Electron Hydrodynamics with X-momentum conservation

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Thesis Defended: 04/09/2024

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1 Introduction

1.1 Electron Hydrodynamics in Experiment

The flow of electrons in most materials is nearly Ohmic – that is the current density is uniform and proportional to the applied voltage. In these materials the contributions from electron-electron collisions are negligible when compared to electron-ion collisions and even moreso when compared to the contributions from electron-impurity collisions [1]. Historically, the approach in solid-state physics has been to treat these contributions collectively neglecting the nuance between momentum relaxation and conservation – embodied in the Drude model under a single collision time τ [2]. However, as far back as the 1960s it has been suggested that hydrodynamic flow characterized by viscous effects may be observed in ultra-pure, low temperature metals when electron-electron interactions again become significant [3]. A schematic depicting these two types of flows is shown in Figure 1, in the hydrodynamic flow the velocity is maximal at the center and suppressed along the edges. This phenomena went mostly ignored until the early 21st century when rapid advancement in the fabrication of ultra-pure materials enabled experimental detection of these effects. Since then signatures of electron hydrodynamics has been detected in a variety of correlated electron materials such as graphene [4], WTe_2 [5], $PdCoO_2$ [6], WP_2 [7]. The purity of these materials may be observed by the fact that the resistance of the materials is directly proportional to their scattering rates. Experimental evidence for this flow is shown in Figure 2 . This technological advancement and the commercial success of materials such as graphene has created renewed interest in this field and necessitated the development of theories which accommodate the effects of hydrodynamic effects. In this paper we will develop one such theory – but first we will provide a primer on modern hydrodynamics.

1.2 Modern Perspective on Hydrodynamics

Historically hydrodynamics has been synonymous with fluid dynamics – governed by the application of the Navier-Stokes equations. These equations have seen great success seeing applications ranging from conventional fluids to quark-gluon plasma [9]. However, the Navier-Stokes equations are just one of a more universal set of effective field theories which govern the thermalization in a given chaotic many-body system [10]. The Navier-Stokes equations describe a system in which charge particle-number, energy, and momentum are conserved. In their non-relativistic form the equations are Galilean-symmetric – that is they are invariant under spatial translations, rotations, and boosts. By writing down the most general equations with these quantities conserved and symmetries respected we can arrive at the form of the Navier-Stokes equations up to phenomenological constants [11]. This more general approach allows us to qualitatively infer the large-scale and long-time dynamics of non-trivial topologies. In section 3.1 we will consider the conserved quantities and symmetries of our specific system and write down the most general form of equations consistent with these. More technical applications of this approach can be found in [12], [13].

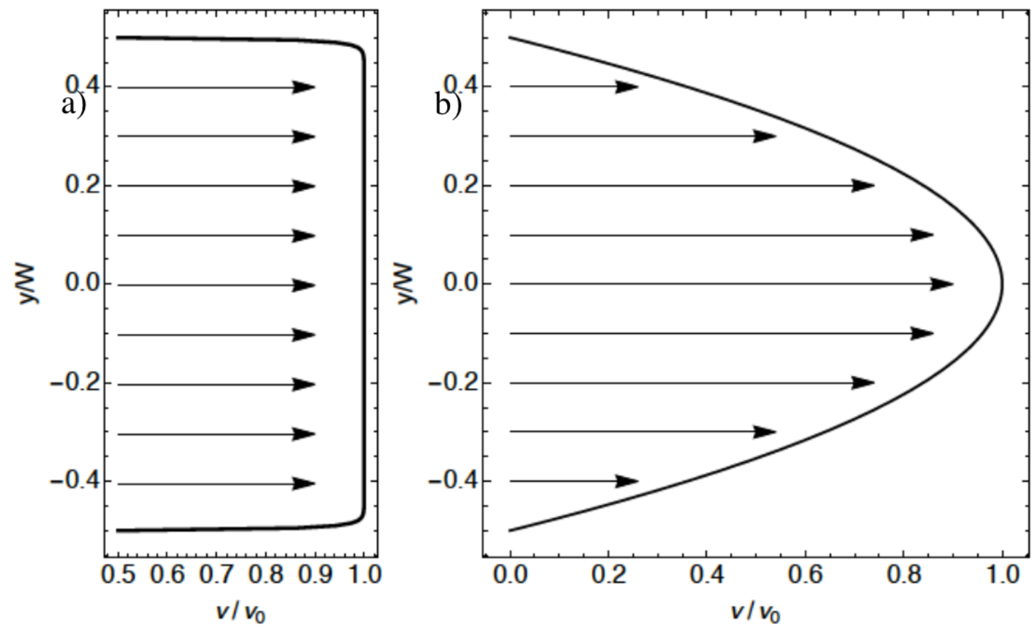

Figure 1: Normalized velocity profile in a channel is depicted for purely Ohmic flow (left) and purely Hydrodynamic flow (right). Image credit [8].

1.3 Hydrodynamics from Kinetic Theory

To go along with the modern zoom-out approach to hydrodynamics we may also consider the zoom-in approach of deriving hydrodynamics from kinetic theory. The starting point for this is the Boltzmann Equation:

$$\frac{\partial f}{\partial t} = \dot{\vec{x}} \cdot \frac{\partial f}{\partial \vec{x}} + \dot{\vec{p}} \cdot \frac{\partial f}{\partial \vec{p}} + \hat{C} f = \{\hat{H}, f\} + \hat{C} f \tag{1}$$

This equation governs the evolution of a phase-space distribution $f(\vec{x}, \vec{p}, t)$ under a microscopic Hamiltonian $\hat{H}$. The quantity $\{\hat{H}, f\}$ is referred to as the streaming term and follows from demanding that the distribution f is locally conserved. The quantity $\hat{C}f$ is referred to as the collisional term and is a non-linear integral operator. Thus in order to make this problem tractable we use the Chapman-Enskog expansion [14]. Begin by considering the equilibrium distribution $f^{(0)}$ given by $\hat{C}f^{(0)} = 0$. Next write your full distribution by including a dummy variable as follows:

$$f = f^0 + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + ... \tag{2}$$

We may then re-write the modified Boltzmann equation using this dummy variable.

$$\frac{\partial f}{\partial t} = \{\hat{H}, f\} + \frac{1}{\epsilon} \hat{C} f \tag{3}$$

This allows us to compute $\hat{C}f$ by considering how the modes interact namely that:

$$\hat{C}[f^{(0)}, f^{(0)}] = 0 \tag{4}$$

$$\hat{C}[f^{(0)}, f^{(n)}] = \frac{1}{2} \left(\partial_t f^{(n-1)} + \{H, f^{(n-1)}\} - \sum_{m=1}^{n-1} \hat{C}[f^{(n)}, f^{(n-m)}] \right) \tag{5}$$

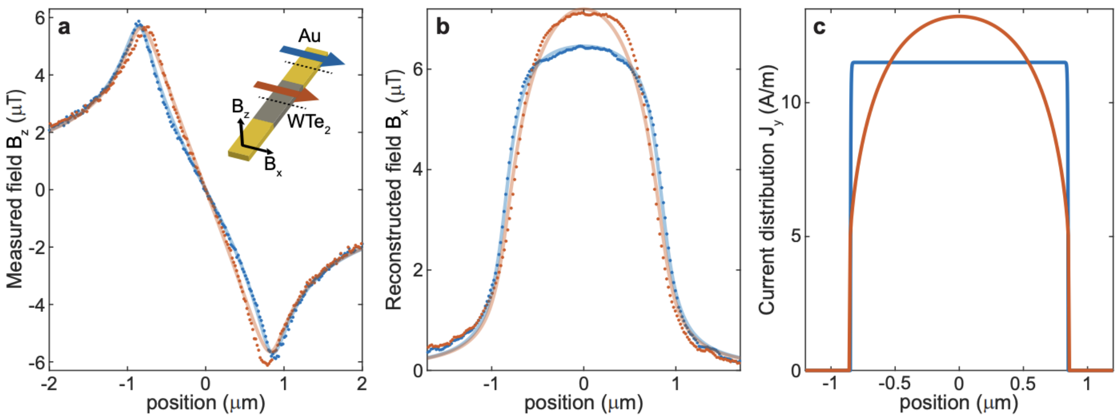

Figure 2: Experimental results for Au (blue) and WTe_2 (orange). The measured magnetic field from the currents along the z-axis just above the device (a). The reconstructed magnetic field along the x-axis of the device (b). The inferred current distribution (c). Image credit [5].

We then consider our conserved quantities and integrate over momentum space $\dot{\vec{x}}$:

$$n(\vec{x},t) = \int f d\dot{\vec{x}} \tag{6}$$

$$p(\vec{x},t) = nv_0 = \int f\dot{\vec{x}}d\dot{\vec{x}} \tag{7}$$

$$\varepsilon(\vec{x},t) = nT = \int f\frac{m\dot{\vec{x}}^2}{3k_B}d\dot{\vec{x}} \tag{8}$$

Taken together the above yield an expression for our first correction $f^{(1)}$ which depends on $f^{(0)}$:

$$f^{(1)} = \left(-\frac{1}{n}\left(\frac{2k_BT}{m}\right)^{\frac{1}{2}}\mathbb{A}\cdot\nabla ln(T) - \frac{2}{n}\sum_i\sum_k B_{ij}\partial_j v_k\right)f^{(0)} \tag{9}$$

Note that in the above expression $\mathbb{A}$ is a vector and $\mathbb{B}$ is a tensor which satisfy the given relations.

In section 3.2 we will apply a similar expansion using a parameterized fermi-surface.

2 Motivation for the Theory

In this paper we will be developing a hydrodynamic theory for a 2-D material in which only x-component of momentum is conserved. To understand why such a theory may be useful we must first begin with some introductory solid state physics. Metals are crystalline structures with a periodic arrangement of nuclei which characterize the cyrstal lattice. This forms a periodic potential as seen by the electrons – solutions to the Schrödinger equation in a period potential are plane waves [15]:

$$\Phi(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}u(\vec{x}) \tag{10}$$

The important thing to note here is that due to the periodic nature of the lattice the wave-function itself is also periodic meaning that $\tilde{\Phi}(\vec{k}+\vec{K}) = \tilde{\Phi}(\vec{k})$ for any state that differ by a reciprocal lattice

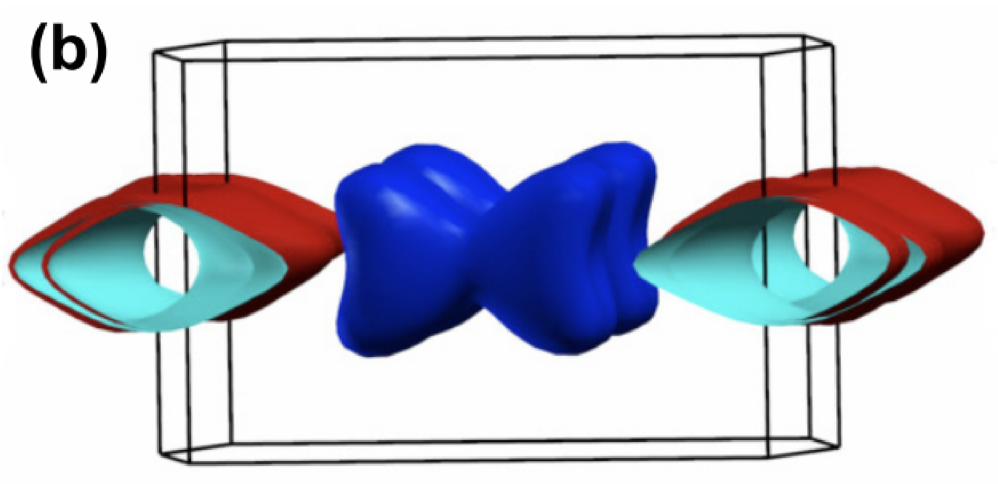

Figure 3: Fermi-surface of WP_2 material, Image credit [18].

vector $\vec{K}$. Let us then consider the collision integral from earlier which capture particle interactions:

$$\hat{C}f = \pi\beta \int \frac{d^2_{p_1} d^2_{p_2} d^2_{p_3} d^2_{p}}{(2\pi\hbar)^8} \delta(\sum p)\delta(\sum \epsilon)\Phi(p)[\Phi(p) + \Phi(p_3) - \Phi(p_1) - \Phi(p_2)] \\ \times |M_{if}|^2 f_F(p) f_F(p_3)(1 - f_F(p_1))(1 - f_F(p_2)) \quad (11)$$

The important thing to note here is the delta-function term capturing momentum conservation:

$$\delta(\sum p) \sim \delta(\vec{k} + \vec{k_1} - \vec{k_2} - \vec{k_3}) \quad (12)$$

However we recall that the wave vector $\vec{k}$ is the same as $\vec{k} + \vec{K}$ thus our equation reads:

$$\delta(\vec{K} + \vec{k} + \vec{k_1} - \vec{k_2} - \vec{k_3}) \quad (13)$$

Thus we can imagine a scattering event in which one of the resultant particles gets kicked out of the reciprocal lattice cell (Brillouin zone) but is reflected back in. While ordinarily disallowed by momentum conservation it is permitted under these conditions. Such events are referred to as Umklapp scattering and may directly relax momentum [16].

It has been previously conjectured that this Umklapp process may be efficient at relaxing momentum near the Fermi-surface (the surface of constant energy below which all states are filled and above which are empty) via electron-electron collisions [17]. Additionally, modern experimental techniques have allowed us to image the Fermi-surface. In materials such as WP_2 the Fermi-surface wraps the Brillouin zone in a single spatial dimension as shown in Figure 3. In such an anisotropic material this process would be efficient at relaxing momentum along this direction (which we take to be y) while allowing for momentum to be conserved along in the other direction (x).

3 Hydrodynamics

3.1 Landau Effective Theory

We consider the hydrodynamics of a 2D system in which momentum is only conserved in the x-direction. Thus our conserved quantities are total particle number N and x-momentum P_x. Examining the symmetries of these unusual conservation laws imposes restrictions on the forms of the allowable terms which may appear in our kinetic theory. First we note that although we are dealing with a discrete lattice we may consider the low-energy limit in which $dx \gg \frac{1}{l}$ where l is the lattice spacing. In this limit we may assume an effective continuous spatial translational symmetry. We also note that because only one dimension of momentum is conserved that we may not impose continuous rotational symmetry as that would violate this non-conservation. Thus, we begin by using the Landau procedure of writing down the most general terms which may appear that respect the following symmetries:

Spatial Parity (P):

$$\partial_t n = A\partial_x v_x + \sigma_{xx}\partial_x^2\mu + \sigma_{yy}\partial_y^2\mu + ... \tag{14}$$

The LHS of the above equation has no spatial dependence thus it is even under the parity transformation $x \to -x$. The terms on the RHS have zero or two factors which depend on x thus they are also even under this parity transformation. Our theory may also contain higher order derivatives which satisfy this constraint.

$$\partial_t \pi_x = B\partial_x \mu + \eta_{xx}\partial_x^2 v_x + \eta_{yy}\partial_y^2 v_x + ... \tag{15}$$

The LHS of the above equation has one term that depends on x making it is odd under parity transformation. The terms on the RHS have either one or three factors which depend on x thus they are also odd under parity transformation. Note that the coefficients A,B,σ_{xx},σ_{yy},η_{xx},η_{yy} will depend on the specific system studied.

Time Reversal (T):

$$\partial_t n = A\partial_x v_x + ... \tag{16}$$

The LHS of the above equation has one factor that depends on time making it odd under time reversal. The first term on the RHS is also odd under time reversal but the other terms from (14) are even under time reversal thus they do not respect this symmetry. Therefore they will not appear in the ideal hydro equations and may only appear from the collision integral.

$$\partial_t \pi_x = B\partial_x \mu + ... \tag{17}$$

The LHS of the above equation is even under time reversal. As before, the first term on the RHS is also even under time reversal and should the other odd terms appear they must appear in the collision integral.

3.2 Kinetic Theory

We want to study what happens in 2D systems in which only the x-component of momentum is conserved. We do this by solving the vectorized Boltzmann equation:

$$\partial_t|\Phi\rangle + L|\Phi\rangle + W|\Phi\rangle = 0 \tag{18}$$

We decompose this perturbation into $|\Phi\rangle = |\Phi\rangle_f + |\Phi\rangle_s$ where $W|\Phi\rangle_s = 0$ and $|\Phi\rangle_s$ corresponds to our conserved quantities n and p_x.

Under this decomposition our equation becomes:

$$\partial_t \begin{pmatrix} |\Phi\rangle_s \\ |\Phi\rangle_f \end{pmatrix} + \begin{pmatrix} L_{ss} & L_{sf} \\ L_{fs} & L_{ff} \end{pmatrix} \begin{pmatrix} |\Phi\rangle_s \\ |\Phi\rangle_f \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & W_{ff} \end{pmatrix} \begin{pmatrix} |\Phi\rangle_s \\ |\Phi\rangle_f \end{pmatrix} = 0 \tag{19}$$

We then make the assumption that our fast modes decay on short time scales, meaning that they vanish on long time scales, ie, $\partial_t|\Phi\rangle_f \approx 0$ and thus:

$$L_{fs}|\Phi\rangle_s + (L_{ff} + W_{ff})|\Phi\rangle_f = 0 \implies |\Phi\rangle_f = -(L_{ff} + W_{ff})^{-1} L_{fs}|\Phi\rangle_s \tag{20}$$

Substituting back into the decomposed Boltzmann equation for the slow sub-space we find that:

$$\partial_t|\Phi\rangle_s + L_{ss}|\Phi\rangle_s - L_{sf}(L_{ff} + W_{ff})^{-1} L_{fs}|\Phi\rangle_s \tag{21}$$

This gives us a streaming operator L_{ss} and a collisional operator $W' = -L_{sf}(L_{ff} + W_{ff})^{-1} L_{fs}$.

Utilizing the approach outlined in [19] we begin by considering the particular polygonal Fermi-surface shown in Figure 4. We encode each edge as a vector $|n, m\rangle$. We may then write $|\Phi\rangle = \sum_n \sum_m \phi_{nm}|n, m\rangle$. Let m index each edge as follows: $m = \pm$ 1, 2 , 3, 4 where vertical (horizontal) edges to the right (above) of the origin O are taken to be positive. Let the $n = 0$ mode correspond to the next number of quasi particles on a particular edge, and the $n = 1$ mode correspond to first order excitations across and edge.

We then use this parameterization to encode the particle number vector $|N\rangle$ and x-momentum vector $|P_x\rangle$ where:

$$|N\rangle = \sum_{m=1}^{4} \sqrt{\nu}\sqrt{l_m}|0, \pm m\rangle \tag{22}$$

$$|P_x\rangle = \sum_{m=1}^{2} P_{x,m}\sqrt{\nu}\sqrt{l_m}|0, \pm m\rangle + h \sum_{m=3}^{4} |1, \pm m\rangle \tag{23}$$

In order to fix the value of the constant ν we construct a current density vector $|J_x\rangle \propto v_f|N\rangle$ and demand that the following relation it satisfied:

$$\langle P_x|J_x\rangle = n_0 \tag{24}$$

This results in the following form for ν:

$$\nu = \frac{n_0(2(a + b) + c)}{2v_f(2ad + c(b + d))} \tag{25}$$

Now that ν has been fixed, we do the same for the constant h quantifying the differential contribution of the $n = 1$ Legendre modes as compared to the $n = 0$ modes. As defined in (23), h may be written as

$$h = \langle 1, m|P_x\rangle \tag{26}$$

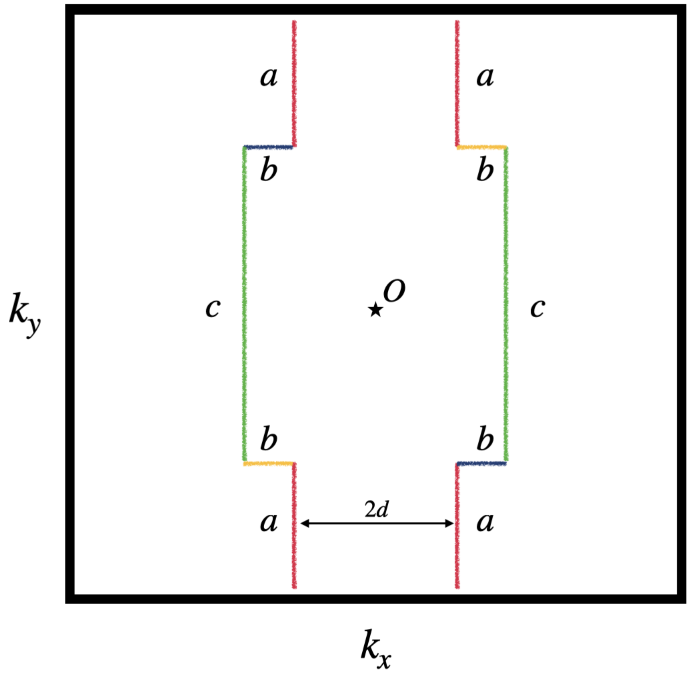

Figure 4: Toy Fermi-surface

Centering the origin at the center of a certain horizontal edge, we may then temporarily understand $|1,m\rangle$ to be given by:

$$|1,m\rangle \equiv \sqrt{\tilde{\nu}} L_1\left(\frac{p_x}{b/2}\right)\left(\frac{b}{2}\right) \tag{27}$$

where $\tilde{\nu}$ is a coefficient we must determine by demanding $\langle 1,m|1,m\rangle = 1$. The extra factor of $b/2$ follows from changing the integration variable, $\int \mathrm{d}z\, L_1(z)(\ldots) \to \int \mathrm{d}p_x\, L_1(p_x/(b/2)) \times b/2(\ldots)$. We can normalize it by working out

$$1 = \int_{-b/2}^{b/2} \mathrm{d}p_x \left[\sqrt{\tilde{\nu}} L_1\left(\frac{p_x}{b/2}\right)\frac{b}{2}\right]^2 = \tilde{\nu}\int_{-b/2}^{b/2} \mathrm{d}p_x\, p_x^2 = \frac{\tilde{\nu} b^3}{6} \implies \tilde{\nu} = \frac{6}{b^3}. \tag{28}$$

Armed with this correct normalization of $|1,m\rangle$, we may work out the inner product (27) as

$$\begin{aligned} h = \langle 1,m|P_x\rangle &= \int_{-b/2}^{b/2} \mathrm{d}p_x \left[\sqrt{\tilde{\nu}}\left(\frac{b}{2}\right) L_1\left(\frac{p_x}{b/2}\right)\right] p_x \\ &= \int_{-b/2}^{b/2} \mathrm{d}p_x \left[\sqrt{\tilde{\nu}}\frac{b}{2}\left(\frac{p_x}{b/2}\right)\right] p_x \\ &= \int_{-b/2}^{b/2} \mathrm{d}p_x\, p_x^2 = 2\sqrt{\tilde{\nu}} \times \frac{2}{3}\left(\frac{b}{2}\right)^3 = \sqrt{\frac{b^3}{6}} = h. \end{aligned} \tag{29}$$

In the last equality, we used the expression for the normalization constant $\tilde{\nu}$ given in (28).

Note that under this construction our perturbation vector takes the form:

$$|\Phi\rangle = \mu|N\rangle + v_x|P_x\rangle + |\Phi\rangle_f \tag{30}$$

We then construct the collision integral W_0 using the relaxation time approximation $W|\Phi\rangle \approx \gamma = \frac{1}{\tau}$:

$$W_0 = \sum_{m=1}^{2} \gamma_s|0, \pm m\rangle + \sum_{m=3}^{4} \gamma_f|0, \pm m\rangle + \sum_{m=1}^{4} \gamma_f|1, \pm m\rangle \tag{31}$$

Next we project this operator onto the fast modes using projector P_f creating a collision operator W guaranteed to annihilate the slow modes where:

$$P_f = \mathbb{I} - \frac{|P_x\rangle\langle P_x|}{\langle P_x|P_x\rangle} - \frac{|N\rangle\langle N|}{\langle N|N\rangle} \tag{32}$$

$$W = P_f W_0 P_f^{-1} \tag{33}$$

We then construct the streaming operator L_0 where:

$$L_0 = I \otimes \left(\sum_{m=1}^{2} \pm v_f \partial_x|n, \pm m\rangle + \sum_{m=3}^{4} \pm v_f \partial_y|n, \pm m\rangle \right) \tag{34}$$

Next we introduce a projector onto the slow modes $P_s = I - P_f$ and determine the form of L_0 in the slow-slow L_{ss} slow-fast L_{sf} and fast-slow L_{fs} sub-spaces by conjugating L_0 as follows:

$$L_{sf} = P_f L_0 P_s \tag{35}$$

$$L_{fs} = P_s L_0 P_f \tag{36}$$

We then return the the form of our effective collision operator $W' = -L_{sf}(L_{ff} + W_{ff})^{-1}L_{fs}$. We recognize that $L_{ff} \propto v_f|\nabla|$ and $W_{ff} \propto \gamma_f$ and under our assumption that these fast modes decay very quickly $\gamma_f = \frac{1}{\tau_f}$ will be very large, ie, $\gamma_f \gg v_f|\nabla|$ and we can take the effective collision operator to be:

$$W' = -L_{sf} W L_{fs} \tag{37}$$

Now that we have all tools we need let us determine the equations of motion. We begin by recalling the expanded form of the perturbation $|\Phi\rangle$:

$$|\Phi\rangle = \mu|N\rangle + v_x|P_x\rangle + |\Phi\rangle_f \tag{38}$$

We then inner product our vectorized Boltzmann equation (4) with $\langle N|$ to get:

$$\partial_t n = \partial_t \mu \langle N|N\rangle = -\langle N|L_{ss}|N\rangle - \langle N|W'|N\rangle = A\partial_x v_x + \sigma_{xx}\partial_x^2 \mu + \sigma_{yy}\partial_y^2 \mu \tag{39}$$

We then inner product our vectorized Boltzmann equation (4) with $\langle P_x|$ to get:

$$\partial_t \pi_x = \partial_t v_x \langle P_x|P_x\rangle = -\langle P_x|L_{ss}|P_x\rangle - \langle P_x|W'|P_x\rangle = B\partial_x \mu + \eta_{xx}\partial_x^2 v_x + \eta_{yy}\partial_y^2 v_x \tag{40}$$

Here the hydrodynamical coefficients A,B,σ_{xx},σ_{yy},η_{xx},η_{yy} depend on the specific geometrical factors but for this example take on the explicit form:

$$A = -n_0 \tag{41}$$

$$\sigma_{xx} = \frac{bn_0 v_f \left(12bd(2a+c) + b(2a(5b+6c)+5bc) + 12d^2(2a+c)\right)}{\gamma_s \left(6d^2(2(a+b)+c) + b^2(5b+6c) + 12bd(b+c)\right)(2ad + c(b+d))} \tag{42}$$

$$\sigma_{yy} = \frac{2bn_0 v_f}{\gamma_s(2ad + c(b+d))} \tag{43}$$

$$B = -n_0 \tag{44}$$

$$\eta_{xx} = \frac{2bn_0 v_f \left(a\left(bc+2d^2\right) + c(b+d)^2\right)}{\gamma_s(2(a+b)+c)(2ad+c(b+d))} \tag{45}$$

$$\eta_{yy} = \frac{bn_0 v_f \left(5b^2 + 12bd + 12d^2\right)}{6\gamma_s(2ad + c(b+d))} \tag{46}$$

We then consider the continuity equations to identify expressions for current density $\vec{J}$ and the stress-energy tensor τ:

$$\partial_t n = -\nabla \cdot \vec{J} = -\partial_x J_x - \partial_y J_y \tag{47}$$

$$\partial_t \pi_x = -\nabla \cdot \vec{\tau} = -\partial_x \tau_{xx} - \partial_y \tau_{yx} \tag{48}$$

Comparing these expressions to (39) and (40) results in the following expressions for $J_x, J_y, \tau_{xx}, \tau_{yx}$:

$$J_x = n_0 v_x - \sigma_{xx}(\partial_x \mu) \tag{49}$$

$$J_y = -\sigma_{yy}(\partial_y \mu) \tag{50}$$

$$\tau_{xx} = n_0 \mu - \eta_{xx}(\partial_x v_x) \tag{51}$$

$$\tau_{yx} = -\eta_{yy}(\partial_y v_x) \tag{52}$$

4 Flow Patterns

Next we will consider the effects of placing the material in a long channel and applying an electric field. The channel and applied electric field will be oriented at an angle θ relative to this fermi-surface as shown in the figure below. This results in the introduction of coordinates $\tilde{x} = x\cos\theta + y\sin\theta$ and $\tilde{y} = -x\sin\theta + y\cos\theta$ in which this electric field is given by $\vec{E} = E_0\hat{\tilde{y}}$.

We must also modify our equations of motion (39) and (40) to account for this new gradient. This introduces an additional term to the chemical potential and $\mu \to \tilde{\mu}(x) + E_{\tilde{y}}$. We are seeking stationary solutions in the rotated frame, ie, solutions for which $\partial_t \rho \to 0$. Employing the chain rule $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial \tilde{x_i}}\frac{\partial \tilde{x_i}}{\partial x_i}$ giving us that $\partial_x = \partial_{\tilde{x}}\cos\theta + \partial_{\tilde{y}}\sin\theta$ and $\partial_y = -\partial_{\tilde{x}}\sin\theta + \partial_{\tilde{y}}\cos\theta$. Note that since $v = \tilde{v}(x)$ we will drop the subscript and consider $v = v_x$. Using these substitutions (39) becomes:

$$0 = (\partial_{\tilde{x}}\cos\theta + \partial_{\tilde{y}}\sin\theta)n_0 v - \sigma_{xx}(\partial_{\tilde{x}}\cos\theta + \partial_{\tilde{y}}\sin\theta)^2\tilde{\mu} - \sigma_{yy}(-\partial_{\tilde{x}}\sin\theta + \partial_{\tilde{y}}\cos\theta)^2\tilde{\mu} \tag{53}$$

We will assume that the channel is sufficiently long and that no gradients exist along the direction of the channel $\tilde{y}$ implying $\partial_{\tilde{y}} \to 0$ allowing us to rewrite (53) as the second order differential equation:

$$\cos\theta n_0 \partial_{\tilde{x}}\tilde{v} = (\sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta)\partial_{\tilde{x}}^2\tilde{\mu} \tag{54}$$

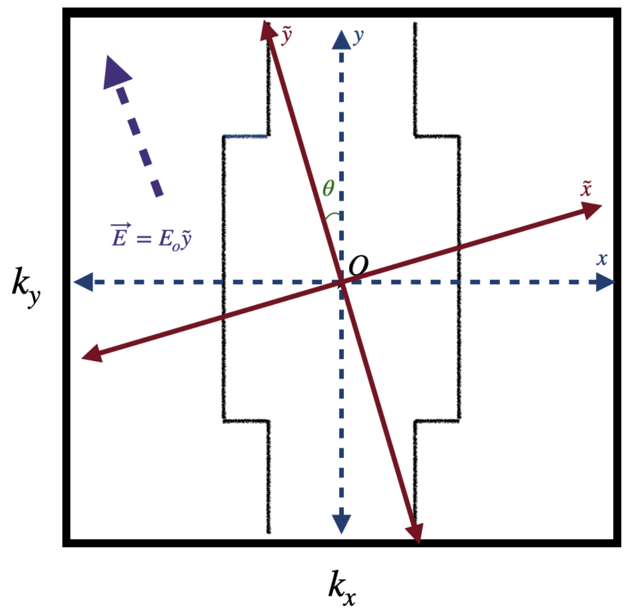

Figure 5: Fermi-surface oriented at an angle θ to an applied electric field.

$$\cos\theta n_0 \partial_{\tilde{x}}\tilde{v} = (\sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta)\partial^2_{\tilde{x}}\tilde{\mu} \tag{55}$$

$$\partial_{\tilde{x}}\tilde{v} = \frac{\tilde{\sigma}_{xx}\partial^2_{\tilde{x}}\tilde{\mu}}{n_0\cos\theta} \tag{56}$$

Note that in the above expression we have defined $\tilde{\sigma}_{xx} = \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta$.

Applying the previously mentioned substitutions to (40) we obtain the following expression:

$$0 = n_0(\partial_{\tilde{x}}\cos\theta + \partial_{\tilde{y}}\sin\theta)\tilde{\mu} + -\eta_{xx}(\partial_{\tilde{x}}\cos\theta + \partial_{\tilde{y}}\sin\theta)^2\tilde{v} - \eta_{yy}(-\partial_{\tilde{x}}\sin\theta + \partial_{\tilde{y}}\cos\theta)^2\tilde{v} + n_0 E_0\sin\theta \tag{57}$$

After expanding and demanding that $\tilde{y}$ gradients vanish we get:

$$n_0\cos\theta\partial_{\tilde{x}}\tilde{\mu} = (\eta_{xx}\cos^2\theta + \eta_{yy}\sin^2\theta)\partial^2_{\tilde{x}}\tilde{v} + n_0E_0\sin\theta \tag{58}$$

$$\partial_{\tilde{x}}\tilde{\mu} = \frac{\tilde{\eta}_{xx}\partial^2_{\tilde{x}}\tilde{v}}{n_0\cos\theta} + E_0\tan\theta \tag{59}$$

As before we have defined a new quantity $\tilde{\eta}_{xx} = (\eta_{xx}\cos^2\theta + \eta_{yy}\sin^2\theta)$.

We then substitute the expression (59) into (56) to obtain the following:

$$\partial_{\tilde{x}}\tilde{v} = \frac{\tilde{\sigma}_{xx}\tilde{\eta}_{xx}\partial^3_{\tilde{x}}\tilde{v}}{n_0^2\cos^2\theta} \tag{60}$$

Next we define $w = \partial_x v$ and $r(\theta) = \frac{n_0\cos\theta}{\sqrt{\tilde{\sigma}_{xx}\tilde{\eta}_{xx}}}$ to obtain:

$$\partial^2_x w = r(\theta)w \tag{61}$$

We recognize the above as the ODE with solutions:

$$\partial_{\tilde{x}} v = w = \tilde{C_+} e^{r(\theta)x} - \tilde{C_-} e^{-r(\theta)x} \tag{62}$$

Integrating and absorbing a factor of $\frac{1}{r(\theta)}$ into our constants we find $\tilde{v}$ of the form:

$$\tilde{v} = C_+ e^{r(\theta)x} + C_- e^{-r(\theta)x} + v_0 \tag{63}$$

Substituting (63) back into (59) and substituting for $r(\theta)$ we obtain the following expression:

$$\partial_{\tilde{x}} \tilde{\mu} = \frac{\tilde{\eta}_{xx} \partial^2_{\tilde{x}}}{n_0 \cos\theta} (C_+ e^{r(\theta)x} + C_- e^{-r(\theta)x}) + E_0 \tan\theta \tag{64}$$

$$\partial_{\tilde{x}} \tilde{\mu} = \frac{\tilde{\eta}_{xx} r^2(\theta)}{n_0 \cos\theta} (C_+ e^{r(\theta)x} + C_- e^{-r(\theta)x}) + E_0 \tan\theta \tag{65}$$

Integrating we arrive at a solution for $\tilde{\mu}$ of the following form:

$$\tilde{\mu} = \frac{n_0 \cos\theta}{\tilde{\sigma}_{xx}} (C_+ e^{r(\theta)x} - C_- e^{-r(\theta)x}) + E_0 x \tan\theta \tag{66}$$

Let us now consider the changes to the current density $\vec{J}$ given by equations (49) and (50) in the new frame our vector $\tilde{\vec{J}}$ will be given by the following expression:

$$\tilde{\vec{J}} = R(\theta)\vec{J} \tag{67}$$

$$\tilde{\vec{J}} = R(\theta) n_0 v + R(\theta) \vec{\sigma} R^T(\theta) \tilde{\vec{E}} \tag{68}$$

$$\begin{bmatrix} \tilde{J}_x \\ \tilde{J}_y \end{bmatrix} = n_0 v \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{yy} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} -\partial_{\tilde{x}} \tilde{\mu} \\ E_0 \end{bmatrix} \tag{69}$$

$$\begin{bmatrix} \tilde{J}_x \\ \tilde{J}_y \end{bmatrix} = n_0 v \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \begin{bmatrix} \sigma_{xx} \cos^2(\theta) + \sigma_{yy} \sin^2(\theta) & (\sigma_{yy} - \sigma_{xx}) \sin(\theta)\cos(\theta) \\ (\sigma_{yy} - \sigma_{xx}) \sin(\theta)\cos(\theta) & \sigma_{xx} \sin^2(\theta) + \sigma_{yy} \cos^2(\theta) \end{bmatrix} \begin{bmatrix} -\partial_{\tilde{x}} \tilde{\mu} \\ E_0 \end{bmatrix} \tag{70}$$

$$\begin{bmatrix} \tilde{J}_x \\ \tilde{J}_y \end{bmatrix} = n_0 v \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} + \begin{bmatrix} \tilde{\sigma}_{xx} & \tilde{\sigma}_{xy} \\ \tilde{\sigma}_{yx} & \tilde{\sigma}_{yy} \end{bmatrix} \begin{bmatrix} -\partial_{\tilde{x}} \tilde{\mu} \\ E_0 \end{bmatrix} \tag{71}$$

Note that above $R(\theta)$ is the rotation matrix and we have defined $\tilde{\sigma_{ij}}$ for ease of notation.

Now let us attempt to fix the constants v_0, C_+, C_- we will do this by assuming no slip boundary conditions. That is that the velocity vanishes at the boundaries, ie, $\tilde{v}(L) = \tilde{v}(-L) = 0$.

Under the condition that $\tilde{v}(L) = \tilde{v}(-L)$ we have:

$$C_+ e^{r(\theta)L} + C_- e^{-r(\theta)L} = C_+ e^{-r(\theta)L} + C_- e^{r(\theta)L} \tag{72}$$

$$C_+ \sinh[r(\theta)L] = C_- \sinh[r(\theta)L] \tag{73}$$

$$C \propto C_+ = C_- \tag{74}$$

Under the condition that $\tilde{v}(L) = 0$ we have that:

$$\tilde{v_0} = -C\cosh[r(\theta)L] \tag{75}$$

We now have an expression for $\tilde{v}$ of the form:

$$\tilde{v} = C(\cosh[r(\theta)x] - \cosh[r(\theta)L]) \tag{76}$$

We now need to fix the constant C by considering our expression for J_x from (71):

$$J_{\tilde{x}} = n_0\tilde{v}\cos\theta - \tilde{\sigma}_{xx}\partial_{\tilde{x}}\mu + E_0\tilde{\sigma}_{xy} \tag{77}$$

We demand that $J_{\tilde{x}} = 0$ as there cannot be a flow through the channel walls upon rearranging the above expression we obtain:

$$\cos\theta\partial_{\tilde{x}}\mu = \frac{\cos\theta}{\tilde{\sigma}_{xx}}(n_0\cos\theta\tilde{v} + E_0\tilde{\sigma}_{xy}) \tag{78}$$

After substituting in (76) we have:

$$\cos\theta\partial_{\tilde{x}}\mu = \frac{\cos\theta}{\tilde{\sigma}_{xx}}(n_0\cos\theta[C(\cosh[r(\theta)x] - \cosh[r(\theta)L]]) + E_0\tilde{\sigma}_{xy}) \tag{79}$$

Next we consider (59) which we rewrite as:

$$\cos\theta\partial_{\tilde{x}}\tilde{\mu} = \frac{n_0\cos^2\theta\partial_{\tilde{x}}^2\tilde{v}}{r^2(\theta)\tilde{\sigma}_{xx}} + E_0\sin\theta \tag{80}$$

Substituting in our expression for $\tilde{v}$ (76) and differentiating we obtain:

$$\cos\theta\partial_{\tilde{x}}\tilde{\mu} = \frac{n_0\cos^2\theta}{\tilde{\sigma}_{xx}}C\cosh[r(\theta)x] + E_0\sin\theta \tag{81}$$

Setting (79) equal to (81) we get the following:

$$\frac{\cos\theta}{\tilde{\sigma}_{xx}}(n_0\cos\theta[C(\cosh[r(\theta)x] - \cosh[r(\theta)L]]) + E_0\tilde{\sigma}_{xy}) = \frac{n_0\cos^2\theta}{\tilde{\sigma}_{xx}}C\cosh[r(\theta)x] + E_0\sin\theta \tag{82}$$

After canceling like terms and solving for C we find that:

$$C = \frac{E_0(\cos\theta\tilde{\sigma}_{xy} - \sin\theta\tilde{\sigma}_{xx})}{n_0\cos^2\theta\cosh[r(\theta)L]} \tag{83}$$

We can further simplify using the definitions from (71) to obtain:

$$C = \frac{E_0\sigma_{yy}\sin\theta}{n_0\cos^2\theta\cosh[r(\theta)L]} \tag{84}$$

Thus after substituting in our expression for C (76) for $\tilde{v}$ becomes:

$$\boxed{\tilde{v} = \frac{E_0\sigma_{yy}\sin\theta}{n_0\cos^2\theta}\left(1 - \frac{\cosh[r(\theta)x]}{\cosh[r(\theta)L]}\right)} \tag{85}$$

We see a flow profile that takes the form shown in Figure 4. Before moving forward let us quickly address the apparent divergence of the maximal-velocity at $\tilde{v}(x = 0)$ expected as $\cos(\theta) \to 0$.

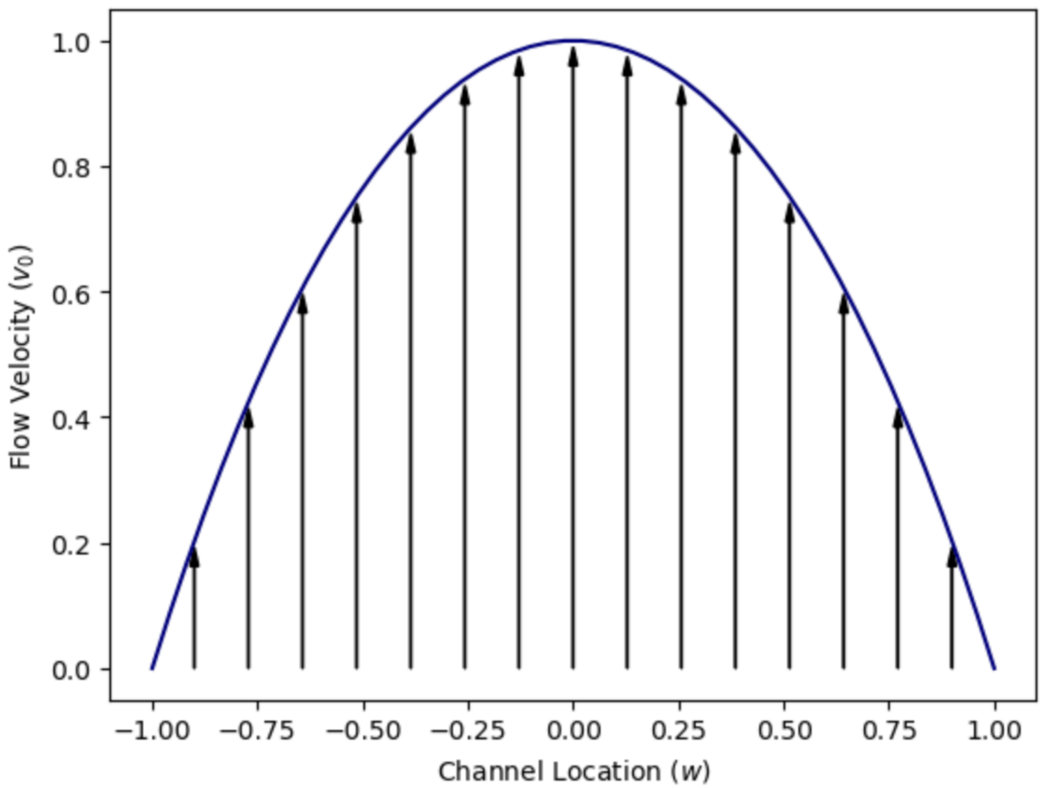

Figure 6: Flow profile in a channel

$$\tilde{v} = \frac{E_0 \sigma_{yy} \sin\theta}{n_0 \cos^2\theta} \left(1 - \frac{1}{\cosh\left[r(\theta)L\right]}\right) \tag{86}$$

We note that when this occurs $r \to 0$ as well, thus Taylor expanding in r we obtain a bounded expression for $\tilde{v}$:

$$\tilde{v} = \frac{E_0 \sigma_{yy} \sin\theta}{n_0 \cos^2\theta} \left(1 - (1 - \frac{1}{2}(rL)^2\right) = E_0 n_0 \frac{L^2}{2} \frac{\sigma_{yy} \sin(\theta)}{\sigma_{xx} \eta_{xx}} \tag{87}$$

Next let us consider what transverse Hall-Voltage would arise as a result of our theory. We again use the fact that $J_x = 0$ to and solve (77) for $\partial_{\tilde{x}} \tilde{\mu}$ to obtain:

$$\partial_{\tilde{x}} \tilde{\mu} = E_0 \frac{\tilde{\sigma}_{xy}}{\tilde{\sigma}_{xx}} + \frac{n_0 \cos(\theta) \tilde{v}}{\tilde{\sigma}_{xx}} \tag{88}$$

Substituting in our expression for $\tilde{v}$ we get:

$$\partial_{\tilde{x}} \tilde{\mu} = E_0 \frac{\tilde{\sigma}_{xy}}{\tilde{\sigma}_{xx}} + E_0 n_0 \sigma_{yy} \tan(\theta) \left(1 - \frac{\cosh\left[r(\theta)x\right]}{\cosh\left[r(\theta)L\right]}\right) \tag{89}$$

Integrating the above we obtain the following expression:

$$\tilde{\mu} = E_0 \left[x \frac{\tilde{\sigma}_{xy}}{\tilde{\sigma}_{xx}} + \sigma_{yy} \tan(\theta) \left(x - \frac{\sinh\left[r(\theta)x\right]}{r \cosh\left[r(\theta)L\right]}\right)\right] \tag{90}$$

Taking $V_H = \Delta\tilde{\mu}$ across the channel and substituting in our original hydrodynamic coefficients we obtain the expression:

$$\boxed{V_H = 2E_0 \left[\frac{L(\sigma_{xx} - \sigma_{yy})\sin(\theta)\cos(\theta)}{\sigma_{xx}\cos(\theta)^2 + \sigma_{yy}\sin(\theta)^2} + \sigma_{yy}\tan(\theta)\left(L - \frac{1}{r}\tanh[r(\theta)L]\right)\right]} \tag{91}$$

Next we will consider the conductance $G = I_y/(E_0 d)$ we might expect for a channel for length d. First we must obtain an expression for I_y which we do by integrating the expression for J_y in (71):

$$I_y = \int_{-L}^{+L} dx\ J_y \tag{92}$$

$$I_y = \int_{-L}^{+L} dx\ dx\ n_0\tilde{v}\sin(\theta) - \tilde{\sigma}_{yx}\partial_{\tilde{x}}\tilde{\mu} + E_0\tilde{\sigma}_{yy} \tag{93}$$

Substituting in our expression for $\tilde{v}$ and recognizing the second term as V_H we obtain:

$$I_y = E_0 \int_{-L}^{+L} dx\ \sigma_{yy}\tan^2(\theta)\left(1 - \frac{\cosh[r(\theta)x]}{\cosh\left[r(\theta)L\right]}\right) + \tilde{\sigma}_{yy} - \tilde{\sigma}_{yx}V_H \tag{94}$$

Integrating the above expression and substituting in V_H we obtain:

$$I_y = 2E_0 \left[\sigma_{yy}\tan(\theta)\left(L - \frac{1}{r}\tanh[r(\theta)L]\right)(\tan(\theta) - \tilde{\sigma}_{xy}) + L\left(\tilde{\sigma}_{yy} - \frac{\tilde{\sigma}_{xy}^2}{\tilde{\sigma}_{xx}}\right)\right] \tag{95}$$

Thus after substituting in for the hydrodynamic coefficients in our original frame we obtain an expression for the conductance G of the following form:

$$\boxed{G = \frac{2}{d}\left[\left(L - \frac{1}{r}\tanh[r(\theta)L]\right)\left(\sigma_{yy}(\tan^2(\theta) - \sin^2(\theta)) + \sigma_{xx}\sin^2(\theta)\right) + L\left(\frac{\sigma_{xx}\sigma_{yy}}{\sigma_{xx}\cos^2(\theta) + \sigma_{yy}\sin^2(\theta)}\right)\right]} \tag{96}$$

5 Conclusion

Above we have described the hydrodynamics of a 2D material with only one spatial-dimension momentum conservation. The material WP_2 shows signatures of hydrodynamic flow and its exotic Fermi-surface makes it an ideal candidate to test out this theory. In the presence of an applied electric field our theory predicts hydrodynamic flow with a strong dependence on orientation. In particular, our theory predicts a minimal current when the electric field is oriented along the y-axis of the Fermi-surface and a maximal current when aligned perpendicular to the y-axis of the surface. In addition to the abnormal flow conditions we also predict the presence of a transverse voltage which we suggestively refer to as a Hall-Voltage that also appears as a result of our theory. In this paper we have only considered the effects an electric-field on flow in a narrow channel. Additional predictions of the theory may be made by considering flow through other geometries as well as the behaviour of the material in the presence of an applied magnetic field.

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