Generalized Supercharacter Theories and Schur Rings for Hopf Algebras

by

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Generalized Supercharacter Theories and Schur Rings for Hopf Algebras

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The character theory for semisimple Hopf algebras with a commutative representation ring has many similarities to the character theory of finite groups. We extend the notion of supercharacter theory to this context, and define a corresponding algebraic object that generalizes the Schur rings of the group algebra of a finite group. We show the existence of Hopf-algebraic analogues for the most common supercharacter theory constructions, specificially the wedge product and supercharacter theories arising from the action of a finite group. In regards to the action of the Galois group of the field generated by the entries of the character table, we show the existence of a unique finest supercharacter theory with integer entries, and describe the superclasses for abelian groups and the family $GL_2(q)$.

Dedication

For Angela.

Acknowledgements

Without the support of my family, I never would have begun. Without Nat's continuous help, wisdom, advice, direction, and patience, I never could have finished. Without Richard's careful eye, the number (and variety) of errors would be truly astonishing; if any remain, they are completely due to myself. Ben, Amy, and John were always available to listen. Angela made everything possible.

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Chapter 1

Introduction

1.1 The representation ring

The character theory of semisimple Hopf algebras largely mirrors that of finite groups. The goal in each case is to more deeply understand an algebra by studying how it can act on a vector space, i.e. by describing the category of A-modules, or equivalently, the representations of A. For a finite group G, the algebra under consideration is $A = \mathbb{C}G$, the complex group algebra. The category of A-modules is highly structured, in the sense that for two A-modules V and W, we have that the underlying field $\mathbb{C}$, the tensor product $V \otimes W$, and the dual space $V^* = \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ all have a natural A-module structure, and every finite dimensional A-module is semisimple.

Hopf algebras are a class of algebras that abstract the essential structure of group algebras in a way that ensures this desired structure on their categories of modules. The class of Hopf algebras includes, for example, the universal enveloping algebras of Lie algebras, providing a unifying framework for these two seemingly disparate structures. Indeed, the applications of Hopf algebras are extremely pervasive, impacting many areas of mathematics and physics. For example, Hopf algebras have their origins [3] in algebraic topology and the theory of algebraic groups; in combinatorics [12], they provide algebraic frameworks for combining and decomposing sets of common combinatorial objects; they provide solutions to the Yang–Baxter equation [16] of statistical mechanics and quantum field theory; and so on.

The category of H-modules for a semisimple Hopf algebra H is associated to a certain ring $R(H)$, called the representation ring, whose objects are formal linear combinations of (isomorphism clases of) simple H-modules, with a multiplication determined by tensor product and addition determined by direct sum. Understanding the structure constants of this ring in the basis of simple modules determines how tensor products of simple modules decompose as direct sums, which is one of the main goals of representation theory in this setting.

The task of understanding the representation ring is further simplified by passing from modules to characters. For any algebra A, every A-module V affords an algebra homomorphism $\rho_V\colon A \to \mathrm{End}_{\mathbb{C}}(V)$ (the representation of A on V). By composing ρ with the trace map $\mathrm{End}_{\mathbb{C}}(V) \to \mathbb{C}$, a character $\chi_V\colon A \to \mathbb{C}$ is associated to the module V. Returning to the case of a semisimple Hopf algebra H, this character map $V \mapsto \chi_V$ extends linearly to take the representation ring $R(H)$ to the span of the characters $C(H)$ in H^*. The space of functions H^* has an induced algebra structure from the Hopf structure of H, and the image $C(H)$ of the character map is isomorphic to the representation ring $R(H)$. Since $R(H) \cong C(H)$, we refer to $C(H)$ as the representation ring, and now we have reduced the problem of understanding the category of H-modules to the problem of understanding a certain algebra of functions with a distinguished basis.

When $H = \mathbb{C}G$ is the group algebra of a finite group, the characters, by virtue of their construction from the matrix trace, are constant on conjugacy classes. Consequently, the value of a character on a group element $g \in G$ is the same as its value on the element

$$c = |\mathscr{C}(g)|^{-1} \sum_{h \in \mathscr{C}(g)} h \in \mathbb{C}G,$$

where $\mathscr{C}(g)$ is the conjugacy class of g. These elements form a basis for the center $Z(\mathbb{C}G)$, and so we have a pairing between the algebras $Z(\mathbb{C}G)$ and $C(\mathbb{C}G)$, by evaluating the distinguished basis of irreducible characters on the distinguished basis of conjugacy class averages. The (square) matrix that records these entries is the character table of G, and this contains a remarkable amount of information about the group G. For example, from the character table one can reconstruct the lattice of normal subgroups of G.

1.2 Recent trends in character theory

When the representation ring $C(H)$ is commutative, the analogy to the character theory of finite groups is even stronger, recovering the notion of character table. In this case, Witherspoon [27] defines a certain basis of the center $Z(H)$, which in the case $H = \mathbb{C}G$ for a finite group G is precisely the basis of conjugacy class averages up to normalization. This basis arises from the set of characters of the commutative semisimple algebra $C(H)$, viewed by double-duality as elements of H. Cohen and Westreich [9] take an equivalent approach from the point of view of Frobenius algebras, and bring the terminology of conjugacy classes and conjugacy class sums to the Hopf setting. They provide an example of a character table (in the sense of Witherspoon) that is not the character table of any finite group.

In a different direction, Diaconis and Isaacs [11] define supercharacter theories for finite groups. A supercharacter theory consists of a collection of 'approximately' irreducible characters satisfying certain axioms. These characters provide a way to simplify or approximate the usual character table of a finite group, while potentially preserving enough information for certain applications. The notion of a general supercharacter theory arises from the work of André [1] and Yan [28] on the representation theory of the unitriangular group; both these theories arrive at a particular supercharacter theory as a manageable alternative to the otherwise intractable character theory, the complexity of which is studied by Gudivok et al. [13]. In the same way that the center of the group algebra and the representation ring determine the character theory of a finite group, supercharacter theories are determined by (and are in one-to-one correspondence with) central Schur rings [14]. An early generalization of the center of the group algebra studied by Schur and later by Wielandt [26], in the context of permutation groups. Recently, interest in Schur rings has revived somewhat as the result of connections to association schemes, supercharacter theories, and circulant graphs.

1.3 Organization

Our main goal is to combine these frameworks, extending the definition of supercharacter theories and Schur rings from finite groups to semisimple Hopf algebras. We then use these definitions to generalize some constructions that are common to the finite group case, and provide some new examples.

Chapters 2 and 3 are preliminary. Chapter 2 introduces the main objects of interest in the finite group case, namely supercharacters and Schur rings. We discuss the corresponence between these objects, and some of the important constructions, in particular the wedge product of Leung and Man [19], which is used to combine Schur rings from smaller groups. Chapter 3 begins with an introduction to the diagrammatic definitions of algebras and coalgebras, with the goal of defining Hopf algebras and summarizing some useful results concerning the structure of semisimple Hopf algebras. We also review Frobenius algebras, a necessary precursor of the character theory of semisimple Hopf algebras with a commutative representation ring.

Chapter 4 introduces generalized supercharacter theories and Schur rings, and shows that more general formulations of the results and constructions from Chapter 2 still hold in this setting. In Chapter 5, we focus specifically on constructing supercharacter theories from group actions, by considering an algebraic characterization of this action on the corresponding Schur rings. We then use these tools to prove in Chapter 6 that every supercharacter theory can be minimally coarsened to a supercharacter theory that has an integer-valued supercharacter table. We describe the superclasses of this minimal rational supercharacter theory for abelian groups and for $GL_2(\mathbb{F}_q)$.

Chapter 2

Supercharacter theories and Schur rings

In this chapter we review the main objects under consideration in the finite group case. We start with the supercharacter theory of an arbitrary finite group in §2.1, as introduced by Diaconis and Isaacs [11], focusing on some of the properties of the usual character theory that are modeled by supercharacter theories. We then give an overview of the much older notion of Schur ring, from the perspective of approximating the center of the group algebra in §2.2, leading up to the recent correspondence between supercharacter theories and central Schur rings in §2.3, as described by Hendrickson [14]. Finally, §2.4–2.5 describes Hendrickson's $*$-product construction [14] for supercharacter theories as a special case of the wedge product construction for Schur rings of Leung and Man [19].

2.1 Supercharacter theories

We begin with some definitions and basic results regarding supercharacter theories of finite groups, first introduced in their present form by Isaacs and Diaconis [11], by abstracting the work of Andre [1, 2] and Yan [28] concerning families of unipotent groups. We let $\mathrm{Irr}(G)$ denote the set of irreducible characters of G and $\mathrm{Cl}(G)$ the set of conjugacy classes of G. The identity of G will be denoted 1_G and the character of the principal representation by ε_G, so that $\varepsilon_G(g) = 1$ for all $g \in G$. To each subset X of $\mathrm{Irr}(G)$ we associate a character σ_X, defined to be the following linear combination of irreducible characters:

$$\sigma_X := \sum_{\psi \in X} \psi(1)\psi.$$

For each subset K of $\mathrm{Cl}(G)$, we will denote the union of the conjugacy classes in K as

$$\mathscr{C}_K := \bigcup_{C \in K} C.$$

Definition 2.1.1. A **supercharacter theory** of G is a pair $(\mathcal{X}, \mathcal{K})$, where $\mathcal{X}$ is a partition of $\mathrm{Irr}(G)$ and $\mathcal{K}$ is a partition of $\mathrm{Cl}(G)$, satisfying:

(1) $|\mathcal{X}| = |\mathcal{K}|$, and

(2) σ_X is constant on $\mathscr{C}_K$ for all $X \in \mathcal{X}$ and $K \in \mathcal{K}$.

The characters σ_X and the sets $\mathscr{C}_K$ are **supercharacters** and **superclasses**, respectively, and the **rank** of $(\mathcal{X}, \mathcal{K})$ is $|\mathcal{X}|$.

Remark 2.1.2. Diaconis and Isaacs [11] take $\mathcal{K}$ to be a partition of G rather than $\mathrm{Cl}(G)$, but show that $\mathcal{K}$ partitions G more coarsely than the partition of G into conjugacy classes. These definitions are therefore equivalent, by replacing our partition $\mathcal{K}$ of $\mathrm{Cl}(G)$ with the partition $\{\mathscr{C}_K \mid K \in \mathcal{K}\}$ of G. The reason for this departure is to facilitate the generalizations of the next chapter, where the group G will no longer have a direct analogue in the larger context of Hopf algebras, but $\mathrm{Cl}(G)$ will.

Definition 2.1.3. The **supercharacter table** corresponding to the supercharacter theory $(\mathcal{X}, \mathcal{K})$ of rank n, is the $n \times n$ matrix T with rows and columns indexed by $\mathcal{X}$ and $\mathcal{K}$ respectively, and entries

$$T_{XK} = \sigma_X(g_K)$$

where g_K is any representative of the superclass $\mathscr{C}_K$.

Example 2.1.4. The following partitions are always supercharacter theories of G.

(1) Take $\mathcal{X}$ and $\mathcal{K}$ to be the partitions of $\mathrm{Irr}(G)$ and $\mathrm{Cl}(G)$, into singletons:

$$\mathcal{X} = \{\{\chi\} \mid \chi \in \mathrm{Irr}(G)\} \quad \text{and} \quad \mathcal{K} = \{\{C\} \mid C \in \mathrm{Cl}(G)\}.$$

In this case, for each singleton $X = \{\chi\}$, the supercharacter $\sigma_X = \chi(1_G)\chi$ is a positive integer multiple of the irreducible character χ. The superclasses are the usual conjugacy classes. The corresponding supercharacter table is the matrix DT where T is the usual character table of G, and D is the diagonal matrix $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ where $d_i = \chi_i(1_G)$ is the degree of the character χ_i in the ith row of T.

(2) A supercharacter theory of rank two is obtained by taking

$$\mathcal{X} = \{\mathrm{Irr}(G) - \{\varepsilon_G\}, \{\varepsilon_G\}\} \quad \text{and} \quad \mathcal{K} = \{\mathrm{Cl}(G) - \{1_G\}, \{1_G\}\}.$$

The corresponding supercharacter table T is given by

T	K_0	K_1
σ_0	1	1
σ_1	$\lvert G\rvert - 1$	-1

$$\begin{aligned} K_0 &= \{1_G\} \\ K_1 &= G - \{1_G\} \\ \sigma_0 &= \varepsilon_G \\ \sigma_1 &= \rho_G - \varepsilon_G \end{aligned}$$

where

$$\rho_G = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1_G)\chi$$

is the character of the regular representation.

A function $G \to \mathbb{C}$ (or equivalently, a linear map $\mathbb{C}G \to C$) that is constant on superclasses is a **superclass function**. The following theorem shows how supercharacters and superclass functions retain some of the essential properties of the ordinary irreducible characters and class functions of G.

Proposition 2.1.5 (Diaconis–Isaacs [11, Prop. 2.2]). *Let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory of G. The following are true.*

(1) *The singletons $\{\varepsilon_G\}$ and $\{1_G\}$ are members of $\mathcal{X}$ and $\mathcal{K}$, respectively.*

(2) *The set* $\{\sigma_X \mid X \in \mathcal{X}\}$ *is a basis for the space of superclass functions.*

(3) *The set* $\{\widehat{\mathscr{C}_K} \mid K \in \mathcal{K}\}$ *is a basis for a subalgebra of* $Z(\mathbb{C}G)$*, where*

$$\widehat{\mathscr{C}_K} = \sum_{g \in \mathscr{C}_K} g.$$

(4) *The partitions* $\mathcal{X}$ *and* $\mathcal{K}$ *uniquely determine each other.*

(5) *Each automorphism of* $\mathbb{C}$ *induces a permutation of* $\mathcal{X}$*.*

(6) *Each automorphism of* $\mathbb{C}$ *induces a permutation of* $\mathcal{K}$*.*

□

There are several obvious choices for a partial order on the set of supercharacter theories of G, using the fact that these are pairs of set partitions. Denote the set of all partitions of a given set X by $\mathrm{Part}(X)$. It is well-known that $\mathrm{Part}(X)$ is partially ordered by 'refinement', in the sense of the following definition.

Definition 2.1.6. Given a finite set X, and partitions $\mathcal{P}, \mathcal{Q} \in \mathrm{Part}(X)$, we say that $\mathcal{P}$ **refines** $\mathcal{Q}$ (equivalently, P partitions X more finely, or less coarsely, than Q) if every $X \in \mathcal{P}$ is a subset of of some $Y \in \mathcal{Q}$. We denote this by $\mathcal{P} \leq \mathcal{Q}$.

Remark 2.1.7. It follows that $\mathcal{P}$ refines $\mathcal{Q}$ if and only if every $X \in \mathcal{Q}$ is a union of elements of P.

If A and B are two finite sets, a partial order $\leq$ can be determined on $\mathrm{Part}(A) \times \mathrm{Part}(B)$ using refinement in several ways. Some examples given below.

(1) Set $(\mathcal{X}, \mathcal{P}) \leq (\mathcal{Y}, \mathcal{Q})$ whenever $\mathcal{X} \leq \mathcal{Y}$.

(2) Set $(\mathcal{X}, \mathcal{P}) \leq (\mathcal{Y}, \mathcal{Q})$ whenever $\mathcal{P} \leq \mathcal{Q}$.

(3) Set $(\mathcal{X}, \mathcal{P}) \leq (\mathcal{Y}, \mathcal{Q})$ whenever $\mathcal{X} \leq \mathcal{Y}$ and $\mathcal{P} \leq \mathcal{Q}$.

In fact, these orders all coincide when we restrict them from $\mathrm{Part}(\mathrm{Irr}(G)) \times \mathrm{Part}(\mathrm{Cl}(G))$ to the set of supercharacter theories of G, as shown by Hendrickson [14] in the following result.

Proposition 2.1.8 (Hendrickson [14, Cor. 3.4])**.** *Let $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{Y}, \mathcal{L})$ be supercharacter theories of G. Then $\mathcal{X} \leq Y$ if and only if $\mathcal{K} \leq \mathcal{L}$.* □

Example 2.1.9. The supercharacter theories of Example 2.1.4 are the smallest (1) and largest (2) possible supercharacter theories, respectively, in this partial order.

2.2 Schur rings

In this section, we will assume that k is an algebraically closed field of characteristic zero. Schur rings are certain subalgebras of the group algebra kG that have a special basis arising from a set partition of the group G. In the following definition, we employ some notation: if X is a subset of G, then we write

$$\widehat{X} := \sum_{g \in X} g \in kG \quad \text{and} \quad X^{-1} = \{g^{-1} \mid g \in X\}.$$

Definition 2.2.1. A subalgebra $A \subseteq kG$ is a **Schur ring** of G if there exists a partition $\mathcal{P}$ of G satisfying

(1) the set $\{\widehat{X} \mid X \in \mathcal{P}\}$ is a basis for A,

(2) if $X \in \mathcal{P}$, then $X^{-1} \in \mathcal{P}$.

If A is a Schur ring, the partition $\mathcal{P} = \mathcal{P}(A)$ is uniquely determined by A, and $\mathcal{P}$ is called a **Schur partition**.

Example 2.2.2. The following subalgebras are always Schur rings of G.

(1) The algebra kG itself with partition $\mathcal{P}(kG) = \{\{g\} \mid g \in G\}$.

(2) The center $Z(kG)$ with partition $\mathcal{P}(Z(kG)) = \{C \mid C \in \mathrm{Cl}(G)\}$.

(3) The subalgebra $A = k\text{-span}\{1_G, \widehat{G}\}$ with partition $\mathcal{P}(A) := \{\{1_G\}, G - \{1_G\}\}$.

Not every partition is a Schur partition. For instance, a partition $\mathcal{P}$ cannot be a Schur partition unless $\{1_G\} \in \mathcal{P}$, since otherwise the span of the basis $\{\widehat{X} \mid X \in \mathcal{P}\}$ would not contain

1_G, and consequently this would not be a subalgebra of kG. It is not a coincidence that this condition resembles the property that $\{1_G\} \in \mathcal{K}$ for any supercharacter theory $\mathcal{K}$. In fact, we will see that the partition of G into superclasses is always a Schur partition. The converse is true whenever the associated Schur ring A is contained in $Z(kG)$, as we will discuss in §2.3.

2.2.1 The lattice of Schur rings

There is a relationship between certain set-algebraic operations on the set of Schur rings of G and the order-theoretic operations on their associated partitions, using the partial order inherited from $\mathrm{Part}(G)$. By set-algebraic operations, we mean that given Schur rings A, B, we can form new Schur rings by taking their intersection $A \cap B$, and the Schur ring that they generate, i.e. the smallest Schur ring containing both A and B. This follows immediately from an alternate characterization [24] of Schur rings in terms of the so called **Hadamard product** or 'circle product' on kG. Let $x, y \in kG$ so that

$$x = \sum_{g\in G} a_g g \quad \text{and} \quad y = \sum_{g\in G} b_g g$$

for some $a_g, b_g \in k$. Then the vector space kG is an algebra with multiplication

$$x \circ y = \sum_{g\in G} a_g b_g g$$

and identity $\widehat{G}$. As an algebra, this is isomorphic to the direct sum of $|G|$ copies of k. The following characterization can be found e.g., in a recent survey of Schur rings by Muzychuk and Ponomarenko [24].

Lemma 2.2.3 (Muzychuk–Ponomarenko [24]). *Let A be a subalgebra of kG. Then A is a Schur ring of G if and only if*

(1) A is closed under the Hadamard product and contains $\widehat{G}$, and

(2) if $x = \sum_{g\in G} a_g g \in A$, then $x^{(-1)} := \sum_{g\in G} a_g g^{-1} \in A$.

From this characterization, we have the following corollary, as observed by Muzychuck and Ponomarenko [24].

Corollary 2.2.4. *Let A, B be Schur rings of G. Then $A \cap B$ is a Schur ring of G.*

Proof. Since A, B are subalgebras of kG with respect to the usual multiplication, and also with respect to the Hadamard product, $A \cap B$ is again a subalgebra with respect to both algebra structures on kG. Let $x \in A \cap B$. Then $x^{(-1)} \in A$ and $x^{(-1)} \in B$, so $x^{(-1)} \in A \cap B$ and it follows from Lemma 2.2.3 that $A \cap B$ is a Schur ring of G. □

It follows that finite intersections of Schur rings are Schur rings. Since there are only finitely many partitions of G, there are only finitely many Schur rings of G. Therefore, given two Schur rings A and B, there exists a smallest Schur ring containing both (there is at least one Schur ring containing both, which is G itself), which we will refer to as the Schur ring generated by A and B. The set of Schur rings of G is partially ordered by inclusion, and this partial order becomes a lattice with respect to intersection and generation. We now consider a related lattice structure on $\mathrm{Part}(G)$.

For an arbitrary finite set X, recall that $\mathrm{Part}(X)$ is partially-ordered by refinement. In fact, the partial order on $\mathrm{Part}(X)$ determines a lattice structure on $\mathrm{Part}(X)$ as follows. The meet (greatest lower bound) $\mathcal{P} \wedge \mathcal{Q}$ of two partitions $\mathcal{P}, \mathcal{Q} \in \mathrm{Part}(X)$ is the coarsest partition that refines both $\mathcal{P}$ and $\mathcal{Q}$. Concretely, this is given by the formula

$$\mathcal{P} \wedge \mathcal{Q} = \{X \cap Y \mid X \in \mathcal{X}, Y \in \mathcal{Y}\}.$$

The join (least upper bound) $\mathcal{P} \vee \mathcal{Q}$ of two partitions $\mathcal{P}, \mathcal{Q} \in \mathrm{Part}(X)$ is the finest partition refined by both $\mathcal{P}$ and $\mathcal{Q}$. Proposition 2.2.6 relates the intersection of Schur rings to the join of the associated partitions. We will use the following lemma, proved by Hendrickson [14].

Lemma 2.2.5 (Hendrickson [14, Lem. 3.2])**.** *Let $\mathcal{P}, \mathcal{Q} \in \mathrm{Part}(G)$. Then*

$$k\text{-span}\left\{\widehat{Z} \mid Z \in \mathcal{P} \vee \mathcal{Q}\right\} = k\text{-span}\left\{\widehat{X} \mid X \in \mathcal{P}\right\} \cap k\text{-span}\left\{\widehat{Y} \mid Y \in \mathcal{Q}\right\}.$$

□

Proposition 2.2.6. *Let A, B be Schur rings of G with associated partitions $\mathcal{P}$ and $\mathcal{Q}$. Then $A \subseteq B$ if and only if $\mathcal{Q} \leq \mathcal{P}$, and $A \cap B$ has partition $\mathcal{P} \vee \mathcal{Q}$.*

Proof. Let A, B be Schur rings of G. Then A has basis $\mathcal{B}_A = \{\widehat{X} \mid X \in \mathcal{P}\}$ and B has basis $\mathcal{B}_B = \{\widehat{Y} \mid Y \in \mathcal{Q}\}$. Suppose $A \subseteq B$. Then each element $\widehat{X}$ of the basis $\mathcal{B}_A$ is a linear combination of the elements of $\mathcal{B}_B$. By the linear independence of the elements of G, the coefficients in this expansion must all be 0 or 1. It follows that X the union of the sets Y such that $\widehat{Y}$ appears in this decomposition with coefficient 1, again by the linear independence of the elements of G in kG. Then since each $Y \in \mathcal{Q}$ is a subset of some $X \in \mathcal{P}$, we have that $\mathcal{Q} \leq \mathcal{P}$. For the reverse direction, if $\mathcal{Q} \leq \mathcal{P}$, we have that every element $X \in \mathcal{P}$ is a union of elements $Y \in \mathcal{Q}$. It follows that each basis element $\widehat{X} \in \mathcal{B}_A$ is a simple sum of elements of the basis $\mathcal{B}_B$, so $A \subseteq B$. The fact that $\mathcal{P} \vee \mathcal{Q}$ is the Schur partition corresponding to $A \cap B$ now follows from Lemma 2.2.5. □

Remark 2.2.7. It is not the case in general that the Schur ring generated by two Schur rings A and B has associated partition $P \wedge Q$. This is only the case when the $A + B$ is a Schur ring, as in the wedge product construction discussed in §2.4.3, which makes the wedge product a valuable construction from the point of view of supercharacter theories.

2.3 Supercharacter theories correspond to Schur rings

We return to the case $k = \mathbb{C}$ for the remainder of this section, in order to discuss the close relationship between supercharacter theories and Schur rings. In particular, each Schur ring contained in $Z(\mathbb{C}G)$ encodes the information contained in either of the partitions $(\mathcal{X}, \mathcal{K})$ of a unique supercharacter theory, and all supercharacter theories can be described in this manner. A description of this correspondence is given by Hendrickson [14].

Proposition 2.3.1 (Hendrickson [14, Prop. 2.4])**.** *Let G be a finite group. There is a one-to-one* *correspondence*

$$\left\{\begin{array}{c}\textit{supercharacter}\\ \textit{theories of } H\end{array}\right\} \quad\longleftrightarrow\quad \left\{\begin{array}{c}\textit{central}\\ \textit{Schur rings of } H\end{array}\right\}$$

$$(\mathcal{X}, \mathcal{K}) \quad\longmapsto\quad A_{\mathcal{K}} := k\text{-span}\left\{\widehat{\mathscr{C}_K} \mid K \in \mathcal{K}\right\}.$$

such that $(\mathcal{X}, \mathcal{K}) \leq (\mathcal{Y}, \mathcal{L})$ *if and only if* $A_{\mathcal{L}} \subseteq A_{\mathcal{K}}$. □

Example 2.3.2. Under this correspondence, the largest central Schur ring, the center $Z(\mathbb{C}G)$ itself, corresponds to the finest supercharacter theory, which is the usual character theory (up to rescaling the irreducible characters). The smallest Schur ring $\mathbb{C}$-span$\{1_G, \widehat{G}\}$ is always central, and this corresponds to the coarsest supercharacter theory, which is the unique supercharacter theory of rank 2 from Example 2.1.4.

2.4 Schur ring lifts and products

In this section, let G be a finite group with subgroup $H \leq G$ and normal subgroup $N \lhd G$. Using the fact that the group algebra kH is a subalgebra of kG, we observe that the Schur rings of H are 'almost' Schur rings of G, and they can be extended to Schur rings of G by a process similar to adjoining an identity to a non-unital ring. By a similar construction, every Schur ring of G/N also gives rise to a Schur ring of G. These two constructions taken together give a way of forming Schur rings of G using the information from a Schur ring of N and a Schur ring of G/N.

2.4.1 Lifting Schur rings from a subgroup

When $H \leq G$, Schur rings can be lifted from H to G, by adjoining the element $\widehat{G} \in G$.

Lemma 2.4.1. *Let* $H \leq G$ *and let* A *be a Schur ring of* H *associated to the partition* P. *Then*

$$A' = k\text{-span}\{A \cup \{\widehat{G}\}\}$$

is a Schur ring of G *with associated partition* $\mathcal{P}' = \mathcal{P} \cup \{G - H\}$.

Proof. Let $\mathcal{P}'$ be as above. Since $\mathcal{P}$ is a partition of H, $\mathcal{P}'$ is clearly a partition of G. Furthermore, $g \in G - H$ implies $g^{-1} \in G - H$, so $\mathcal{P}'$ has the desired property that $X \in \mathcal{P}'$ implies $X^{-1} \in \mathcal{P}'$. Since $\widehat{G} = \widehat{H} + \widehat{G - H}$, and $\widehat{H} \in A$, we can express A' as

$$A' = k\text{-span}\{A \cup \{\widehat{G}\}\} = k\text{-span}\{A \cup \{\widehat{G-H}\}\} = k\text{-span}\{\widehat{X} \mid X \in \mathcal{P}'\}.$$

The space A' is closed under multiplication, since $x\widehat{G} = \varepsilon_G(x)\widehat{G}$ for all $x \in kG$, and $1_G \in A \subseteq A'$, so A' is a subalgebra of kG. It follows that A' is a Schur ring with associated partition P'. □

2.4.2 Lifting Schur rings from a quotient group

When $N \leq G$ is normal, Schur rings can also be lifted from G/N to G. We can identify kG/N as a subspace of kG by the injective map

$$i(gN) = |N|^{-1}\widehat{gN} = |N|^{-1}g\widehat{N}$$

for all $g \in G$. The image of this map is an algebra, under the inherited multiplication and scalar multiplication of kG, but with identity $|N|^{-1}\widehat{N}$, and as such it is isomorphic to the algebra kG/N. To see this, we note that $(\widehat{N})^2 = |N|\widehat{N}$ and

$$i(gN)i(hN) = (|N|^{-1}g\widehat{N})(|N|^{-1}h\widehat{N}) = |N|^{-2}gh(\widehat{N})^2 = |N|^{-1}gh\widehat{N} = i(ghN)$$

for all $g, h \in G$.

Suppose $\pi\colon G \to G/N$ is the natural map to the quotient and let $\overline{\pi}$ be the linear extension $\overline{\pi}\colon kG \to kG/N$. Then $\overline{\pi}$ restricted to the subspace $i(G/N)$ gives a linear map

$$|N|^{-1}g\widehat{N} \mapsto gN$$

for all $g \in G$. Thus we have $\overline{\pi}(i(gN)) = gN$. If $\mathcal{P}$ is a partition of G/N, we can construct the partition

$$\pi^{-1}(\mathcal{P}) := \{\pi^{-1}(X) \mid X \in \mathcal{P}\}$$

of G, which is refined by the partition of G into cosets of N.

Lemma 2.4.2. *Let N be a normal subgroup of G, and let A be a Schur ring of G/N with associated partition $\mathcal{P}$. Identify kG/N as a subspace of kG via $gN \mapsto |N|^{-1}g\widehat{N}$ for all $g \in G$. Then*

$$A' = k\text{-span}\{A \cup \{1_G\}\}$$

is a Schur ring of G with associated partition $\mathcal{P}' = (\pi^{-1}(\mathcal{P}) - \{N\}) \cup \{\{1_G\}, \{N - \{1_H\}\}\}$.

Proof. Let P' be as above. Since $\pi^{-1}(\mathcal{P})$ is a partition of G, so is P'. Furthermore, $g \in N - \{1_G\}$ implies $g^{-1} \in N - \{1_G\}$, so P' has the desired property that $X \in P'$ implies $X^{-1} \in P'$. Since $\widehat{N} = \widehat{1}_G + \widehat{N - \{1_G\}}$, and $\widehat{N} \in A$, we can express A' as

$$A' = k\text{-span}\{A \cup \{1_G\}\} = k\text{-span}\{A \cup \{\widehat{N - \{1\}}\}\} = k\text{-span}\{\widehat{X} \mid X \in P'\}.$$

The space A' is clearly closed under multiplication, and $1_G \in A'$ by construction, so A' is a subalgebra of kG. It follows that A' is a Schur ring with associated partition P'. $\square$

Thus we can lift Schur rings from a subgroup N or quotient groups G/N, by viewing them as subspaces of kG and adjoining an appropriate element of kG, which increases the dimension by at most 1. In the next section, we combine these methods to give a description of the wedge product construction of Leung and Man [18, 19].

2.4.3 Wedge products

Let $N \lhd G$ be normal, and suppose $N \leq K \leq G$. It is possible to combine the information of a Schur ring A of K with that of a Schur ring B of G/N, to get a a new Schur ring $A \wedge B$ of G, using the lifts from subgroups and quotient groups. When $N = K$, this is an algebraic version of the $*$-product, via the correspondence of Proposition 2.3.1, which will be reviewed in §2.5.3.

One can always form the Schur ring generated by the lifts A' and B', in the sense of the smallest Schur ring containing both A' and B'. The problem is that it would be difficult to describe the associated partition in terms of $\mathcal{P}$ and $\mathcal{Q}$, the partitions associated to A and B. In particular, it would be hard to decide if the Schur ring generated by A' and B' is not all of kG. The following proposition shows that under certain conditions, the Schur ring generated by A' and B' is no larger than their linear span, and the associated partition is just the mutual refinement $\mathcal{P} \wedge \mathcal{Q}$ of $\mathcal{P}$ and $\mathcal{Q}$. The proposition has been restated in the context of the lifts $A \mapsto A'$ and $B \mapsto B'$ of the associated Schur rings, to emphasize the connection with the join of the corresponding partitions.

Proposition 2.4.3 (Leung–Man [19]). *Suppose $N \leq K \leq G$ and that N is normal in G. Let A be a Schur ring of K with partition $\mathcal{P}$, and let B be a Schur ring of G/N with partition $\mathcal{Q}$, such that $\widehat{N} \in A$ and that $\overline{\pi}(A) = B \cap kK/H$. Then $A' + B'$ is a Schur ring of G with associated partition $\mathcal{P}' \wedge \mathcal{Q}'$.* □

Definition 2.4.4. In the situation of Proposition 2.4.3, the Schur ring $A' + B'$ is denoted by $A \wedge B$ and called the **wedge product** of A and B.

When it is defined, the wedge product is uniquely characterized as the smallest Schur ring S such that $S \cap kK = A$ and $\pi(S) = B$. The fact that $A \wedge B$ has this property is shown by Leung and Man [19]. To see that $A \wedge B$ is the smallest such Schur ring, suppose S is another Schur ring such that $S \cap kK = A$ and $\pi(S) = B$. Since $S \cap kK = A$, we have that S must contain A'. Similarly, since $\pi(S) = B$, so S must contain B'. It follows that S also contains $A \wedge B = A' + B'$.

Using the correspondence of Proposition 2.3.1 we consider the supercharacter theory constructions that correspond to the above. In the case of the wedge product, we restrict to the case of $N = K$, which gives the $*$-product described by Hendrickson [14].

2.5 Supercharacter theory lifts and products

Let G be a finite group with normal subgroup $N \leq G$ and let i and π be the natural inclusion and projection

$$N \xrightarrow{i} G \xrightarrow{\pi} G/N.$$

For each of the constructions in §2.4, a corresponding construction for supercharacter theories is found by restricting to the case of central Schur rings, and applying the correspondence of Proposition 2.3.1.

2.5.1 Lifting supercharacter theories from a subgroup

We are interested in characters in this section, so we restrict to the case $k = \mathbb{C}$. We will lift supercharacter theories from a subgroup N by lifting the associated central Schur rings. To ensure that a central Schur ring in $\mathbb{C}N$ is also central in $\mathbb{C}G$, a necessary and sufficient condition is that the superclasses of N are all unions of conjugacy classes of G. Since in particular this implies that N itself is a union of conjugacy classes of G, we only consider lifts from normal subgroups.

Lemma 2.5.1. *Suppose $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of the normal subgroup N of G such that the superclass $\mathscr{C}_K$ is a union of conjugacy classes of G for all $K \in \mathcal{K}$. Then there exists a supercharacter theory $(\mathcal{X}', \mathcal{K}')$ of G where*

$$\mathcal{K}' = \{K' \mid K \in \mathcal{K}\} \cup \{K_{G \setminus N}\},$$

with $K' = \{C \in \mathrm{Cl}(G) \mid C \subseteq \bigcup K\}$ and $K_{G\setminus N} = \{C \in \mathrm{Cl}(G) \mid C \cap N = \varnothing\}$.

Proof. Since each superclass of $(\mathcal{X}, \mathcal{K})$ is a union of conjugacy classes of G, the associated Schur ring A is contained in $Z(kG)$. Since $\widehat{G} \in Z(kG)$, we also have that the lift A' is contained in $Z(kG)$, and it is easy to check that $(\mathcal{X}', \mathcal{K}')$ is the corresponding supercharacter theory. $\square$

2.5.2 Lifting supercharacter theories from a quotient group

Now we consider lifts of supercharacter theories from G/N where N is a normal subgroup of G. If we have a partition $\mathcal{K}$ of the conjugacy classes of G/N, this already gives a partition $\mathcal{K}'$ of the conjugacy classes of G in a natural way, i.e. we can simply combine the cosets of N according to $\mathcal{P}$, since the cosets of N are already unions of conjugacy classes. The partition $\mathcal{K}'$ is almost a partition of $\mathrm{Cl}(G)$ into superclasses, as the next lemma shows.

Lemma 2.5.2. *Suppose $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of the quotient G/N for some normal subgroup N of G, with natural map $\pi\colon G \to G/N$. Let $K_1 = \{1_G\}$ be the conjugacy class of the identity and let*

$$\mathcal{K}' = \{K' \mid K \in \mathcal{K}\} \cup \{K_1\}.$$

Then there exists a supercharacter theory $(\mathcal{X}', \mathcal{K}')$ of G.

Proof. Viewing kG/N as a subspace of kG, the Schur ring $\mathcal{A}$ associated to $(\mathcal{X}, \mathcal{K})$ has a partition coarser than the partition of G into conjugacy classes, so $A \subseteq Z(kG)$. Since $1_G \in Z(kG)$, it follows that the lift A' is contained in $Z(kG)$, and $(\mathcal{X}', \mathcal{K}')$ is the associated supercharacter theory. □

2.5.3 The $*$-product

We can now describe the $*$-product introduced by Hendrickson [14] as way of combining supercharacter theories of N and G/N. This is the supercharacter theory corresponding to the wedge product of the corresponding central Schur rings A of N and B of G/N in the special case where the subgroup K containing N is chosen to be N itself.

Proposition 2.5.3 (Hendrickson [14])**.** *Let N be a normal subgroup of G. Suppose $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of N for which each superclass is a union of conjugacy classes of G, and suppose that $(\mathcal{Y}, \mathcal{L})$ is a supercharacter theory of G/N. Then $(\mathcal{X}' \wedge \mathcal{Y}', \mathcal{K}' \wedge L')$ is a supercharacter theory of G, where $(\mathcal{X}', \mathcal{K}')$ and $(\mathcal{Y}', \mathcal{L}')$ are the lifts to G of $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{Y}, \mathcal{L})$, respectively.*

Proof. Since N is normal in G, let $K = N$ and note that this is a subgroup of G containing N. Let A and B be the Schur rings corresponding to $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{Y}, \mathcal{L})$, respectively. Then A may be viewed as a supercharacter theory of K that contains $\widehat{N}$. Furthermore, since $\overline{\pi}(A) = k\text{-span}\{1_{G/N}\}$, it is certainly true that $B \cap kK/N = B \cap kN/N = k\text{-span}\{1_{G/N}\} = \overline{\pi}(A)$. Thus we may form the wedge product $A \wedge B = A' + B'$. This Schur ring is central in kG, since A' and B' are central in kG. The corresponding supercharacter theory has the partition $\mathcal{K}' \wedge \mathcal{L}'$, and it follows that the partition of characters must be $\mathcal{X}' \wedge \mathcal{Y}'$. □

Chapter 3

Hopf algebras

In this expository chapter, we give the diagrammatic definitions of algebra and coalgebra in §3.1 with the goal of defining Hopf algebras in §3.2, as in e.g. Montgomery [22], or Dăscălescu et al. [10]. We then recall some of the main structure results for semisimple Hopf algebras as we review integrals, Hopf ideals, normal Hopf subalgebras, and Hopf quotients. Frobenius algebras are briefly reviewed in §3.3, in view of the natural Frobenius structure of semisimple Hopf algebras. This Frobenius structure is then used in §3.4 to present the character theory for semisimple Hopf algebras developed by Witherspoon [27] and Cohen and Westreich [8, 9].

3.1 Algebras and coalgebras

In this section, k is an arbitrary field. Most of the objects we consider will have an underlying vector space structure, and most of the maps we consider will be linear maps. Therefore, we take a moment to recall some of the common notions from linear algebra that will be used frequently in what follows.

3.1.1 Vector spaces

All vector spaces are k-vector spaces unless otherwise specified, and $\otimes = \otimes_k$. The class of vector spaces are the objects of the category $\mathbf{Vect}_k$. Given vector spaces V and W, the morphisms $\mathrm{Hom}_k(V, W)$ are the linear maps $V \to W$. Evaluation gives a linear map $\langle \cdot, \cdot \rangle \colon V^* \otimes V \to k$ defined by $\langle \phi, v \rangle = \phi(v)$.

The **dual space** V^* is the vector space $\mathrm{Hom}_k(V,k)$ with pointwise addition and scalar multiplication. Given a linear map $f\colon V \to W$, the **transpose** of f is the linear map $f^*\colon W^* \to V^*$ defined by $\langle f^*(\phi), v\rangle = \langle \phi, f(v)\rangle$ for all $\phi \in W^*$, $v \in V$. A contravariant functor $\mathbf{Vect}_k \to \mathbf{Vect}_k$ is obtained by the mapping $V \to V^*$ and $f \mapsto f^*$ for all vector spaces V and linear maps f. When V is finite dimensional, there exists a natural isomorphism $\mathrm{ev}\colon V \mapsto V^{**}$ defined by

$$\langle \mathrm{ev}(v), \phi\rangle = \langle \phi, v\rangle$$

for all $v \in V$, $\phi \in V^*$, and it will sometimes be convenient to identify $V = V^{**}$ via evaluation in this case.

The twist map $\tau(V,W)\colon V \otimes W \to W \otimes V$ is a linear isomorphism defined by $w \otimes v \mapsto v \otimes w$. If the spaces V, W are clear from context, then we simply write $\tau = \tau(V,W)$. The map $V^* \otimes V^* \to (V \otimes V)^*$ defined by $(\phi \otimes \psi)(u \otimes v) = \phi(u)\psi(v)$ for all $\phi, \psi \in V^*$, $u, v \in V$ is injective, so we view this as an inclusion map, identifying $V^* \otimes V^*$ as a subspace of $(V \otimes V)^*$. When V is finite dimensional, we make the identification $V^* \otimes V^* = (V \otimes V)^*$, since in this case the map is an isomorphism.

3.1.2 Algebras and modules

In this section we review the basic definitions and terminology of k-algebras and their modules for the purpose of dualizing these objects in §3.1.3 and setting some notation.

Definition 3.1.1. An **algebra** (or k-algebra) is a tuple (A, m, u) consisting of a vector space A together with a **multiplication** $m\colon A \otimes A \to A$ and **unit** $u\colon k \to A$, which are linear maps such that the following diagrams commute.

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes m"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[d, "m"] & \\
k \otimes A \arrow[ur, "u \otimes \mathrm{id}"] \arrow[r] & A & A \otimes k \arrow[ul, "\mathrm{id} \otimes u"'] \arrow[l]
\end{tikzcd}$$

These properties of (A, m, u) are referred to as associativity and the unit law, respectively. The maps $k \otimes A \to A$ and $A \otimes k \to A$ are linear isomorphisms determined by scalar multiplication, i.e. by $1 \otimes a \mapsto a$ and $a \otimes 1 \mapsto a$ for all $a \in A$.

Remark 3.1.2. When no confusion is likely, we denote the algebra (A, m, u) by A, the multiplication $m(x, y)$ by xy, and $u(1)$ by 1_A.

Definition 3.1.3. Given an algebra A, a **left A-module** is a pair (M, φ), consisting of a vector space M, together with a linear map $\varphi\colon A \otimes M \to M$ called the **structure map**, such that the following diagrams commute.

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "m \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes \varphi"'] & A \otimes M \arrow[d, "\varphi"] \\ A \otimes M \arrow[r, "\varphi"] & M \end{tikzcd} \qquad \begin{tikzcd} k \otimes M \arrow[r, "u \otimes \mathrm{id}"] \arrow[dr] & A \otimes M \arrow[d, "\varphi"] \\ & M \end{tikzcd}$$

Similarly, a **right A-module** is a pair (M, φ), consisting of a vector space M, together with a structure map $\varphi\colon M \otimes A \to M$, such that the following diagrams commute.

$$\begin{tikzcd} M \otimes A \otimes A \arrow[r, "\mathrm{id} \otimes m"] \arrow[d, "\varphi \otimes \mathrm{id}"'] & M \otimes A \arrow[d, "\varphi"] \\ M \otimes A \arrow[r, "\varphi"] & M \end{tikzcd} \qquad \begin{tikzcd} M \otimes k \arrow[r, "\mathrm{id} \otimes u"] \arrow[dr] & M \otimes A \arrow[d, "\varphi"] \\ & M \end{tikzcd}$$

The maps $k \otimes M \to M$ and $M \otimes k \to M$ are linear isomorphisms determined by $1 \otimes m \mapsto m$ and $m \otimes 1 \mapsto 1$ as in Definition 3.1.1.

Remark 3.1.4. When the structure map is clear from context, we denote (M, φ) by M. When it is necessary to emphasize that M is a left (respectively right) A-module, then M is written as ${}_AM$ (respectively M_A). Equivalently, we say that A acts on M by φ, on the left (for a left A-module) or the right (for a right A-module).

Example 3.1.5. Viewing the multiplication $m\colon A \otimes A \to A$ as the structure map for an A-module, we obtain a left A-module ${}_AA$ or a right A-module A_A, which are called the left and right regular modules, respectively. Let $L_x\colon A \to A$ and $R_x\colon A \to A$ be left and right multiplication by an element $x \in A$. Taking the transpose of these maps, we get linear maps $L_x^*\colon A^* \to A^*$ and $R_x^*\colon A^* \to A^*$, from which we obtain right and left actions, respectively, of A on A^*. The left action $\rightharpoonup\colon A \otimes A^* \to A^*$ arises from the transpose of right multiplication. For $x \in A$ and $\phi \in A^*$ the action is defined by $x \rightharpoonup \phi = R_x^*(\phi)$, or equivalently by

$$\langle x \rightharpoonup \phi, y \rangle = \langle \phi, yx \rangle.$$

Similarly, the right action $\leftharpoonup\colon A^* \otimes A \to A^*$ arises from the transpose of left multiplication. For $x \in A$ and ϕ in A^*, the action is defined by $\phi \leftharpoonup x = L_x^*(\phi)$, or equivalently by

$$\langle \phi \leftharpoonup x, y \rangle = \langle \phi, xy \rangle.$$

The associated A-modules are denoted by ${}_AA^*$ and A^*_A.

Definition 3.1.6. An A-module M is **simple** if whenever N is a submodule of M, either $N = M$ or $N = 0$. An A-module is **semisimple** if it decomposes as a (finite) direct sum of simple modules. An algebra A is semisimple if ${}_AA$ is semisimple.

We will be primarily concerned with semisimple algebras. In this case the center has a canonical basis of orthogonal idempotents, a fact that we will make frequent use of in what follows.

Theorem 3.1.7 (Wedderburn)**.** *Let A be a finite dimensional semisimple algebra. Then every A-module is semisimple, and the regular module decomposes as*

$${}_AA = \bigoplus_{i=0}^{n} L_i$$

where each $L_i = Ae_i$ is a simple submodule for some $e_i \in A$ satisfying

(1) $e_ie_j = 0$ if $i \neq j$,

(2) $e_i^2 = e_i$ for all i,

(3) $\sum_{i=1}^n e_i = 1$.

Let A be an algebra. An element $e \in A$ is an **idempotent** if $e^2 = e$. A set $\{e_i\}$ of idempotents of A is **orthogonal** if $e_ie_j = 0$ whenever $i \neq j$. A set of orthogonal idempotents is **complete** if $\sum_i e_i = 1$. An idempotent is **primitive** if it cannot be written as a sum of two orthogonal idempotents. An idempotent is **central** if it is contained in $Z(A)$, and is **primitive central** if it is central, but cannot be written as a sum of two orthogonal central idempotents.

Proposition 3.1.8 (Wedderburn)**.** *Let A be a simple algebra over k. Then $A \cong M_n(k)$ where $M_n(k)$ is the full matrix ring of $n \times n$ matrices over k, for some n.*

This gives the following well-known result, of which we will make frequent use, that the primitive central idempotents are a basis for the center. In particular, when we have a commutative semisimple algebra A, the primitive idempotents and primitive central idempotents coincide, so that $A = Z(A)$ has a basis of primitive idempotents.

Proposition 3.1.9. *Let A be a finite dimensional semisimple algebra. The primitive central idempotents of A are a basis for $Z(A)$.*

Proof. Let A be a finite dimensional simisimple algebra. Then ${}_AA = \bigoplus_{i=0}^N L_i \cong \bigoplus_{i=0}^N M_{n_i}(k)$ for some N, since each L_i is simple. The center of $\bigoplus_{i=0}^N M_{n_i}(k)$ is the set of block-scalar matrices, with block sizes given by $n_1, n_2, \ldots, n_N$. Let E_i be the matrix that is the $n_i \times n_i$ identity matrix in the ith block, and zero in all other blocks. This is a basis for the block-scalar matrices, hence a basis for the center. Each E_i is idempotent, and any central idempotent must be a sum of the E_i, so these are the primitive central idempotents. The algebra isomorphism above must map the center of A to the center of $\bigoplus_{i=0}^N M_{n_i}(k)$, and it must map the set of primitive central idempotents of A to the matrices E_i. It follows that the primitive central idempotents of A are basis for $Z(A)$. $\square$

3.1.3 Coalgebras and comodules

In this section we introduce coalgebras and comodules, which are are obtained by dualizing the definitions in §3.1.2. These objects will be necessary for our definition of Hopf algebras in §3.2.

Definition 3.1.10. A **coalgebra** (or k-coalgebra) is a tuple (C, Δ, ε) consisting of a vector space C together with a **comultiplication** $\Delta \colon C \to C \otimes C$ and **counit** $\varepsilon \colon C \to k$, which are linear maps such that the following diagrams commute.

$$
\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "\Delta \otimes \mathrm{id}"'] \\
C \otimes C \arrow[u, "\mathrm{id} \otimes \Delta"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C \otimes C \arrow[dl, "\varepsilon \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes \varepsilon"] & \\
k \otimes C & & C \otimes k \\
 & C \arrow[ul] \arrow[ur] \arrow[uu, "\Delta"] &
\end{tikzcd}
$$

These properties of (C, Δ, ε) are referred to as coassociativity and the counit law, respectively. The maps $C \to k \otimes C$ and $C \to C \otimes k$ are determined linearly by $c \to 1 \otimes c$ and $c \to c \otimes 1$, respectively.

When working with a coalgebra C, we employ Sweedler notation by writing the result of applying the coproduct to an element $x \in C$ as

$$\Delta(x) = \sum_x x_{(1)} \otimes x_{(2)}.$$

The subscripts are symbolic, so that $x_{(1)}$ and $x_{(2)}$ are not specific elements of C, but placeholders for the first and second parts of the simple tensors arising after applying Δ to x, and the sum is over all such simple tensors. When no other subscripts are present so that confusion is unlikely, we will simply write $\sum x_1 \otimes x_2$ for $\Delta(x)$. The fact that $(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta$ is the same map by coassociativity can be expressed in this notation as

$$\sum (x_1)_1 \otimes (x_1)_2 \otimes x_2 = \sum x_1 \otimes (x_2)_1 \otimes (x_2)_2.$$

Therefore, to further simplify we denote either map as $\Delta_2 \colon H \to H \otimes H \otimes H$ and we denote the unique result of applying it to an element of $x \in C$ simply by

$$\Delta_2(x) = \sum x_1 \otimes x_2 \otimes x_3.$$

Iterating this process n times, we write

$$\Delta_n(x) = \sum x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1}$$

representing (for example) the application of the composition

$$\Delta_n = (\Delta \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}) \circ \cdots \circ (\Delta \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\Delta \otimes \mathrm{id}) \circ \Delta$$

to $x \in C$, but where the tensor factor of $C^{\otimes r}$ to which Δ is applied at each stage is irrelevant as a result of coassociativity.

Definition 3.1.11. A coalgebra C is **cocommutative** if $\Delta = \tau \circ \Delta$, i.e. if

$$\sum x_1 \otimes x_2 = \sum x_2 \otimes x_1$$

for all $x \in C$.

Example 3.1.12. Let G be a finite group. The vector space kG is a (cocommutative) coalgebra with coproduct and counit linearly determined by

$$\Delta(g) = g \otimes g \quad \text{and} \quad \varepsilon(g) = 1.$$

Definition 3.1.13. Let $(B, \Delta_B, \varepsilon_B)$ and $(C, \Delta_C, \varepsilon_C)$ be coalgebras. A linear map $f\colon B \to C$ is a coalgebra homomorphism if

(1) $(f \otimes f) \circ \Delta_B = \Delta_C \circ f$, and

(2) $\varepsilon_B = \varepsilon_C \circ f$.

Definition 3.1.14. Given a coalgebra C, a **left C-comodule** is a pair (M, φ), consisting of a vector space M, together with a linear map $\varphi\colon M \to C \otimes M$ called the **structure map**, such that the following diagrams commute.

$$\begin{tikzcd} C \otimes C \otimes M & C \otimes M \arrow[l, "\Delta \otimes \mathrm{id}"'] \\ C \otimes M \arrow[u, "\mathrm{id} \otimes \varphi"'] & M \arrow[l, "\varphi"'] \arrow[u, "\varphi"'] \end{tikzcd} \qquad \begin{tikzcd} k \otimes M & C \otimes M \arrow[l, "\varepsilon \otimes \mathrm{id}"] \\ & M \arrow[u, "\varphi"] \arrow[ul] \end{tikzcd}$$

Similarly, a **right C-comodule** is a pair (M, φ), consisting of a vector space M, together with a structure map $\varphi\colon M \otimes M \to C$, such that the following diagrams commute.

$$\begin{tikzcd} M \otimes C \otimes C & M \otimes C \arrow[l, "\mathrm{id} \otimes \Delta"'] \\ M \otimes C \arrow[u, "\varphi \otimes \mathrm{id}"'] & M \arrow[l, "\varphi"'] \arrow[u, "\varphi"'] \end{tikzcd} \qquad \begin{tikzcd} M \otimes k & M \otimes C \arrow[l, "\mathrm{id} \otimes \varepsilon"] \\ & M \arrow[u, "\varphi"] \arrow[ul] \end{tikzcd}$$

The maps $M \to k \otimes M$ and $M \to M \otimes k$ are determined as in Definition 3.1.10.

Remark 3.1.15. When the structure map is clear from context, denote (M, φ) by M. When it is necessary to emphasize that M is a left (respectively right) C-comodule, then we write ${}^{C}M$ (respectively M^{C}).

3.1.4 Duality

Finite dimensional algebras and coalgebras are dual objects in the sense of the following proposition, which is well-known.

Proposition 3.1.16. *Let A be a finite dimensional vector space. Then (A, m, u) is an algebra if and only if (A^*, m^*, u^*) is a coalgebra. If A is an algebra, then A is commutative if and only if the coalgebra A^* is cocommutative.*

Proof. The proof is a straightforward application of the definitions, after making the identification $A^* \otimes A^* = (A \otimes A)^*$. □

3.2 Hopf algebras

Combining the structures of algebras and coalgebras in a compatible way gives a self-dual object called a bialgebra. Hopf algebras are a particular class of bialgebra, which have a natural notion of a 'dual representation' precisely as we do in the finite group case. The basic definitions and examples of this section can be found in an introduction to Hopf algebras, such as Montgomery [22] or Dăscălescu et al. [10].

Definition 3.2.1. A **bialgebra** is a tuple $(B, m, u, \Delta, \varepsilon)$ such that (B, m, u) is an algebra, (B, Δ, ε) is a coalgebra, and the following (equivalent) conditions hold:

(1) m, u are coalgebra homomorphisms,

(2) Δ, ε are algebra homomorphisms.

If (A, m, u) is an algebra and (C, Δ, ε) is a coalgebra, then there exists an algebra structure on the vector space $\mathrm{Hom}_k(C, A)$, with multiplication given by the **convolution product**, defined by the composition

$$f * g = m \circ (f \otimes g) \circ \Delta$$

with unit element $u \circ \varepsilon$. To verify that $u \circ \varepsilon$ is the multiplicative identity, let $f \in \mathrm{Hom}(C, A), x \in C$ and note that

$$f * (u \circ \varepsilon)(x) = \sum \varepsilon(x_2) f(x_1) 1_A = f\left(\sum \varepsilon(x_2) x_1\right) = f(x)$$

by the counit law, and a similar calculation shows that $(u \circ \varepsilon) * f = f$.

Example 3.2.2. If C is a coalgebra, then there is a convolution product on C^*, since $C^* = \mathrm{Hom}(C, k)$, and k is naturally a k-algebra. This is the transpose of the comultiplication of C restricted to the subspace $C^* \otimes C^*$ of $(C \otimes C)^*$.

Definition 3.2.3. A **Hopf algebra** is a tuple $(H, m, u, \Delta, \varepsilon, S)$ such that $(H, m, u, \Delta, \varepsilon)$ is a bialgebra, and $S\colon H \to H$ is an algebra antihomomorphism, called the **antipode**, such that the following diagram commutes.

$$\begin{tikzcd}
H \otimes H \arrow[d, "S \otimes \mathrm{id}"] & H \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "u \circ \epsilon"] & H \otimes H \arrow[d, "\mathrm{id} \otimes S"] \\
H \otimes H \arrow[r, "m"] & H & H \otimes H \arrow[l, "m"']
\end{tikzcd}.$$

Example 3.2.4. If $(H, m, u, \Delta, \epsilon, S)$ is a finite dimensional Hopf algebra, then so is

$$H^* = (H^*, \Delta^*, \epsilon^*, m^*, u^*, S^*),$$

after identifying $(H \otimes H)^* \cong H^* \otimes H^*$. The Hopf algebra H^* is called the dual of H.

Remark 3.2.5. When confusion is possible, we will use a subscript to indicate the Hopf algebra to which an operation belongs. For example, S_H denotes the antipode of H, while S_{H^*} denotes the the antipode of H^*, which happens to be the transpose $(S_H)^*$.

Example 3.2.6. Let G be a finite group. Then the group algebra kG is a finite dimensional cocommutative Hopf algebra with comultiplication, counit, and antipode determined linearly by

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad \text{and} \quad S(g) = g^{-1}$$

for all $g \in G$.

Example 3.2.7. The dual of the group algebra kG^* is a finite dimensional commutative Hopf algebra. Let ϕ_g be the basis of kG^* dual to $\{g \mid g \in G\}$. Then the operations of kG^* are linearly determined by

(1) $\phi_g \phi_h = \delta_{gh} \phi_g$,

(2) $\Delta(\phi_g) = \sum_{h \in G} \phi_{gh^{-1}} \otimes \phi_h$,

(3) $u(1) = \sum_{g \in G} \phi_g$,

(4) $\epsilon(\phi_g) = 1$,

(5) $S(\phi_g) = \phi_{g^{-1}}$,

for all $g \in G$, where δ is the Kronecker delta.

Definition 3.2.8. Given Hopf algebras H and K, a linear map $f \colon H \to K$ is a **Hopf algebra homomorphism** (or Hopf algebra map) if f is a homomorphism of H and K as bialgebras.

Remark 3.2.9. If $f \colon H \to K$ is a Hopf algebra homomorphism, then $f \circ S_H = S_K \circ f$.

3.2.1 Integrals

Let H be a Hopf algebra. We recall the definition and basic properties of integrals, which are certain elements that are of key importance to the Frobenius algebra structure of H in the semisimple case. Montgomery [22] and Dăscălescu et al. [10] again provide good resources for the results of this section.

Definition 3.2.10. An element $x \in H$ is a left (respectively right) **integral** if $hx = \varepsilon(h)x$ for all $h \in H$ (respectively $xh = \epsilon(h)x$).

The set of left (respectively right) integrals is an ideal of H, denoted by I_L (respectively I_R). To see this, suppose $\Lambda \in \mathcal{I}_L$, and let $x \in H$. Then $x\Lambda$ is again a left integral, since for all $h \in H$

we have

$$h(x\Lambda) = h(\varepsilon(x)\Lambda) = \varepsilon(x)h\Lambda = \varepsilon(x)\varepsilon(h)\Lambda = \varepsilon(h)(\varepsilon(x)\Lambda) = \varepsilon(h)(x\Lambda).$$

But Λx is also a left integral, since

$$h(\Lambda x) = (h\Lambda)x = \varepsilon(h)(\Lambda x).$$

When $\mathcal{I}_L = \mathcal{I}_R$, then H is said to be **unimodular**. We will mainly be concerned with semisimple Hopf algebras, which are finite dimensional (here semisimple simply means semisimple as an algebra). A result of Larson–Sweedler shows that this condition is sufficient to ensure that there is a unique left and right integral, up to rescaling.

Proposition 3.2.11 (Larson–Sweedler)**.** *Let H be a finite dimensional Hopf algebra. Then*

$$\dim(\mathcal{I}_L) = \dim(\mathcal{I}_R) = 1$$

and $S_H(\mathcal{I}_L) = (\mathcal{I}_R)$.

The integrals of a finite dimensional Hopf algebra H control whether or not H is semisimple, by the following generalization of Maschke's theorem.

Proposition 3.2.12 (Larson–Sweedler)**.** *Let H be a finite dimensional Hopf algebra. Then the following are equivalent:*

(1) H is semisimple;

(2) $\varepsilon_H(\mathcal{I}_L) \neq 0$;

(3) $\varepsilon_H(\mathcal{I}_R) \neq 0$.

Example 3.2.13. If $H = kG$ for an arbitrary field k of characteristic p and finite group G, then $\Lambda_H = \widehat{G} \neq 0$ is a left and right integral of H, so $\mathcal{I}_L = \mathcal{I}_R = H\Lambda_H$. Since $\varepsilon_H(\Lambda_H) = |G|$ in this case, the condition $\varepsilon_H(\mathcal{I}_L) \neq 0$ is equivalent to the condition that p does not divide $|G|$. Thus the above states that kG is semisimple whenever p does not divide $|G|$, which is one way of formulating Maschke's theorem for finite groups.

If H is not only finite dimensional but also semisimple, then a corollary of Maschke's theorem gives that $\mathcal{I}_L = \mathcal{I}_R$, so that we no longer need to distinuish between left and right integrals. When H is semisimple, it is also cosemisimple; i.e. H^* is semisimple. In this case we will use λ_H to denote the unique idempotent of integral of H^*. That is, we fix the integral λ_H by choosing any nonzero integral and rescaling so that $\epsilon_{H^*}(\lambda_H) = \lambda_H(1) = 1$, which in turn gives $\lambda_H^2 = \lambda_H(1)\lambda_H = \lambda_H$. After fixing $\lambda_H \in H^*$, we fix an integral $\Lambda_H \in H$ by rescaling any nonzero integral so that $\lambda_H(\Lambda_H) = 1$.

3.2.2 Hopf subalgebras, ideals, and quotients

In the next chapter, we will be concerned with Hopf subalgebras and quotients of semisimple Hopf algebras. As with much of the theory of semisimple Hopf algebras, this proceeds similarly to the corresponding theory for finite groups their group algebras.

Definition 3.2.14. Let H be a Hopf algebra. A subalgebra $K \subseteq H$ satisfying $\Delta_H(K) \subseteq K \otimes K$ and $S_H(K) \subseteq (K)$ is a **Hopf subalgebra** of H.

Remark 3.2.15. In general if C is a coalgebra, then a subspace D with $\Delta(D) \subseteq D \otimes D$ is itself a coalgebra, with counit map ε_D given by the restriction of ε_C. In the above definition then, K is a subalgebra and a subcoalgebra, which is closed under the antipode. The restrictions of these operations from H to K give K the structure of a Hopf algebra.

Example 3.2.16. If $H = kG$ is the group algebra of a finite group, the only Hopf subalgebras are the subalgebras of the form kK, where K is a subgroup of G, and we can identify kK with the group algebra of the group K by linearly extending the inclusion $K \to G$.

For a finite group G, the group algebra kG acts on itself by linearly extending the action of G on itself by conjugation. This is called the left or right **adjoint action**, depending on whether we extend the left action $g \cdot x = gxg^{-1}$ or the right action $x \cdot g = g^{-1}xg$ for all $x, g \in G$. This is a special case of a more general adjoint action of any Hopf algebra on itself. These left and right adjoint actions are given by

$$h \cdot x = \sum h_1 x S_H(h_2) \quad \text{and} \quad x \cdot h = \sum S_H(h_1) x h_2$$

respectively for all $x, h \in H$. In the case $H = kG$ the only Hopf subalgebras that are closed under the adjoint action are the Hopf subalgebras kN, where $N \triangleleft G$ is a normal subgroup. This motivates the following definition.

Definition 3.2.17. Let H be a Hopf algebra. A Hopf subalgebra N is **left** (respectively **right**) **normal** if it is closed under the left (respectively right) adjoint action of H on itself.

While normal subgroups are precisely the kernels (in the sense of group theory) of group homomorhpisms, the kernel (in the sense of linear algebra) of a Hopf algebra homomorphism $f \colon H \to K$ must be an ideal with additional structure, since H is an algebra and f respects the algebra structure. But a normal Hopf algebra that is properly contained in H contains 1_H, and therefore cannot be a proper ideal, so normal Hopf algebras are not quite the kernels of the Hopf algebra morphisms. Nevertheless, they are closely related in the semisimple case. The kernels of Hopf algebra homomorphisms are the Hopf ideals, and are defined as follows.

Definition 3.2.18. Let H be a Hopf algebra. An ideal I of the algebra H satisfying

(1) $I \subseteq \ker(\varepsilon_H)$,

(2) $\Delta_H(I) \subseteq I \otimes H + H \otimes I$, and

(3) $S_H(I) \subseteq I$

is a **Hopf ideal** of H.

Remark 3.2.19. A subspace I satisfying conditions (1) and (2) in an arbitrary coalgebra is called a **coideal** and these are precisely the kernels of coalgebra homomorphisms. Thus a Hopf ideal is a subspace that is an ideal, a coideal, and is closed under the antipode.

When H is semisimple, the relationship between normal Hopf subalgebras and Hopf ideals is particularly nice. If K is any Hopf algebra, the space $K^+ = \ker(\varepsilon_K)$ is a Hopf ideal, called the augmentation ideal. When K is a Hopf subalgebra of H, this can be extended to a left or right ideal of H in the usual way, by forming the space HK^+ or K^+H.

Proposition 3.2.20. *If N is a normal Hopf subalgebra of H, then*

(1) $HN^+ = N^+H$, and

(2) $I = HN^+$ is a Hopf ideal of H.

When H is semisimple, every Hopf ideal arises in this way, as the next proposition shows.

Proposition 3.2.21. *Let H be a semisimple Hopf algebra. If I is a Hopf ideal, then $I = HN^+$ for some normal Hopf subalgebra N of H.*

When I is a Hopf ideal, the quotient space H/I of cosets of I can be given the structure of a Hopf algebra as follows. Since I is an ideal, the cosets of I have have an algebra structure by multiplying representatives, with identity $1_H + I$. Since I is a coideal, the cosets of I have a coalgebra structure by

$$\Delta(x + I) = \sum (x_1 + I) \otimes (x_2 + I)$$

for all $x \in H$. Finally, the antipode of H/I is given by

$$S(x + I) = S(x) + I$$

for all $x \in H$.

3.3 Frobenius algebras

We recall the basic structure and characterizations of Frobenius algebras, since in what follows we will make frequent use of the natural Frobenius structure of semisimple Hopf algebras.

Proposition 3.3.1. *Let A be a finite dimensional algebra. The following are equivalent.*

(1) *There exists a module isomorphism* ${}_AA \to {}_AA^*$.

(2) *There exists a functional* $\lambda \in A^*$ *such that* $\ker(\lambda)$ *contains no nonzero left ideals.*

(3) *There exists a nondegenerate bilinear form* $(\cdot,\cdot)\colon A \otimes A \to k$ *that is associative, i.e. such that* $(xy, z) = (x, yz)$ *for all* $x, y, z \in A$.

Proof. (1) $\Rightarrow$ (2). Let $\mathcal{F}\colon {}_AA \to {}_AA^*$ be an isomorphism of left A-modules. Set $\lambda = \mathcal{F}(1)$. Then

$$\mathcal{F}(x) = \mathcal{F}(x \cdot 1) = x \rightharpoonup F(1) = x \rightharpoonup \lambda$$

for all $x \in A$. Let I be an ideal of A contained in $\ker(\lambda)$ and let $x \in I$. Then for all $y \in A$ we have

$$\langle \mathcal{F}(x), y\rangle = \langle x \rightharpoonup \lambda, y\rangle = \langle \lambda, yx\rangle = 0$$

since $yx \in I$. Thus $\mathcal{F}(x)$ is identically zero, and by the injectivity of $\mathcal{F}$ it follows that $x = 0$ and hence $I = 0$.

(2) $\Rightarrow$ (3). Suppose $\lambda \in A^*$ and that λ contains no nonzero left ideals. Define a bilinear form on A by $(x, y) = \langle y \rightharpoonup \lambda, x\rangle = \lambda(xy)$ for all $x, y \in A$. Let $0 \neq y \in A$ and suppose that $(x, y) = 0$ for all $x \in A$. Then $\lambda(xy) = 0$ for all $x \in A$, and hence $I = Ay$ is a left ideal contained in $\ker(\lambda)$. It follows that $y = 0$, and hence the form is nondegenerate. For all $x, y, z \in A$ we have

$$(xy, z) = \langle z \rightharpoonup \lambda, xy\rangle = \lambda(xyz) = \langle yz \rightharpoonup \lambda, x\rangle = (x, yz)$$

so this form is associative.

(3) $\Rightarrow$ (1). Suppose that there exists a nondegenerate associative bilinear form $(\cdot,\cdot)$ on A. Then define a map $\mathcal{F}\colon A \to A^*$ by $\mathcal{F}(x)(y) = (y, x)$ for all $x, y \in A$. The nondegeneracy of $(\cdot,\cdot)$ ensures that $\mathcal{F}$ is a bijection. Let $a \in A$. Then for all $y \in A$ we have

$$\langle \mathcal{F}(ax), y\rangle = (y, ax) = (ya, x) = \langle a \rightharpoonup \mathcal{F}(x), y\rangle.$$

Thus $\mathcal{F}$ is a module homomorphism ${}_AA \to {}_AA^*$. $\square$

Definition 3.3.2. A **Frobenius algebra** is a tuple (A, m, u, λ) such that (A, m, u) is an algebra, and $\lambda \in A^*$, called the **Frobenius homomorphism**, is a functional whose kernel contains no nonzero left ideals. The associated module isomorphism $\mathcal{F}\colon {}_AA \to_A A^*$ defined by

$$\langle \mathcal{F}(x), y \rangle = \langle \lambda, yx \rangle$$

for all $x, y \in A$ is called the **Frobenius isomorphism**. If $\lambda(xy) = \lambda(yx)$ for all $x, y \in A$, then A is called a **symmetric algebra**.

Remark 3.3.3. The Frobenius algebra (A, m, u, λ) may be denoted by (A, λ) if the algebra structure is clear from context, or simply A if the Frobenius structure is clear. The associated bilinear form from Proposition 3.3.1 will be denoted by $(\cdot, \cdot)$.

Proposition 3.3.4. *Let (A, λ) be a Frobenius algebra. Then there exist bases $\{\ell_i\}$ and $\{r_i\}$ of A such that*

$$\sum_i (x, r_i)\ell_i = x = \sum_i (\ell_i, x) r_i.$$

Proof. Let $\{\ell_i\}$ be any basis of A and let $\{\ell_i^*\} \subseteq A^*$ be the dual basis, i.e. for each i, $\ell_i^* \in A^*$ is uniquely determined by $\ell_i^*(\ell_j) = \delta_{ij}$ for all j. Then set $r_i = \mathcal{F}^{-1}(\ell_i^*)$. Since $\mathcal{F}$ is a linear isomorphism, $\{r_i\}$ is a basis of A. For the first equality, we have for all $x \in A$

$$\sum_i (x, r_i)\ell_i = \sum_i \lambda(x r_i)\ell_i = \sum_i \mathcal{F}(r_i)(x)\ell_i = \sum_i \ell_i^*(x)\ell_i = x.$$

For the second equality, let $\mathrm{ev}\colon A \to A^{**}$ be the canonical isomorphism where $\mathrm{ev}(x)(f) = f(x)$ for all $x \in A$, $f \in A^*$. Then $\mathrm{ev}(\ell_i) = \ell_i^{**}$. For all $x \in A$ we have

$$\mathcal{F}\left(\sum_i (\ell_i, x) r_i\right) = \sum_i \lambda(\ell_i x)\ell_i^* = \sum_i \langle \mathcal{F}(x), \ell_i \rangle \ell_i^* = \sum_i \langle \ell_i^{**}, \mathcal{F}(x) \rangle \ell_i^* = \mathcal{F}(x)$$

and hence

$$\sum_i (\ell_i, x) r_i = x.$$

$\square$

Definition 3.3.5. Bases $\{\ell_i\}$ and $\{r_i\}$ of a Frobenius algebra (A, λ) satisfying Proposition 3.3.4 are called **dual with respect to** λ.

Example 3.3.6. Let $(H, m, u, \Delta, \varepsilon, S)$ be a semisimple Hopf algebra with nonzero integral $\Lambda \in H$ and nonzero integral $\lambda \in H^*$. Then (H, m, u, λ) and $(H^*, \Delta^*, \varepsilon^*, \Lambda_{\text{ev}})$ are symmetric Frobenius algebras, where $\Lambda_{\text{ev}} \in H^{**}$ is evaluation at Λ.

The following examples are special cases of Example 3.3.6, where $H = kG$ or $H = kG^*$. To ensure semisimplicity, we will assume that the characteristic of k does not divide $|G|$.

Example 3.3.7. The group algebra (kG, λ) is a Frobenius algebra, with Frobenius homormophism is $\lambda = \phi_1$. The associated bilinear form is determined by

$$(g, h) = \lambda(gh) = \phi_1(gh) = (g \rightharpoonup \phi_1)(h) = \phi_{g^{-1}}(h) = \delta_{g^{-1}h}$$

for all $g, h \in G$. Since $\delta_{g^{-1}h} = \delta_{h^{-1}g}$, this form is symmetric, kG is a symmetric algebra. The map $\mathcal{F}$ is determined by

$$\mathcal{F}(g) = g \rightharpoonup \phi_1 = \phi_{g^{-1}}.$$

Example 3.3.8. The dual $(kG^*, \Lambda_{\text{ev}})$ of the group algebra is a Frobenius algebra, where

$$\Lambda = \sum_{g \in G} g.$$

The associated bilinear form is determined by

$$(\phi_g, \phi_h) = \Lambda_{\text{ev}}(\phi_g \phi_h) = \Lambda_{\text{ev}}(\delta_{gh}\phi_g) = \delta_{gh}\phi_g(\Lambda) = \delta_{gh}.$$

Since $\delta_{gh} = \delta_{hg}$, this is a symmetric algebra. The map $\mathcal{F}$ is determined by

$$\mathcal{F}(\phi_g) = \phi_g \rightharpoonup \Lambda_{\text{ev}} = g.$$

3.4 Character theory

In character theory, we are interested in modules and their associated characters, so we review these objects for an arbitrary k-algebra, and recall the theory of semisimple algebras and central idempotents. We then specialize to the case of a semisimple Hopf algebra over $\mathbb{C}$ with a commutative representation ring and review the character table construction of Witherspoon [27], using the conjugacy classes of Cohen and Westreich [8].

Definition 3.4.1. Let V be an A-module with structure map $\varphi\colon A \otimes V \to V$. Define a map $\rho\colon A \to \mathrm{End}_k(V)$ by

$$\rho(x)(v) = xv.$$

The map ρ is the **representation** of A afforded by V.

It is trivial to check that ρ is an algebra homomorphism. Conversely, any algebra homomorphism $\rho\colon A \to \mathrm{End}_k(V)$ for a vector space V gives V an A-module structure by taking the structure map $\varphi\colon A \otimes V \to V$ to be $\varphi(a \otimes v) = \rho(a)(v)$ for all $a \in A$, $v \in V$.

Definition 3.4.2. Given a representation ρ of A afforded by the A-module V, the **character** $\chi\colon A \to k$ of ρ (equivalently, the character of V) is defined by $\chi = \mathrm{tr} \circ \rho$ where $\mathrm{tr}\colon \mathrm{End}_k(V) \to k$ is the usual trace of a linear map. A character is **irreducible** if the associated module is simple.

3.4.1 The representation ring

We now return to the case of a semisimple Hopf algebra, where the set of isomorphism classes of H-modules is highly structured. Since H is semisimple, there are finitely many isomorphism classes of irreducible left H-modules. Let $\widehat{H}$ be an index set for this set of isomorphism classes. Then the left regular H-module decomposes into irreducible H-modules as

$$H = \bigoplus_{\alpha \in \widehat{H}} V_\alpha^{\oplus m_\alpha}$$

where $m_\alpha = \dim V_\alpha$. Let $[V_\alpha]$ denote the isomorphism class of V_α. Then $\{[V_\alpha] \mid \alpha \in \widehat{H}\}$ is the complete set of isomorphism classes of irreducible H-modules. Since H is semisimple, the set $M(H)$ of all isomorphism classes of H-modules forms a commutative monoid under direct sum. This can be extended to the Grothendieck group $G_0(H)$, a free abelian group with generating set $\{[V_i]\}$. By tensoring with k we get a vector space $R(H) = k \otimes G_0(H)$, with scalar multiplication

$x(y \otimes V) = xy \otimes V$ for all $x, y \in k$ and $V \in r(H)$. This makes $(R(H), \otimes, u)$ an algebra, where the multiplication is obtained by linearly extending the map

$$[V] \otimes [W] = [V \otimes W],$$

where V, W are H-modules and $V \otimes W$. The unit is determined by the trivial module, $u(1) = [V_0]$. The algebra $(R(H), \otimes, u)$ is called the **representation ring** of H.

Let U, V be H-modules, with associated characters χ_U and χ_V. Then $\chi_U = \chi_V$ if and only if $U \cong V$. It follows that the character map : $R(H) \to H^*$, which linearly extends the map taking an isomorphism class $[V] \in R(H)$ to the character χ_V of a representative $V \in [V]$, is well-defined. In fact, the character map is an injective algebra homomorphism. The image of $R(H)$ is a subalgebra $C(H)$ of H^*, and consequently $C(H) \cong R(H)$. For this reason we will also refer to $C(H)$ as the representation ring of H, and from this point forward we will primarily deal with characters rather than modules. This is a key motivation for studying characters: knowledge of the structure constants of the character ring with respect to the basis of irreducible characters is equivalent to understanding the decomposition of tensor products of simple H-modules into simple modules.

3.4.2 The character table

Let H be a semisimple Hopf algebra with $k = \mathbb{C}$, and further assume that the representation ring $C(H)$ is commutative. This condition is satisfied by a large class of Hopf algebras, known as quasitriangular Hopf algebras, and every Hopf algebra can be embedded as a Hopf subalgebra of a quasitraingular Hopf algebra. In this case, one can define conjugacy classes of H, from which the usual notion of conjugacy class can be recovered in the case $H = \mathbb{C}G$, which will be used to define the character table of H. We will need the following result of Zhu [30].

Proposition 3.4.3 (Zhu [30, Lem. 2])**.** *Let H be a semisimple Hopf algebra. Then the representation ring $C(H)$ is semisimple.* □

So together with our assumption that $C(H)$ is commutative, it follows that $C(H)$ has a basis of primitive central idempotents $\{E_1, E_2, \cdots E_n\}$. We label these so that $E_1 = \lambda_H$. We have already seen that λ_H is idempotent by construction, and λ_H is primitive since $C(H)\lambda_H = \mathbb{C}\text{-span}\{\lambda_H\}$ is a one dimensional (hence irreducible) submodule of the regular $C(H)$-module.

Definition 3.4.4 (Cohen–Westreich [8])**.** For each primitive idempotent E_i of $C(H)$, the space

$$\mathscr{C}_i := S_H(\mathcal{F}_H^{-1}(H^* E_i))$$

is called a **conjugacy class** of H. The **conjugacy class sum** associated to $\mathscr{C}_i$ is

$$C_i := S_H(\mathcal{F}_H^{-1}(E_i))$$

and the **normalized conjugacy class sum** (or conjugacy class average) is the element $c_i := m_i^{-1} C_i$ where $m_i = \dim(H^* E)$.

Example 3.4.5. Suppose $H = \mathbb{C}G$ for a finite group G. Let $\{g_i\}$ be a set of congugacy class representatives. The primitive idempotents of $C(H)$ are given by

$$E_i = \sum_{g \in \mathrm{Cl}(g_i)} \phi_g.$$

where $\{\phi_g \mid g \in G\}$ is the dual basis to G. It follows that $g_i \in \mathscr{C}_i$ for all i, and the usual conjugacy class of G is recovered as the set $\mathscr{C}_i \cap G$. Alternatively, if $\mathrm{Cl}(g_i)$ is the conjugacy class of g_i in G, then $\mathscr{C}_i$ is just the linear span of $\mathrm{Cl}(g)$ in $\mathbb{C}G$. The integer m_i is equal to $|\,\mathrm{Cl}(g_i)|$, so the notion of conjugacy class sum and normalized conjugacy sum are precisely recovered by

$$C_i = \sum_{g \in \mathrm{Cl}(g_i)} g \quad \text{and} \quad c_i = |\,\mathrm{Cl}(g_i)|^{-1} \sum_{g \in \mathrm{Cl}(g_i)} g.$$

Remark 3.4.6. We will let $\mathrm{Cl}(H)$ denote the set of normalized conjugacy class sums, rather than the set of conjugacy classes, to more easily define supercharacter theories of H in Chapter 4.

In the finite group case, the character table is given (up to reordering rows and columns) by the matrix $T_{ij} = \chi_i(g_j)$, where χ_i is the ith irreducible character and g_j is a representative of the jth conjugacy class, for some ordering of $\mathrm{Irr}(G)$ and $\mathrm{Cl}(g)$. Since characters are constant on conjugacy classes, we have that

$$\chi_i(g_j) = |\,\mathrm{Cl}(g_j)|^{-1} \sum_{g_j \in \mathrm{Cl}(g_j)} \chi_i(g_j) = \chi_i(c_j),$$

which motivates the following definition.

Definition 3.4.7. Let H be a semisimple Hopf algebra with a commutative representation ring. The **character table** of H is the square matrix T (up to reordering rows and columns) with entries

$$T_{ij} = \langle \chi_i, c_j \rangle$$

where $\mathrm{Irr}(H) = \{\chi_1, \ldots, \chi_n\}$ is the set of irreducible characters of H and where $\mathrm{Cl}(H) = \{c_1, \ldots, c_n\}$ is the set of normalized conjugacy class sums.

When H is semisimple, the Frobenius map $\mathcal{F}_H$ maps $Z(H)$ to $C(H)$, so the conjugacy class sums $\{C_i\}$ are a basis for $Z(H)$. This gives two bases for both $C(H)$ and $Z(H)$. In each case, one basis consists of the primitive idempotents, the other basis being the normalized conjugacy class sums (for $Z(H)$) and the irreducible characters (for $C(H)$). Each non-idempotent basis is, up to rescaling, the dual basis of one of the bases of primitive idempotents, as described in the next lemma.

Lemma 3.4.8. *Let $\{E_i\}$ and $\{e_i\}$ be the primitive idempotents of $C(H)$ and $Z(H)$, respectively. Let $\{c_i\}$ and $\{\chi_i\}$ be the class averages and irreducible characters of H, respectively. Then we have:*

(1) $\langle E_i, c_j \rangle = \delta_{ij}$,

(2) $\langle \chi_i, e_j \rangle = \delta_{ij} d_i$,

(3) $S_H(\mathcal{F}_H^{-1}(E_i)) = m_i c_i$, *and*

(4) $|H| F_H(e_i) = d_i \chi_i$,

where $m_i = \dim(H^* E_i)$ *and* $d_i = \dim(He_i)$.

The character table T can also be realized as (the transpose of) the change of basis matrix from the basis $\{E_i\}$ of $C(H)$ to the basis $\{\chi_i\}$. To see this, write

$$\chi_i = \sum_j \alpha_{ij} E_j,$$

so that (α_{ij}) is the transpose of the change of basis matrix. Since the bases $\{E_i\}$ and $\{c_j\}$ are dual by the above lemma, we have

$$T_{ij} = \langle \chi_i, c_j \rangle = \sum_k \alpha_{ik} \langle E_k, c_j \rangle = \sum_k \delta_{kj} \alpha_{ik} = \alpha_{ij}.$$

Chapter 4

Generalized supercharacter theories and Schur rings

4.1 Generalized supercharacter theories

We extend the notion of supercharacter theory from the group algebra of a finite group to semisimple Hopf algebras with a commutative representation ring, using the character theory of §3.4. Recall that we have two bases of the character ring $C(H)$, denoted by

$$\begin{aligned} \mathrm{Irr}(H) = \{\chi_i\} &= \{\text{irreducible characters of } H\} \\ \{E_i\} &= \{\text{primitive idempotents of } C(H)\} \end{aligned}$$

as well as two bases of the center $Z(H)$, denoted by

$$\begin{aligned} \mathrm{Cl}(H) = \{c_i\} &= \{\text{normalized conjugacy class sums of } H\} \\ \{e_i\} &= \{\text{primitive central idempotents of } H\}. \end{aligned}$$

Let $d_i = \dim(He_i)$ and let $m_i = \dim(H^*E_i)$. Recall that these bases are related by

$$\begin{aligned} \langle \chi_i, e_j \rangle &= \delta_{ij} d_i, & |H| F_H(e_i) &= d_i \chi_i, \\ \langle E_i, c_j \rangle &= \delta_{ij}, & \mathcal{S}_H(F_H^{-1}(E_i)) &= m_i c_i. \end{aligned}$$

To each subset X of $\mathrm{Irr}(H)$ we associate a character σ_X, defined to be the following linear combination of irreducible characters:

$$\sigma_X := \sum_{\chi_i \in X} \chi_i(1_H) \chi_i.$$

We also let

$$d_X := \sum_{\chi_i \in X} d_i^2 = \sum_{\chi_i \in X} (\chi_i(1_H))^2$$

so that $d_X = \sigma_X(1_H)$.

Recall that a conjugacy class $\mathscr{C}_i$ of H is a subspace of the form $\mathscr{C}_i = S_H(\mathcal{F}_H^{-1}(H^* E_i))$ for some primitive idempotent E_i of $C(H)$. Then for each subset $K \subseteq \mathrm{Cl}(H)$, we can form the subspace

$$\mathscr{C}_K := k\text{-span}\left\{ \bigcup_{c_i \in K} \mathscr{C}_i \right\} = \bigoplus_{c_i \in K} \mathscr{C}_i.$$

We also define

$$m_K := \sum_{c_i \in K} m_i \quad \text{and} \quad c_K := \frac{1}{m_K} \sum_{c_i \in K} m_i c_i$$

so that $\dim(\mathscr{C}_K) = m_K$, and $\mathscr{C}_K = c_K \leftharpoonup H^*$.

Definition 4.1.1. Let H be a semisimple Hopf algebra with a commutative representation ring. A **supercharacter theory** of H is a pair $(\mathcal{X}, \mathcal{K})$, where $\mathcal{X}$ is a partition of $\mathrm{Irr}(H)$ and $\mathcal{K}$ is a partition of $\mathrm{Cl}(H)$, such that

(1) $|\mathcal{X}| = |\mathcal{K}|$,

(2) σ_X is constant on K for all $X \in \mathcal{X}$ and $K \in \mathcal{K}$.

Each σ_X is a **supercharacter** and each space $\mathscr{C}_K$ is the **superclass** associated to the **normalized superclass sum** c_K. The **rank** of $(\mathcal{X}, \mathcal{K})$ is $|\mathcal{X}|$.

Example 4.1.2. The following partitions are always supercharacter theories of H.

(1) Take $\mathcal{X}$ and $\mathcal{K}$ to be the finest possible partitions of $\mathrm{Irr}(H)$ and $\mathrm{Cl}(H)$, into singletons:

$$\mathcal{X} = \{\{\chi\} \mid \chi \in \mathrm{Irr}(H)\} \quad \text{and} \quad \mathcal{K} = \{\{c\} \mid c \in \mathrm{Cl}(H)\}.$$

In this case, for each singleton $X = \{\chi\}$, the supercharacter $\sigma_X = \chi(1_H)\chi$ is just a positive integer multiple of the irreducible character χ, and the normalized superclass sums are the same as the normalized class sums.

(2) A supercharacter theory of rank two is obtained by taking the coarsest possible partitions of $\mathcal{X}$ and $\mathcal{K}$ that satisfy the condition $\{\varepsilon_H\} \in \mathcal{X}$ and $\{1_H\} \in \mathcal{K}$:

$$\mathcal{X} = \{\mathrm{Irr}(H) - \{\varepsilon_H\}, \{\varepsilon_H\}\} \quad \text{and} \quad \mathcal{K} = \{\mathrm{Cl}(H) - \{1_H\}, \{1_H\}\}.$$

4.1.1 Idempotents

The supercharacters and normalized superclass sums are related to certain idempotents in $Z(H)$ and $C(H)$. Let $(\mathcal{X}, \mathcal{K})$ be a pair of partitions of $\mathrm{Irr}(H)$ and $\mathrm{Cl}(H)$, respectively. Then for all $X \in \mathcal{X}$ and all $K \in \mathcal{K}$, define

$$e_X := \sum_{\chi_i \in X} e_i \quad \text{and} \quad E_K := \sum_{c_i \in K} E_i.$$

Then by the orthogonality of the e_i and E_i, the sets $\{e_X\}$ and $\{E_K\}$ are also orthogonal, and

$$\sum_{X \in \mathcal{X}} e_X = \sum_i e_i = 1_H \quad \text{and} \quad \sum_{K \in \mathcal{K}} E_K = \sum_i E_i = \varepsilon_K$$

using the fact that $\mathcal{X}$ and $\mathcal{K}$ are partitions.

The sets $\{e_X\}$ and $\{E_K\}$ are taken by $\mathcal{F}_H$ to the sets of supercharacters $\{\sigma_X\}$ and normalized superclass sums $\{c_K\}$, respectively, and they are dual with respect to the evaluation pairing in the following sense.

Lemma 4.1.3. *Let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory of H. Then with notation as above, we have*

(1) $\langle E_K, c_L \rangle = \delta_{KL}$,

(2) $\langle \sigma_X, e_Y \rangle = \delta_{XY} d_X$,

(3) $S_H(\mathcal{F}_H^{-1}(E_K)) = m_K c_K$, *and*

(4) $|H| \mathcal{F}_H(e_X) = \sigma_X$.

Proof. In each case, the result follows from a short calculation.

(1) Using the fact that $\{E_i\}$ and $\{c_i\}$ are dual bases, we compute

$$\langle E_K, c_L \rangle = \left\langle \sum_{c_i \in K} E_i, m_L^{-1} \sum_{c_j \in L} m_j c_j \right\rangle = \delta_{KL} m_L^{-1} \sum_{\substack{c_i \in K \\ c_j \in L}} \delta_{ij} m_j = \delta_{KL} m_L^{-1} m_K = \delta_{KL}.$$

(2) Using the fact that $\{\chi_i\}$ and $\{e_j\}$ are dual bases up to rescaling, we have

$$\langle \sigma_X, e_Y \rangle = \left\langle \sum_{\chi_i(1_H)\chi_i \in X} \chi_i(1_H)\chi_i, \sum_{\chi_j \in Y} e_j \right\rangle = \delta_{XY} \sum_{\substack{\chi_i \in X \\ \chi_j \in Y}} \delta_{ij}(\chi_i(1_H))^2 = \delta_{XY} d_X.$$

(3) Since $S_H \circ \mathcal{F}_H^{-1}$ takes the basis $\{E_K\}$ to the basis $\{c_K\}$, we have

$$\begin{aligned} S_H(\mathcal{F}_H^{-1}(E_K)) &= S_H\left(\mathcal{F}_H^{-1}\left(\sum_{c_i \in K} E_i\right)\right) \\ &= \sum_{c_i \in K} S_H(\mathcal{F}_H^{-1}(E_i)) \\ &= \sum_{c_i \in K} m_i c_i \\ &= m_K\left(m_K^{-1} \sum_{c_i \in K} m_i c_i\right) \\ &= m_K c_K. \end{aligned}$$

(4) Finally, we use the fact that $\mathcal{F}_H$ takes the basis $\{|H|e_i\}$ to the basis $\{d_i\chi_i\}$, to get

$$|H|\mathcal{F}_H(e_X) = |H|\mathcal{F}_H\left(\sum_{\chi_i \in X} e_i\right) = \sum_{\chi_i \in X} d_i\chi_i = \sum_{\chi_i \in X} \chi_i(1_H)\chi_i = \sigma_X.$$

$\square$

In the same way that the idempotent bases $\{e_i\}$ and $\{E_i\}$ play an important role in the character theory of H, the idempotents $\{e_X\}$ and and $\{E_K\}$ will play an analogous role for certain algebras associated to $(\mathcal{X}, \mathcal{K})$.

4.1.2 Superclass functions

Given a supercharacter theory $(\mathcal{X}, \mathcal{K})$ of H, an element of $\psi \in C(H)$ is a **superclass function** if ψ is constant on each subset $K \in \mathcal{K}$. Note that K can be recovered from the superclass $\mathscr{C}_K$ by $K = \mathscr{C}_K \cap \mathrm{Cl}(H)$. The set of superclass functions is clearly a subspace of $C(H)$, but we show that this subspace is in fact a subalgebra, as in the case of finite groups. We will need the following lemma.

Lemma 4.1.4. *Let H be a semisimple Hopf algebra, and let* $\mathrm{ev}\colon H \to H^{**}$ *be the canonical isomorphism. Then the following diagram commutes:*

$$\begin{tikzcd} H \arrow[r, "\mathcal{F}_H"] \arrow[d, "|H|^{-1}S_H"'] & H^* \arrow[d, "\mathcal{F}_{H^*}"] \\ H \arrow[r, "\mathrm{ev}"] & H^{**} \end{tikzcd}$$

Proof. We compare the result of applying $\mathcal{F}_{H*} \circ \mathcal{F}_H$ and $\mathrm{ev} \circ S_H$ to an element $x \in H$, and then evaluate these elements of H^{**} at a function $g \in H^*$. We make use of the fact that $\lambda_{H^*} = |H|^{-1}\Lambda_{H^{**}} = |H|^{-1}\,\mathrm{ev}(\Lambda_H)$, and write $\Delta(\Lambda_H) = \sum \Lambda_1 \otimes \Lambda_2$. We compute

$$\begin{aligned}
\langle(\mathcal{F}_{H*} \circ \mathcal{F}_H)(x), g\rangle &= \langle (x \rightharpoonup \lambda_H) \rightharpoonup \lambda_{H^*}, g\rangle \\
&= \lambda_{H^*}(g(x \rightharpoonup \lambda_H)) && \text{def. of } H^* \rightharpoonup H^{**} \\
&= |H|^{-1}\langle g(x \rightharpoonup \lambda_H), \Lambda_H\rangle && \lambda_{H^*} = |H|^{-1}\,\mathrm{ev}(\Lambda_H) \\
&= |H|^{-1}\sum \langle g, \Lambda_H\rangle\langle x \rightharpoonup \lambda_H, \Lambda_2\rangle && \text{convolution product} \\
&= |H|^{-1}\sum \langle g, \Lambda_H\rangle\langle \lambda_H, \Lambda_2 x\rangle && \text{def. of } H \rightharpoonup H^* \\
&= |H|^{-1}\sum \langle g, \Lambda_1\rangle\langle \lambda_H, S_H(x)S_H(\Lambda_2)\rangle && \lambda_H = S_{H^*}(\lambda_H) \\
&= |H|^{-1} g\left(\sum \lambda_H(S_H(x)S_H(\Lambda_2))\Lambda_1\right) && \text{linearity} \\
&= g(|H|^{-1}S_H(x)) && \{\Lambda_1\},\ \{S_H(\Lambda_2)\} \text{ dual w.r.t. } \lambda_H \\
&= \langle(\mathrm{ev} \circ |H|^{-1}S_H)(x), g\rangle.
\end{aligned}$$

□

Lemma 4.1.5. *Given a supercharacter theory $(\mathcal{X}, \mathcal{Y})$, the subspace of superclass functions is a subalgebra of $C(H)$.*

Proof. Let χ, ψ be superclass functions in $C(H)$. Choose $K \in \mathcal{K}$ and let $c_i, c_j \in K$. Then

$$\langle \chi, c_i\rangle = \langle \chi, c_j\rangle \quad \text{and} \quad \langle \psi, c_i\rangle = \langle \psi, c_j\rangle.$$

Since c_i and E_i are dual bases for $Z(H)$ and $C(H)$, for any $\psi \in C(H)$, write $\psi = \sum_i \alpha_i E_i$ for some scalars α_i. Then by the orthogonality of the E_i we have

$$\psi E_i = \sum_j \alpha_i E_j E_i = \alpha_i E_i,$$

and also

$$\langle \psi, c_i \rangle = \left\langle \sum_j \alpha_i E_j, c_i \right\rangle = \sum_j \delta_{ij} \alpha_j = \alpha_i,$$

so that $\psi E_i = \langle \psi, c_i \rangle E_i$. We compute

$$\begin{aligned}
\langle \chi\psi, c_i \rangle &= \langle \chi, \psi \rightharpoonup c_i \rangle \\
&= m_i^{-1} \langle \chi, \psi \rightharpoonup m_i c_i \rangle \\
&= m_i^{-1} \langle \chi, \psi \rightharpoonup S_H(\mathcal{F}_H^{-1}(E_i)) \rangle \\
&= m_i^{-1} \langle \chi, \psi \rightharpoonup |H| \mathcal{F}_{H^*}(E_i) \rangle \\
&= m_i^{-1} \langle \chi, |H| \mathcal{F}_{H^*}(\psi E_i) \rangle \\
&= m_i^{-1} \langle \chi, |H| \mathcal{F}_{H^*}(\langle \psi, c_i \rangle E_i) \rangle \\
&= m_i^{-1} \langle \chi, |H| \mathcal{F}_{H^*}(E_i) \rangle \langle \psi, c_i \rangle \\
&= m_i^{-1} \langle \chi, S_H(\mathcal{F}_H^*(E_i)) \rangle \langle \psi, c_i \rangle \\
&= m_i^{-1} \langle \chi, m_i c_i \rangle \langle \psi, c_i \rangle. \\
&= \langle \chi, c_i \rangle \langle \psi, c_i \rangle.
\end{aligned}$$

By the same argument, we have $\langle \chi\psi, c_j \rangle = \langle \chi, c_j \rangle \langle \psi, c_j \rangle$ so

$$\langle \chi\psi, c_i \rangle = \langle \chi, c_i \rangle \langle \psi, c_i \rangle = \langle \chi, c_j \rangle \langle \psi, c_j \rangle = \langle \chi\psi, c_j \rangle.$$

It remains to determine whether the principal character $\varepsilon_H = 1_{H^*}$ is a superclass function. Since $\varepsilon_H = \sum E_i$, for all j we have that

$$\langle \varepsilon_H, c_j \rangle = \sum_i \langle E_i, c_j \rangle = \sum_i \delta_{ij} = 1.$$

Thus ε_H is constant on $\mathrm{Cl}(H)$, hence is constant on all K in the partition $\mathcal{K}$. $\square$

4.1.3 Properties of supercharacter theories

We now show that the general properties of Proposition 2.1.5 still hold in this setting. Given a supercharacter theory $(\mathcal{X}, \mathcal{K})$, it will be useful to introduce the subspaces

$$\begin{aligned} A_{\mathcal{X}} &= k\text{-span}\{e_X \mid X \in \mathcal{X}\}, \\ A_{\mathcal{K}} &= k\text{-span}\{c_K \mid K \in \mathcal{K}\}, \\ B_{\mathcal{X}} &= k\text{-span}\{\sigma_X \mid X \in \mathcal{X}\}, \\ B_{\mathcal{K}} &= k\text{-span}\{E_K \mid K \in \mathcal{K}\}, \end{aligned}$$

so that $A_{\mathcal{X}}, A_{\mathcal{K}}$ are subspaces of $Z(H)$, and $B_{\mathcal{X}}, B_{\mathcal{K}}$ are subspaces of $C(H)$. Since $A_{\mathcal{X}}$ and $B_{\mathcal{K}}$ have bases of orthogonal idempotents which sum to unity, these are subalgebras of $Z(H)$ and $C(H)$, respectively.

Theorem 4.1.6. *Let H be a semisimple Hopf algebra with a commutative representation ring $C(H)$ and let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory of H. Then the following are true.*

(1) *The supercharacters $\{\sigma_X\}$ are a basis for the algebra of superclass functions in $C(H)$.*

(2) *The normalized superclass sums $\{c_K\}$ are a basis for a subalgebra of $Z(H)$.*

(3) *The partitions $\mathcal{X}$ and $\mathcal{K}$ uniquely determine each other.*

(4) *The singletons $\{\varepsilon_H\}$ and $\{1_H\}$ are elements of $\mathcal{X}$ and $\mathcal{K}$, respectively.*

(5) *Each automorphism of $\mathbb{C}$ induces a permutation of $\mathcal{X}$.*

(6) *Each automorphism of $\mathbb{C}$ induces a permutation of $\mathcal{K}$.*

Proof. (1) Any function in $C(H)$ is determined by its values on the basis $\{c_i\}$ of $Z(H)$, since we have seen that the values $\langle \psi, c_i \rangle$ are the coefficients of ψ in the basis $\{E_i\}$ for all $\psi \in C(H)$. A simple choice for the a basis for the algebra of superclass functions in $C(H)$ are the characteristic functions $\{\phi_K\}$ determined by

$$\langle \phi_K, c_i \rangle = \begin{cases} 1 & c_i \in K \\ 0 & c_i \notin K \end{cases}$$

since these are clearly linearly independent and span the algebra of superclass functions. But we have that

$$\langle E_K, c_i\rangle = \left\langle \sum_{c_j\in K} E_j, c_i \right\rangle = \begin{cases} 1 & c_i \in K \\ 0 & c_i \notin K \end{cases}$$

so in fact $\phi_K = E_K$ for all $K \in \mathcal{K}$. Thus the span $A_{\mathcal{K}}$ of the idempotents $\{E_K\}$ is the algebra of superclass functions. Since supercharacters are superclass functions by definition, we have that $B_{\mathcal{X}} \subseteq A_{\mathcal{K}}$. The fact that $|X| = |K|$ means the $\{\sigma_X\}$ is in fact a basis for algebra of superclass functions.

(2) We have shown that $A_{\mathcal{K}} = B_{\mathcal{X}}$. Since the antipode of H^* takes an irreducible character χ to the irreducible character $\overline{\chi}$, i.e. χ composed with complex conjugation, it is clear that $S_{H^*}(\sigma_X)$ is again a class function, for each supercharacter σ_X. It follows that $S_{H^*}(A_{\mathcal{K}}) = A_{\mathcal{K}}$, since the supercharacters are a basis for the algebra of class functions, and the antipode is bijective. Therefore we have

$$B_{\mathcal{K}} = S_H\mathcal{F}_H^{-1}(A_{\mathcal{K}}) = \mathcal{F}_H^{-1}(S_{H^*}(A_{\mathcal{K}})) = \mathcal{F}_H^{-1}(B_{\mathcal{X}}) = \mathcal{A}_{\mathcal{X}}.$$

Since $\mathcal{A}_X$ is a subalgebra of $Z(H)$, the result follows.

(3) Let $\mathcal{X}$ be given and suppose that there exist supercharacter theories $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{X}, \mathcal{K}')$. By (1) we have

$$A_{\mathcal{K}} = B_{\mathcal{X}} = A_{\mathcal{K}'}.$$

The set of primitive central idempotents of $A_{\mathcal{K}}$ is $\{E_K \mid K \in \mathcal{K}\}$ and the set of primitive central idempotents in $A_{\mathcal{K}'}$ is $\{E_K \mid K \in \mathcal{K}'\}$. Since these are the same algebra, we have $\mathcal{K} = \mathcal{K}'$.

Now let $\mathcal{K}$ be given and suppose that there exist supercharacter theories $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{X}', \mathcal{K})$. By (2) we have

$$A_{\mathcal{X}} = B_{\mathcal{K}} = A_{\mathcal{X}'}.$$

The set of primitive central idempotents of $A_{\mathcal{X}}$ is $\{e_X \mid X \in \mathcal{X}\}$ and the set of primitive idempotents in $A_{\mathcal{X}'}$ is $\{e_X \mid X \in \mathcal{X}'\}$. Since these are the same algebra, $\mathcal{X} = \mathcal{X}'$.

(4) The irreducible character $\chi_1 = \varepsilon_H$ is a superclass function. Since $A_{\mathcal{K}} = B_{\mathcal{X}}$, this character

can be written uniquely in the basis $\{\sigma_X\}$ as

$$\varepsilon_H = \sum_{X \in \mathcal{X}} a_X \sigma_X.$$

The above decomposition can be extended to a decomposition of ε into irreducible characters, namely

$$\varepsilon_H = \sum_{X \in \mathcal{X}} \sum_{\chi \in X} a_X \chi(1)\chi.$$

Since irreducible characters are linearly independent, we must have that

(1) $X = \{\varepsilon_H\}$ for some $X \in \mathcal{X}$,

(2) $a_X = 1$, and

(3) $a_Y = 0$ for all $Y \in \mathcal{X}$ such that $Y \neq X$.

On the other hand, we have that

$$1_H = \sum_{X \in \mathcal{X}} e_X \in k\text{-span}\{c_K\},$$

and so we can decompose 1_H uniquely as

$$1_H = \sum_{K \in \mathcal{K}} a_K c_K$$

for some $a_K \in k$. Expanding this sum in the basis $\{c_i\}$ of $Z(H)$ gives

$$1_H = \sum_{K \in \mathcal{K}} \sum_{c_i \in K} \frac{a_K m_i}{m_K} c_i.$$

But $1_H = c_1$ is itself a normalized conjugacy class sum, and normalized conjugacy class sums are linearly independent. So we must have $K = \{1_H\}$ for some $K \in \mathcal{K}$. Furthermore, since $m_1 = 1$, this gives $m_K = 1$, $a_K = 1$, and $a_L = 0$ for all $L \in \mathcal{K}$ such that $L \neq K$.

(5) Let $\alpha \in \mathrm{Aut}(\mathbb{C})$ be a field automorphism of $\mathbb{C}$. The set $\mathcal{X}$ is in one-to-one correspondence with the primitive idempotents $\{e_X\}$ of the algebra $A_{\mathcal{X}}$. Since $A_{\mathcal{X}} \subseteq Z(H)$, the algebra $A_{\mathcal{X}}$ is commutative and semisimple, and so its irreducible representations are precisely the algebra maps $\rho_X \colon A_{\mathcal{X}} \to \mathbb{C}$, defined by

$$z e_X = \rho_X(z) e_X$$

for all $X \in \mathcal{X}$ and all $z \in A_{\mathcal{X}}$. The composition

$$A_{\mathcal{X}} \xrightarrow{\rho_X} \mathbb{C} \xrightarrow{\alpha} \mathbb{C}$$

is again an algebra map $A_{\mathcal{X}} \to \mathbb{C}$, and therefore gives an irreducible representation with index $X_\alpha \in \mathcal{X}$ determined by α. Since α is invertible, and $(X_\alpha)_{\alpha^{-1}} = X$, the map $X \mapsto X_\alpha$ is a permutation of the set $\mathcal{X}$.

(6) Again let $\alpha \in \mathrm{Aut}(\mathbb{C})$. The set $\mathcal{K}$ is in one-to-one correspondence with the primitive idempotents $\{E_K\}$ of the algebra $A_{\mathcal{K}}$. Since $A_{\mathcal{K}} \subseteq C(H)$, the algebra A is commutative and semisimple, and so its irreducible representations are precisely the algebra maps $\tau_K \colon A_{\mathcal{K}} \to \mathbb{C}$, defined by

$$\psi E_K = \tau_K(\psi) E_K$$

for all $K \in \mathcal{K}$ and all $\psi \in A$. The composition

$$A_{\mathcal{K}} \xrightarrow{\tau_K} \mathbb{C} \xrightarrow{\alpha} \mathbb{C}$$

is again an algebra map $A_{\mathcal{K}} \to \mathbb{C}$, and therefore gives an irreducible representation with index $K_\alpha \in \mathcal{K}$ determined by α. As before, since α is invertible, and $(K_\alpha)_{\alpha^{-1}} = K$, the map $K \mapsto K_\alpha$ is a permutation of the set $\mathcal{K}$. $\square$

4.1.4 Partial order

As in the finite group case, supercharacter theories can be partially ordered according to the partial order on set partitions using the partial order on $\mathrm{Part}(\mathrm{Irr}(H))$ or $\mathrm{Part}(\mathrm{Cl}(H))$, and this choice yields the same result.

Proposition 4.1.7. *Let $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{Y}, \mathcal{L})$ be supercharacter theories of H. Then $\mathcal{X} \leq \mathcal{Y}$ if and only if $\mathcal{Y} \leq \mathcal{L}$.*

Proof. Suppose $\mathcal{X} \leq \mathcal{Y}$ as set partitions, so that $\mathcal{X}$ refines $\mathcal{Y}$. This is equivalent to the condition that the idempotents of $A_{\mathcal{Y}}$ can be formed as simple sums of the idempotents of $A_{\mathcal{X}}$, which happens if and only if $A_{\mathcal{Y}} \subseteq A_{\mathcal{X}}$, since subalgebras of $Z(H)$ are commutative and semisimple. Applying the bijection $\mathcal{F}_H$, this is the same as $A_{\mathcal{L}} \subseteq A_{\mathcal{K}}$. Using the fact that subalgebras of $C(H)$ are commutative and semisimple, this occurs if and only if the primitive idempotents of $A_{\mathcal{L}}$ are simple sums of the primitive idempoents of $A_{\mathcal{L}}$, which is equivalent to $\mathcal{K} \leq \mathcal{L}$. □

4.2 Generalized Schur rings

When passing from the character theory of a finite group G to a supercharacter theory $(\mathcal{X}, \mathcal{K})$, the associated central Schur ring A played a role analogous to that of the center $Z(kG)$. We now consider certain subalgebras of a semisimple Hopf algebra H, which generalize Schur rings. We will show that these subalgebras play the same role for the supercharacter theories of H as the Schur rings did for the supercharacter theories of a finite group G.

Definition 4.2.1. Let H be a semisimple Hopf algebra. Then a subspace A of H is a **Schur ring** of H if the following hold:

(1) A is a subalgebra of H,

(2) $\mathcal{F}_H(A)$ is a subalgebra of H^*,

(3) $S_H(A) = A$.

The following are the Hopf-algebraic analogues of the fundamental examples from the finite group case.

Example 4.2.2. The following are always Schur rings of H:

(1) the algebra $A = H$ itself, with $\mathcal{F}_H(A) = H^*$,

(2) the center $A = Z(H)$, with $\mathcal{F}_H(A) = C(H)$,

(3) the algebra k-span$\{1_H, \Lambda_H\}$, with $\mathcal{F}_H(A) = k$-span$\{\varepsilon_H, \lambda_H\}$.

4.3 Generalized correspondence

The correspondence between central Schur rings and supercharacter theories of finite groups extends to the Hopf algebra setting.

Theorem 4.3.1. *Let H be a semisimple Hopf algebra with a commutative representation ring. There is a one-to-one correspondence*

$$\begin{array}{ccc} \left\{ \begin{array}{c} \textit{supercharacter} \\ \textit{theories of } H \end{array} \right\} & \longleftrightarrow & \left\{ \begin{array}{c} \textit{central} \\ \textit{Schur rings of } H \end{array} \right\} \\ (\mathcal{X}, \mathcal{K}) & \longmapsto & A_{\mathcal{X}}. \end{array}$$

such that $(\mathcal{X}, \mathcal{K}) \leq (\mathcal{Y}, \mathcal{L})$ if and only if $A_{\mathcal{Y}} \subseteq A_{\mathcal{X}}$.

Proof. Let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory of H. Then by Proposition 4.1.6, the space $A_{\mathcal{X}}$ is a subalgebra of $Z(H)$, hence of H. The image under the Frobenius map is $A_{\mathcal{K}}$, which is a subalgebra of $C(H)$, hence of H^*. It follows that $S_H(A) = A$.

To show that this is a one-to-one correspondence, we construct the inverse map. Let A be a central Schur ring of H. Since A is semisimple and commutative, it has a basis of primitive central idempotents, which must be simple sums of the primitive central idempotents $\{e_i\}$ of H. Each idempotent e_i corresponds to an irreducible character $\chi_i \in \mathrm{Irr}(H)$, so the idempotents of A are of the form

$$e_X = \sum_{\chi_i \in X} e_i$$

for each X in a partition $\mathcal{X}$ of $\mathrm{Irr}(H)$. Now, $B = \mathcal{F}_H(A)$ is a subalgebra of $C(H)$, so B is semisimple and commutative, and therefore has a basis of primitive central idempotents, which must be simple sums of the primitive central idempotents $\{E_i\}$ of $C(H)$. Each idempotent E_i corresponds to an element $c_i \in Z(H)$, so the idempotents of B are of the form

$$E_K = \sum_{c_i \in K} E_i$$

for each K in a partition $\mathcal{K}$ of $\mathrm{Cl}(H)$. The map $\mathcal{F}_H$ is a linear isomorphism, so we have

$$|\mathcal{X}| = \dim(A) = \dim(B) = |\mathcal{K}|,$$

since $\mathcal{X}$ indexes a basis of A and $\mathcal{K}$ indexes a basis of B. It remains to check that σ_X is constant on K for each $K \in \mathcal{K}$. Let $X \in \mathcal{X}$ and $K \in \mathcal{K}$ be given. Since $\mathcal{F}_H(|H|e_X) = \sigma_X$, we have that $\sigma_X \in B$, so it can be written as a linear combination

$$\sigma_X = \sum_{L\in\mathcal{K}} a_L E_L = \sum_{L\in\mathcal{K}} \sum_{c_i\in L} a_L E_i.$$

Then since the bases $\{E_i\}$ and $\{c_i\}$ are dual bases, we have

$$\langle \sigma_X, c_i\rangle = \sum_{L\in\mathcal{K}} \sum_{c_\ell\in L} a_L \langle E_\ell, c_i\rangle = a_K = \sum_{L\in\mathcal{K}} \sum_{c_\ell\in L} a_L \langle E_\ell, c_j\rangle = \langle \sigma_X, c_j\rangle$$

for all $c_i, c_j \in K$. It follows that $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of H.

Finally, we have seen that $A_{\mathcal{Y}} \subseteq A_{\mathcal{X}}$ if and only if $\mathcal{X} \le \mathcal{Y}$ as set partitions, and by definition the partial order on supercharacter theories is the same as the partial on $\operatorname{Part}(\operatorname{Irr}(H))$ by comparing the first component. $\square$

Example 4.3.2. The following are the extreme examples of this order-reversing correspondence.

(1) The largest central Schur ring $Z(H)$, corresponds to the finest superchararacter theory

$$\begin{array}{ccc} & & A_{\mathcal{X}} = Z(H) \\ \mathcal{X} = \{\{\chi\} \mid \chi \in \operatorname{Irr}(H)\} & & \\ & \longleftrightarrow & \downarrow \mathcal{F}_H \\ \mathcal{K} = \{\{c\} \mid c \in \operatorname{Cl}(H)\} & & \\ & & A_{\mathcal{K}} = C(H) \end{array}$$

(2) The smallest central Schur ring k-span$\{1_H, \Lambda_H\}$ corresponds to the coarsest supercharacter theory

$$\begin{array}{ccc} & & A_{\mathcal{X}} = k\text{-span}\{1_H, \Lambda_H\} \\ \mathcal{X} = \{\operatorname{Irr}(H) - \{\varepsilon_H\}, \{\varepsilon_H\} & & \\ & \longleftrightarrow & \downarrow \mathcal{F}_H \\ \mathcal{K} = \{\operatorname{Cl}(H) - \{1_H\}, \{1_H\}\} & & \\ & & A_{\mathcal{K}} = k\text{-span}\{\varepsilon_H, \lambda_H\} \end{array}$$

Let G be a finite group. Following his proof of the above correspondence [14] in the finite group case, Hendrickson remarks that if A is any central subalgebra with basis arising from a partition $\mathcal{P}$ of G, then $\mathcal{P}$ is automatically a Schur partition, since the condition $X \in \mathcal{P}$ if and only if $X^{-1} \in \mathcal{P}$ is not used in the proof of the correspondence. This is true in the present context as well, for the analogous condition that $A = S_H(A)$, which reduces to the condition $X \in \mathcal{P}$ if and only if $X^{-1} \in \mathcal{P}$ in the case $H = \mathbb{C}G$.

Corollary 4.3.3. *Let H be a semisimple Hopf algebra over $\mathbb{C}$ with a commutative representation ring $C(H)$. Let A be a subalgebra of $Z(H)$ such that $\mathcal{F}_H(A)$ is a subalgebra $C(H)$. Then $S_H(A) = A$.*

Proof. Let A be as above. Then $A = A_{\mathcal{X}}$ for some partition $\mathcal{X}$ of $\mathrm{Irr}(H)$ and $\mathcal{F}_H(A) = A_{\mathcal{K}}$ for some partition $\mathcal{K}$ of $\mathrm{Cl}(H)$, determined so that $\{e_X \mid X \in \mathcal{X}\}$ and $\{E_K \mid K \in \mathcal{K}\}$ are the respective bases of primitive idempotents. From the proof of Theorem 4.3.1, it follows that $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of H. Thus the algebra $A_{\mathcal{K}}$ of superclass functions is closed under S_{H^*} and consequently $A = A_{\mathcal{X}} = \mathcal{F}_H^{-1}$ is closed under S_H. □

4.3.1 The central supercharacter theory

The following is an immediate application of the correspondence theorem for Hopf algebras. Birciu [5] considers the central subalgebra $\widehat{Z}(H)$, defined to be the intersection $Z(H) \cap C(H^*)$. Here $C(H^*)$ is the character ring of H^*, viewed as a subalgebra of H by the isomorphism $H \cong H^{**}$. These are the 'central characters' of H^*, i.e. those characters that are central in H. The map $\mathcal{F}_H$ takes $\widehat{Z}(H)$ to $\widehat{Z}(H^*) = Z(H^*) \cap C(H)$, the algebra of central characters of H.

The algebra $\widehat{Z}(H)$ is always a central Schur ring of H, since $\widehat{Z}(H)$ and its image under the Frobenius map $\widehat{Z}(H^*)$ are subalgebras of H and H^*, respectively. When $C(H)$ is commutative, the correspondence theorem yields a supercharacter theory with the property that all supercharacters are central in H^*, since the space of superclass functions is precisely the space of central characters. Furthermore, this must be smallest supercharacter theory such that all supercharacters are central characters. In the case $H = kG$, we have $\widehat{Z}(H) = Z(H)$, so this is a supercharacter theory that appears only for non-trivial Hopf algebras. The usefulness of this supercharacter theory is made evident by the following proposition of Birciu, which shows that the supercharacters are sufficient to determine all normal Hopf subalgebras of H.

Proposition 4.3.4 (Birciu [4, Thm. 3.8]). *Let H be a finite dimensional semisimple Hopf algebra. Any normal Hopf subalgebra of N of H is the kernel of a central character of H.* □

If χ is the character of an H-module V, the kernel of χ is the largest Hopf subalgebra A such that $x \cdot v = \varepsilon_H(x)v$ for all $x \in A$.

4.4 Schur ring lifts and products

In this section we describe a process for lifting Schur rings from Hopf subalgebras and quotient Hopf algebras, which reduces to the corresponding construction from Chapter 2 in the finite group case.

4.4.1 Lifting Schur rings from Hopf subalgebras

Let H be a semisimple Hopf algebra with Hopf subalgebra K. We will use need the following lemma of Birciu [6], which shows that as algebras, $\mathcal{F}_H(K) \cong K^*$.

Lemma 4.4.1 (Birciu [6, Lem. 1.6]). *Let H be a semisimple Hopf algebra with Frobenius map $\mathcal{F}_H\colon H \to H^*$. Let K be a Hopf subalgebra with Frobenius map $\mathcal{F}_K\colon K \to K^*$ and canonical inclusion $i\colon K \to H$. The following diagram commutes:*

$$\begin{array}{ccc} H & \xrightarrow{\mathcal{F}_H} & H^* \\ \uparrow{\scriptstyle i} & & \downarrow{\scriptstyle i^*} \\ K & \xrightarrow{\mathcal{F}_K} & K^* \end{array}$$

□

Corollary 4.4.2. *The injective map $\widehat{i}\colon K^* \to H^*$ defined by*

$$\widehat{i} := \mathcal{F}_H \circ i \circ \mathcal{F}_K^{-1}$$

respects multiplication and comultiplication. In particular, $\widehat{i}\colon K^* \to \mathcal{F}_H(K)$ is an algebra isomorphism.

Proof. Since $\mathcal{F}_H$ and $\mathcal{F}_K$ are linear isomorphisms, the map $\hat{i}$ is injective with image $\mathcal{F}_H$. After restricting i^* in Lemma 4.4.1 to $\mathcal{F}_H(K)$, we get a bijective map which respects the multiplication and comultiplication of H^* and K^*. The map $\widehat{i}$ is the inverse of this restriction by Lemma 4.4.1 and therefore also respects the multiplication and comultiplication. In particular, this gives $\mathcal{F}_H(K)$ the structure of an algebra with multiplication inherited from H^* and identity

$$\widehat{i}(1_{K^*}) = \mathcal{F}_H(\mathcal{F}_K^{-1}(1_{K^*})) = \mathcal{F}_H(\Lambda_K).$$

□

Lemma 4.4.3. *Let K be a Hopf subalgebra of the semisimple Hopf algebra H. If $K \neq H$, then $\Lambda_H \notin K$.*

Proof. We have that $\lambda_K = \lambda_H \mid_K$, and $\varepsilon_K = \varepsilon_H \mid_K$. There is a unique idempotent $\Lambda_K \in K$ such that $\lambda_K(\Lambda_K) = 1$, so that $\varepsilon_K(\Lambda_K) = |K|$. Suppose $\Lambda_H \in K$. Then λ_H is an idempotent of K for which

$$\lambda_K(\Lambda_H) = \lambda_H(\Lambda_H) = 1,$$

so $\Lambda_H = \Lambda_K$. On the other hand, we have

$$\varepsilon_K(\Lambda_H) = \varepsilon_H(\Lambda_H) = |H| > |K|.$$

This is a contradiction, and the result follows. □

Proposition 4.4.4. *Let K be a Hopf subalgebra of a semisimple Hopf algebra H and let A be a Schur ring of K. Then*

$$\mathcal{I}(A) := A + \mathcal{I}_H$$

is a Schur ring of H. Furthermore, if $K \neq H$ then sum is direct and $\dim(\mathcal{I}(A)) = \dim(A) + 1$.

Proof. If $K = H$, then $\mathcal{I}_H = \mathcal{I}_K \subseteq A$, so $\mathcal{I}(A) = A$, in which case the result follows immediately. Suppose then, that $K \neq H$. Lemma 4.4.3 implies that $A \cap \mathcal{I}_H = 0$, so we may consider the subspace $\mathcal{I}(A) = A \oplus \mathcal{I}_H \subseteq H$ of dimension $\dim(A) + 1$.

It remains to show that $\mathcal{I}(A)$ is a Schur ring of H. The space A is a subalgebra of K, hence of H, and $\mathcal{I}_H$ is an ideal of H, so $\mathcal{I}(A)$ is a subalgebra of H. The map $\mathcal{F}_H$ is a linear isomorphism, so

$$\mathcal{F}_H(\mathcal{I}(A)) = \mathcal{F}_H(A) \oplus \mathcal{F}_H(\mathcal{I}_H).$$

Since $\mathcal{F}_H(A)$ is the image of $\mathcal{F}_K(A)$ via the composition

$$\mathcal{F}_K(A) \xrightarrow{\mathcal{F}_K^{-1}} A \xrightarrow{i} A \xrightarrow{\mathcal{F}_H} \mathcal{F}_H(A),$$

it follows from Corollary 4.4.2 that $\mathcal{F}_H(A)$ is multiplicatively closed in H^*. Furthermore, $\mathcal{F}(\mathcal{I}_H) = k\text{-span}\{\varepsilon_H\}$. Since ε_H is the identity of H^*, the space $\mathcal{F}_H(\mathcal{I}(A)) = \mathcal{F}_H(A) \oplus k\text{-span}\{\varepsilon_H\}$ is also multiplicatively closed, and contains the identity, so it is a subalgebra of H^*.

Finally, since $S_K = S_H\mid_K$, the subalgebra A of H is closed under S_H. Since $S(\Lambda_H) = \Lambda_H$, we also have that $\mathcal{I}_H$ is closed (in fact, fixed pointwise) by S_H. Thus $\mathcal{I}(A) = A \oplus \mathcal{I}_H$ is also closed under S_H, and is therefore a Schur ring of H. □

Example 4.4.5. Let K be a Hopf subalgebra of H. Then K itself is a Schur ring of K, so $K + \mathcal{I}_H$ is a Schur ring of H.

4.4.2 Lifting Schur rings from Hopf quotients

We describe a way to lift Schur rings from Hopf quotients, that is in some sense dual to the process of lifting from Hopf subalgebras. To accomplish this, we first need some general duality results about Schur rings of Hopf algebras.

Let H be a semisimple Hopf algebra. Since this implies that H^* is also semisimple, it makes sense to consider the Schur rings of H^*.

Proposition 4.4.6. *Let H be a semisimple Hopf algebra. Then A is a Schur ring of H if and only if $\mathcal{F}_H(A)$ is a Schur ring of H^*.*

Proof. Since A is an Schur ring of H, $B = \mathcal{F}_H(A)$ is a subalgebra of H^*. The subalgebra B is closed under the antipode of H^*, since

$$\begin{aligned} S_{H^*}(B) &= (\mathcal{F}_H \circ S_H \circ \mathcal{F}_H^{-1})(B) \\ &= (\mathcal{F}_H \circ S_H)(A) \\ &= \mathcal{F}_H(A) \\ &= B. \end{aligned}$$

Finally, by Lemma 4.1.4, we have that

$$\mathcal{F}_{H^*}(B) = S_H(A)) = \mathrm{ev}(A),$$

after identifying H with H^{**}. The reverse direction follows by replacing H with H^* and identifying H with H^{**}. $\square$

Suppose H is self-dual, so that $H \cong H^*$ as Hopf algebras, and fix an isomorphism $f\colon H \to H^*$. Then each Schur ring A of H can be paired with another 'dual' Schur ring A^f in H, given by $A^f = f^{-1}(\mathcal{F}_H(A))$ (of course it may be the case that $A = A^f$.)

Now suppose $H = \mathbb{C}G$ for a finite group G. Then H is self-dual, by Pontryagin duality, which identifies the elements $g \in G$ with the irreducible characters of $\mathbb{C}G$. In this case $Z(H) = H$ and $C(H) = H^*$, so that every Schur ring is central. Suppose A is a Schur ring of G, with corresponding supercharacter theory $(\mathcal{X}, \mathcal{K})$. Then after identifying $\mathrm{Irr}(H)$ and $\mathrm{Cl}(H)$ via the isomorphism f, the supercharacter theory corresponding to A^f has the partitions $(\mathcal{K}, \mathcal{X})$. But note that even after this identification, it is not necessarily the case that $\mathcal{X} = \mathcal{K}$, so that supercharacter theories also come in dual pairs (given a choice f).

Recall the situation $\pi\colon H \to Q$. In this case, the transpose $\pi^*\colon Q^* \to H^*$ is an inclusion of Hopf algebras. Consider the following diagram.

$$\begin{tikzcd} H^* \arrow[r, "\mathcal{F}_{H^*}"] & H^{**} \arrow[r, "\mathrm{ev}_H^{-1}"] \arrow[d, "\pi^{**}"] & H \arrow[r, "S_H"] \arrow[d, "\pi"] & H \arrow[d, "\pi"] \\ Q^* \arrow[u, "\pi^*"] \arrow[r, "\mathcal{F}_{K^*}"] & Q^{**} \arrow[r, "\mathrm{ev}_Q^{-1}"] & Q \arrow[r, "S_Q"] & Q \end{tikzcd}$$

The commutativity of the above diagram follows (from left to right) by (1) Lemma 4.4.1, (2) the fact that $\mathrm{ev}\colon \mathbf{Vect}_k \to \mathbf{Vect}_k$ restricts to a natural isomorphism of finite dimensional Hopf algebras, and (3) the fact that π is a Hopf algebra map (and thus respects the antipodes of H and Q). Now, by Lemma 4.1.4 we have that

$$\mathcal{F}_{H^*} \circ \mathrm{ev}_H^{-1} \circ S_H = |H|^{-1}\mathcal{F}_H^{-1},$$

and similarly

$$\mathcal{F}_{Q^*} \circ \mathrm{ev}_Q^{-1} \circ S_Q = |Q|^{-1}\mathcal{F}_Q^{-1}.$$

Thus the following diagram commutes:

$$\begin{tikzcd}
H^* \arrow[r, "|H|^{-1}\mathcal{F}_H^{-1}"] & H \arrow[d, "\pi"] \\
Q^* \arrow[u, "\pi^*"] \arrow[r, "|Q|^{-1}\mathcal{F}_Q^{-1}"] & Q
\end{tikzcd}$$

Proposition 4.4.7. *Let Q be a Hopf quotient of a semisimple Hopf algebra H, with natural map $\pi\colon H \to Q$ and let A be a Schur ring of Q. Then*

$$\mathcal{J}(A) := \mathcal{F}_H^{-1} \circ \mathcal{I} \circ \mathcal{F}_Q$$

is a Schur ring of H. Furthermore, when $Q \neq H$, we have $\dim(\mathcal{J}(A)) = \dim(A) + 1$.

Proof. We have that $\mathcal{F}_Q(A)$ is a Schur ring of Q^* of dimension $\dim(A)$. Since $\pi^*\colon Q^* \to H^*$ is an inclusion of Hopf algebras, we have that $\mathcal{I}(\mathcal{F}_Q(A))$ is a Schur ring of H^*. The dimension of this Schur ring is either $\dim(A)$ if $Q = H$ or $\dim(A)+1$ if Q^* is a proper subalgebra of H^*, i.e. whenever $Q \neq H$. It then follows that $\mathcal{F}_H(\mathcal{I}(\mathcal{F}_Q(A)))$ is a Schur ring of H of the desired dimension. □

4.4.3 Generalized wedge products

We now assume that N, K are Hopf subalgebras of the semisimple Hopf algebra H, that N is normal, and that $N \subseteq K$. Let $Q = H/HN^+$ be the resulting quotient with natural map $\pi\colon H \to Q$, and let $i\colon K \to H$ be the canonical inclusion. In this section we give a method for constructing Schur rings of H given Schur rings of K and Q, which generalizes the wedge product.

We will assume we have a Schur ring A of K and a Schur ring B of $Q = H/HN^+$, and we will assume the conditions $\Lambda_N \in A$ and $\pi(A) = B \cap (K/KN^+)$, which reduce to the assumptions made in the finite group case when $H = kG$. In this situation, we will show that $\mathcal{I}(A) + \mathcal{J}(B)$ is a Schur ring of H with the desired properties. Specifically, we will show that the space $A \wedge B := \mathcal{I}(A) + \mathcal{J}(B)$ is a Schur ring of H such that $(A \wedge B) \cap K = A$ and $\pi(A \wedge B) = B$. While the proof of this result in the finite group case deals extensively with partitions of G and uses some technical results from group theory, our proof will be Hopf-theoretic. The following lemmas will be useful.

Lemma 4.4.8. *Let H be a semisimple Hopf algebra, N a normal Hopf subalgebra with inclusion map $i\colon N \to H$ and $Q = H/HN^+$ the quotient with natural map $\pi\colon H \to Q$. Let*

$$\widehat{i}\colon N^* \to H^* \qquad \text{and} \qquad \widehat{\pi}\colon Q \to H$$
$$\widehat{i} = \mathcal{F}_H \circ i \circ \mathcal{F}_N^{-1} \qquad\qquad \widehat{\pi} = \mathcal{F}_H^{-1} \circ \pi^* \circ \mathcal{F}_Q$$

Then we have

(1) $\widehat{i}(\varepsilon_N) = \mathcal{F}_H(\Lambda_N)$,

(2) $\widehat{i}(\lambda_N) = \lambda_H$,

(3) $\widehat{\pi}(1_Q) = |N|^{-1}\Lambda_N$, *and*

(4) $\widehat{\pi}(\Lambda_Q) = \Lambda_H$.

Proof. (1) We have for $x \in N$ that

$$\mathcal{F}_N(\Lambda_N)(x) = \lambda_N(x\Lambda_N) = \varepsilon_N(x)\lambda_N(\Lambda_N) = \varepsilon_N(x).$$

Then $\mathcal{F}_N^{-1}(\varepsilon_N) = \Lambda_N$, so after including N into H and applying $\mathcal{F}_H$, we have $\widehat{i}(\varepsilon_N) = \mathcal{F}_H(\Lambda_N)$.

(2) Observe that $\mathcal{F}_N(1_N) = \lambda_N$, and $\mathcal{F}_H(1_H) = \lambda_H$. Since the inclusion map takes 1_N to 1_H, we have that $\widehat{i}(\lambda_N) = \lambda_H$.

(3) It is enough to show that $|H|\mathcal{F}_{H^*}^{-1}(\Lambda_N) = \pi^*(|Q|\mathcal{F}_{Q^*}^{-1}(1_Q))$, or equivalently, that $\mathcal{F}_H(\Lambda_N) = \pi^*(\mathcal{F}_Q(1_Q)) = \pi^*(\lambda_Q)$. To show this, we observe that $H = H\Lambda_N \oplus HN^+$ as H-modules. Thus, the H modules $H\Lambda_N$ and Q are isomorphic via π, where Q is viewed as an H-module by $h \cdot x =$

$\pi(h)x$, for all $h \in H$ and $x \in Q$. It follows that $H\Lambda_N$ is isomorphic to the lift of the left-regular Q-module via the surjective algebra map π. Then the character of $H\Lambda_N$ in H^* is π^* applied to the character of the left regular Q-module in Q^*. Since $|N|^{-1}\Lambda_N$ is the central idempotent of H corresponding to $H\Lambda_N$, we have that $\dim(H\Lambda_N)|H|\mathcal{F}_H(|N|^{-1}\Lambda_H)$ is the character of this module. Since $\dim(H\Lambda_N) = |Q| = |H||N|^{-1}$, it follows that this character is precisely $F_H(\Lambda_N)$, so $\mathcal{F}_H(\Lambda_N) = \pi^*(\lambda_Q)$ as required.

(4) We note that $\mathcal{F}_Q(\Lambda_Q) = \varepsilon_Q$, which is the identity of Q^*. Since π^* is an injective algebra homomorphism, $\pi^*(\varepsilon_Q) = \varepsilon_H$. Applying $\mathcal{F}_H^{-1}$ gives Λ_H. □

Lemma 4.4.9. *Under the above assumptions, the space $A \wedge B := \mathcal{I}(A) + \mathcal{J}(B)$ is a subalgebra of H with $(A \wedge B) \cap K = A$ and $\pi(A \wedge B) = B$.*

Proof. We have the commutative diagram

$$\begin{tikzcd} H^* \arrow[r, "|H|^{-1}\mathcal{F}_H^{-1}"] & H \arrow[d, "\pi"] \\ Q^* \arrow[u, "\pi^*"] \arrow[r, "|Q|^{-1}\mathcal{F}_Q^{-1}"] & Q \end{tikzcd}$$

Both H and Q have a canonical $\mathcal{I}(A)$ module structure, since $\mathcal{I}(A)$ is a subalgebra of H, and π restricted to $\mathcal{I}(A)$ is an algebra map. Concretely, we have that H is an $\mathcal{I}(A)$-module by restricting the action of the regular module, to obtain the action

$$a \cdot h = ah$$

for all $a \in \mathcal{I}(A)$, $h \in H$. Since $\pi(\mathcal{I}(A))$ is a subalgebra of Q, we can pull back the same module structure to $\mathcal{I}(A)$, to obtain the action

$$a \cdot x = \pi(a)x$$

for all $a \in \mathcal{I}(A)$ and $x \in Q$. The map π is an $\mathcal{I}(A)$-module homomorphism, since

$$\pi(a \cdot h) = \pi(ah) = \pi(a)\pi(h) = a \cdot \pi(h).$$

Now let $\widehat{\pi}$ be the composition

$$Q \xrightarrow{|Q|\mathcal{F}_H} Q^* \xrightarrow{\pi^*} H^* \xrightarrow{|H|^{-1}\mathcal{F}(H)} H.$$

If we restrict π to $\widehat{\pi}(Q)$, it still respects the $\mathcal{I}(A)$ module structures, but the restriction of π is invertible with inverse $\widehat{\pi}$, by the commutativity of the above diagram. It follows that $\widehat{\pi}$ is a $\mathcal{I}(A)$-module homomorphism, and that $\widehat{\pi}(Q)$ is a $\mathcal{I}(A)$-submodule of H.

Now we consider the subalgebra $B \subseteq Q$. Since $\pi(|N|^{-1}\Lambda_N) = \pi(1_H) = 1_Q$, we have that $\pi(\mathcal{I}(A)) = \pi(A) = B \cap (K/KN^+)$, with the last equality holding by our assumption. In particular then, we have that $\pi(\mathcal{I}(A))$ is a subalgebra of B, so that B is a submodule of ${}_{\mathcal{I}(A)}Q$. Since $\widehat{\pi}$ is an injective $\mathcal{I}(A)$-module homomorphism, it follows that $\widehat{\pi}(B)$ is a submodule of ${}_{\mathcal{I}(A)}H$, i.e. that the space $\widehat{\pi}(B)$ is closed under left multiplication by $\mathcal{I}(A)$.

By construction, $\mathcal{J}(B) = \widehat{\pi}(B) + k\text{-span}\{1_H\}$, so for $a \in \mathcal{I}(A)$ and $b \in \mathcal{J}(B)$ we can write $b = b' + \beta 1_H$ where $b' \in \widehat{\pi}(B)$ and $\beta \in k$. We then compute

$$ab = ab' + \beta a \in \mathcal{I}(A) + \mathcal{J}(B)$$

since $ab' \in \widehat{\pi}(B) \subseteq \mathcal{J}(B)$ and $\beta a \in \mathcal{I}(A)$. Thus we have that $\mathcal{I}(A)$ and $\mathcal{J}(B)$ are each individually closed under multiplication, and products of the form ab where $a \in \mathcal{I}(A)$ and $b \in \mathcal{J}(B)$ are contained in the sum $\mathcal{I}(A) + \mathcal{J}(B)$. This sum also contains products of the form ba, by repeating the above argument replacing the left $\mathcal{I}(A)$-module structures arising from left multiplication, with right $\mathcal{I}(A)$-module structures arising from right multiplication. Thus the space $\mathcal{I}(A) + \mathcal{J}(B)$ is multiplicatively closed, and since $1_H \in \mathcal{I}(A) \cap \mathcal{J}(B) \subseteq \mathcal{I}(A) + \mathcal{J}(B)$, so $A \wedge B$ is a subalgebra of H.

Next, we show that $(A \wedge B) \cap K = A$. Clearly $A \subseteq (A \wedge B)$ and $A \subseteq K$, so $A \subseteq (A \wedge B) \cap K$. For the opposite inclusion, we have that

$$A \wedge B = A + k\text{-span}(\Lambda_H) + \widehat{\pi}(B) + k\text{-span}(1_H) = A + \widehat{\pi}(B)$$

since $1_H \in A$ and $\Lambda_H \in k\text{-span}\{\widehat{\pi}(1_Q)\} \subseteq \widehat{\pi}(B)$. Thus it suffices to show that $\widehat{\pi}(B) \cap K \subseteq A$.

To show this, we begin by writing the map $\widehat{\pi}$ explicitly. Starting with $x \in Q$ we write $x = \pi(h)$ for some $h \in H$ and compute

$$\begin{aligned}
\widehat{\pi}(x) &= |Q||H|^{-1}\mathcal{F}_H^{-1}(\pi^*(\mathcal{F}_Q(x))) \\
&= |Q||H|^{-1}\mathcal{F}_H^{-1}(\pi^*(x \rightharpoonup \lambda_Q)) \\
&= |Q||H|^{-1}\mathcal{F}_H^{-1}(\pi^*(x \rightharpoonup \lambda_Q)) \\
&= |Q||H|^{-1}\mathcal{F}_H^{-1}(h \rightharpoonup \pi^*(\lambda_Q)) \\
&= |Q||H|^{-1}\mathcal{F}_H^{-1}(h \rightharpoonup \mathcal{F}_H(\Lambda_N)) \\
&= |Q||H|^{-1}\mathcal{F}_H^{-1}(\mathcal{F}_H(h\Lambda_N)) \\
&= |Q||H|^{-1}h\Lambda_N \\
&= |N|^{-1}h\Lambda_N.
\end{aligned}$$

Suppose that $b \in \widehat{\pi}(B) \cap K$. Then we have that $b \in \widehat{\pi}(Q) = H\Lambda_N$. But then we have $H\Lambda_N \cap K = K\Lambda_N$, so that $b \in K\Lambda_N = \widehat{\pi}(K/KN^+)$. It follows that $b \in \widehat{\pi}(B) \cap \widehat{\pi}(K/KN^+) = \widehat{\pi}(B \cap K/KN^+) = \widehat{\pi}(\pi(A)) = A\Lambda_N \subseteq A$, since we assumed that $\Lambda_N \in A$. Thus we have $(A \wedge B) \cap K \subseteq A$, and so in fact we have equality.

Finally, since $\pi(A) \subseteq B$ and $\pi(\Lambda_H) \in k\text{-span}\{\Lambda_Q\} \subseteq B$, we have $\pi(\mathcal{I}(A)) \subseteq B$. Since $\pi(1_H) = 1_Q \in B$, we also have that $\pi(\mathcal{J}(B)) \subseteq B$. Then $\pi(A \wedge B) \subseteq B$. But $\pi(\widehat{\pi}(B)) = B$ and $\widehat{\pi}(B) \subseteq \mathcal{J}(B)$, so $\pi(A \wedge B)$ is indeed all of B. □

Proposition 4.4.10. *Suppose H is a semisimple Hopf algebra with subalgebras $N \subseteq K \subseteq H$ with N normal in H and $Q = H/HN^+$. Let A be a Schur ring of K and let B be an Schur ring of Q, and suppose that $\Lambda_N \in K$ and that $\pi(A) \cap Q = B$. Then*

$$A \wedge B := \mathcal{I}(A) + \mathcal{J}(B)$$

is a Schur ring of H of such that $(A \wedge B) \cap K = A$ and $\pi(A \wedge B) = B$.

Proof. We will use the notation from Lemma 4.4.9, regarding the inclusion $i\colon K \to H$, the natural map $\pi\colon H \to Q$. By Lemma 4.4.9, $A \wedge B$ is a subalgebra of H, with the properties $(A \wedge B) \cap K = A$

and $\pi(A \wedge B) = B$. As a sum of two Schur rings, $A \wedge B$ is closed under S_H. It remains to show that $\mathcal{F}(A \wedge B)$ is a subalgebra of H^*.

This follows by considering the sequence

$$Q^* \xrightarrow{\pi^*} H^* \xrightarrow{i^*} K^*.$$

The injective map π^* may be viewed as the inclusion of Q^* as a normal Hopf subalgebra of H^* and i^* is surjective. We view these Hopf algebras as left Q^* modules, via left multiplication by Q^* for Q^* and H^*, and via $\phi \cdot \psi = i^*(\phi)\psi$ in the case of K^* for all $\phi \in Q^*$ and $\psi \in K^*$. Repeating the argument of the previous lemma shows that $\mathcal{F}_H(\mathcal{I}(A))$ is a $\mathcal{F}_H(\mathcal{J}(B))$-module, with respect to left and right multiplication. Together with the fact that $\mathcal{F}_H(\mathcal{I}(A))$ and $\mathcal{F}_H(\mathcal{J}(B))$ are individually closed under multiplication, this implies that the space $\mathcal{F}_H(A \wedge B)$ is multiplicatively closed in H^*. Since $1_{H^*} = 1_{Q^*} \in \mathcal{F}_H(J(B))$, this completes the proof. $\square$

4.5 Supercharacter theory lifts and products

4.5.1 Lifting supercharacter theories from Hopf subalgebras

Throughout §4.5, H is a semisimple Hopf algebra with a commutative representation ring. Since a Schur ring A of a Hopf subalgebra K of H must contain Λ_K, the Schur ring A can only be central in H if Λ_K is central in H. When H is semisimple, this is equivalent to the condition that K be normal in H. Therefore, it makes sense when lifting supercharacter theories to restrict our attention to normal Hopf subalgebras with a commutative representation ring.

Proposition 4.5.1. *Suppose $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of the normal Hopf subalgebra N of H such that the superclass $\mathscr{C}_K$ is a union of conjugacy classes of H, for all $K \in \mathcal{K}$. Then there exists a supercharacter theory $(\mathcal{X}', \mathcal{K}')$ of H where*

$$\mathcal{K}' = \{K' \mid K \in \mathcal{K}\} \cup \{K_{G \setminus N}\}$$

with $K' = \mathrm{Cl}(H) \cap \mathscr{C}_K$ and $K_{H \setminus N} = \{c \in \mathrm{Cl}(H) \mid c \notin N\}$.

Proof. Let A be the Schur ring of N associated to $(\mathcal{X}, \mathcal{K})$, and consider the Schur ring $\mathcal{I}(A)$ of H. For all $K \in \mathcal{K}$, the superclass $\mathscr{C}_K$ is a union of conjugacy classes, so we have

$$c_K = m_K^{-1} \sum_{c_i \in K'} m_i c_i \in Z(H).$$

It follows that A is contained in $Z(H)$. The ideal $\mathcal{I}_H$ is contained in $Z(H)$, so $\mathcal{I}(A) = A + \mathcal{I}_H$ is a central Schur ring of H. The central Schur ring $\mathcal{I}(A)$ corresponds to the supercharacter theory with superclasses determined by $\mathcal{K}'$. □

4.5.2 Lifting supercharacter theories from Hopf quotients

Now we consider lifts of supercharacter theories from $Q = H/HK^+$ where K is a normal Hopf subalgebra of H. As we will see in the following proof, $C(Q)$ is a subalgebra of $C(H)$, so it is sufficient to assume that $C(H)$ is commutative to have supercharacter theories of Q.

Proposition 4.5.2. *Suppose $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of the quotient $Q = H/HK^+$ for some normal Hopf subalgebra K of H with natural map $\pi\colon H \to Q$. Let $K' = \{c \in \mathrm{Cl}(H) \mid \pi(c) \in K\}$ and let $K_1 = \{1_H\}$. Then there exists a supercharacter theory $(\mathcal{X}', \mathcal{K}')$ of H where*

$$\mathcal{K}' = \{K' \mid K \in \mathcal{K}\} \cup \{K_1\}.$$

Proof. Let A be the central Schur ring of Q corresponding to $(\mathcal{X}, \mathcal{K})$. The map $\pi^*\colon Q^* \to H^*$ takes the character ring of Q into the character ring of H^*, by mapping an irreducible character χ of Q to the irreducible character $\chi \circ \pi$ of H. It follows that the Schur ring $\mathcal{I}(\mathcal{F}_Q(A))$ of H^* is contained in $C(H)$. As a result, the Schur ring $J(A) = \mathcal{F}_H^{-1}(I(A))$ is contained in $Z(H)$. The supercharacter theory corresponding to $J(A)$ has superclasses determined by $\mathcal{K}'$. □

4.5.3 The generalized $*$-product

Using the wedge product in the case $N = K$, we describe how to combine a supercharacter theory of a normal Hopf subalgebra N with a supercharacter theory of $Q = H/HN^+$ to produce a nontrivial supercharacter theory of H.

Proposition 4.5.3. *With notation as above, suppose that $(\mathcal{X}, \mathcal{K})$ is a supercharacter theory of N, such that for all $K \in \mathcal{K}$, the superclass $\mathscr{C}_K$ is a union of conjugacy classes of H. Suppose also that $(\mathcal{Y}, \mathcal{L})$ is a supercharacter theory of G/N. Then $(\mathcal{X}' \wedge \mathcal{Y}', \mathcal{K}' \wedge L')$ is a supercharacter theory of H, where $(\mathcal{X}', \mathcal{K}')$ and $(\mathcal{Y}', \mathcal{L}')$ are the lifts to H of $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{Y}, \mathcal{L})$, respectively.*

Proof. Let A and B be the central Schur rings of N and Q, associated to $(\mathcal{X}, \mathcal{K})$ and $(\mathcal{Y}, \mathcal{L})$, respectively. Then viewing $K = N$ as a subalgebra containing N, we have that A is a Schur ring of K containing Λ_N, and it is certainly true that $\pi(A) = B \cap H/HN^+$, since both sides of the equality are k-span$\{1_Q\}$. Then we can form the wedge product $A \wedge B$, which is a Schur ring of H. We have that $\mathcal{I}(A)$ and $\mathcal{J}(B)$ are central, so $A \wedge B$ is central. Since $A \wedge B = \mathcal{I}(A) + \mathcal{J}(B)$, it follows that the corresponding supercharacter theory has superclasses determined by the partition $\mathcal{K}' \wedge \mathcal{L}'$. $\square$

Chapter 5

Group actions on supercharacter theories

Let H be a semisimple Hopf algebra. We wish to generalize the process by which supercharacter theories of groups are formed from orbits of group actions. To this end, we first develop some algebraic machinery in the form of group actions on Schur rings. As a result of the correspondence, we focus primarily on central Schur rings. We then use this machinery to formulate the notion of group actions on supercharacter theories. In Chapter 6, we focus on the action of a particular Galois group that gives rise to a supercharacter theory of special interest, which we then examine in more detail.

5.1 Group actions on Schur rings

5.1.1 Schur ring maps

In order to define what it means for a group to act on a Schur ring of H, it is helpful to have a notion of a Schur ring morphism.

Definition 5.1.1. Let A, B be Schur rings of H. A linear map $f\colon A \to B$ is a **Schur ring map** provided that

(1) f is an algebra homomorphism $A \to B$,

(2) $\mathcal{F}_H(f) := \mathcal{F}_H \circ f \circ \mathcal{F}_H^{-1}$ is an algebra homomorphism $\mathcal{F}_H(A) \to \mathcal{F}_H(B)$,

(3) $f \circ S_H = S_H \circ f$.

An immediate consequence of this definition is that Schur ring maps are injective, as the next proposition shows.

Proposition 5.1.2. *Let A, B be Schur rings of H and let $f\colon A \to B$ be a Schur ring map. Then* $\ker(f) = 0$.

Proof. If $|H| = 1$, then $A = B = H$, so we assume $|H| \geq 2$. We first establish that $f(A)$ is a Schur ring of H. The image $f(A) \subseteq B$ is multiplicatively closed and contains 1_H, since f is an algebra homomorphism. Likewise $\mathcal{F}_H(f(A)) = \mathcal{F}_H(f)(\mathcal{F}_H(A))$ is the image of the algebra map $\mathcal{F}_H(f)$ with respect to the multiplication and unit inherited from H^*, so $\mathcal{F}_H(A)$ is a subalgebra of H^*. Finally, $f(A)$ is closed under the antipode, since

$$S_H(f(A)) = f(S_H(A)) = f(A).$$

Thus $f(A)$ is an Schur ring of H. As such, $\dim(f(A)) \geq 2$, since $f(A)$ must contain both 1_H and Λ_H.

We next show that $\ker(f) \subseteq \ker(\lambda) \cap A$. Suppose $x \in \ker(f)$. Then $\psi = \mathcal{F}_H(x) \in \ker \mathcal{F}_H(f)$. But since $\ker(\mathcal{F}_H(f))$ is an ideal of $\mathcal{F}_H(A)$, we must have

$$\phi\lambda_H = \varepsilon_{H^*}(\phi)\lambda_H = \langle \phi, 1_H \rangle \lambda_H \in \ker(\mathcal{F}_H(f)).$$

Since $\mathcal{F}_H(f)(\lambda_H) = 1_H \neq 0$, we must have $\langle \phi, 1_H \rangle = 0$. But then

$$0 = \langle \phi, 1_H \rangle = \langle \mathcal{F}_H(x), 1_H \rangle = \langle \lambda, x \rangle$$

so $x \in \ker \lambda$. Thus $\ker(f) \subseteq \ker(\lambda) \cap A$.

Since $\ker(\lambda) \cap A = \ker(\lambda|_A)$ contains no nontrivial ideals, we must have either $\ker(f) = \ker(\lambda) \cap A$ or $\ker(f) = 0$. Suppose $\ker(f) = \ker(\lambda) \cap A$. Since $\lambda\colon H \to k$, we would have $\dim(\ker(f)) \geq \dim(A) - 1$ and consequently

$$\dim(A) = \dim(\ker(f)) + \dim(f(A)) \geq (\dim(A) - 1) + 2 = \dim(A) + 1,$$

a contradiction. Thus $\ker(f) = 0$. □

The fact that all Schur ring maps are injective may seem to be overly restrictive, but as the previous proposition shows, this fact is forced upon us if we wish to respect both algebra structures. This is consistent with the literature, for example the survey of Muzychuk and Ponomarenko [24], which mentions three notions of Schur ring isomorphism, but does consider structure preserving maps between Schur rings that are not injective. These notions are Cayley isomorphisms, combinatorial isomorphisms, and the algebraic isomorphisms, each of which is increasingly general, so that an algebraic isomorphism is the most general notion of a Schur ring map that is discussed. When $H = kG$, every Schur ring map in the sense of Definition 5.1.1 is an algebraic isomorphism.

Algebraic isomorphisms are not explicitly required to respect the antipode, and can be characterized as those maps satisfying conditions (1) and (2) our definition. Nevertheless, they respect the antipode as a consequence of (1) and (2) in the finite group case, as we briefly show. Suppose A, B are Schur rings of the finite group G and let X be an element of the associated partition $\mathcal{P}$. Let $f\colon A \to B$ be an algebraic isomorphism, so that f is an algebra isomorphism that takes the standard basis of A to the standard basis of B. Then we have that

$$f(S(\widehat{X})) = f(\widehat{X^{-1}}) = \sum_{g \in X} f(g^{-1}) = \sum_{g \in X} f(g)^{-1} = \sum_{g \in f(X)} g^{-1} = S(\widehat{f(X)}).$$

It follows that $f \circ S = S \circ f$, so that our definition coincides with the notion of algebraic isomorphism in the case $H = kG$.

Now, even though we require Schur ring maps to respect the antipode, it may be that this third axiom is a consequence of the other two. We show that this is at least the case for central Schur rings, as might be expected from Corollary 4.3.3. To accomplish this recall that if χ is a character of H afforded by the H-module V, then χ^* denotes the character of the dual H-module V^*, and we have $\chi^* = S_{H^*}(\chi) = \overline{\chi}$ where $\overline{\chi}$ denotes complex conjugation. We are now in a position to prove the following proposition.

Proposition 5.1.3. *Let H be a semisimple Hopf algebra over $\mathbb{C}$ with a commutative representation ring $C(H)$, with central Schur ring A. Suppose $f\colon A \to A$ and $\mathcal{F}_H(f)\colon \mathcal{F}_H(A) \to \mathcal{F}_H(A)$ are algebra automorphisms. Then $S_H \circ f = f \circ S_H$.*

Proof. Let A be a central Schur ring of H with corresponding supercharacter theory $(\mathcal{X}, \mathcal{K})$. Suppose $f\colon A \to A$ is an algebra map such that $\mathcal{F}_H(f)\colon \mathcal{F}_H(A) \to \mathcal{F}_H(A)$ is also an algebra map. We must show that $S_H \circ f = f \circ S_H$. Let $g = \mathcal{F}_H(f)$. Then after conjugation by $\mathcal{F}_H$, it suffices to show $S_{H^*} \circ g = g \circ S_{H^*}$.

Recall that $\{e_X\}$ and $\{d_X^{-1}\sigma_X\}$ are dual bases of A and $\mathcal{F}_H(A)$, respectively. Since g is an algebra automorphism, it permutes the basis $\{E_K\}$ of primitive idempotents. Since f is an algebra automorphism of A, we have that g also permutes the basis $\{\sigma_X\}$. Also, note that $f(e_X) = e_Y$ if and only if $g(\sigma_X) = \sigma_Y$, by the definition of g. Then since f and g act identically on the set $\mathcal{X}$, we will write either $f(X)$ or $g(X)$ for the index Y, whenever $f(e_X) = e_Y$ or $g(\sigma_X) = \sigma_Y$. Equivalently, in either case we can write X as $f^{-1}(Y)$ or $g^{-1}(Y)$. Thus for all $X, Y \in \mathcal{X}$ we have

$$\delta_{g(X)Y} = \delta_{Xf^{-1}(Y)}$$

since $g(X) = Y$ if and only if $X = f^{-1}(Y)$. Similarly, S_{H^*} permutes the supercharacters since S_H is a Schur ring map, so let $X^* \in \mathcal{X}$ denote the index of $S_{H^*}(\sigma_X)$.

Now for all $X, Y \in \mathcal{X}$, we compute

$$\begin{aligned}
\langle g(S_{H^*}(\sigma_X)), e_Y\rangle &= \langle g(\sigma_{X^*}), e_Y\rangle \\
&= \delta_{g(X^*)Y} \\
&= \delta_{X^*f^{-1}(Y)} \\
&= \langle \sigma_{X^*}, f^{-1}(e_Y)\rangle \\
&= \langle S_{H^*}(\sigma_X), f^{-1}(e_Y)\rangle \\
&= \overline{\langle \sigma_X, f^{-1}(e_Y)\rangle} \\
&= \overline{\delta_{Xf^{-1}(Y)}} \\
&= \overline{\delta_{g(X)Y}} \\
&= \overline{\langle g(\sigma_X), e_Y\rangle} \\
&= \langle S_{H^*}(g(\sigma_X)), e_Y\rangle.
\end{aligned}$$

It follows that $g \circ S_{H^*} = S_{H^*} \circ g$ as required. $\square$

5.1.2 The category of Schur rings

We define a small category $\mathbf{Sring}_H$, which has the set of Schur rings of H as objects. Given two Schur rings A and B of H, the morphisms $\mathrm{Hom}(A, B)$ are the Schur ring maps. By Proposition 5.1.2, the Schur ring maps $A \to A$ are all injective, hence are all isomorphisms, so we define $\mathrm{Aut}(A) = \mathrm{Hom}(A, A)$.

Proposition 5.1.4. *The map $\mathcal{F}_H$ defines a functor*

$$\mathcal{F}_H \colon \mathbf{Sring}_H \to \mathbf{Sring}_{H^*}$$

$$A \mapsto \mathcal{F}_H(A)$$

$$f \mapsto \mathcal{F}_H(f)$$

for all Schur rings A and all Schur ring maps f. The functor $\mathcal{F}_H$ is an equivalence of categories.

Proof. We will show that the functor

$$\mathcal{F}_H \colon \mathbf{Sring}_H \to \mathbf{Sring}_{H^*}$$

has inverse

$$\mathcal{F}_{H^*} \colon \mathbf{Sring}_{H^*} \to \mathbf{Sring}_H$$

so that $\mathcal{F}_{H^*} \circ \mathcal{F}_H$ is the identity functor on $\mathbf{Sring}_H$ and $\mathcal{F}_H \circ \mathcal{F}_{H^*}$ is the identity functor on $\mathbf{Sring}_{H^*}$. Ordinarily, an equivalence of categories only requires that these compositions be naturally isomorphic to the respective identity functors, but in this case we obtain the identity functors precisely.

As maps on Hopf algebras, we have that $\mathcal{F}_{H^*} \circ \mathcal{F}_H = |H|^{-1} S_H$, so for any Schur ring A of H, it follows that

$$(\mathcal{F}_{H^*} \circ \mathcal{F}_H)(A) = |H|^{-1} S_H(A) = |H|^{-1} A = A.$$

Let A, B be Schur rings of H and suppose $f\colon A \to B$ is a Schur ring map. Then we compute

$$\begin{aligned}
(\mathcal{F}_{H^*} \circ \mathcal{F}_H)(f) &= \mathcal{F}_{H^*}(\mathcal{F}_{H^{-1}} \circ f \circ \mathcal{F}_H) \\
&= \mathcal{F}_{H^*}^{-1} \circ (\mathcal{F}_H^{-1} \circ f \circ \mathcal{F}_H) \circ \mathcal{F}_{H^*} \\
&= (\mathcal{F}_H \circ \mathcal{F}_{H^*})^{-1} \circ f \circ (\mathcal{F}_H \circ \mathcal{F}_{H^*}) \\
&= (|H|^{-1} S_H)^{-1} \circ f \circ (|H|^{-1} S_H) \\
&= |H| S_H \circ f \circ |H|^{-1} S_H \\
&= S_H \circ f \circ S_H \\
&= f.
\end{aligned}$$

The same argument applied to H^* instead of H completes the result. □

5.1.2.1 Group actions and fixed Schur rings

Definition 5.1.5. Let A be an Schur ring of H. We will say that a finite group G acts on the Schur ring A if G acts on the underlying set A and for all $g \in G$, the map $\varphi_g\colon A \to A$ defined by $\varphi_g(a) = g \cdot a$ is a Schur ring map.

Remark 5.1.6. View the group G as a category $\mathbf{G}$, with one object, and with morphisms identified with the elements of G, such that composition $g \circ h$ is determined by the group operation. Then a group action of G on a Schur ring A is equivalent to the existence of a functor $\mathbf{G} \to \mathbf{Sring}_H$ which maps the unique object of $\mathbf{G}$ to the object A.

Given an action of G on a Schur ring A, we can define a new Schur ring A^G by

$$A^G = \{a \in A \mid g \cdot a = a \text{ for all } g \in G\}.$$

We verify that A^G is a Schur ring of H. We immediately have that A^G is a subalgebra of H, since A^G is the subalgebra of A of points fixed by the algebra automorphisms $\{\varphi_g \mid g \in G\}$. We must also check that $\mathcal{F}_H(A^G)$ is a subalgebra of H^*, but this is the set of fixed points of the algebra

$\mathcal{F}_H(A)$ fixed by the algebra automorphisms $\{\mathcal{F}_H(\varphi_g) \mid g \in G\}$. Finally, A^G is closed under the antipode, since for all $g \in G$ and all $x \in A^G$ we have

$$\varphi_g(S_H(x)) = S_H(\varphi_g(x)) = S_H(x).$$

5.2 Group actions on supercharacter theories

5.2.1 Supercharacter theories from orbits

Throughout this section, we assume H is a semisimple Hopf algebra with a commutative representation ring $C(H)$, set of irreducible characters $\mathrm{Irr}(H)$, and set of normalized conjugacy class sums $\mathrm{Cl}(H)$. We consider the situation where a group G acts on the sets $\mathcal{X}$ and $\mathcal{K}$ of a supercharacter theory $(\mathcal{X}, \mathcal{K})$ of H in a compatible way, to produce a new supercharacter theory. Whenever G acts on a finite set Ω we will use $\mathrm{Orbits}_G(\Omega)$ to denote the set of orbits.

Definition 5.2.1. Let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory of H, and suppose that G acts on the sets $\mathcal{X}$ and $\mathcal{K}$. Then we say these actions are **compatible** if

$$\langle \chi_{g \cdot X}, c_{g \cdot K} \rangle = \langle \chi_X, c_K \rangle$$

for all $g \in G$, $X \in \mathcal{X}$, and $K \in \mathcal{K}$. We describe this situation by saying that G acts on $(\mathcal{X}, \mathcal{K})$.

Proposition 5.2.2. *Suppose G acts on the supercharacter theory $(\mathcal{X}, \mathcal{K})$. Then there is an induced action of G on the corresponding Schur ring A, determined linearly for all $g \in G$ and $X \in \mathcal{X}$ by*

$$g \cdot e_X = e_{g \cdot X}$$

where e_X is the primitive idempotent of A corresponding to σ_X.

Proof. Suppose G acts on $(\mathcal{X}, \mathcal{G})$. Then G permutes the basis $\{e_X\}$ of A by $g \cdot e_X = e_{g \cdot X}$. Linear extension gives an action of G on the vector space A, so that $g \in G$ acts by the linear map $\varphi_g \colon A \to A$. Since it permutes the idempotent basis, φ_g is an algebra map. To see this let $x, y \in A$. In the idempotent basis, we have

$$x = \sum_{X \in \mathcal{X}} a_X e_X \quad \text{and} \quad y = \sum_{Y \in \mathcal{X}} b_Y e_Y$$

for some $a_X, b_Y \in k$. Then φ_g preserves the multiplication

$$\begin{aligned}
\varphi_g(x)\varphi_g(y) &= \sum_{X\in\mathcal{X}} a_X e_{g\cdot X} \sum_{Y\in\mathcal{X}} b_Y e_{g\cdot Y} \\
&= \sum_{X\in\mathcal{X}} \sum_{Y\in\mathcal{X}} a_X b_Y e_{g\cdot X} e_{g\cdot Y} \\
&= \sum_{X\in\mathcal{X}} \sum_{Y\in\mathcal{X}} a_X b_Y e_{g\cdot X} \delta_{XY} \\
&= \sum_{X\in\mathcal{X}} a_X b_X e g \cdot Y \\
&= \varphi_g(xy)
\end{aligned}$$

and unit

$$\varphi_g(1) = \sum_{X\in\mathcal{X}} e_{g\cdot X} = \sum_{X\in\mathcal{X}} e_X = 1.$$

Next we show that $\mathcal{F}(\varphi_g) = \mathcal{F} \circ \varphi_g \circ \mathcal{F}^{-1}$ is an algebra map of $\mathcal{F}(A)$. Using the same argument as above, it suffices to show that $\mathcal{F}(\varphi_g)$ permutes the idempotent basis $\{E_K\}$. Now, since $\{\sigma_X\}$ is a basis for $\mathcal{F}(A)$, the map $\mathcal{F}(\varphi_g)$ is determined linearly by

$$\mathcal{F}(\varphi_g)(\sigma_X) = |H|\mathcal{F}_H(\varphi_g(e_X)) = |H|\mathcal{F}_H(e_{g\cdot X}) = \sigma_{g\cdot X}.$$

On the other hand, consider the map ψ_g that permutes the $\{F_K\}$ basis according to the action of G on $\mathcal{K}$, which is defined by

$$\psi_g \colon \mathcal{F}(A) \to \mathcal{F}(A)$$

$$\psi_g(E_K) = E_{g\cdot K}.$$

This map permutes the basis $\{\sigma_X\}$ in precisely the same way as φ_g, as a consequence of the compatibilty condition on the actions of G on $\mathcal{X}$ and $\mathcal{K}$, as shown by the calculation

$$\begin{aligned}
\psi_g(\sigma_X) &= \sum_{K\in\mathcal{K}} \langle \sigma_X, c_K\rangle \psi_g(E_K) \\
&= \sum_{K\in\mathcal{K}} \langle \sigma_X, c_K\rangle E_{g\cdot K} \\
&= \sum_{K\in\mathcal{K}} \langle \sigma_X, c_{g^{-1}\cdot K}\rangle E_K \\
&= \sum_{K\in\mathcal{K}} \langle \sigma_{g\cdot X}, c_K\rangle E_K \\
&= \sigma_{g\cdot X}.
\end{aligned}$$

Since $\mathcal{F}(\varphi_g)$ and ψ_g agree on the basis $\{\sigma_X\}$, they are equal, so $\mathcal{F}(\varphi_g)$ is an algebra map as required. Since A is a central Schur ring, this is enough to conclude that φ_g is a Schur ring map. $\square$

When G acts on a supercharacter theory $(\mathcal{X}, \mathcal{K})$, we may now consider the fixed Schur ring

$$A^G = \{a \in A \mid g \cdot a = a \text{ for all } g \in G\}.$$

Since G permutes the $\{e_X\}$ basis, a basis for A^G is given by the orbits of the basis elements

$$A^G = k\text{-span}\left\{\sum_{X\in\mathcal{O}} e_X \mid \mathcal{O} \in \mathrm{Orbits}_G(\mathcal{X})\right\}$$

and this is clearly the basis of primitive orthogonal idempotents of A^G. By the same argument, the primitive idempotent basis of $\mathcal{F}_H(A^G)$ is given by

$$\mathcal{F}_H(A^G) = k\text{-span}\left\{\sum_{K\in\mathcal{O}} E_K \mid \mathcal{O} \in \mathrm{Orbits}_G(\mathcal{K})\right\}.$$

Having identified the primitive idempotent bases, the supercharacter theory corresponding to A^G is therefore $(\mathcal{X}^{\mathcal{G}}, \mathcal{K}^{\mathcal{G}})$, where

$$\begin{aligned}
\mathcal{X}^{\mathcal{G}} &= \left\{\bigcup_{X\in\mathcal{O}} X \mid \mathcal{O} \in \mathrm{Orbits}_{\mathcal{G}}(\mathcal{X})\right\}, \text{ and} \\
\mathcal{K}^{\mathcal{G}} &= \left\{\bigcup_{K\in\mathcal{O}} K \mid \mathcal{O} \in \mathrm{Orbits}_{\mathcal{G}}(\mathcal{K})\right\}.
\end{aligned}$$

We have proved the following theorem.

Theorem 5.2.3. *Let G be a finite group, H a semisimple Hopf algebra with a commutative representation ring and $(\mathcal{X}, \mathcal{K})$ a supercharacter theory of H. Suppose G acts on $(\mathcal{X}, \mathcal{K})$. Then $(\mathcal{X}^{\mathcal{G}}, \mathcal{K}^{\mathcal{G}})$ is a supercharacter theory of H where*

$$\mathcal{X}^{\mathcal{G}} = \left\{ \bigcup_{X \in \mathcal{O}} X \mid \mathcal{O} \in \mathrm{Orbits}_{\mathcal{G}}(\mathcal{X}) \right\}, \text{ and}$$
$$\mathcal{K}^{\mathcal{G}} = \left\{ \bigcup_{K \in \mathcal{O}} K \mid \mathcal{O} \in \mathrm{Orbits}_{\mathcal{G}}(\mathcal{K}) \right\}.$$

The following are direct consequences of the above theorem.

Corollary 5.2.4. *Let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory of H, and suppose that G acts on $(\mathcal{X}, \mathcal{K})$. Then the separate actions of G on $\mathcal{X}$ and $\mathcal{K}$ have equal numbers of orbits.*

Corollary 5.2.5. *Let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory of H, and suppose that G acts on $(\mathcal{X}, \mathcal{K})$. Then G fixes $\{1\} \in \mathcal{X}$ and $\{\epsilon\} \in \mathcal{K}$.*

Chapter 6

Rational supercharacter theories

Throughout Chapter 6 we fix a semisimple Hopf algebra H with a commutative representation ring $C(H)$ and character table T. We will assume from now on that $k = \mathbb{C}$. Among supercharacter theories arising from group actions, of particular interest is the supercharacter theory corresponding to the central Schur ring fixed by a Galois group associated to each supercharacter table. We show that this supercharacter theory has the property that all supercharacters are rational-valued (hence integer-valued), and futhermore, it is the unique minimal supercharacter theory with this property. We describe this supercharacter theory for an arbitrary abelian group, and compute the number of supercharacters in the case of $GL_2(\mathbb{F}_q)$.

6.1 The Galois group $\mathcal{G}(T)$

Let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory of H, and let T be the supercharacter table. The entries of the matrix T are algebraic integers, and they lie in the cyclotomic field $\mathbb{Q}_N = \mathbb{Q}(\zeta^N)$ where N is the exponent of H. Let $\mathbb{Q}(T)$ be the smallest normal subfield of $\mathbb{Q}$ containing the entries of T. Then we have $\mathbb{Q} \subseteq \mathbb{Q}(T) \subseteq \mathbb{Q}_N$. We then define $\mathcal{G}(T)$ to be the Galois group $\mathrm{Gal}(\mathbb{Q}(T)/\mathbb{Q})$. The group $\mathcal{G}(T)$ is the Galois group of the intermediate field $\mathbb{Q}(T)$, so we can view it as a quotient of the group $\mathrm{Gal}(\mathbb{Q}_N/\mathbb{Q})$, where the natural map $\pi\colon \mathrm{Gal}(\mathbb{Q}_N/\mathbb{Q}) \to \mathcal{G}$ is restriction to $\mathbb{Q}(T)$.

The elements of $\mathrm{Gal}(\mathbb{Q}_N/\mathbb{Q})$ are precisely the maps determined by $\zeta \mapsto \zeta^r$ where $1 \le r \le N-1$ and $\gcd(r, N) = 1$. As such, $\mathrm{Gal}(\mathbb{Q}_N/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times$. Each such r determines an element of $\mathcal{G}$, by restriction of the automorphism $\zeta \mapsto \zeta^r$ to $\mathbb{Q}(T)$. Given an element of $\mathcal{G}(T)$, the surjectivity of π implies that there is always some choice of r, though this choice is not unique. We will write $\alpha_r \in \mathcal{G}$ to mean the restriction of $\zeta \mapsto \zeta^r$ to $\mathbb{Q}(T)$, and as we just noted, every element of $\mathcal{G}(T)$ can be expressed (though not uniquely expressed) in this form.

6.1.1 The action of $\mathcal{G}(T)$ on A

Fix a supercharacter theory $(\mathcal{X}, \mathcal{K})$ of H, with corresponding central Schur ring A, and character table T. Let $\mathcal{G} = \mathcal{G}(T)$. Then $\mathcal{G}$ acts on the Schur ring A as follows. Let e_X be an idempotent of A. Then associated to e_X, and hence to X, is an irreducible character ψ_X of A, defined by

$$ae_X = \psi_X(a)e_X$$

for all $a \in A$. The irreducible characters of A are all of the form ψ_X for some $X \in \mathcal{X}$, and these are precisely the algebra homomorphisms $A \to \mathbb{C}$. For each $\alpha \in \mathcal{G}$, choose an extension $\widehat{\alpha} \colon \mathbb{C} \to \mathbb{C}$ to an automorphism of $\mathbb{C}$. Then for each $X \in \mathcal{X}$, the composition $\widehat{\alpha} \circ \psi_X$ is again an automorphism $A \to \mathbb{C}$, and therefore must be ψ_Y for some $Y \in \mathcal{X}$. Since ψ_X and ψ_Y are both completely determined by their restriction to $\mathbb{Q}$-span$\{c_K \mid K \in \mathcal{K}\}$, where they take values only in $\mathbb{Q}(T)$, we have that Y is determined by X and α independently of the choice of $\widehat{\alpha}$. It can be checked that this defines a (left) action of $\mathcal{G}$ on the set $\mathcal{X}$, by $\alpha \cdot X = Y$ for all $\alpha \in \mathcal{G}$ and $X \in \mathcal{X}$, where $\psi_Y = \widehat{\alpha} \circ \psi_X$. Since $\mathcal{G}$ acts on X, it acts on A by algebra automorphisms, by linearly extending the permutation

$$\alpha \cdot e_X = e_{\alpha \cdot X}$$

for all $\alpha \in \mathcal{G}$ and all $X \in \mathcal{X}$.

6.1.2 The action of $\mathcal{G}(T)$ on $\mathcal{F}_H(A)$

We now show that the action just described is in fact an action of $\mathcal{G}$ on A as a Schur ring. It is enough to find an action of $\mathcal{G}$ on $\mathcal{K}$ that is compatible with the action on $\mathcal{X}$. Let E_K be an idempotent of $\mathcal{F}_H(A)$. Then associated to E_K, and hence to K, is an irreducible character τ_K of

$\mathcal{F}_H(A)$, defined by

$$\phi E_K = \tau_K(\phi) E_K$$

for all $\phi \in \mathcal{F}_H(A)$. Then for all $\alpha \in \mathcal{G}, K \in \mathcal{K}, X \in \mathcal{X}$ we claim that

$$\tau_K(\widehat{\alpha}^{-1} \circ \sigma_X) = \tau_L(\sigma_X)$$

for some $L \in \mathcal{K}$ depending on $\widehat{\alpha}$ and K. To see this, we observe that

$$\begin{aligned}
\langle \tau_K, \widehat{\alpha}^{-1} \circ \sigma_X \rangle &= \langle \widehat{\alpha}^{-1} \circ \sigma_X, c_K \rangle \\
&= \widehat{\alpha}^{-1}(\langle \sigma_X, c_K \rangle) \\
&= \widehat{\alpha}^{-1}(\langle \tau_K, \sigma_X \rangle) \\
&= \langle \widehat{\alpha}^{-1} \circ \tau_K, \sigma_X \rangle.
\end{aligned}$$

Since the composition $\widehat{\alpha}^{-1} \circ \tau_K$ is an algebra homomorphism $\mathcal{F}_H(A) \to \mathbb{C}$, it must be an irreducible character τ_L for some $L \in \mathcal{K}$. By the same argument as before, τ_K and τ_L are both determined by their values on $\mathbb{Q}$-span$\{\sigma_X \mid X \in \mathcal{X}\}$, which lie in $\mathbb{Q}(T)$, so L depends only on α and K and not on the choice of extension $\widehat{\alpha}$. Thus we can define a (left) action of $\mathcal{G}$ on $\mathcal{K}$ by

$$\langle \tau_{\alpha \cdot K}, \sigma_X \rangle = \langle \tau_K, \widehat{\alpha}^{-1} \circ \sigma_X \rangle$$

for all $\alpha \in \mathcal{G}$, $X \in \mathcal{X}$, $K \in \mathcal{K}$.

Consequently, if φ_α is the automorphism of A induced by $\alpha \in \mathcal{G}$ that permutes the idempotents $\{e_X\}$ of A according to

$$\varphi_\alpha(e_X) = e_{\alpha \cdot X}$$

then $\mathcal{F}_H(\varphi_\alpha)$ permutes the idempotents $\{E_K\}$ of $\mathcal{F}_H(A)$ according to

$$\mathcal{F}_H(\varphi_\alpha)(E_K) = E_{\alpha \cdot K}.$$

6.2 Rational supercharacter theories

Definition 6.2.1. A supercharacter theory $(\mathcal{X}, \mathcal{K})$ of H is **rational** if $\sigma_X(c_K) \in \mathbb{Q}$ for all $X \in \mathcal{X}$ and all $K \in \mathcal{K}$.

Remark 6.2.2. Since supercharacter values are algebraic integers, the supercharacter table of a rational supercharacter theory is a matrix over $\mathbb{Z}$.

Proposition 6.2.3. *Let $(\mathcal{X}, \mathcal{K})$ be the finest supercharacter theory, and let $(\mathcal{Y}, \mathcal{L})$ be a rational supercharacter theory. Then $(\mathcal{X}^{\mathcal{G}}, \mathcal{K}^{\mathcal{G}}) \leq (\mathcal{Y}, \mathcal{L})$.*

Proof. The Schur ring corresponding to $(\mathcal{X}, \mathcal{K})$ is $Z(H)$. Let A be the central Schur ring corresponding to $(\mathcal{Y}, \mathcal{L})$. It suffices to show that $A \subseteq Z(H)^{\mathcal{G}}$. Let $B = \mathcal{F}(A)$. By applying $\mathcal{F}$, it is equivalent to show that $B \subseteq C(H)^{\mathcal{G}}$. Now $B = k\text{-span}\{\sigma_Y \mid Y \in \mathcal{Y}\}$. Let $Y \in \mathcal{Y}$. Then it remains only to show that $\sigma_Y \in C(H)^{\mathcal{G}}$. But $C(H)^{\mathcal{G}}$ contains all characters that are fixed by the action of $\mathcal{G}$. Since σ_Y is rational valued, we have for each $\alpha \in \mathcal{G}$ and each $c \in \mathrm{Cl}(H)$ that

$$\begin{aligned}
\langle \alpha_r \cdot \sigma_Y, c\rangle &= \sum_{\chi_i \in Y} \chi_i(1)\langle \alpha \cdot \chi_i, c\rangle \\
&= \sum_{\chi_i \in Y} \alpha_r(\chi_i(1))\alpha(\langle \chi_i, c\rangle) \\
&= \sum_{\chi_i \in Y} \alpha(\chi_i(1)\langle \chi_i, c\rangle) \\
&= \alpha(\langle \sigma_Y, c\rangle) \\
&= \langle \sigma_Y, c\rangle.
\end{aligned}$$

Thus $\sigma_Y \in C(H)^{\mathcal{G}}$ as required. □

Proposition 6.2.4. *Let $(\mathcal{X}, \mathcal{K})$ be the finest supercharacter theory. Then the supercharacter theory $(\mathcal{X}^{\mathcal{G}}, \mathcal{K}^{\mathcal{G}})$ is rational, so every entry of the supercharacter table lies in $\mathbb{Z}$.*

Proof. Let $X \in \mathcal{X}^{\mathcal{G}}$ and let $c \in \mathrm{Cl}(H)$. Then for each $\alpha \in \mathcal{G}$ we have that

$$\begin{aligned}
\alpha_r(\langle \sigma_X, c\rangle) &= \sum_{\chi_i \in X} \alpha(\chi_i(1)\langle \chi_i, c\rangle) \\
&= \sum_{\chi_i \in X} \alpha(\chi_i(1))\alpha_r(\langle \chi_i, c\rangle) \\
&= \sum_{\chi_i \in X} \chi_i(1)\alpha(\langle \chi_i, c\rangle) \\
&= \sum_{\chi_i \in X} \chi_i(1)\langle \alpha \cdot \chi_i, c\rangle \\
&= \langle \alpha \cdot \sigma_X, c\rangle \\
&= \langle \sigma_X, c\rangle.
\end{aligned}$$

Since the fixed field of $\mathcal{G}$ is $\mathbb{Q}$, the result follows. □

6.3 Abelian groups

Let G be an abelian group, and let $H = kG$ be the group Hopf algebra. We consider the ordinary character theory $(\mathcal{X}, \mathcal{K})$, so that

$$\mathcal{K} = \{\{g\} \mid g \in G\}$$

and consequently the set of normalized class sums $\mathrm{Cl}(G)$ is G itself. We will identify the superclasses K of the partition $\mathcal{K}^{\mathcal{G}}$ of the minimal rational supercharacter theory $(\mathcal{X}^{\mathcal{G}}, \mathcal{K}^{\mathcal{G}})$. We need the following lemma.

Lemma 6.3.1. *Given $g \in G$, and $\alpha_r \in \mathcal{G}$, there exists $1 \le r' < |G|$ such that*

(1) $\alpha_r = \alpha_{r'}$,

(2) $\gcd(r', |G|) = 1$.

Proof. We have that $1 \le r \le N-1$ and $\gcd(1, N) = 1$, where N is the exponent of G. Let

$$|G| = p_1^{\ell_1} p_2^{\ell_2} \cdots p_s^{\ell_s}$$

be the unique prime factorization of $|G|$, where $\ell_i > 0$ and write

$$N = p_1^{m_1} p_2^{m_2} \cdots p_s^{m_s}$$

where $0 \leq m_i \leq \ell_i$ since N divides $|G|$. Let

$$M_1 = \prod_{m_i \neq 0} p_i^{\ell_i} \quad \text{and} \quad M_2 = \prod_{m_i = 0} p_i^{\ell_i}$$

so that $G = M_1 M_2$, $\gcd(M_1, M_2) = 1$, and $N \mid M_1$. Then by the Chinese remainder theorem, there exists r' such that $1 \leq r' \leq M_1 M_2 = |G|$ satisfying

$$r' \equiv r \mod M_1$$

$$r' \equiv 1 \mod M_2.$$

Since $N \mid M_1$, we have that $r' \equiv r \mod N$, so that $\alpha_r = \alpha_{r'}$. If p is any prime dividing both r' and M_1, then p must also divide r, since $r = r' - tM_1$ for some integer t. This contradicts the fact that $\gcd(r, N) = 1$, so we must have $\gcd(r', N) = 1$. By definition of M_1, this also gives $\gcd(r', M_1) = 1$, and clearly $\gcd(r', M_2) = 1$. Thus we have that $\gcd(r', |G|) = 1$. $\square$

Lemma 6.3.2. *Given* $1 \leq r' \leq |G|$ *with* $\gcd(r', |G|) = 1$*, there exists* $1 \leq r < N$ *such that*

(1) $\gcd(r, N) = 1$

(2) $\alpha_r = \alpha_{r'}$.

Proof. Choose $r \equiv r' \mod N$ so that $1 \leq r \leq N - 1$. It is clear that $\gcd(r', N) = 1$, since $N \mid |G|$. Thus $\gcd(r, N) = 1$, since any prime dividing both r and N would also divide r', a contradiction. Since $r \equiv r' \mod N$, we have that $\alpha_r = \alpha_{r'}$. $\square$

By Lemma 6.3.1 and Lemma 6.3.2, it follows that for any $\alpha_r \in \mathcal{G}$, we may assume that $\gcd(r, |G|) = 1$, since if this were not the case, we could find a new representative $1 \leq r' < |G|$ for which this is the case. Since G is abelian, every character χ of kG is a homomorphism $\mathbb{C}G \to \mathbb{C}$, so we have that

$$\langle \chi, g^r \rangle = \langle \chi, g \rangle^r$$

for all positive integers r, $\chi \in \mathrm{Irr}(\mathbb{C}G)$, and $g \in G$. Thus we have that

$$\langle \chi, g^r \rangle = \langle \chi, g \rangle^r = \alpha_r(\langle \chi, g \rangle) = \langle \alpha_r \cdot \chi, g \rangle = \langle \chi, (\alpha_r)^{-1} \cdot g \rangle.$$

Proposition 6.3.3. *Let $g, g' \in G$. Then $\langle g' \rangle = \langle g \rangle$ if and only if there exists $\alpha \in \mathcal{G}$ such that $\alpha \cdot g = g'$.*

Proof. Suppose $\langle g' \rangle = \langle g \rangle$. This is equivalent to the existence of a positive integer r such that $\gcd(r, |G|) = 1$, and $g^r = g'$, which is to say

$$(\alpha_r)^{-1} \cdot g = g'$$

so take $\alpha = (\alpha_r)^{-1}$. □

Corollary 6.3.4. *Let G be an abelian group and let $(\mathcal{X}, \mathcal{K})$ be the minimal rational supercharacter theory. The superclasses are in one-to-one correspondence with the cyclic subgroups of G, so that the cyclic subgroup C corresponds to the superclass*

$$K = \{g \mid C = \langle g \rangle\}.$$

Example 6.3.5. Let $C_n = \langle g \rangle$ be a cyclic group of order n. The n irreducible characters are the algebra homomorphisms $kC_n \to \mathbb{C}$, so they are all of the form $\chi_i(g) = \zeta^i$ for a primitive nth root of unity ζ. The character table is (up to reordering), the $n \times n$ matrix

$$T_{ij} = \langle \chi_i, g^j \rangle = \zeta^{ij}$$

for $0 \leq i, j \leq n-1$. Let

$$X_d = \{\{\chi_i\} \mid |\chi_i(g)| = d\} \quad \text{and} \quad K_d = \{\{g^i\} \mid |g^i| = d\}$$

where d ranges over the divisors of n. Then $K_1 = \{1\}$, $X_1 = \{\varepsilon\}$, K_n is the set of generators of C_n, and X_n is the set of characters taking g to a primitive nth root of unity. The the minimal rational supercharacter theory is given by

$$\mathcal{X} = \{X_d \mid d \text{ divides } n\} \quad \text{and} \quad \mathcal{K} = \{K_d \mid d \text{ divides } n\}.$$

Let $\sigma_s = \sigma_{X_s}$ and $c_t = c_{K_t}$ denote the supercharacters and normalized superclass sums, where s and t range over the divisors of n. Using $g^{n/t}$ as a representative of the superclass containing elements of order t, the supercharacter table (T_{st}) is the matrix with entries

$$T_{st} = \langle \sigma_s, c_t \rangle = \left\langle \sum_{|\chi_i(g)|=s} \chi_i, g^{n/t} \right\rangle = \sum_{|\chi_i(g)|=s} \zeta^{(n/s)(n/t)} = \mathrm{Tr}\left(\zeta^{(n/s)(n/t)}\right)$$

where $\mathrm{Tr}\colon \mathbb{Q}(\zeta) \to \mathbb{Q}$ is the usual field trace

$$\mathrm{Tr}(x) = \sum_{\alpha \in \mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})} \alpha(x).$$

Example 6.3.6. Let G be the group $C_p \times C_p$ for some prime p. Then the character table is a $p^2 \times p^2$ matrix. The supercharacter table T has only

$$\frac{p^2-1}{p-1} = p+2$$

rows and columns, and is the integer valued matrix with $T_{1i} = 1$, $T_{i1} = p-1$ for $i \neq 1$, and

$$T_{ij} = \begin{cases} p-1 & i = j \\ -1 & i \neq j \end{cases}$$

for $i, j \neq 1$. With $p = 5$, we get that the supercharacter table is the 7×7 matrix

$$T = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 4 & 4 & -1 & -1 & -1 & -1 & -1 \\ 4 & -1 & 4 & -1 & -1 & -1 & -1 \\ 4 & -1 & -1 & 4 & -1 & -1 & -1 \\ 4 & -1 & -1 & -1 & 4 & -1 & -1 \\ 4 & -1 & -1 & -1 & -1 & 4 & -1 \\ 4 & -1 & -1 & -1 & -1 & -1 & 4 \end{pmatrix},$$

instead of the 25×25 matrix that represents the full character table. In some sense this matrix reflects much of symmetry of the original group, while removing the symmetry that results from choosing roots of unity.

6.4 The family $GL_n(\mathbb{F}_q)$

Let $k = \mathbb{F}_q$ be the finite field with q elements. We begin by describing the conjugacy classes of $GL_n(k)$ in §6.4.1, a description which can be found e.g., in Macdonald [21]. In §6.4.2, we place some restrictions on which classes can belong to the same superclass in the minimal rational supercharacter theory, to produce a lower bound on the rank. Finally, in §6.4.3 we compute the rank of the minimal rational supercharacter theory of $GL_n(\mathbb{F}_q)$ explicitly.

6.4.1 Conjugacy classes and Φ-Partitions

Let $g \in GL_n(k)$, and consider the action of the polynomial ring $k[t]$ in the indeterminate t on the vector space $V = k^n$, where t acts by the automorphism g. That is, V is a $k[t]$ module by extending the action $t \cdot v = gv$ for all $v \in V$.

Lemma 6.4.1 (e.g., [21])**.** *The isomorphism classes of GL_n are in one-to-one correspondence with isomorphism classes of $k[t]$-modules V, such that*

(1) $\dim V = n$*, and*

(2) $t \cdot v = 0$ *implies* $v = 0$ *for all* $v \in V$*.*

□

Now let $M = (\overline{k})^\times$ be the multiplicative group of the algebraic closure of k. The map $x \mapsto x^q$ is an field automorphism of $\overline{k}$, hence is a group automorphism of M. Let Φ be the set of orbits.

Lemma 6.4.2 (e.g., [21])**.** *Every orbit in Φ of size $d \in \{0, 1, 2, \ldots\}$ is of the form*

$$\{\alpha, \alpha^q, \alpha^{q^2}, \ldots, \alpha^{q^{d-1}}\}$$

for some $\alpha \in M$ with $\alpha^{q^d} = \alpha$ and the polynomial

$$f = \prod_{i=0}^{d-1} (t - \alpha^{q^i}) \in k[t]$$

of degree d is irreducible. Furthermore, every irreducible polynomial $f \neq t$ is of this form, for some orbit in Φ of size $d = \deg(f)$. □

Thus the orbits of Φ can be identified with the corresponding irreducible polynomial, so that we denote by f either an orbit in M or the irreducible polynomial whose roots are determined by that orbit, depending on the context.

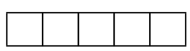
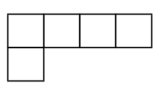
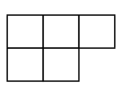
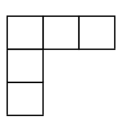
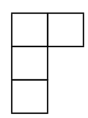
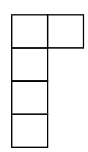
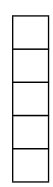
Proposition 6.4.3 (Macdonald [21])**.** *Suppose that V is a $k[t]$-module satisfying the conditions of Lemma 6.4.1. For a polynomial $f \in k[t]$, let (f) be the ideal of $k[t]$ generated by f. Then V has a primary decomposition*

$$V \cong \bigoplus_{f,i} k[t]/(f)^{\mu_i(f)}$$

for some positive integers $\mu_i(f)$, where each $f \neq t$. $\square$

Definition 6.4.4. Let n be a nonnegative integer. A **partition** μ of n of **rank** k is a (possibly empty) list of nonnegative integers $(\mu_1, \mu_2, \dots \mu_k)$ such that $\mu_1 \geq \mu_2 \geq \cdots \mu_k$ and $\sum_i \mu_i = n$. The size of μ is n and we denote this by $|\mu|$.

Example 6.4.5. Partitions are often depicted by rows of boxes called Young diagrams. We list all non-empty partitions of $n = 5$ with the associated diagram:

μ	Young diagram
(5)	
$(4,1)$	
$(3,2)$	
$(3,1,1)$	
$(2,2,1)$	
$(2,1,1,1)$	
$(1,1,1,1,1)$	

Proposition 6.4.3 shows that to each f appearing in the decomposition of a $k[t]$-module V,

there is an associated partition $\mu(f) = (\mu_1(f), \mu_2(f), \dots, \mu_k(f)$, and these partitions satisfy

$$\sum_{\mu(f)\neq\varnothing} \deg(f)|\mu(f)| = n.$$

The functions f appearing in the primary decomposition of V, together with the partitions $\mu(f)$, uniquely determine the isomorphism class of V. This motivates the following definition, where $\mathscr{P}$ denotes the set of partitions of all sizes.

Definition 6.4.6. A Φ-partition $\boldsymbol{\mu}$ of rank n is a function $\boldsymbol{\mu}\colon \Phi \to \mathscr{P}$ satisfying

$$\sum_{f\in\Phi} \deg(f)|\boldsymbol{\mu}(f)| = n.$$

Given any Φ-partition of rank n, we can create work backwards to form the associated $k[t]$-module V, and these are in turn in one-to-one correspondence with the conjugacy classes of $GL_n(k)$. Thus the conjugacy classes of $GL_n(k)$ are indexed by the Φ-partitions of rank n.

Example 6.4.7. The (generalized) Jordan canonical form of a matrix $g \in GL_n(k)$ may be taken as a representative of the conjugacy class of g. Let $f \in \Phi$, with $f = t^d + a_1 t^{d-1} + \cdots + a_d$ and define $J(f)$ to be the $d \times d$ companion matrix

$$J(f) = \begin{pmatrix} 0 & 0 & \cdots & 0 & -a_1 \\ 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 1 & \cdots & 0 & -a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_d \end{pmatrix}.$$

Set $J_m(f)$ to be the block diagonal matrix with m diagonal blocks of $J(f)$. Then the block diagonal matrix with blocks $J_{\boldsymbol{\mu}_i(f)}$ is the generalized Jordan canonical form of the conjugacy class indexed by the Φ-partition $\boldsymbol{\mu}$.

6.4.2 A lower bound on the rank of $(\mathcal{X}^{\mathcal{G}}, \mathcal{K}^{\mathcal{G}})$

We define the type of a Φ-partition and show that the number of types of Φ-partitions of rank n is a lower bound for the number of superclasses of the minimal rational supercharacter theory of

$GL_n(k)$. In particular, this is always a nontrivial supercharacter theory (neither the maximal nor minimal supercharacter theory) when $n \geq 2$ and $q \geq 3$.

Definition 6.4.8. Given a Φ-partition $\boldsymbol{\mu}$, the **type** of $\boldsymbol{\mu}$ is a tuple with entries (up to reordering) indexed by $f \in \Phi$ such that $\boldsymbol{\mu}(f) \neq \varnothing$, where the entry in the position indexed by f is written $\deg(f)^{\boldsymbol{\mu}(f)}$.

Example 6.4.9. The polynomials $t+4$ and t^2+4t+2 are irreducible over $\mathbb{F}_5$. The Φ-partition $\boldsymbol{\mu}$ mapping

$$(t+4) \mapsto (2,2,1) \quad \text{and} \quad t^2+4t+2 \mapsto (1)$$

has rank $(1)(5)+(2)(1) = 7$, so it indexes a conjugacy class of $GL_7(\mathbb{F}_5)$. The type of $\boldsymbol{\mu}$ is $(1^{(2,2,1)}, 2^{(1)})$.

Let $J_m(\alpha)$ be the $m \times m$ Jordan block where each diagonal entry is α, each superdiagonal entry is 1, and all other entries are zero. We will need the following lemma, working over $\overline{k}$.

Lemma 6.4.10. *For $0 \neq \alpha \in \overline{k}$, the Jordan canonical form of the matrix $J_m(\alpha)^r$ is $J_m(\alpha^r)$.*

Proof. Write $J_m(\alpha) = \alpha I_m + N$, where I_m is the $m \times m$ identity matrix, and N is the $m \times m$ matrix with each superdiagonal matrix equal to 1, and all other entries equal to zero. Then we have

$$J_m(\alpha)^r = (\alpha I_m + N)^r = \alpha^r I_m + N'$$

where N' is nilpotent and has rank m, so that $J_m(\alpha)^r$ has rank m, and eigenvalue α^r with multiplicity m. It follows that the Jordan form of $J_m(\alpha)^r$ is $J_m(\alpha^r)$. □

Let $f \in \Phi$ of degree d. Since f has distinct roots in $\overline{k}$, the Jordan canonical form of the companion matrix $J(f)$ is the diagonal matrix with diagonal entries given by the orbit f. By block multiplication, the matrix $J_m(f)$ is conjugate to the block upper-triangular matrix that replaces companion matrix $J(f)$ with its diagonalized Jordan form. By permuting rows and columns, this matrix is similar to the block upper triangular matrix with blocks $J_m(\alpha)$, for each $\alpha \in f$. Thus we have that the Jordan form of $J_m(f)$ is the block diagonal matrix with a block $J_m(\alpha)$ for each $\alpha \in f$. We are ready to prove the following.

Lemma 6.4.11. *Let* $f = \{\alpha, \alpha^q, \ldots, \alpha^{q^d}\} \in \Phi$ *with* $d \leq n$ *and choose* $1 \leq r \leq N$ *satisfying* $\gcd(r, N) = 1$ *where*

$$N = |GL_n(\mathbb{F}_q)| = q^n - 1(q^n - q) \cdots (q^n - q^{n-1}).$$

Then $J_m(f)^r$ *is conjugate to* $J_m(g)$ *in* $GL_n(\mathbb{F}_q)$ *where* $g = \{\beta, \beta^q, \ldots, \beta^{q^d}\}$ *and* $\beta = \alpha^r$.

Proof. First, since $\deg(f) = d$, f is an orbit in $F_{q^d}^{\times}$. Since $d \leq n$, we have that $|F_{q^d}^{\times}| = q^d - 1$ divides N. It follows that $\gcd(r, q^d - 1) = 1$, and so the map $x \mapsto x^r$ is a group automorphism of the multiplicative group $F_{q^d}^{\times}$ taking f to g. It follows that g is another orbit of the map $x \mapsto x^q$, and so $g \in \Phi$ is again an irreducible polynomial of degree d, and g has the correct form.

The following diagram commutes, since exponentiation commutes with conjugation:

$$\begin{tikzcd}
GL_{md}(\mathbb{F}_q) \arrow[r, "X \mapsto TXT^{-1}"] \arrow[d, "X \mapsto X^r"'] & GL_{md}(\overline{\mathbb{F}_q}) \arrow[d, "X \mapsto X^r"] \\
GL_{md}(\mathbb{F}_q) & GL_{md}(\overline{\mathbb{F}_q}) \arrow[l, "X \mapsto T^{-1}XT"']
\end{tikzcd}$$

Starting at the top left, we follow the matrix $J_m(f)$, and take $T \in GL_n(\overline{\mathbb{F}_q})$ to be the matrix such that $TJ_m(f)T^{-1}$ is the Jordan canonical form $\bigoplus_{\alpha \in f} J_m(\alpha)$. Raising to the r power gives $\bigoplus_{\alpha \in f} J_m(\alpha^r)$. Conjugation by T^{-1} must give $J_m(f)^r \in GL_{md}(\mathbb{F}_q)$ by commutativity. But this shows that $J_m(f)^r$ has Jordan canonical form in $GL_{md}(\overline{\mathbb{F}_q})$ given by

$$\bigoplus_{\alpha \in f} J_m(\alpha^r) = \bigoplus_{\alpha \in g} J_m(\alpha),$$

which is the Jordan canonical form in $GL_{md}(\overline{\mathbb{F}_q})$ of the matrix $J_m(g)$. Since $J_m(g)$ and $J_m(f)^r$ are conjugate in $GL_{md}(\overline{\mathbb{F}_q})$, they are also conjugate in $GL_{md}(\mathbb{F}_q)$. $\square$

Extending this result to block diagonal matrices with blocks of the form $J_m(f)$, we have the following corollary.

Corollary 6.4.12. *Let* $g, g' \in GL_n(\mathbb{F}_q)$ *in conjugacy classes indexed by* $\boldsymbol{\mu}$ *and* $\boldsymbol{\mu}'$*, respectively. If* $g^r = g'$ *for some power* $1 \leq r \leq N$ *with* $\gcd(r, N) = 1$*, then* $\boldsymbol{\mu}$ *and* $\boldsymbol{\mu}'$ *have identical types.*

Since superclasses in the minimal rational supercharacter theory are unions conjugacy classes over orbits of the action $g \mapsto g^r$, it follows that any conjugacy classes in the same superclass must be indexed by the Φ-partitions of the same type. Thus the number of superclasses is at least the number of possible types. This ensures that this is not the maximal supercharacter theory as soon as $n \geq 2$, in which case we already have at least three distinct types: $1^{(1,1)}$, $1^{(2)}$, and $2^{(1)}$. Since $\mathbb{F}_q^\times$ is a quotient of $GL_n(\mathbb{F}_q)$ (by $SL_n(\mathbb{F}_q)$), once $q \geq 3$ we have that the character table of $\mathbb{F}_q^\times$ contains irrational entries. By lifting characters of $\mathbb{F}_q^\times$, the same is true for the character table of $GL_n(\mathbb{F}_q)$. Thus for $q \geq 3$, the minimal rational supercharacter theory is not the minimal supercharacter theory. Combining these facts, we have that the minimal rational supercharacter theory is neither the largest nor smallest supercharacter theory for $n \geq 2$ and $q \geq 3$.

6.4.3 Superclasses in $GL_2(\mathbb{F}_q)$

Let $n = 2$ and suppose $q \geq 3$. Then the only possible non-empty partitions appearing in a Φ-partition of rank n are $\mu = (2)$ and $\mu = (1)$. Similarly, the only possible irreducible polynomials f that can be mapped to a non-empty partition are polynomials of degree 1 or 2. Then the only possible types of Φ-partitions of rank 2 are given below.

Form of $\boldsymbol{\mu}$	No. of classes
$\left(1^{(2)}\right)$	$q-1$
$\left(1^{(1,1)}\right)$	$q-1$
$\left(1^{(1)}, 1^{(1)}\right)$	$\frac{1}{2}(q-1)(q-2)$
$\left(2^{(1)}\right)$	$\frac{1}{2}q(q-1)$

To count the total number of classes of each type, we count the number of polynomials in Φ of the appropriate degree. For the first row and second row, there are $q-1$ irreducbile polynomials of degree 1 (excluding $f = t$), since these all are of the form $(t - \alpha)$ for $0 \neq \alpha \in \mathbb{F}_q$. For the third row, we need to choose two such polynomials in any order, which gives $(q-1)(q-2)/2$ choices. For the fourth row, we can choose any polynomial of degree 2. These are orbits of the form (α, α^q) where each orbit is contained in $\mathbb{F}_{q^2}^\times - \mathbb{F}_q^\times$. This gives $(q^2 - 1) - (q - 1) = q(q - 1)$ elements, and these are partitioned into orbits of size two. Thus there are $q(q-1)/2$ choices for this polynomial.

From our lower bound, we know that the rank of the minimal rational supercharacter theory is at least four, so that it must be nontrivial. In fact, the rank will generally be much larger, as we will show. Since all conjugacy classes in a superclass are indexed by Φ-partitions of the same type, it makes sense to talk of the type of the superclass. We consider each type separately and count the number of superclasses of that type.

Lemma 6.4.13. *Let $\tau(n)$ denote the number of divisors of n. The types $1^{(2)}$ and $1^{(1,1)}$ each belong to $\tau(q-1)$ superclasses.*

Proof. For the types $1^{(2)}$ and $1^{(1,1)}$, raising to the r power takes $J_2(f)$ to $J_2(g)$ in the former case or $J_1(f) \to J_1(g)$ in the latter case. So in either case, the orbits of conjugacy classes are in this in one-to-one correspondence with the orbits of $\mathbb{F}_q^\times$ under the action $x \mapsto x^r$. This is a cyclic group of size $q-1$, and we are acting by the full automorphism group of $\mathbb{F}_q^\times$, so we are precisely in the case of counting the superclasses in a cyclic group. We have seen that these are in one-to-one correspondence with the number of divisors of the size of the group, in this case $q-1$. Thus the number of superclasses of type $1^{(2)}$ or $1^{(1,1)}$ is the number of divisors of $q-1$, or $\tau(q-1)$. □

Lemma 6.4.14. *There are $\tau(q^2-1) - \tau(q-1)$ superclasses of type $2^{(1)}$.*

Proof. For the type $2^{(1)}$, raising to the r power takes $J_1(f)$ to $J_1(g)$, so we need to count orbits in $\mathbb{F}_{q^2}^\times - \mathbb{F}_q^\times$ under the action $x \mapsto x^r$. The total number of orbits in $\mathbb{F}_{q^2}^\times$ is $\tau(q^2-1)$ and $\tau(q-1)$ of these are contained in $\mathbb{F}_q^\times$. If $f = (\alpha, \alpha^q)$, then α and α^q lie in distinct orbits under the action $x \mapsto x^r$, since $\gcd(r, q) = 1$ for each r. It follows that there are $\tau(q^2-1) - \tau(q-1)$ distinct orbits of f, since each orbit of f corresponds to an orbit of α. □

Lemma 6.4.15. *There are*

$$\frac{1}{\phi(q-1)} \sum_{\substack{1 \le r \le q-1 \\ \gcd(r,q-1)=1}} \left[\binom{\gcd(r-1, q-1)}{2} + \frac{\gcd(r^2-1, q-1) - \gcd(r-1, q-1)}{2} \right]$$

superclasses of type $(1^{(1)}, 1^{(1)})$.

Proof. The conjugacy classes of this type have Jordan blocks $J_1(f) \oplus J_1(g)$ with $f \neq g$ and $\deg(f) = \deg(g) = 1$. Each degree one polynomial $f \neq 0$ determines an element of $\mathbb{F}_q^\times$. Then after applying the map $x \mapsto x^r$, we are interested in orbits in $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$, discounting those orbits generated by elements of the form (α, α), since we cannot have $f \neq g$.

Using Burnside's lemma, this is the same as the average number of fixed points. It suffices to consider $1 \leq r \leq q$, since $x^q = x$ for all $x \in \mathbb{F}_q^\times$, and we are acting by all such r with $\gcd(r, q) = 1$. For a fixed r, we need to count the fixed points $(\alpha, \beta) \in \mathbb{F}_q^\times \times \mathbb{F}_q^\times$, up to reordering the pair, and with $\alpha \neq \beta$. We divide these fixed points into two cases.

For the first case, we could have $(\alpha^r, \beta^r) = (\alpha, \beta)$ because the pair is fixed pointwise. Up to reordering, this is the same as choosing two fixed points of $\mathbb{F}_q^\times$. Suppose $\alpha \in \mathbb{F}_q^\times$ is fixed, so that $\alpha^r = \alpha$. This occurs if and only if $|\alpha|$ divides $r - 1$. The number of elements in the cyclic group $\mathbb{F}_q^\times$ of order $q - 1$ with order dividing $r - 1$ is $\gcd(r - 1, q - 1)$. Thus the total number of fixed points of this type is $\binom{\gcd(r-1,q-1)}{2}$.

For the second case, we could have that $(\alpha^r, \beta^r) = (\alpha, \beta)$ (up to reordering) because $\alpha^r = \beta$ and $\beta^r = \alpha$. This is possible if and only if $\alpha^{r^2} = \alpha$, i.e. if $|\alpha|$ divides $r^2 - 1$. As before, the number of elements with this property is $\gcd(r^2 - 1, q - 1)$. Now, since we require $\alpha \neq \beta$, we need to subtract the number of pairs (α, α) with this property, but this is equivalent to finding the number of fixed points of $\mathbb{F}_q^\times$, which we computed to be $\gcd(r - 1, q - 1)$. Then the total number of fixed points (α, β) of this type for which $\alpha \neq \beta$ is $\gcd(r^2 - 1, q - 1) - \gcd(r - 1, q - 1)$, and up to reordering, is

$$\frac{1}{2}(\gcd(r^2 - 1, q - 1) - \gcd(r - 1, q - 1)).$$

For a given r, the number of fixed points of the map $x \mapsto x^r$ (applied pointwise) is therefore

$$\binom{\gcd(r - 1, q - 1)}{2} + \frac{\gcd(r^2 - 1, q - 1) - \gcd(r - 1, q - 1)}{2}$$

and the result follows by averaging over the possible values of r. $\square$

Combining these results, we have proved the following.

Theorem 6.4.16. *The rank of the minimal rational supercharacter theory for $GL_2(\mathbb{F}_q)$ is*

$$\tau(q-1)+\tau(q^2-1)+N(q)$$

where

$$N(q)=\frac{1}{\phi(q-1)}\sum_{\substack{1\le r\le q-1\\ \gcd(r,q-1)=1}}\left[\binom{\gcd(r-1,q-1)}{2}+\frac{\gcd(r^2-1,q-1)-\gcd(r-1,q-1)}{2}\right],$$

$\tau(n)$ is the number of divisors of n, and ϕ is the Euler totient function.

Example 6.4.17. We list the number $|\mathcal{K}|$ of congjugacy classes, the number $|\mathcal{K}^{\mathcal{G}}|$ of minimal rational superclasses, and the ratio (to three decimals) for $GL_2(\mathbb{F}_q)$ for a few small values of $q \geq 3$.

q	$\|\mathcal{K}\|$	$\|\mathcal{K}^{\mathcal{G}}\|$	$\|\mathcal{K}^{\mathcal{G}}\|/\|\mathcal{K}\|$
3	8	7	0.875
4	15	8	0.533
5	24	15	0.625
7	48	23	0.479
8	63	12	0.190
9	80	25	0.313
11	120	33	0.275
13	168	47	0.280
16	255	30	0.118
17	288	47	0.163
19	360	63	0.175

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