RESTRICTIONS OF RAINBOW SUPERCHARACTERS AND POSET BINOMIALS

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Abstract. A supercharacter theory of a finite group is a natural approximation to the ordinary character theory. There is a particularly nice supercharacter theory for U_n, the group of unipotent upper triangular matrices over a finite field, that has a rich combinatorial structure based on set partitions. Various representation theoretic constructions such as restriction and induction have supercharacter theoretic analogues. In the case of U_n, restrictions give rise to families of coefficients that are not well understood in general. This paper constructs a family of modules for U_n, and uses these modules to describe the coefficients arising from restrictions of certain supercharacters. This description involves introducing certain q-analog binomial coefficients associated to a finite poset.

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1. Introduction

A supercharacter theory of a finite group is a sort of generalized character theory that shares many of its nice properties, while hopefully being easier to compute. In a supercharacter theory, irreducible characters are replaced by supercharacters, and conjugacy classes are replaced by superclasses. The supercharacters and superclasses must satisfy certain compatibility conditions. In particular, the character theory of a finite group is also a supercharacter theory for that group, and a group may have many different supercharacter theories.

This paper is on a particular supercharacter theory for the group $U_n(q)$ of unipotent upper-triangular $n \times n$ matrices with entries in the finite field $\mathbb{F}_q$ (unipotent means that the matrices have 1's on the diagonal). The representation theory of $U_n(q)$ is known to be "wild". This means that, in some precise sense, determining the irreducible characters and conjugacy classes of $U_n(q)$, for all n and q, is hopeless. In [2], André introduced a supercharacter theory of $U_n(q)$ that carries enough information to be useful in applications that previously required the more difficult ordinary character theory. For example, it is used in [3] to study a certain class of random walks. This supercharacter theory is also interesting on its own. The supercharacters and superclasses of this supercharacter theory can be indexed in a natural way by set partitions, and there is a nice formula for the value of a supercharacter on a superclass in terms of their associated set partitions. In some ways, this situation is analogous to the classical character theory of the symmetric group, and its dependence on integer partitions. For the rest of this paper, we fix a q, and write U_n for $U_n(q)$.

For $m < n$, there are various embeddings of U_m as a subgroup of U_n. Just as in ordinary character theory, it turns out that the restriction of a supercharacter of U_n to the subgroup U_m can be written as a sum of supercharacters of U_m. Classically, the decompositions of restrictions of irreducible characters in terms of irreducible characters of the subgroup are given by *branching rules*. Branching rules for the supercharacter theory of U_n have been studied [9] and [10]. In [9], an algorithm is given for computing arbitrary restrictions of supercharacters. However, the resulting coefficients are not well understood in general.

Motivated by this problem, this paper constructs a family modules of $U_n(q)$, whose characters are constant on superclasses. The main result of this paper is a formula for evaluating these characters at elements of U_n, as well as their decomposition in terms of supercharacters. In order to describe these characters, we introduce certain q-analog binomial coefficients associated to a poset. Finally, we use the characters to solve a special case of the restriction problem. In particular, we give a nice description of the coefficients arrising from restrictions of "rainbow" supercharacters.

2. Supercharacter Theory

The goal of this section is to give the definition of a supercharacter theory of a finite group. To motive this definition, we briefly review some properties of the character theory of a finite group. Both [7] and [6] contain all of the following material, and much more. For the remainder of this section, we fix a finite group G. A (finite dimensional, complex) *representation* φ of G on a finite dimensional, complex vector space V is a group homomorphism

$$\varphi : G \longrightarrow \mathrm{GL}(V)$$

from G to the group of linear automorphisms of V. One typically uses either φ or V to refer to a representation if the other is understood from context. If a subspace $W \subset V$ satisfies $\varphi(g)(W) \subset W$ for all $g \in G$, we say that W is a *subrepresentation* of V. If a representation V has no subrepresentations other than 0 and V, we say that V is *irreducible*. For each representation φ of G, consider the function

$$\chi_\varphi : G \longrightarrow \mathbb{C}, \qquad g \longmapsto \mathrm{tr}(\varphi(g)).$$

We call χ_φ the *character* of the representation V. It turns out that, under our assumptions, a representation is uniquely determined (up to isomorphism) by its character. This is rather surprising, as taking the trace of a matrix involves ignoring most of its entries. Define the set of *irreducible characters* $\mathrm{Irr}(G)$ of G by

$$\mathrm{Irr}(G) = \{\chi_\varphi \mid \varphi \text{ is an irreducible representation of } G\},$$

and write $\mathrm{Conj}(G)$ for the set of conjugacy classes of G. The irreducible characters of a group have many important properties, some of which we record below. Recall that a *class function* on G is a complex-valued function on G that is constant on conjugacy classes, and that the set of class functions on G forms a vector space with respect to pointwise addition and scalar multiplication.

- The number of irreducible characters of G is equal to the number of conjugacy classes of G.
- Each irreducible character is constant on the conjugacy classes of G. Furthermore, distinct irreducible characters are linearly independent, and therefore the set of irreducible characters is a basis for the space of class functions on G.
- In fact, the irreducible characters form an orthonormal basis for the space of class functions on G with respect to the standard inner product

  $$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_1(g)\overline{\chi_2(g)}.$$

  on the space of class functions on G.

We can view all of the irreducible characters of a group simultaneously as a square array of numbers with rows indexed by irreducible characters and columns indexed by conjugacy classes. This array is referred to as the *character table* of G. The character table of a group encodes a lot of information about the group. For instance, given the character table of a group, one can determine the orders of the conjugacy classes of the group, as well as all of its normal subgroups. However, the character table does not determine a group up to isomorphism; for instance the dihedral group D_8 and the quaternion group Q_8 have the same character table.

We will now define the notion of a supercharacter theory of a finite group. There are several equivalent definitions; the following one appears in [4]. For a set $X \subset \mathrm{Irr}(G)$ of irreducible characters of G, let $\chi_X = \sum_{\psi \in X} \psi(1)\psi$. We define a *supercharacter theory* to be a pair $(\mathcal{X}, \mathcal{K})$, where $\mathcal{X}$ is a partition of $\mathrm{Irr}(G)$ and $\mathcal{K}$ is a partition of G, such that

(1) $|\mathcal{X}| = |\mathcal{K}|$,
(2) for each $X \in \mathcal{X}$, the character χ_X is constant on the the elements of each $K \in \mathcal{K}$, and
(3) the set $\{1\}$ is in $\mathcal{K}$, and the set containing only the trivial character, $\{\mathbb{1}\}$, is in $\mathcal{X}$.

It can be shown that these assumptions imply that each $K \in \mathcal{K}$ is a union of conjugacy classes, making this definition quite symmetric (see [4]). Given a supercharacter theory $(\mathcal{X}, \mathcal{K})$ of G, we refer to the sets $\cup_{C \in K} C$ for $K \in \mathcal{K}$ as *superclasses*. For each $X \in \mathcal{X}$, it is typical to pick a $c_X \in \mathbb{Q}_{>0}$ such that $c_X \chi_X$ is a character. The characters $c_X \chi_X$ are then referred to as *supercharacters*. We record the following facts in direct analogy with the character theory of G.

- The number of supercharacters of G is equal to the number of superclasses classes of G.
- Each supercharacter is constant on the superclasses of G. Furthermore, distinct supercharacters are linearly independent, and therefore the set of supercharacters is a basis for the space of superclass functions on G.
- The supercharacters are orthogonal with respect to the standard inner product on G. However, they are not orthonormal in general, as a supercharacter may have more than one irreducible constituent.

Example 2.1. Every group has at least two supercharacter theories. The ordinary character theory is a supercharacter theory, given by the partitions

$$\mathcal{X} = \{\text{irreducible characters of } G\} \quad \text{and} \quad \mathcal{K} = \{\text{conjugacy classes of } G\}.$$

This is refered to as the *trivial* supercharacter theory. The partitions

$$\mathcal{X} = \{\{\mathbb{1}\}, \mathrm{Irr}(G) - \{\mathbb{1}\}\} \quad \text{and} \quad \mathcal{K} = \{\{1\}, G - \{1\}\}$$

give another supercharacter theory, called the *maximal* supercharacter theory. In both cases, the axioms of a supercharacter theory are easy to check.

It is an open problem to determine whether every finite group has supercharacter theories beyond these two, although most do.

Example 2.2. The smallest group with a supercharacter theory not of this form is the Klein group $V_4 = \langle x, y \mid x^2 = y^2 = (xy)^2 = 1 \rangle$. We record the character table of V_4.

Table 1. Character Table of V_4

	$\{1\}$	$\{x\}$	$\{y\}$	$\{xy\}$
$\mathbb{1}$	1	1	1	1
χ_1	1	1	-1	-1
χ_2	1	-1	1	-1
χ_3	1	-1	-1	1

There is a supercharcter theory of V_4 with superclasses $\{1\},\{x,y\}$ and $\{xy\}$, and supercharacters $\mathbb{1},\chi_1+\chi_2,\chi_3$. The supercharacter table of this supercharacter theory is

Table 2. Supercharacter Table of V_4

	$\{1\}$	$\{x,y\}$	$\{xy\}$
$\mathbb{1}$	1	1	1
$\chi_1+\chi_2$	2	0	-2
χ_3	1	-1	1

There are many ways to construct supercharacter theories. For instance, automorphisms of G give rise to supercharacter theories.

Example 2.3. Recall that the automorphism group $\mathrm{Aut}(G)$ of G acts on G by

$$\sigma\cdot g=\sigma(g)\quad \text{for } \sigma\in\mathrm{Aut}(G), g\in G,$$

and on $\mathrm{Irr}(G)$ by

$$(\sigma\cdot\chi)(h)=\chi(\sigma^{-1}(h))\quad \text{for } \sigma\in\mathrm{Aut}(G),\chi\in\mathrm{Irr}(G),h\in G.$$

Let $A\leq\mathrm{Aut}(G)$ be a subgroup of the automorphism group of G. Note that the action of A permutes the conjugacy classes of G, as $g=xhx^{-1}$ if and only if $\sigma(g)=\sigma(x)\sigma(h)\sigma(x)^{-1}$, for $g,h,x\in G$ and $\sigma\in\mathrm{Aut}(G)$. Therefore, A also acts on $\mathrm{Conj}(G)$, and hence A partitions both $\mathrm{Irr}(G)$ and $\mathrm{Conj}(G)$ into orbits. In fact, these orbits give a supercharacter theory for G. The statement of the following result appears in [4].

Proposition 2.4. *Suppose $A\leq\mathrm{Aut}(G)$. Then, the partitions*

$$\mathcal{X}=\{A\chi\mid\chi\in\mathrm{Irr}(G)\}\qquad\text{and}\qquad\mathcal{K}=\left\{\bigcup_{C'\in AC}C'\mid C\in\mathrm{Conj}(G)\right\}$$

form a supercharacter theory of G.

Proof. We have to check three conditions. For the first, we must show that the number of orbits of A on $\mathrm{Irr}(G)$ is equal to the number of orbits of A on G. By Burnside's Lemma, it will be sufficient to show that each element of A fixes equal numbers of irreducible characters and conjugacy classes of G. For $g,h\in G$, let C_g denote the conjugacy class of g in G, and recall the *column orthogonality relation* (see [6],[7])

$$\sum_{\chi\in\mathrm{Irr}(G)}\chi(g)\overline{\chi(h)}=\begin{cases}\frac{|G|}{|C_g|}, & \text{if } C_g=C_h,\\ 0, & \text{otherwise.}\end{cases}$$

We fix a $\sigma \in A$, and compute:

$$
\begin{aligned}
\#\left\{\chi \in \mathrm{Irr}(G) \mid \chi = \sigma \cdot \chi\right\} &= \sum_{\chi \in \mathrm{Irr}(G)} \langle \chi, \sigma \cdot \chi \rangle \\
&= \sum_{\chi \in \mathrm{Irr}(G)} \left(\frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi(\sigma^{-1}(g))} \right) \\
&= \frac{1}{|G|} \sum_{g \in G} \sum_{\chi \in \mathrm{Irr}(G)} \chi(g) \overline{\chi(\sigma^{-1}(g))} \\
&= \frac{1}{|G|} \sum_{g \in G} \frac{|G|}{|C_g|} \delta_{C_g, C_{\sigma^{-1}(g)}} \\
&= \#\left\{\text{Conjugacy classes } C \text{ of } G \mid \sigma(C) = C\right\}.
\end{aligned}
$$

Thus, the numbers of orbits are indeed equal. For the second condition, let $\chi \in \mathrm{Irr}(G)$, and let $X = A\chi$ be the orbit of χ. Suppose that $g, h \in G$ are in the same orbit. Then, there is a $\sigma \in A$ such that $\sigma(g) = h$. Now,

$$
\begin{aligned}
\chi_X(\sigma(g)) &= \sum_{\psi \in X} \psi(1)\psi(\sigma(g)) \\
&= \frac{|X|}{|A|} \sum_{\gamma \in A} (\gamma \cdot \psi)(1)(\gamma \cdot \psi)(\sigma(g)) \\
&= \frac{|X|}{|A|} \sum_{\gamma \in A} \psi(1)\psi(\gamma^{-1}(\sigma(g))) \\
&= \frac{|X|}{|A|} \sum_{\gamma' \in A} \psi(1)\psi(\gamma'(h)) \\
&= \chi_X(h).
\end{aligned}
$$

Finally, the identity and the trivial character are fixed by the action of A. Thus, the orbits of A on $\mathrm{Irr}(G)$ and G satisfy the requirements for a supercharacter theory. □

Note that if $A \subset \mathrm{Inn}(G)$, then this supercharacter theory is in fact the trivial supercharacter theory.

We mention one final source of supercharacter theories. Given a normal subgroup $N \lhd G$ and supercharacter theories of N and G/N, it is possible to stitch them together to get a supercharacter theory for G. This construction gives a non-trivial, non-maximal supercharacter theory for any non-simple group. There are many supercharacter theories that do not come from any of the constructions we have just described, but instead depend on the unique properties of the group in question. We describe such a supercharacter theory in the next section. Very little is known in general about the possible supercharacter theories of a group. In [5], the author constructs all of the supercharacter theories of cyclic p-groups. Even in this very restrictive setting, the answer is non-trivial.

3. A Supercharacter Theory for U_n

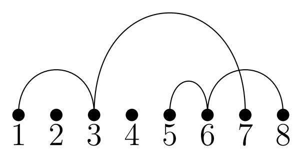
The goal of this section is to define a particular supercharacter theory of U_n. This supercharacter theory has a natural description in terms of set partition combinatorics. In some ways, this description is analagous to the classical representation theory of the symmetric group, and its dependence on the combinatorics of integer partitions. We begin with some necessary background on set partitions, and then give two constructions of the supercharacter theory.

3.1. **Set Partition Combinatorics.** We define a *set partition* λ of $\{1,2,\ldots,n\}$ to be a set of pairs of integers (i,j) such that $1 \le i < j \le n$, and such that, if $(i,j),(k,l) \in \lambda$, then $i = k$ if and only if $j = l$. We refer to these pairs as the *arcs* of λ, and write $i \frown j$ for the pair (i,j).

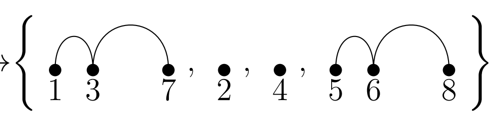
Let $\mathcal{S}_n = \{\text{set partitions of } \{1,\ldots,n\}\}$. A typical element of $\mathcal{S}_8$ is

$$\lambda = \{1\frown 3, 3\frown 7, 5\frown 6, 6\frown 8\} \tag{3.1}$$

We can write elements of $\mathcal{S}_n$ in a compact way by drawing a row of n dots, and putting an arc from the ith dot to the jth dot if the arc $i\frown j \in \lambda$. For example, we depict the above λ as

$\lambda = \{1\frown 3, 3\frown 7, 5\frown 6, 6\frown 8\} \longleftrightarrow$ 

This definition of a set partition is essentially the same as the usual notion of a set partition of n elements. To see this, take an element λ of $\mathcal{S}_n$, and consider the relation on $\{1,\ldots,n\}$ given by $i \sim j$ if $i \frown j \in \lambda$. The transitive closure of this relation gives an equivalence relation on $\{1,\ldots,n\}$, and these equivalence classes give a (usual) set partition. Moreover, this correspondence is bijective. We refer to these equivalence classes as *blocks*, and write $\mathrm{bl}(\lambda)$ for the set of blocks of λ. For example, the λ in 3.1 has

$\mathrm{bl}(\lambda) = \{\,\{1,3,7\},\{2\},\{4\},\{5,6,8\}\} \longleftrightarrow$ 

We will define some statistics on set partitions. For $\lambda \in \mathcal{S}_n$, define the *dimension* of λ by

$$\dim(\lambda) = \sum_{i\frown j\in\lambda} j - i. \tag{3.2}$$

For $\lambda,\mu \in \mathcal{S}_n$, we define a *crossing* in λ to be a pair $((i\frown j, k\frown l) \in \lambda\times\lambda$ such that $i < k < j < l$, and a *nesting* of μ in λ to be a pair $(i\frown j, k\frown l) \in \lambda\times\mu$ such that $i < k < l < j$. Let

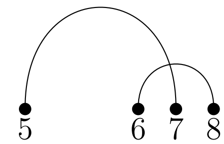
$$\mathrm{crs}(\lambda) = \#\{(i\frown j, k\frown l) \in \lambda\times\lambda \mid i<k<j<l\} \tag{3.3}$$

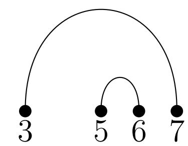
and

$$\mathrm{nst}^{\lambda}_{\mu} = \#\{(i\frown j, k\frown l) \in \lambda\times\mu \mid i<k<l<j\}. \tag{3.4}$$

For example, for the λ defined in 3.1,

$$\dim(\lambda) = 2 + 4 + 1 + 2 = 9,$$

$$\mathrm{crs}(\lambda) = 1 = \#\big\{(3\frown 7, 6\frown 8)\big\} \longleftrightarrow \#\Bigg\{$$  $$\Bigg\}, \text{ and}$$

$$\mathrm{nst}_\lambda^\lambda = 1 = \#\big\{(3\frown 7, 5\frown 6)\big\} \longleftrightarrow \#\Bigg\{$$  $$\Bigg\}.$$

We are now in a position to define our supercharacter theory.

3.2. **The Supercharacter Theory.** In [4], the authors define our supercharacter theory as a special case of a construction for a more general class of groups, called algebra groups. We first give this definition, and then describe the connection to set partition combinatorics.

Let $J_n = \{u - 1 \mid u \in U_n\} = U_n - 1$ be the space of strictly upper triangular matrices with entries in $\mathbb{F}_q$ (where 1 is the $n \times n$ identity matrix). The group U_n acts on J_n on the left by left multiplication, and on the right by right multiplication. These actions are compatible, in the sense that $(ax)b = a(xb)$ for all $x \in J_n$ and $a, b \in U_n$. Consequently, the notation $U_n x U_n$ for the two-sided orbit of $x \in J_n$ is unambiguous. Consider the dual space J_n^* of linear functions from J_n to $\mathbb{F}_q$. There is a corresponding two-sided action of U_n on J_n^*, given by

$$(a\gamma b)(u - 1) = \gamma(a^{-1}(u - 1)b^{-1}), \quad \text{for } \gamma \in J_n^*, a, b, u \in U_n$$

There is a supercharacter theory whose superclasses are parameterized by the two-sided orbits of U_n on J_n, and whose supercharacters are parameterized by the two-sided orbits of U_n on J_n^*. More specifically, the superclasses are the sets

$$U_n x U_n + 1, \quad \text{for } x \in J_n.$$

That is, two elements $u, v \in U_n$ are in the same superclass if there exists $a, b \in U_n$ such that $u = a(v - 1)b + 1$. To describe the supercharacters, fix a non-trivial linear character ρ of the group $\mathbb{F}_q^+$. We can make a function $\gamma \in J_n^*$ into a function $\widehat{\gamma} : U_n \longrightarrow \mathbb{C}$ by setting

$$\widehat{\gamma}(u) = \rho(\gamma(u - 1)), \quad \text{for } u \in U_n.$$

The supercharacters are the functions

$$\frac{|G\lambda|}{|G\lambda G|} \sum_{\varphi \in U_n \gamma U_n} \widehat{\varphi}, \quad \text{for } \gamma \in J_n^*.$$

Note that we have not shown that this choice of supercharacters and superclasses does indeed form a supercharacter theory. We can verify that they satisfy some of the properties of a supercharacter theory without too much difficulty. For instance, consider the orbits $U_n 1 U_n$ and $U_n \mathbb{1} U_n$ of the identity and the trivial character respectively. Note that, for any $a, b \in U_n$, $a(1 - 1)b + 1 = 1$, and therefore the identity is in its own orbit. Likewise, note that $\mathbb{1} = \widehat{\mathbf{O}}$, where $\mathbf{O} \in J_n^*$ is the zero function. Now, for any $a, b, u \in U_n$, $(a\mathbf{O}b)(u - 1) =$

$\mathbf{O}(a^{-1}(u-1)b^{-1}) = 0$, so $\mathbb{1}$ is in its own orbit. The proofs of the other properties are contained in Lemma 4.1, Theorem 5.5, and Theorem 5.8 of [4].

This supercharacter theory is closely related to the one we want to construct. Consider the group $T_n \subset \mathrm{GL}_n$ of invertible diagonal $n \times n$ matrices over $\mathbb{F}_q$. Then, T_n acts on U_n by conjugation. To see this, take $u \in U_n$ and $t \in T_n$, and set $u - 1 = x \in J_n$. We have

$$t \cdot u = tut^{-1} = t(x+1)t^{-1} = txt^{-1} + 1 \in J_n + 1 = U_n.$$

There is a dual action of T_n on the space of functions $U_n \longrightarrow \mathbb{C}$ that sends a function $f : U_n \longrightarrow \mathbb{C}$ to the function given by

$$(t \cdot f)(u) = f(t^{-1} \cdot u) = f(t^{-1}ut), \quad \text{for } t \in T_n, u \in U_n.$$

As in Example 2.3, this action permutes the superclasses and supercharacters of the above supercharacter theory. Thus, the orbits of this action clump together certain superclasses and supercharacters into orbits, and it follows from Proposition 2.4 that these orbits determine a supercharacter theory. For the remainder of this paper, whenever we refer to a superclass or a supercharacter, it will be understood that we mean a superclass or supercharacter of this particular supercharacter theory that we have just constructed.

We now describe our supercharacter theory from a combinatorial point of view. It turns out that the number of superclasses (or supercharacters) of U_n is precisely $|\mathcal{S}_n|$. In fact, for each superclass of U_n, there exists a partition $\mu \in \mathcal{S}_n$ and a distinguished element $u_\mu \in U_n$ of the superclass such that

$$(u_\mu)_{ij} = \begin{cases} 1, & \text{if } i \frown j \in \mu \text{ or } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

For example, note that the identity is in the superclass corresponding to the empty partition $\emptyset$.

Furthermore, there is a compatible way to index the supercharacters by the elements of $\mathcal{S}_n$. Under this indexing, there is a nice formula for the value of the supercharacter χ^λ indexed by $\lambda \in \mathcal{S}_n$ at an element of the superclass indexed by $\mu \in \mathcal{S}_n$. Recall definitions 3.2 and 3.4. Then, using the shorthand $\chi^\lambda(\mu) = \chi^\lambda(u_\mu)$,

$$\chi^\lambda(\mu) = \begin{cases} \frac{(q-1)^{|\lambda-\mu|} q^{\dim(\lambda)-|\lambda|)} (-1)^{|\lambda\cap\mu|}}{\mathrm{nst}^\lambda_\mu}, & \text{if } i \frown k \in \lambda \text{ implies } i \frown j, j \frown k \notin \mu \\ & \text{for all } j \text{ with } i < j < k, \\ 0, & \text{otherwise.} \end{cases}$$

For instance, the degree of a character is given by

$$\chi^\lambda(1) = \chi^\lambda(\emptyset) = (q-1)^{|\lambda|} q^{\dim(\lambda)-|\lambda|}.$$

By the definition of a supercharacter theory, it is clear that the supercharacters are orthogonal with respect to the standard inner product. However, as they are not irreducible in general, they are not orthonormal. The inner product of two supercharacters turns out to have a nice expression [1] as

$$\langle \chi^\lambda, \chi^\mu \rangle = (q-1)^{|\lambda|} q^{\mathrm{crs}(\lambda)} \delta_{\lambda,\mu}.$$

In particular, this implies that the only supercharacter that is an irreducible character is the trivial supercharacter. We mention that the factor of $(q-1)^{|\lambda|}$ arises from the T_n action in our definition, and that the finer supercharacter theory that omits this action has many supercharacters that are irreducible characters. This concludes our description of the basic properties of the supercharacter theory.

4. A Family of Modules for U_n and Their Characters

This section contains results on certain modules for U_n. These modules arise when computing restrictions of certain supercharacters, which was the original motivation for their study. We include this as an application in the next section. We begin this section by defining poset binomials, which are certain combinatorial data associated to posets that arise naturally when describing these modules. Next, we define the modules, and compute the traces of elements of U_n acting on them. These characters will turn out to be constant on superclasses. This suggests the problem of finding their decompositions as sums of supercharacters. The solution of this problem is the final part of this section, and the main result of the paper.

4.1. **Poset Binomials.** Let $\mathcal{P}$ be a poset with n elements. For $x \in \mathcal{P}$, define the *weight* of x by $w(x) = \#\left\{y \in \mathcal{P} \mid y \succ x\right\}$. For a subset $S \subset \mathcal{P}$, we set

$$w(S) = \sum_{x \in S} w(x).$$

For $k \in \mathbb{N}$, we define a q-analog of the binomial coefficients by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \sum_{\substack{S \subset \mathcal{P} \\ |S|=k}} q^{w(S)} \in \mathbb{N}\left[q\right].$$

We refer to these as the *poset binomials* associated to $\mathcal{P}$. There is a standard q-analog of the binomial coefficients, called the *Gaussian binomial coefficients*, defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q!\,[n-k]_q!} = \frac{[n]_q\,[n-1]_q \ldots [1]_q}{[k]_q \ldots [1]_q\,[n-k]_q \ldots [1]_q}, \quad \text{for } n \geq k,$$

where

$$[m]_q = \frac{q^m - 1}{q-1}$$

is a q-integer. The Gaussian binomials have various interpretations. For instance, $\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q$ is the number of k-dimensional subspaces of $\mathbb{F}_q^n$ (see [8], chapter 3). The poset binomials interpolate between the ordinary binomial coefficients and Gaussian binomial coefficients in the following sense. If $\mathcal{P}$ is the poset with no relations, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = \binom{n}{k},$$

and if $\mathcal{P}$ is a total order, then

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{P}} = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Of particular importance to us are the poset binomials of a total order. Note that there is some redundancy in our notation, as n must be equal to $|\mathcal{P}|$. Bearing this in mind, whenever total order binomials occur we will simply write the total order as $\mathcal{T}$, with it understood that $|\mathcal{T}|$ is whatever it has to be so that the poset binomial makes sense.

Another source of posets for us is the poset $\mathcal{P}(\lambda)$ associated to a set partition λ, which we will describe next. Define the set of *crossing-free* partitions

$$\mathcal{NCS}_n = \{\lambda \in \mathcal{S}_n \mid \operatorname{crs}(\lambda) = 0\}.$$

For $\lambda \in \mathcal{S}_n$, we have the set of *left-endpoints*, $L^\lambda = \{a \mid a \frown b \in \lambda \text{ for some } b\}$, and the set of *right-endpoints*, $R^\lambda = \{b \mid a \frown b \in \lambda \text{ for some } a\}$. We prove the following lemma.

Lemma 4.1. *For each $\lambda \in \mathcal{S}_n$, there is a unique partition $\mu \in \mathcal{NCS}_n$ such that*

$$L^\lambda = L^\mu \quad \textit{and} \quad R^\lambda = R^\mu.$$

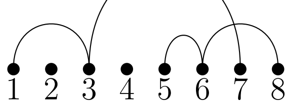
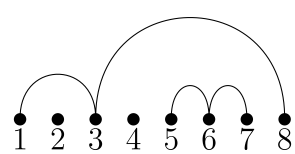
Proof. We first show existence, then uniqueness. Suppose $(i \frown k, j \frown l)$ is a crossing in λ, with $i < j < k < l$. We can replace this crossing with a nesting by defining $\lambda' = (\lambda - \{i \frown k, j \frown l\}) \cup \{i \frown l, j \frown k\}$. Then, $L^\lambda = L^{\lambda'}$ and $R^\lambda = R^{\lambda'}$. Also, $\operatorname{crs}(\lambda') = \operatorname{crs}(\lambda) - 1$. Thus, iteratively uncrossing each crossing in λ gives a crossing-free partition with the same right and left endpoints as λ.

Next, we show uniqueness. That is, we show that the order in which the crossings are uncrossed does not matter. If $\lambda = \emptyset$ the result is clear. Suppose $\lambda \neq \emptyset$. Let b be the leftmost right-endpoint of an arc in λ. In order for an arc $a \frown b$ to not cross any other arcs, it must be that a is the closest left-endpoint to the left of b. The same reasoning applies with L^λ replaced by $L^\lambda - a$ and R^λ replaced by $R^\lambda - b$. Thus, μ is uniquely determined among those partitions with the same left and right endpoints as λ by the requirement that $\operatorname{crs}(\mu) = \emptyset$. $\square$

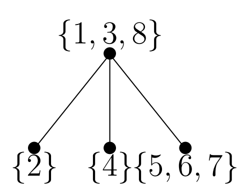
We use this result to define a function

$$\operatorname{uncr} : \mathcal{S}_n \longrightarrow \mathcal{NCS}_n,$$

where $\operatorname{uncr}(\lambda)$ is the unique partition in $\mathcal{NCS}_n$ such that $L^{\operatorname{uncr}(\lambda)} = L^\lambda$ and $R^{\operatorname{uncr}(\lambda)} = R^\lambda$. For example, for the λ defined in 3.1, we have

$\lambda =$  and $\operatorname{uncr}(\lambda) =$ 

Let $\mathcal{P}(\lambda)$ be the poset on $\operatorname{bl}(\operatorname{uncr}(\lambda))$ with order $\succ$, where $a \succ b$ if either $|a| > 1$ and there exists $j, k \in a$ and $i, l \in b$ such that $i < j < k < l$, or $|a| = \{j\}$ and there exists $i, j \in b$ such that $i < j < k$. Note that, as $\operatorname{uncr}(\lambda)$ has no crossings, each connected component of this poset is a tree with a unique maximal element. Thus, the above λ has

$\mathcal{P}(\lambda) =$ 

.

From this picture, it is easy to see that

$$w(\{1,3,8\})=0, \quad \text{and} \quad w(\{2\})=w(\{4\})=w(\{5,6,7\})=1.$$

4.2. **The Modules.** For $u\in U_n$ and a subset $A\subset\{1,\dots,n\}$, let u_A be the submatrix of u obtained by taking only those rows indexed by A. For $k\in[0,n]$, let

$$U_{k\times n}=\bigsqcup_{\substack{A\subset\{1,\dots,n\}:\\ |A|=k}}\{u_A\mid u\in U_n\}.$$

Define right modules for U_n by

$$V^{k\times n}=\mathbb{C}\text{-span}\left\{v\mid v\in U_{k\times n}\right\}.$$

with action given by right multiplication. As special cases, note that $V^{0\times n}=\{0\}$ is the trivial module, and $V^{n\times n}=\mathbb{C}U_n$ is the (right) regular module of U_n. We will prove a formula for the trace of the action of an element of U_n on $V^{k\times n}$. First, we need the following two lemmas.

Lemma 4.2. *For $k\in[0,n]$,*

$$|U_{k\times n}|=\begin{bmatrix}n\\k\end{bmatrix}_{\mathcal{T}}.$$

Proof. Let v be a $k\times n$ matrix over $\mathbb{F}_q$ of rank k that is in reduced row echelon form. Then, there are precisely $q^{\binom{k}{2}}$ elements of $U_{k\times n}$ whose reduced row echelon form is v. To see this, note that each $u\in U_{k\times n}$ is already almost in reduced row echelon form. We only need to account for its possible nonzero entries in the spots above the $1's$ coming from the diagonal entries of U_n. Moving from left to right, there are no spots above the first 1, one spot above the second 1, and so on, for a total of $0+1+2+\cdots+(k-1)=\binom{k}{2}$ spots. This is easy to see from an example. For instance, consider the following matrices in $U_{4\times 6}$ and their reduced row echelon form (we write $*$ for an arbitrary element of $\mathbb{F}_q$).

$$U_{4\times 6}\ni\begin{pmatrix}1&*&*&*&*&*\\ &1&*&*&*&*\\ &&&1&*&*\\ &&&&&1\end{pmatrix}\longleftrightarrow\begin{pmatrix}1&0&*&0&*&0\\ &1&*&0&*&0\\ &&&1&*&0\\ &&&&&1\end{pmatrix}$$

There are $0+1+2+3=\binom{4}{2}$ spots that are forced to be 0 in the reduced row echelon form, as expected. The number of $k\times n$ matrices of rank k in reduced row echelon form is precisely the number of k-dimensional subspaces of $\mathbb{F}_q^n$, which is given by the Gaussian binomial coefficient $\begin{bmatrix}n\\k\end{bmatrix}_q$. The lemma follows. $\square$

We also show the following.

Lemma 4.3. *The function $U_n\longrightarrow\mathbb{C}$ defined by*

$$u\longmapsto \operatorname{tr}(u,V^{k\times n})$$

is a superclass function.

Proof. We will show that, for $u, a, b \in U_n$,

$$\operatorname{tr}(u, V^{k\times n}) = \operatorname{tr}(a(u-1)b+1, V^{k\times n}).$$

The vector space $V^{k\times n}$ is spanned by the elements of $U_{k\times n}$, and the U_n action permutes these basis vectors. Thus,

$$\operatorname{tr}(u, V^{k\times n}) = \#\left\{v \in U_{k\times n} \mid v \cdot u = v\right\} = \#\left\{v \in U_{k\times n} \mid v(u-1) = 0\right\}.$$

As U_n acts by permutations, we have that, for each $a \in U_n$,

$$\#\left\{v \in U_{k\times n} \mid v(u-1) = 0\right\} = \#\left\{v \in U_{k\times n} \mid (v \cdot a)(u-1) = 0\right\}.$$

Futhermore, for any $b \in U_n$, $(v \cdot a)(u-1) = 0$ if and only if $(v \cdot a)(u-1)b = 0$ if and only if $v(a(u-1)b+1) = v$. Thus $\operatorname{tr}(u, V^{k\times n})$ is indeed constant on superclasses. □

We use the previous two lemmas to prove the following proposition.

Proposition 4.4. *Let $k \in [0, n]$ and $\mu \in \mathcal{S}_n$. Then, for each element $u \in U$ in the superclass indexed by μ,*

$$\operatorname{tr}(u, V^{k\times n}) = \begin{bmatrix} n - |\mu| \\ k \end{bmatrix}_{\mathcal{T}}.$$

Proof. Suppose $u \in U_n$ is in the superclass indexed by μ. We compute:

$$\begin{aligned}
\operatorname{tr}(u, V^{k\times n}) &= \operatorname{tr}(u_\mu, V^{k\times n}) \\
&= \#\left\{v \in U_{k\times n} \mid v \cdot u_\mu = v\right\} \\
&= \#\left\{v \in U_{k\times n} \mid j \frown l \in \mu \text{ implies } v_{ij} = 0 \text{ for all } 1 \le i \le k\right\} \\
&= \left|U_{k\times n-|\mu|}\right| \\
&= \begin{bmatrix} n - |\mu| \\ k \end{bmatrix}_{\mathcal{T}},
\end{aligned}$$

where the first equality is by Lemma 4.3, and the last equality is by Lemma 4.2. The result follows. □

By Lemma 4.3, the character $\operatorname{tr}(u, V^{k\times n})$ is a superclass function, so we can ask for its decomposition in terms of supercharacters.

4.3. **Decomposition into Supercharacters.** We will prove the following theorem.

Theorem 4.5. *Let $u \in U_n$ be in the superclass indexed by $\mu \in \mathcal{S}_n$. Then, for each $k \in [0, n]$,*

$$\operatorname{tr}(u, V^{k\times n}) = \sum_{\lambda \in \mathcal{S}_n} q^{nst_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \chi^\lambda.$$

To that end, we make the definition

$$\psi_k^n = \sum_{\lambda \in \mathcal{S}_n} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \chi^\lambda.$$

Bearing in mind Proposition 4.4, we will work towards showing the equivalent statement

$$\psi_k^n(\mu) = \begin{bmatrix} n - |\mu| \\ k \end{bmatrix}_{\mathcal{T}}.$$

The proof is somewhat involved, so we outline its structure before we begin. There are three main steps.

Step 1: First, we prove Proposition 4.6, which give the value of ψ_k^n at the identity. Most of the work in this step is contained in Lemma 4.7.
Step 2: Next, we prove Proposition 4.8, which states that $\psi_k^n(\mu)$ depends only on $|\mu|$. This proof involves two lemmas; Lemma 4.10, a technical result about certain sums of supercharacters, and Lemma 4.9, which states that $\psi_k^n(\mu)$ is invariant under "small" changes to μ.
Step 3: Finally, we use Proposition 4.8 to prove Proposition 4.11, which is a recurrence for $\psi_k^n(\mu)$ that allows us to reduce to the case of evaluating at the identity.

We begin with the first step. (Note that the identity is the unique element in the superclass indexed by the empty partition $\emptyset$).

Proposition 4.6. *For each $k \in [0, n]$,*

$$\psi_k^n(1) = \begin{bmatrix} n \\ k \end{bmatrix}_{\mathcal{T}}.$$

We require the following definitions. Each element of $U_{k\times n}$ is of the form u_A, for some $A \subset \{1, \ldots, n\}$ with $|A| = k$. For each $k \in [0, n]$, there is an injection $\iota_k : U_{k\times n} \longrightarrow U_n$ given by

$$\iota_k(u_A)_{ij} = \begin{cases} u_{ij}, & \text{if } i \in A, \\ 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Define a bijection $\theta : J_n^* \longrightarrow U_n$ by

$$\theta(\gamma)_{ij} = \begin{cases} \gamma(e_{ij}), & \text{if } 1 \le i < j < n, \\ 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

where e_{ij} is the matrix with a 1 in the (i, j) position, and zeroes everywhere else. For $\lambda \in \mathcal{S}_n$, define the function $\gamma_\lambda \in J_n^*$ by

$$\gamma_\lambda(u - 1) = \sum_{i \frown j \in \lambda} u_{ij}, \quad \text{for } u \in U_n,$$

and let

$$U_{k\times n}^\lambda = \left\{ u \in U_{k\times n} \mid \theta^{-1}(\iota_k(u)) \in U_n \gamma_\lambda U_n \right\}.$$

We have the following lemma.

Lemma 4.7. *For each $k \in [0, n]$ and $\lambda \in \mathcal{S}_n$,*

$$\left| U_{k\times n}^\lambda \right| = q^{nst_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \chi^\lambda(1).$$

Proof. We first consider the case when $\mathrm{crs}(\lambda)=\emptyset$. Fix such a λ, and define

$$\mathcal{D}_\lambda = \{e_{jk} \mid i\frown k \in \lambda \text{ for some } i \leq j < k \leq n\} \cup \{e_{ij} \mid i\frown k \in \lambda \text{ for some } i \leq j < k \leq n\}.$$

Recall that $L^\lambda = \{i \mid i\frown j \in \lambda\}$. Suppose A satisfies $L^\lambda \subset A \subset \{1,\dots,n\}$. Then,

$$\{u_A \mid u \in U_n\} \cap U^\lambda_{k\times n} = \left\{u \in U_n \mid \theta^{-1}(\iota(u_A)) \in U_n\gamma_\lambda U_n\right\}.$$

The elements of $U_n\gamma_\lambda U_n$ are precisely the functions $\gamma : J_n^* \longrightarrow \mathbb{F}_q$ that satisfy $\mathrm{supp}(\gamma) \subset \{e_{jk} \mid i \leq j < k \leq l \text{ for some } i\frown l \in \lambda\}$ and $i\frown j \in \lambda \Rightarrow \gamma(e_{ij}) \in \mathbb{F}_q^\times$. Therefore,

$$\left|\{u_A \mid u \in U_n\} \cap U^\lambda_{k\times n}\right| = (q-1)^{|\lambda|} q^{|\{e_{ij}\in\mathcal{D}_\lambda | i\in A\}|-|\lambda|}.$$

Consequently,

$$\begin{aligned}
\left|U^\lambda_{k\times n}\right| &= \sum_{\substack{L^\lambda\subset A\subset\{1,\dots,n\}\\ |A|=k}} \left|\{u_A \mid u \in U_n\} \cap U^\lambda_{k\times n}\right| \\
&= \sum_{\substack{L^\lambda\subset A\subset\{1,\dots,n\}\\ |A|=k}} (q-1)^{|\lambda|} q^{|\{e_{ij}\in\mathcal{D}_\lambda | i\in A\}|-|\lambda|}.
\end{aligned}$$

Now, $|\{e_{ij} \in \mathcal{D}_\lambda \mid i \in A\}| - |\lambda| = |\{e_{ij} \in \mathcal{D}_\lambda \mid i \in A, i\frown j \notin \lambda\}|$, and

$$\begin{aligned}
\{e_{ij} \in \mathcal{D}_\lambda \mid i \in A, i\frown j \notin \lambda\} &= \left\{e_{ij} \in \mathcal{D}_\lambda \mid i \in L^\lambda, i\frown k \in \lambda, j<k\right\} \\
&\sqcup \left\{e_{ik} \in \mathcal{D}_\lambda \mid i \in L^\lambda, i\frown j \in \lambda, j<k\right\} \\
&\sqcup \left\{e_{ij} \in \mathcal{D}_\lambda \mid i \in L^\lambda - A\right\}.
\end{aligned}$$

We compute

$$\begin{aligned}
\left|\left\{e_{ij} \in \mathcal{D}_\lambda \mid i \in L^\lambda, i\frown k \in \lambda, j<k\right\}\right| &= \prod_{i\frown j\in\lambda} (j-i-1) = \dim(\lambda) - |\lambda|, \\
\left|\left\{e_{ik} \in \mathcal{D}_\lambda \mid i \in L^\lambda, i\frown j \in \lambda, j<k\right\}\right| &= \mathrm{nst}^\lambda_\lambda, \quad \text{and} \\
\left|\left\{e_{ij} \in \mathcal{D}_\lambda \mid i \in L^\lambda - A\right\}\right| &= \sum_{j\in A-L^\lambda} |\{i\frown k \in \lambda \mid i<j<k\}| \\
&= \sum_{j\in A-L^\lambda} w(\mathrm{bl}(j)),
\end{aligned}$$

where $\mathrm{bl}(j)$ is the block of λ containing j. Therefore,

$$\begin{aligned}
\left|U^\lambda_{k\times n}\right| =&(q-1)^{|\lambda|}\sum_{\substack{L^\lambda\subset A\subset\{1,\ldots,n\}\\|A|=k}} q^{|\{e_{ij}\in\mathcal{D}_\lambda | i\in A\}|-|\lambda|}\\
=&(q-1)^{|\lambda|}q^{\dim(\lambda)-|\lambda|}q^{\mathrm{nst}^\lambda_\lambda}\sum_{\substack{L^\lambda\subset A\subset\{1,\ldots,n\}\\|A|=k}} q^{w(A-L^\lambda)}\\
=&\chi^\lambda(1)q^{\mathrm{nst}^\lambda_\lambda}\sum_{\substack{A\subset\{1,\ldots,n\}-L^\lambda\\|A|=k-|\lambda|}} q^{w(A)}\\
=&\chi^\lambda(1)q^{\mathrm{nst}^\lambda_\lambda}\begin{bmatrix}n-|\lambda|\\k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}.
\end{aligned}$$

Now, we prove the result for arbitrary λ. Fix a $\lambda\in\mathcal{S}_n-\mathcal{NCS}_n$. Our strategy will be to show that both

$$\left|U^\lambda_{k\times n}\right| \quad\text{and}\quad \chi^\lambda(1)q^{\mathrm{nst}^\lambda_\lambda}\begin{bmatrix}n-|\lambda|\\k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}$$

change by the same amount when we uncross a maximal crossing in λ. Suppose $(a\frown c, b\frown d)\in\mathrm{crs}(\lambda)$, with $c-b$ maximal, and write $\mu=(\lambda-\{a\frown c,b\frown d\})\cup\{a\frown d,b\frown c\}$. We will first show that

$$\left|U^\lambda_{k\times n}\right| = \frac{1}{q}\left|U^\mu_{k\times n}\right|.$$

Fix an A with $L^\lambda=L^\mu\subset A\subset\{1,\ldots,n\}$ and $|A|=k$. We know

$$\left|\{u_A\mid u\in U_n\}\cap U^\lambda_{k\times n}\right| = (q-1)^{|\lambda|}q^{|\{e_{ij}\in\mathcal{D}_\lambda|i\in A\}|-|\lambda|},$$

and $|\lambda|=|\mu|$, so it will be sufficient to show that

$$|\{e_{ij}\in\mathcal{D}_\lambda\mid i\in A\}| = |\{e_{ij}\in\mathcal{D}_\mu\mid i\in A\}|-1.$$

We compute

$$\begin{aligned}
|\{e_{ij}\in\mathcal{D}_\lambda\mid i\in A\}-\{e_{ij}\in\mathcal{D}_\mu\mid i\in A\}| &= \#\{e_{a'c}\mid a'\in A, a<a'<b\}\\
&\quad\cup\{e_{bc'}\mid b<c'<d\}\\
&=|\{a'\in A\mid a<a'<b\}|+d-b-1
\end{aligned}$$

and

$$\begin{aligned}
|\{e_{ij}\in\mathcal{D}_\mu\mid i\in A\}-\{e_{ij}\in\mathcal{D}_\lambda\mid i\in A\}| &= \#\{e_{a'd}\mid a'\in A, a<a'<b\}\\
&\quad\cup\{e_{ac'}\mid b<c'\le d\}\\
&=|\{a'\in A\mid a<a'<b\}|+d-b.
\end{aligned}$$

Therefore,

$$\begin{aligned}\left|U^{\lambda}_{k\times n}\right| =& \sum_{\substack{L^{\lambda}\subset A\subset\{1,\dots,n\}\\|A|=k}} \left|\{u_A \mid u\in U_n\}\cap U^{\lambda}_{k\times n}\right|\\ =&\frac{1}{q}\sum_{\substack{L^{\mu}\subset A\subset\{1,\dots,n\}\\|A|=k}} \left|\{u_A \mid u\in U_n\}\cap U^{\mu}_{k\times n}\right| = \frac{1}{q}\left|U^{\mu}_{k\times n}\right|.\end{aligned}$$

To complete the proof, note that, by maximality of our choice of crossing $\mathrm{nst}^{\lambda}_{\lambda} = \mathrm{nst}^{\mu}_{\mu} - 1$. As $\mathcal{P}(\lambda) = \mathcal{P}(\mu)$ and $\chi^{\lambda}(1) = \chi^{\mu}(1)$, we conclude that

$$\chi^{\lambda}(1)q^{\mathrm{nst}^{\lambda}_{\lambda}}\begin{bmatrix} n-|\lambda| \\ k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)} = \frac{1}{q}\chi^{\mu}(1)q^{\mathrm{nst}^{\mu}_{\mu}}\begin{bmatrix} n-|\mu| \\ k-|\mu|\end{bmatrix}_{\mathcal{P}(\mu)}.$$

We can reach any λ by starting with $\mathrm{uncr}(\lambda)\in\mathcal{NCS}_n$ and adding maximal crossings one at a time. We just showed that both sides of the equality in Lemma 4.7 are multiplied by a factor of $\frac{1}{q}$ when we add a crossing, so they are equal for all $\lambda\in\mathcal{S}_n$, and the proof is finished. □

We are now able to prove Proposition 4.6.

Proof of Proposition 4.6. Note that each γ_λ is in a distinct orbit of J^*_n. Thus, the orbits $U_n\gamma_\lambda U_n$ partition J^*_n, so applying θ gives us a partition of U_n,

$$U_n = \bigsqcup_{\lambda\in\mathcal{S}_n}\theta(U_n\gamma_\lambda U_n).$$

This gives us a partition

$$U_{k\times n} = \bigsqcup_{\substack{\lambda\in\mathcal{S}_n:\\|\lambda|\le k}} U^{\lambda}_{k\times n}.$$

The result follows, as by Lemma 4.7

$$\psi^n_k(1) = \sum_{\lambda\in\mathcal{S}_n} q^{\mathrm{nst}^{\lambda}_{\lambda}}\begin{bmatrix} n-|\lambda| \\ k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}\chi^{\lambda}(1) = \sum_{\lambda\in\mathcal{S}_n}\left|U^{\lambda}_{k\times n}\right| = |U_{k\times n}| = \begin{bmatrix} n \\ k\end{bmatrix}_{\mathcal{T}}.$$

□

This concludes the first step of the proof of Theorem 4.5. We move on to step two, which is the following proposition.

Proposition 4.8. *Let $k\in[0,n]$, and $\mu,\nu\in\mathcal{S}_n$. If $|\mu| = |\nu|$, then*

$$\psi^n_k(\mu) = \psi^n_k(\nu).$$

We will prove this by first showing that $\psi^n_k(\mu)$ is invariant under small changes to μ. Specifically, for $l\in[1,n-1]$, define

$$\mathcal{T}_l = \{\lambda\in\mathcal{S}_n \mid l\frown(l+1)\in\lambda\},$$

and consider the bijections

$$\epsilon^R_l : \mathcal{S}_n - \mathcal{T}_l \longrightarrow \mathcal{S}_n - \mathcal{T}_l \quad\text{and}\quad \epsilon^L_l : \mathcal{S}_n - \mathcal{T}_l \longrightarrow \mathcal{S}_n - \mathcal{T}_l$$

where $\epsilon_l^R(\lambda)$ (respectively $\epsilon_l^R(\lambda)$) is the partition obtained by applying the transposition $(l, l+1)$ to the right (respectively left) endpoints of the arcs in λ. We will prove the following lemma.

Lemma 4.9. *Fix $l \in [1, n-1]$, and let $k \in [0, n]$. Then, for all $\mu \in \mathcal{S}_n - \mathcal{T}_l$,*

$$\psi_k^n(\epsilon_l^R(\mu)) = \psi_k^n(\epsilon_l^L(\mu)) = \psi_k^n(\mu).$$

We prove the ϵ_l^R case. The argument for ϵ_l^L is completely symmetric. Write $\epsilon = \epsilon_l^R$. We will use the following technical result to facilitate our computations.

Lemma 4.10. *Fix $1 \leq i < l \leq n$. Suppose the function $\theta : \mathcal{S}_n \longrightarrow \mathbb{C}$ satisfies, for some $z \in \mathbb{C}$,*

$$\theta(\epsilon(\lambda)) = \begin{cases} zq\theta(\lambda), & \text{for all } \lambda \in \mathcal{S}_n \text{ such that } i \frown l, j \frown (l+1) \in \lambda \text{ for some } i < j < l, \\ q\theta(\lambda), & \text{for all } \lambda \in \mathcal{S}_n \text{ such that } i \frown (l+1) \in \lambda, j \frown l \notin \lambda \forall j < l, \end{cases}$$

and

$$\theta(\epsilon(\lambda) \cup l \frown (l+1)) = z(q-1)\theta(\lambda) \quad \text{for all } \lambda \in \mathcal{S}_n \text{ such that } i \frown (l+1) \in \lambda, j \frown l \notin \lambda \forall j < l.$$

Then, for all $i < j < l$,

$$\sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown l \in \lambda \\ j \frown (l+1) \notin \lambda \forall j < i}} q^{nst_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda) = z \sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown (l+1) \in \lambda \\ j \frown l \notin \lambda \forall j < i}} q^{nst_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda).$$

Proof. We split the sum on the left hand side into two parts as

$$\begin{aligned} &\sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown l \in \lambda \\ j \frown (l+1) \notin \lambda \forall j < i}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda) \\ &\qquad = \sum_{j=i+1}^{l+1} \sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown l \in \lambda \\ j \frown (l+1) \in \lambda}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda) + \sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown l \in \lambda \\ j \frown (l+1) \notin \lambda \forall j < i}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda). \end{aligned}$$

Consider the first sum. As $q^{\mathrm{nst}_{\epsilon(\lambda)}^{\epsilon(\lambda)}} = q^{\mathrm{nst}_\lambda^\lambda + 1}$ and $\mathcal{P}(\epsilon(\lambda)) = \mathcal{P}(\lambda)$, we have that

$$\sum_{j=i+1}^{l+1} \sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown l \in \lambda \\ j \frown (l+1) \in \lambda}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda) = z \sum_{j=i+1}^{l+1} \sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown (l+1) \in \lambda \\ j \frown l \in \lambda}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda),$$

by our hypothesis on θ. We next consider the second sum. As

$$z \sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown (l+1) \in \lambda \\ j \frown l \notin \lambda \forall j < i}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda)$$

$$= z \sum_{j=i+1}^{l+1} \sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown (l+1) \in \lambda \\ j \frown l \in \lambda}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda) + z \sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown (l+1) \in \lambda \\ j \frown l \notin \lambda \forall j < i}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda),$$

it will be sufficient to show that

$$\sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown l \in \lambda \\ j \frown (l+1) \notin \lambda \forall j < i}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda) = z \sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown (l+1) \in \lambda \\ j \frown l \notin \lambda \forall j < i}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda).$$

Define the set

$$\mathcal{B} = \left\{ \lambda - \{i \frown l\} \mid \lambda \in \mathcal{S}_n, i \frown l \in \lambda, \text{ and } j \frown (l+1) \notin \lambda \forall j \leq l \right\}.$$

For each $\lambda \in \mathcal{B}$, note that both $\lambda' = \lambda \cup i \frown l$ and $\lambda'' = \lambda \cup i \frown l \frown (l+1)$ are valid set partitions in $\mathcal{S}_n$. Furthermore,

$$\{\lambda \in \mathcal{S}_n \mid i \frown l \in \lambda, j \frown (l+1) \notin \lambda \forall j < i\} = \{\lambda' \mid \lambda \in \mathcal{B}\} \sqcup \{\lambda'' \mid \lambda \in \mathcal{B}\}.$$

Therefore, we can rewrite the above sum as

$$\sum_{\substack{\lambda \in \mathcal{S}_n \\ i \frown l \in \lambda \\ j \frown (l+1) \notin \lambda \forall j < i}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda) = \sum_{\lambda \in \mathcal{B}} \left(q^{\mathrm{nst}_{\lambda'}^{\lambda'}} \begin{bmatrix} n - |\lambda'| \\ k - |\lambda'| \end{bmatrix}_{\mathcal{P}(\lambda')} \theta(\lambda') + q^{\mathrm{nst}_{\lambda''}^{\lambda''}} \begin{bmatrix} n - |\lambda''| \\ k - |\lambda''| \end{bmatrix}_{\mathcal{P}(\lambda'')} \theta(\lambda'') \right)$$

$$= \sum_{\lambda \in \mathcal{B}} \left(q^{\mathrm{nst}_{\lambda'}^{\lambda'}} \begin{bmatrix} n - |\lambda| - 1 \\ k - |\lambda| - 1 \end{bmatrix}_{\mathcal{P}(\lambda')} z\theta(\epsilon(\lambda')) \right.$$
$$\left. + q^{\mathrm{nst}_{\lambda''}^{\lambda''}} \begin{bmatrix} n - |\lambda| - 2 \\ k - |\lambda| - 2 \end{bmatrix}_{\mathcal{P}(\lambda'')} z(q-1)\theta(\epsilon(\lambda')) \right),$$

by our assumption on θ. We claim that

$$q^{\mathrm{nst}_{\lambda'}^{\lambda'}} \begin{bmatrix} n - |\lambda| - 1 \\ k - |\lambda| - 1 \end{bmatrix}_{\mathcal{P}(\lambda')} + (q-1) q^{\mathrm{nst}_{\lambda''}^{\lambda''}} \begin{bmatrix} n - |\lambda| - 2 \\ k - |\lambda| - 2 \end{bmatrix}_{\mathcal{P}(\lambda'')} = q^{\mathrm{nst}_{\epsilon(\lambda')}^{\epsilon(\lambda')}} \begin{bmatrix} n - |\lambda| - 1 \\ k - |\lambda| - 1 \end{bmatrix}_{\mathcal{P}(\epsilon(\lambda'))}$$

To see this, write $a = \{l\} \in \mathcal{P}(\epsilon(\lambda))$ and let $b \in \mathcal{P}(\lambda')$ be the block containing $l+1$. Note that $w(a) = w(b) + 1$. Set $x = k - |\lambda| - 1$. Then,

$$\begin{aligned}
\begin{bmatrix} n-|\lambda|-1 \\ x \end{bmatrix}_{\mathcal{P}(\epsilon(\lambda'))} &= \sum_{\substack{S \subset \mathcal{P}(\epsilon(\lambda)) \\ |S|=x}} q^{w(S)} = \sum_{\substack{S \subset \mathcal{P}(\epsilon(\lambda)) \\ |S|=x, a\in S}} q^{w(S)} + \sum_{\substack{S \subset \mathcal{P}(\epsilon(\lambda)) \\ |S|=x, a\notin S}} q^{w(S)} \\
&= q \sum_{\substack{S \subset \mathcal{P}(\lambda) \\ |S|=x, b\in S}} q^{w(S)} + \sum_{\substack{S \subset \mathcal{P}(\lambda) \\ |S|=x, b\notin S}} q^{w(S)} \\
&= (q-1) \sum_{\substack{S \subset \mathcal{P}(\lambda) \\ |S|=x, b\in S}} q^{w(S)} + \sum_{\substack{S \subset \mathcal{P}\lambda) \\ |S|=x}} q^{w(S)} \\
&= (q-1) q^{w(b)} \sum_{\substack{S \subset \mathcal{P}(\lambda) \\ |S|=x, b\in S}} q^{w(S)-w(b)} + \begin{bmatrix} n-|\lambda|-1 \\ x \end{bmatrix}_{\mathcal{P}(\lambda')} \\
&= (q-1) q^{w(b)} \begin{bmatrix} n-|\lambda|-2 \\ x-1 \end{bmatrix}_{\mathcal{P}(\lambda'')} + \begin{bmatrix} n-|\lambda|-1 \\ x \end{bmatrix}_{\mathcal{P}(\lambda')}.
\end{aligned}$$

Noting that $q^{\mathrm{nst}_{\epsilon(\lambda)}^{\epsilon(\lambda)}+w(b)} = q^{\mathrm{nst}_{\lambda''}^{\lambda''}}$ and $q^{\mathrm{nst}_{\epsilon(\lambda)}^{\epsilon(\lambda)}} = q^{\mathrm{nst}_{\lambda'}^{\lambda'}}$, the claim follows. Thus, we can write

$$\begin{aligned}
\sum_{\substack{\lambda \in \mathcal{S}_n \\ i\frown l \in \lambda \\ j \frown (l+1) \notin \lambda \forall j<i}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n-|\lambda| \\ k-|\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda) &= \sum_{\lambda \in \mathcal{B}} q^{\mathrm{nst}_{\epsilon(\lambda')}^{\epsilon(\lambda')}} \begin{bmatrix} n-|\lambda|-1 \\ k-|\lambda|-1 \end{bmatrix}_{\mathcal{P}(\epsilon(\lambda'))} z\theta(\epsilon(\lambda)) \\
&= \frac{z}{q} \sum_{\substack{\lambda \in \mathcal{S}_n \\ i\frown (l+1) \in \lambda \\ j \frown l \notin \lambda \forall j<l}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n-|\lambda| \\ k-|\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \theta(\lambda).
\end{aligned}$$

The lemma follows. □

We now proceed with the proof of Lemma 4.9.

Proof of Lemma 4.9. We identify three cases.

Case 1: $\epsilon(\mu) = \mu$
Case 2: There exists $i \in [1, l-1]$ such that one of $i \frown l$ or $i \frown (l+1)$ is in μ.
Case 3: There exist $i, j \in [1, l-1]$ such that $i \frown l$ and $j \frown (l+1)$ are in μ.

Case 1. This is obvious.

Case 2. We can assume without loss of generality that $i \frown l \in \mu$ (and thus $i \frown (l+1) \in \epsilon(\mu)$). Let $\nu = \mu - i \frown l$, and note that $\epsilon(\mu) = \nu \cup i \frown (l+1)$ by assumption. Consider

$$\begin{aligned}
\psi_k^n(\mu) - \psi_k^n(\epsilon(\mu)) &= \sum_{\lambda \in \mathcal{S}_n} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n-|\lambda| \\ k-|\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} (\chi^\lambda(\mu) - \chi^\lambda(\epsilon(\mu))) \\
&= \sum_{\lambda \in \mathcal{S}_n} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n-|\lambda| \\ k-|\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} (\chi^\lambda(\nu \cup i \frown l) - \chi^\lambda(\nu \cup i \frown (l+1))).
\end{aligned}$$

Consider the six disjoint subsets of $\mathcal{S}_n$ defined by

$$\begin{aligned}
\mathcal{A}_1 &= \{\lambda \in \mathcal{S}_n \mid i \frown (l+1) \in \lambda \text{ and } i' \frown l \in \lambda \text{ for some } i' < i\} \\
\mathcal{A}_2 &= \{\lambda \in \mathcal{S}_n \mid i \frown l \in \lambda \text{ and } i' \frown (l+1) \in \lambda \text{ for some } i' < i\} \\
\mathcal{B}_1 &= \{\lambda \in \mathcal{S}_n \mid i' \frown l \in \lambda \text{ for some } i' < i, i'' \frown (l+1) \notin \lambda, i'' \leq i, i \frown l' \notin \lambda, l' > l+1\} \\
\mathcal{B}_2 &= \{\lambda \in \mathcal{S}_n \mid i' \frown (l+1) \in \lambda \text{ for some } i' < i, i'' \frown l \notin \lambda, i'' \leq i, i \frown l' \notin \lambda, l' > l+1\} \\
\mathcal{C}_1 &= \{\lambda \in \mathcal{S}_n \mid i \frown l \in \lambda, i'' \frown (l+1) \notin \lambda, i'' \leq i\} \\
\mathcal{C}_2 &= \{\lambda \in \mathcal{S}_n \mid i \frown (l+1) \in \lambda, i'' \frown l \notin \lambda, i'' \leq i\}.
\end{aligned}$$

Define $\theta(\lambda) = \chi^\lambda(\nu \cup i \frown l) - \chi^\lambda(\nu \cup i \frown (l+1))$. We apply the character formula in each case to obtain that

$$\theta(\lambda) = \begin{cases} q^{-\mathrm{nst}^\lambda_{i\frown(l+1)}}(q-1)^{-1}\chi^\lambda(\nu) & \text{if } \lambda \in \mathcal{A}_1 \\ -q^{-\mathrm{nst}^\lambda_{i\frown l}}(q-1)^{-1}\chi^\lambda(\nu) & \text{if } \lambda \in \mathcal{A}_2 \\ -q^{-\mathrm{nst}^\lambda_{i\frown(l+1)}}\chi^\lambda(\nu) & \text{if } \lambda \in \mathcal{B}_1 \\ q^{-\mathrm{nst}^\lambda_{i\frown l}}\chi^\lambda(\nu) & \text{if } \lambda \in \mathcal{B}_2 \\ -q^{-\mathrm{nst}^\lambda_{i\frown l}}q(q-1)^{-1}\chi^\lambda(\nu) & \text{if } \lambda \in \mathcal{C}_1 \\ q^{-\mathrm{nst}^\lambda_{i\frown(l+1)}}(q-1)^{-1}\chi^\lambda(\nu) & \text{if } \lambda \in \mathcal{C}_2 \\ 0 & \text{otherwise.} \end{cases}$$

We will show that

$$\sum_{\lambda \in \mathcal{A}_1 \cup \mathcal{A}_2} q^{\mathrm{nst}^\lambda_\lambda} \begin{bmatrix} n-|\lambda| \\ k-|\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} (\chi^\lambda(\nu \cup i \frown l) - \chi^\lambda(\nu \cup i \frown (l+1)) = 0,$$

and likewise for $\mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{C}_1 \cup \mathcal{C}_2$. For $\mathcal{A}_1 \cup \mathcal{A}_2$, note that ϵ gives a bijection between $\mathcal{A}_1$ and $\mathcal{A}_2$. Furthermore, for $\lambda \in \mathcal{A}_1$,

$$\theta(\epsilon(\lambda)) = -q^{-\mathrm{nst}^\lambda_{i\frown l}}(q-1)^{-1}\chi^{\epsilon(\lambda)}(\nu) = \frac{-1}{q}\theta(\lambda).$$

As $q^{\mathrm{nst}^{\epsilon(\lambda)}_{\epsilon(\lambda)}} = q^{\mathrm{nst}^\lambda_\lambda + 1}$, the claim follows. For both $\mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{C}_1 \cup \mathcal{C}_2$, we will show that, in each case, θ satisfies the conditions of Lemma 4.10. Note that ϵ gives a bijection between $\mathcal{B}_1$ and $\mathcal{B}_2$. Furthermore, for $\lambda \in \mathcal{B}_2$, we compute

$$\begin{aligned}
\theta(\epsilon(\lambda)) &= -q^{-\mathrm{nst}^{\epsilon(\lambda)}_{i\frown(l+1)}}\chi^{\epsilon(\lambda)}(\nu) = -q^{-\mathrm{nst}^\lambda_{i\frown l}+1}\chi^{\epsilon(\lambda)}(\nu) \\
&= \begin{cases} -q\theta(\lambda), & \text{if } i' \frown (l+1), j \frown l \in \lambda \text{ for some } i' < i < j < l, \\ -\theta(\lambda), & \text{if } i' \frown (l+1) \in \lambda \text{ for some } i' < i, j \frown l \notin \lambda \text{ for all } j < l. \end{cases}
\end{aligned}$$

Furthermore, for λ such that $i' \frown (l+1) \in \lambda$ for some $i' < i, j \frown l \notin \lambda$ for all $j < l$,

$$\theta(\epsilon(\lambda) \cup l \frown (l+1)) = \theta(\epsilon(\lambda)) = -\theta(\lambda).$$

Therefore, Lemma 4.10 implies that

$$\sum_{\lambda \in \mathcal{B}_1 \cup \mathcal{B}_2} q^{\mathrm{nst}^\lambda_\lambda} \begin{bmatrix} n-|\lambda| \\ k-|\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} (\chi^\lambda(\nu \cup i \frown l) - \chi^\lambda(\nu \cup i \frown (l+1)) = 0.$$

Finally, note that ϵ gives a bijection between $\mathcal{C}_1$ and $\mathcal{C}_2$. For $\lambda \in \mathcal{C}_2$, we compute

$$\begin{aligned}\theta(\epsilon(\lambda)) &= -q^{-\mathrm{nst}^{\epsilon(\lambda)}_{i\frown l}+1}(q-1)^{-1}\chi^{\epsilon(\lambda)}(\nu) = -q^{-\mathrm{nst}^{\lambda}_{i\frown(l+1)}+1}\chi^{\epsilon(\lambda)}(\nu)\\ &= \begin{cases} -q\theta(\lambda), & \text{if } i\frown(l+1), j\frown l \in \lambda \text{ for some } i<j<l,\\ -\theta(\lambda), & \text{if } i\frown(l+1)\in\lambda, j\frown l\notin\lambda \text{ for all } j<l.\end{cases}\end{aligned}$$

Furthermore, for λ such that $i\frown(l+1)\in\lambda, j\frown l\notin\lambda$ for all $j<l$,

$$\theta(\epsilon(\lambda)\cup l\frown(l+1)) = \theta(\epsilon(\lambda)) = -\theta(\lambda).$$

Thus, by Lemma 4.10,

$$\sum_{\lambda\in\mathcal{C}_1\cup\mathcal{C}_2} q^{\mathrm{nst}^{\lambda}_{\lambda}}\begin{bmatrix} n-|\lambda| \\ k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}(\chi^{\lambda}(\nu\cup i\frown l) - \chi^{\lambda}(\nu\cup i\frown(l+1)) = 0.$$

Therefore,

$$\psi^n_k(\mu) - \psi^n_k(\epsilon(\mu)) = \sum_{\lambda\in\mathcal{S}_n} q^{\mathrm{nst}^{\lambda}_{\lambda}}\begin{bmatrix} n-|\lambda| \\ k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}(\chi^{\lambda}(\nu\cup i\frown l) - \chi^{\lambda}(\nu\cup i\frown(l+1))) = 0,$$

and the second case is finished.

Case 3. We proceed similarly to the previous case. Assume without loss of generality that $i\frown l, j\frown(l+1)\in\mu$ for some $i<j<l$. Let $\nu = \mu - \{i\frown l, j\frown(l+1)\}$. Consider

$$\psi^n_k(\mu) - \psi^n_k(\epsilon(\mu)) = \sum_{\lambda\in\mathcal{S}_n} q^{\mathrm{nst}^{\lambda}_{\lambda}}\begin{bmatrix} n-|\lambda| \\ k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}(\chi^{\lambda}(\mu) - \chi^{\lambda}(\epsilon(\mu)))$$

As in the previous case, consider the six disjoint subsets of $\mathcal{S}_n$ defined by

$$\begin{aligned}\mathcal{A}_1 &= \{\lambda\in\mathcal{S}_n \mid i\frown l, j\frown(l+1)\in\lambda\}\\ \mathcal{A}_2 &= \{\lambda\in\mathcal{S}_n \mid i\frown(l+1), j\frown l\in\lambda\}\\ \mathcal{B}_1 &= \{\lambda\in\mathcal{S}_n \mid i\frown l\in\lambda \text{ and } j'\frown(l+1)\notin\lambda\forall j'\le j\}\\ \mathcal{B}_2 &= \{\lambda\in\mathcal{S}_n \mid i\frown(l+1)\in\lambda \text{ and } j'\frown l\notin\lambda\forall j'\le j\}\\ \mathcal{C}_1 &= \{\lambda\in\mathcal{S}_n \mid j\frown l\in\lambda, i'\frown(l+1)\notin\lambda\forall i'\le i\}\\ \mathcal{C}_2 &= \{\lambda\in\mathcal{S}_n \mid j\frown(l+1)\in\lambda, i'\frown l\notin\lambda\forall i'\le i\}.\end{aligned}$$

Let $\theta(\lambda) = \chi^{\lambda}(\nu\cup i\frown l\cup j\frown(l+1)) - \chi^{\lambda}(\nu\cup i\frown(l+1)\cup j\frown l)$. We apply the character formula in each case to obtain that

$$\theta(\lambda) = \begin{cases} q^{-\mathrm{nst}^{\lambda}_{i\frown l\cup j\frown(l+1)}}(q-1)^{-2}\chi^{\lambda}(\nu) & \text{if } \lambda\in\mathcal{A}_1\\ -q^{-\mathrm{nst}^{\lambda}_{i\frown(l+1)\cup j\frown l}}(q-1)^{-2}\chi^{\lambda}(\nu) & \text{if } \lambda\in\mathcal{A}_2\\ -q^{-\mathrm{nst}^{\lambda}_{i\frown l\cup j\frown(l+1)}}(q-1)^{-1}\chi^{\lambda}(\nu) & \text{if } \lambda\in\mathcal{B}_1\\ q^{-\mathrm{nst}^{\lambda}_{i\frown(l+1)\cup j\frown l}}(q-1)^{-1}\chi^{\lambda}(\nu) & \text{if } \lambda\in\mathcal{B}_2\\ q^{-\mathrm{nst}^{\lambda}_{i\frown(l+1)\cup j\frown l}}q(q-1)^{-1}\chi^{\lambda}(\nu) & \text{if } \lambda\in\mathcal{C}_1\\ -q^{-\mathrm{nst}^{\lambda}_{i\frown l\cup j\frown(l+1)}}(q-1)^{-1}\chi^{\lambda}(\nu) & \text{if } \lambda\in\mathcal{C}_2\\ 0 & \text{otherwise.}\end{cases}$$

We will show that

$$\sum_{\lambda\in\mathcal{A}_1\cup\mathcal{A}_2} q^{\mathrm{nst}_\lambda^\lambda}\begin{bmatrix}n-|\lambda|\\k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}(\chi^\lambda(\nu\cup i\frown l\cup j\frown(l+1))-\chi^\lambda(\nu\cup i\frown(l+1)\cup j\frown l)=0,$$

and likewise for $\mathcal{B}_1\cup\mathcal{B}_2$ and $\mathcal{C}_1\cup\mathcal{C}_2$. For $\mathcal{A}_1\cup\mathcal{A}_2$, note that ϵ gives a bijection between $\mathcal{A}_1$ and $\mathcal{A}_2$. Furthermore, for $\lambda\in\mathcal{A}_1$,

$$\theta(\epsilon(\lambda))=-q^{-\mathrm{nst}_{i\frown l}^{\epsilon(\lambda)}}(q-1)^{-2}\chi^{\epsilon(\lambda)}(\nu)=\frac{-1}{q}\theta(\lambda).$$

As $q^{\mathrm{nst}_{\epsilon(\lambda)}^{\epsilon(\lambda)}}=q^{\mathrm{nst}_\lambda^\lambda+1}$, the claim follows. For both $\mathcal{B}_1\cup\mathcal{B}_2$ and $\mathcal{C}_1\cup\mathcal{C}_2$, we will show that, in each case, θ satisfies the conditions of Lemma 4.10. Note that ϵ gives a bijection between $\mathcal{B}_1$ and $\mathcal{B}_2$. Furthermore, for $\lambda\in\mathcal{B}_2$, we compute

$$\begin{aligned}\theta(\epsilon(\lambda))&=-q^{-\mathrm{nst}_{i\frown l\cup j\frown(l+1)}^{\epsilon(\lambda)}}(q-1)^{-1}\chi^{\epsilon(\lambda)}(\nu)=-q^{-\mathrm{nst}_{i\frown(l+1)\cup j\frown l}^{\lambda}+1}(q-1)^{-1}\chi^{\epsilon(\lambda)}(\nu)\\&=\begin{cases}-q\theta(\lambda), & \text{if } i\frown(l+1)\in\lambda, j'\frown l\in\lambda \text{ for some } j<j'<l,\\ -\theta(\lambda), & \text{if } i\frown(l+1)\in\lambda, j'\frown l\notin\lambda \text{ for all } j'<l.\end{cases}\end{aligned}$$

Furthermore, for λ such that $i\frown(l+1)\in\lambda, j'\frown l\notin\lambda$ for all $j'<l$,

$$\theta(\epsilon(\lambda)\cup l\frown(l+1))=\theta(\epsilon(\lambda))=\theta(\lambda).$$

Therefore, Lemma 4.10 implies that

$$\sum_{\lambda\in\mathcal{B}_1\cup\mathcal{B}_2} q^{\mathrm{nst}_\lambda^\lambda}\begin{bmatrix}n-|\lambda|\\k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}\theta(\lambda)=0.$$

Finally, note that ϵ gives a bijection between $\mathcal{C}_1$ and $\mathcal{C}_2$. For $\lambda\in\mathcal{C}_2$, we compute

$$\begin{aligned}\theta(\epsilon(\lambda))&=q^{-\mathrm{nst}_{i\frown l\cup j\frown(l+1)}^{\epsilon(\lambda)}+1}(q-1)^{-1}\chi^{\epsilon(\lambda)}(\nu)=q^{-\mathrm{nst}_{i\frown(l+1)\cup j\frown l}^{\lambda}+1}\chi^{\epsilon(\lambda)}(\nu)\\&=\begin{cases}-q\theta(\lambda), & \text{if } j\frown(l+1), i'\frown l\in\lambda \text{ for some } i<i'<l,\\ -\theta(\lambda), & \text{if } j\frown(l+1)\in\lambda, i'\frown l\notin\lambda \text{ for all } i'<l.\end{cases}\end{aligned}$$

Furthermore, for λ such that $j\frown(l+1)\in\lambda, i'\frown l\notin\lambda$ for all $i'<l$,

$$\theta(\epsilon(\lambda)\cup l\frown(l+1))=\theta(\epsilon(\lambda))=\theta(\lambda).$$

Thus, by Lemma 4.10,

$$\sum_{\lambda\in\mathcal{C}_1\cup\mathcal{C}_2} q^{\mathrm{nst}_\lambda^\lambda}\begin{bmatrix}n-|\lambda|\\k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}\theta(\lambda)=0.$$

Therefore,

$$\psi_k^n(\mu)-\psi_k^n(\epsilon(\mu))=\sum_{\lambda\in\mathcal{S}_n} q^{\mathrm{nst}_\lambda^\lambda}\begin{bmatrix}n-|\lambda|\\k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}\theta(\lambda)=0.$$

$\square$

We can now prove Proposition 4.8

Proof of Proposition 4.8. Fix $\mu \in \mathcal{S}_n$, and let $m = |\mu|$. Let

$$\nu_m = 1 \frown 2 \frown \ldots \frown m \frown (m+1).$$

We will show that

$$\psi_k^n(\mu) = \psi_k^n(\nu_m).$$

Consider the arc $a \frown b \in \mu$ with a minimal. Then,

$$1 \frown 2 = \epsilon_2^R \circ \epsilon_3^R \circ \cdots \circ \epsilon_{b-1}^R \circ \epsilon_1^L \circ \epsilon_2^L \circ \cdots \circ \epsilon_{a-1}^L (a \frown b).$$

Moreover, by Lemma 4.9

$$\psi_k^n(\mu) = \psi_k^n(\epsilon_2^R \circ \cdots \circ \epsilon_{b-1}^R \circ \epsilon_1^L \circ \cdots \circ \epsilon_{a-1}^L(\mu)).$$

We may repeat this procedure to move the arc in μ with the next smallest left endpoint to the arc $2 \frown 3$, and so on. Therefore,

$$\psi_k^n(\mu) = \psi_k^n(1 \frown 2 \frown \ldots \frown m \frown (m+1)) = \psi_k^n(\nu_m),$$

as desired. □

We now state and prove Proposition 4.11.

Proposition 4.11. *Suppose $\mu \in \mathcal{S}_n$ and $\nu \in \mathcal{S}_{n-1}$ satisfy $|\mu| = |\nu| - 1$. Then,*

$$\psi_k^n(\mu) = \psi_k^{n-1}(\nu).$$

Proof of Proposition 4.11. Let $\mu \in \mathcal{S}_n$, and set $m = |\mu|$. By Proposition 4.8, it suffices to prove the result for

$$\nu = 1 \frown 2 \frown \ldots \frown (m-1) \frown m \text{ and } \mu = \nu \cup (n-1) \frown n.$$

Consider the sum

$$\psi_k^n(\mu) = \sum_{\lambda \in \mathcal{S}_n} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \chi^\lambda(\mu) = \sum_{\lambda \in \mathcal{S}_n} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \chi^\lambda(\nu \cup (n-1) \frown n).$$

Each $\lambda \in \mathcal{S}_n$ falls into exactly one of the following three cases. In each case, we apply the character formula to relate $\chi^\lambda(\mu) = \chi^\lambda(\nu \cup (n-1) \frown n)$ and $\chi^\lambda(\nu)$.

Case 1: If $(n-1) \frown n \in \lambda$, then $\chi^\lambda(\nu \cup (n-1) \frown n) = -1/(q-1)\chi^\lambda(\nu)$.
Case 2: If $j \frown n \in \lambda$ for some $j < n-1$, then $\chi^\lambda(\nu \cup (n-1) \frown n) = \chi^\lambda(\nu) = 0$.
Case 3: If $j \frown n \notin \lambda$ for all $j < n$, then $\chi^\lambda(\nu \cup (n-1) \frown n) = \chi^\lambda(\nu)$.

Thus, we can rewrite the sum as

$$\psi_k^n(\mu) = \frac{-1}{q-1} \sum_{\substack{\lambda \in \mathcal{S}_n: \\ (n-1) \frown n \in \lambda}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \chi^\lambda(\nu) + \sum_{\substack{\lambda \in \mathcal{S}_n: \\ j \frown n \notin \lambda \forall j}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \chi^\lambda(\nu).$$

We rewrite the first sum as

$$\frac{-1}{q-1} \sum_{\substack{\lambda \in \mathcal{S}_n: \\ (n-1) \frown n \in \lambda}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| \\ k - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \chi^\lambda(\nu) = - \sum_{\lambda \in \mathcal{S}_{n-1}} q^{\mathrm{nst}_\lambda^\lambda} \begin{bmatrix} n - |\lambda| - 1 \\ k - |\lambda| - 1 \end{bmatrix}_{\mathcal{P}(\lambda \cup (n-1) \frown n)} \chi^\lambda(\nu)$$

and the second sum as

$$\sum_{\substack{\lambda\in\mathcal{S}_n:\\ j\frown n\notin\lambda\forall j}} q^{\mathrm{nst}_\lambda^\lambda}\begin{bmatrix} n-|\lambda| \\ k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}\chi^\lambda(\nu)=\sum_{\lambda\in\mathcal{S}_{n-1}} q^{\mathrm{nst}_\lambda^\lambda}\left(\begin{bmatrix} n-|\lambda|-1 \\ k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}+\begin{bmatrix} n-|\lambda|-1 \\ k-|\lambda|-1\end{bmatrix}_{\mathcal{P}(\lambda)}\right)\chi^\lambda(\nu).$$

Noting that $\mathcal{P}(\lambda\cup(n-1)\frown n)$ and $\mathcal{P}(\lambda)$ have the same poset binomial coefficients, we conclude that

$$\psi_k^n(\mu)=\sum_{\lambda\in\mathcal{S}_{n-1}} q^{\mathrm{nst}_\lambda^\lambda}\begin{bmatrix} n-|\lambda|-1 \\ k-|\lambda|\end{bmatrix}_{\mathcal{P}(\lambda)}\chi^\lambda(\nu)=\psi_k^{n-1}(\nu),$$

as desired. □

Theorem 4.5 follows immediately.

Proof of Theorem 4.5. Let $\mu\in\mathcal{S}_n$. Then, Proposition 4.11 implies

$$\psi_k^n(\mu)=\psi_k^{n-|\mu|}(1),$$

and by Proposition 4.6,

$$\psi_k^n(\mu)=\psi_k^{n-|\mu|}(1)=\begin{bmatrix} n-|\mu| \\ k\end{bmatrix}_{\mathcal{T}},$$

as desired □

5. Restrictions of Rainbow Supercharacters

Given an $m<n$ and a subset $A\subset\{1,\dots,n\}$ with $|A|=m$, there is a natural embedding of U_m as a subgroup of U_n given by the function

$$\iota: U_m\longrightarrow U_n,\quad \iota(u)_{ij}=\begin{cases} u_{ij}, & \text{if } i,j\in A \text{ or } i=j,\\ 0, & \text{otherwise.}\end{cases}$$

When we are considering such subgroups of U_n, we use the notation U_A to indicate a copy of U_m equipped with the above map into U_n. Given a supercharacter χ^λ of U_n, we consider the problem of writing the restriction $\mathrm{Res}_{U_A}^{U_n}(\chi^\lambda)$ as a sum of supercharacters of U_A. In [9], an algorithm is given for computing such restrictions. However, the resulting coefficients are not well understood in general. We use the characters ψ_k^n to describe the coefficients arrising from restrictions of a particular family of supercharacters, which we call rainbow supercharacters. These supercharacters are of interest, as they seem to exhibit many of the features that make the general restriction problem hard. The rainbow supercharacter with l arcs is the (pointwise) product

$$\chi^{1\frown(n+2)}\odot\cdots\odot\chi^{1\frown(n+2)}=\chi^{1\frown(n+2)\odot l}\in\mathcal{S}_{n+2}.$$

The following theorem is a consequence of Theorem 4.5.

Theorem 5.1. *For each $l\in\mathbb{N}$,*

$$\mathrm{Res}_{U_{[2,n+1]}}^{U_{[1,n+2]}}(\chi^{1\frown(n+2)\odot l})=(q-1)^l\sum_{j=0}^{l}\left(\prod_{i=0}^{j-1}(q^{l-i}-1)\right)\psi_j^n.$$

To prove this, we will use the following lemma, which appears in chapter 4 of [8].

Lemma 5.2. *Let $n \in \mathbb{N}$. Then, the identity*

$$x^n = \sum_{j=0}^{l} \left(\prod_{i=0}^{j-1} (x - q^i) \right) \begin{bmatrix} n \\ j \end{bmatrix}_q$$

holds in the polynomial ring $\mathbb{Z}[x, q]$.

Proof of Theorem 5.1. Fix a $\mu \in \mathcal{S}_n$. We evaluate both sides of the claimed equality at μ. By the character formula,

$$\mathrm{Res}^{U_{[1,n+2]}}_{U_{[2,n+1]}}(\chi^{1\frown(n+2)\odot l})(\mu) = (q-1)^l q^{l(n-|\mu|)}.$$

On the other side, we have

$$\begin{aligned}(q-1)^l \sum_{j=0}^{l} \left(\prod_{i=0}^{j-1} (q^{l-i} - 1) \right) \psi^n_j(\mu) &= (q-1)^l \sum_{j=0}^{l} \left(\prod_{i=0}^{j-1} (q^{l-i} - 1) \right) \begin{bmatrix} n - |\mu| \\ j \end{bmatrix} \\ &= (q-1)^l \sum_{j=0}^{l} \left(\prod_{i=0}^{j-1} (q^{l} - q^i) \right) \begin{bmatrix} n - |\mu| \\ j \end{bmatrix}_q .\end{aligned}$$

By Lemma 5.2, we can expand the monomial $x^{n-|\mu|}$ as

$$x^{n-|\mu|} = \sum_{j=0}^{l} \left(\prod_{i=0}^{j-1} (x - q^i) \right) \begin{bmatrix} n - |\mu| \\ j \end{bmatrix}_q .$$

Therefore,

$$(q-1)^l \sum_{j=0}^{l} \left(\prod_{i=0}^{j-1} (q^{l} - q^i) \right) \begin{bmatrix} n - |\mu| \\ j \end{bmatrix}_q = (q-1)^l q^{l(n-|\mu|)},$$

as desired. □

Theorem 4.5 also gives the following expression for the decomposition of the restriction of rainbow supercharacters.

Corollary 5.3. *For each l, the restriction of the rainbow supercharacter with l arcs has the following expansion in terms of the supercharacters of U_n.*

$$\mathrm{Res}^{U_{[1,n+2]}}_{U_{[2,n+1]}}(\chi^{1\frown(n+2)\odot l}) = (q-1)^l \sum_{\lambda \in \mathcal{S}_n} \left(\sum_{j=0}^{l} \prod_{i=0}^{j-1} (q^{l-i} - 1) q^{nst^{\lambda}_{\lambda}} \begin{bmatrix} n - |\lambda| \\ j - |\lambda| \end{bmatrix}_{\mathcal{P}(\lambda)} \right) \chi^{\lambda}.$$

References

[1] Marcelo Aguiar, Carlos André, Carolina Benedetti, Nantel Bergeron, Zhi Chen, Persi Diaconis, Anders Hendrickson, Samuel Hsiao, I. Martin Isaacs, Andrea Jedwab, Kenneth Johnson, Gizem Karaali, Aaron Lauve, Tung Le, Stephen Lewis, Huilan Li, Kay Magaard, Eric Marberg, Jean-Christophe Novelli, Amy Pang, Franco Saliola, Lenny Tevlin, Jean-Yves Thibon, Nathaniel Thiem, Vidya Venkateswaran, C. Ryan Vinroot, Ning Yan and Mike Zabrocki, Supercharacters, symmetric functions in noncommuting variables, and related Hopf algebras, *Advances in Mathematics*, **229** (2012) 2310–2337

[2] Carlos A.M. André, Basic Characters of the Unitriangular Group, *Transactions of the American Mathematical Society*, **130** (2002), 1943–1954.

[3] Ery Arias-Castro, Persi Diaconis and Richard Stanley, A Super-Class Walk on Upper-Triangular Matrices, *Journal of Algebra*, **278** (2004), 739–765.
[4] Persi Diaconis and I.M. Isaacs, Supercharacters and Superclasses for Algebra Groups, *Transactions of the American Mathematical Society*, **360** (2008), 2359–2392.
[5] Ander Hendrickson, Supercharacter theories of cyclic p-groups, http://www.cord.edu/faculty/ahendric/ (2008).
[6] I. Martin Isaacs, *Character Theory of Finite Groups*, Dover, New York (1994).
[7] Gordon James and Martin W. Liebeck, *Representations and Characters of Finite Groups*, Cambridge (2001).
[8] V. Kac and P. Cheung, *Quantum Calculus*, Universitext, Springer (2002).
[9] Nathaniel Thiem, Branching rules in the ring of superclass functions of unipotent upper-triangular matrices, *Journal of Algebraic Combinatorics*, **31** (2010), 267–298.
[10] Nathaniel Thiem and V. Venkateswaran, Restricting supercharacters of the finite group of unipotent uppertriangular matrices, *Electronic Journal of Combinatorics*, **16** (2009), paper number 23.