Scientific Report No. 53

ON OPTIMUM EXCITATION OF GROUND WAVES

by

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December 1979

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Acknowledgement

The authors certainly enjoyed the numerous discussions they had with Prof. J. R. Wait on this subject. It is his earlier calculation of fields on a spherical earth due to a horizontal dipole that motivated us to look into this problem.

This project was sponsored by Rome Air Development Center.
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1. **Introduction**

It is well known that at radio frequencies, a flat earth as modelled by a homogeneous, conducting half-space, admits a particular solution called the Zenneck wave. This is a bound wave which propagates along the earth surface with a complex propagation constant given by $k_0 \lambda_p$; $\lambda_p = n/(1+n^2)^{1/2}$ where $k_0 = \omega(\mu_0 \varepsilon_0)^{1/2}$ is the free-space wave number and $n = (\varepsilon_r - i\sigma/\omega\varepsilon_0)^{1/2}$ is the complex refractive index of earth, having a relative permittivity $\varepsilon_r$, conductivity $\sigma$, and permeability $\mu_0$ [Collin, 1960; Barlow and Brown, 1962]. A time factor of $\exp(i\omega t)$ has been assumed. For a sufficiently low angular frequency $\omega$, attenuation of this wave along the earth surface being equal to the imaginary part of $k_0 \lambda_p$, can be very small. The fields of this wave are confined to a region near the earth surface as the field strength associated with this wave can be shown to decay exponentially in air at a rate of $\text{Im}(k_0/n)$ away from the earth surface. Because of these remarkable properties, many suggestions have been made in the past for launching of the Zenneck wave [Goubau, 1951; Barlow, 1967; Miller and Deadrick, 1973; Miller et al., 1977; Hill and Wait, 1978; Wait and Hill, 1979]. Related launching problems have been considered by Millar [1968]. Questions remain however, as to whether the Zenneck wave mode is the optimum situation to be operated at, and furthermore, whether one can actually devise a realistic scheme for an efficient, if not outright exclusive, excitation of this mode. It is our purpose in this paper to provide a partial answer to this complicated, yet intriguing problem.
2. **Zenneck waves and ground waves**

We know theoretically that a Zenneck wave mode is discrete and orthogonal to the continuous modes of the radiation spectrum, and as such, can be excited at least in principle by an impressed field over a vertical aperture of infinite height [Goubau, 1951; Hill and Wait, 1978]. However, because the wave is guided by an open structure and because it spreads out indefinitely (although decaying exponentially) into both the air and earth regions, a physically realizable source distribution of finite nature can only provide partial excitation of this mode and must also excite modes in the radiation spectrum. Furthermore, because the field associated with the radiation spectrum characteristically decays in an algebraic manner while the Zenneck wave field decays exponentially, the importance of the latter is restricted to a finite range, since the former will eventually prevail. Admittedly, this range depends upon how efficient the excitation is as well as the decay rate.

Looking at the question realistically, it is not difficult to understand why a vertical aperture excitation is not very attractive because of the physical dimensions involved. We can take the case of earth with conductivity $\sigma = 10^{-2}$ mhos/m and relative permittivity $\varepsilon_r = 10$ as an example. The spread of the Zenneck wave field in the air region is approximately $d = [\text{Im}(k_0/n)]^{-1} = 42.7$ meters for $f = 10$ MHz, and 904.4 meters at 1 MHz. (This dimension is about 20 times larger for seawater where $\sigma = 4$ mhos/m.) Because of this, practical excitation schemes are likely to involve line-sources (or arrays thereof) rather than two-dimensional apertures.
Departing from this line of reasoning for a moment, we may note that the most elementary excitation in an earth-related problem is perhaps that of an electrically small Hertzian dipole. Regardless of the dipole orientation, it is known in this case that the total field consists of both the Zenneke wave and radiation contributions. In fact, because the excitation is not particularly tailored to either type of wave, it is customary to consider the field as a whole without alluding to the individual contributions. In the region near the earth surface, the vertical electric field in the far zone is expressible in terms of the potential \( V(\vec{r}, \vec{r}') \) [Baños, 1966]

\[
V(\vec{r}, \vec{r}') \approx \left( \frac{2e^{-ik_0 R}}{R} \right) W(p) ;
\]

where \( W(p) \) is an attenuation function defined as

\[
W(p) = 1 - i(\pi p)^{\frac{1}{2}} e^{-p} \text{erfc}(ip^{\frac{1}{2}})
\]

where

\[
R = |\vec{r} - \vec{r}'| = [(\vec{\rho} - \vec{\rho}')^2 + (z + z')^2]^{\frac{1}{2}} ; \quad p = -ik_0[R - \lambda_p|\vec{\rho} - \vec{\rho}'| + (\lambda_p/n)(z+ z')]
\]

are respectively the physical distance and the so-called Sommerfeld-Norton numerical distance between the observation point \( \vec{r} \) and the image of the source whose location is \( \vec{r}' = (\rho', z') \); \( |\vec{\rho} - \vec{\rho}'| \) is the radial distance between the projections of these two points in the horizontal plane; \( \text{erfc} \) is the complementary error function and \( k_0 = \frac{\lambda_p}{n(1 + n^2)^{\frac{1}{2}}} \) is the complex propagation constant of the Zenneke wave. In the region where the numerical distance is moderate, the field expression given by (1) is commonly referred to as the ground wave. Although the field structure of the total ground wave bears little resemblance to that of a Zenneke wave, the influence of
the latter exhibits itself unmistakably in the expression for the numerical
distance, and hence in the ground wave.

This discussion points up the important fact that, although the Zenneck
wave can, in principle, exist alone, it is usually accompanied by an entire
spectrum of radiation modes (i.e., a contribution from a branch cut integral
in the complex \( \lambda \)-plane) to the extent that the total field no longer possesses
the same characteristics. We may infer from this that so long as the physical
constraints or the size of the source remain a key concern, a more relevant
question to pose is not how to excite the Zenneck wave itself, but whether
the total ground wave field can be enhanced with certain source distributions.

To obtain some physical insight into the structure of the ground-wave
of a dipole source, let us consider how the potential \( V(\vec{r}, \vec{r}') \) varies in the
far field. Suppose that the coordinates \( \vec{r}' = (\vec{\rho}', z') \) of the source point
are in the neighborhood of the origin, while the observation point,
\( r = (\vec{\rho}, z) \) is allowed to vary in the neighborhood of some fixed point
\( \vec{r}_0 = (\vec{\rho}_0, z) \) in the same horizontal plane. We define the far field such
that

\[
R_o >> |\vec{r}'|; \quad R_o >> |\vec{\rho} - \vec{\rho}_0|
\]

where \( R_o = [|\vec{\rho}_0|^2 + (z + z')^2]^{1/2} \). Then, as is shown in Appendix A,

\[
\begin{align*}
V(\vec{r}, \vec{r}') &= \left(2eR_o \right)^{-i\lambda o R_o} \left\{ e^{-ik_o\theta_o \vec{\rho}_0 \cdot \vec{\xi}} + V_z e^{-ik_o\lambda \vec{\rho}_0 \cdot \vec{\xi}} \\
&+ \int_0^1 \frac{1}{V_c(u)e^{i(k_o[2u^2\cos\theta_o + \lambda (1-u^2)]\vec{\rho}_0 \cdot \vec{\xi})}} du \right\} 
\end{align*}
\]

where
\[ V_z = -i(\pi p_0)^{\frac{1}{2}} e^{-p_0} \]
\[ V_c(u) = -2p_0 e^{-p_0(1-u^2)} \]
\[ \theta_o = \tan^{-1}[(z+z')/|\vec{\rho}_o|] \]

and
\[ p_0 = -ik_o[R_o - \lambda_p|\vec{\rho}_o| + (\lambda_p/n)(z + z')] \]

is the numerical distance corresponding to the point \( \vec{r}_o \). Finally, 
\[ \xi = \vec{\rho} - \vec{\rho}_o - \vec{\rho}' \text{ and } \vec{\rho}_o = \vec{\rho}/|\vec{\rho}_o|. \]
We interpret equation (5) as follows: the dipole source produces a ground wave made up of i) a space wave whose (horizontal) propagation constant is \( k_o \cos \theta \); ii) a Zenneck wave with a propagation constant \( k_o \lambda_p \), and a continuous bundle of waves with propagation constants varying from \( k_o \lambda_p \) (at \( u = 0 \)) to \( k_o \cos \theta \) (at \( u = 1 \)). Since for large \( |n| \) and near-endfire observation points we will have both \( |\vec{\rho}' - 1| \ll 1 \) and \( |\cos \theta - 1| \ll 1 \), all spectral contributions to the ground wave are spaced closely together.

Now, conventional waveguide theory suggests that strong excitation of mode fields is possible when a travelling-wave current source of nearly the same propagation constant is employed. Thus, a horizontal linear current source with a propagation constant between \( \lambda_p \) and \( \cos \theta \) seems a likely means to provide the desired field enhancement in the ground wave. Indeed, an early method used to obtain low-angle space wave radiation was the so-called Beverage or wave antenna [Beverage et al., 1923]. This antenna consists of a finite (but long) horizontal wire of length \( 2\lambda \) located close to the earth's surface (Fig. 1). The excitation of the ground wave by this antenna for the case when \( 2\lambda \) is not too large has been investigated approximately by Jenssen [1950], Wait [1954], and Than [1957], but no indication is given as to how
large $2\lambda$ is permitted to be. We shall examine the excitation of the ground wave by this antenna in somewhat more detail in this paper.

3. **Ground wave excitation by a finite, horizontal current line source**

Assume that the form of the current in the wire of Fig. 1 is $\tilde{a}_x I_0 \exp(-ik_0 c x')$ on $-\lambda \leq x' \leq \lambda$ at $y' = 0$ and a height $z'$. Here $\tilde{a}_x$ is the unit vector in the $x$-direction, $I_0$ is the (complex) amplitude of the current and $k_0 \alpha$ is the (as yet unspecified) complex propagation constant of the current wave. Note that we have assumed that any reflected wave on the wire has been suppressed by an appropriate termination. If desired, such a reflected wave can be accounted for in a straightforward manner by superposition of a term with $\alpha \rightarrow -\alpha$.

From [Baños, 1966] and eqn. (5) we obtain the following expression for the vertical electric field in the endfire direction ($y = 0$) and on the surface of the earth ($z = 0$):

$$E_z = \left( \frac{k_0 I_0}{4\pi \varepsilon_0} \right) \frac{\partial^2}{\partial x \partial z} V_H(\tilde{\rho}, z + z') \bigg|_{y=0}^{y=0}$$

where

$$V_H(\tilde{\rho}, z + z') = \int_{-\lambda}^{\lambda} e^{-ik_0 \alpha x'} V(\tilde{r}, \tilde{r}') \bigg|_{y'=0} \, dx'$$

(9)

If the length of the line source is small compared to the observation distance, i.e., $|\tilde{r}| \gg \lambda$, then we may use the approximate expression for $V$ given in (5). Taking the observation point $\tilde{\rho} = \tilde{\rho}_0$ for simplicity, we have

$$V(\tilde{r}, \tilde{r}') \bigg|_{y'=0} = \left( \frac{2e^{-ik_0 R_0}}{k_0 R_0} \right) \left\{ e^{ik_0 x' \cos \theta} + V_z e^{ik_0 \lambda \rho x'} + \int_0^1 V_c(u)e^{ik_0 [u^2 \cos \theta + (1-u^2) \lambda \rho] x'} \, du \right\}$$

(10)
where now $|\rho_0| = x$, so that

$$R_0 = [x^2 + (z + z')^2]^{1/2}; \quad p_0 = -ik_0[R_0 - \lambda_p x + (\lambda_p/n)(z + z')]$$

$$\theta = \tan^{-1}[(z + z')/x]$$

(10a)

provided that $x > 0$, so that $R_0$ is the distance of the observation point from the image of the center of the line source, $(0,0,-z')$. Substitution of (10) into (9) yields, after some manipulation, the following result:

$$V_H(\rho;z + z') = \left[ 4\kappa \frac{e^{-ik_0 R_0}}{k_0 R_0} \right] \left[ \text{sinc}[k_0 \kappa(\alpha - \cos \theta)] - \frac{1}{\kappa_0} \text{sinc}[k_0 \kappa(\alpha - \lambda_p)] \right]$$

$$- \sum_{m=0}^{\infty} \left[ \frac{k_0 \kappa(\lambda_p - \cos \theta)}{(2m+1)!} \right]^{2m+1} A_{2m}(\tau;p_0) \right]$$

$$\tau = (\lambda_p \cos \phi - \alpha)/(\lambda_p - \cos \theta)$$

(11)

where $\text{sinc} x = (\sin x)/x$ and $A_j(\tau;p_0)$ for $j = 0,1,2\ldots$ are defined as

$$A_j(\tau;p_0) = i2 \left( \kappa_0 \right)^{1/2} \int_0^1 e^{\rho_0 u^2} (u^2 - \tau)^j du$$

(12)

In particular, $A_0 = \text{erf} (ip_0^{1/2})$ where $\text{erf}$ is the error function [Abramowitz and Stegun, 1964]. Higher order $A_{2m}$ can be obtained in terms of $A_0$ from the recursion formula

$$2p_0 A_{j+1} = [(2j+1) + 2\tau p_0]A_j - 2j \tau A_{j-1} - i \left( \frac{p_0}{\kappa_0} \right)^{1/2} p_0(\tau-1)^j$$

(13)

provided that $A_{-1}$ is defined as 0.

If we assume $|n^2| >> 1$ and $\cos \theta = 1$, it is not difficult to show that the leading terms of $E_z$, as we substitute (11) into (8), come mainly from
differentiation of the two exponentials, \( \exp(-ik_0 R_o) \) and \( \exp(-ip_0 - ik_0 R_o) \). Noting that

\[
\frac{\partial^2}{\partial x \partial z} e^{-ik_0 R_o} = -k_0^2 \sin \theta \cos \theta e^{-ik_0 R_o},
\]

\[
\frac{\partial^2}{\partial x \partial z} e^{-p_0 - ik_0 R_o} = \frac{k_0^2 R_0^2}{n} e^{-p_0 - ik_0 R_o},
\]

we can obtain an approximate expression for \( E_z \) as,

\[
E_z = \frac{4}{n^2} \omega \varepsilon_0 k_0^2 I_0[W_d(p_0) + \Delta W(p_0; \alpha)] \left( \frac{e^{-ik_0 R_o}}{R_o} \right)
\]

(14)

where \( \varepsilon_0 = 120 \pi \) ohms is the free-space characteristic impedance;

\[
W_d(p_0) = -\sin \theta \cos \theta - i \frac{\lambda_p^2}{n} e^{-p_0} [1 - \text{erf}(ip_0)],
\]

(15)

is the attenuation function of a horizontal dipole source; and

\[
\Delta W(p_0; \alpha) = [1 - \text{sinc} k_0 \lambda(\alpha - \cos \theta)] \sin \theta \cos \theta
\]

\[
+ \frac{i \lambda_p^2}{n} (\pi p_0)^{1/2} e^{-p_0} [1 - \text{sinc} k_0 \lambda(\alpha - \lambda_p)]
\]

\[
+ \sum_{m=1}^{\infty} [ik_0 \lambda(\lambda_p - \cos \theta)]^{2m} \frac{1}{(2m+1)!} A_{2m}(\tau; R_0)
\]

(16)

is the deviation from the dipole result due to a current travelling wave with a propagation constant \( \alpha \). Note that we should set \( z = 0 \) in \( R_o \), \( p_0 \) and \( \theta \) as they appear in (10a). In many cases of practical interest, we may find the terms \( |k_0 \lambda(\alpha - \cos \theta)| \) and \( |k_0 \lambda(\alpha - \lambda_p \cos \theta)| \) to be equal to or less than one, even though the source itself may be many wavelengths in length, i.e. \( k_0 \lambda \gg 1 \). Provided that this is the case, \( \Delta W \) can be further simplified to
\[ \Delta W(\alpha) = \frac{k_0^2 \chi^2}{6} \{ (\alpha - \cos \theta)^2 \sin \theta \cos \theta \]
\[ + i \frac{\lambda^2}{n} (\pi \rho_0)^{\frac{1}{2}} e^{-\rho_0[2(\alpha - \lambda_p)^2 - (\lambda_p - \cos \theta)^2 A_2(\tau; \rho_0)]} \} \] (17)

where \( A_2 \) is again defined according to (12).

Equation (14), without the \( \Delta W \) term present, is obtained by Jenssen [1950], Wait [1954] and Thän [1957]. Expression (17) enables us to determine the conditions under which their approximations will be accurate.

4. Field enhancement

We now proceed to investigate the possibility of field enhancement by varying the propagation constant \( \alpha \) of the current wave. Defining an amplitude function \( F(\alpha; \rho_0) = |W_d + \Delta W_0|^2 \), we can show without any difficulty that the condition \( \frac{\partial F}{\partial \alpha} = 0 \) for a local maximum or minimum reduces to

\[ \text{Re}[W_d + \Delta W]^* \frac{\partial}{\partial \alpha} \Delta W = 0 \] (18)

where "*" is the complex conjugate. An expression for \( (\Delta W)' \) can be obtained directly from the differentiation of (17). After some considerable amount of manipulation, together with the use of the identity given in (13), the following expression is obtained.

\[ (\Delta W)' = \frac{\partial}{\partial \alpha} (\Delta W) = \frac{k_0^2 \chi^2}{3} W_d (\alpha - \alpha_s) ; \] (19)

where

\[ \alpha_s = [Q \cos \theta + (1 - Q) \lambda_p] \] (20)

\[ Q = 1 - \frac{\lambda^2}{n W_d} \{ 1 - i(\pi \rho_0)^{\frac{1}{2}} e^{-\rho_0[2\text{erfc}(i \rho_0)^{\frac{1}{2}} - \frac{1}{2 \rho_0} \text{erfc}(i \rho_0^{\frac{1}{2}})]} \} \] (21)
and \( \text{erfc} = 1 - \text{erf} \) is the complementary error function. We note that in the complex \( \alpha \) plane, \( \alpha_s \) represents a stationary point of the complex function \( \Delta W \) since the first derivative of this function vanishes. Substitution of (19) into (18) then yields, in principle, the necessary condition for determining the optimum value(s) of \( \alpha \). Particularly, if our interest is limited to the region near the stationary point \( \alpha_s \), we can expand the term \( (W_d + \Delta W) \) in a Taylor series around \( \alpha_s \) to obtain the following approximate condition:

\[
\arg(\alpha - \alpha_s) = \frac{\pi}{2} + \nu_s; \quad \nu_s = \arg(W_d + \Delta W_s) - \arg W_d \quad (22)
\]

where \( \arg(Z) \) denotes the argument of a complex constant \( Z \) and \( \Delta W_s \) is a value of \( \Delta W \) evaluated at \( \alpha = \alpha_s \).

As shown in Fig. 2, Equation (22) can be represented by a straight line (labelled SAP) in the complex \( \alpha \) domain. Thus, in addition to the real solution \( \alpha = \alpha_0 \), other complex values of \( \alpha \) along the line SAP in the fourth quadrant can also satisfy the requirement that \( \partial F / \partial \alpha = 0 \). In order to determine whether current waves with these (complex) propagation constants actually produce an optimum field strength, we can first define \( \nu \) as the phase angle between the stationary point \( \alpha_s \) and a neighboring point \( \alpha \), i.e. \( \nu = \arg(\alpha - \alpha_s) \) and show that the Taylor series expansion of the amplitude function \( F(\alpha; p_0) \) around \( \alpha_s \) is given approximately as

\[
F(\alpha; p_0) = |W_d + \Delta W_s|^2 - \frac{k_0^2}{3} |W_d(W_d + \Delta W_s)|^2 \cos 2(\nu - \nu_s); \quad \eta = |\alpha - \alpha_s| \quad (23)
\]

Thus, depending upon the phase angle of \( (\alpha - \alpha_s) \), the stationary point \( \alpha_s \) can be either a local minimum or a local maximum. Similar to the
saddle point in the asymptotic evaluation of complex integrals, the region surrounding \( \alpha_s \) can be actually divided by the lines \( \nu = \nu_s \pm \pi/4 \) into descending and ascending regions as labelled by \( S_d \) and \( S_a \) in Fig. 2. In the descending region, \( F(\alpha;p_o) \) reaches a local maximum as \( \alpha \) approaches \( \alpha_s \), while in the ascending region \( F(\alpha;p_o) \) is a local minimum. As expected the steepest ascent path (SAP) is given by the condition \( 2(\nu - \nu_s) = \pm \pi \) which is precisely the same condition as we previously derived in (22) by requiring that \( \partial F/\partial \alpha = 0 \). Specifically, the amplitude function in this case can be written as

\[
F(\alpha;p_o) = |W_d + \Delta W_s|^2 + k_o^2 \xi^2 \eta^2 |W_d(W_d + \Delta W_s)|/3 .
\] (24)

From the above discussion, it is now apparent that field enhancement indeed can be achieved by a travelling current wave having a propagation constant given as

\[
\alpha = \alpha_s + i\eta \exp(i\nu_s) ; \text{ for a real and positive } \eta
\] (25)

We note that in the case when the deviation from the dipole attenuation function is small, i.e. \( |\Delta W| \ll |W_d| \), the value of \( \nu_s \) is also close to zero. Consequently, the value of \( \eta \) is directly related to the rate of decay of the current wave. According to (23), we may therefore conclude that a greater enhancement of the field can be achieved by a larger decay rate, while maintaining the same propagation constant.

5. Optimum excitation: Zenneck wave or space wave?

As is evident in (20) and (21), the location of the stationary point \( \alpha_s \) depends not only on the space wave number \( k_o \cos \theta \) and the Zenneck wave number \( k_o \lambda_p \), but also the actual numerical distance \( p_o \) of the
observation point. For a large numerical distance \(|p_0| \gg 1\), a large argument expansion of \(Q\) yields \(Q \approx 1\) so that

\[
\alpha_s = \cos \theta
\]

which is the component of the space-wave propagation vector along the wire. On the other hand, for a small numerical distance \(|p_0| \leq 1\), a small argument expansion of \(Q\) yields the expression

\[
Q = \frac{n \sin \theta \cos \theta}{n \sin \theta \cos \theta + i\lambda_p^2 (\pi p_0)^\frac{1}{2}}
\]

For most cases of practical interest, the value of \(|n|^2 \sin \theta\) is typically less than one. Consequently, the value of \(Q\) is close to zero so that

\[
\alpha = \lambda_p
\]

which is the propagation constant of the Zenneck wave. Thus, the factor \(Q\) can be considered as a weighting factor which has the effect of shifting the location of the stationary point \(\alpha_s\) (and hence, the condition for field enhancement) from the wave number of a Zenneck wave for a small numerical distance, to the wave number of a space wave for a large numerical distance. We may note in passing that current excitation with a propagation constant of a space wave yields a greater field strength in the region where \(p_0\) is large than that of a Zenneck wave as has also been observed by Wait and Hill [1979] in their treatment of the spherical earth problem.

At this point, one may ask: to what extent can field enhancement be achieved by the method of analysis outlined above? Such a question unfortunately, cannot be answered in a definite manner without extensive numerical computation. What we can discuss however, are the conditions for
field enhancement to occur. From the expression given in (24), it is clear that if \( F(\alpha; p_0) \) is to be significantly different from the dipole result, the magnitude of \( \Delta W_s \) must be comparable to \( W_d \) and/or the decay constant \( \eta \) must not be small. In addition, we must be able to maintain the same propagation constant as determined by the values \( \nu_s \) and \( \alpha_s \) from (25). While the realizability of a current source with a specified propagation and decay constant will be discussed in the next section, some conclusions can be made regarding the magnitude of \( \Delta W \). In particular, we can show from (17) that the expression for \( \Delta W \) at \( \alpha = \alpha_s \) is

\[
\Delta W_s = \frac{k_0^2 \omega^2}{6} (\lambda_p - \cos \theta)^2 \left[(1-Q)^2 \sin \theta \cos \theta + \frac{i \lambda_p^2}{n} (\pi p_o)^{1 \frac{1}{4}} e^{-p_o [Q^-A_2(Q;p_0)]} \right]
\]

which, upon the use of the identity in (13) and after some considerable algebra, can be rewritten in the following form

\[
\Delta W_s = -\lambda_p^2 \left[ -\frac{\lambda_p^2}{n} + (Q + \frac{3}{2p_o}) W_c Q + (1 + \frac{3}{2p_o}) \sin \theta \cos \theta \right]
\]

(26)

where

\[
\lambda_p^2 = -ik_0 \lambda(\lambda_p - \cos \theta)
\]

(27)

which can be interpreted roughly as a numerical distance associated with the source length, \( \lambda \). As we mentioned before, for the observation range where the Sommerfeld-Norton numerical distance is large, i.e. \( |p_o| \gg 1 \), the value of \( Q \) approaches unity so that \( \Delta W_s \) as determined from (26) is simply

\[
\Delta W_s = \frac{\lambda_p^2}{3} \left( \frac{\lambda_p^2}{n} \right)
\]

(28)

In deriving (28), the large argument expansion \( W_d = -\sin \theta \cos \theta - \lambda_p^2/n \)
was used. Now since \( |n^2 \sin \theta| \) is typically small, the ratio of \( \Delta W_s \) to \( W_d \) approximately equals \( -\frac{\lambda^2}{3} \). Hence, the enhancement of field indeed seems possible if the numerical length of the source is equal to or greater than unity. Physically, this means the phase difference between a space wave and a Zennecke wave over the length of the source must not be small. On the other hand, if the observation is made over a range where the Sommerfeld-Norton numerical distance is in the order of unity, i.e. \( |p_0| \approx 1 \), the value of \( Q \) can become small so that from (26), we have

\[
\frac{\Delta W_s}{W_d} = i \frac{\lambda^2}{6} \left( \frac{\lambda^2}{p_0} \right) (\pi p_0)^{-\frac{1}{4}}
\]

(29)

where the condition \( |n^2 \sin \theta| \ll 1 \), is again used. Now since \( |p_0| \approx 1 \) and \( |\lambda^2/p_0| = \lambda/R_0 \) where \( R_0 \) is the physical distance of the observation point, we conclude that the value \( |\Delta W_s| \) in this case is necessarily very small compared with the dipole attenuation function, \( W_d \). Consequently, any enhancement in the field strength has to be very marginal.

6. **Propagation and attenuation of a current wave: A capacitive leaky wave cable**

The remaining question about field enhancement by a horizontal current wave concerns the realizability of a current source with prescribed propagation and attenuation constants. Since we have concluded that most enhancement comes from the space wave region where the Sommerfeld-Norton numerical distance is large, the specific requirement for the current wave is to have a propagation constant \( \alpha \) equal to \( \cos \theta \) (\( \approx 1 \) in the endfire direction). The amplitude function associated with this current can be obtained from (23) as
\[
F(\alpha_s; \rho_o) = \left| W_d \right|^2 \chi_0 (\chi_0 + k_0^2 \eta^2 / 3); \quad \chi_0 = \left| 1 + \frac{k_0^2 \eta^2}{3(\lambda_p - \cos \theta)^2} \right|
\]

where \( \eta \) is the attenuation constant of the current wave. Since the value \( k_0 \eta \) is typically very large, a small increase in \( \eta \) could result in a substantial increase in the amplitude of the electric field in the far zone.

In order to gain some insight into what type of propagation and attenuation value one can expect from a source structure, we have computed the discrete modes of a horizontal, capacitive leaky-wave coaxial cable. Specifically, the theoretical model we used is assumed to consist of a perfectly conducting inner conductor of radius \( b \) (meters); a leaky outer conductor of radius \( a \) (meters) which has a transfer impedance of \( Z_T = (+i\omega C)^{-1} \); an insulator of dielectric constant \( \eta \) between the two conductors. The actual value of the capacitance depends, of course, upon the detailed coupling mechanism of a slotted cable. The cable is located \( h \) meters above the earth and the earth's refractive index is \( n \). All fields vary as \( \exp(-i\omega k_0 z + i\omega t) \) where \( \alpha = \alpha_r - i\alpha_i \) is the normalized (complex) propagation constant of a discrete mode. Its value is obtained by solving the modal equation [Plate et al., 1978]

\[
0 = M(\alpha) = \zeta_1^2[H_0^{(2)}(\zeta_1 A) - H_0^{(2)}(\zeta_1 H)] + P(\alpha; H) - \alpha^2 Q(\alpha; H) + 4\omega \varepsilon_0 Z(\alpha) / k_0^2
\]

where

\[
P(\alpha; H) = \frac{i2}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-u_1 H)}{u_1 + u_2} \, d\lambda
\]

\[
Q(\alpha; H) = \frac{i2}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-u_1 H)}{u_1^2 + u_2^2} \, d\lambda
\]
\[
\begin{align*}
\zeta_1 &= (1 - \alpha^2)^{\frac{1}{2}}; \quad \zeta_2 = (n^2 - \alpha^2)^{\frac{1}{2}}; \quad 0 < \arg \zeta_1, \zeta_2 < \pi \\
A &= k_0 a; \quad H = 2k_0 h \\
Z(\alpha) &= \frac{Z_T Z_b(\alpha)}{Z_T + Z_b(\alpha)} \\
Z_T &= (+ i\omega C)^{-1} \\
Z_b(\alpha) &= \frac{i(n_b^2 - \alpha^2)}{2\pi \varepsilon_0 \omega n_b} \ln \left( \frac{a}{b} \right)
\end{align*}
\]

We calculated the value of the propagation constant $\alpha$ for the parameters: $f = 10 \text{ MHz}$; $a = 10 \text{ cm}$; $b = 4 \text{ cm}$; $n_b = 1.01$; and $n = 5.3 + 10.95$. Figure 3 shows a plot of $\alpha$ as $C$ is varied over a wide range for $h = 5$ meters. We note that in the absence of the earth we would expect to find a bifilar mode at $\alpha = 1.01$ for $C = \infty$; this decreases as $C$ is decreased and eventually crosses the branch cut at unity to the improper Riemann sheet to become an improper or leaky mode.

In the presence of the earth we found three modes at $C = \infty$. One mode is the bifilar mode at $\alpha = 1.01$. The other two modes are the modes that would exist on a bare wire over the earth. From Figure 3 it is apparent that as $C$ is decreased the bifilar mode approaches unity but also becomes lossy and thus remains a proper mode for all values of $C$. One of the other modes does cross the branch cut to become improper which is expected since only two modes may exist at $C = 0$. Since this mode becomes improper it is of little interest to us. The other two modes are labeled "internal coaxial" and "external, earth attached" modes because of their characteristics at $C = \infty$. 
Table 1 gives the values of the current on the inner conductor $I_i$ and the current on the outer conductor $I_o$ for a total external current on the cable $I_T$ of unity at a cable height of 5 meters. The field strength outside the cable is approximately proportional to $|I_T|$ while the field strength inside the cable is proportional to $|I_i|$. Hence the relative value of the field strength inside the cable to that on the outside of the cable is $|I_i|$. This is a measure of the bifilar nature of a mode. We note that for $C = 0.8$ pf, the propagation constant of the internal coaxial mode is close to unity, and the associated attenuation constant is about $2 \times 10^{-3}$. Thus, for a current source 3 km in length, the improvement due to the attenuation term in (30) has the value of $k_0^2 \lambda^2 \eta^2 / 3 \approx 0.53$ which could be significant in practice.

In closing, we should remark that the above result is based upon the assumption that the transfer impedance of the leaky cable is capacitive and its value can vary as desired. We have not investigated however how certain capacitance values can be achieved using a realistic structure with periodic slots. The result reported here merely indicates what could be expected when a realistic structure is used.
TABLE I

Innur and outer currents ($I_i$ and $I_o$) for a total external current of unity on a
capacitive leaky coaxial cable located above a dissipative earth. Here $f = 10$ MHz,
a = 10 cm, $b = 4$ cm, $b = 1.01$, $n = 5.5 + 10.95$, and $h = 5$ meters.

<table>
<thead>
<tr>
<th>C (pF)</th>
<th>$I_i$</th>
<th>$I_o$</th>
<th>$I_i$</th>
<th>$I_o$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>25.44</td>
<td>.29</td>
<td>.14</td>
<td>.05</td>
</tr>
<tr>
<td>1.0</td>
<td>11.26</td>
<td>.32</td>
<td>.29</td>
<td>.07</td>
</tr>
<tr>
<td>0.8</td>
<td>8.32</td>
<td>.34</td>
<td>.37</td>
<td>.09</td>
</tr>
<tr>
<td>0.4</td>
<td>2.71</td>
<td>.26</td>
<td>.83</td>
<td>.20</td>
</tr>
</tbody>
</table>
APPENDIX A

In this appendix, we derive the far-field expression for the variation of the potential function $V(\bar{r}, \bar{r}')$ as $\bar{r}'$ varies near the origin, and $\bar{r} = (\bar{\rho}, z)$ varies near $\bar{r}_0 = (\bar{\rho}_0, z)$. Define the radial distances

$$R = \left[ |\bar{\rho} - \bar{\rho}'|^2 + (z + z')^2 \right]^{\frac{1}{2}},$$

$$R_0 = \left[ |\bar{\rho}_0|^2 + (z + z')^2 \right]^{\frac{1}{2}}$$

(A.1)

Then if $R_0 \gg |\bar{r}'|$ and $R_0 \gg |\bar{\rho} - \bar{\rho}_0|$, 

$$|\bar{\rho} - \bar{\rho}'| \approx |\bar{\rho}_0| + \bar{a}_{\rho_0} \cdot (\bar{\rho} - \bar{\rho}_0 - \bar{\rho}')$$

(A.2)

where $\bar{a}_{\rho_0}$ is the (cylindrical) radial unit vector at the point $\bar{\rho}_0$, and therefore

$$R = R_0 + \bar{a}_{\rho_0} \cdot (\bar{\rho} - \bar{\rho}_0 - \bar{\rho}') \cos \theta_0$$

(A.3)

where $\theta_0$ is the vertical elevation angle of $\bar{r}_0$:

$$\cos \theta_0 = |\bar{\rho}_0|/R_0$$

(A.4)

The numerical distance $p$ from (3) becomes

$$p = p_0 - i k_0 (\cos \theta_0 - \lambda_p) \bar{a}_{\rho_0} \cdot (\bar{\rho} - \bar{\rho}_0 - \bar{\rho}')$$

(A.5)

where $p_0$ is the numerical distance corresponding to $\bar{r}_0$:

$$p_0 = -i k_0 [R_0 - \lambda_p |\bar{\rho}_0| + (\lambda_p/n)(z + z')]$$

(A.6)

Now, from (2),

$$W(p) = 1 - i (\pi p)^{\frac{1}{2}} e^{-p} [1 - \text{erf}(ip^{\frac{1}{2}})]$$

(A.7)
where \( \text{erf}(x) = 1 - \text{erfc}(x) \) is the error function. Using the integral representation for \( \text{erf}(x) \), [Abramowitz and Stegun, 1964],

\[
W(p) - 1 = i(\pi p)^{\frac{1}{2}} e^{-\hat{p} \cdot \hat{r}} - 2p \int_0^1 e^{-p(1-u^2)} du \quad (A.8)
\]

Using (1), (2), (A.3), (A.5) and (A.8), we find the far-field expression for \( V(\vec{r}, \vec{r}') \):

\[
V(\vec{r}, \vec{r}') = \left( \frac{2e^{-ik_o R_o}}{R_o} \right) \left\{ \begin{aligned} &-ik_o \cos \theta_o \bar{a}_{\rho_o} \cdot \vec{\xi} + V_Z e^{-i k_o \rho_o \rho_o} \cdot \vec{\xi} \\ &+ \int_0^1 V_c(u) e^{ik_o [u^2 \cos \theta_o + \lambda_p (1-u^2)] \bar{a}_{\rho_o} \cdot \vec{\xi}} \, du \end{aligned} \right\} \quad (A.9)
\]

where

\[
\vec{\xi} = \vec{\rho} - \vec{\rho}_o - \vec{\rho}'
\]

\[
V_Z = -i(\pi p_o)^{\frac{1}{2}} e^{-p_o}
\]

\[
V_c(u) = -2p_o e^{-p_o (1-u^2)}
\]
REFERENCES


Barlow, H.M. (1967), "Launching a surface wave over the earth," Electron. Lett., 3, 304-305. See also discussions on this paper by J.R. Wait, ibid., 396-397; by P. Knight, G.D. Monteath and H. Page, ibid., 432-433; and replies by the author, ibid., 397, 433.


Fig. 1: Geometry of the Beverage antenna
Fig. 2: Steepest-ascent path (SAP) in the complex $\alpha$-plane
Fig. 3: Mode trajectories as a function of the transfer capacitance of the outer conductor of a coaxial cable for a height of 0.5 meters.