  is real, and from (B.1) we see that the circuit is resonant when

$$R_2 \geq \sqrt{\frac{L}{C}}$$

As  $R_2 \rightarrow \infty$

$$Z \rightarrow j\omega L + R_1 + \frac{1}{j\omega C}$$

and

$$\omega_o \rightarrow \frac{1}{\sqrt{LC}}.$$

We assume that the circuit is fed by a transmission line having a characteristic impedance  $Z_0$ , so the reflection coefficient  $\rho(\omega)$  is

$$\rho = \frac{-LCR_2\omega^2 + j\omega(L + CR_1R_2 - CR_2Z_0) + (R_1 + R_2 - Z_0)}{-LCR_2\omega^2 + j\omega(L + CR_1R_2 + CR_2Z_0) + (R_1 + R_2 + Z_0)}$$

We define the following notations:

$$\begin{aligned}\alpha_1 &= LCR_2 & \beta_1 &= LCR_2 \\ \alpha_2 &= L + CR_1R_2 - CR_2Z_0 & \beta_2 &= L + CR_1R_2 + CR_2Z_0 \\ \alpha_3 &= R_1 + R_2 - Z_0 & \beta_3 &= R_1 + R_2 + Z_0\end{aligned}$$

We now choose to match the transmission line to the circuit so that it matches the circuit at resonance, i.e., we choose  $Z_0 = R_0$  so that  $\rho(\omega_0) = 0$ . The above parameters become

$$\begin{aligned}\alpha_1 &= LCR_2 & \beta_1 &= LCR_2 \\ \alpha_2 &= 0 & \beta_2 &= 2(L + CR_1R_2) \\ \alpha_3 &= \frac{R_2^2C - L}{R_2C} & \beta_3 &= \frac{R_2^2C + 2R_1R_2C + L}{R_2C}\end{aligned}\tag{B.2}$$

The time-averaged stored energy is contained in the capacitor and the inductor of the circuit.

We determine the time-averaged stored energy  $\tilde{u}_C$  within the capacitor by

$$\tilde{u}_C = \frac{1}{4}C|V|^2$$

and similarly for the inductor:

$$\tilde{u}_L = \frac{1}{4}L|I|^2$$

It is clear from the circuit (see Figure 1.2) that the voltage across the capacitor  $C$  is

$$V_C = \frac{1}{j\omega C}I_2$$

Therefore,

$$\begin{aligned}\tilde{u}_C &= \frac{1}{4}C \left| \frac{1}{j\omega C} I_2 \right|^2 \\ &= \frac{1}{4}C \frac{|V|^2}{\left( \frac{R_1 + R_2}{R_2} - \omega^2 LC \right)^2 + \omega^2 \left( \frac{L}{R_2} + R_1 C \right)^2}\end{aligned}$$

and

$$\tilde{u}_L = \frac{1}{4}L \frac{\left( \omega^2 C^2 + \frac{1}{R_2^2} \right) |V|^2}{\left( \frac{R_1 + R_2}{R_2} - \omega^2 LC \right)^2 + \omega^2 \left( \frac{L}{R_2} + R_1 C \right)^2}$$

The total time-averaged stored energy  $\tilde{u}_T = \tilde{u}_C + \tilde{u}_L$  within the circuit is thus

$$\tilde{u}_T = \frac{1}{4} |V|^2 \frac{C + L \left( \omega^2 C^2 + \frac{1}{R_2^2} \right)}{\left( \frac{R_1 + R_2}{R_2} - \omega^2 LC \right)^2 + \omega^2 \left( \frac{L}{R_2} + R_1 C \right)^2}$$

To put the voltage  $V$  in terms of the incident forward traveling voltage  $V_+$  on the transmission line connected to these circuits (see the first circuit example of Chapter 4), we note that

$$\frac{2Z}{Z + Z_o} V_+ = V$$

and

$$\frac{Z}{Z + Z_o} = \frac{\frac{L}{R_2 C} + R_1 + j \left( \left( \omega L - \frac{1}{\omega C} \right) - \frac{R_1}{\omega R_2 C} \right)}{\frac{L}{R_2 C} + R_1 + Z_o + j \left( \left( \omega L - \frac{1}{\omega C} \right) - \frac{R_1}{\omega R_2 C} - \frac{Z_o}{\omega R_2 C} \right)}$$

The magnitude squared of the voltage can then be written as

$$|V|^2 = |2V_+|^2 \frac{\left( \frac{R_2 + R_1}{R_2} - \omega^2 LC \right)^2 + \omega^2 \left( \frac{L}{R_2} + R_1 C \right)^2}{\left( \frac{R_2 + R_1 + Z_o}{R_2} - \omega^2 LC \right)^2 + \omega^2 \left( \frac{L}{R_2} + C(R_1 + Z_o) \right)^2}$$

and it follows that

$$\tilde{E}_{stored}(\omega) = |V_+|^2 \frac{C + L \left( \omega^2 C^2 + \frac{1}{R_2^2} \right)}{\left( \frac{R_2 + R_1 + Z_o}{R_2} - \omega^2 LC \right)^2 + \omega^2 \left( \frac{L}{R_2} + C(R_1 + Z_o) \right)^2}$$

In terms of the incident power  $P_{in} = |V_+|^2 / 2Z_o$ , the time-averaged stored energy within the circuit is

$$\tilde{E}_{stored}(\omega) = 2CZ_oP_{in} \frac{1 + CL \left( \omega^2 + \frac{1}{C^2R_2^2} \right)}{\left( \frac{R_2 + R_1 + Z_o}{R_2} - \omega^2LC \right)^2 + \omega^2 \left( \frac{L}{R_2} + C(R_1 + Z_o) \right)^2}$$

We use the expression (4.63) to calculate the recoverable energy  $\tilde{E}_{rec}$ . In terms of the parameters (B.2), the magnitude squared of the reflection coefficient is

$$|\rho(\omega)|^2 = \frac{(\alpha_3 - \alpha_1\omega^2)^2 + \omega^2\alpha_2^2}{(\beta_3 - \alpha_1\omega^2)^2 + \omega^2\beta_2^2}$$

We notice in the list of parameters that  $\alpha_1 = \beta_1$ . Factoring  $\rho(\omega)$ , we find that

$$\rho(\omega) = \frac{\alpha_1 \left( \omega - \frac{1}{2\alpha_1} \left( j\alpha_2 - \sqrt{4\alpha_1\alpha_3 - \alpha_2^2} \right) \right) \left( \omega - \frac{1}{2\alpha_1} \left( j\alpha_2 + \sqrt{4\alpha_1\alpha_3 - \alpha_2^2} \right) \right)}{\beta_1 \left( \omega - \frac{1}{2\beta_1} \left( j\beta_2 - \sqrt{4\beta_1\beta_3 - \beta_2^2} \right) \right) \left( \omega - \frac{1}{2\beta_1} \left( j\beta_2 + \sqrt{4\beta_1\beta_3 - \beta_2^2} \right) \right)}$$

From here we can determine that

$$\text{Im} \left[ \sum_{i=1}^2 \frac{1}{\omega - \alpha_i} \right] = \alpha_2 \frac{(\omega^2\alpha_1 + \alpha_3)}{(\alpha_3 - \alpha_1\omega^2)^2 + \omega^2\alpha_2^2}$$

and

$$\text{Im} \left[ \sum_{i=1}^{M_\rho} \frac{1}{\omega - \beta_i} \right] = \beta_2 \frac{(\omega^2\beta_1 + \beta_3)}{(\beta_3 - \beta_1\omega^2)^2 + \omega^2\beta_2^2}$$

so that

$$\text{Im} \left[ \sum_{i=1}^{N_\rho} \frac{|\rho(\omega)|^2}{\omega - \alpha_i} - \sum_{i=1}^{M_\rho} \frac{1}{\omega - \beta_i} \right] = \frac{\alpha_2(\omega^2\alpha_1 + \alpha_3) - \beta_2(\omega^2\beta_1 + \beta_3)}{(\beta_3 - \beta_1\omega^2)^2 + \omega^2\beta_2^2}$$

Since  $\alpha_2 = 0$ , we have

$$\text{Im} \left[ \sum_{i=1}^{N_\rho} \frac{|\rho(\omega)|^2}{\omega - \alpha_i} - \sum_{i=1}^{M_\rho} \frac{1}{\omega - \beta_i} \right] = -\beta_2 \frac{(\omega^2\beta_1 + \beta_3)}{(\beta_3 - \beta_1\omega^2)^2 + \omega^2\beta_2^2}$$

or

$$\text{Im} \left[ \sum_{i=1}^{N_\rho} \frac{|\rho(\omega)|^2}{\omega - \alpha_i} - \sum_{i=1}^{M_\rho} \frac{1}{\omega - \beta_i} \right] = \frac{-2CR_o \left( 1 + LC \left( \omega^2 + \frac{1}{R_2^2C^2} \right) + \frac{2R_1}{R_2} \right)}{\left( \frac{R_1 + R_2 + R_o}{R_2} - LC\omega^2 \right)^2 + \omega^2 \left( \frac{L}{R_2} + C(R_1 + R_o) \right)^2}$$

To determine  $\kappa(\omega)$ , its magnitude squared is

$$\begin{aligned}
|\kappa(\omega)|^2 &= 1 - |\rho(\omega)|^2 \\
&= \frac{(\beta_3 - \beta_1\omega^2)^2 + \omega^2\beta_2^2 - (\alpha_3 - \alpha_1\omega^2)^2}{(\beta_3 - \beta_1\omega^2)^2 + \omega^2\beta_2^2} \\
&= \frac{(\beta_1^2 - \alpha_1^2)\omega^4 + (\beta_2^2 - 2(\beta_1\beta_3 - \alpha_1\alpha_3))\omega^2 + (\beta_3^2 - \alpha_3^2)}{(\beta_3 - \beta_1\omega^2)^2 + \omega^2\beta_2^2} \\
&= \frac{(\beta_2^2 - 2\alpha_1(\beta_3 - \alpha_3))\omega^2 + (\beta_3^2 - \alpha_3^2)}{(\beta_3 - \beta_1\omega^2)^2 + \omega^2\beta_2^2}
\end{aligned}$$

and factoring this, we have

$$|\kappa(\omega)|^2 = \frac{D \left( \omega + j\sqrt{\frac{(\beta_3^2 - \alpha_3^2)}{(\beta_2^2 - 2\alpha_1(\beta_3 - \alpha_3))}} \right) \left( \omega - j\sqrt{\frac{(\beta_3^2 - \alpha_3^2)}{(\beta_2^2 - 2\alpha_1(\beta_3 - \alpha_3))}} \right)}{\left| \left( \omega - \frac{1}{2\beta_1} \left( j\beta_2 - \sqrt{4\beta_1\beta_3 - \beta_2^2} \right) \right) \left( \omega - \frac{1}{2\beta_1} \left( j\beta_2 + \sqrt{4\beta_1\beta_3 - \beta_2^2} \right) \right) \right|^2}$$

where

$$D = \frac{(\beta_2^2 - 2\alpha_1(\beta_3 - \alpha_3))}{\beta_1^2}$$

Selecting the part of this last equation that is analytic in the lower complex plane, and has zeros only in the upper, we find that

$$\kappa = \sqrt{(\beta_2^2 - 2\alpha_1(\beta_3 - \alpha_3))} \frac{\left( \omega - j\sqrt{\frac{(\beta_3^2 - \alpha_3^2)}{(\beta_2^2 - 2\alpha_1(\beta_3 - \alpha_3))}} \right)}{-\beta_1\omega^2 + j\omega\beta_2 + \beta_3}$$

Clearly,

$$\text{Im} \left[ \sum_{i=1}^{M_\rho} \frac{1}{\omega - \xi_i} \right] = \text{Im} \left[ \frac{\omega + j\sqrt{\frac{(\beta_3^2 - \alpha_3^2)}{(\beta_2^2 - 2\alpha_1(\beta_3 - \alpha_3))}}}{\omega^2 + \frac{(\beta_3^2 - \alpha_3^2)}{(\beta_2^2 - 2\alpha_1(\beta_3 - \alpha_3))}} \right]$$

or

$$\text{Im} \left[ \sum_{i=1}^{M_\rho} \frac{1}{\omega - \xi_i} \right] = \frac{\sqrt{(\beta_2^2 - 2\alpha_1(\beta_3 - \alpha_3))} (\beta_3^2 - \alpha_3^2)}{(\beta_2^2 - 2\alpha_1(\beta_3 - \alpha_3)) \omega^2 + (\beta_3^2 - \alpha_3^2)}$$

In terms of the circuit parameters

$$\text{Im} \left[ \sum_{i=1}^{M_\rho} \frac{|\kappa(\omega)|^2}{\omega - \xi_i} \right] = \frac{4 \frac{C}{R_2} R_o \sqrt{R_1 (R_1 + R_2)}}{\left( \frac{R_1 + R_2 + Z_o}{R_2} - LC\omega^2 \right)^2 + \omega^2 \left( \frac{L}{R_2} + C (R_1 + Z_o) \right)^2}$$

From the calculations above, we find that the time-averaged recoverable energy for the circuit of Figure 1.2 is

$$\tilde{E}_{rec}(\omega) = 2CR_oP_{in} \frac{1 + LC \left( \omega^2 + \frac{1}{R_2^2 C^2} \right) - \frac{2}{R_2} \left( \sqrt{R_1 (R_1 + R_2)} - R_1 \right)}{\left( \frac{R_1 + R_2 + R_o}{R_2} - LC\omega^2 \right)^2 + \omega^2 \left( \frac{L}{R_2} + C (R_1 + R_o) \right)^2}$$

Comparing  $\tilde{E}_{rec}$  to  $\tilde{E}_{stored}$ , we can write  $\tilde{E}_{rec}$  as

$$\tilde{E}_{rec}(\omega) = \tilde{E}_{stored}(\omega) - \frac{4CR_oP_{in} \frac{1}{R_2} \left( \sqrt{R_1 (R_1 + R_2)} - R_1 \right)}{\left( \frac{R_1 + R_2 + R_o}{R_2} - LC\omega^2 \right)^2 + \omega^2 \left( \frac{L}{R_2} + C (R_1 + R_o) \right)^2}$$

It is convenient, for the purpose of plotting all the quantities in this appendix, to choose the normalizations

$$\begin{aligned} w &= \frac{\omega}{\omega_o} \\ \zeta &= \sqrt{\frac{L}{R_1^2 C}} \\ \tau &= CR_1 + \frac{L}{R_2} \end{aligned}$$

and

$$\theta = R_2 \sqrt{\frac{C}{L}}$$

With these normalizations, we see that the circuit will be resonant if

$$\theta \geq 1$$

The stored energy and the recoverable energy may now be written as

$$\tilde{E}_{stored}(w) = \frac{2\tau P_{in} \left[ w^2 \left( 1 - \frac{1}{\theta^2} \right) + \left( 1 + \frac{1}{\theta^2} \right) \right]}{\left( 1 + \frac{1}{\theta^2} + 2\frac{1}{\theta\zeta} - w^2 \left( 1 - \frac{1}{\theta^2} \right) \right)^2 + 4w^2 \left( 1 - \frac{1}{\theta^2} \right) \left( \frac{1}{\theta} + \frac{1}{\zeta} \right)^2}$$

and

$$\tilde{E}_{rec}(w) = \tilde{E}_{stored}(w) \frac{4\tau P_{in} \left( \sqrt{\frac{1}{\theta\zeta} \left( \frac{1}{\theta\zeta} + 1 \right)} - \frac{1}{\theta\zeta} \right)}{\left( 1 + \frac{1}{\theta^2} + 2\frac{1}{\theta\zeta} - w^2 \left( 1 - \frac{1}{\theta^2} \right) \right)^2 + 4w^2 \left( 1 - \frac{1}{\theta^2} \right) \left( \frac{1}{\theta} + \frac{1}{\zeta} \right)^2}$$

To calculate

$$Q = \frac{\omega_o \tilde{E}_{stored}(\omega_0)}{P}$$

the power  $P$  dissipated in the system is

$$\begin{aligned} P &= P_{in} |\kappa(\omega)|^2 \\ &= P_{in} \frac{(\beta_2^2 - 2\alpha_1(\beta_3 - \alpha_3))\omega^2 + (\beta_3^2 - \alpha_3^2)}{(\beta_3 - \beta_1\omega^2)^2 + \omega^2\beta_2^2} \end{aligned}$$

The  $Q$  in terms of normalized variables is thus

$$Q = \frac{\zeta \sqrt{\theta^2 - 1}}{\theta + \zeta}$$

and defining

$$Q_{rec} = \frac{\omega_o \tilde{E}_{rec}(\omega_0)}{P}$$

we find that

$$Q_{rec} = \left[ \left( \frac{1}{\theta\zeta} + 1 \right) - \sqrt{\frac{1}{\theta\zeta} \left( \frac{1}{\theta\zeta} + 1 \right)} \right] Q$$