

**Clones of Finite Idempotent Algebras with Strictly Simple
Subalgebras**

by

Topaz Dent

B.A., University of California, Davis, 2002

A thesis submitted to the
Faculty of the Graduate School of the
University of Colorado in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
Department of Mathematics

2011

This thesis entitled:
Clones of Finite Idempotent Algebras with Strictly Simple Subalgebras
written by Topaz Dent
has been approved for the Department of Mathematics

Ágnes Szendrei

Professor Keith Kearnes

Date _____

The final copy of this thesis has been examined by the signatories, and we find that both the content and the form meet acceptable presentation standards of scholarly work in the above mentioned discipline.

Dent, Topaz (Ph.D., Mathematics)

Clones of Finite Idempotent Algebras with Strictly Simple Subalgebras

Thesis directed by Professor Ágnes Szendrei

Abstract: We determine the clone of a finite idempotent algebra \mathbb{A} that is not simple and has a unique nontrivial subalgebra \mathbb{S} with more than two elements. Under these conditions, the proper subalgebras and the quotient algebra of \mathbb{A} are finite idempotent strictly simple algebras of size at least 3 and it is known that such algebras are either affine, quasiprimal, or of a third classification. We focus on the first two cases. By excluding binary edge blockers from the relational clone when \mathbb{S} is affine and by excluding ternary edge blockers from the relational clone together with an additional condition on the subuniverses of \mathbb{A}^2 when \mathbb{S} is quasiprimal, we give a nice description of the generating set of the relational clone of \mathbb{A} . Thus, by the Galois connection between operations and relations, we determine the clone of \mathbb{A} .

Dedication

Por la piña y Bustelo, one who is coming and one who has gone.

Acknowledgements

I owe my deep gratitude to my advisor, Ágnes Szendrei, for her invaluable instruction and guidance and the exceptional generosity with which she gave me her time. I would like to thank my family for their tireless support and encouragement through these many years. Finally, I am profoundly grateful to my partner, Joshua Wiscons, whose steadfast confidence in me always emboldened me.

Contents

Chapter	
1 Introduction	1
2 Preliminaries	3
2.1 Algebras, Operations, and Clones	3
2.2 Compatible Relations and Relational Clones	5
2.3 The Galois Connection	12
2.4 Finite Idempotent Strictly Simple Algebras	14
2.5 Crosses	17
3 The Subuniverses of \mathbb{A}^2	19
3.1 A Description of the Subuniverses of \mathbb{A}^2	21
3.2 Crosses Among the Subuniverses of \mathbb{A}^2	36
4 Edge Blockers	48
4.1 Marković–Maróti–McKenzie Edge Blockers	48
4.2 The Edge Blockers Λ and \mathcal{K}_b	50
4.3 The Existence of an Algebra with a Relation Λ or \mathcal{K}_b	52
4.4 Algebras with Small Arity Edge Blockers	57
5 The Clone of \mathbb{A} When \mathbb{S} is Quasiprimal or Affine	64
5.1 Reductions	66

5.2	H.D.-Automorphisms of \mathbb{A} when \mathbb{S} is Affine.	70
5.3	Compatible Subuniverses that Indicate a Ternary Edge Blocker	81
5.4	Reduced Subuniverses Are θ -Closed in their A -coordinates	85
5.5	The Clone of \mathbb{A}	109

Bibliography		129
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Chapter 1

Introduction

In this dissertation we will determine the clones of finite idempotent algebras with certain restrictions on subalgebras. In particular, we will consider algebras that satisfy the following assumption.

Assumption 1. \mathbb{A} is a finite idempotent algebra with a unique proper nontrivial subalgebra \mathbb{S} such that $|S| > 2$ and $|A \setminus S| > 1$.

Algebras that satisfy Assumption 1 have the property that their proper subalgebras and quotient algebras are strictly simple of size at least 3. In particular, the unique nontrivial subalgebra \mathbb{S} is a finite idempotent strictly simple algebra with more than two elements. Such algebras were classified by Szendrei [Sze87, Theorem 2.1] to be in one of three categories: quasiprimal, affine, or a third category which is described in Theorem 2.4.4. We will focus our investigation on the cases when \mathbb{S} is quasiprimal or affine.

To determine the clone of an algebra \mathbb{A} it suffices, by the Galois connection, to determine the relational clone of \mathbb{A} . Informally, the relational clone of \mathbb{A} is the collection of all subuniverses of finite powers of \mathbb{A} . Therefore, our goal is to determine a set of subuniverses of finite powers of \mathbb{A} that generates the relational clone of \mathbb{A} .

Our examination of the subuniverses of \mathbb{A}^n , $n \geq 1$, brought to light a family of binary and ternary relations on \mathbb{A} which, when included in the relational clone of \mathbb{A} , will prevent \mathbb{A} from having an edge operation (equivalently, a cube operation; equivalently, a parallelogram operation). We will call these relations *edge blockers*. It has been shown by Aichinger, McKenzie, and Mayr [AMM]

that a finite algebra with an edge operation is finitely related, that is, has a finitely generated relational clone.

While excluding the binary and ternary edge blockers from the relational clone of \mathbb{A} will not imply that \mathbb{A} has an edge operation, it will allow us to find a nice description for the relational clone, hence the clone of \mathbb{A} . This description, which is the main result of this dissertation, is stated in Theorem 5.5.9.

Chapter 2

Preliminaries

In this section we will describe the notation and conventions that we will use throughout the paper. We will also list useful facts, some of which are well-known and some that are, perhaps, less familiar.

For an integer n , we let $\bar{n} := \{1, \dots, n\}$. For a nonempty set A , and a tuple $(a_1, \dots, a_n) \in A^n$, we will sometimes write \bar{a} rather than the tuple (a_1, \dots, a_n) . Though the notation is similar, it will always be clear from context whether we are referring a set of integers or tuple.

2.1 Algebras, Operations, and Clones

For a nonempty set A and a collection of finitary operations, F , on A , the algebra with *underlying set* A and *basic operations* F is denoted by $\mathbb{A} = (A; F)$. We will denote that \mathbb{B} is a *subalgebra* of \mathbb{A}^n , $n \geq 1$, by $\mathbb{B} \leq \mathbb{A}^n$. We will say that the underlying set, B , of \mathbb{B} is an *n -ary compatible relation* of \mathbb{A} , or equivalently, B is a *subuniverse* of \mathbb{A}^n , which we will denote by $B \leq \mathbb{A}^n$. We will call the one-element subuniverses of \mathbb{A} the *singleton subuniverses* of \mathbb{A} , or the *trivial subuniverses* of \mathbb{A} .

A *clone* on the set A is a set of finitary operations on A that is closed under compositions and contains the projections. The clone of an algebra $\mathbb{A} = (A; F)$ is the least clone on A that contains F , we say that this clone is the *clone generated by F* and we denote this clone by $\text{Clo}(\mathbb{A})$. The operations in the clone of \mathbb{A} , which we call *term operations*, are exactly the operations that are interpretations of terms in the language of the algebra. Two algebras are called *term equivalent* if

they have the same clones, that is, if they have the same term operations.

For $n \geq 1$, an n -ary operation, f , on A is an *idempotent operation* if $f(x, \dots, x) = x$, for any $x \in A$. The algebra $\mathbb{A} = (A, F)$ is an *idempotent algebra* if and only if $\{a\}$ is a one-element subuniverse of \mathbb{A} , for every $a \in A$.

For any $n \geq 1$ we will denote $\bar{n} := \{1, 2, \dots, n\}$. If θ is a congruence on \mathbb{A} and $(a, b) \in \theta$, then we say that a is θ -related to b , and we will denote the relationship of a and b by $a\theta b$. If θ is the equality relation on \mathbb{A} , then will write \mathbb{A} instead of \mathbb{A}/θ .

Let \mathbb{A}_i be a collection of algebras that have a common language, for all $i \in \bar{n}$. Let θ_i be an equivalence relation on A_i , for each i . Then the map on the product of A_1, \dots, A_n , given by,

$$\Pi A_i \rightarrow \Pi(A_i/\theta_i) : (a_1, \dots, a_n) \mapsto (a_1/\theta_1, \dots, a_n/\theta_n),$$

will be called the *natural map*. If θ_i is a congruence on \mathbb{A}_i , for all $1 \leq i \leq n$, then this map is the *natural homomorphism* from ΠA_i to $\Pi(A_i/\theta_i)$. We will always specify the domain and the codomain of a natural map (homomorphism), however we will omit stating the map on the elements of the domain since this assignment is clear from the domain, the codomain, and the definition of a natural map (homomorphism).

Theorem 2.1.1. *If the variety \mathcal{V} is congruence distributive or congruence permutable, then \mathcal{V} is congruence modular.*

The following theorem of Mal'cev characterizes congruence modular varieties.

Theorem 2.1.2 ([Mal54]). *The variety \mathcal{V} is congruence permutable if and only if there is a term $p(x, y, z)$ in its language such that these equations hold for \mathcal{V} .*

$$(1) \quad p(x, x, y) = y,$$

$$(2) \quad p(y, x, x) = y.$$

The following theorem of Gumm characterizes congruence modular varieties.

Theorem 2.1.3 ([Gum83]). *A variety \mathcal{V} is congruence modular iff for some $n \geq 0$ there are terms $d_0(x, y, z), \dots, d_n(x, y, z), p(x, y, z)$ in its language such that these equations hold in \mathcal{V} .*

- (1) $d_0(x, y, z) \approx x, d_i(x, y, x) \approx x$ for $1 \leq i \leq n$,
- (2) $d_i(x, y, y) \approx d_{i+1}(x, y, y)$ for even $i < n$,
- (3) $d_i(x, x, y) \approx d_{i+1}(x, x, y)$ for odd $i < n$,
- (4) $d_n(x, y, y) \approx p(x, y, y), p(x, x, y) \approx y$.

2.2 Compatible Relations and Relational Clones

Let A be a fixed set. For an n -ary operation f and an m -ary relation ρ on A we say that f preserves ρ or ρ is invariant under f if for any $\bar{a}_i = (a_{(i,1)}, \dots, a_{(i,m)}) \in A^m, 1 \leq i \leq n$,

$$\rho \ni f(\bar{a}_1, \dots, \bar{a}_n) = f \left(\begin{pmatrix} a_{(1,1)} \\ a_{(1,2)} \\ \vdots \\ a_{(1,m)} \end{pmatrix}, \dots, \begin{pmatrix} a_{(n,1)} \\ a_{(n,2)} \\ \vdots \\ a_{(n,m)} \end{pmatrix} \right) = \begin{pmatrix} f(a_{(1,1)}, \dots, a_{(n,1)}) \\ f(a_{(1,2)}, \dots, a_{(n,2)}) \\ \vdots \\ f(a_{(1,m)}, \dots, a_{(n,m)}) \end{pmatrix}$$

A *relational clone* on the set A can be defined by two equivalent notions. The first, and one we will use more often, is that a relational clone on A is a set of relations that contains the equality relation and is closed under taking Cartesian products, intersections, projections, and permuting coordinates. The second definition is that a set of relations K on A is a relational clone on A if K is closed under primitive-positive definability, that is, if K is a set of relations on A such that $\rho \in K$ holds for every relation ρ on A which is definable by a primitive-positive formula (pp-formula) in the relational structure $\langle A, K \rangle$, where a pp-formula in the language of $\langle A, K \rangle$ is a first-order formula using only the logical symbols $\exists, \wedge, =$, and the symbols for the relations in K .

For an algebra $\mathbb{A} = (A; F)$, the relational clone $\text{RClo}(\mathbb{A})$ on \mathbb{A} is the relational clone on A that contains all relations that are invariant under the operations in $\text{Clo}(\mathbb{A})$. Notice that the subuniverses of finite powers of \mathbb{A} are relations that are invariant under $\text{Clo}(\mathbb{A})$, in other words,

the relational clone of \mathbb{A} is exactly the collection of all subuniverses of finite powers of \mathbb{A} . Recall that we defined a subuniverse of \mathbb{A}^n to be an n -ary compatible relation, thus for an operation f and a relation ρ on A , we can say that ρ is invariant under f or ρ is *compatible* with f . If the relational clone of \mathbb{A} is generated by (taking products, intersections, projections, and permuting the coordinates of) a set R of relations on A , then we will write $\text{RClo}(\mathbb{A}) = \langle R \rangle_{\text{RClo}}$.

Definition 2.2.1. The relational clone of a finite algebra \mathbb{A} is *finitely related* if its relational clone is determined by finitely many relations.

In other words, the relational clone of \mathbb{A} is finitely related if there exists some finite set $\sigma_1, \dots, \sigma_n$ of relations A such that $\text{RClo}(\mathbb{A}) = \langle \sigma_1, \dots, \sigma_n \rangle_{\text{RClo}}$.

We will now give some examples of compatible relations. First, we will give some notation for constructions that yield subuniverses from subuniverses. Suppose that $D \leq \mathbb{A}^m$, for some $m \geq 1$. Let $I = \{i_1, \dots, i_m\}$. We define the notation

$$D(x_{i_1}, \dots, x_{i_m}) := \{(a_{i_1}, \dots, a_{i_m}) \in A^I : (a_1, \dots, a_m) \in D \text{ where } a_1 = a_{i_1}, \dots, a_m = a_{i_m}\}.$$

Thus the i_k^{th} variable of $D(x_{i_1}, \dots, x_{i_m})$ corresponds to the k^{th} variable of D , for all $1 \leq k \leq m$. Though $D(x_{i_1}, \dots, x_{i_m})$ is a subuniverse of A^I and D is a subuniverse of A^m , we will consider them equal.

Now suppose that B is a subuniverse of \mathbb{A}^n , for some $n \geq 1$. Suppose that $I \subseteq \bar{n}$, $I = \{i_1, \dots, i_m\}$, $J \subseteq I$, where $i_1 < \dots < i_m$. Then the *projection of B onto its coordinates in I* , is the subuniverse of \mathbb{A}^I , denoted $\text{pr}_I B$, that is defined by,

$$\text{pr}_I B := \{(x_{i_1}, \dots, x_{i_m}) : (x_1, \dots, x_{i_1-1}, x_{i_1}, x_{i_1+1}, \dots, x_{i_m-1}, x_{i_m}, x_{i_m+1}, \dots, x_n) \in B\}.$$

If D is a subset of A^m such that $\text{pr}_I B = D$, then from the definition of the projection we get that $\text{pr}_J(\text{pr}_I B) = \text{pr}_J B \leq \mathbb{A}^J$, thus $\text{pr}_J(\text{pr}_I B) = \text{pr}_J D$.

If \bar{a} is a tuple in \mathbb{A}^n , then the projection of \bar{a} onto its i^{th} -coordinates, for all $i \in I$, will be denoted by $\bar{a}_I := \text{pr}_I \bar{a}$.

Let $\bar{a} \in A^{\bar{n} \setminus I}$. Then the subset of A^I arising from B and the tuple \bar{a} is defined by

$$\begin{aligned} & B(a_1, \dots, a_{i_1-1}, x_{i_1}, a_{i_1+1}, \dots, a_{i_m-1}, x_{i_m}, a_{i_m+1}, \dots, a_n) \\ & := \{(x_{i_1}, \dots, x_{i_m}) \in \mathbb{A}^I : (a_1, \dots, a_{i_1-1}, x_{i_1}, a_{i_1+1}, \dots, a_{i_m-1}, x_{i_m}, a_{i_m+1}, \dots, a_n) \in B\}. \end{aligned}$$

Furthermore, if $B(a_1, \dots, a_{i_1-1}, x_{i_1}, a_{i_1+1}, \dots, a_{i_m-1}, x_{i_m}, a_{i_m+1}, \dots, a_n) = D$, for some $D \subseteq A^m$, then

$$\text{pr}_J B(a_1, \dots, a_{i_1-1}, x_{i_1}, a_{i_1+1}, \dots, a_{i_m-1}, x_{i_m}, a_{i_m+1}, \dots, a_n) = \text{pr}_J D \leq \mathbb{A}^J.$$

Proposition 2.2.2. *Let B be a subset of A^n such that $\text{pr}_1 B = \{a\}$, for some $a \in A$. Then B is in the relational clone of \mathbb{A} if and only if $\{a\}$ and $\text{pr}_{2, \dots, n} B$ are in the relational clone of \mathbb{A} .*

Proposition 2.2.3. *For an algebra \mathbb{A} and an automorphism, π , of \mathbb{A} , the set of fixed points of π is a subuniverse of \mathbb{A} .*

Proposition 2.2.4. *Let $n \geq 2$. If \mathbb{A} is an algebra, then the set $\{(x, \dots, x, y) : x, y \in A\}$ is a subuniverse of \mathbb{A}^n .*

Proposition 2.2.5. *Let θ be an equivalence relation on \mathbb{A} . Then θ is a subuniverse of \mathbb{A}^2 if and only if θ is a congruence on \mathbb{A} .*

Proof. Suppose that θ is an equivalence relation on \mathbb{A} . Then θ is a subset of A^2 . Let f be an m -ary term operation on \mathbb{A} and let $\bar{a}_j := \langle a_j, a'_j \rangle$ be an element of θ , for all $1 \leq j \leq m$.

(\Rightarrow) Suppose that θ is a subuniverse of \mathbb{A}^2 . Then θ is closed under the term operations of \mathbb{A} .

In particular, θ contains

$$\begin{aligned} f(\bar{a}_1, \dots, \bar{a}_m) &= f(\langle a_1, a'_1 \rangle, \dots, \langle a_m, a'_m \rangle) \\ &= (f(a_1, \dots, a_m), f(a'_1, \dots, a'_m)) \end{aligned}$$

Therefore $f(a_1, \dots, a_m)$ is θ -related to $f(a'_1, \dots, a'_m)$ whenever $\bar{a}_j \in \theta$. Since θ is an equivalence relation, $\bar{a}_j \in \theta$ implies that a_j is θ -related to a'_j . It follows from the definition of a congruence that θ is a congruence on \mathbb{A} .

(\Leftarrow) Now suppose that θ is a congruence on \mathbb{A} . Since θ is an equivalence relation, $\bar{a}_j \in \theta$ implies a_j is θ -related to a'_j , for all $1 \leq j \leq m$. Furthermore, since θ is a congruence on \mathbb{A} and f is a term operation on \mathbb{A} , we get that $f(a_1, \dots, a_m)$ is θ -related to $f(a'_1, \dots, a'_m)$. Hence θ contains $(f(a_1, \dots, a_m), f(a'_1, \dots, a'_m)) = f(\bar{a}_1, \dots, \bar{a}_m)$. Therefore θ is closed under f and hence, θ is a subalgebra of \mathbb{A}^2 . \square

Proposition 2.2.6. *If \mathbb{A} is an idempotent algebra, then every congruence class is a subuniverse of \mathbb{A} .*

Proof. Let \mathbb{A} be an idempotent algebra, where θ is a nontrivial congruence on \mathbb{A} and g is an n -ary term operation on \mathbb{A} . Let $a_1, \dots, a_n, a \in A$ be elements of the congruence class $a/\theta \in A/\theta$. Thus $a_1\theta a, \dots, a_n\theta a$. Since θ is a congruence on \mathbb{A} we get that $g(a_1, \dots, a_n)\theta g(a, \dots, a)$. Furthermore, since \mathbb{A} is idempotent, $g(a, \dots, a) = a$. Thus, $g(a_1, \dots, a_n)\theta a$, which means $g(a_1, \dots, a_n) \in a/\theta$. Therefore, the congruence class a/θ is preserved by g . Since a was an arbitrary element of \mathbb{A} and g was an arbitrary term operation of \mathbb{A} , we get that every congruence class is a subuniverse of \mathbb{A} . \square

Definition 2.2.7. For $1 \leq i \leq n$, let \mathbb{A}_i be a collection of algebras that share a common language, and let θ_i be a congruence on \mathbb{A}_i . Let B be a subuniverse of $\mathbb{A}_1 \times \dots \times \mathbb{A}_n$. We will say that B is θ_i -closed in its i^{th} -coordinate, for some $i \in \bar{n}$, if

$$(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \in B \text{ and } a_i\theta_i a'_i \implies (a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n) \in B.$$

We will say that B is $\theta_1 \times \dots \times \theta_n$ -closed if, for all $1 \leq i \leq n$,

$$(a_1, \dots, a_n) \in B \text{ and } a_i\theta_i a'_i \implies (a'_1, \dots, a'_n) \in B.$$

Proposition 2.2.8. *For $1 \leq i \leq n$, let \mathbb{A}_i be a collection of algebras that share a common language, let θ_i be a congruence on \mathbb{A}_i , and let $\rho : \prod \mathbb{A}_i \rightarrow \prod \mathbb{A}_i/\theta_i$ be the natural map. If $\mathbb{B} \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$, then TFAE.*

- (a) For each i , B is θ_i -closed in its i^{th} coordinate.
- (b) B is $\theta_1 \times \dots \times \theta_n$ -closed.

$$(c) \ B = \rho^{-1}(\rho(B)).$$

Proof. (a) \implies (b) We will show this implication by inducting on n . If $n = 1$ and B be a subuniverse of \mathbb{A}_1 that is θ_1 -closed, then (a) clearly implies (b).

Suppose that every subuniverse of $\prod_{i=1}^{n-1} \mathbb{A}_i$ that is θ_i -closed in its i^{th} coordinate, for each $1 \leq i \leq n-1$, is also $\theta_1 \times \cdots \times \theta_{n-1}$ -closed.

Let B be a subuniverse of $\prod_{i=1}^n \mathbb{A}_i$ that is θ_i -closed in its i^{th} -coordinate, for each $1 \leq i \leq n$. Let $(a_1, \dots, a_{n-1}, a_n) \in B$ and suppose that $a_i \theta_i a'_i$, for each $1 \leq i \leq n$. Then, for any $i \in \overline{n-1}$, B contains the tuple $(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_{n-1}, a_n)$. Thus $B(x_1, \dots, x_{n-1}, a_n)$ is a subuniverse of $\prod_{i=1}^{n-1} \mathbb{A}_i$ that contains $(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_{n-1})$ and $(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_{n-1})$, which means $B(x_1, \dots, x_{n-1}, a_n)$ is θ_i -closed in its i^{th} coordinate. Since i was arbitrary in $\overline{n-1}$, we get that $B(x_1, \dots, x_{n-1}, a_n)$ is θ_i -closed in its i^{th} coordinate, for each $1 \leq i \leq n-1$, thus by the induction hypothesis, $B(x_1, \dots, x_{n-1}, a_n)$ is $\theta_1 \times \cdots \times \theta_{n-1}$ -closed. Then $a_i \theta_i a'_i$, for each $1 \leq i \leq n-1$, implies $(a'_1, \dots, a'_{n-1}) \in B(x_1, \dots, x_{n-1}, a_n)$. Hence $(a'_1, \dots, a'_{n-1}, a_n) \in B$. Since B is θ_n -closed in its n^{th} -coordinate and $a_n \theta_n a'_n$, we get that $(a'_1, \dots, a'_{n-1}, a'_n) \in B$. Therefore, B is $\theta_1 \times \cdots \times \theta_n$ -closed.

(b) \implies (c) By the definition of ρ it is clear that $B \subseteq \rho^{-1}(\rho(B))$. Let $(a_1, \dots, a_n) \in \rho^{-1}(\rho(B))$. Then $(a_1/\theta_1, \dots, a_n/\theta_n) \in \rho(B)$ which means, for each $1 \leq i \leq n$, there exists some $a'_i \in A_i$ such that $a'_i \theta_i a_i$ and $(a'_1, \dots, a'_n) \in B$. Since B is $\theta_1 \times \cdots \times \theta_n$ -closed and $(a'_1, \dots, a'_n) \in B$ with $a'_i \theta_i a_i$, for each $1 \leq i \leq n$, we get that $(a_1, \dots, a_n) \in B$. Hence $B \supseteq \rho^{-1}(\rho(B))$.

(c) \implies (a) Let $i \in \bar{n}$. Suppose $(a_1, \dots, a_{i-1}, a_i, a_{i+1}, \dots, a_n) \in B = \rho^{-1}(\rho(B))$ and $a_i \theta_i a'_i$. By the definition of ρ , the tuple $(a_1/\theta_1, \dots, a_{i-1}/\theta_{i-1}, a_i/\theta_i, a_{i+1}/\theta_{i+1}, \dots, a_n/\theta_n) \in \rho(B)$. Then, since $\rho^{-1}(\rho(B))$ is the full inverse image of $\rho(B)$ under ρ , it follows from the definition of ρ and $a_i \theta_i a'_i$, that $(a_1, \dots, a_{i-1}, a'_i, a_{i+1}, \dots, a_n) \in \rho^{-1}(\rho(B)) = B$. Since i was an arbitrary element of \bar{n} , this completes the proof. \square

Proposition 2.2.9. *Let \mathbb{A} and \mathbb{B} be algebras with a common language. Let $\alpha : \mathbb{A} \rightarrow \mathbb{B}$ be a homomorphism. Let A' be any subuniverse of \mathbb{A} and let B' be any subuniverse of \mathbb{B} . Then $\alpha(A')$ is a subuniverse of \mathbb{B} and $\alpha^{-1}(B')$ is a subuniverse of \mathbb{A}*

Proposition 2.2.10. *Let \mathbb{A}_i be a collection of algebras that have a common language, for all $1 \leq i \leq n$. Let θ_i be a congruence on \mathbb{A}_i , for each i . Let ρ denote the natural map $\Pi\mathbb{A}_i \rightarrow \Pi(\mathbb{A}_i/\theta_i)$. Let $C \subseteq \Pi(\mathbb{A}_i/\theta_i)$ and let $B = \rho^{-1}(C)$. Then*

$$B \text{ is a subuniverse of } \Pi\mathbb{A}_i \Leftrightarrow C \text{ is a subuniverse of } \Pi(\mathbb{A}_i/\theta_i).$$

Proposition 2.2.11. *Let $\pi_i : \mathbb{A}_i \rightarrow \mathbb{B}_i$ be an isomorphism, for all $1 \leq i \leq n$. Then the map, $\Pi\pi_i$, defined by*

$$\Pi\pi_i : \Pi_{i=1}^n \mathbb{A}_i \rightarrow \Pi_{i=1}^n \mathbb{B}_i : (a_1, \dots, a_n) \mapsto (\pi_1(a_1), \dots, \pi_n(a_n))$$

is an isomorphism.

Corollary 2.2.12. *Let $\pi_i : \mathbb{A}_i \rightarrow \mathbb{B}_i$ be an isomorphism, for all $1 \leq i \leq n$. Let $\Pi\pi_i$ be the map defined in Proposition 2.2.11. Then,*

$$R \leq \Pi\mathbb{A}_i \Leftrightarrow \Pi\pi_i(R) \leq \Pi\mathbb{B}_i.$$

Proof. (\Rightarrow) This implication clearly holds by Proposition 2.2.9 and since $\Pi\pi_i$ is a homomorphism.

(\Leftarrow) Suppose that $\Pi\pi_i(R)$ is a subuniverse of $\Pi\mathbb{B}_i$. By Proposition 2.2.9 and since $\Pi\pi_i$ is a homomorphism we have that the inverse image of $\Pi\pi_i(R)$ under $\Pi\pi_i$ is a subuniverse of $\Pi\mathbb{A}_i$. Proposition 2.2.11 states that, in fact, $\Pi\pi_i$ is an isomorphism, thus the inverse image of $\Pi\pi_i(R)$ under $\Pi\pi_i$ is equal to R . Therefore, R is a subuniverse of $\Pi\mathbb{A}_i$. \square

Proposition 2.2.13. *Let \mathbb{A}_i be a collection of algebras that have a common language, for all $1 \leq i \leq n$. Then the following implications hold.*

(i) *If B and C be subuniverses of $\mathbb{A}_1 \times \dots \times \mathbb{A}_n$, then $B \cap C$ is a subuniverse of $\mathbb{A}_1 \times \dots \times \mathbb{A}_n$.*

(ii) *If $\mathbb{T}_i \leq \mathbb{A}_i$ for all $1 \leq i \leq n$, then $T_1 \times \dots \times T_n$ is a subuniverse of $\Pi\mathbb{A}_i$.*

(iii) *If $\mathbb{T}_i \leq \mathbb{A}_i$ for all $1 \leq i \leq n$ and $\mathbb{B} \leq \mathbb{A}_1 \times \dots \times \mathbb{A}_n$, then $B \cap T_1 \times \dots \times T_n$ is a subuniverse of $\mathbb{A}_1 \times \dots \times \mathbb{A}_n$.*

Proposition 2.2.14. *Let $\mathbb{A}_1, \mathbb{A}_2$ be algebras that have a common language. Let ϕ be a function from A_1 to A_2 . Then*

$$\phi : \mathbb{A}_1 \rightarrow \mathbb{A}_2 \text{ is a homomorphism} \Leftrightarrow \text{the graph of } \phi \text{ is a subuniverse of } \mathbb{A}_1 \times \mathbb{A}_2.$$

In particular, if ϕ is a bijection, then

$$\phi : \mathbb{A}_1 \rightarrow \mathbb{A}_2 \text{ is an isomorphism} \Leftrightarrow \text{the graph of } \phi \text{ is a subuniverse of } \mathbb{A}_1 \times \mathbb{A}_2.$$

Proposition 2.2.15. *Let \mathbb{A}_1 and \mathbb{A}_2 be algebras that have a common language. Let θ_i be an equivalence relation on A_i , $i = 1, 2$. Let ρ be the natural map,*

$$\rho : A_1 \times A_2 \rightarrow A_1/\theta_1 \times A_2/\theta_2.$$

Let ϕ be a bijection from A_1/θ_1 to A_2/θ_2 , and let $B = \rho^{-1}(\phi)$. If B is a subuniverse of $\mathbb{A}_1 \times \mathbb{A}_2$, then

- (i) θ_i is a congruence on \mathbb{A}_i , for $i = 1, 2$, and
- (ii) ϕ is an isomorphism $\mathbb{A}_1/\theta_1 \rightarrow \mathbb{A}_2/\theta_2$.

Proof. To show (i), for $i = 1$, we will show that $\theta_1 = B \circ B^{-1}$. Then, since $B \circ B^{-1}$ is a subalgebra of \mathbb{A}_1^2 , so is θ_1 , hence, by Proposition 2.2.5, θ_1 is a congruence on \mathbb{A}_1 .

(\supseteq) Let $(x, z) \in B \circ B^{-1}$. Then there exists some $y \in A_2$ such that $(x, y) \in B$ and $(z, y) \in B$. From the assumption that $B = \rho^{-1}(\phi)$ we get that $\rho((x, y)) = (x/\theta_1, y/\theta_2)$ and $\rho((z, y)) = (z/\theta_1, y/\theta_2)$ are elements of the graph of ϕ . Since ϕ is a bijection, this means that $x/\theta_1 = z/\theta_1$. Therefore x is θ_1 -related to z , and hence $(x, z) \in \theta_1$.

(\subseteq) Let $(u, v) \in \theta_1$. Then $u/\theta_1 = v/\theta_1$. Since ϕ is a bijection, it follows that $\phi(u/\theta_1) = \phi(v/\theta_1)$. Let $a_2 \in A_2$ be such that $a_2/\theta_2 = \phi(u/\theta_1) = \phi(v/\theta_1)$. Then $(u/\theta_1, a_2/\theta_2) = (v/\theta_1, a_2/\theta_2)$ are elements in the graph of ϕ . Therefore $(u, a_2), (v, a_2) \in \rho^{-1}(\phi) = B$. Thus $(u, v) \in B \circ B^{-1}$.

To show (i), for $i = 2$, it is enough to note that $B^{-1} = \rho^{-1}(\phi^{-1})$, where $\phi^{-1} : A_2/\theta_2 \rightarrow A_1/\theta_1$ is a bijection and B^{-1} is a subuniverse of $\mathbb{A}_2/\theta_2 \times \mathbb{A}_1/\theta_1$. These conditions satisfy the assumptions of (i), therefore θ_2 is a congruence on \mathbb{A}_2 .

By Proposition 2.2.14, to show (ii) holds it is enough to show that the graph of ϕ is a subuniverse of $\mathbb{A}_1/\theta_1 \times \mathbb{A}_2/\theta_2$. First note that, by (i), θ_i is a congruence on \mathbb{A}_i , $i = 1, 2$, thus \mathbb{A}_i/θ_i is an algebra. Our assumptions on B state that $B = \rho^{-1}(\phi)$ and B is a subuniverse of $\mathbb{A}_1 \times \mathbb{A}_2$, so applying Proposition 2.2.10 gives that the graph of ϕ is a subuniverse of $\mathbb{A}_1/\theta_1 \times \mathbb{A}_2/\theta_2$. By assumption we have that ϕ is a bijection, therefore, it follows from Proposition 2.2.14 that ϕ is an isomorphism \mathbb{A}_1/θ_1 to \mathbb{A}_2/θ_2 . \square

Definition 2.2.16. Let $\mathbb{A}_1, \mathbb{A}_2, \theta_1, \theta_2$ be as in Proposition 2.2.15. If B is a subuniverse of $\mathbb{A}_1 \times \mathbb{A}_2$ such that $B = \rho^{-1}(\phi)$ for some bijection $\phi : \mathbb{A}_1/\theta_1 \rightarrow \mathbb{A}_2/\theta_2$ where $\rho : \mathbb{A}_1 \times \mathbb{A}_2 \rightarrow \mathbb{A}_1/\theta_1 \times \mathbb{A}_2/\theta_2$ is the natural map, then we will call B an *isomorphism from \mathbb{A}_1/θ_1 to \mathbb{A}_2/θ_2* .

2.3 The Galois Connection

Let A be a finite set. Let Op be the set of all finitary operations on A and let Rel be the set of all finitary relations on A . Then there is a correspondence between the subsets of Op and the subsets of Rel under notions of invariance and preservation that defines the following Galois connection,

$$l \text{Op} \rightleftharpoons \text{Rel}$$

$$F \rightarrow F^\perp = \{\rho \in \text{Rel} : \rho \text{ is invariant under } f \text{ for all } f \in F\}$$

$$R^\perp = \{f \in \text{Op} : f \text{ preserves } \rho \text{ for all } \rho \in R\} \leftarrow R,$$

where

$$(1) F_1 \subseteq F_2 \implies F_1^\perp \supseteq F_2^\perp \text{ for all } F_1, F_2 \subseteq \text{Op},$$

$$(2) R_1 \subseteq R_2 \implies R_1^\perp \supseteq R_2^\perp \text{ for all } R_1, R_2 \subseteq \text{Rel},$$

$$(3) F \subseteq F^{\perp\perp} \text{ for all } F \in \text{Op},$$

$$(4) R \subseteq R^{\perp\perp} \text{ for all } R \in \text{Rel},$$

$$(5) F^{\perp\perp\perp} = F^\perp \text{ for all } F \in \text{Op},$$

(6) $R^{\perp\perp\perp} = R^\perp$ for all $R \in \text{Rel}$.

The Galois connection induces the Galois closure operation on the subsets of Op given by $F \mapsto F^{\perp\perp}$ for all $F \in \text{Op}$ (respectively, the Galois connection induces the Galois closure operation on the subsets of Rel given by $R \mapsto R^{\perp\perp}$ for all $R \in \text{Op}$). Thus, a set $C \subseteq \text{Op}$ of operations is *Galois closed* if $C = C^{\perp\perp}$, and a set $K \subseteq \text{Rel}$ of relations is *Galois closed* if $K = K^{\perp\perp}$.

The next two theorems come from Theorems 2.9.1, 2.9.2 in Part II of [Lau06].

Theorem 2.3.1 ([Lau06]). *Let A be a finite set. TFAE for arbitrary $C \subseteq \text{Op}$.*

- (a) C is Galois closed.
- (b) C is a clone.
- (c) C is the clone $\text{Clo}(\mathbb{A})$ of term operations of an algebra $\mathbb{A} = (A; F)$.

Thus, the Galois closure of a subset $F \subseteq \text{Op}$ is the clone generated by F .

Theorem 2.3.2 ([Lau06]). *Let A be a finite set. TFAE for arbitrary $K \subseteq \text{Rel}$.*

- (a) K is Galois closed.
- (b) K is a relational clone.

Under the Galois connection there is a one-to-one correspondence between clones and relational clones.

Let $\mathbb{A} = (A; F)$. Then $F^{\perp\perp} = \text{Clo}(\mathbb{A})$ and, by property (5), $(F^\perp)^{\perp\perp} = F^\perp$, thus $F^\perp = \text{RClo}(\mathbb{A})$. This says that if R is a generating set for the relational clone $\text{RClo}(\mathbb{A})$ of \mathbb{A} , then this set describes the clone $\text{Clo}(\mathbb{A})$ of \mathbb{A} in the sense that $\text{Clo}(\mathbb{A}) = (\langle R \rangle_{\text{RClo}})^\perp$.

Definition 2.3.3. The clone of a finite algebra is *finitely related* if its relational clone is finitely generated.

2.4 Finite Idempotent Strictly Simple Algebras

Definition 2.4.1. A finite algebra $\mathbb{A} = (A; F)$ is called *quasiprimal* if every operation preserving all isomorphisms between subalgebras of \mathbb{A} is a term operation of \mathbb{A} .

Definition 2.4.2. A finite algebra $\mathbb{A} = (A; F)$ is called *affine* with respect to an abelian group $\mathbb{B} = (A, +, -, 0)$ if the Mal'cev operation $x - y + z$ is a term operation of \mathbb{A} and every operation of \mathbb{A} commutes with $x - y + z$.

Definition 2.4.3. An algebra is called *strictly simple* if it is simple and has no nontrivial proper subalgebras.

The finite idempotent strictly simple algebras with more than two elements are classified by Szendrei in [Sze87] as stated in the theorem below. For a permutation group G on A , let $\mathcal{I}_A(G)$ be the clone of all idempotent operations on A commuting with every member of G . If $0 \in A$, $k \geq 2$, then let \mathcal{F}_k^0 denote the clone of all idempotent operations on A that preserves the relation $\mathbf{X}_k^0 = \{(x_1, \dots, x_k) \in A^k : x_i = 0 \text{ for some } 1 \leq i \leq k\}$. Let $\mathcal{F}_\omega^0 = \bigcap_{2 \leq k < \omega} \mathcal{F}_k^0$.

Theorem 2.4.4 ([Sze87, Theorem 2.1]). *Let $\mathbb{A} = (A; F)$ be a finite idempotent strictly simple algebra, $|A| \geq 3$. Then \mathbb{A} is term equivalent to one of the following algebras:*

- (i) $(A; \mathcal{I}_A(G))$ for a permutation group G acting on A such that every nonidentity member of G has at most one fixed point,
- (ii) the full idempotent reduct of the module $(\text{End}_K A)A$ for some vector space ${}_K A = (A; +; K)$ over a finite field K ,
- (iii) $(A; \mathcal{I}_A(G) \cap \mathcal{F}_k^0)$ for some $2 \leq k \leq \omega$, some element $0 \in A$, and a permutation group G acting on A such that 0 is the unique fixed point of every nonidentity member of G .

Note that a finite idempotent strictly simple algebra with more than two elements that satisfies (i) or (ii) of Theorem 2.4.4 is quasiprimal or affine, respectively. If \mathbb{A} is affine, then ${}_K A$ will be called the vector space associated to \mathbb{A} .

The next three propositions concern the subuniverses of finite powers of \mathbb{A} when \mathbb{A} is a finite idempotent strictly simple algebra, $|A| > 2$. The first proposition is a special case of Theorem 4.2 in [Sze86].

Proposition 2.4.5 ([Sze86]). *Let \mathbb{A} be a finite idempotent strictly simple quasiprimal algebra, $|A| > 2$. A subuniverse $B \leq \mathbb{A}^n$ ($n \geq 2$) may have unary projections that are singletons, or binary projections that are automorphisms of \mathbb{A} , or if there are no such unary and binary projections, then $B = A^n$.*

The next proposition follows from Lemma 4.4 in [Sze86], combined with the remark at the bottom of page 98 in [Sze86]

Proposition 2.4.6 ([Sze86]). *Let \mathbb{A} be a finite idempotent strictly simple affine algebra, $|A| > 2$, and ${}_K A$ be the associated vector space. Then, up to permutation of coordinates, every subuniverse of \mathbb{A}^n ($n \geq 2$) has the form,*

$$\{(x_1, x_2, \dots, x_s, \sum_{i=1}^s c_{(s+1,i)}x_i + \delta_{s+1}, \sum_{i=1}^s c_{(s+2,i)}x_i + \delta_{s+2}, \dots, \sum_{i=1}^s c_{(n,i)}x_i + \delta_n) \in A^n : x_1, \dots, x_s \in A\},$$

for some $\delta_{s+1}, \delta_{s+2}, \dots, \delta_n \in A$ and $c_{(s+1,i)}, c_{(s+2,i)}, \dots, c_{(n,i)} \in K$, $1 \leq i \leq s$.

It is helpful to notice that from the above description we get that a subuniverse B of \mathbb{A}^n ($n \geq 2$), where \mathbb{A} is a finite idempotent strictly simple affine algebra, $|A| > 2$, either has a unary projection that is a singleton, an m -ary projection ($m \geq 2$) of the form (up to permutation of coordinates) $\text{pr}_{1, \dots, m} B = \{(x_1, \dots, x_{m-1}, \sum_{i=1}^{m-1} c_i x_i + \delta) : x_1, \dots, x_{m-1} \in A\}$ which is the graph of a function $\mathbb{A}^{m-1} \rightarrow \mathbb{A} : (x_1, \dots, x_{m-1}) \mapsto \sum_{i=1}^{m-1} c_i x_i + \delta$ for some $\delta \in A$ and $c_i \in K$, $1 \leq i \leq m-1$, or if there are no such unary and m -ary projections, then $B = A^n$.

Proposition 2.4.7 ([Sze87]). *Let \mathbb{A} be a finite idempotent strictly simple algebra, $|A| > 2$, such that \mathbb{A} is term equivalent to the algebra in case (iii) of Theorem 2.4.4. Let $B \leq_{s,d} \mathbb{A}^n$ ($n \geq 2$) such that no binary projection $\text{pr}_{i,j} B$ ($1 \leq i < j \leq n$) is a permutation of A . Then for some $0 \in A$,*

$$B = \{(x_1, \dots, x_n) \in \mathbb{A}^n : \bar{x}_I \in \text{pr}_I B \text{ for all } I \in P\}.$$

where P is the family of subsets of \bar{n} such that $I \in P$ if and only if $|I| \geq 2$ and $\text{pr}_I B = \mathbf{X}_{|I|}^0$.

In the next proposition we will denote the set of automorphisms of \mathbb{A} by $\text{Aut}(\mathbb{A})$.

Proposition 2.4.8. *Let \mathbb{A} be a finite idempotent strictly simple algebra, $|A| > 2$.*

- (i) *If \mathbb{A} is quasiprimal, then $\{\{a\} : a \in A\} \cup \text{Aut}(\mathbb{A})$ is a generating set for $\text{RClo}(\mathbb{A})$.*
- (ii) *If \mathbb{A} is affine, then $\{\{a\} : a \in A\} \cup \text{Aut}(\mathbb{A}) \cup \{(x, y, z, x - y + x) : x, y, z \in A\}$ is a generating set for $\text{RClo}(\mathbb{A})$.*
- (iii) *If \mathbb{A} is term equivalent to the algebra in case (iii) of Theorem 2.4.4, then there exists some $3 \leq n \leq \omega$ such that $\{\{a\} : a \in A\} \cup \text{Aut}(\mathbb{A}) \cup \{\mathbf{X}_k^0 : 2 \leq k < n\}$ is a generating set for $\text{RClo}(\mathbb{A})$.*

Thus, the relational clone of \mathbb{A} is finitely related if \mathbb{A} is affine or quasiprimal. It may or may not be finitely related in the third case.

Corollary 2.4.9. *Let \mathbb{A} be a finite idempotent strictly simple algebra, $|A| > 2$. Then \mathbb{A} is either quasiprimal, or affine, or $\mathbf{X}_2^0 = (A \times \{0\}) \cup (\{0\} \times A)$ is a subuniverse of \mathbb{A}^2 for some $0 \in A$.*

Proof. This follows immediately from Theorem 2.4.4 and Proposition 2.4.8 □

Theorem 2.4.10 ([Sze88]). *For an idempotent strictly simple algebra $\mathbb{A} = (A; F)$, one of the following conditions holds:*

- (i) $\mathcal{V}(\mathbb{A})$ is congruence distributive, or
- (ii) \mathbb{A} is term equivalent to the full idempotent reduct of the module $(\text{End}_K A)A$ for some vector space ${}_K A = (A; +, K)$, or
- (iii) \mathbb{A} is a 2-element algebra term equivalent to a semilattice or to a left zero semigroup on A .

Corollary 2.4.11. *The variety generated by an idempotent strictly simple algebra \mathbb{A} , where $|A| > 2$, is congruence modular.*

Proof. Let \mathbb{A} be an idempotent strictly simple algebra where $|A| > 2$. Then either statement (i) or (ii) of Theorem 2.4.10 holds. In the latter case, the variety generated by \mathbb{A} has a term that satisfies the Mal'cev identities, therefore by Mal'cev's Theorem, Theorem 2.1.2, the variety is congruence permutable. Thus, for in either case, it follows from Theorem 2.1.1 that the variety generated by \mathbb{A} is congruence modular. \square

2.5 Crosses

Definition 2.5.1. For $i \in \{1, 2\}$, let A_i be sets, let θ_i be an equivalence relation on A_i , and let $a_i \in A_i$. The *thick* $(A_1 \times A_2)$ -cross $[A_1/\theta_1, A_2/\theta_2, a_1/\theta_1, a_2/\theta_2]$ is defined to be the set

$$[A_1/\theta_1, A_2/\theta_2, a_1/\theta_1, a_2/\theta_2] := \{(x_1, x_2) \in A_1 \times A_2 : x_1\theta_1 a_1 \text{ or } x_2\theta_2 a_2\}.$$

Proposition 2.5.2. For $i \in \{0, 1, 2, 3\}$, let A_i be sets, let θ_i be an equivalence relation on A_i , and let $b_0 \in A_0$, $a_1 \in A_1$, $a_2, b_2 \in A_2$, $a_3 \in A_3$.

(i) If $(a_2, b_2) \notin \theta_2$, then

$$[A_1/\theta_1, A_2/\theta_2, a_1/\theta_1, a_2/\theta_2] \circ [A_2/\theta_2, A_3/\theta_3, b_2/\theta_2, a_3/\theta_3] = [A_1/\theta_1, A_3/\theta_3, a_1/\theta_1, a_3/\theta_3].$$

(ii) If $\Phi_i \subseteq A_i \times A_{i+1}$ is the graph of a bijection $\phi_i : A_i/\theta_i \rightarrow A_{i+1}/\theta_{i+1}$, for $i = 0, 2$, then

$$\Phi_0^{-1} \circ [A_0/\theta_0, A_2/\theta_2, b_0/\theta_0, b_2/\theta_2] \circ \Phi_2 = [A_1/\theta_1, A_3/\theta_3, \phi_0(b_0/\theta_0), \phi_2(b_2/\theta_2)].$$

Proof. Let $i \in \{0, 1, 2, 3\}$, let A_i be sets, let θ_i be an equivalence relation on A_i , and let $b_0 \in A_0$, $a_1 \in A_1$, $a_2, b_2 \in A_2$, $a_3 \in A_3$.

[(i)] To show property (i), suppose that $a_2/\theta_2 \neq b_2/\theta_2$. Let $D := [A_1/\theta_1, A_2/\theta_2, a_1/\theta_1, a_2/\theta_2]$, $E := [A_2/\theta_2, A_3/\theta_3, b_2/\theta_2, a_3/\theta_3]$, and $F := [A_1/\theta_1, A_3/\theta_3, a_1/\theta_1, a_3/\theta_3]$. Let $C = D \circ E$. Then we want to show that $C = F$. Let $(x_1, x_3) \in C$. Then there exists some $x_2 \in A_2$ such that $(x_1, x_2) \in D$ and $(x_2, x_3) \in E$. To show that $(x_1, x_3) \in F$, we must show that at least one of the following are true: either $x_1\theta_1 a_1$ or $x_3\theta_3 a_3$. Suppose, for contradiction, that $x_1/\theta_1 \neq a_1/\theta_1$ and $x_3/\theta_3 \neq a_3/\theta_3$. Then $(x_1, x_2) \in D$ and $x_1/\theta_1 \neq a_1/\theta_1$ implies $x_2\theta_2 a_2$. Similarly, $(x_2, x_3) \in E$ and $x_3/\theta_3 \neq a_3/\theta_3$

implies $x_2\theta_2b_2$. Then $x_2\theta_2a_2$ and $x_2\theta_2b_2$ implies $a_2\theta_2b_2$, which contradicts $a_2/\theta_2 \neq b_2/\theta_2$. Therefore, either $x_1\theta_1a_1$ or $x_3\theta_3a_3$ or both statements are true, which means $(x_1, x_3) \in F$. Hence $C \subseteq F$.

Now suppose that $(x_1, x_3) \in F$. Then either $x_1\theta_1a_1$ or $x_3\theta_3a_3$. If $x_1\theta_1a_1$, then $(x_1, b_2) \in D$ and $(b_2, x_3) \in E$, hence $(x_1, x_3) \in C$. If $x_3\theta_3a_3$, then $(x_1, a_2) \in D$ and $(a_2, x_3) \in E$, hence $(x_1, x_3) \in C$. Therefore, $F \subseteq C$. This completes the proof of the statement.

[(ii)] Suppose $\Phi_i \subseteq A_i \times A_{i+1}$ is the graph of a bijection $\phi_i : A_i/\theta_i \rightarrow A_{i+1}/\theta_{i+1}$, for $i = 0, 2$. Let $R := [A_0/\theta_0, A_2/\theta_2, b_0/\theta_0, b_2/\theta_2]$. Then,

$$\begin{aligned}
\Phi_0^{-1} \circ R \circ \Phi_2 &= \{(y_1, y_3) \in A_1 \times A_3 : \text{there exists some } x_0 \in A_0, x_2 \in A_2 \text{ such that } (y_1, x_0) \in \Phi_0^{-1}, \\
&\quad (x_0, x_2) \in R, (x_2, y_3) \in \Phi_2\} \\
&= \{(y_1, y_3) \in A_1 \times A_3 : \text{there exists some } x_0 \in A_0, x_2 \in A_2 \text{ such that} \\
&\quad \phi_0(x_0/\theta_0) = y_1/\theta_1, \phi_2(x_2/\theta_2) = y_3/\theta_3, \text{ and either } x_0\theta_0b_0 \text{ or } x_2\theta_2b_2\} \\
&= \{(y_1, y_3) \in A_1 \times A_3 : \text{there exists some } x_0 \in A_0, x_2 \in A_2 \text{ such that} \\
&\quad y_1/\theta_1 = \phi_0(x_0/\theta_0) = \phi_0(b_0/\theta_0) \text{ or } y_3/\theta_3 = \phi_2(x_2/\theta_2) = \phi_2(b_2/\theta_2)\} \\
&= \{(y_1, y_3) \in A_1 \times A_3 : y_1\theta_1\phi_0(b_0/\theta_0) \text{ or } y_3\theta_3\phi_2(b_2/\theta_2)\} \\
&= [A_1/\theta_1, A_3/\theta_3, \phi_0(b_0/\theta_0), \phi_2(b_2/\theta_2)].
\end{aligned}$$

This completes the proof of this statement. □

Chapter 3

The Subuniverses of \mathbb{A}^2

In this section we start our investigation of finite idempotent algebras \mathbb{A} that satisfy Assumption 1.

The main result of this chapter is Theorem 3.1.5 in Section 3.1, which describes the possible binary relations that can be subuniverses of \mathbb{A}^2 for such an algebra \mathbb{A} . In Section 3.2 we study how these binary relations compose, and which of them can occur simultaneously as subuniverses of \mathbb{A}^2 .

So, throughout this chapter we will let \mathbb{A} be a fixed algebra that satisfies Assumption 1. It follows that \mathbb{S} is an idempotent algebra that contains no nontrivial proper subalgebras. Thus, by Proposition 2.2.6, \mathbb{S} has no nontrivial congruences. Then \mathbb{S} is a finite simple idempotent algebra of size greater than 2, equivalently, \mathbb{S} is a finite strictly simple idempotent algebra, $|S| > 2$.

Definition 3.0.3. Define θ to be the equivalence relation on A given by $\theta := S^2 \cup \{(b, b) : b \in A \setminus S\}$. Let $\bar{s} := s/\theta$ for any $s \in S$ and $\bar{b} := b/\theta$ for any $b \in A \setminus S$.

A picture of θ can be found on page 23.

Proposition 3.0.4. *If \mathbb{A} is not simple, then θ is the unique nontrivial congruence on \mathbb{A} and \mathbb{A}/θ has no nontrivial proper subalgebras.*

Proof. Suppose that \mathbb{A} is not simple and let Γ be a nontrivial congruence on \mathbb{A} . By Proposition 2.2.6, every congruence class of Γ is a subuniverse of \mathbb{A} , hence a/Γ is either a singleton or S for each $a \in A$. Therefore $\Gamma = \theta$ proving that θ is the unique nontrivial congruence on \mathbb{A} .

Furthermore, if C is a subuniverse of \mathbb{A}/θ , then by Proposition 2.2.10, $B = \rho^{-1}(C)$ is a subuniverse of \mathbb{A} , where $\rho : A \rightarrow A/\theta$ is the natural map. By our assumptions on the subalgebras of \mathbb{A} , B is either A , S , or a singleton $\{a\}$ for some $a \in A$. Hence $C = \rho(B)$ is A/θ , $\{\overline{S}\}$, or $\{a/\theta\}$, where $a \in A \setminus S$. In particular, if C is a proper subuniverse of \mathbb{A}/θ then B is a proper subuniverse of \mathbb{A} . It follows that \mathbb{A}/θ has no nontrivial proper subuniverses. □

Proposition 3.0.5. *If σ is an automorphism of \mathbb{A} , then $\sigma|_S$ is an automorphism of \mathbb{S} .*

Proof. Suppose that σ is an automorphism of \mathbb{A} . Since \mathbb{S} is a subalgebra of \mathbb{A} , it follows from the properties of homomorphisms that $\sigma(S)$ is a subuniverse of \mathbb{A} . Furthermore, since σ is bijective, we can infer that $|\sigma(S)| = |S|$. By assumption, \mathbb{S} is the unique nontrivial subalgebra of \mathbb{A} , therefore, $\sigma(S) = S$. Hence $\sigma|_S$ is an automorphism of \mathbb{S} . □

It follows from Proposition 3.0.5 that the (the graphs of) automorphism of \mathbb{A} are subsets of $S^2 \cup (A \setminus S)^2$.

Proposition 3.0.6. *If σ_1 and σ_2 are automorphisms of \mathbb{A} such that $\sigma_1|_{A \setminus S} = \sigma_2|_{A \setminus S}$, then $\sigma_1 = \sigma_2$.*

Proof. Suppose $\sigma_1, \sigma_2 \in \text{Aut}(\mathbb{A})$ and $\sigma_1|_{A \setminus S} = \sigma_2|_{A \setminus S}$. Let R be the graph of the automorphism $\mathbb{A} \rightarrow \mathbb{A}$, $a \mapsto \sigma_2^{-1}(\sigma_1(a))$, of \mathbb{A} . To prove the proposition it is enough to show that $\Delta \subseteq R$ where $\Delta = \{(x, x) : x \in A\}$ is the graph of the identity automorphism of \mathbb{A} .

Since Δ and R are graphs of an automorphisms of \mathbb{A} , they are subuniverses of \mathbb{A}^2 . Hence $R \cap \Delta \leq \mathbb{A}^2$ and $\text{pr}_1 R \cap \Delta \leq \mathbb{A}$. Let $a \in A \setminus S$. Then $\sigma_1|_{A \setminus S} = \sigma_2|_{A \setminus S}$ implies that $\sigma_2^{-1}(\sigma_1(a)) = a$, therefore $(a, a) \in R$. Since a was an arbitrary element of $A \setminus S$, this means that $(a, a) \in R$ for all $a \in A \setminus S$. Then $R \cap \Delta$ contains $\{(a, a) : a \in A \setminus S\}$, therefore $A \setminus S \subseteq \text{pr}_1 R \cap \Delta \leq \mathbb{A}$. Since $\text{pr}_1 R \cap \Delta \leq \mathbb{A}$ is a subuniverse of \mathbb{A} it follows from our assumptions on the subuniverses of \mathbb{A} that $\text{pr}_1 R \cap \Delta \leq \mathbb{A} = A$. Thus $\Delta \subseteq R$. This completes the proof of the proposition. □

3.1 A Description of the Subuniverses of \mathbb{A}^2

Understanding the subuniverses of \mathbb{A}^2 is essential in determining the subuniverses of finite powers of \mathbb{A} . In this section we will describe the possible binary relations that can be subuniverses of \mathbb{A}^2 . We start with some notation and terminology.

Definition 3.1.1. The relations $\{(a, a')\}$, where $a, a' \in A$, will be called *points*. The relations $\{a\} \times S$, $\{a\} \times A$ and their inverses will be called *lines*.

Definition 3.1.2. For $s, s' \in S$ and $a, a' \in A$, let

$$\nu_{s,s'} := (\{s\} \times S) \cup (S \times \{s'\}),$$

$$\mu_{a,a'} := (\{a\} \times A) \cup (A \times \{a'\}),$$

$$\kappa_{a,s} := (\{a\} \times S) \cup (A \times \{s\}),$$

$$\lambda_{S,s} := S^2 \cup (A \times \{s\}),$$

$$\chi_{s,s'} := S^2 \cup (\{s\} \times A) \cup (A \times \{s'\}),$$

$$\chi_{S,s} := S^2 \cup (S \times A) \cup (A \times \{s\}) = (S \times A) \cup (A \times \{s\}),$$

$$\chi_{S,S} := S^2 \cup (S \times A) \cup (A \times S) = (S \times A) \cup (A \times S).$$

The relations $\nu_{s,s'}$ will be called (S, S) -crosses since their unary projections are equal to S . Similarly, the relations $\mu_{a,a'}$ will be called (A, A) -crosses and the relations $\kappa_{a,s}$ will be called (A, S) -crosses. We will call the remaining relations *thick* crosses since they contain S^2 . Thus the relations $\lambda_{S,s}$ are thick (A, S) -crosses and the remaining relations are thick (A, A) -crosses. If $s = s'$, then we will denote the (S, S) -crosses $\nu_{s,s}$ by ν_s . If $a = a'$, then we will denote (A, A) -crosses $\mu_{a,a}$ by μ_a .

Pictorial examples of the relations in Definition 3.1.2 can be found on pages 23-24.

If θ is a congruence on \mathbb{A} , then we will use the following notation for some relations of $A/\theta \times S$ and $(A/\theta)^2$.

Definition 3.1.3. For $s \in S$ and $\bar{a} \in A/\theta$, let

$$\begin{aligned}\eta_{\bar{a}} &:= (\{\bar{a}\} \times A/\theta) \cup (A/\theta \times \{\bar{a}\}) \\ \varkappa_{\bar{a},s} &:= (\{\bar{a}\} \times S) \cup (A/\theta \times \{s\}) \\ \varkappa_{s,\bar{a}} &:= (\{s\} \times A/\theta) \cup (S \times \{\bar{a}\}).\end{aligned}$$

Each relation $\eta_{\bar{a}}$ will be called an $(A/\theta, A/\theta)$ -cross since its unary projections are equal to A/θ . Similarly, each relation $\varkappa_{\bar{a},s}$ will be called an $(A/\theta, S)$ -cross and each relation $\varkappa_{s,\bar{a}}$ will be called an $(S, A/\theta)$ -cross. Note that $\varkappa_{\bar{a},s} = \varkappa_{s,\bar{a}}^{-1}$.

Definition 3.1.4. Let $s, s' \in S$ and let τ be a permutation of $A \setminus S$. Then

$$\nu_{s,s'}^\tau := \nu_{s,s'} \cup \{(x, \tau(x)) : x \in A \setminus S\}.$$

If $s = s'$, then we will denote $\nu_{s,s'}^\tau$ by ν_s^τ .

A pictorial example of a relation ν_s^τ can be found on page 24.

If θ is a congruence on \mathbb{A} , then we will use the terminology introduced in Definition 2.2.16 to describe an isomorphism from \mathbb{A}_1/θ_1 to \mathbb{A}_2/θ_2 where $A_i \in \{S, A\}$ and $\theta_i \in \{\text{id}_S, \text{id}_A, \theta\}$. Namely, if B is a subuniverse of $\mathbb{A} \times \mathbb{S}$ such that B is the full inverse image of a bijection $\phi : A/\theta \rightarrow S$ under the natural map $\rho : A \times S \rightarrow A/\theta \times S$, then we will call B an isomorphism from \mathbb{A}/θ to \mathbb{S} . A symmetric definition is given for an isomorphism from \mathbb{S} to \mathbb{A}/θ . If B is a subuniverse of \mathbb{A}^2 such that B is the full inverse image of an automorphism ϕ of \mathbb{A}/θ under the natural map $\rho : A \times A \rightarrow A/\theta \times A/\theta$, then we will call B an *automorphism of \mathbb{A}/θ* .

One should note that if θ is a congruence on \mathbb{A} (that is, θ is a subuniverse of \mathbb{A}^2), then θ is an automorphism of \mathbb{A}/θ in this sense, namely the identity automorphism of \mathbb{A}/θ .

The next theorem is the main result of this section.

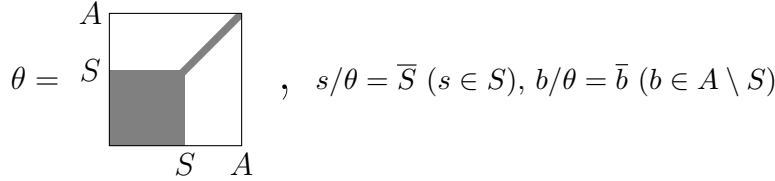
Theorem 3.1.5. *Every subuniverse of \mathbb{A}^2 is one of the following:*

- a direct product of subuniverses of \mathbb{A} : a point, a line, S^2 , $A \times S$, $S \times A$, or A^2 ,

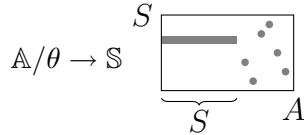
- an automorphism of \mathbb{S} , or an automorphism of \mathbb{A} ,
- an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, an isomorphism $\mathbb{S} \rightarrow \mathbb{A}/\theta$, or an automorphism of \mathbb{A}/θ (hence θ is a congruence on \mathbb{A}),
- a cross: $\nu_s, \mu_a, \kappa_{a,s}$, or $(\kappa_{a,s})^{-1}$, for some $s \in S, a \in A$,
- a thick cross: $\lambda_{S,s}, (\lambda_{S,s})^{-1}, \chi_{s,s'}, \chi_{S,s}, (\chi_{S,s})^{-1}$, or $\chi_{S,S}$, for some $s, s' \in S$,
- ν_s^τ , for some $s \in S$ and some fixed-point free permutation τ of $A \setminus S$ (hence θ is a congruence on \mathbb{A}).

Using Definitions 3.0.3, 2.2.16, 3.1.2, and 3.1.4 we will depict some examples of what the possible subuniverses of \mathbb{A}^2 look like. Let $s, s' \in S, a \in A$.

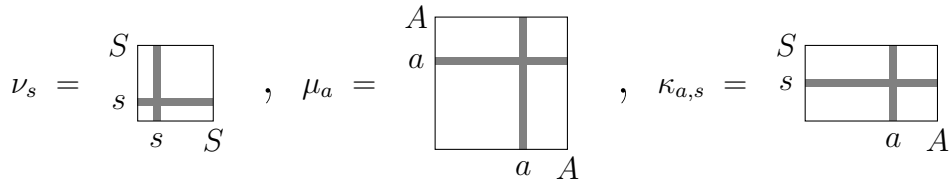
- An automorphism of \mathbb{A}/θ



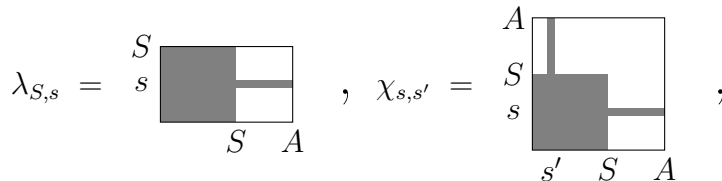
- An isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$



- A cross



- A thick cross



$$\begin{array}{c}
\chi_{S,s} = \begin{array}{|c|c|} \hline A & \\ \hline s & \\ \hline S & A \\ \hline \end{array} \quad , \quad \chi_{s,S} = \begin{array}{|c|c|} \hline A & \\ \hline S & \\ \hline s & A \\ \hline \end{array} \quad , \quad \chi_{S,S} = \begin{array}{|c|c|} \hline A & \\ \hline S & \\ \hline S & A \\ \hline \end{array} \\
\nu_s^\tau = \begin{array}{|c|c|} \hline A & \\ \hline s & \\ \hline s & A \\ \hline \end{array}
\end{array}$$

Corollary 3.1.6. *Suppose that θ is a congruence on \mathbb{A} . Let $B \leq_{s.d.} \mathbb{B}_1 \times \mathbb{B}_2$ where $\mathbb{B}_1, \mathbb{B}_2 \in \{\mathbb{S}, \mathbb{A}\}$. For $i = 1, 2$ let $\mathbb{A}_i = \mathbb{B}_i / \Theta_i$ where Θ_i is the equality relation if $\mathbb{B}_i = \mathbb{S}$ and is θ if $\mathbb{B}_i = \mathbb{A}$. Furthermore, let ρ be the natural homomorphism $\mathbb{B}_1 \times \mathbb{B}_2 \rightarrow \mathbb{A}_1 \times \mathbb{A}_2$, and let $B' = \rho(B)$. Then B' is one of the following:*

- a direct product of subuniverses of \mathbb{S} and \mathbb{A}/θ : a point, a line, S^2 , $A/\theta \times S$, $S \times A/\theta$, or $(A/\theta)^2$,
- an automorphism of \mathbb{S} , an automorphism of \mathbb{A}/θ , an isomorphism from \mathbb{S} to \mathbb{A}/θ , or an isomorphism from \mathbb{A}/θ to \mathbb{S} ,
- a cross: ν_s , $\eta_{\bar{a}}$, $\varkappa_{\bar{a},s}$, or $\varkappa_{s,\bar{a}}$, for some $s \in S$, $\bar{a} \in A/\theta$.

Proof. The corollary follows directly from Theorem 3.1.5. □

Theorem 3.1.5 will be proved by a sequence of lemmas, and the proof will occupy the rest of this section.

Lemma 3.1.7. *Let B be a subuniverse of \mathbb{A}^2 . If $(a, b_1), (a, b_2) \in B$ for some distinct $b_1, b_2 \in A$, $a \in A$, then $\{a\} \times S \subseteq B$. Furthermore, if $b_i \in A \setminus S$, for some $i \in \{1, 2\}$, then $\{a\} \times A \subseteq B$.*

Proof. Let B be a subuniverse of \mathbb{A}^2 such that $(a, b_1), (a, b_2) \in B$ for distinct $b_1, b_2 \in A$ and $a \in A$. Then $B(a, x_2)$ is a subuniverse of \mathbb{A} that contains $\{b_1, b_2\}$ which implies $|B(a, x_2)| \geq 2$. Since \mathbb{S} and \mathbb{A} are the only nontrivial subalgebras of \mathbb{A} , it follows that $B(a, x_2)$ contains S . Hence $\{a\} \times S \subseteq B$. Furthermore, if $b_i \in A \setminus S$, for some $i \in \{1, 2\}$, then $B(a, x_2) \neq S$, in this case $B(a, x_1) = A$, therefore $\{a\} \times A \subseteq B$. □

Lemma 3.1.8. *Let B be a subuniverse of \mathbb{A}^2 . If $(b_1, a), (b_2, a) \in B$, for $a, b_1, b_2 \in A$, $b_1 \neq b_2$, then $S \times \{a\} \subseteq B$. Furthermore, if $b_i \in A \setminus S$, for some $i \in \{1, 2\}$, then $A \times \{a\} \subseteq B$.*

Proof. This follows from Lemma 3.1.7 and the fact that, under the assumptions of the lemma, $(a, b_1), (a, b_2) \in B^{-1}$ for distinct b_1, b_2 . \square

Lemma 3.1.9. *Let B be a subuniverse of \mathbb{A}^2 . If $(\{a_1\} \times S) \cup (\{a_2\} \times S) \subseteq B$ for distinct $a_1, a_2 \in A$, then $S^2 \subseteq B$. Furthermore, if $a_i \in A \setminus S$ for some $i \in \{1, 2\}$, then $A \times S \subseteq B$.*

Proof. Let B be a subuniverse of \mathbb{A}^2 such that $(\{a_1\} \times S) \cup (\{a_2\} \times S) \subseteq B$ for distinct $a_1, a_2 \in A$. Then for all $s \in S$ the subuniverse $B(x_1, s)$ contains both a_1 and a_2 , therefore $S \subseteq B(x_1, s)$, which means $S \times \{s\} \subseteq B$. Furthermore, if one of a_1, a_2 is in $A \setminus S$, then $B(x_1, s) = A$, thus $A \times \{s\} \subseteq B$. \square

Lemma 3.1.10. *Let B be a subuniverse of \mathbb{A}^2 . If $(S \times \{a_1\}) \cup (S \times \{a_2\}) \subseteq B$ for distinct $a_1, a_2 \in A$, then $S^2 \subseteq B$. Furthermore, if $a_i \in A \setminus S$ for some $i \in \{1, 2\}$, then $S \times A \subseteq B$.*

Proof. This follows from Lemma 3.1.9 and the fact that, under the assumptions of the lemma, $\{a_1\} \times S \cup \{a_2\} \times S \subseteq B^{-1}$ for distinct $a_1, a_2 \in A$. \square

Lemma 3.1.11. *The following implications hold for $s, s' \in S$, $a, a', b \in A$.*

(i) *If $\mu_{a, a'}$ is a subuniverse of \mathbb{A}^2 , then $a = a'$.*

(ii) *If $\nu_{s, s'}$ is a subuniverse of \mathbb{A}^2 , then $s = s'$.*

Proof. Each statements (i) and (ii) follows from the fact that the intersection of subuniverses of \mathbb{A}^2 is a subuniverse of \mathbb{A}^2 and the unary projection of a subuniverse of \mathbb{A}^2 is a subuniverse of \mathbb{A} .

[(i)] If $\mu_{a, a'} \leq \mathbb{A}^2$ for some $a, a' \in A$, $a \neq a'$, then $\mu_{a, a'} \cap \mu_{a, a'}^{-1} = \{(a, a), (a', a')\}$. Hence $\text{pr}_1(\mu_{a, a'} \cap \mu_{a, a'}^{-1}) = \{a, a'\}$ is a two-element subuniverse of \mathbb{A} which contradicts the assumptions on the subalgebras of \mathbb{A} . Hence $a = a'$.

[(ii)] The proof is similar to the proof of (i), replace $\mu_{a, a'}$ with $\nu_{s, s'}$, and a with s .

\square

We will now begin the proof of Theorem 3.1.5. We will start by considering those subuniverses of \mathbb{A}^2 that have trivial unary projections.

Lemma 3.1.12. *Let B be subuniverse of \mathbb{A}^2 . If B has a trivial unary projection, then B is a point or a line.*

Proof. Let B be a subuniverse of \mathbb{A}^2 such that the unary projection of B onto its i^{th} -coordinate is trivial, for some $i \in \{1, 2\}$. WLOG, suppose $\text{pr}_1 B = \{a\}$ for some $a \in A$. Then $B = \{a\} \times \text{pr}_2 B$. By our assumptions on the subalgebras of \mathbb{A} , the projection of B onto its second coordinate is either a singleton, or S or A . It follows that B is either a point or a line. \square

Lemma 3.1.13. *If B is a subuniverse of \mathbb{A}^2 whose unary projections are equal to S , then B is either an automorphism of \mathbb{S} , or $B = \nu_s$ for some $s \in S$, or $B = S^2$.*

Proof. Let B be a subuniverse of \mathbb{A}^2 such that $\text{pr}_i B = S$, $i = 1, 2$. Then $B \subseteq S^2$. Fix $s \in S$. Then $B(s, x_2)$ is subuniverse of \mathbb{A} that is contained in S . Since $s \in S = \text{pr}_1 B$ and $\text{pr}_2 B = S$, we get that there exists some $y \in S$ such that $(s, y) \in B$, thus $y \in B(s, x_2) \subseteq S$. It follows from our assumptions on the subalgebras of \mathbb{A} that $B(s, x_2)$ is either a singleton or S . By a symmetric argument we get that $B(x_1, s)$ is either a singleton or S . Furthermore, we chose $s \in S$ arbitrarily, therefore each of $B(s, x_2)$ and $B(x_1, s)$ is either a singleton or S for every $s \in S$.

First suppose that $B(s, x_2)$ is a singleton for every $s \in S$. This condition means that B is (the graph of) a function $\phi : S \rightarrow S$, where ϕ is defined by $B(s, x_2) = \{\phi(s)\}$ for all $s \in S$. Since $\text{pr}_2 B = S$, we have that ϕ is an onto function $S \rightarrow S$. As S is finite, ϕ is also one-to-one. Thus B is (the graph of) a permutation of S , and hence by Proposition 2.2.14, B is an automorphism of \mathbb{S} . Similarly, if $B(x_1, s)$ is a singleton for every $s \in S$, then it follows that B is an automorphism of \mathbb{S} .

It remains to consider the case where $B(s, x_2) = S$ and $B(x_1, s') = S$ for at least one $s \in S$ and at least one $s' \in S$. Then $B \supseteq \nu_{s, s'}$. If $B = \nu_{s, s'}$, then we get from statement (ii) of Lemma 3.1.11 that $s = s'$ and $B = \nu_s$. If $B \neq \nu_{s, s'}$, then let $(t, t') \in B \setminus \nu_{s, s'}$. Clearly $t, t' \in S$ and $t \neq s$, $t' \neq s'$. Thus $(t, t') \in B$ and $(t, s') \in \nu_{s, s'} \subseteq B$, $t' \neq s'$ implies, by Lemma 3.1.7, that $\{t\} \times S \subseteq B$. Then,

by Lemma 3.1.9 and $\{s\} \times S \subseteq \nu_{s,s'} \subseteq B$, $s \neq t$, we get that $S^2 \subseteq B$. Recall that $B \subseteq S^2$, therefore it follows that $B = S^2$. This completes the proof of the lemma. \square

We will use the previous lemma to determine that subdirect subalgebras of $\mathbb{A} \times \mathbb{S}$.

Lemma 3.1.14. *If B is a subuniverse of \mathbb{A}^2 such that $\text{pr}_1 B = A$ and $\text{pr}_2 B = S$, then B is one of the following:*

- an isomorphism from \mathbb{A}/θ to \mathbb{S} (hence θ is a congruence on \mathbb{A}),
- $\kappa_{a,s}$, for some $a \in A$, $s \in S$,
- $\lambda_{S,s}$, for some $s \in S$, or
- $A \times S$.

Proof. Let B be a subuniverse of \mathbb{A}^2 such that $\text{pr}_1 B = A$ and $\text{pr}_2 B = S$. Then $B \cap S^2$ is a subuniverse of \mathbb{S}^2 . Since $S \subseteq \text{pr}_1 B$ and $\text{pr}_2 B = S$ we have, for each $t \in S$, that there exists some $c_t \in S$ such that $(t, c_t) \in B$, so $(t, c_t) \in B \cap S^2$, therefore, $\text{pr}_1(B \cap S^2) = S$. Furthermore, since $\text{pr}_2(B \cap S^2) \subseteq \text{pr}_2 B = S$, we have that $\text{pr}_2(B \cap S^2)$ is a nonempty subuniverse of \mathbb{A} that is contained in S . Therefore $\text{pr}_2(B \cap S^2)$ is either a singleton or S . We will consider these two cases separately.

Case 1. First suppose that $\text{pr}_2(B \cap S^2) = \{s\}$ for some $s \in S$. Then $\text{pr}_1(B \cap S^2) = S$ implies $S \times \{s\} = B \cap S^2$. Recall that $\text{pr}_2 B = S$. Therefore, for each $t \in S \setminus \{s\}$ there exists some $b_t \in A$ such that $(b_t, t) \in B$ and, since $\text{pr}_2(B \cap S^2) = \{s\}$ and $t \neq s$, we can infer that $b_t \in A \setminus S$. Thus $B(x_1, t)$ is a nonempty subuniverse of \mathbb{A} that contains $b_t \in A \setminus S$. It follows from our assumptions on the subalgebras of \mathbb{A} that $B(x_1, t) = \{b_t\}$ or A . However the latter cannot hold, otherwise we get that $A \times \{t\} \subseteq B$, hence $S \times \{t\} \subseteq B$, which means $t \in \text{pr}_2(B \cap S^2) = \{s\}$, where $t \neq s$, a contradiction. Therefore $B(x_1, t) = \{b_t\}$ for each $t \in S \setminus \{s\}$. We now have two subcases to consider. Either there exists distinct $t, t' \in S \setminus \{s\}$ such that $b_t = b_{t'}$ or $b_t \neq b_{t'}$ for all distinct $t, t' \in S$.

Subcase 1.1. Suppose that there exists distinct $t, t' \in S$ such that $b_t = b_{t'}$, let $b := b_t$. Then $(b, t), (b, t') \in B$, and $t \neq t'$ implies, by Lemma 3.1.7, that $\{b\} \times S \subseteq B$. In particular $(b, s) \in B$.

Recall that $S \times \{s\} \subseteq B$. Therefore $S \cup \{b\} \subseteq B(x_1, s)$, where $b \in A \setminus S$, which means $B(x_1, s) = A$, thus $A \times \{s\} \subseteq B$. Then we have found that $B \supseteq A \times \{s\} \cup \{b\} \times S = \kappa_{b,s}$. Suppose, for contradiction, that there exists some $(c, c') \in B \setminus \kappa_{b,s}$. Then $c' \in S$, $c' \neq s$, and $c \neq b$. This means that (c, s) and (c, c') are distinct elements in B , thus it follows from Lemma 3.1.7 that $\{c\} \times S \subseteq B$. Then $\{b\} \times S \subseteq \kappa_{b,s} \subseteq B$ and $\{c\} \times S \subseteq B$, where $c \neq b$ and $b \in A \setminus S$ implies, by Lemma 3.1.9, that $A \times S \subseteq B$. However this is a contradiction to the assumption that $\text{pr}_2(B \cap S^2) = \{s\}$. Therefore $B = \kappa_{b,s}$.

Subcase 1.2. Now suppose that $b_t \neq b_{t'}$ for all distinct $t, t' \in S \setminus \{s\}$. We claim that this property implies that $S \not\subseteq B(a, x_2)$ for each $a \in A \setminus S$. Suppose, for contradiction, that there exists some $a \in A \setminus S$ such that $S \subseteq B(a, x_2)$. Then $\{a\} \times S \subseteq B$ and for distinct $t, t' \in S \setminus \{s\}$ we get that $(a, t), (a, t') \in B$. However this means that $b_t = B(x_1, t) = a = B(x_1, t') = b_{t'}$ for distinct $t, t' \in S \setminus \{s\}$ which is a contradiction to the assumptions of this subcase. Then $S \not\subseteq B(a, x_2)$ for each $a \in A \setminus S$.

Now we will show that this means that $A \times \{s'\} \not\subseteq B$ for any $s' \in S$. Suppose not. Then $A \times \{s'\} \subseteq B$ for some $s' \in S$. Let $t \in S \setminus \{s, s'\}$, such an element exists since $|S| > s$. We saw that there exists some $b_t \in A \setminus S$ such that $(b_t, t) \in B$. Then $(b_t, s') \in A \times \{s'\} \subseteq B$, $t \neq s'$, and Lemma 3.1.7 imply that $\{b_t\} \times S \subseteq B$. Therefore, $B(b_t, x_2) \supseteq S$ where $b_t \in A \setminus S$, which contradicts $S \not\subseteq B(a, x_2)$ for each $a \in A \setminus S$. Therefore $A \times \{s'\} \not\subseteq B$ for any $s' \in S$.

Recall that $\text{pr}_1 B = A$ and $\text{pr}_2 B = S$, therefore for each $a \in A \setminus S$ there exists some $c_a \in S$ such that $(a, c_a) \in B$, thus $B(a, x_2)$ is a nonempty subuniverse of \mathbb{A} that contains c_a but does not contain S . It follows that $B(a, x_2) = \{c_a\}$ for each $a \in A \setminus S$. We claim that, in fact, $c_a \in S \setminus \{s\}$ for each $a \in A \setminus S$. Suppose not. Then there exists some $a \in A \setminus S$ such that $c_a = s$, thus $(a, s) \in B$. Then $(s, s) \in S \times \{s\} \subseteq B$, $a \in A \setminus S$, and $a \neq s$ implies, by Lemma 3.1.8, that $A \times \{s\} \subseteq B$, which is a contradiction. Hence, for each $a \in A \setminus S$, there exists some $c_a \in S \setminus \{s\}$ such that $B(a, x_2) = \{c_a\}$.

This property, together with the assumption that $S \times \{s\} \subseteq B$ implies that $B = \rho^{-1}(\phi)$,

where ρ is the natural map $\rho : A \times S \rightarrow A/\theta \times S$ and ϕ is the function,

$$\phi : A/\theta \rightarrow S : \bar{S} \mapsto s, \bar{a} \mapsto c_a,$$

for each $a \in A \setminus S$, $\bar{a} = a/\theta$, $\bar{S} = s'/\theta$ for some $s' \in S$. Since $S \times \{s\} \cup \{(b_t, t) : t \in S \setminus \{s\}\} \subseteq B$, then $\phi(\bar{S}) = s$ and $\phi(b_t) = t$ for all $t \in A \setminus \{s\}$, therefore ϕ is onto. We claim that ϕ is one-to-one. Suppose not. Then there exist distinct $\bar{u}, \bar{v} \in A/\theta$ such that $\phi(\bar{u}) = s' = \phi(\bar{v})$ for some $s' \in S$. Thus $B \supseteq \rho^{-1}(\{(\bar{u}, s'), (\bar{v}, s')\})$, which means there exist distinct $u, v \in A$, where $u/\theta = \bar{u}$ and $v/\theta = \bar{v}$ such that $(u, s'), (v, s') \in B$. Furthermore, since $\bar{u} \neq \bar{v}$, it follows that at least one of u or v is in $A \setminus S$. Therefore, it follows from Lemma 3.1.8 that $A \times \{s'\} \subseteq B$, which contradicts $A \times \{s'\} \not\subseteq B$ for any $s' \in S$. We have shown that $B = \rho^{-1}(\phi)$ where ϕ is a bijection from A/θ to S , thus, by Definition 2.2.16, we have that B is an isomorphism from \mathbb{A}/θ to \mathbb{S} .

It remains to show that θ is a congruence on \mathbb{A} . Recall that B is a subalgebra of \mathbb{A}^2 and in fact $B \leq_{s.d.} \mathbb{A} \times \mathbb{S}$. We showed that $B = \rho^{-1}(\phi)$ where $\phi : A/\theta \rightarrow S$ is a bijection. Then it follows from statement (i) of Proposition 2.2.15 that θ is a congruence on \mathbb{A} .

Case 2. We will now consider the second case, namely that when $\text{pr}_2(B \cap S^2) = S$. In this case we get that $B \cap S^2$ is a subuniverse of \mathbb{A}^2 such that $\text{pr}_1(B \cap S^2) = S = \text{pr}_2(B \cap S^2)$. Hence we can apply Proposition 3.1.13 to conclude that $B \cap S^2$ is either an automorphism of \mathbb{S} , or an (A, S) -cross ν_s for some $s \in S$, or S^2 . We will consider these three subcases separately. In all of the subcases the assumption that $\text{pr}_1 B = A$ and $\text{pr}_2 B = S$ implies that for each $b \in A \setminus S$, there exists some $s_b \in S$ such that $(b, s_b) \in B$.

Subcase 2.1. First suppose that $B \cap S^2 = \sigma \in \text{Aut}(\mathbb{S})$. Let $b \in A \setminus S$, and let $s := s_b \in S$ where $(b, s) \in B$. By our assumptions on $B \cap S^2$ we also have that $(\sigma^{-1}(s), s) \in B$, clearly $\sigma^{-1}(s) \in S$. Then applying Lemma 3.1.8 to $(b, s), (\sigma^{-1}(s), s)$, where $b \in A \setminus S$ and $b \neq \sigma^{-1}(s)$ gives that $A \times \{s\} \subseteq B$, thus $S \times \{s\} \subseteq B \cap S^2 = \sigma$, which is a contradiction. Therefore this case cannot occur.

Subcase 2.2. Now let us suppose that $B \cap S^2 = \nu_s$. If $t \in S \setminus \{s\}$, then $B(x_1, t)$ is a subuniverse of \mathbb{A} such that $B(x_1, t) \cap S = (B \cap S^2)(x_1, t) = \nu_s(x_1, t) = \{s\}$, which forces that $B(x_1, t) = \{s\}$.

This implies that $B \subseteq \kappa_{s,s}$ and also that $s_b = s$ for all $b \in A \setminus S$. Thus $(b, s) \in B$ for all $b \in A \setminus S$. Since we also have that $\nu_s \subseteq B$, we get that $\kappa_{s,s} \subseteq B$, and hence $B = \kappa_{s,s}$.

Subcase 2.3. Finally let us suppose that $B \cap S^2 = S^2$. For arbitrary $(b, t) \in B$, $b \in A \setminus S$, we have also that $(t, t) \in S^2 \subseteq B$. Then $(b, t), (t, t) \in B$, $b \in A \setminus S$, $b \neq t$, implies, by Lemma 3.1.8, that $A \times \{t\} \subseteq B$. If there is a unique element $s \in S$ such that $A \times \{s\} \subseteq B$, then this argument shows that $B \subseteq \lambda_{S,s}$ and also that $s_b = s$ for all $b \in A \setminus S$. Thus $(b, s) \in B$ for all $b \in A \setminus S$. Since $S^2 \subseteq B$, we get that $\lambda_{S,s} \subseteq B$ and hence $B = \lambda_{S,s}$. If there are at least two distinct elements $s, s' \in S$ such that $A \times \{s\}, A \times \{s'\} \subseteq B$, then, by Lemma 3.1.10 we get that $B = A \times S$. This completes the proof of the lemma. \square

Corollary 3.1.15. *If B is a subuniverse of \mathbb{A}^2 such that $\text{pr}_1 B = S$ and $\text{pr}_2 B = A$, then B is one of the following:*

- an isomorphism from \mathbb{S} to \mathbb{A}/θ (hence θ is a congruence on \mathbb{A}),
- $\kappa_{a,s}^{-1}$, for some $a \in A$, $s \in S$,
- $\lambda_{S,s}^{-1}$, for some $s \in S$, or
- $S \times A$.

Proof. This follows directly by applying Lemma 3.1.14 to B^{-1} . \square

Lemma 3.1.16. *If B is a subuniverse of \mathbb{A}^2 such that $\text{pr}_1 B = A = \text{pr}_2 B$, then B is one of the following:*

- an automorphism of \mathbb{A} ,
- an automorphism of \mathbb{A}/θ (hence θ is a congruence on \mathbb{A}),
- μ_a , for some $a \in A$,
- $\chi_{s,s'}$, $\chi_{S,s}$, $(\chi_{S,s})^{-1}$, or $\chi_{S,S}$, for some $s, s' \in S$

- ν_s^τ , for some $s \in S$ and some fixed-point free permutation τ of $A \setminus S$ (hence θ is a congruence on \mathbb{A}), or
- A^2 .

Proof. Suppose that B is a subuniverse of \mathbb{A}^2 such that $\text{pr}_1 B = A = \text{pr}_2 B$. Then for each $a \in A$ we get that there exists some $b_a, c_a \in A$ such that $(a, c_a), (b_a, a) \in B$. Then $B(a, x_2) = \{c_a\}$, or S , or A . Similarly $B(x_1, a) = \{b_a\}$, or S , or A . Let $D = B \cap (A \times S)$. Then either some projection of D is a singleton, or $\text{pr}_1 D = S = \text{pr}_2 D$, or $\text{pr}_1 D = A$ and $\text{pr}_2 D = S$. We will consider these three cases separately.

Case 1. Suppose that some projection of D is a singleton. Then Lemma 3.1.12 implies that D is either a point or a line. If $D = \{(u, v)\}$ for some $u \in A, v \in S$, or if $D = S \times \{v\}$ for some $v \in S$ or if $D = A \times \{v\}$ for some $v \in S$, then $\text{pr}_2(B \cap (A \times S)) = \text{pr}_2 D = \{v\}$, hence $\text{pr}_2 B \subseteq \{v\} \cup (A \setminus S)$, which contradicts $\text{pr}_2 B = A$. This forces $D = \{u\} \times S$ for some $u \in A$. Then for each $a \in A \setminus \{u\}$, we get that $B(a, x_2) \cap S = (B \cap (A \times S))(a, x_2) = \emptyset$, which means $B(a, x_2) = \{c_a\}$. Furthermore, $S \subseteq B(u, x_2)$ implies that either $B(u, x_2) = S$ or $B(u, x_2) = A$. We will consider these two cases separately.

Subcase 1.1. First suppose that $B(u, x_2) = A$. Then $u \times A \subseteq B$. Let $a \in A \setminus (S \cup \{u\})$, such an element exists since $|A \setminus S| > 1$. We showed that there exists some $c_a \in A \setminus S$ such that $B(a, x_2) = \{c_a\}$. Then we can infer from $(a, c_a) \in B$, $(u, c_a) \in \{u\} \times A \subseteq B$, $a \neq u$, $a \in A \setminus S$ and Lemma 3.1.8 that $A \times \{c_a\} \subseteq B$. Let $c := c_a$. Then $A \times \{c\} \cup \{u\} \times A \subseteq B$ implies $\mu_{u,c} \subseteq B$. Furthermore, we showed, for each $a \in A \setminus \{u\}$, that $B(a, x_2)$ is a singleton. Thus $(a, c) \in B$ implies $B(a, x_2) = \{c\}$ for every $a \in A \setminus \{u\}$. Then $B(u, x_2) = A$ implies $B \subseteq \mu_{u,c}$, hence $B = \mu_{u,c}$. Since B is a subuniverse of \mathbb{A}^2 , it follows from statement (i) of Lemma 3.1.11 that $u = c$. Therefore $B = \mu_u$.

Subcase 1.2. Now suppose that $B(u, x_2) = S$. Then $(u, c) \notin B$ if $c \in A \setminus S$, hence $A \times \{c\} \not\subseteq B$ for any $c \in A \setminus S$. We also have $A \times \{c\} \not\subseteq B$ for any $c \in S$, because $B \cap (A \times S) = \{u\} \times S$. Therefore, $A \times \{c\} \not\subseteq B$ for any $c \in A$.

Notice that there must exist distinct $a, a' \in A \setminus \{u\}$ such that $c_a = c_{a'}$, otherwise $\text{pr}_1 B = A = \text{pr}_2 B$ and $B(a, x_2) = \{c_a\}$ for each $a \in A \setminus \{u\}$ and some $c_a \in A \setminus S$ implies that $|A \setminus \{u\}| = |A \setminus S|$, which contradicts the assumption that $|S| > 2$. Let $a, a' \in A$ such that $c := c_a = c_{a'}$. Then Lemma 3.1.8 and $(a, c), (a', c) \in B$, with $a \neq a'$, implies that $S \times \{c\} \subseteq B$ and if one of a or a' is in $A \setminus S$, then $A \times \{c\} \subseteq B$. Since $A \times \{c\} \subseteq B$ contradicts $A \times \{c\} \not\subseteq B$ for any $c \in A$, it must be that $a, a' \in S$. Then $S \times \{c\} \subseteq B$ means that $B(s, x_2) = \{c\}$ for all $s \in S$.

Furthermore, this forces $u \in A \setminus S$, otherwise $(u, s) \in \{u\} \times S \subseteq B$ and $(u, c) \in S \times \{c\}$, where $c \in A \setminus S$ and $c \neq s$ implies, by Lemma 3.1.7, that $\{u\} \times A \subseteq B$, then $B(u, x_2) = A$, which contradicts the assumption that $B(u, x_2) = S$.

Finally, we claim that $c_a \neq c_{a'}$ for distinct $a, a' \in A \setminus (S \cup \{u\})$, otherwise, $(a, c_a), (a', c_{a'}) = (a', c_a) \in B$ implies that $A \times \{c_a\} \subseteq B$, which contradicts $A \times \{c\} \not\subseteq B$ for any $c \in A$.

We have shown that $(\{u\} \times S) \cup (S \times \{c\}) \subseteq B$ for some $u, c \in A \setminus S$. Also, $B(a, x_2) = \{c_a\}$ for each $a \in A \setminus (S \cup \{u\})$, where $c_a \neq c_{a'}$ for distinct $a, a' \in A \setminus (S \cup \{u\})$. For $s \in S$, let $s/\theta = \bar{s}$. Let $a/\theta = \bar{a}$ for each $a \in A \setminus S$. Then we have that $B = \rho^{-1}(\phi)$ where ρ is the natural map $\rho: A \times A \rightarrow A/\theta \times A/\theta$ and ϕ is the function

$$\phi: A/\theta \rightarrow A/\theta, \quad \bar{u} \mapsto \bar{S}, \quad \bar{S} \mapsto \bar{c}, \quad \bar{a} \mapsto \bar{c}_a, \quad \text{for all } a \in A \setminus (S \cup \{u\}).$$

Since $c_a \neq c_{a'}$ for distinct a, a' we get that $\bar{c}_a \neq \bar{c}_{a'}$ for distinct $\bar{a}, \bar{a}' \in A/\theta \setminus \{\bar{S}, \bar{u}\}$, therefore ϕ is a one-to-one function. Since A/θ is finite we get that ϕ is a bijection. Therefore, by Definition 2.2.16, B is an automorphism of \mathbb{A}/θ .

It remains to show that θ is a congruence on \mathbb{A} . Recall that B is a subalgebra of \mathbb{A}^2 and we showed that $B = \rho^{-1}(\phi)$ where $\phi: A/\theta \rightarrow A/\theta$ is a bijection. Then it follows from statement (i) of Proposition 2.2.15 that θ is a congruence on \mathbb{A} .

Case 2. Suppose that $\text{pr}_1 D = S = \text{pr}_2 D$. Then for each $a \in A \setminus S$ we have that $B(a, x_2) \cap S = (B \cap (A \times S))(a, x_2) = D(a, x_2) = \emptyset$. Also, $\text{pr}_1 B = A$ implies that there exists some $c_a \in A$ such that $(a, c_a) \in B$, therefore it follows that $B(a, x_2) = \{c_a\}$ for some $c_a \in A \setminus S$.

Since $\text{pr}_1 D = S = \text{pr}_2 D$ it is clear that $\text{pr}_1(B \cap (S \times A)) = S$ and $\text{pr}_2(B \cap (S \times A)) \supseteq S$.

Thus, either $\text{pr}_2(B \cap (S \times A)) = S$ or $\text{pr}_2(B \cap (S \times A)) = A$. We will consider these two cases separately.

Subcase 2.1. First suppose that $\text{pr}_2(B \cap (S \times A)) = A$. Let $a \in A \setminus S$. Then there exists some $s \in S$ such that $(s, a) \in B$. Furthermore, $\text{pr}_1 D = S = \text{pr}_2 D$ implies that there exists some $s' \in S$ such that $(s, s') \in B$. From $(s, a), (s, s') \in B$, $a \in A \setminus S$, $s' \in S$, and Lemma 3.1.7 we can infer that $\{s\} \times A \subseteq B$.

Now recall that for $a \in A \setminus S$ we have that $B(a, x_2) = \{c_a\}$ for some $c_a \in A \setminus S$. Let $c := c_a$. Then $(a, c) \in B$ and $(s, c) \in \{s\} \times A \subseteq B$, where $a \in A \setminus S$ and $s \in S$ implies, by Lemma 3.1.7, that $A \times \{c\} \subseteq B$. This means that $B(a, x_2) = \{c\}$ for all $a \in A \setminus S$ and $(b, b) \notin B$ for any $b \in A \setminus (S \cup \{c\})$.

Let $\Delta = \{(x, x) : x \in A\}$. Then Δ is a subuniverse of \mathbb{A}^2 , therefore $\Delta \cap B \leq \mathbb{A}^2$ and $\text{pr}_1(\Delta \cap B) \leq \mathbb{A}$. Note that $\{s\} \times A \cup A \times \{c\} \subseteq B$ implies that $\{(s, s), (c, c)\} \in \Delta \cap B$. Furthermore, since $(b, b) \notin B$ for any $b \in A \setminus (S \cup \{c\})$, we have that $(b, b) \notin \Delta \cap B$ for any $b \in A \setminus (S \cup \{c\})$. Then $\text{pr}_1(\Delta \cap B)$ is a proper nontrivial subuniverse of \mathbb{A} that contains $c \in A \setminus S$, which contradicts our assumptions on the subalgebras of \mathbb{A} . Hence, this case fails.

Subcase 2.2. Now suppose that $\text{pr}_2(B \cap (S \times A)) = S$. Then for each $a \in A \setminus S$ we have that $B(x_1, a) \cap S = (B \cap (S \times A))(x_1, a) = \emptyset$. Since $\text{pr}_2 B = A$ we know that there exists some $b_a \in A$ such that $(b_a, a) \in B$ therefore $B(x_1, a) = \{b_a\}$ and $b_a \in A \setminus S$. Recall, for each $a \in A \setminus S$, that there exists some $c_a \in A \setminus S$ such that $B(a, x_2) = \{c_a\}$. Then these conditions imply that B contains (the graph of) an onto function $\tau : A \setminus S \rightarrow A \setminus S : a \mapsto c_a$ where $A \setminus S$ is finite. Therefore, τ is a permutation of $A \setminus S$ that contains (the graph of) a bijection from $A \setminus S$ to $A \setminus S$. Furthermore, B is the union of D and the graph of τ . Under the assumption that $\text{pr}_1 D = S = \text{pr}_2 D$ we know, by Lemma 3.1.13, that D is either an automorphism of \mathbb{S} , or $D = \nu_s$ for some $s \in S$, or $D = S^2$.

If D is an automorphism of \mathbb{S} , then we have that $B(a, x_1)$ is a singleton for every $a \in A$. This condition means that B is (the graph of) a function $\phi : A \rightarrow A$ where ϕ is defined by $B(a, x_1) = \{\phi(a)\}$ for all $a \in A$. Since $\text{pr}_2 B = A$, we have that ϕ is onto. Furthermore, A is finite, therefore B is (the graph of) a permutation of A . Hence by Proposition 2.2.14, B is an

automorphism of \mathbb{A} .

Suppose that $D = \nu_s$ for some $s \in S$. Then it follows from the above discussion that $B = \nu_s^\tau$. We claim that τ is fixed-point free. Suppose not. Let $b \in A \setminus S$ such that $\tau(b) = b$, then $(b, b) \in B$. Let $\Delta = \{(x, x) : x \in A\}$. Then Δ is a subuniverse of \mathbb{A}^2 which means $\Delta \cap B \leq \mathbb{A}^2$ and $\text{pr}_1(\Delta \cap B) \leq \mathbb{A}$. Since $(b, b) \in B$ and $(s, s) \in \nu_s \subseteq B$, we have that $\Delta \cap B \supseteq \{(s, s), (b, b)\}$. Furthermore, since $(s', s') \notin \nu_s = B \cap (S \times S)$ for all $s' \in S \setminus \{s\}$, it follows that $(s', s') \notin \Delta \cap B$. Hence $\text{pr}_1(\Delta \cap B)$ is a proper nontrivial subuniverse of \mathbb{A} that contains $b \in A \setminus S$, which contradicts our assumptions on the subuniverses of \mathbb{A} . Therefore τ is fix-point free. Lastly, we claim that if $B = \nu_s^\tau$ is a subuniverse of \mathbb{A}^2 , then θ is a congruence on \mathbb{A} . Let $R := B \circ B^{-1} = \{(x, z) : \text{there exists some } y \in A \text{ such that } (x, y), (z, y) \in B\}$. Since relational clones are closed under composition, we get that R is a subuniverse of \mathbb{A}^2 . We will show that $R = \theta$. Recall that $\theta = S^2 \cup \{(x, x) : x \in A \setminus S\}$. Let $(u, v) \in \theta$. If $(u, v) \in S^2$ then we get that $(u, s), (v, s) \in \nu_s \subseteq B$, therefore, $(u, v) \in R$. If $(u, v) \in (A \setminus S)^2$, then $u = v$ which means $(u, \tau(u)), (v, \tau(u)) = (u, \tau(u)) \in B$, thus $(u, v) \in R$. Hence $\theta \subseteq R$. Now suppose that $(x, z) \in R$. Then there exists some $y \in A$ such that $(x, y), (z, y) \in B$. If $y \in S$, then $(x, z) \in S^2 \subseteq \theta$. Suppose that $y \in A \setminus S$. Then $x, z \in A \setminus S$ and $x = \tau^{-1}(y) = z$. Therefore $(x, z) \in \theta$. We have shown that $R \subseteq \theta$, so we may conclude that $R = \theta$. Therefore θ is a subuniverse of \mathbb{A}^2 , from Proposition 2.2.5 it follows that θ is a congruence on \mathbb{A} .

Finally, suppose that $D = S^2$. Let $s/\theta = \bar{S}$ for all $s \in S$ and $a/\theta = \bar{a}$ for all $a \in A \setminus S$. Then $B = \rho^{-1}(\phi)$, where ρ is the natural map, $\rho : A \times A \rightarrow A/\theta \times A/\theta$, and ϕ is the function given by

$$\phi : A/\theta \rightarrow A/\theta, \quad \bar{S} \mapsto \bar{S}, \quad \bar{a} \mapsto \tau(\bar{a}), \quad \text{for all } \bar{a} \in (A/\theta) \setminus \{\bar{S}\}.$$

Since $\text{pr}_2 B = A$ we have that ϕ is onto. As, A/θ is finite, ϕ is also one-to-one. Then by Definition 2.2.14 we get that B is an automorphism of \mathbb{A}/θ .

It remains to show that θ is a congruence on \mathbb{A} . Recall that B is a subuniverse of \mathbb{A}^2 and we showed that $B = \rho^{-1}(\phi)$ where $\phi : A/\theta \rightarrow A/\theta$ is a bijection. Then it follows from statement (i) of Proposition 2.2.15 that θ is a congruence on \mathbb{A} .

Case 3. Suppose that $\text{pr}_1 D = A$ and $\text{pr}_2 D = S$. Then it follows from Lemma 3.1.14 that D is either an isomorphism from \mathbb{A}/θ to \mathbb{S} , or $D = \kappa_{a,s}$ for some $a \in A$, $s \in S$, or $D = \lambda_{S,s}$ for some $s \in S$, or $D = A \times S$. In each of these four cases we get that for each $a \in A$, $B(a, x_2)$ is either a singleton, or $B(a, x_2) = S$, or $B(a, x_2) = A$.

We first claim that D cannot be an isomorphism from \mathbb{A}/θ to \mathbb{S} . Suppose D is an isomorphism from \mathbb{A}/θ to \mathbb{S} . Then for each $a \in A$ we have that $B(a, x_2) \cap S = (B \cap (A \times S))(a, x_2) = D(a, x_2) = \{s_a\}$ for some $s_a \in A$. Therefore $B(a, x_2) = \{s_a\}$ for all $a \in A$. However this contradicts $\text{pr}_2 B = A$, so this case cannot occur.

Suppose that $D = \kappa_{b,s}$ for some $b \in A$, $s \in S$. Then for each $a \in A \setminus \{b\}$ we get that $B(a, x_2) \cap S = (B \cap (A \times S))(a, x_2) = D(a, x_2) = \{s\}$. Therefore $B(a, x_2) = \{s\}$ for all $a \in A \setminus \{b\}$. Since $\text{pr}_2 B = A$, this forces $B(b, x_2) = A$. Hence $B \subseteq \mu_{b,s}$. Furthermore, $B(a, x_2) = A$ implies $\{a\} \times A \subseteq B$. We also have that $\{s\} \times A \subseteq \kappa_{b,s} \subseteq B$, therefore $B \subseteq \mu_{b,s}$, which means $B = \mu_{b,s}$. Since B is a subuniverse of \mathbb{A}^2 , it follows from statement (i) of Lemma 3.1.11 that $b = s$. Therefore $B = \mu_s$.

Now suppose that $D = \lambda_{S,s}$ for some $s \in S$. Then for each $a \in A \setminus S$ we get that $B(a, x_2) \cap S = (B \cap (A \times S))(a, x_2) = D(a, x_2) = \{s\}$. Therefore $B(a, x_2) = \{s\}$ for all $a \in A \setminus S$. Since $S^2 \subseteq B$ we have that $S \subseteq B(t, x_2)$ for all $t \in S$. Thus either $B(t, x_2) = S$ or $B(t, x_2) = A$ for each $t \in S$. Furthermore, $\text{pr}_2 B = A$ implies that there exists at least one $t \in S$ such that $B(t, x_2) = A$. Suppose that t is the unique element of S such that $B(t, x_2) = A$. Then $B(t, x_2) = A$, $B(t', x_2) = S$ for all $t' \in S \setminus \{t\}$ and $B(a, x_2) = \{s\}$ for all $a \in A \setminus S$ implies $B \subseteq \chi_{t,s}$. Furthermore, $B(t, x_2) = A$ implies that $\{t\} \times A \subseteq B$ and we have that $\lambda_{S,s} \subseteq B$, therefore $\chi_{t,s} \subseteq B$, hence equality holds. Now suppose that there exists distinct $t, t' \in S$ such that $B(t, x_2) = A$ and $B(t', x_2) = A$. Then $(\{t\} \times A) \cup (\{t'\} \times A) \subseteq B$, thus it follows from Lemma 3.1.9 and from $t, t' \in S$, $t \neq t'$, that $S \times A \subseteq B$. Therefore $S \times A \subseteq B$ and $B(a, x_2) = \{s\}$ for all $a \in A \setminus S$ implies $B = \chi_{S,s}$.

Finally suppose that $D = S \times A$. Then $S \subseteq B(a, x_2)$ for all $a \in A \setminus S$, which means either $B(a, x_2) = S$ or $B(a, x_2) = A$. Since $\text{pr}_2 B = A$, we get that there must exist some $a \in A$ such that $B(a, x_2) = A$.

Suppose that a is the unique element of A with this property. Then $B(a, x_2) = A$ and $B(a', x_2) = S$ for all $a' \in A \setminus \{a\}$ implies $B \subseteq \chi_{S,a}^{-1}$. Furthermore, $S \times A \subseteq B$ and $B(a, x_2) = A$ implies $\{a\} \times A \subseteq B$, therefore $\chi_{S,a}^{-1} \subseteq B$, thus equality holds. We claim that $a \in S$. Suppose not. Recall that $\Delta = \{(x, x) : x \in A\}$ is a subuniverse of \mathbb{A}^2 , therefore $\Delta \cap B \leq \mathbb{A}^2$ and $\text{pr}_1(\Delta \cap B) \leq \mathbb{A}$. If $a \in A \setminus S$, then $\Delta \cap B = \{(s, s) : s \in S\} \cup \{(a, a)\}$. We are assuming that $|A \setminus S| > 1$, therefore, $\text{pr}_1(\Delta \cap B) = S \cup \{a\}$ is a proper nontrivial subuniverse of \mathbb{A} that contains $a \in A \setminus S$, which contradicts our assumptions on the subalgebras of \mathbb{A} . Hence $B = \chi_{S,a}^{-1}$ and $a \in S$.

Now suppose that there exists distinct $a, a' \in A$ such that $B(a, x_2) = A$ and $B(a', x_2) = A$. If there exists some $b \in A \setminus S$ such that $B(b, x_2) = A$, then choose $a = b$. Then we have that $(\{a\} \times A) \cup (\{a'\} \times A) \subseteq B$. If $a \in A \setminus S$ then it follows from Lemma 3.1.7 and $a \neq a'$ that $A \times A \subseteq B$, therefore $B = A^2$. However, if $a, a' \in S$, then we can conclude from Lemma 3.1.7 that $S \times A \subseteq B$. Then $S \times A \subseteq B$ and $B(b, x_2) = S$ for all $b \in A \setminus S$ implies that $B = \chi_{S,S}$.

We have shown that either B is an automorphism of \mathbb{A} , or B is an automorphism of \mathbb{A}/θ and θ is a congruence on \mathbb{A} , or $B = \mu_a$ for some $a \in A$, or B is one of the thick (A, A) -crosses, $\chi_{S,S}$, or $\chi_{S,s}$ or $\chi_{S,s}^{-1}$ for some $s \in S$, or $\chi_{s,t}$ for some $s, t \in S$. This completes the proof of the lemma. \square

Proof of Theorem 3.1.5. The result of this theorem follows from Lemmas 3.1.12-3.1.16. \square

3.2 Crosses Among the Subuniverses of \mathbb{A}^2

We will now apply Proposition 2.5.2 to the (thick) cross relations occurring in Theorem 3.1.5. Using the notation of Definition 2.5.1 these relations can be rewritten as follows. Let $a, b \in A$ and $s \in S$, then

- $\nu_s = [S, S, s, s]$,
- $\mu_a = [A, A, a, a]$, and if $b \in A \setminus S$, then $\mu_b = [A/\theta, A/\theta, \bar{b}, \bar{b}]$,
- $\kappa_{a,s} = [A, S, a, s]$, and if $b \in A \setminus S$, then $\kappa_{b,s} = [A/\theta, S, \bar{b}, s]$,

- $\lambda_{S,s} = [A/\theta, S, \overline{S}, s]$,
- $\chi_{S,s} = [A/\theta, A, \overline{S}, s]$,
- $\chi_{S,S} = [A/\theta, A/\theta, \overline{S}, \overline{S}]$.

Since the thick (A, A) -crosses $\chi_{s,s'}$, where $s, s' \in S$, are not θ -closed in their i^{th} -coordinate ($i = 1, 2$) for the equivalence relation θ on A , therefore these thick crosses do not fit Definition 2.5.1.

Proposition 3.2.1. *The following implications hold for all $s, s' \in S$, $a, a' \in A$, $b \in A \setminus S$, $\bar{a}, \bar{a}' \in A/\theta$:*

- (i) *If μ_a and $\mu_{a'}$ are subuniverses of \mathbb{A}^2 , then $a = a'$.*
- (ii) *If ν_s and $\nu_{s'}$ are subuniverse of \mathbb{A}^2 , then $s = s'$.*
- (iii) *If $\eta_{\bar{a}}$ and $\eta_{\bar{a}'}$ are subuniverses of $(\mathbb{A}/\theta)^2$, then $\bar{a} = \bar{a}'$.*
- (iv) *If $\kappa_{a,s}$ and $\kappa_{a',s'}$ are subuniverses of \mathbb{A}^2 , then $a = a'$ or $s = s'$.*
- (v) *If $\varkappa_{\bar{a},s}$ and $\varkappa_{\bar{a}',s'}$ are subuniverses of $\mathbb{A}/\theta \times \mathbb{S}$, then $\bar{a} = \bar{a}'$ or $s = s'$.*
- (vi) *If μ_s is a subuniverses of \mathbb{A}^2 , then ν_s is a subuniverse of \mathbb{A}^2 .*
- (vii) *If ν_s^τ is a subuniverse of \mathbb{A}^2 for some fixed-point free permutation τ of $A \setminus S$, then ν_s is a subuniverse of \mathbb{A}^2 .*
- (viii) *If $\kappa_{a,s}$ is a subuniverse of \mathbb{A}^2 and $a \in S$, then $a = s$ and ν_s is a subuniverse of \mathbb{A}^2 .*
- (ix) *If $\chi_{s',s}$ or $\chi_{S,s}$ is a subuniverse of \mathbb{A}^2 , then $\lambda_{S,s}$ is a subuniverse of \mathbb{A}^2 .*
- (x) *If $\chi_{S,S}$, $\chi_{S,s}$, or $\chi_{s,s'}$ is a subuniverse of \mathbb{A}^2 , then μ_b is not a subuniverse of \mathbb{A}^2 . Conversely, if μ_b is a subuniverse of \mathbb{A}^2 , then neither $\chi_{S,S}$, nor $\chi_{S,s}$, nor $\chi_{s,s'}$ is a subuniverse of \mathbb{A}^2 .*

Proof. Each statement (i)-(x) follows from the fact that the intersection of subuniverses of $(\mathbb{A}/\theta_1)^2$ is a subuniverse of $(\mathbb{A}/\theta_1)^2$ and the unary projection of a subuniverse of $(\mathbb{A}/\theta_1)^2$ is a subuniverse of \mathbb{A}/θ_1 , where $\theta_1 \in \{\text{id}_A, \theta\}$.

[(i)] If $\mu_a, \mu_{a'} \leq \mathbb{A}^2$ for some $a, a' \in A$, $a \neq a'$, then $\mu_a \cap \mu_{a'} = \{(a, a'), (a', a)\}$. Hence $\text{pr}_1(\mu_a \cap \mu_{a'}) = \{a, a'\}$ is a two-element subuniverse of \mathbb{A} which contradicts the assumptions on the subalgebras of \mathbb{A} . Hence $a = a'$.

[(ii)] The proof is similar to the proof of (i), replace μ_a with ν_s , and $\mu_{a'}$ with $\nu_{s'}$.

[(iii)] The proof is similar to the proof of (i), replace μ_a with $\eta_{\bar{a}}$, and $\mu_{a'}$ with $\eta_{\bar{a}'}$.

[(iv)] If $\kappa_{a,s}, \kappa_{a',s'} \leq \mathbb{A}^2$ and $a \neq a'$ and $s \neq s'$ then $\kappa_{a,s} \cap \kappa_{a',s'} = \{(a', s), (a, s')\}$. Therefore $\text{pr}_1(\kappa_{a,s} \cap \kappa_{a',s'}) = \{a, a'\}$ is a two-elements subuniverse of \mathbb{A} which is a contradiction. Similarly $\text{pr}_2(\kappa_{a,s} \cap \kappa_{a',s'}) = \{s, s'\}$ is a two-elements subuniverse of \mathbb{S} which is a contradiction. Therefore, either $a = a'$ or $s = s'$.

[(v)] The proof is similar to the proof of (iv), replace $\kappa_{a,s}$ with $\varkappa_{\bar{a},s}$, and $\kappa_{a',s'}$ with $\varkappa_{\bar{a}',s'}$. Recall that \mathbb{A}/θ has only trivial proper subuniverses and $|A/\theta| > 2$.

[(vi)] If $\mu_s \leq \mathbb{A}^2$, then $\mu_s \cap S^2 = \nu_s$. Hence $\nu_s \leq \mathbb{A}^2$.

[(vii)] If $\nu_s^\tau \leq \mathbb{A}^2$ for some fixed-point free permutation τ of $A \setminus S$, then $\nu_s^\tau \cap S^2 = \nu_s$. Hence $\nu_s \leq \mathbb{A}^2$.

[(viii)] If $\kappa_{a,s} \leq \mathbb{A}^2$ and $a \in S$, then $\kappa_{a,s} \cap S^2 = \nu_{a,s}$. Therefore, by statement (ii) of Lemma 3.1.11, we get that $a = s$ and hence $\nu_s \leq \mathbb{A}^2$.

[(ix)] Suppose that $B \leq \mathbb{A}^2$, where $B \in \{\chi_{s',s}, \chi_{S,s}\}$. Then $B \cap (A \times S) = \lambda_{S,s}$. Hence $\lambda_{S,s} \leq \mathbb{A}^2$.

[(x)] We can show that both implications hold by assuming that μ_b and at least one of the thick (A, A) -crosses, $\chi_{S,S}$, $\chi_{S,s}$, or $\chi_{s,s'}$, are simultaneously subuniverses of \mathbb{A}^2 , and thus arrive at a contradiction.

If $\mu_b, \chi_{S,S} \leq \mathbb{A}^2$, then $\mu_b \cap \chi_{S,S} = (\{b\} \times S) \cup (S \times \{b\})$. This, together with the assumption, $|A \setminus S| > 1$ implies $\text{pr}_1(\mu_b \cap \chi_{S,S}) = \{b\} \cup S$ is a proper nontrivial subuniverse of \mathbb{A} that contains $b \in A \setminus S$, which is a contradiction to our assumptions on the subalgebras of \mathbb{A} .

If $\mu_b, \chi_{S,s} \leq \mathbb{A}^2$, then $\mu_b \cap \chi_{S,s} = \{(b, s)\} \cup S \times \{b\}$ which means $\text{pr}_1(\mu_b \cap \chi_{S,s}) = \{b\} \cup S$ is a subuniverse of \mathbb{A} , which leads to the same contradiction as above.

Finally, if $\mu_b, \chi_{s,s'} \leq \mathbb{A}^2$, then $\mu_b \cap \chi_{s,s'} = \{(b, s')\} \cup \{(s, b)\}$. Then $b \in A \setminus S$ and $s \in S$

implies $\text{pr}_1(\mu_b \cap \chi_{s,s'}) = \{b, s\}$ a two elements subuniverse of \mathbb{A} which contradicts the assumptions on the subuniverses of \mathbb{A}^2 . This completes the proof of the proposition. \square

Proposition 3.2.2. *The following implications hold for all $s, s' \in S$, $a, b \in A$, and $\bar{a}, \bar{b} \in A/\theta$:*

- (i) *If $\nu_s, \kappa_{a,s'} \leq \mathbb{A}^2$ and $s \neq s'$, then $\kappa_{a,s} \leq \mathbb{A}^2$,*
- (ii) *If $\nu_s, \lambda_{S,s'} \leq \mathbb{A}^2$ and $s \neq s'$, then $\lambda_{S,s} \leq \mathbb{A}^2$,*
- (iii) *If $\mu_b, \kappa_{a,s} \leq \mathbb{A}^2$ and $a \neq b$, then $\kappa_{b,s} \leq \mathbb{A}^2$,*
- (iv) *If $\mu_b, \lambda_{S,s} \leq \mathbb{A}^2$ and $b \in A \setminus S$, then $\kappa_{b,s} \leq \mathbb{A}^2$,*
- (v) *If $\kappa_{a,s}, \kappa_{b,s} \leq \mathbb{A}^2$ and $a \neq b$, then $\nu_s \leq \mathbb{A}^2$,*
- (vi) *If $\kappa_{a,s}, \kappa_{a,s'} \leq \mathbb{A}^2$ and $s \neq s'$, then $\mu_a \leq \mathbb{A}^2$,*
- (vii) *If $\kappa_{b,s}, \lambda_{S,s} \leq \mathbb{A}^2$ and $b \in A \setminus S$, then $\nu_s \leq \mathbb{A}^2$,*
- (viii) *If $\kappa_{a,s}, \lambda_{S,s'} \leq \mathbb{A}^2$ and $s \neq s'$, then $a = s$ and $\chi_{S,s} \leq \mathbb{A}^2$,*
- (ix) *If $\lambda_{S,s}, \lambda_{S,s'} \leq \mathbb{A}^2$ and $s \neq s'$, then $\chi_{S,S} \leq \mathbb{A}^2$.*
- (x) *If $\kappa_{a,s}, \chi_{S,S} \leq \mathbb{A}^2$ and $a \in A \setminus S$, then $\lambda_{S,s} \leq \mathbb{A}^2$.*
- (xi) *If $\lambda_{S,s}, \mu_b \leq \mathbb{A}^2$ and $b \in A \setminus S$, then $\kappa_{b,s} \leq \mathbb{A}^2$.*
- (xii) *If $\varkappa_{s,\bar{a}}, \varkappa_{s,\bar{b}} \leq \mathbb{S} \times \mathbb{A}/\theta$ and $\bar{a} \neq \bar{b}$, then $\nu_s \leq \mathbb{S}^2$.*
- (xiii) *If $\varkappa_{\bar{a},s} \leq \mathbb{S} \times \mathbb{A}/\theta$, $\eta_{\bar{b}} \leq (\mathbb{A}/\theta)^2$, and $\bar{a} \neq \bar{b}$, then $\varkappa_{\bar{b},s} \leq \mathbb{A}/\theta \times S$.*

Proof. Let $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3 \in \{\mathbb{S}, \mathbb{A}, \mathbb{A}/\theta\}$. Each statement (i)-(ix) follows from a special case of Proposition 2.5.2 (i) indicated below and the fact that the composition of a subuniverse of $\mathbb{A}_1 \times \mathbb{A}_2$ with a subuniverse of $\mathbb{A}_2 \times \mathbb{A}_3$ is a subuniverse of $\mathbb{A}_1 \times \mathbb{A}_3$. Let $s, s' \in S$, $a, b \in A$, and $\bar{a}, \bar{b} \in \mathbb{A}/\theta$.

[(i)] If $\nu_s, \kappa_{a,s'} \leq \mathbb{A}^2$ and $s \neq s'$, then $\kappa_{a,s'} \circ \nu_s = [A, S, a, s'] \circ [S, S, s, s] = [A, S, a, s] = \kappa_{a,s} \leq \mathbb{A} \times \mathbb{S}$.

[(ii)] If $\nu_s, \lambda_{S,s'} \leq \mathbb{A}^2$ and $s \neq s'$, then $\lambda_{S,s'} \circ \nu_s = [A/\theta, S, \bar{S}, s'] \circ [S, S, s, s] = [A/\theta, S, \bar{S}, s] = \lambda_{S,s} \leq \mathbb{A} \times \mathbb{S}$.

[(iii)] If $\mu_b, \kappa_{a,s} \leq \mathbb{A}^2$ and $a \neq b$, then $\mu_b \circ \kappa_{a,s} = [A, A, b, b] \circ [A, S, a, s] = [A, S, b, s] = \kappa_{b,s} \leq \mathbb{A} \times \mathbb{S}$.

[(iv)] If $\mu_b, \lambda_{S,s} \leq \mathbb{A}^2$ and $b \in A \setminus S$, then $\mu_b \circ \lambda_{S,s} = [A/\theta, A/\theta, \bar{b}, \bar{b}] \circ [A/\theta, S, \bar{S}, s] = [A/\theta, S, \bar{b}, s] = \kappa_{b,s} \leq \mathbb{A} \times \mathbb{S}$.

[(v)] If $\kappa_{a,s}, \kappa_{b,s} \leq \mathbb{A}^2$ and $a \neq b$, then $\kappa_{a,s}^{-1} \circ \kappa_{b,s} = [S, A, s, a] \circ [A, S, b, s] = [S, S, s, s] = \nu_s \leq \mathbb{S}^2$.

[(vi)] If $\kappa_{a,s}, \kappa_{a,s'} \leq \mathbb{A}^2$ and $s \neq s'$, then $\kappa_{a,s} \circ \kappa_{a,s'}^{-1} = [A, S, a, s] \circ [S, A, s', a] = [A, A, a, a] = \mu_a \leq \mathbb{A}^2$.

[(vii)] If $\kappa_{b,s}, \lambda_{S,s} \leq \mathbb{A}^2$ and $b \in A \setminus S$, then $\kappa_{b,s}^{-1} \circ \lambda_{S,s} = [S, A/\theta, s, \bar{b}] \circ [A/\theta, S, \bar{S}, s] = [S, S, s, s] = \nu_s \leq \mathbb{S}^2$.

[(viii)] If $\kappa_{a,s}, \lambda_{S,s'} \leq \mathbb{A}^2$ and $s \neq s'$, then $\lambda_{S,s'} \circ \kappa_{a,s}^{-1} = [A/\theta, S, \bar{S}, s'] \circ [S, A, s, a] = [A/\theta, A, \bar{S}, a]$ is a subuniverse of \mathbb{A}^2 . By Definition 2.5.1, $[A/\theta, A, \bar{S}, a] = S \times A \cup A \times \{a\}$, thus Theorem 3.1.5 yields that $a \in S$. Then statement (viii) of Proposition 3.2.1 and $\kappa_{a,s} \leq \mathbb{A}^2$ with $a \in S$ implies $a = s$. Hence $\lambda_{S,s'} \circ \kappa_{a,s}^{-1} = [A/\theta, A, \bar{S}, s] = \chi_{S,s} \leq \mathbb{A}^2$.

[(ix)] If $\lambda_{S,s}, \lambda_{S,s'} \leq \mathbb{A}^2$ and $s \neq s'$, then $\lambda_{S,s} \circ \lambda_{S,s'}^{-1} = [A/\theta, S, \bar{S}, s] \circ [S, A/\theta, s', \bar{S}] = [A/\theta, A/\theta, \bar{S}, \bar{S}] = \chi_{S,S} \leq \mathbb{A}^2$.

[(x)] If $\kappa_{a,s}, \chi_{S,S} \leq \mathbb{A}^2$ and $a \in A \setminus S$, then $\bar{a} \neq \bar{S}$ and $\chi_{S,S} \circ \kappa_{a,s} = [A/\theta, A/\theta, \bar{S}, \bar{S}] \circ [A/\theta, S, \bar{a}, s] = [A/\theta, S, \bar{S}, s] = \lambda_{S,s} \leq \mathbb{A} \times \mathbb{S}$.

[(xi)] If $\lambda_{S,s}, \mu_b \leq \mathbb{A}^2$ and $b \in A \setminus S$, then $\bar{b} \neq \bar{S}$ and $\mu_b \circ \lambda_{S,s} = [A/\theta, A/\theta, \bar{b}, \bar{b}] \circ [A/\theta, S, \bar{S}, s] = [A/\theta, S, \bar{b}, s] = \kappa_{b,s} \leq \mathbb{A} \times \mathbb{S}$.

[(xii)] If $\varkappa_{s,\bar{a}}, \varkappa_{s,\bar{b}} \leq \mathbb{S} \times \mathbb{A}/\theta$ and $\bar{a} \neq \bar{b}$, then $\varkappa_{s,\bar{a}} \circ \varkappa_{s,\bar{b}}^{-1} = [S, A/\theta, s, \bar{a}] \circ [A/\theta, S, \bar{b}, s] = [S, S, s, s] = \nu_s \leq \mathbb{S}^2$.

[(xiii)] If $\varkappa_{\bar{a},s} \leq \mathbb{S} \times \mathbb{A}/\theta$, $\eta_{\bar{b}} \leq (\mathbb{A}/\theta)^2$, and $\bar{a} \neq \bar{b}$, then $\eta_{\bar{b}} \circ \varkappa_{\bar{a},s} = [A/\theta, A/\theta, \bar{b}, \bar{b}] \circ [A/\theta, S, \bar{a}, s] = [A/\theta, S, \bar{b}, s] = \varkappa_{\bar{b},s} \leq \mathbb{A}/\theta \times S$. \square

Corollary 3.2.3. *The following implications hold for all $s, s' \in S$, $a, b \in A$.*

- (i) *If $\mu_b, \kappa_{a,s} \leq \mathbb{A}^2$, $b \in A \setminus S$, and $a \neq b$, then $\nu_s \leq \mathbb{A}^2$.*
- (ii) *If $\mu_b, \lambda_{S,s} \leq \mathbb{A}^2$, and $b \in A \setminus S$, then $\nu_s \leq \mathbb{A}^2$.*

Proof. Let $s, s' \in S$, $a, b \in A$.

[(i)] Suppose $\mu_b, \kappa_{a,s} \leq \mathbb{A}^2$, $b \in A \setminus S$, and $a \neq b$. Then by statement (iii) of Proposition 3.2.2, $\mu_b, \kappa_{a,s} \leq \mathbb{A}^2$ implies that $\kappa_{b,s} \leq \mathbb{A}^2$. Since $\kappa_{a,s}, \kappa_{b,s} \leq \mathbb{A}^2$, $a \neq b$, it follows from statement (v) of Proposition 3.2.2 that $\nu_s \leq \mathbb{A}^2$.

[(ii)] Suppose $\mu_b, \lambda_{S,s} \leq \mathbb{A}^2$, and $b \in A \setminus S$. Then statement (iv) of Proposition 3.2.2, implies that $\kappa_{b,s} \leq \mathbb{A}^2$. Furthermore, since $\kappa_{b,s}, \lambda_{S,s} \leq \mathbb{A}^2$, we get from statement (vii) of Proposition 3.2.2 that $\nu_s \leq \mathbb{A}^2$. □

Proposition 3.2.4. *The following implications hold for all $s, s' \in S$, $a \in A$, $b \in A \setminus S$*

- (i) *If $\mu_a \leq \mathbb{A}^2$ and $\sigma \in \text{Aut}(\mathbb{A})$, then $\mu_{\sigma(a)} \leq \mathbb{A}^2$ and every automorphism of \mathbb{A} fixes a .*
- (ii) *If $\nu_s \leq \mathbb{A}^2$ and $\pi \in \text{Aut}(\mathbb{S})$, then $\nu_{\pi(s)} \leq \mathbb{A}^2$ and every automorphism of \mathbb{S} fixes s .*
- (iii) *If $\kappa_{a,s} \leq \mathbb{A}^2$ and $\pi \in \text{Aut}(\mathbb{S})$, then $\kappa_{a,\pi(s)} \leq \mathbb{A}^2$.*
- (iv) *If $\lambda_{S,s} \leq \mathbb{A}^2$ and $\pi \in \text{Aut}(\mathbb{S})$, then $\lambda_{S,\pi(s)} \leq \mathbb{A}^2$.*
- (v) *If $\mu_b \leq \mathbb{A}^2$, θ is a congruence on \mathbb{A} , and $\Phi \in \text{Aut}(\mathbb{A}/\theta)$, then $\mu_{\Phi(b)} \leq \mathbb{A}^2$ and every automorphism of \mathbb{A}/θ fixes \bar{b} .*
- (vi) *If $\chi_{S,S}$, $\chi_{S,s}$, or $\chi_{s,s'}$ is a subuniverse of \mathbb{A}^2 , θ is a congruence on \mathbb{A} , and $\Phi \in \text{Aut}(\mathbb{A}/\theta)$, then every automorphism of \mathbb{A}/θ fixes \bar{S} .*
- (vii) *If $\eta_{a/\theta} \leq (\mathbb{A}/\theta)^2$, θ is a congruence on \mathbb{A} , and $\Phi \in \text{Aut}(\mathbb{A}/\theta)$, then $\eta_{\Phi(a/\theta)} \leq (\mathbb{A}/\theta)^2$ and every automorphism of \mathbb{A}/θ fixes a/θ .*
- (viii) *If $\mu_b \leq \mathbb{A}^2$, θ is a congruence on \mathbb{A} , and Ψ is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then $\nu_{\Psi(b)} \leq \mathbb{A}^2$ and every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ maps \bar{b} to $\Psi(b)$.*

- (ix) If $\chi_{S,S} \leq \mathbb{A}^2$, θ is a congruence on \mathbb{A} , and Ψ is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then $\nu_{\Psi(\bar{S})} \leq \mathbb{A}^2$, and every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ maps \bar{S} to $\Psi(\bar{S})$.
- (x) If $\nu_s \leq \mathbb{A}^2$, θ is a congruence on \mathbb{A} , and Ψ is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then either $\chi_{S,S} \leq \mathbb{S}^2$ and every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ maps \bar{S} to s or $\mu_c \leq \mathbb{A}^2$ for some $c \in A \setminus S$, and every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ maps \bar{c} to s .
- (xi) If $\kappa_{b,s} \leq \mathbb{A}^2$, θ is a congruence on \mathbb{A} , and Ψ is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then $\nu_s \leq \mathbb{A}^2$ and every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ maps \bar{b} to s .
- (xii) If $\lambda_{S,s} \leq \mathbb{A}^2$, θ is a congruence on \mathbb{A} , and Ψ is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then $\nu_s \leq \mathbb{A}^2$ and every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ maps \bar{S} to s .
- (xiii) If θ is a congruence on \mathbb{A} , $\varkappa_{s,\bar{a}} \leq \mathbb{S} \times \mathbb{A}/\theta$, and Φ is an automorphism of \mathbb{A}/θ , then $\varkappa_{s,\Phi(\bar{a})} \leq \mathbb{S} \times \mathbb{A}/\theta$.
- (xiv) If θ is a congruence on \mathbb{A} , $\varkappa_{s,\bar{a}} \leq \mathbb{S} \times \mathbb{A}/\theta$, and Ψ is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then $\nu_s \leq \mathbb{A}^2$ and every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ maps \bar{a} to s .

Proof. Each statement (i)–(xiv) follows from a special case of Proposition 2.5.2 (ii) indicated below and the fact that the composition of a subuniverse of $\mathbb{A}_1 \times \mathbb{A}_2$ with a subuniverse of $\mathbb{A}_2 \times \mathbb{A}_3$ is a subuniverse of $\mathbb{A}_1 \times \mathbb{A}_3$. Let $s, s' \in S$, $a \in A$, $b \in A \setminus S$, and $\bar{a} \in A/\theta$.

[(i)] If $\mu_a \leq \mathbb{A}^2$ and $\sigma \in \text{Aut}(\mathbb{A})$, then

$$\begin{aligned} \sigma^{-1} \circ \mu_a \circ \sigma &= \sigma^{-1} \circ [A, A, a, a] \circ \sigma \\ &= [A, A, \sigma(a), \sigma(a)] = \mu_{\sigma(a)} \leq \mathbb{A}^2. \end{aligned}$$

Furthermore $\mu_a, \mu_{\sigma(a)} \leq \mathbb{A}^2$ and statement (i) of Lemma 3.1.11 implies that $\sigma(a) = a$. Therefore the automorphisms of \mathbb{A} fix a .

[(ii)] If $\nu_s \leq \mathbb{A}^2$ and $\pi \in \text{Aut}(\mathbb{S})$, then

$$\begin{aligned} \pi^{-1} \circ \nu_s \circ \pi &= \pi^{-1} \circ [S, S, s, s] \circ \pi \\ &= [S, S, \pi(s), \pi(s)] = \nu_{\pi(s)} \leq \mathbb{A}^2. \end{aligned}$$

Furthermore $\nu_s, \nu_{\pi(s)} \leq \mathbb{A}^2$ and statement (ii) of Lemma 3.1.11 implies that $\pi(s) = s$. Therefore the automorphisms of \mathbb{S} fix s .

[(iii)] If $\kappa_{a,s} \leq \mathbb{A}^2$ and $\pi \in \text{Aut}(\mathbb{S})$, then

$$\begin{aligned} \text{id}_{\mathbb{A}} \circ \kappa_{a,s} \circ \pi &= \text{id}_{\mathbb{A}} \circ [A, S, a, s] \circ \pi \\ &= [A, S, a, \pi(s)] = \kappa_{a,\pi(s)} \leq \mathbb{A}^2, \end{aligned}$$

where $\text{id}_{\mathbb{A}}$ is the identity automorphism of \mathbb{A} .

[(iv)] If $\lambda_{S,s} \leq \mathbb{A}^2$ and $\pi \in \text{Aut}(\mathbb{S})$, then

$$\begin{aligned} \text{id}_{\mathbb{A}/\theta} \circ \lambda_{S,s} \circ \pi &= \text{id}_{\mathbb{A}/\theta} \circ [A/\theta, S, \bar{S}, s] \circ \pi \\ &= [A/\theta, S, \bar{S}, \pi(s)] = \lambda_{S,\pi(s)} \leq \mathbb{A}^2, \end{aligned}$$

where $\text{id}_{\mathbb{A}/\theta}$ is the identity automorphism of \mathbb{A}/θ .

[(v)] If $\mu_b \leq \mathbb{A}^2$, θ is a congruence on \mathbb{A} , and $\Phi \in \text{Aut}(\mathbb{A}/\theta)$, then

$$\begin{aligned} \Phi^{-1} \circ \mu_b \circ \Phi &= \Phi^{-1} \circ [A/\theta, A/\theta, \bar{b}, \bar{b}] \circ \Phi \\ &= [A/\theta, A/\theta, \Phi(\bar{b}), \Phi(\bar{b})] = \begin{cases} \chi_{S,S}, & \text{if } \Phi(\bar{b}) = \bar{S}, \\ \mu_c, & \text{if } \Phi(\bar{b}) = \bar{c}, \end{cases} \end{aligned}$$

for some $c \in A \setminus S$, where $\bar{c} = \Phi(\bar{b})$. We have from statement (x) of Proposition 3.2.1 that μ_b and $\chi_{S,S}$ cannot simultaneously be subuniverses of \mathbb{A}^2 , therefore, it follows that $\Phi^{-1} \circ \mu_b \circ \Phi = \mu_c \leq \mathbb{A}^2$, for some $c \in A \setminus S$. Furthermore, $\mu_b, \mu_c \leq \mathbb{A}^2$ and statement (i) of Lemma 3.1.11 implies that $b = c$ where $\bar{c} = \Phi(\bar{b})$. Hence $\bar{b} = \Phi(\bar{b})$ which means the automorphisms of \mathbb{A}/θ fix \bar{b} .

[(vi)] Suppose that $\chi_{S,S}$, $\chi_{S,s}$, or $\chi_{s,s'}$ is a subuniverse of \mathbb{A}^2 , θ is a congruence on \mathbb{A} , and $\Phi \in \text{Aut}(\mathbb{A}/\theta)$. Since θ is a congruence on \mathbb{A} it follows that the θ -closure in both coordinates of a subuniverse of \mathbb{A}^2 is also a subuniverse of \mathbb{A}^2 . By Proposition 2.2.8 the θ -closure in both coordinates of $\chi_{S,s}$ (or $\chi_{s,s'}$) is $B = \rho^{-1}(\rho(\chi_{S,s}))$ (respectively, $B = \rho^{-1}(\rho(\chi_{s,s'}))$) where ρ is the natural homomorphism $\rho : \mathbb{A}^2 \rightarrow (\mathbb{A}/\theta)^2$. Since $B = \chi_{S,S}$ we get that $\chi_{S,S}$ is a subuniverse of \mathbb{A}^2 .

Then

$$\begin{aligned}\Phi^{-1} \circ \chi_{S,S} \circ \Phi &= \Phi^{-1} \circ [A/\theta, A/\theta, \bar{S}, \bar{S}] \circ \Phi \\ &= [A/\theta, A/\theta, \Phi(\bar{S}), \Phi(\bar{S})] = \begin{cases} \chi_{S,S}, & \text{if } \Phi \text{ fixes } \bar{S}, \\ \mu_b, & \text{if } \Phi(\bar{S}) = \bar{b}, \end{cases}\end{aligned}$$

for some $b \in A \setminus S$. Property (x) of Proposition 3.2.1 states that μ_b and $\chi_{S,S}$ cannot simultaneously be subuniverses of \mathbb{A}^2 , therefore $\Phi^{-1} \circ \chi_{S,S} \circ \Phi = \chi_{S,S}$. Hence Φ fixes \bar{S} .

[(vii)] Suppose $\eta_{a/\theta} \leq (\mathbb{A}/\theta)^2$, θ is a congruence on \mathbb{A} , and $\Phi \in \text{Aut}(\mathbb{A}/\theta)$. Let $\rho : A^2 \rightarrow (A/\theta)^2$ be the natural map. If $a/\theta = \bar{S}$, then $\rho^{-1}(\eta_{a/\theta}) = \chi_{S,S}$, thus the statement follows from statement (vi). Otherwise, $\rho^{-1}(\eta_{a/\theta}) = \mu_b$ and the statement follows from statement (v).

[(viii)] If $\mu_b \leq \mathbb{A}^2$, θ is a congruence on \mathbb{A} , and Ψ is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then

$$\begin{aligned}\Psi^{-1} \circ \mu_b \circ \Psi &= \Psi^{-1} \circ [A/\theta, A/\theta, \bar{b}, \bar{b}] \circ \Psi \\ &= [S, S, \Psi(\bar{b}), \Psi(\bar{b})] = \nu_t\end{aligned}$$

for some $t \in S$, where $\Psi(\bar{b}) = t$. Property (ii) of Lemma 3.1.11 implies that there is exactly one such element $t \in S$. Therefore every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ maps \bar{b} to t .

[(ix)] If $\chi_{S,S} \leq \mathbb{A}^2$, θ is a congruence on \mathbb{A} , and Ψ is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then

$$\begin{aligned}\Psi^{-1} \circ \chi_{S,S} \circ \Psi &= \Psi^{-1} \circ [A/\theta, A/\theta, \bar{S}, \bar{S}] \circ \Psi \\ &= [S, S, \Psi(\bar{S}), \Psi(\bar{S})] = \nu_t\end{aligned}$$

for some $t \in S$, where $\Psi(\bar{S}) = t$. Property (ii) of Lemma 3.1.11 implies that there is exactly one such element $t \in S$. Therefore every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ maps \bar{S} to t .

[(x)] If $\nu_s \leq \mathbb{A}^2$, θ is a congruence on \mathbb{A} , and Ψ is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then

$$\begin{aligned}\Psi \circ \nu_s \circ \Psi^{-1} &= \Psi \circ [S, S, s, s] \circ \Psi^{-1} \\ &= [A/\theta, A/\theta, \Psi^{-1}(s), \Psi^{-1}(s)] = \begin{cases} \chi_{S,S}, & \text{if } \Psi(\bar{S}) = s, \\ \mu_c, & \text{if } \Psi(\bar{c}) = s, \end{cases}\end{aligned}$$

for some $c \in A \setminus S$. Property (x) of Proposition 3.2.1 states that $\chi_{S,S}$ and μ_c cannot simultaneously be subuniverses of \mathbb{A}^2 . Therefore either every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ satisfies the first case or every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ satisfies the second case. In the first case we get that every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ maps \bar{S} to s . In the second case, Property (i) of Lemma 3.1.11 implies that there is exactly one such element $c \in A$, thus every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ maps \bar{c} to s .

[(xi)] If $\kappa_{b,s} \leq \mathbb{A}^2$, θ is a congruence on \mathbb{A} , and Ψ is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then

$$\begin{aligned} \Psi^{-1} \circ \kappa_{b,s} \circ \text{id}_S &= \Psi^{-1} \circ [A/\theta, S, \bar{b}, s] \circ \text{id}_S \\ &= [S, S, \Psi(\bar{b}), s] = \nu_{\Psi(\bar{b}),s} \leq \mathbb{A}^2, \end{aligned}$$

where id_S is the identity automorphism of \mathbb{S} . By property (ii) of Lemma 3.1.11 it follows that $\Psi(\bar{b}) = s$. Hence $\Psi^{-1} \circ \kappa_{b,s} \circ \text{id}_S = \nu_s \leq \mathbb{A}^2$ and $\Psi(\bar{b}) = s$. Property (ii) of Lemma 3.1.11 states that there is exactly one such element $s \in S$. Therefore every isomorphism maps \bar{b} to s .

[(xii)] If $\lambda_{S,s} \leq \mathbb{A}^2$, θ is a congruence on \mathbb{A} , and Ψ is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then

$$\begin{aligned} \Psi^{-1} \circ \lambda_{S,s} \circ \text{id}_S &= \Psi^{-1} \circ [A/\theta, S, \bar{S}, s] \circ \text{id}_S \\ &= [S, S, \Psi(\bar{S}), s] = \nu_{\Psi(\bar{S}),s} \leq \mathbb{A}^2, \end{aligned}$$

where id_S is the identity automorphism of \mathbb{S} . By property (ii) of Lemma 3.1.11 it follows that $\Psi(\bar{S}) = s$. Hence $\Psi^{-1} \circ \lambda_{S,s} \circ \text{id}_S = \nu_s \leq \mathbb{A}^2$ and $\Psi(\bar{S}) = s$. Property (ii) of Lemma 3.1.11 states that there is exactly one such element $s \in S$. Therefore every isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ maps \bar{S} to s .

[(xiii)] If θ is a congruence on \mathbb{A} , $\varkappa_{s,\bar{a}} \leq \mathbb{S} \times \mathbb{A}/\theta$, and Φ is an automorphism of \mathbb{A}/θ , then

$$\begin{aligned} \text{id}_S \circ \varkappa_{s,\bar{a}} \circ \Phi &= \text{id}_S \circ [S, A/\theta, s, \bar{a}] \circ \Phi \\ &= [S, A/\theta, s, \Phi(\bar{a})] = \varkappa_{s,\Phi(\bar{a})} \leq \mathbb{S} \times \mathbb{A}/\theta, \end{aligned}$$

where id_S is the identity automorphism of \mathbb{S} .

[(xiv)] If θ is a congruence on \mathbb{A} , $\varkappa_{s,\bar{a}} \leq \mathbb{S} \times \mathbb{A}/\theta$, and Ψ is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then

$$\begin{aligned} \text{id}_S \circ \varkappa_{s,\bar{a}} \circ \Psi &= \text{id}_S \circ [S, A/\theta, s, \bar{a}] \circ \Psi \\ &= [S, S, s, \Psi(\bar{a})] = \nu_{s,\Psi(\bar{a})} \leq \mathbb{A}^2, \end{aligned}$$

where id_S is the identity automorphism of S . By statement (ii) of Lemma 3.1.11 it follows that $\Psi(\bar{a}) = s$. Hence $\text{id}_S \circ \varkappa_{s,\bar{a}} \circ \Psi = \nu_s \leq \mathbb{A}^2$ and $\Psi(\bar{a}) = s$. Property (ii) of Lemma 3.1.11 states that there is exactly one such element $s \in S$. Therefore every isomorphism $\mathbb{A}/\theta \rightarrow S$ maps \bar{a} to s .

□

If S is either quasiprimal or affine, then it follows from Propositions 2.4.5 and 2.4.6 that there is no (S, S) -cross among the subuniverses of \mathbb{A}^2 . Therefore, the next corollary follows directly from the above investigation of the subuniverses of \mathbb{A}^2 that are crosses.

Corollary 3.2.5. *Suppose that either S is quasiprimal, or S is affine, or that there is no (S, S) -cross among the subuniverses of \mathbb{A}^2 . Let $a, a' \in A$, $s, s' \in S$, $b \in A \setminus S$.*

- (i) *If $\kappa_{a,s} \leq \mathbb{A}^2$, then $a \in A \setminus S$.*
- (ii) *If $\kappa_{a,s} \leq \mathbb{A}^2$, then $\lambda_{S,s'} \not\leq \mathbb{A}^2$.*
- (iii) *If $\kappa_{a,s} \leq \mathbb{A}^2$, then $\chi_{S,S} \not\leq \mathbb{A}^2$.*
- (iv) *If $\kappa_{a,s}, \kappa_{a',s'} \leq \mathbb{A}^2$, then $a = a'$.*
- (v) *If θ is a congruence on \mathbb{A} and there exists an isomorphism from \mathbb{A}/θ to S , then $\chi_{S,S} \not\leq \mathbb{A}^2$.*
- (vi) *If θ is a congruence on \mathbb{A} and there exists an isomorphism from \mathbb{A}/θ to S , then $\kappa_{a,s} \not\leq \mathbb{A}^2$.*
- (vii) *If θ is a congruence on \mathbb{A} and there exists an isomorphism from \mathbb{A}/θ to S , then $\lambda_{S,s} \not\leq \mathbb{A}^2$.*
- (viii) *If θ is a congruence on \mathbb{A} and there exists an isomorphism from \mathbb{A}/θ to S , then $\varkappa_{a/\theta,s} \not\leq \mathbb{A}/\theta \times S$.*
- (ix) *If θ is a congruence on \mathbb{A} and Φ is an automorphism of \mathbb{A}/θ , then $\varkappa_{s,\bar{a}} \leq S \times \mathbb{A}/\theta$ implies Φ fixes \bar{a} .*
- (x) *If θ is a congruence on \mathbb{A} , $\eta_{\bar{a}} \leq (\mathbb{A}/\theta)^2$, and $\varkappa_{\bar{b},s} \leq S \times \mathbb{A}/\theta$, for some $\bar{a}, \bar{b} \in A/\theta$, then $\bar{a} = \bar{b}$.*

Proof. If \mathbb{S} is quasiprimal or affine, then there is no (S, S) -cross among the subuniverses of \mathbb{A}^2 . Then, each of statements (i)-(x) follows from the the assumption that there is no (S, S) -cross among the subuniverses of \mathbb{A}^2 .

[(i)] Follows directly from property (viii) of Proposition 3.2.1.

[(ii)] Suppose $\kappa_{a,s}, \lambda_{S,s'} \leq \mathbb{A}^2$. Then property (i) of this proposition implies $a \in A \setminus S$. If $s = s'$, then property (vii) of Proposition 3.2.2 implies $\nu_s \leq \mathbb{A}^2$, which is a contradiction. If $s \neq s'$, then property (viii) of Proposition 3.2.2 implies $a = s \in S$, which contradicts $a \in A \setminus S$.

[(iii)] Suppose $\kappa_{a,s}, \chi_{S,S} \leq \mathbb{A}^2$. Then property (x) of Proposition 3.2.2 implies $\lambda_{S,s} \leq \mathbb{A}^2$, which contradicts property (ii)

[(iv)] Suppose $\kappa_{a,s}, \kappa_{a',s'} \leq \mathbb{A}^2$. If $a \neq a'$, then by property (iv) of Proposition 3.2.1 we get that $s = s'$. Thus property (v) of Proposition 3.2.2 implies $\nu_s \leq \mathbb{A}^2$, which is a contradiction.

[(v)] Follows directly from property (ix) of Proposition 3.2.4.

[(vi)] Follows directly from property (i) above and property (xi) of Proposition 3.2.4.

[(vii)] Follows directly from property (xii) of Proposition 3.2.4.

[(viii)] Suppose $\varkappa_{a/\theta,s} \leq \mathbb{A}/\theta \times \mathbb{S}$ and there exists an isomorphism from \mathbb{A}/θ to \mathbb{S} . Let $\rho : A \times S \rightarrow A/\theta \times S$ be the natural map. Then $\rho^{-1}(\varkappa_{a/\theta,s})$ is a (thick) (A, S) -cross. Therefore, $\varkappa_{a/\theta,s} \leq \mathbb{A}/\theta \times \mathbb{S}$ implies that there exists a (thick) (A, S) -cross among the subuniverses of \mathbb{A}^2 , which gives a contradiction to statements (vi) and (vii).

[(ix)] Suppose θ is a congruence on \mathbb{A} and $\Phi \in \text{Aut}(\mathbb{A}/\theta)$. Then by property (xiii) of Proposition 3.2.4, $\varkappa_{s,\bar{a}} \leq \mathbb{S} \times \mathbb{A}/\theta$ implies $\varkappa_{s,\Phi(\bar{a})} \leq \mathbb{S} \times \mathbb{A}/\theta$. Suppose that $\Phi(\bar{a}) \neq \bar{a}$. Then $\varkappa_{s,\bar{a}}, \varkappa_{s,\Phi(\bar{a})} \leq \mathbb{S} \times \mathbb{A}/\theta$ and statement (xii) of Proposition 3.2.2 implies $\nu_s \leq \mathbb{A}^2$, which is a contradiction. Hence Φ fixes \bar{a} .

[(x)] Suppose θ is a congruence on \mathbb{A} , $\eta_{\bar{a}} \leq (\mathbb{A}/\theta)^2$, and $\varkappa_{s,\bar{b}} \leq \mathbb{S} \times \mathbb{A}/\theta$, for some $\bar{a}, \bar{b} \in A/\theta$. Suppose, for contradiction, that $\bar{a} \neq \bar{b}$. Then statement (xiii) of Proposition 3.2.2 implies $\varkappa_{\bar{a},s} \leq \mathbb{A}/\theta \times S$. Thus $\varkappa_{\bar{a},s}, \varkappa_{\bar{b},s} \leq \mathbb{A}/\theta \times S$ and $\bar{a} \neq \bar{b}$ implies, by statement (xii) of Proposition 3.2.2, that $\nu_s \leq \mathbb{S}^2$, which is a contradiction. Hence, $\bar{a} = \bar{b}$.

□

Chapter 4

Edge Blockers

In their manuscript [MMM10], Marković, Maróti, and McKenzie state a necessary and sufficient condition for a finite idempotent algebra \mathbb{A} to have no edge operation. As we will show in Proposition 4.1.4 below, this condition is equivalent to the existence of an infinite sequence of relations in the relational clone of \mathbb{A} .

In this chapter we will exhibit binary and ternary relations R such that if \mathbb{A} is a finite idempotent algebra that satisfies our usual Assumption 1, and R is in the relational clone of \mathbb{A} , then \mathbb{A} has no edge operation. These small arity edge blockers arose while investigating the subuniverses of finite powers of \mathbb{A} when \mathbb{S} is either quasiprimal or affine. We shall see in Chapter 6 that in all cases when \mathbb{S} is affine and in almost all cases when \mathbb{S} is quasiprimal, if we restrict the relational clone of \mathbb{A} so that it does not contain these small arity edge blockers, then we have a nice description for the relational clone, and hence for the clone of \mathbb{A} .

4.1 Marković–Maróti–McKenzie Edge Blockers

Definition 4.1.1. For an algebra \mathbb{A} and proper subset $G \subsetneq A$ we say that a k -ary operation f is *G -absorbing in its i^{th} -variable*, for some $1 \leq i \leq k$, if whenever $\bar{a} = (a_1, \dots, a_i, \dots, a_k) \in A^k$ with $a_i \in G$, then $f(\bar{a}) \in G$.

We will often apply this definition to a subalgebra \mathbb{A}' of \mathbb{A} and a proper subset $G \subsetneq A'$.

Definition 4.1.2. Let \mathbb{A} be a finite idempotent algebra, $\mathbb{A}' \leq \mathbb{A}$, $G \subsetneq A'$, and $n \geq 1$. The

n -dimensional cross on A' at G is

$$X_n^{A',G} := \{(a_1, \dots, a_n) \in (\mathbb{A}')^n : \text{there exists } i \text{ such that } a_i \in G\}.$$

In the case that $\mathbb{A}' = \mathbb{A}$, we will simply write X_n^G .

Notice that we will allow the 1-dimensional cross, $X_1^{A',G} = G$. We will often consider higher dimensional crosses where $\mathbb{A} = \mathbb{A}'$.

Theorem 4.1.3 ([MMM10]). *Let \mathbb{A} be a finite idempotent algebra. TFAE.*

- (a) \mathbb{A} has no edge operation.
- (b) There exists $\mathbb{A}' \leq \mathbb{A}$ and a nonempty proper subset $G \subsetneq A'$ such that for all $k \geq 1$ and $f \in \text{Clo}^k(\mathbb{A})$, the restriction $f|_{A'}$ is G -absorbing in its i^{th} variable, for some $1 \leq i \leq k$.

We will show that the second condition of Theorem 4.1.3 can be equivalently stated in terms of the relations $X_n^{A',G}$, for all $n \geq 1$.

Proposition 4.1.4. *Let \mathbb{A} be a finite idempotent algebra. Then for each subalgebra $\mathbb{A}' \leq \mathbb{A}$ and nonempty proper subset $G \subsetneq A'$, TFAE.*

- (a) For all $k \geq 1$ and $f \in \text{Clo}^k(\mathbb{A})$, there exists some $1 \leq i \leq k$ such that $f|_{A'}$ is G -absorbing in its i^{th} variable.
- (b) The term operations of \mathbb{A} preserve the relation $X_n^{A',G}$, for all $n \geq 1$.

Proof. (a) \implies (b) Let $R := X_n^{A',G}$ for arbitrary $1 \leq n < \omega$. For any $k \geq 1$ and $f \in \text{Clo}^k(\mathbb{A})$ we have, by (a), that $f|_{A'}$ is G -absorbing in its i^{th} -variable, for some $1 \leq i \leq k$. WLOG, assume that $f|_{A'}$ is G -absorbing in its first variable. Let $\bar{a}_1, \dots, \bar{a}_k \in R$ where $\bar{a}_j = (a_{j,1}, \dots, a_{j,n})$, $1 \leq j \leq k$. Recall that R is an n -ary relation on A' and $\mathbb{A}' \leq \mathbb{A}$, thus $(A')^n \ni f(\bar{a}_1, \dots, \bar{a}_k) = f|_{A'}(\bar{a}_1, \dots, \bar{a}_k)$, where

$$f|_{A'}(\bar{a}_1, \dots, \bar{a}_k) = f|_{A'} \left(\begin{pmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,n} \end{pmatrix}, \dots, \begin{pmatrix} a_{k,1} \\ a_{k,2} \\ \vdots \\ a_{k,n} \end{pmatrix} \right) = \begin{pmatrix} f|_{A'}(a_{1,1}, \dots, a_{k,1}) \\ f|_{A'}(a_{1,2}, \dots, a_{k,2}) \\ \vdots \\ f|_{A'}(a_{1,n}, \dots, a_{k,n}) \end{pmatrix}.$$

By definition of R , $\bar{a}_1 \in R$ implies that there exists some $1 \leq l \leq n$ such that $a_{1,l} \in G$. Since $f|_{A'}$ is G -absorbing in its first variable and $a_{1,l} \in G$ we get that $f|_{A'}(a_{1,1}, \dots, a_{k,1}) \in G$. Thus, the l^{th} -coordinate of the column vector $f|_{A'}(\bar{a}_1, \dots, \bar{a}_k)$ is in G which means $f(\bar{a}_1, \dots, \bar{a}_k) = f|_{A'}(\bar{a}_1, \dots, \bar{a}_k) \in R$. Therefore, R is closed under the term operations of \mathbb{A} .

To prove (b) \implies (a) we will show that the contrapositive holds. Suppose that there exists an operation $f \in \text{Clo}^k(\mathbb{A})$, for some $k \geq 1$, such that $f|_{A'}$ is not G -absorbing in any of its variables. We will show that $R := X_k^{A',G}$ is not preserved by f .

Because $f|_{A'}$ is not G -absorbing in any variable, we have, for each $1 \leq i \leq k$, that there exists some $g_i \in G$ and $a_{i,1}, \dots, a_{i,i-1}, a_{i,i+1}, \dots, a_{i,k} \in A'$ such that $f|_{A'}(a_{i,1}, \dots, a_{i,i-1}, g_i, a_{i,i+1}, \dots, a_{i,k}) = h_i$, for some $h_i \in A' \setminus G$. It is clear that the tuples

$$\begin{pmatrix} g_1 \\ a_{2,1} \\ \vdots \\ a_{k,1} \end{pmatrix}, \begin{pmatrix} a_{1,2} \\ g_2 \\ \vdots \\ a_{k,2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ g_k \end{pmatrix} \in R \subseteq (A')^k.$$

However,

$$f|_{A'} \left(\begin{pmatrix} g_1 \\ a_{2,1} \\ \vdots \\ a_{k,1} \end{pmatrix}, \begin{pmatrix} a_{1,2} \\ g_2 \\ \vdots \\ a_{k,2} \end{pmatrix}, \dots, \begin{pmatrix} a_{1,k} \\ a_{2,k} \\ \vdots \\ g_k \end{pmatrix} \right) = \begin{pmatrix} f|_{A'}(g_1, a_{1,2}, \dots, a_{1,k}) \\ f|_{A'}(a_{2,1}, g_2, \dots, a_{2,k}) \\ \vdots \\ f|_{A'}(a_{k,1}, a_{k,2}, \dots, g_k) \end{pmatrix} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_k \end{pmatrix},$$

where $h_i \notin G$ for all $1 \leq i \leq k$, therefore $(h_1, h_2, \dots, h_k) \notin R$. Therefore $f|_{A'}$ does not preserve R .

This completes the proof of the proposition. \square

4.2 The Edge Blockers Λ and \mathcal{K}_b

Let \mathbb{A} be a finite idempotent algebra that satisfies Assumption 1.

Definition 4.2.1. For b a fixed element in $A \setminus S$, and for $\sigma \in \text{Aut}(S)$ we define the following subsets

of A^3 :

$$\Lambda_\sigma := S^3 \cup \{(x, y, \sigma(x)) : x \in S, y \in A\},$$

$$\mathcal{K}_{b,\sigma} := \{(x, y, \sigma(x)) : x \in S, y \in A\} \cup \{(x, b, y) : x, y \in S\}.$$

We will write Λ for Λ_{id_S} and \mathcal{K}_b for $\mathcal{K}_{b,\text{id}_S}$, for any $b \in A \setminus S$.

Lemma 4.2.2. *For any $\pi \in \text{Aut}(S)$, $\Lambda_\pi \leq A^3$ if and only if $\Lambda \leq A^3$.*

Proof. Suppose that $\pi \in \text{Aut}(S)$. For $\pi_1 = \text{id}_S: S \rightarrow S$, $\pi_2 = \text{id}_A: A \rightarrow A$, and $\pi_3 = \pi^{-1}: S \rightarrow S$, the product isomorphism, $\prod_{i=1}^3 \pi_i$, maps Λ_π onto Λ . Therefore, by Corollary 2.2.12, $\Lambda_\pi \leq A^3$ if and only if $\Lambda \leq A^3$. \square

Lemma 4.2.3. *For any $\pi \in \text{Aut}(S)$ and $b \in A \setminus S$, $\mathcal{K}_{b,\pi} \leq A^3$ if and only if $\mathcal{K}_b \leq A^3$.*

Proof. The proof is similar to the proof of Lemma 4.2.2 \square

Therefore, by Lemmas 4.2.2 and 4.2.3, whenever Λ_π or $\mathcal{K}_{b,\pi}$ is a subuniverse of A^3 , for some $\pi \in \text{Aut}(S)$, $b \in A \setminus S$, we may assume that π is the identity automorphism of S .

Lemma 4.2.4. *If A satisfies Assumption 1, then at most one of the relations \mathcal{K}_b for some $b \in A \setminus S$ or Λ is a subuniverse of A^3 .*

Proof. For contradiction, first suppose that there exists some distinct $b, b' \in A \setminus S$ such that $\mathcal{K}_b, \mathcal{K}_{b'} \leq A^3$. Let $B := \mathcal{K}_b, B' := \mathcal{K}_{b'}$. Since $|S| > 2$ we have that there exists distinct elements $s, s' \in S$. Then it follows from Definition 4.2.1 that $B(s, x_2, x_3) = \kappa_{b,s}$ is a subuniverse of A^2 and $B'(s', x_1, x_2) = \kappa_{b',s'}$ is a subuniverse of A^2 . Then $\kappa_{b,s}, \kappa_{b',s'} \leq A^2$ implies, by statement (iv) of Proposition 3.2.1 that either $b = b'$ or $s = s'$, which contradicts our assumption that b and b' are distinct and also s and s' are distinct.

Now suppose, for contradiction, that $\mathcal{K}_b, \Lambda \leq A^2$, for some $b \in A \setminus S$. Let $B := \mathcal{K}_b, C := \Lambda$, and let s, s' be distinct elements in S . Then $B(s, x_2, x_3) = \kappa_{b,s}$ is a subuniverse of A^2 and $C(s', x_2, x_3) = \lambda_{S,s'}$ is a subuniverse of A^2 . Hence statement (viii) of Proposition 3.2.2 implies that $s = b$, which contradicts the assumptions that $s \in S$ and $b \in A \setminus S$. This completes the proof of the lemma. \square

4.3 The Existence of an Algebra with a Relation Λ or \mathcal{K}_b

We will now show that there exists an algebra \mathbb{A} that satisfies Assumption 1 and has among the subuniverses of \mathbb{A}^3 either Λ or \mathcal{K}_b .

Proposition 4.3.1. *Let A and S be finite sets such that $S \subseteq A$ and $1 < |S| < |A|$. Let $\hat{\mathbb{S}}$ be a strictly simple idempotent algebra on S . Let*

$$G = \begin{cases} S, \text{ or} \\ \{b\}, \text{ for some } b \in A \setminus S. \end{cases}$$

Let $B := \Lambda$ if $G = S$ and let $B := \mathcal{K}_b$ if $G = \{b\}$. Then there exists a finite idempotent algebra \mathbb{A} on A such that

- \mathbb{A} has a subalgebra \mathbb{S} on S such that $\text{Clo}(\mathbb{S}) = \text{Clo}(\hat{\mathbb{S}})$, and \mathbb{S} is the unique proper nontrivial subalgebra of \mathbb{A} ,
- θ is a congruence on \mathbb{A} , and
- $B \leq \mathbb{A}^3$.

Proof. It is enough to construct a finite idempotent algebra that satisfies these conditions. Let $k \geq 1$. For each operation $f \in \text{Clo}^k(\hat{\mathbb{S}})$ we will define a k -ary operation, F_f , on A by

$$F_f(\bar{x}) = \begin{cases} f(\bar{x}), \text{ if } \bar{x} \in S^k, \\ x, \text{ if } x_1 = \dots = x_k = x, \\ g, \text{ otherwise,} \end{cases}$$

where g is some fixed element of G .

For any $a, a_1 \in A$, $a_2 \in A \setminus G$, such that $a_1/\theta \neq a_2/\theta$, define the binary operation, $f_{(a_1, a_2, a)}$, on A by

$$f_{(a_1, a_2, a)}(x, y) = \begin{cases} a, \text{ if } x \in a_1/\theta, y \in a_2/\theta \\ y, \text{ otherwise.} \end{cases}$$

We claim that the algebra,

$$\mathbb{A} = (A; \{F_f | f \in \text{Clo}(\hat{\mathbb{S}})\} \cup \{f_{(a_1, a_2, a)} | a, a_1 \in A, a_2 \in A \setminus G, \text{ and } a_1/\theta \neq a_2/\theta\})$$

satisfies the statements of the proposition.

From their definitions, it is clear that F_f and $f_{(a_1, a_2, a)}$ are idempotent operations. Since $F_f|_S = f \in \text{Clo}(\hat{\mathbb{S}})$ and $f_{(a_1, a_2, a)}|_S$ is the projection onto the second variable, it is trivial to see that these operations preserve S , and that the subalgebra \mathbb{S} of \mathbb{A} on S has the same clone as $\hat{\mathbb{S}}$.

We claim that \mathbb{S} is the unique nontrivial proper subalgebra of the algebra \mathbb{A} . Suppose not. Then there exists some $\mathbb{Q} \leq \mathbb{A}$, $\mathbb{Q} \neq A$, $\mathbb{Q} \neq S$, and $|\mathbb{Q}| > 1$. Recall that \mathbb{S} is strictly simple, therefore \mathbb{S} has no nontrivial proper subalgebras, which means \mathbb{Q} is clearly not a proper subset of S . Therefore, there exist distinct elements $q_1, q_2 \in \mathbb{Q}$ such that $q_1/\theta \neq q_2/\theta$. This means, since $G = S$ or $\{b\}$, that $\{q_1, q_2\} \cap A \setminus G \neq \emptyset$. WLOG, suppose that $q_2 \in A \setminus G$. Let $a \in A \setminus \mathbb{Q}$, such an element exists since $\mathbb{Q} \subsetneq A$. Then $f_{(q_1, q_2, a)}(q_1, q_2) = a \notin \mathbb{Q}$, which contradicts the assumption that \mathbb{Q} is a subuniverse of \mathbb{A} . Hence \mathbb{S} is the unique nontrivial proper subalgebra of \mathbb{A} .

We will now show that the operations F_f and $f_{(a_1, a_2, a)}$ preserve θ and B .

Claim 4.3.1.1. *Let $k \geq 1$, $f \in \text{Clo}^k(\hat{\mathbb{S}})$. Then the k -ary operation F_f preserves θ .*

Proof. Let $f \in \text{Clo}^k(\hat{\mathbb{S}})$. Suppose that $\bar{x}, \bar{y} \in A^k$ such that $\bar{x}\theta\bar{y}$. We must show that $F_f(\bar{x})\theta F_f(\bar{y})$. By the definition of θ , it is easy to see that $\bar{x}\theta\bar{y}$ implies that

$$\bar{x} \in S^n \Leftrightarrow \bar{y} \in S^n.$$

Therefore, either $\bar{x}, \bar{y} \in S^k$ or $\bar{x}, \bar{y} \notin S^k$. If $\bar{x}, \bar{y} \in S^k$, then we get that $F_f(\bar{x}) = f(\bar{x}) \in S$ and $F_f(\bar{y}) = f(\bar{y}) \in S$, thus $F_f(\bar{x})\theta F_f(\bar{y})$.

Suppose that $\bar{x}, \bar{y} \notin S^n$. Again, by the definition of θ , it is easy to see that $\bar{x}\theta\bar{y}$ implies that

$$x_1 = \cdots = x_n = x \in A \setminus S \Leftrightarrow y_1 = \cdots = y_n = x \in A \setminus S.$$

Hence if $x_1 = \cdots = x_n = x \in A \setminus S$ and $y_1 = \cdots = y_n = x \in A \setminus S$, then $F_f(\bar{x}) = x = F_f(\bar{y})$. Thus $F_f(\bar{x})\theta F_f(\bar{y})$.

Finally, suppose that $\bar{x}, \bar{y} \notin S^n$ and $x_i \neq x_j$, for some $1 \leq i < j \leq k$. Then, as we saw in the preceding paragraph, $\bar{x}\theta\bar{y}$ implies that $y_i \neq y_j$ for the same $1 \leq i < j \leq k$. By the definition of F_f we get that $F_f(\bar{x}) = g = F_f(\bar{y})$. Therefore, in all cases, $F_f(\bar{x})\theta F_f(\bar{y})$, hence F_f preserves θ . \square

Claim 4.3.1.2. *Let $k \geq 1$, $f \in \text{Clo}^k(\hat{S})$. Then the k -ary operation F_f preserves B .*

Proof. Let $f \in \text{Clo}^k(\hat{S})$, $\bar{u}_1, \dots, \bar{u}_k \in B$, where $\bar{u}_i = (u_{i,1}, u_{i,2}, u_{i,3})$, for $1 \leq i \leq k$. By the definition of B , we have that $u_{i,1}, u_{i,3} \in S$, for all $1 \leq i \leq k$. Then, for some $s_1, s_2 \in S$,

$$\begin{aligned} F_f(\bar{u}_1, \dots, \bar{u}_k) &= F_f \left(\left(\begin{array}{c} u_{1,1} \\ u_{1,2} \\ u_{1,3} \end{array} \right), \dots, \left(\begin{array}{c} u_{k,1} \\ u_{k,2} \\ u_{k,3} \end{array} \right) \right) \\ &= \left(\begin{array}{c} F_f(u_{1,1}, \dots, u_{k,1}) \\ F_f(u_{1,2}, \dots, u_{k,2}) \\ F_f(u_{1,3}, \dots, u_{k,3}) \end{array} \right) \\ &= \left(\begin{array}{c} s_1 \\ F_f(u_{1,2}, \dots, u_{k,2}) \\ s_2 \end{array} \right). \end{aligned}$$

If $F_f(u_{1,2}, \dots, u_{k,2}) \in G$, then by the definition of B it is clear that $F_f(\bar{u}_1, \bar{u}_2, \bar{u}_3) \in B$. Let us suppose that $F_f(u_{1,2}, \dots, u_{k,2}) \notin G$. We claim that $F_f(u_{1,2}, \dots, u_{k,2}) \in A \setminus G$ implies $u_{i,1} = u_{i,3}$, for all $1 \leq i \leq k$. For contradiction, suppose that there exists some $1 \leq i \leq k$ such that $u_{i,1} \neq u_{i,3}$. WLOG, suppose that $i = 1$. Then $(u_{1,1}, u_{1,2}, u_{1,3}) = \bar{u}_1 \in B$, $u_{1,1} \neq u_{1,3}$ implies that $u_{1,2} = g'$, for some $g' \in G$. Thus, for some $s' \in S$,

$$F_f(u_{1,2}, u_{2,2}, \dots, u_{k,2}) = F_f(g', u_{2,2}, \dots, u_{k,2}) = \begin{cases} s', & \text{if } g', u_{2,2}, \dots, u_{k,2} \in S \\ g', & \text{if } g' = u_{2,2} = \dots = u_{k,2} \\ g, & \text{otherwise.} \end{cases}$$

Recall that $g, g' \in G$, and observe that if the first case occurs, then $g' \in S$ implies that $G = S \ni s'$. Thus, in all cases we get that $G \ni F_f(g', u_{2,2}, \dots, u_{k,2}) = F_f(u_{1,2}, u_{2,2}, \dots, u_{k,2})$, which

contradict the assumption that $F_f(u_{1,2}, \dots, u_{k,2}) \in A \setminus G$. Therefore, $F_f(u_{1,2}, \dots, u_{k,2}) \in A \setminus G$ implies $u_{i,1} = u_{i,3}$, for all $1 \leq i \leq k$, which means $s_1 = F_f(u_{1,1}, \dots, u_{k,1}) = F_f(u_{1,3}, \dots, u_{k,3}) = s_2$. Hence $F_f(\bar{u}_1, \bar{u}_2, \bar{u}_3) = (s_1, F_f(u_{1,2}, \dots, u_{k,2}), s_2) = (s_1, F_f(u_{1,2}, \dots, u_{k,2}), s_1) \in B$. In all cases we get that $F_f(\bar{u}_1, \bar{u}_2, \bar{u}_3) \in B$, thus F_f preserves B . This completes the proof of the claim. \square

Claim 4.3.1.3. *Let $a, a_1 \in A$, $a_2 \in A \setminus G$, such that $a_1/\theta \neq a_2/\theta$. Then $f_{(a_1, a_2, a)}$ preserves θ .*

Proof. Let $x, x', y, y' \in A$ such that $x\theta x'$ and $y\theta y'$. We must show that $f_{(a_1, a_2, a)}(x, y)\theta f_{(a_1, a_2, a)}(x', y')$. Either $x \in a_1/\theta$, $y \in a_2/\theta$ or not.

First suppose that $x \in a_1/\theta$ and $y \in a_2/\theta$. Then $x\theta x'$ and $y\theta y'$ implies that $x' \in a_1/\theta$ and $y' \in a_2/\theta$. Hence $f_{(a_1, a_2, a)}(x, y) = a = f_{(a_1, a_2, a)}(x', y')$.

Now suppose that either $x \notin a_1/\theta$ or $y \notin a_2/\theta$. Then $x\theta x'$ and $y\theta y'$ implies that either $x' \notin a_1/\theta$ or $y' \notin a_2/\theta$, respectively. Thus $f_{(a_1, a_2, a)}(x, y) = y$ and $f_{(a_1, a_2, a)}(x', y') = y'$. Since $y\theta y'$, we get that $f_{(a_1, a_2, a)}(x, y)\theta f_{(a_1, a_2, a)}(x', y')$. Therefore $f_{(a_1, a_2, a)}$ preserves θ . \square

Claim 4.3.1.4. *Let $a, a_1 \in A$, $a_2 \in A \setminus G$, such that $a_1/\theta \neq a_2/\theta$. Then $f_{(a, a_1, a_2)}$ preserves B .*

Proof. Let $\bar{u}, \bar{v} \in B \subseteq A^3$. Then, for $i = 1, 3$, $u_i, v_i \in S$ which means $u_i\theta v_i$ and, by the definition of $f_{(a_1, a_2, a)}$, we get that $f_{(a_1, a_2, a)}(u_i, v_i) = v_i$. Therefore,

$$f_{(a_1, a_2, a)}(\bar{u}, \bar{v}) = f_{(a_1, a_2, a)}\left(\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}\right) = \begin{pmatrix} f_{(a_1, a_2, a)}(u_1, v_1) \\ f_{(a_1, a_2, a)}(u_2, v_2) \\ f_{(a_1, a_2, a)}(u_3, v_3) \end{pmatrix} = \begin{pmatrix} v_1 \\ f_{(a_1, a_2, a)}(u_2, v_2) \\ v_3 \end{pmatrix}.$$

Furthermore,

$$f_{(a_1, a_2, a)}(u_2, v_2) = \begin{cases} a, & \text{if } u_2 \in a_1/\theta \text{ and } v_2 \in a_2/\theta \\ v_2, & \text{otherwise.} \end{cases}$$

If $f_{(a_1, a_2, a)}(u_2, v_2) = v_2$, then $f_{(a_1, a_2, a)}(\bar{u}, \bar{v}) = \bar{v} \in B$. Suppose that $f_{(a_1, a_2, a)}(u_2, v_2) = a$. Then $v_2 \in a_2/\theta$ and $a_2 \in A \setminus G$ implies that $v_2 \in A \setminus G$. Since $\bar{v} \in B$, it follows from $v_2 \in A \setminus G$ that $v_1 = v_3$. Hence $f_{(a_1, a_2, a)}(\bar{u}, \bar{v}) = (v_1, f_{(a_1, a_2, a)}(u_2, v_2), v_3) = (v_1, f_{(a_1, a_2, a)}(u_2, v_2), v_1) \in B$. Therefore $f_{(a_1, a_2, a)}$ preserves B . This completes the proof of the claim. \square

It follows from claims 4.3.1.1 and 4.3.1.3 that θ is a congruence on \mathbb{A} . Furthermore, from claims 4.3.1.2 and 4.3.1.4, we get that B is a subuniverse of \mathbb{A}^3 . \square

We now state a version of Proposition 4.3.1 where, under the added assumption that clone of $\hat{\mathbb{S}}$ is finitely related, we prove that \mathbb{A} can be chosen so that its clone is finitely related. Recall, from Proposition 2.4.8 that there exist finite idempotent strictly simple algebras that have finitely related relational clones, thus finitely related clones.

Corollary 4.3.2. *Let A and S be finite sets such that $S \subseteq A$ and $1 < |S| < |A|$. Let $\hat{\mathbb{S}}$ be a strictly simple idempotent algebra on S such that the clone of $\hat{\mathbb{S}}$ is finitely related. Let*

$$G = \begin{cases} S, \text{ or} \\ \{b\}, \text{ for some } b \in A \setminus S. \end{cases}$$

Let $B := \Lambda$ if $G = S$ and let $B := \mathcal{K}_b$ if $G = \{b\}$. Then there exists a finite idempotent algebra \mathbb{A} on A such that

- \mathbb{A} has a subalgebra \mathbb{S} on S such that $\text{Clo}(\mathbb{S}) = \text{Clo}(\hat{\mathbb{S}})$, and \mathbb{S} is the unique proper nontrivial subalgebra of \mathbb{A} ,
- θ is a congruence on \mathbb{A} ,
- $B \leq \mathbb{A}^3$, and
- the clone of \mathbb{A} is finitely related.

Proof. Since the clone of $\hat{\mathbb{S}}$ is finitely related, there exists finitely many relations $\sigma_1, \dots, \sigma_t$ that determine the clone of $\hat{\mathbb{S}}$. Let $\mathbb{A} = (A; \mathcal{F})$, where \mathcal{F} is the set of all operations that preserve every relation in $\mathcal{R} := \{\{a\} : a \in A\} \cup \{S, \theta, B, \sigma_1, \dots, \sigma_t\}$. Then the clone of \mathbb{A} is determined by \mathcal{R} , so it is finitely related. Also, the fact that the relations $\{a\}$ for all $a \in A$, S , θ , and B belong to \mathcal{R} implies that the algebra \mathbb{A} is idempotent, S is a subuniverse of \mathbb{A} , θ is a congruence on \mathbb{A} , and $B \leq \mathbb{A}^3$. The fact that the relations $\sigma_1, \dots, \sigma_t$ belong to \mathcal{R} forces that for the the subalgebra \mathbb{S} of \mathbb{A} on S

we have that $\text{Clo}(\mathbb{S}) \subseteq \text{Clo}(\hat{\mathbb{S}})$. This proves all required properties, except that (i) $\text{Clo}(\mathbb{S}) \supseteq \text{Clo}(\hat{\mathbb{S}})$, and that (ii) \mathbb{S} is the unique nontrivial proper subalgebra of \mathbb{A} .

For the proof of (i) and (ii) we will make use of the algebra

$$(A; \{\{F_f | f \in \text{Clo}(\hat{\mathbb{S}})\} \cup \{f_{(a_1, a_2, a)} | a, a_1 \in A, a_2 \in A \setminus G, \text{ and } a_1/\theta \neq a_2/\theta\}\})$$

constructed in the proof of Proposition 4.3.1, which we will call now \mathbb{A}' . It was shown in Proposition 4.3.1 that every operation of \mathbb{A}' is idempotent and preserves S , θ , and B . Furthermore, it was shown that $\text{Clo}(\mathbb{S}') = \text{Clo}(\hat{\mathbb{S}})$ holds for the subalgebra \mathbb{S}' of \mathbb{A}' on S . These properties imply that every operation of \mathbb{A}' preserves all relations in \mathcal{R} , and hence is an operation of \mathbb{A} . Thus $\text{Clo}(\mathbb{A}') \subseteq \text{Clo}(\mathbb{A})$ and $\text{Clo}(\mathbb{S}') \subseteq \text{Clo}(\mathbb{S})$. The second inclusion, together with the equality $\text{Clo}(\mathbb{S}') = \text{Clo}(\hat{\mathbb{S}})$ implies (i). The first inclusion implies that every subalgebra of \mathbb{A} is a subalgebra of \mathbb{A}' , therefore (ii) follows from the analogous property of \mathbb{A}' . \square

4.4 Algebras with Small Arity Edge Blockers

Throughout this section we will assume that \mathbb{A} satisfies Assumption 1. In this section we will show that if one of Λ or \mathcal{K}_b for some $b \in A \setminus S$ is a subuniverse of \mathbb{A}^3 , then \mathbb{A} does not have an edge operation. In fact, we will show that \mathbb{A} does not belong to a congruence modular variety. In the case when the subalgebra \mathbb{S} is affine, we will show that the same conclusions follow even if \mathbb{A} satisfies the weaker assumption that one of the binary crosses $\lambda_{S,s}$ or $\kappa_{b,s}$ is a subuniverse of \mathbb{A}^2 for some $s \in S$, $b \in A \setminus S$.

Lemma 4.4.1. *If \mathbb{S} is a simple affine algebra and $0 \in S$ is the additive identity of the vector space associated to \mathbb{S} , then for arbitrary $s \in S$, $b \in A \setminus S$,*

$$(i) \quad \lambda_{S,s} \leq \mathbb{A}^2 \text{ implies that } \lambda_{S,0} \leq \mathbb{A}^2.$$

$$(ii) \quad \kappa_{b,s} \leq \mathbb{A}^2 \text{ implies that } \kappa_{b,0} \leq \mathbb{A}^2.$$

Proof. If \mathbb{S} is a simple affine algebra, then we have from statement (i) of Proposition 2.4.8 that the automorphisms of \mathbb{S} are in the relational clone of \mathbb{S} , thus they are compatible relations, therefore,

for any $s \in S$, the translation $\pi(x) = x - s$ of the vector space associated to \mathbb{S} is an automorphism of \mathbb{S} . Then by statement (iv) of Proposition 3.2.4, if $\lambda_{S,s} \leq \mathbb{A}^2$, then $\lambda_{S,0} = \lambda_{S,\pi(s)} \leq \mathbb{A}^2$, which proves (i). Similarly, by statement (iii) of Proposition 3.2.4, if $\kappa_{b,s} \leq \mathbb{A}^2$, then $\kappa_{b,0} = \kappa_{b,\pi(s)} \leq \mathbb{A}^2$, which proves (ii). \square

If \mathbb{S} is a simple affine algebra, then ν_0 is not a subuniverse of \mathbb{S}^2 . Therefore it follows from statements (v) and (vii) of Proposition 3.2.2 that at most one of the crosses $\lambda_{S,0}$ and $\kappa_{b,0}$ ($b \in A \setminus S$) is a subuniverse of \mathbb{A}^2 . Hence, Lemma 4.4.1 implies that if $\lambda_{S,s} \leq \mathbb{A}^2$ for some $s \in S$, then $\kappa_{b,t} \not\leq \mathbb{A}^2$ for all $b \in A \setminus S$, $t \in S$. Similarly, if $\kappa_{b,s} \leq \mathbb{A}^2$ for some $b \in A \setminus S$, $s \in S$, then $\kappa_{b',t} \not\leq \mathbb{A}^2$ for all $b' \neq b$, $b' \in A \setminus S$, $t \in S$.

Proposition 4.4.2. *Suppose that \mathbb{S} is affine and either $\lambda_{S,s} \leq \mathbb{A}^2$ or $\kappa_{b,s} \leq \mathbb{A}^2$, for some $s \in S$, $b \in A \setminus S$. Let $G = S$, if $\lambda_{S,s} \leq \mathbb{A}^2$, and let $G = \{b\}$, if $\kappa_{b,s} \leq \mathbb{A}^2$. Let $k \geq 1$, $f \in \text{Clo}^k(\mathbb{A})$, and $i \in \bar{k}$. If f is not G -absorbing in its i^{th} variable, then $f|_S$ does not depend on its i^{th} variable.*

Proof. Under the assumptions of the proposition, we have that either $\lambda_{S,s} \leq \mathbb{A}^2$ or $\kappa_{b,s} \leq \mathbb{A}^2$. Then, by Lemma 4.4.1, we have that either $\kappa_{b,0} \leq \mathbb{A}^2$ or $\lambda_{S,0} \leq \mathbb{A}^2$, respectively, where 0 is the additive identity the vector space associated to \mathbb{S} . By the remark preceding the proposition, exactly one of the crosses $\lambda_{S,0}$, $\kappa_{b,0}$, for some $b \in A \setminus S$, is a subuniverse of \mathbb{A}^2 . Hence G , as defined in the proposition, is uniquely determined. Let,

$$B := \begin{cases} \lambda_{S,0}, & \text{if } G = S \\ \kappa_{b,0}, & \text{if } G = \{b\}. \end{cases}$$

Then B is a subuniverse of \mathbb{A}^2 . Let $k \geq 1$, $f \in \text{Clo}^k(\mathbb{A})$ and $i \in \bar{k}$. Suppose that f is not G -absorbing in its i^{th} variable. WLOG, suppose that $i = 1$. Then there exists some $g \in G$, $a_2, \dots, a_n \in A$ such that $f(g, a_2, \dots, a_n) = h$, where $h \in A \setminus G$. Note that $B \supseteq \{g\} \times S \cup A \times \{0\}$. Therefore, for any $t \in S$,

$$B \ni \begin{pmatrix} g \\ t \end{pmatrix}, \begin{pmatrix} a_2 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} a_k \\ 0 \end{pmatrix},$$

and, since $f \in \text{Clo}(\mathbb{A})$ and $B \leq \mathbb{A}^2$, we get that

$$B \ni f \left(\begin{pmatrix} g \\ t \end{pmatrix}, \begin{pmatrix} a_2 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} a_k \\ 0 \end{pmatrix} \right) = \begin{pmatrix} f(g, a_2, \dots, a_k) \\ f(t, 0, \dots, 0) \end{pmatrix} = \begin{pmatrix} h \\ f(t, 0, \dots, 0) \end{pmatrix}.$$

Since $t, 0 \in S$ we have that $f(t, 0, \dots, 0) = f|_S(t, 0, \dots, 0)$. Recall that \mathbb{S} is an idempotent affine subalgebra of \mathbb{A} , therefore $f|_S$ is a term operation on \mathbb{S} and thus $f|_S(x_1, \dots, x_k) = \sum_{i=1}^k \alpha_i x_i$, for some endomorphism α_i of the vector space associated to \mathbb{S} , $1 \leq i \leq k$, where $\sum_{i=1}^k \alpha_i = 1$. Thus $f(t, 0, \dots, 0) = f|_S(t, 0, \dots, 0) = \alpha_1 t$ and $B \ni \begin{pmatrix} h \\ f(t, 0, \dots, 0) \end{pmatrix} = \begin{pmatrix} h \\ \alpha_1 t \end{pmatrix}$, which means, since $h \notin G$, that $\alpha_1 t = 0$. Since t was an arbitrary element of S , $|S| > 2$, it follows that α_1 is the zero endomorphism. Thus $f|_S(x_1, \dots, x_k) = \sum_{i=2}^k \alpha_i x_i$, hence $f|_S$ does not depend on its first variable. \square

In the next proposition we will make no assumption on the subalgebra \mathbb{S} of \mathbb{A} , and will prove that the conclusions of the previous proposition hold if we assume that one of the relations Λ or \mathcal{K}_b , for some $b \in A \setminus S$, is a subuniverse of \mathbb{A}^3 . Recall from Proposition 4.2.4 that no two of these relations can simultaneously be subuniverses of \mathbb{A}^3 .

Proposition 4.4.3. *Suppose that either $\Lambda \leq \mathbb{A}^3$ or $\mathcal{K}_b \leq \mathbb{A}^3$, for some $b \in A \setminus S$. Let $G = S$, if $\Lambda \leq \mathbb{A}^3$, and let $G = \{b\}$, if $\mathcal{K}_b \leq \mathbb{A}^3$. Let $k \geq 1$, $f \in \text{Clo}^k(\mathbb{A})$, and $i \in \bar{k}$. If f is not G -absorbing in its i^{th} variable, then $f|_S$ does not depend on its i^{th} variable.*

Proof. Under the assumptions of the proposition, we have that either $\Lambda \leq \mathbb{A}^3$ or $\mathcal{K}_b \leq \mathbb{A}^3$ for some $b \in A \setminus S$. It follows from our remark preceding the proposition that no two of these relations can simultaneously be subuniverses of \mathbb{A}^3 . Thus G , as defined in the proposition, is uniquely determined.

Let,

$$B := \begin{cases} \Lambda, & \text{if } G = S \\ \mathcal{K}_b, & \text{if } G = \{b\}. \end{cases}$$

Then B is a subuniverse of \mathbb{A}^3 . Let $k \geq 1$, $f \in \text{Clo}^k(\mathbb{A})$ and $i \in \bar{k}$. Suppose that f is not G -absorbing in its i^{th} variable. WLOG, suppose that $i = 1$. Then there exists some $g \in G$, $a_2, \dots, a_n \in A$ such

that $f(g, a_2, \dots, a_n) = h$, where $h \in A \setminus G$. Note that $B \supseteq (S \times \{g\} \times S) \cup \{(x, y, x) : x \in S, y \in A\}$.

Therefore, for any $t, t', s_2, \dots, s_k \in S$,

$$B \ni \begin{pmatrix} t \\ g \\ t' \end{pmatrix}, \begin{pmatrix} s_2 \\ a_2 \\ s_2 \end{pmatrix}, \dots, \begin{pmatrix} s_k \\ a_k \\ s_k \end{pmatrix},$$

and, since $f \in \text{Clo}(\mathbb{A})$ and $B \leq \mathbb{A}^3$, we get that

$$B \ni f \left(\begin{pmatrix} t \\ g \\ t' \end{pmatrix}, \begin{pmatrix} s_2 \\ a_2 \\ s_2 \end{pmatrix}, \dots, \begin{pmatrix} s_k \\ a_k \\ s_k \end{pmatrix} \right) = \begin{pmatrix} f(t, s_2, \dots, s_k) \\ f(g, a_2, \dots, a_k) \\ f(t', s_2, \dots, s_k) \end{pmatrix} = \begin{pmatrix} f(t, s_2, \dots, s_k) \\ h \\ f(t', s_2, \dots, s_k) \end{pmatrix}.$$

Then $\begin{pmatrix} f(t, s_2, \dots, s_k) \\ h \\ f(t', s_2, \dots, s_k) \end{pmatrix} \in B$, where $h \in A \setminus G$, which implies that $f(t, s_2, \dots, s_k) = f(t', s_2, \dots, s_k)$.

The elements t, t', s_2, \dots, s_k were arbitrarily chosen from S , hence $f|_S$ does not depend on its 1st-variable. \square

Proposition 4.4.4. *Let $s \in S$, $b \in A \setminus S$. Suppose that one of the following conditions holds.*

(I) \mathbb{S} is affine and either $\lambda_{S,s} \leq \mathbb{A}^2$ or $\kappa_{b,s} \leq \mathbb{A}^2$, or

(II) $\Lambda \leq \mathbb{A}^3$ or $\mathcal{K}_b \leq \mathbb{A}^3$.

Let $G = S$, if $\lambda_{S,s} \leq \mathbb{A}^2$ or $\Lambda \leq \mathbb{A}^3$. Let $G = \{b\}$, if $\kappa_{b,s} \leq \mathbb{A}^2$ or $\mathcal{K}_b \leq \mathbb{A}^3$. Then the following statements hold.

(i) If $k \geq 1$ and $f \in \text{Clo}^k(\mathbb{A})$, then there exists some $1 \leq i \leq k$ such that f is G -absorbing in its i^{th} -variable.

(ii) The clone of \mathbb{A} preserves X_n^G , for all $n \geq 1$.

(iii) \mathbb{A} does not have an edge operation.

(iv) *The variety generated by \mathbb{A} is not congruence modular.*

Proof. Let $s \in S$, $b \in A \setminus S$. Suppose that one of conditions (I) or (II) holds.

[(i)] Let $k \geq 1$ and $f \in \text{Clo}^k(\mathbb{A})$. Then f is an idempotent term operation on \mathbb{A} , which means $f|_S$ is idempotent, therefore there exists at least one $1 \leq i \leq k$ such that $f|_S$ is dependent on its i^{th} -variable. Then, if condition (I) or (II) holds, we get from Propositions 4.4.2 and 4.4.3, respectively, that f is G -absorbing in its i^{th} -variable. This completes the proof of statement (i).

[(ii)] This follows directly from statement (i) and Proposition 4.1.4, where $A = A'$. This completes the proof of statement (ii).

[(iii)] From statement (ii) we have that the idempotent operations on \mathbb{A} preserve X_n^G , for arbitrary n , this yields, from Proposition 4.1.4, that statement (a) of Proposition 4.1.4 holds which in turn implies that statement (b) of Theorem 4.1.3 holds. Then we can conclude from Theorem 4.1.3 that statement (a) of Theorem 4.1.3 holds, hence \mathbb{A} does not have an edge operation.

[(iv)] Suppose, for contradiction, that the variety generated by \mathbb{A} is congruence modular. Then, by Theorem 2.1.3, for some $n \geq 0$, there exists term operations $d_0, \dots, d_n, p \in \text{Clo}^3(\mathbb{A})$ such that identities (1) - (4) of Theorem 2.1.3 hold. Then $d_0|_S, \dots, d_n|_S, p|_S$ also satisfy identities (1) - (4) of Theorem 2.1.3.

Claim 4.4.4.1. *The term operation $d_i|_S$ does not depend on its second variable, for any $0 \leq i \leq n$.*

Proof. Let $g \in G$ and $h \in A \setminus G$. Then by identity (1) of Theorem 2.1.3, for any $0 \leq i \leq n$, we get that $d_i(h, g, h) = h \in A \setminus G$. Thus d_i is not G -absorbing in its second variable. We are assuming condition (I) or (II) holds, therefore it follows from Proposition 4.4.2 or 4.4.3, respectively, that $d_i|_S$ does not depend on its second variable, for any $0 \leq i \leq n$. \square

Then $d_i|_S(x, y, z) = d_i|_S(x, y', z)$, for arbitrary $x, y, y', z \in S$.

Claim 4.4.4.2. *The term operation $p|_S$ does not depend on its first or second variable.*

Proof. Let $g \in G$ and $h \in A \setminus G$. Then by identity (4) of Theorem 2.1.3 we get that $p(g, g, h) = h \in A \setminus S$. Hence p is not G -absorbing in its first or second variable. We are assuming condition (I)

or (II) holds, therefore it follows from Proposition 4.4.2 or 4.4.3, respectively, that $p|_S$ does not depend on its first or second variable. \square

Then $p|_S(x, y, z) = p|_S(x', y', z)$, for arbitrary $x, x', y, y', z \in S$.

Claim 4.4.4.3. For all $1 \leq i < n$, $d_i|_S(x, y, z) = d_{i+1}|_S(x, y, z)$, for arbitrary $x, y, z \in S$.

Proof. Let $x, y, z \in S$ be arbitrary. If $i < n$ is even, then by Claim 4.4.4.1 and identity (2) of Theorem 2.1.3 we get that $d_i|_S(x, y, z) = d_i|_S(x, z, z) = d_{i+1}|_S(x, z, z) = d_{i+1}|_S(x, y, z)$. If $i < n$ is odd, then by Claim 4.4.4.1 and identity (3) of Theorem 2.1.3 we get that $d_i|_S(x, y, z) = d_i|_S(x, x, z) = d_{i+1}|_S(x, x, z) = d_{i+1}|_S(x, y, z)$. This completes the proof of the claim. \square

From identity (1) of Theorem 2.1.3 we have that $d_0(x, y, z) = x$. Thus, it follows from repeated application of Claim 4.4.4.3 that $d_n(x, y, z) = x$. This, together with identity (4) of Theorem 2.1.3 and Claims 4.4.4.1 and 4.4.4.2 imply that

$$x = d_n|_S(x, y, z) = d_n|_S(x, z, z) = p|_S(x, z, z) = p|_S(x, x, z) = z,$$

for arbitrary $x, y, z \in S$. Since $|S| > 2$, the statement $x = z$, for all $x, z \in S$, gives a contradiction and completes the proof of statement (iv). This completes the proof of the proposition. \square

Corollary 4.4.5. *There exists a finite idempotent algebra \mathbb{A} that has a congruence θ such that the congruence classes, as algebras, generate congruence modular varieties, and \mathbb{A}/θ generates a congruence modular variety, but \mathbb{A} does not generate a congruence modular variety.*

Proof. Let A and S be finite sets such that $S \subseteq A$ and $2 < |S| < |A|$. Let \mathbb{S} be a strictly simple idempotent algebra on S . From Proposition 4.3.1 we have that there exists an algebra \mathbb{A} such that \mathbb{S} is the unique proper nontrivial subalgebra on \mathbb{A} , θ is a congruence on \mathbb{A} , \mathbb{A}/θ is strictly simple, and $B = \Lambda$ or $B = \mathcal{K}_b$ is a subuniverse of \mathbb{A}^3 , for some fixed $b \in A \setminus S$. Then the congruence classes of \mathbb{A} are $\mathbb{S}, \{\{a\} : a \in A\}$. It is clear that the one-element algebras generate a congruence modular variety. Furthermore, under these assumptions, statement (iv) of Proposition 4.4.4 implies that the variety generated by \mathbb{A} is not congruence modular. Finally, since \mathbb{S} and \mathbb{A}/θ are idempotent

strictly simple algebras with more than two elements, it follows from Corollary 2.4.11 that \mathbb{S} and \mathbb{A}/θ generate a congruence modular variety. This completes the proof of the corollary. \square

Finally, recall that in Corollary 4.3.2 we constructed finite idempotent algebras \mathbb{A} satisfying Assumption 1 and condition (II) from Proposition 4.4.4 such that the clone of \mathbb{A} is finitely related. Thus the conclusions of Proposition 4.4.4 hold for \mathbb{A} ; in particular, \mathbb{A} has no edge operation, and also the variety generated by \mathbb{A} is not congruence modular. These algebras \mathbb{A} lend support to the following conjecture of M. Valeriote.

Conjecture 4.4.6 ([MMM10]). Let \mathbb{A} be a finite idempotent algebra. If $\text{Clo}(\mathbb{A})$ is finitely related and \mathbb{A} generates a congruence modular variety, then \mathbb{A} has an edge operation.

Chapter 5

The Clone of \mathbb{A} When \mathbb{S} is Quasiprimal or Affine

Throughout this chapter \mathbb{A} will denote an algebra that satisfies Assumption 1.

In this chapter we will describe the clone of \mathbb{A} when θ is a congruence of \mathbb{A} , \mathbb{S} is either quasiprimal or affine, and no small arity edge blockers occur in the relational clone of \mathbb{A} . If \mathbb{S} is quasiprimal, we will also assume that the subuniverses of \mathbb{A}^2 have a θ -closure property that we will define below. We accomplish this description by finding a transparent generating set for the relational clone of \mathbb{A} and then using the Galois connection from Section 2.3 to describe the clone of \mathbb{A} .

There are three integral steps to finding a generating set for the relational clone of \mathbb{A} . The first is a reduction step that allows us to describe a relation in the relational clone of \mathbb{A} by isomorphisms between subalgebras and quotients of \mathbb{A} and higher dimensional analogs of such isomorphisms, and by a smaller arity relation that cannot be further decomposed in this way, and will therefore be called reduced. This means that later on, it suffices to focus on the reduced subuniverses of finite powers of \mathbb{A} . Secondly, we show that the reduced subuniverses of all finite powers of \mathbb{A} satisfy the θ -closure property mentioned above. This allows us to recover a subuniverse from its image under a quotient map. In the third step we describe the image of such a subuniverse, and use this description to find a generating set for the relational clone of \mathbb{A} .

Definition 5.0.7. Let B be a subuniverse of \mathbb{A}^n , $1 \leq i \leq n$. We will call the i^{th} -coordinate of B an A -coordinate if $\text{pr}_i B = A$.

In the next definition we use terminology from Definition 2.2.7 and Proposition 2.2.8

Definition 5.0.8. Let $n \geq 1$. We will say that a subuniverse B of \mathbb{A}^n is θ -closed in its A -coordinates if B is $\theta_1 \times \cdots \times \theta_n$ -closed, where $\theta_i = \theta$ for all $i \in \bar{n}$ such that the i^{th} -coordinate of B is an A -coordinate and θ_i is the equality relation otherwise.

Thus, the θ -closure property that we alluded to above is the property that a reduced subuniverse of a finite power of \mathbb{A} is θ -closed in its A -coordinates. In our exploration of the subuniverses of finite powers of \mathbb{A} , we discovered a family of relations whose members do not satisfy this property. We call these relations *higher dimensional automorphism* and we will define them here.

Definition 5.0.9. Let \mathbb{G} be a finite, idempotent, strictly simple affine algebra, and let $B \leq_{s.d} \mathbb{G}^n$. We will call B a *higher dimensional automorphism*, or *h.d.-automorphism*, of \mathbb{G} , if $n \geq 3$ and B satisfies the following conditions:

- (i) for every $i \in \bar{n}$, and for every $(n-1)$ -tuple, $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \text{pr}_{1, \dots, i-1, i+1, \dots, n} B$ there exists a unique element $x_i \in \text{pr}_i B$ such that $(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \in B$, and
- (ii) no projection $\text{pr}_I B$ of B where I is a proper subset of \bar{n} has this property.

If \mathbb{A} satisfies Assumption 1 and \mathbb{S} is affine, then a subuniverse $B \leq_{s.d} \mathbb{A}^n$ will be called a *higher dimensional automorphism*, or *h.d.-automorphism*, of \mathbb{A} , if $n \geq 3$, $B|_{\mathbb{S}}$ is an h.d.-automorphism of \mathbb{S} , and B satisfies condition (i) above.

For a finite, idempotent strictly simple algebra \mathbb{G} let $\text{Aut}_{h.d.}(\mathbb{G})$ denote the set of h.d.-automorphism of \mathbb{G} if \mathbb{G} is affine, and let $\text{Aut}_{h.d.}(\mathbb{G}) = \emptyset$ otherwise. Similarly, if \mathbb{A} satisfies Assumption 1, let $\text{Aut}_{h.d.}(\mathbb{A})$ denote the set of h.d.-automorphism of \mathbb{A} if \mathbb{S} is affine, and let $\text{Aut}_{h.d.}(\mathbb{A}) = \emptyset$ otherwise.

It is not hard to see, using Propositions 2.4.5 and 2.4.7, that if \mathbb{G} is not affine, then for $n \geq 3$, \mathbb{G}^n has no subuniverse that satisfies conditions (i) and (ii). This is why h.d.-automorphisms of \mathbb{G} are defined only when \mathbb{G} is affine, and h.d.-automorphisms of \mathbb{A} are defined only when \mathbb{S} is affine. If \mathbb{S} is affine, then the h.d.-automorphisms of \mathbb{A} will also satisfy condition (ii), because they contain an h.d.-automorphism of \mathbb{S} which satisfies (ii).

Notice also that condition (i) in the definition says that if B is an h.d.-automorphism of \mathbb{G} or \mathbb{A} , then for all $i \in \bar{n}$ there exists a function,

$$f_i : \text{pr}_{\bar{n} \setminus \{i\}} B \rightarrow \text{pr}_i B : (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \mapsto x_i,$$

where $x_i \in \text{pr}_i B$ is the unique element such that $(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \in B$.

5.1 Reductions

In this section we will complete the first step of determining a generating set for the relational clone of \mathbb{A} .

Definition 5.1.1. Let

$$\mathcal{T}_{\mathbb{A}} = \{\{a\} : a \in A\} \cup \text{Aut}(\mathbb{S}) \cup \text{Aut}(\mathbb{A}) \cup \text{Aut}_{\text{h.d.}}(\mathbb{S}) \cup \text{Aut}_{\text{h.d.}}(\mathbb{A}).$$

Let $n \geq 1$. We will say that a subuniverse B of \mathbb{A}^n is *reduced* if no projection of B is in the set $\mathcal{T}_{\mathbb{A}}$.

Our goal is to show that every subuniverse B of \mathbb{A}^n is contained in the relational clone generated by $\mathcal{T}_{\mathbb{A}}$ and a projection of B that is reduced. For the case when θ is a congruence on \mathbb{A} , we will prove an analogous result for every subuniverse B' of $\prod_{i=1}^n \mathbb{A}_i$, where $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$, $1 \leq i \leq n$. This will imply that determining the relational clone of \mathbb{A} will depend on determining the reduced subuniverses of \mathbb{A}^n and $\prod_{i=1}^n \mathbb{A}_i$.

Both kinds of reductions rely on the following lemma.

Lemma 5.1.2. *For $n \geq 2$, let $B \leq_{s.d.} \prod_{i=1}^n \mathbb{A}_i$, where $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}\}$, for all $1 \leq i \leq n$, or $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$, for all $1 \leq i \leq n$. Suppose that there exists some $2 \leq k \leq n$ such that $\text{pr}_{\bar{k}} B = \{(f(x_2, \dots, x_k), x_2, \dots, x_k) : (x_2, \dots, x_k) \in \text{pr}_{2, \dots, k} B\}$, where $f : \text{pr}_{2, \dots, k} B \rightarrow \text{pr}_1 B$ is a function. Then the following are true:*

$$(i) \quad \bar{x} \in B \Leftrightarrow \bar{x}_{\bar{n} \setminus \{1\}} \in \text{pr}_{\bar{n} \setminus \{1\}} B \text{ and } \bar{x}_{\bar{k}} \in \text{pr}_{\bar{k}} B,$$

$$(ii) \quad \langle B \rangle_{RClone} = \langle \text{pr}_{\bar{n} \setminus \{1\}} B, \text{pr}_{\bar{k}} B \rangle_{RClone}.$$

Proof. [(i)] Clearly, if $\bar{x} \in B$, then $\bar{x}_{\bar{n} \setminus \{1\}} \in \text{pr}_{\bar{n} \setminus \{1\}} B$ and $\bar{x}_{\bar{k}} \in \text{pr}_{\bar{k}} B$. We will show the reverse implication. Under the assumptions of the proposition, suppose that $\bar{x} \in \prod_{i=1}^n \mathbb{A}_i$ is such that $(x_1, \dots, x_k) \in \text{pr}_{\bar{k}} B$ and $(x_2, \dots, x_n) \in \text{pr}_{\bar{n} \setminus \{1\}} B$. Then $(x_2, \dots, x_n) \in \text{pr}_{\bar{n} \setminus \{1\}} B$ implies that there exists some $a \in \text{pr}_1 B$ such that $(a, x_2, \dots, x_n) \in B$. This means that $(a, x_2, \dots, x_k) \in \text{pr}_{\bar{k}} B$ and by our assumptions on $\text{pr}_{\bar{k}} B$ we get that $a = f(x_2, \dots, x_k)$. Furthermore the assumption $(x_1, \dots, x_k) \in \text{pr}_{\bar{k}} B$ implies that $x_1 = f(x_2, \dots, x_k)$. Since f is a function it must be that $a = x_1$. Hence $B \ni (a, x_2, \dots, x_n) = (x_1, x_2, \dots, x_n)$.

[(ii)] Since relational clones are closed under projections, it is clearly the case that $\langle B \rangle_{RClone} \supseteq \{\text{pr}_{\bar{n} \setminus \{1\}} B, \text{pr}_{\bar{k}} B\}$, and hence, $\langle B \rangle_{RClone} \supseteq \langle \text{pr}_{\bar{n} \setminus \{1\}} B, \text{pr}_{\bar{k}} B \rangle_{RClone}$.

To see the reverse inclusion, let $\alpha := \text{pr}_{\bar{n} \setminus \{1\}} B$ and $\beta := \text{pr}_{\bar{k}} B$. Then by property (i), we have that B is the set defined by the p.p. formula $\alpha(x_2, \dots, x_n) \wedge \beta(x_1, \dots, x_k)$, and thus, $B \in \langle \alpha, \beta \rangle_{RClone} = \langle \text{pr}_{\bar{n} \setminus \{1\}} B, \text{pr}_{\bar{k}} B \rangle_{RClone}$. Hence, $\langle B \rangle_{RClone} \subseteq \langle \text{pr}_{\bar{n} \setminus \{1\}} B, \text{pr}_{\bar{k}} B \rangle_{RClone}$. \square

Proposition 5.1.3. *Let $B \leq_{s.d.} \prod_{i=1}^n \mathbb{A}_n$, for some $n \geq 1$, where $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}\}$, for all $1 \leq i \leq n$. Then there exists a nonempty subset $I \subseteq \bar{n}$ such that $\text{pr}_I B$ is reduced and $B \in \langle \text{pr}_I B, \mathcal{T}_{\mathbb{A}} \rangle_{RClone}$.*

Proof. Let I be a minimal nonempty subset of \bar{n} such that $B \in \langle \text{pr}_I B, \mathcal{T}_{\mathbb{A}} \rangle_{RClone}$. Such a subset exists since $B \in \langle B, \mathcal{T}_{\mathbb{A}} \rangle_{RClone} = \langle \text{pr}_{\bar{n}} B, \mathcal{T}_{\mathbb{A}} \rangle_{RClone}$.

We want to show that $\text{pr}_I B$ is reduced. This is clear if $|I| = 1$, because $\text{pr}_i B = A_i \in \{S, A\}$ for all $1 \leq i \leq n$, and $S, A \notin \mathcal{T}_{\mathbb{A}}$. Therefore let $|I| > 1$. For contradiction, suppose $\text{pr}_I B$ is not reduced. Then some projection of $\text{pr}_I B$ is in $\mathcal{T}_{\mathbb{A}}$. Let J be nonempty subset of I such that $\text{pr}_J(\text{pr}_I B) \in \mathcal{T}_{\mathbb{A}}$. Notice that $\text{pr}_{J'}(\text{pr}_I B) = \text{pr}_{J'} B$ for all $J' \subseteq I$.

Claim 5.1.3.1. $B \in \langle \text{pr}_{I \setminus \{j\}} B, \mathcal{T}_{\mathbb{A}} \rangle_{RClone}$, for any $j \in J$.

Proof. Let $\hat{B} = \text{pr}_I B$. We saw that $\text{pr}_J \hat{B} = \text{pr}_J B$, and that for a one-element set J we have $\text{pr}_J B \notin \mathcal{T}_{\mathbb{A}}$. Therefore $\text{pr}_J \hat{B} \in \mathcal{T}_{\mathbb{A}}$ implies that $|J| > 1$. WLOG, suppose that $J = \{1, \dots, |J|\}$. Then $\text{pr}_J \hat{B} \in \mathcal{T}_{\mathbb{A}}$ implies that $\text{pr}_J \hat{B} = \{(f(x_2, \dots, x_{|J|}), x_2, \dots, x_{|J|}) : (x_2, \dots, x_{|J|}) \in \text{pr}_{2, \dots, |J|} \hat{B}\}$, for some function $f : \text{pr}_{2, \dots, |J|} \hat{B} \rightarrow \text{pr}_1 \hat{B}$. Therefore, by property (ii) of Lemma 5.1.2 we get that

$\langle \hat{B} \rangle_{RClone} = \langle \text{pr}_{I \setminus \{1\}} \hat{B}, \text{pr}_J \hat{B} \rangle_{RClone}$. Since $\hat{B} = \text{pr}_I B$ and $J \subseteq I$, this means that

$$\langle \text{pr}_I B \rangle_{RClone} = \langle \hat{B} \rangle_{RClone} = \langle \text{pr}_{I \setminus \{1\}} \hat{B}, \text{pr}_J \hat{B} \rangle_{RClone} = \langle \text{pr}_{I \setminus \{1\}} B, \text{pr}_J B \rangle_{RClone}.$$

Therefore, the assumptions $B \in \langle \text{pr}_I B, \mathcal{T}_{\mathbb{A}} \rangle_{RClone}$ and $\text{pr}_J B = \text{pr}_J \hat{B} \in \mathcal{T}_{\mathbb{A}}$, imply that

$$B \in \langle \text{pr}_I B, \mathcal{T}_{\mathbb{A}} \rangle_{RClone} = \langle \text{pr}_{I \setminus \{1\}} B, \text{pr}_J B, \mathcal{T}_{\mathbb{A}} \rangle_{RClone} = \langle \text{pr}_{I \setminus \{1\}} B, \mathcal{T}_{\mathbb{A}} \rangle_{RClone}.$$

This completes the claim. \square

Hence $B \in \langle \text{pr}_{I'} B, \mathcal{T}_{\mathbb{A}} \rangle_{RClone}$, for $I' = I \setminus \{j\}$, which contradicts the minimality of I . Therefore, the assumption that $\text{pr}_I B$ not reduced is false. This completes the proof. \square

Proposition 5.1.4. *Let B be a reduced subuniverse of \mathbb{A}^n for some $n \geq 1$. Then the k -ary projections of B are reduced, for all $1 \leq k \leq n$.*

Proof. Let B be a reduced subuniverse of \mathbb{A}^n , let $1 \leq k \leq n$, and $I \subseteq \bar{n}$ such that $|I| = k$. If $\text{pr}_J(\text{pr}_I B) \in \mathcal{T}_{\mathbb{A}}$ for some $J \subseteq I$, then $\text{pr}_J(\text{pr}_I B) = \text{pr}_J B$ implies that $\text{pr}_J B \in \mathcal{T}_{\mathbb{A}}$. However, this contradicts the assumption that B is reduced. \square

Definition 5.1.5. In case θ is a congruence on \mathbb{A} we define

$$\mathcal{T}'_{\mathbb{A}} = \text{Aut}(\mathbb{A}/\theta) \cup \text{Isom}(\mathbb{S}, \mathbb{A}/\theta) \cup \text{Aut}_{\text{h.d.}}(\mathbb{A}/\theta).$$

Let B' be a subuniverse of $\prod_{i=1}^n \mathbb{A}_i$ for some $n \geq 1$, where $\mathbb{A}_i = \mathbb{B}_i/\Theta_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$ for all $1 \leq i \leq n$ such that $\mathbb{B}_i = \mathbb{A}$ and $\Theta_i = \theta$ if $\mathbb{A}_i = \mathbb{A}/\theta$, and $\mathbb{B}_i = \mathbb{S}$ and Θ_i is the equality relation if $\mathbb{A}_i = \mathbb{S}$. Let $\rho^{-1}(B')$ be the full inverse image of B' under the natural homomorphism $\rho : \prod_{i=1}^n \mathbb{B}_i \rightarrow \prod_{i=1}^n \mathbb{A}_i$. We will say that B' is *reduced* if $\rho^{-1}(B')$ is reduced in the sense of Definition 5.1.1, and no projection of B' is in the set $\mathcal{T}'_{\mathbb{A}}$.

Definitions 5.1.1 and 5.1.5 give two notions of reduced subuniverses: one for the subuniverses of \mathbb{A}^n and one for the subuniverses of $\prod_{i=1}^n \mathbb{A}_i$, where $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$. Since a subuniverse of \mathbb{S}^n is both a subuniverse of \mathbb{A}^n and a subuniverse of $\prod_{i=1}^n \mathbb{A}_i$, we must check that the two definitions are consistent for subuniverses of finite powers of \mathbb{S} .

Let $B \leq \mathbb{S}^n$. Then clearly no projection of B is in the set $\text{Aut}(\mathbb{A}/\theta) \cup \text{Isom}(\mathbb{S}, \mathbb{A}/\theta) \cup \text{Aut}_{\text{h.d.}}(\mathbb{A}/\theta)$ and, by the definition of ρ , we have that $\rho^{-1}(B) = B$. Thus if B is reduced in the sense of Definition 5.1.5 if and only if B is reduced in the sense of Definition 5.1.1.

Proposition 5.1.6. *Let B' be a subuniverse of $\prod_{i=1}^n \mathbb{A}_i$ ($n \geq 1$), where $\mathbb{A}_i = \mathbb{B}_i/\Theta_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$ for all $1 \leq i \leq n$ such that $\mathbb{B}_i = \mathbb{A}$ and $\Theta_i = \theta$ if $\mathbb{A}_i = \mathbb{A}/\theta$, and $\mathbb{B}_i = \mathbb{S}$ and Θ_i is the equality relation if $\mathbb{A}_i = \mathbb{S}$. If the full inverse image $\rho^{-1}(B')$ of B' under the natural homomorphism $\rho : \prod_{i=1}^n \mathbb{B}_i \rightarrow \prod_{i=1}^n \mathbb{A}_i$ is reduced, then there exists some nonempty $I \subseteq \bar{n}$ such that $\text{pr}_I B'$ is reduced and $B' \in \langle \text{pr}_I B', \mathcal{T}'_{\mathbb{A}} \rangle_{RClone}$.*

The proof of this proposition is similar to the proof of Proposition 5.1.3.

Proof. Let I be a minimal nonempty subset of \bar{n} such that $B' \in \langle \text{pr}_I B', \mathcal{T}'_{\mathbb{A}} \rangle_{RClone}$. Such a subset exists since $B' \in \langle B', \mathcal{T}'_{\mathbb{A}} \rangle_{RClone} = \langle \text{pr}_{\bar{n}} B', \mathcal{T}'_{\mathbb{A}} \rangle_{RClone}$. We want to show that $\text{pr}_I B'$ is reduced. Since $\rho^{-1}(\text{pr}_I B') = \text{pr}_I \rho^{-1}(B')$ and our assumption is that $\rho^{-1}(B')$ is reduced, Proposition 5.1.4 shows that $\text{pr}_I \rho^{-1}(B')$, and hence $\rho^{-1}(\text{pr}_I B')$ is reduced.

Therefore it remains to prove that no projection of $\text{pr}_I B'$ is in $\mathcal{T}'_{\mathbb{A}}$. Let $\hat{B} := \text{pr}_I B'$, and suppose, for contradiction, that J is a nonempty subset of I such that $\text{pr}_J(\hat{B}) \in \mathcal{T}'_{\mathbb{A}}$. WLOG, suppose that $I = \{1, \dots, |I|\}$ and $J = \{1, \dots, |J|\}$.

Then $\text{pr}_J \hat{B} \in \mathcal{T}'_{\mathbb{A}}$ implies $\text{pr}_J \hat{B} = \{(f(x_2, \dots, x_{|J|}), x_2, \dots, x_{|J|}) : (x_2, \dots, x_{|J|}) \in \text{pr}_{2, \dots, |J|} \hat{B}\}$, for some function $f : \text{pr}_{J \setminus \{j\}} \hat{B} \rightarrow \text{pr}_1 \hat{B}$. Applying statement (ii) of Lemma 5.1.2 we get that $\langle \hat{B} \rangle_{RClone} = \langle \text{pr}_{I \setminus \{1\}} \hat{B}, \text{pr}_J \hat{B} \rangle_{RClone}$. Since $\hat{B} = \text{pr}_I B'$ and $J \subseteq I$, this means that

$$\langle \text{pr}_I B' \rangle_{RClone} = \langle \hat{B} \rangle_{RClone} = \langle \text{pr}_{I \setminus \{1\}} \hat{B}, \text{pr}_J \hat{B} \rangle_{RClone} = \langle \text{pr}_{I \setminus \{1\}} B', \text{pr}_J B' \rangle_{RClone}.$$

By assumption we have that $B' \in \langle \text{pr}_I B', \mathcal{T}'_{\mathbb{A}} \rangle_{RClone}$ and $\text{pr}_J \hat{B} \in \mathcal{T}'_{\mathbb{A}}$. Therefore,

$$B' \in \langle \text{pr}_I B', \mathcal{T}'_{\mathbb{A}} \rangle_{RClone} = \langle \text{pr}_{I \setminus \{1\}} B', \text{pr}_J B', \mathcal{T}'_{\mathbb{A}} \rangle_{RClone} = \langle \text{pr}_{I \setminus \{1\}} B', \mathcal{T}'_{\mathbb{A}} \rangle_{RClone}.$$

Hence $B' \in \langle \text{pr}_{I'} B', \mathcal{T}'_{\mathbb{A}} \rangle_{RClone}$, for $I' = I \setminus \{1\}$, which contradicts the minimality of I . This completes the proof of the proposition. \square

Proposition 5.1.7. *With the same notation as in Proposition 5.1.6, if B' is a reduced subuniverse of $\prod_{i=1}^n \mathbb{A}_i$, $n \geq 1$, then the k -ary projections of B' are reduced, for all $1 \leq k \leq n$.*

Proof. Let B' be a reduced subuniverse of $\prod_{i=1}^n \mathbb{A}_i$, let $1 \leq k \leq n$, and $I \subseteq \bar{n}$ such that $|I| = k$. The same argument as in the proof of Proposition 5.1.6 shows that $\rho^{-1}(\text{pr}_I B')$ is reduced. Furthermore, if $\text{pr}_J(\text{pr}_I B') \in \mathcal{T}'_{\mathbb{A}}$ for some $J \subseteq I$, then $\text{pr}_J(\text{pr}_I B') = \text{pr}_J B'$ implies that $\text{pr}_J B' \in \mathcal{T}'_{\mathbb{A}}$. However, this contradicts the assumption that B' is reduced. \square

5.2 H.D.-Automorphisms of \mathbb{A} when \mathbb{S} is Affine.

Under Assumption 1 we will determine the general form of a subuniverse $B \leq \mathbb{A}^n$, when \mathbb{S} is affine, $B|_{\mathbb{S}}$ is an h.d.-automorphism of \mathbb{S} , and $B \neq B \cap S^n$. We will see that such a subuniverse is an h.d.-automorphism of \mathbb{A} .

Proposition 5.2.1. *Suppose that \mathbb{G} is a finite idempotent strictly simple affine algebra, and let ${}_K G$ be the associated vector space. If $n \geq 3$ and $B \leq \mathbb{G}^n$ is an h.d.-automorphism of \mathbb{G} , then the following properties hold.*

- (1) *There exist nonzero elements $c_1, \dots, c_{n-1} \in K$ and $g \in G$ such that*

$$B = \{(x_1, \dots, x_{n-1}, \sum_{i=1}^{n-1} c_i x_i + g) \in G^n : x_1, \dots, x_{n-1} \in G\}.$$

- (2) *For every $i \in \bar{n}$, $\text{pr}_{\bar{n} \setminus \{i\}} B = G^{\bar{n} \setminus \{i\}}$.*

- (3) *For every $i \in \bar{n}$,*

$$B = \{(x_1, \dots, x_{i-1}, f_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n), x_{i+1}, \dots, x_n) : \\ x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n \in G\},$$

for some map $f_i : \text{pr}_{1, \dots, i-1, i+1, \dots, n} B \rightarrow \text{pr}_i B : (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \mapsto x_i$, where x_i is the unique element of $\text{pr}_i B$ such that $(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \in B$.

- (4) *Let $\{i, j\} \subseteq \bar{n}$, $i < j$. Then, for any $(c_1, \dots, c_{i-1}, c_{i+1}, \dots, c_{j-1}, c_{j+1}, \dots, c_n) \in G^{\bar{n} \setminus \{i, j\}}$,*

$B(c_1, \dots, c_{i-1}, x_i, c_{i+1}, \dots, c_{j-1}, x_j, c_{j+1}, \dots, c_n) \leq \mathbb{G}^{\{i, j\}}$ is an automorphism of \mathbb{G} .

(5) Let $i \in \bar{n}$, $n \geq 4$. Then for any $c_i \in G^{\{i\}}$, $B(x_1, \dots, x_{i-1}c_i, x_{i+1}, \dots, x_n)$ is an h.d.-automorphism of \mathbb{G} .

Proof. Suppose that $B \leq \mathbb{G}^n$ is an h.d.-automorphism of $\mathbb{G} \in \{\mathbb{S}, \mathbb{A}/\theta\}$, where \mathbb{G} is affine, for some $n \geq 3$.

[(1)] Since \mathbb{G} is affine, this follows directly from Proposition 2.4.6 which describes the subuniverses of finite powers of a finite idempotent strictly simple affine algebra.

[(2)] This is an immediate consequence of (1).

[(3)] This follows directly from Definition 5.0.9 and property (2).

[(4)] Let $\{i, j\} \subseteq \bar{n}$, $i < j$. Since $n \geq 3$, we will assume, WLOG, that $i = 1, j = 2$. Let $(c_3, \dots, c_n) \in G^{\bar{n} \setminus \{1, 2\}}$. It follows from property (2) that $\text{pr}_i B(x_1, x_2, c_3, \dots, c_n) = G$ for $i = 1, 2$. Property (3) implies that $B(x_1, x_2, c_3, \dots, c_n) = \{(x_1, f_2(x_1, c_3, \dots, c_n)) : x_1 \in G\}$ and $B(x_1, x_2, c_3, \dots, c_n) = \{(f_1(x_2, c_3, \dots, c_n), x_2) : x_2 \in G\}$. Thus $B(x_1, x_2, c_3, \dots, c_n)$ is the graph of a permutation of G and since $B(x_1, x_2, c_3, \dots, c_n) \leq \mathbb{G}^2$ we get that $B(x_1, x_2, c_3, \dots, c_n)$ is an automorphism of \mathbb{G} .

[(5)] Let $i \in \bar{n}$, $n \geq 4$. WLOG, suppose $i = n$. Let $c \in G^{\{n\}}$ and let $C := B(x_1, \dots, x_{n-1}, c)$. Then C is a subuniverse of \mathbb{G}^{n-1} . For all $i \in \overline{n-1}$, it follows from property (2) that $\text{pr}_{\bar{n} \setminus \{i\}} B = G^{\bar{n} \setminus \{i\}} = G^{\overline{n-1} \setminus \{i\}} \times G^{\{n\}} \supseteq G^{\overline{n-1} \setminus \{i\}} \times \{c\}$. Thus $G^{\overline{n-1} \setminus \{i\}} \subseteq \text{pr}_{\overline{n-1} \setminus \{i\}} B(x_1, \dots, x_{n-1}, c)$, and hence $\text{pr}_{\overline{n-1} \setminus \{i\}} C = G^{\overline{n-1} \setminus \{i\}}$ for all $i \in \overline{n-1}$. Then property (3) implies that for each $i \in \overline{n-1}$ we have that

$$C = \{(x_1, \dots, x_{i-1}, f_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1}, c), x_{i+1}, \dots, x_{n-1}, c) : \\ x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_{n-1} \in G\}.$$

This shows that C is an h.d.-automorphism of \mathbb{G} . □

Theorem 5.2.2. *Suppose that \mathbb{A} satisfies Assumption 1. Suppose that \mathbb{S} is affine and there is no (thick) (A, S) -cross among the subuniverses of \mathbb{A}^2 . Let $n \geq 3$. Suppose that $B \leq \mathbb{A}^n$ such that $B|_{\mathbb{S}}$ is an h.d.-automorphism of \mathbb{S} . If $B \not\leq S^n$, then θ is a congruence on \mathbb{A} and $B = (B \cap S^n) \cup \sigma$,*

where $\sigma = \{(x, \pi_2(x), \dots, \pi_n(x)) : x \in A \setminus S\}$ and π_i is an automorphism of \mathbb{A}/θ that fixes \bar{S} , for all $2 \leq i \leq n$.

Proof. Under the assumptions of the theorem, let B be a subuniverse of \mathbb{A}^n such that $B|_S$ is an h.d.-automorphism of \mathbb{S} . Suppose that $B \neq B \cap S^n$. We will prove the theorem by inducting on n .

Suppose that $n = 3$. We will first show that $B(s, x_2, x_3), B(x_1, s, x_3), B(x_1, x_2, s) \subseteq S^2$, for all $s \in S$, thus $B \subseteq S^3 \times (A \setminus S)^3$.

We claim that, for each $s \in S$, the subuniverses $B(s, x_2, x_3)$, $B(x_1, s, x_3)$, and $B(x_1, x_2, s)$ of \mathbb{A}^2 are automorphisms of \mathbb{S} . WLOG, we will show the claim for $B(s, x_2, x_3)$.

Claim 5.2.2.1. *For each $s \in S$, $B(s, x_2, x_3)$ is an automorphism of \mathbb{S} .*

Proof of claim. Let $s \in S$. Since $B|_S$ is an h.d.-automorphism of \mathbb{S} and \mathbb{S} is affine we get that $B|_S$ satisfies the assumptions of Proposition 5.2.1. Then property (2) of Proposition 5.2.1 implies that $B(s, x_2, x_3)|_S$ is an automorphism of \mathbb{S} . Therefore, by Theorem 3.1.5, $B(s, x_2, x_3)$ is either an automorphism of \mathbb{S} or an automorphism of \mathbb{A} . Since $s \in S$ is arbitrary, this shows that $B(s, x_2, x_3)$ is either an automorphism of \mathbb{S} or an automorphism of \mathbb{A} , for all $s \in S$.

To prove the claim, we must show that $B(s, x_2, x_3)$ is not an automorphism of \mathbb{A} , for any $s \in S$. To do this, we will first define a subuniverse, $C \leq \mathbb{S} \times \mathbb{A}^2$, and prove two subclaims.

Let C be the subuniverse of $\mathbb{S} \times \mathbb{A}^2$ defined by $C := B \cap S \times \mathbb{A}^2$. Then $B|_S \subseteq C$. Property (2) of Proposition 5.2.1 implies $S^2 \subseteq \text{pr}_{i,j}(B|_S)$, for all $1 \leq i < j \leq 3$, therefore $S^2 \subseteq \text{pr}_{i,j}C$. Furthermore, it is clear from the definition of C that $C = \bigcup_{s \in S} B(s, x_2, x_3)$. Since we showed that, for each $s \in S$, $B(s, x_2, x_3)$ is either an automorphism of \mathbb{S} or an automorphism of \mathbb{A} , it follows that $B(s, x_2, x_3) \subseteq S^2 \times (A \setminus S)^2$ and $C \subseteq S^3 \cup S \times (A \setminus S)^2$. Then $S^2 \subseteq \text{pr}_{2,3}C \subseteq S^2 \times (A \setminus S)^2$ implies, by Theorem 3.1.5, that either θ is a congruence on \mathbb{A} and $\text{pr}_{2,3}C$ is an automorphism of \mathbb{A}/θ that fixes \bar{S} or $\text{pr}_{2,3}C = S^2$.

Subclaim 5.2.2.1.1. *If $B(s, x_2, x_3)$ and $B(s', x_2, x_3)$ are automorphisms of \mathbb{A} , for distinct $s, s' \in S$, then $B(s, x_2, x_3)|_{A \setminus S} = B(s', x_2, x_3)|_{A \setminus S}$.*

Proof of subclaim. Suppose that $B(s, x_2, x_3)$ and $B(s', x_2, x_3)$ are automorphisms of \mathbb{A} , for distinct $s, s' \in S$. Then $\text{pr}_i B(s, x_2, x_3) = A$, for $i = 2, 3$, and since $B(s, x_2, x_3) \subseteq \text{pr}_{2,3} C$ we get that $\text{pr}_{2,3} C \leq_{s.d} \mathbb{A}^2$. Hence $\text{pr}_{2,3} C \neq S^2$, which means θ is a congruence on \mathbb{A} and $\text{pr}_{2,3} C$ is an automorphism of \mathbb{A}/θ that fixes \bar{S} . Finally, since $B(s, x_2, x_3), B(s', x_2, x_3) \subseteq \text{pr}_{2,3} C$, the subclaim follows. \square

Subclaim 5.2.2.1.2. *If $B(s, x_2, x_3)$ is an automorphism of \mathbb{A} , for some $s \in S$, then $B(s', x_2, x_3)$ is an automorphism of \mathbb{A} , for all $s' \in S$.*

Proof of subclaim. Let $s \in S$ and suppose that $B(s, x_2, x_3) = \pi$, where π is an automorphism of \mathbb{A} . Then $\{(s, x, \pi(x)) : x \in A\} \subseteq B$ implies $\{s\} \times A \subseteq \text{pr}_{1,2} B$. Since $s \in S$, it follows from the definition of C that $\{s\} \times A \subseteq \text{pr}_{1,2} C$. Furthermore, we saw that $S^2 \subseteq \text{pr}_{1,2} C$, therefore $\lambda_{S,s}^{-1} = S^2 \cup (\{s\} \times A) \subseteq \text{pr}_{1,2} C$. By assumption there is no thick (A, S) -cross, and thus no thick (S, A) -cross, among the subuniverses of \mathbb{A}^2 , therefore it follows from Theorem 3.1.5, that $\text{pr}_{1,2} C = S \times A$. Then $\{s'\} \times A \subseteq S \times A = \text{pr}_{1,2} C$, for all $s' \in S$. Fix $s' \in S$. For each $a \in A$, there exists some $x_a \in A$ such that $(s', a, x_s) \in C \subseteq B$, thus, $\text{pr}_2 B(s', x_2, x_3) = A$. We showed that $B(s', x_2, x_3)$ is either an automorphism of \mathbb{S} or an automorphism of \mathbb{A} , therefore $\text{pr}_2 B(s', x_2, x_3) = A$ implies that $B(s', x_2, x_3)$ is an automorphism of \mathbb{A} . Since s' was an arbitrary element from S , this completes the proof of the subclaim. \square

We are now ready to complete the proof of the claim. Recall that it remained to show that $B(s, x_2, x_3)$ is not an automorphism of \mathbb{A} , for any $s \in S$. Suppose, for contradiction, that there exists some $s \in S$, such that $B(s, x_2, x_3)$ is an automorphism of \mathbb{A} . Then by Subclaim 5.2.2.1.2, we get that $B(s, x_2, x_3)$ is an automorphism of \mathbb{A} for all $s \in S$. Since $|S| \geq 2$, there exists distinct $s, s' \in S$ and, by Subclaim 5.2.2.1.1, we have that $B(s, x_2, x_3)|_{A \setminus S} = B(s', x_2, x_3)|_{A \setminus S}$. Then we can infer from Lemma 3.0.6 that $B(s, x_2, x_3) = B(s', x_2, x_3)$, which means $B(s, x_2, x_3)|_S = B(s', x_2, x_3)|_S$.

Let $(c_2, c_3) \in B(s, x_2, x_3)|_S$. Then $(s, c_2, c_3), (s', c_2, c_3) \in B|_S$, which means $B(x_1, x_2, c_3)|_S$ is a subuniverse of \mathbb{S}^2 that contains $(s, c_2), (s', c_2)$. Recall that $B|_S$ satisfies the assumptions of Proposition 5.2.1, therefore property (4) implies that $B(x_1, x_2, c_3)|_S = \pi$, for some $\pi \in \text{Aut}(\mathbb{S})$.

Hence $s = \pi^{-1}(c_2) = s'$, which contradicts $s \neq s'$. Hence, the assumption that $B(s, x_1, x_2)$ is an automorphism of \mathbb{A} must be false, which completes the proof of the claim. \square

By symmetric arguments to those given in the proof of Claim 5.2.2.1, we get that for any $s \in S$, $B(s, x_2, x_3)$, $B(x_1, s, x_3)$ and $B(x_1, x_2, s)$ are each a subuniverse of \mathbb{S}^2 . Therefore, there is no tuple in B that has coordinates from both S and $A \setminus S$. In other words, if $\bar{u} \in B$, then either $\bar{u} \in S^3$ or $\bar{u} \in (A \setminus S)^3$. Hence, $B = (B \cap S^3) \cup (B \cap (A \setminus S)^3)$. We want to determine $B \cap (A \setminus S)^3$.

Every binary projection of B is contained in $S^2 \cup (A \setminus S)^2$ and we saw that $S^2 \subseteq \text{pr}_{i,j} B$, for $1 \leq i < j \leq 3$. Since we have assumed that $B \neq B \cap S^3$, there exists some $\bar{u} \in B \cap (A \setminus S)^3$, which means $(u_i, u_j) \in \text{pr}_{i,j} B \cap (A \setminus S)^2$. Then, by Theorem 3.1.5, it must be that θ is a congruence on \mathbb{A} and $\text{pr}_{i,j} B$ is an automorphism of \mathbb{A}/θ that fixes \bar{S} , for all $1 \leq i < j \leq 3$. In particular, for $i = 2, 3$, $\text{pr}_{1,i} B = \pi_i$, where π_i is an automorphism of \mathbb{A}/θ that fixes \bar{S} . Therefore $B \cap (A \setminus S)^3 \subseteq \{(x, \pi_2(x), \pi_3(x)) : x \in A \setminus S\}$

We claim that $B \cap (A \setminus S)^3 = \{(x, \pi_2(x), \pi_3(x)) : x \in A \setminus S\}$. Let $(a_1, \pi_2(a_1), \pi_3(a_1)) \in \{(x, \pi_2(x), \pi_3(x)) : x \in A \setminus S\}$. Since there exists some $\bar{u} \in B \cap (A \setminus S)^3$ and $S^2 \subseteq \text{pr}_{1,2} B$, we have that $S \cup \{u_1\} \in \text{pr}_1 B$, where $u_1 \in A \setminus S$, which implies $\text{pr}_1 B = A$. Then $a_1 \in (A \setminus S) \subseteq \text{pr}_1 B$ implies that there exists some $a_2, a_3 \in A$ such that $(a_1, a_2, a_3) \in B$. Thus, $(a_1, a_2) \in \text{pr}_{1,2} B|_{A \setminus S} = \pi_2|_{A \setminus S}$ implies $a_2 = \pi_2(a_1)$ and $(a_1, a_3) \in \text{pr}_{1,3} B|_{A \setminus S} = \pi_3|_{A \setminus S}$ implies $a_3 = \pi_3(a_1)$. Hence $(a_1, \pi_2(a_1), \pi_3(a_1)) = (a_1, a_2, a_3) \in B$.

Therefore we have shown that θ is a congruence on \mathbb{A} and $B = (B \cap S^3) \cup \sigma$, where $\sigma = \{(x, \pi_2(x), \pi_3(x)) : x \in A \setminus S\}$ and π_2, π_3 are automorphisms of \mathbb{A}/θ that fix \bar{S} . This completes the proof of the theorem for the case when $n = 3$.

Let $n > 3$ and suppose that for any $B \leq \mathbb{A}^{n-1}$, such that $B|_S$ is an h.d.-automorphism of \mathbb{S} and $B \neq B \cap S^{n-1}$, we have that θ is a congruence on \mathbb{A} and $B = (B \cap S^{n-1}) \cup \sigma$, where $\sigma = \{(x, \pi_2(x), \dots, \pi_{n-1}(x)) : x \in A \setminus S\}$ and π_i is an automorphism of \mathbb{A}/θ that fixes \bar{S} , for $1 \leq i \leq n-1$.

Let $B \leq \mathbb{A}^n$. Suppose that $B|_S$ is an h.d.-automorphism of \mathbb{S} and suppose that $B \neq B \cap S^n$. Since \mathbb{S} is affine, we get that $B|_S$ satisfies the assumptions of Proposition 5.2.1. Then property (5) of Proposition 5.2.1 implies that, for all $1 \leq j \leq n$ and for all $s \in S$, $B(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_n)|_S$ is an h.d.-automorphism of \mathbb{S} . We will show that $B(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_n) \subseteq S^{n-1}$, for all $1 \leq j \leq n$ and all $s \in S$. Then it will follow that $B \subseteq S^n \cup (A \setminus S)^n$. WLOG, we will show the claim for $j = 1$.

Claim 5.2.2.2. *For each $s \in S$, $B(s, x_2, \dots, x_n) \subseteq S^{n-1}$.*

Proof of claim. For each $s \in S$, either $B(s, x_2, \dots, x_n) \subseteq S^{n-1}$ or $B(s, x_2, \dots, x_n) \not\subseteq S^{n-1}$. It is clear that $B(s, x_2, \dots, x_n)$ is a subuniverse of \mathbb{A}^{n-1} and, as we discussed in the previous paragraph, $B(s, x_2, \dots, x_n)|_S$ is an h.d.-automorphism of \mathbb{S} , therefore by the induction hypothesis,

$$B(s, x_2, \dots, x_n) = (B(s, x_2, \dots, x_n) \cap S^{n-1}) \cup \sigma_s,$$

where

$$\sigma_s = \begin{cases} \emptyset, & \text{if } B(s, x_2, \dots, x_n) \subseteq S^{n-1} \\ \{(x, \pi_3(x), \dots, \pi_n(x)) : x \in A \setminus S\}, & \text{otherwise,} \end{cases}$$

for some automorphisms, π_i , of \mathbb{A}/θ that fix \bar{S} , $3 \leq i \leq n$.

To prove this claim we must show that $\sigma_s = \emptyset$, for all $s \in S$. First we will define a subuniverse, $C \leq \mathbb{S} \times \mathbb{A}^{n-1}$, and prove two subclaims.

Let C be the subuniverse of $\mathbb{S} \times \mathbb{A}^{n-1}$ defined by $C := B \cap (S \times \mathbb{A}^{n-1})$. Then $B|_S \subseteq C$. Recall that $B|_S$ satisfies the assumptions of Proposition 5.2.1, therefore property (2) implies that $S^{n-1} \subseteq \text{pr}_{\bar{n} \setminus \{i\}} B|_S$, for all $i \in \bar{n}$, thus $S^{n-1} \subseteq \text{pr}_{\bar{n} \setminus \{i\}} C$. Since $n > 3$, this means that $S^2 \subseteq \text{pr}_{i,j} C$, for all $1 \leq i < j \leq n$. Furthermore, it is clear from the definition of C that $C = \bigcup_{s \in S} B(s, x_2, \dots, x_n)$. Since $B(s, x_1, \dots, x_n) \subseteq S^{n-1} \cup (A \setminus S)^{n-1}$ we get that $C \subseteq S^n \cup S \times (A \setminus S)^{n-1}$. Thus, $S^2 \subseteq \text{pr}_{2,i} C \subseteq S^2 \cup (A \setminus S)^2$, for every $3 \leq i \leq n$. It follows from Theorem 3.1.5 that, for each $3 \leq i \leq n$, either θ is a congruence on \mathbb{A} and $\text{pr}_{2,i} C$ is an automorphism of \mathbb{A}/θ that fixes \bar{S} or $\text{pr}_{2,i} C = S^2$.

Subclaim 5.2.2.2.1. *If s, s' are distinct elements of S such that $\sigma_s \neq \emptyset$ and $\sigma_{s'} \neq \emptyset$, then $\sigma_s = \sigma_{s'}$.*

Proof of subclaim. Let $s, s' \in S$, $s \neq s'$, and suppose that $\sigma_s \neq \emptyset$ and $\sigma_{s'} \neq \emptyset$. Let

$$\sigma_s = \{(x, \pi_3(x), \dots, \pi_n(x)) : x \in A \setminus S\}$$

and

$$\sigma_{s'} = \{(x, \gamma_3(x), \dots, \gamma_n(x)) : x \in A \setminus S\},$$

where π_i and γ_i are automorphisms of \mathbb{A}/θ that fix \bar{S} , for all $3 \leq i \leq n$.

Then $\sigma_s \subseteq B(s, x_2, \dots, x_n)$, $\sigma_{s'} \subseteq B(s', x_2, \dots, x_n)$, and $s, s' \in S$ implies $\sigma_s \cup \sigma_{s'} \subseteq \text{pr}_{2, \dots, n} C$. Let $i \in \bar{n} \setminus \{1, 2\}$. Then $\{(x, \pi_i(x)) : x \in A \setminus S\} \cup \{(x, \gamma_i(x)) : x \in A \setminus S\} \subseteq \text{pr}_{2, i} C$ implies $\text{pr}_{2, i} C \neq S^2$, thus $\text{pr}_{2, i} C$ is an automorphism of \mathbb{A}/θ that fixes \bar{S} . Therefore, it must be that $\pi_i(x) = \gamma_i(x)$, for all $x \in A \setminus S$. Since i was an arbitrary element of $\bar{n} \setminus \{1, 2\}$, we get that $\sigma_s = \sigma_{s'}$. \square

Subclaim 5.2.2.2.2. *If $\sigma_s \neq \emptyset$, for some $s \in S$, then $\sigma_{s'} \neq \emptyset$, for every $s' \in S$.*

Proof of subclaim. Let $s \in S$ and suppose that $\sigma_s \neq \emptyset$. Then it follows from the discussion above that $B(s, x_2, \dots, x_n) = (B(s, x_2, \dots, x_n) \cap S^{n-1}) \cup \sigma_s$, where $\sigma_s = \{(x, \pi_3(x), \dots, \pi_n(x)) : x \in A \setminus S\}$, for automorphisms, π_i , of \mathbb{A}/θ that fix \bar{S} , $3 \leq i \leq n$. Let $s' \in S \setminus \{s\}$. We will show that $B(s', x_2, \dots, x_n) \not\subseteq S^{n-1}$. Since $B(s', x_2, \dots, x_n)$ is an $(n-1)$ -dimensional h.d-automorphism of \mathbb{S} and s' is an arbitrary element of S , this proves the subclaim.

Since $\sigma_s \subseteq B(s, x_2, \dots, x_n)$ we get that $\{(s, x, \pi_3(x), \dots, \pi_n(x)) : x \in A \setminus S\} \subseteq B$. Then $s \in S$ implies that $\{(s, x, \pi_3(x), \dots, \pi_n(x)) : x \in A \setminus S\} \subseteq C$, therefore $\{s\} \times A \setminus S \subseteq \text{pr}_{1,2} C$. Furthermore, we showed that $S^2 \subseteq \text{pr}_{1,2} C$. Thus $S^2 \cup (\{s\} \times A \setminus S) \subseteq \text{pr}_{1,2} C$. By assumption, there is no (A, S) -cross among the subuniverses of \mathbb{A}^2 which implies no (S, A) -cross is a subuniverse of \mathbb{A}^2 . Then it follows from Theorem 3.1.5 that $\text{pr}_{1,2} C = S \times A$. This means that for any $b \in A \setminus S$, $(s', b) \in S \times A = \text{pr}_{1,2} C \subseteq \text{pr}_{1,2} B$. Therefore, there exists some tuple $\bar{u} \in A^{n-2}$ such that $(s', b, \bar{u}) \in B$, which means $B(s', x_2, \dots, x_n) \not\subseteq S^{n-1}$, so it must be that $B(s', x_2, \dots, x_n) =$

$(B(s', x_2, \dots, x_n) \cap S^{n-1}) \cup \sigma_{s'}$, where $\sigma_{s'} \neq \emptyset$. Since s' was an arbitrary element of $S \setminus \{s\}$, this completes the proof of the subclaim. \square

We are ready to complete the proof of the claim. To do so, we must first show that $\sigma_s = \emptyset$, for all $s \in S$. Suppose, for contradiction, that there exists some $s \in S$ such that $\sigma_s \neq \emptyset$. Then by Subclaim 5.2.2.2, we get that $\sigma_{s'} \neq \emptyset$, for all $s' \in S$. Thus, it follows from Subclaim 5.2.2.1 that $\sigma_s = \sigma_{s'}$, for all distinct $s, s' \in S$. Therefore, for all $s \in S$,

$$B(s, x_2, \dots, x_n) = (B(s, x_2, \dots, x_n) \cap S^{n-1}) \cup \sigma, \quad (5.1)$$

where $\sigma = \{(x, \pi_3(x), \dots, \pi_n(x)) : x \in A \setminus S\}$ and π_i is an automorphism of \mathbb{A}/θ that fixes \bar{S} , $3 \leq i \leq n$.

Let s, s' be distinct elements in S and define

$$D := \{(x, x') : \text{there exists } \bar{x} \in A^{n-2} \text{ such that } (s, \bar{x}, x), (s', \bar{x}, x') \in B\}.$$

Then D is a subuniverse of \mathbb{A}^2 . Let $\Delta = \{(x, x) : x \in A\}$. We claim that $D \cap \Delta = \{(x, x) : x \in A \setminus S\}$.

Suppose $(a, a') \in D$. Then for some $(c_2, \dots, c_{n-1}) \in A^{n-2}$ we have that $(s, c_2, \dots, c_{n-1}, a), (s', c_2, \dots, c_{n-1}, a') \in B$. This means that $(c_2, \dots, c_{n-1}, a) \in B(s, x_2, \dots, x_n)$ and $(c_2, \dots, c_{n-1}, a') \in B(s', x_1, \dots, x_{n-1})$, which, by (5.1), in turn implies that $(c_2, \dots, c_{n-1}, a), (c_2, \dots, c_{n-1}, a') \in S^{n-1} \cup \sigma \subseteq S^{n-1} \cup (A \setminus S)^{n-1}$ and hence, $(a, a') \in S^2 \cup (A \setminus S)^2$.

First suppose $(c_2, \dots, c_{n-1}, a) \in S^{n-1}$, then $(c_2, \dots, c_{n-1}, a') \in S^{n-1}$ and $(s, c_2, \dots, c_{n-1}, a), (s', c_2, \dots, c_{n-1}, a') \in S^n$. Suppose, for contradiction, that $a = a'$. Then $B(x_1, c_2, \dots, c_{n-1}, x_2)|_S$ is a subuniverse of \mathbb{S}^2 that contains $(s, a), (s', a) = (s', a)$. Since \mathbb{S} is affine and $B|_S$ is an h.d.-automorphism of \mathbb{S} , we get that $B|_S$ satisfies the assumptions of Proposition 5.2.1. Then, by statement (4) of Proposition 5.2.1, we get that $B(x_1, c_2, \dots, c_{n-1}, x_2)|_S = \pi$, for some $\pi \in \text{Aut}(\mathbb{S})$. Thus $(s, a), (s', a) \in B(x_1, c_2, \dots, c_{n-1}, x_2)|_S$ implies $s = \pi^{-1}(a) = s'$, which contradicts the assumption that $s \neq s'$. Therefore, $a \neq a'$ whenever $(a, a') \in S^2$.

If $(c_2, \dots, c_{n-1}, a) \in (A \setminus S)^{n-1}$, then $(c_2, \dots, c_{n-1}, a), (c_2, \dots, c_{n-1}, a') \in \sigma$, hence $a = \pi_n(c_2) = a'$. Therefore $a = a'$ whenever $(a, a') \in (A \setminus S)^2$.

Furthermore, we have that $(x, \pi_2(x), \dots, \pi_n(x)) \in \sigma \subseteq B(s, x_1, \dots, x_n) \cap B(s', x_1, \dots, x_{n-1})$, for any $x \in A \setminus S$. Thus $(s, x, \pi_2(x), \dots, \pi_n(x)), (s', x, \pi_2(x), \dots, \pi_n(x)) \in B$, for any $x \in A \setminus S$, which means $\{(x, x) : x \in A \setminus S\} = \{(\pi_n(x), \pi_n(x)) : x \in A \setminus S\} \subseteq D$.

Therefore we have shown that $D \cap \Delta = \{(x, x) : x \in A \setminus S\}$. Since D and Δ are subuniverses of \mathbb{A}^2 and relational clones are closed under intersections and projections, we get that $\text{pr}_1(D \cap \Delta) = A \setminus S$ is a subuniverse of \mathbb{A} . However \mathbb{S} is the unique nontrivial subalgebra of \mathbb{A} , therefore we have a contradiction. This completes the proof of the claim. \square

By Claim 5.2.2.2, we have that $B(s, x_1, \dots, x_n) = B(s, x_1, \dots, x_n) \cap S^{n-1}$, for all $s \in S$. Applying Claim 5.2.2.2 to the subuniverses of \mathbb{A}^n that are obtained from B by permuting coordinates we get that for any $s \in S$, $B(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_n) = B(x_1, \dots, x_{j-1}, s, x_{j+1}, \dots, x_n) \cap S^{n-1}$, for all $j \in \bar{n}$. Thus, for any element $\bar{u} \in B$, either $\bar{u} \in S^n$, or $\bar{u} \in (A \setminus S)^{n-1}$. Hence $B = (B \cap S^n) \cup (B \cap (A \setminus S)^n)$. We want to determine $B \cap (A \setminus S)^n$.

By assumption, $B \neq B \cap S^n$, therefore $B \cap (A \setminus S)^n \neq \emptyset$. Let $\bar{u} \in B \cap (A \setminus S)^n$. Then for all $1 \leq i < j \leq n$, $(u_i, u_j) \in \text{pr}_{i,j} B \subseteq S^2 \cup (A \setminus S)^2$, where $u_i, u_j \in A \setminus S$. Furthermore, we showed that $S^2 \subseteq \text{pr}_{i,j} B|_S \subseteq \text{pr}_{i,j} B$. Therefore, by Theorem 3.1.5, it follows that θ is a congruence on \mathbb{A} and $\text{pr}_{i,j} B$ is an automorphism of \mathbb{A}/θ that fixes \bar{S} . In particular, for all $i \in \bar{n} \setminus \{1\}$, $\text{pr}_{1,i} B = \pi_i$, where π_i is an automorphism of \mathbb{A}/θ that fixes \bar{S} . Thus $B \cap (A \setminus S)^n \subseteq \{(x, \pi_2(x), \dots, \pi_n(x)) : s \in A \setminus S\}$.

We claim that $B \cap (A \setminus S)^n = \{(x, \pi_2(x), \dots, \pi_n(x)) : s \in A \setminus S\}$. Let $(a_1, \pi_2(a_1), \dots, \pi_n(a_1)) \in \{(x, \pi_2(x), \dots, \pi_n(x)) : s \in A \setminus S\}$. We showed that $S^2 \subseteq \text{pr}_{1,2} B$. Therefore $S \cup \{u_1\} \subseteq \text{pr}_1 B$, $u_1 \in A \setminus S$, which means $\text{pr}_1 B = A$. Then $a_1 \in A \setminus S \subseteq \text{pr}_1 B$, which means there exists some $a_2, \dots, a_n \in A \setminus S$ such that $(a_1, a_2, \dots, a_n) \in B$. For each $2 \leq i \leq n$, $(a_1, a_i) \in (\text{pr}_{1,i} B)|_{A \setminus S} = \pi_i|_{A \setminus S}$, thus $a_i = \pi_i(a_1)$. Therefore $B \ni (a_1, a_2, \dots, a_n) = (a_1, \pi_2(a_1), \dots, \pi_n(a_1))$.

We have shown that θ is a congruence on \mathbb{A} and $B = (B \cap S^n) \cup \{(x, \pi_2(x), \dots, \pi_n(x)) : s \in A \setminus S\}$, where π_i is an automorphism of \mathbb{A}/θ that fixes \bar{S} , $2 \leq i \leq n$. This completes the proof of the theorem. \square

Combining Definition 5.0.9 with Theorem 5.2.2 we get two characterizations of the h.d.-

automorphisms of the algebras \mathbb{A} that satisfy the assumptions of the theorem.

Corollary 5.2.3. *Suppose that \mathbb{A} satisfies Assumption 1, \mathbb{S} is affine, and there is no (thick) (A, S) -cross among the subuniverses of \mathbb{A}^2 . Let $n \geq 3$. The following conditions on $B \leq \mathbb{A}^n$ are equivalent:*

- (a) B is an h.d.-automorphism of \mathbb{A} ,
- (b) $B|_S$ is an h.d.-automorphism of \mathbb{S} and $B \not\leq B \cap S^n$,
- (c) $B|_S$ is an h.d.-automorphism of \mathbb{S} and $B = (B \cap S^n) \cup \sigma$, where $\sigma = \{(x, \pi_2(x), \dots, \pi_n(x)) : x \in A \setminus S\}$ and π_i is an automorphism of \mathbb{A}/θ that fixes \overline{S} , for all $2 \leq i \leq n$.

Proposition 5.2.4. *Suppose that $B \leq \mathbb{S}^n$ for some $n \geq 3$ and there exists a tuple $(a_1, \dots, a_n) \in B$ such that $B(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{n-1}, x_n)$ is an automorphism of \mathbb{S} for all $1 \leq i \leq n-1$. Then*

- (i) \mathbb{S} is not quasiprimal, and
- (ii) if \mathbb{S} is affine, then B is an h.d.-automorphism of \mathbb{S} .

Proof. The subuniverses of \mathbb{S}^n that are defined by $B(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{n-1}, x_n)$, where $1 \leq i \leq n-1$, will be useful in proving this proposition. Therefore we will denote the i^{th} such subuniverse by $B_i = B(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{n-1}, x_n)$. Then by the assumptions of this proposition, we have that $B_i = \sigma_i$ for some $\sigma_i \in \text{Aut}(\mathbb{S})$. Notice that $B_i \subseteq \text{pr}_{i,n} B$ for all $1 \leq i \leq n$. Also, $B_i = \sigma_i \in \text{Aut}(\mathbb{S})$ implies $B \neq S^n$.

[(i)] We will show that no unary projection of B is a singleton and no binary projection of B is a bijection. Since $B \leq \mathbb{S}^n$ and $B \neq S^n$, this will imply, by Proposition 2.4.5 that \mathbb{S} is not quasiprimal.

Let $i \in \overline{n-1}$. Then $B_i = \sigma_i \in \text{Aut}(\mathbb{S})$ implies $\text{pr}_i B_i = S = \text{pr}_n B_i$. Therefore $S = \text{pr}_i B_i \subseteq \text{pr}_i B$ and $S = \text{pr}_n B_i \subseteq \text{pr}_n B$. Since $i \in \overline{n-1}$, this means that no unary projection of B is a singleton.

We will now show that no binary projection $\text{pr}_{i,j} B$ of B is a bijection. We must consider two cases: $1 \leq i < j \leq n-1$ and $1 \leq i \leq n-1, j = n$.

Suppose that $1 \leq i < j \leq n - 1$. Recall that $n \geq 3$. We will suppose, WLOG, that $i = 1$ and $j = 2$. Then $B_2 = \sigma_2 \in \text{Aut}(\mathbb{S})$ implies $(s, \sigma_2(s)) \in B_2$ for all $s \in S$. Thus, $(a_1, s, a_3, \dots, a_{n-1}, \sigma_2(s)) \in B$ for each $s \in S$, which means $\{a_1\} \times S \subseteq \text{pr}_{1,2} B$, so $\text{pr}_{1,2} B$ is not a bijection.

Now suppose, WLOG, that $i = 1$ and $j = n$. Then $B_2 = \sigma_2 \in \text{Aut}(\mathbb{S})$ implies $(s, \sigma_2(s)) \in B_2$ for all $s \in S$, therefore $(a_1, s, a_3, \dots, a_{n-1}, \sigma_2(s)) \in B$. Thus, $\{a_1\} \times S \subseteq \text{pr}_{1,n} B$. Hence $\text{pr}_{1,n} B$ is not a bijection.

We have shown that B is a subuniverse of \mathbb{S}^n such that no unary projection of B is a singleton, no binary projection of B is a bijection, and $B \neq \mathbb{S}^n$. Therefore, by Proposition 2.4.5, \mathbb{S} is not quasiprimal.

[(ii)] Suppose that \mathbb{S} is affine. Since \mathbb{S} is a finite idempotent strictly simple algebra and $B \leq \mathbb{S}^n$, it follows from Proposition 2.4.6 that, up to permutation of coordinates,

$$B = \{(x_1, \dots, x_t, \sum_{i=1}^t c_{(t+1,i)} x_i + \delta_{t+1}, \dots, \sum_{i=1}^t c_{(n,i)} x_i + \delta_n) \in \mathbb{S}^n : x_1, \dots, x_t \in S\},$$

for some $\delta_{t+1}, \dots, \delta_n \in S$ and $c_{(t+1,i)}, \dots, c_{(n,i)} \in K$, $1 \leq i \leq t$, where ${}_K S$ is the associated vector space.

We saw that $B \neq \mathbb{S}^n$, therefore $t \neq n$. We claim that $t = n - 1$. Suppose not. Then $1 \leq t \leq n - 2$. This means that

$$\begin{aligned} \sigma_{n-1} &= B_{n-1} \\ &= B(a_1, \dots, a_t, \dots, a_{n-2}, x_{n-1}, x_n) \\ &= \text{pr}_{n-1,n} \left(\{(a_1, \dots, a_t, \sum_{i=1}^t c_{(t+1,i)} a_i + \delta_{t+1}, \dots, \sum_{i=1}^t c_{(n,i)} a_i + \delta_n)\} \right) \\ &= \left\{ \left(\sum_{i=1}^t c_{(n-1,i)} a_i + \delta_{n-1}, \sum_{i=1}^t c_{(n,i)} a_i + \delta_n \right) \right\}, \end{aligned}$$

which is a contradiction. Therefore,

$$B = \{(x_1, \dots, x_{n-1}, \sum_{i=1}^{n-1} c_i x_i + \delta) \in \mathbb{S}^n : x_1, \dots, x_{n-1} \in S\},$$

for some $\delta \in S$ and $c_i \in K$, $1 \leq i \leq s$.

We claim that $c_i \neq 0$ for all $1 \leq i \leq n-1$. Suppose, for contradiction, that $c_i = 0$ for some $1 \leq i \leq n-1$. WLOG, suppose that $c_1 = 0$. Then

$$\begin{aligned} \sigma_1 &= B_1 \\ &= B(x_1, a_2, \dots, a_{n-1}, x_n) \\ &= \text{pr}_{1,n} \left(\left\{ (x_1, a_2, \dots, a_{n-1}, c_1 x_1 + \sum_{i=2}^{n-1} c_i a_i + \delta) : x_1 \in S \right\} \right) \\ &= S \times \left\{ \sum_{i=2}^{n-1} x_i a_i + \delta \right\}, \end{aligned}$$

where the last equality holds since $c_1 = 0$. This is a contradiction, therefore, $c_i \neq 0$ for all $1 \leq i \leq n-1$.

This means that no projection $\text{pr}_I B$ of B where $I \subsetneq \bar{n}$ is an h.d.-automorphism of \mathbb{S} , therefore B satisfies property (ii) of Definition 5.0.9. We will now show that B satisfies property (i) of Definition 5.0.9.

For each $(x_1, \dots, x_{n-1}) \in \text{pr}_{\overline{n-1}} B$ it is clear that $x_n = \sum_{i=1}^{n-1} c_i x_i + \delta \in \text{pr}_n B$ is the unique element such that $(x_1, \dots, x_{n-1}, x_n) \in B$. Let $i \in \overline{n-1}$. Recall that c_i is in the field K and we showed that $c_i \neq 0$, therefore $c_i^{-1} \in K$. Then for any $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in \text{pr}_{\overline{n} \setminus \{i\}} B$ we have that $x_i = c_i^{-1}(x_n - \delta - \sum_{j \in \overline{n-1} \setminus \{i\}} c_j a_j)$ is the unique element such that $(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_n) \in B$. Therefore, B satisfies property (ii) of Definition 5.0.9, hence, B is an h.d.-automorphism of \mathbb{S} . □

5.3 Compatible Subuniverses that Indicate a Ternary Edge Blocker

The goal of this section is to show that, under Assumption 1, if a subdirect subuniverse B of $\mathbb{S} \times \mathbb{A} \times \mathbb{S}$ contains a triple (a_1, a_2, a_3) such that the tuples $(x_1, x_2, x_3) \in B$ with $x_2 = a_2$ yield an automorphism of \mathbb{S} , while those with $x_1 = a_1$ a (thick) (A, S) -cross, then one of the ternary edge blockers Λ or \mathcal{K}_b ($b \in A \setminus S$) introduced in Definition 4.2.1 is among the subuniverses of \mathbb{A}^3 . This fact will be useful in proving subsequent statements.

Theorem 5.3.1. *Suppose \mathbb{S} is quasiprimal and let $B \leq \mathbb{S} \times \mathbb{A} \times \mathbb{S}$. Suppose that there exists some $(a_1, a_2, a_3) \in B$ such that $B(x_1, a_2, x_3) = \sigma$, for some $\sigma \in \text{Aut}(\mathbb{S})$. Then the following implications hold.*

(i) *If $B(a_1, x_2, x_3) = \kappa_{b, a_3}$, for some $b \in A \setminus S$, then $\mathcal{K}_b \leq \mathbb{A}^3$.*

(ii) *If $B(a_1, x_2, x_3) = \lambda_{S, a_3}$, then $\Lambda \leq \mathbb{A}^3$.*

Proof. To prove this Theorem, we will show that for each $s \in S$, the subuniverse $B(x_1, x_2, s)$ of \mathbb{A}^2 can be described as follows:

$$B(x_1, x_2, s) = \begin{cases} \kappa_{b, \sigma^{-1}(s)}^{-1}, & \text{if } B(a_1, x_2, x_3) = \kappa_{b, a_3}, \\ \lambda_{S, \sigma^{-1}(s)}^{-1}, & \text{if } B(a_1, x_2, x_3) = \lambda_{S, a_3}. \end{cases} \quad (5.2)$$

Therefore, if $B(a_1, x_2, x_3) = \kappa_{b, a_3}$, then $B(x_1, x_2, s) = \kappa_{b, \sigma^{-1}(s)}^{-1} = (\{\sigma^{-1}(s)\} \times A) \cup (S \times \{b\})$ and

$$\begin{aligned} B &= \bigcup_{s \in S} (B(x_1, x_2, s) \times \{s\}) \\ &= \bigcup_{s \in S} ((\{\sigma^{-1}(s)\} \times A \times \{s\}) \cup (S \times \{b\} \times \{s\})) \\ &= (\bigcup_{s \in S} (\{\sigma^{-1}(s)\} \times A \times \{s\})) \cup (S \times \{b\} \times S) \\ &= \{(\sigma^{-1}(x), y, x) : x \in S, y \in A\} \cup (S \times \{b\} \times S) \\ &= \{(x, y, \sigma(x)) : x \in S, y \in A\} \cup (S \times \{b\} \times S) \\ &= \mathcal{K}_{b, \sigma}. \end{aligned}$$

While, if $B(a_1, x_2, x_3) = \lambda_{S, a_3}$, then $B(x_1, x_2, s) = \lambda_{S, \sigma^{-1}(s)}^{-1} = (\{\sigma^{-1}(s)\} \times A) \cup S^2$ and

$$\begin{aligned}
B &= \bigcup_{s \in S} (B(x_1, x_2, s) \times \{s\}) \\
&= \bigcup_{s \in S} ((\{\sigma^{-1}(s)\} \times A \times \{s\}) \cup (S^2 \times \{s\})) \\
&= (\bigcup_{s \in S} (\{\sigma^{-1}(s)\} \times A \times \{s\})) \cup S^3 \\
&= \{(\sigma^{-1}(x), y, x) : x \in S, y \in A\} \cup S^3 \\
&= \{(x, y, \sigma(x)) : x \in S, y \in A\} \cup S^3 \\
&= \Lambda_\sigma.
\end{aligned}$$

Therefore Lemmas 4.2.3 and 4.2.2 imply that $K_b \leq \mathbb{A}^3$ or $\Lambda \leq \mathbb{A}^3$, respectively.

To show (5.2), we will first consider the binary projections $\text{pr}_{1,3} B$ and $\text{pr}_{2,3} B$.

Claim 5.3.1.1. $\text{pr}_{1,3} B = S^2$.

Proof of claim. The assumption that $B \leq \mathbb{S} \times \mathbb{A} \times \mathbb{S}$ implies $\text{pr}_{1,3} B \leq \mathbb{S}^2$. We will show that $(\{a_1\} \times S) \cup \sigma \subseteq \text{pr}_{1,3} B$, therefore, by Theorem 3.1.5, $\text{pr}_{1,3} B = S^2$.

Since $B(a_1, x_2, x_3)$ is a (thick) (A, S) -cross, we get that $\text{pr}_3 B(a_1, x_2, x_3) = S$. Then, for each $s \in S$ there exists some $a_s \in A$ such that $(a_s, s) \in B(a_1, x_2, x_3)$, therefore $(a_1, a_s, s) \in B$, which means $\{a_1\} \times S \subseteq \text{pr}_{1,3} B$. Furthermore, $\text{pr}_{1,3} B \supseteq B(x_1, a_2, x_3) = \sigma$ for some $\sigma \in \text{Aut}(\mathbb{S})$. Thus, $(\{a_1\} \times S) \cup \sigma \subseteq \text{pr}_{1,3} B$, which, as previously noted, completes the proof of the claim. \square

Claim 5.3.1.2. $\text{pr}_{2,3} B = A \times S$.

Proof of claim. The assumption that $B \leq \mathbb{S} \times \mathbb{A} \times \mathbb{S}$ implies $\text{pr}_{2,3} B \leq \mathbb{A} \times \mathbb{S}$. We will show that $\{a_2\} \times S \subseteq \text{pr}_{2,3} B$ and, under the assumption of each statement (i)–(ii) of the theorem, there exists some $u \in A \setminus \{a_2\}$ such that $\{u, a_2\} \cap A \setminus S \neq \emptyset$ and $\{u\} \times S \subseteq \text{pr}_{2,3} B$. Then $(\{a_2\} \times S) \cup (\{u\} \times S) \subseteq \text{pr}_{2,3} B$, $a_2 \neq u$ and $\{u, a_2\} \cap A \setminus S \neq \emptyset$ implies, by Lemma 3.1.9, that $A \times S \subseteq \text{pr}_{2,3} B$, hence we have equality.

Since $B(x_1, a_2, x_3) = \sigma$ we get, for all $s \in S$, that $(\sigma^{-1}(s), s) \in B(x_1, a_2, x_3)$. Then $(\sigma^{-1}(s), a_2, s) \in B$, for each $s \in S$, implies $\{a_2\} \times S \subseteq \text{pr}_{2,3} B$.

Clearly, $B(a_1, x_2, x_3) \subseteq \text{pr}_{2,3} B$. We are assuming that $B(a_1, x_2, x_3)$ is a (thick) (A, S) -cross, therefore there exists some $u \in A$ such that $\{u\} \times S \subseteq B(a_1, x_2, x_3) \subseteq \text{pr}_{2,3} B$.

We claim that $u \neq a_2$ and $\{u, a_2\} \cap A \setminus S \neq \emptyset$. First note that $\{a_2\} \times S \not\subseteq B(a_1, x_2, x_3)$, otherwise we would have that $\{a_1\} \times \{a_2\} \times S \subseteq B$ which implies $\{a_1\} \times S \subseteq B(x_1, a_2, x_3) = \sigma$, a contradiction. Then $\{u\} \times S \subseteq B(a_1, x_2, x_3)$ implies $u \neq a_2$. Furthermore, if $B(a_1, x_2, x_3) = \lambda_{S, a_3}$, then $\{a_2\} \times S \not\subseteq B(a_1, x_2, x_3)$ implies $a_2 \in A \setminus S$. While if $B(a_1, x_2, x_3) = \kappa_{b, a_3}$, then $\{u\} \times S \subseteq B(a_1, x_2, x_3)$ implies $u = b$, where $b \in A \setminus S$. Hence, in both cases we get that $\{u, a_2\} \cap A \setminus S \neq \emptyset$.

We have shown that $(\{a_2\} \times S) \cup (\{u\} \times S) \subseteq \text{pr}_{2,3} B$, with $a_2 \neq u$ and either $u \in A \setminus S$ or $a_2 \in A \setminus S$, which completes the proof of this claim. \square

We will now show that (5.2) holds. Let s be an arbitrary element of S and consider the subuniverse $B(x_1, x_2, s) \subseteq \text{pr}_{1,2} B \leq \mathbb{S} \times \mathbb{A}$. By Claims 5.3.1.1 and 5.3.1.2, we have that $S \times \{s\} \subseteq S^2 = \text{pr}_{1,3} B$ and $A \times \{s\} \subseteq A \times S = \text{pr}_{2,3} B$, therefore $B(x_1, x_2, s) \leq_{s.d.} \mathbb{S} \times \mathbb{A}$. From Theorem 3.1.5, we get that $B(x_1, x_2, s)$ is one of the following,

$$B(x_1, x_2, s) = \begin{cases} S \times A, \\ \text{an isomorphism } \mathbb{S} \rightarrow \mathbb{A}/\theta, \\ \text{an } (S, A)\text{-cross, or} \\ \text{a thick } (S, A)\text{-cross.} \end{cases}$$

First note that $B(x_1, x_2, s) \neq S \times A$. Otherwise, we get that $S \times \{a_2\} \subseteq S \times A = B(x_1, x_2, s)$, which implies $S \times \{a_2\} \times \{s\} \subseteq B$ and thus, $S \times \{s\} \subseteq B(x_1, a_2, x_3) = \sigma$, a contradiction. Furthermore, since $B(a_1, x_2, x_3) = \kappa_{b, a_3} \leq \mathbb{A}^2$ or $B(a_1, x_2, x_3) = \lambda_{S, a_3} \leq \mathbb{A}^2$, then our assumption that \mathbb{S} is quasiprimal and statements (vi) and (vii) of Corollary 3.2.5, respectively, imply that $B(x_1, x_2, s)^{-1}$ is not an isomorphism from \mathbb{A}/θ to \mathbb{S} . Therefore it must be that $B(x_1, x_2, s)$ is a (thick) (S, A) -cross.

Now we will consider the two cases treated in statements (i)–(ii) of the theorem separately. Suppose first that $B(a_1, x_2, x_3) = \kappa_{b, a_3} \leq \mathbb{A}^2$ for some $b \in A \setminus S$. Since \mathbb{S} is quasiprimal it

follows from statements (ii) and (iv) of Corollary 3.2.5 that $B(x_1, x_2, s)^{-1} = \kappa_{b,v}$, for some $v \in S$, thus $B(x_1, x_2, s) = \kappa_{b,v}^{-1}$. Next, suppose that $B(a_1, x_2, x_3) = \lambda_{S,a_3}$. Since \mathbb{S} is quasiprimal we can infer from statement (ii) of Corollary 3.2.5 that $B(x_1, x_2, s)^{-1} = \lambda_{S,v}$, for some $v \in S$, thus $B(x_1, x_2, s) = \lambda_{S,v}^{-1}$.

We claim that, in either case, $v = \sigma^{-1}(s)$. Since $B(x_1, a_2, x_3) = \sigma$, we have that $(\sigma^{-1}(s), s) \in B(x_1, a_2, x_3)$ and $S \times \{s\} \not\subseteq B(x_1, a_2, x_3)$. Therefore, $(\sigma^{-1}(s), a_2, s) \in B$ and $S \times \{a_2\} \times \{s\} \not\subseteq B$, which means $(\sigma^{-1}(s), a_2) \in B(x_1, x_2, s)$ and $S \times \{a_2\} \not\subseteq B(x_1, x_2, s)$. Thus, $B(x_1, x_2, s) = \kappa_{b,v}^{-1}$ or $\lambda_{S,v}^{-1}$ implies $v = \sigma^{-1}(s)$,

Since $s \in S$ was arbitrary, we have shown the following. If $B(a_1, x_2, x_3) = \kappa_{b,a_3}$, then for each $s \in S$, $B(x_1, x_2, s) = \kappa_{b,\sigma^{-1}(s)}^{-1}$. On the other hand, if $B(a_1, x_2, x_3) = \lambda_{S,a_3}$, then for each $s \in S$, $B(x_1, x_2, s) = \lambda_{S,\sigma^{-1}(s)}^{-1}$. Therefore, as we saw at the start of the proof, this completes the proof of the Theorem. \square

Corollary 5.3.2. *Suppose that \mathbb{A} satisfies Assumption 1, \mathbb{S} is quasiprimal, and θ is a congruence on \mathbb{A} . Suppose $B \leq \mathbb{S} \times \mathbb{A} \times \mathbb{S}$ and let $B' = \rho(B)$ be the image of B under the natural homomorphism $\rho : \mathbb{S} \times \mathbb{A} \times \mathbb{S} \rightarrow \mathbb{S} \times \mathbb{A}/\theta \times \mathbb{S}$. Suppose that there exists a tuple $(a_1, a_2/\theta, a_3) \in B'$, where, for some $\bar{b} \in A/\theta$ and some $\sigma \in \text{Aut}(\mathbb{S})$, $B'(a_1, x_2, x_3) = \kappa_{\bar{b},a_3}$ and $B'(x_1, a_2/\theta, x_3) = \sigma$. Then either $\Lambda \leq \mathbb{A}^3$ or $\mathcal{K}_b \leq \mathbb{A}^3$ and $\bar{b} \neq \bar{S}$.*

Proof. Under the assumptions of the corollary, let D be the full inverse image of B' under ρ . Then D is a subuniverse of $\mathbb{S} \times \mathbb{A} \times \mathbb{S}$, $(a_1, a_2, a_3) \in D$, $D(x_1, a_2, x_3) = \sigma \in \text{Aut}(\mathbb{S})$, and

$$D(a_1, x_2, x_3) = \begin{cases} \lambda_{S,a_3}, & \text{if } \bar{b} = \bar{S}, \\ \kappa_{\bar{b},a_3}, & \text{otherwise.} \end{cases}$$

Then D satisfies the assumptions of Theorem 5.3.1, therefore the assertions of the corollary follows. \square

5.4 Reduced Subuniverses Are θ -Closed in their A -coordinates

In this section we will consider algebras \mathbb{A} which satisfy the following assumption.

Assumption 2. \mathbb{A} is a finite idempotent algebra with a unique proper nontrivial subalgebra \mathbb{S} such that $|S| > 2$, $|A \setminus S| > 1$ (that is, \mathbb{A} satisfies Assumption 1), and one of the following conditions holds for \mathbb{S} :

(A) \mathbb{S} is affine and $\lambda_{S,s}, \kappa_{b,s} \not\leq \mathbb{A}^2$ for any $s \in S$, $b \in A \setminus S$;

(Q) \mathbb{S} is quasiprimal, the reduced subuniverses of \mathbb{A}^2 are θ -closed in their A -coordinates, and $\Lambda, \mathcal{K}_b \not\leq \mathbb{A}^3$ for any $b \in A \setminus S$.

The purpose of this section is to accomplish the second step of our strategy for finding a generating set for the relational clone of such an algebra \mathbb{A} . This will be done by proving the following theorem.

Theorem 5.4.1. *If \mathbb{A} satisfies Assumption 2, then the reduced subuniverses of all finite powers of \mathbb{A} are θ -closed in their A -coordinates.*

Proof. Suppose that \mathbb{A} satisfies Assumption 2. By Proposition 2.2.8, the conclusion of the theorem will follow if we show that every reduced subuniverse of a finite power of \mathbb{A} is θ -closed in each A -coordinate. Therefore we will assume that there exists a reduced subuniverse of a finite power of \mathbb{A} that is not θ -closed in some A -coordinate and show that this assumption induces a contradiction.

Let n be minimal such that there exists a reduced subuniverse, B , of \mathbb{A}^n , such that $\text{pr}_i B = A$ for some $1 \leq i \leq n$ and B is not θ -closed in its i^{th} coordinate. We must have $n \geq 2$, because if $n = 1$ and B is reduced, then $B = S$ or $B = A$, so B is θ -closed in its A -coordinates. In fact, we must have $n \geq 3$. If \mathbb{S} is quasiprimal, this is clear, since Assumption 2 (Q) forces that all reduced subuniverses of \mathbb{A}^2 are θ -closed in their A -coordinates. Now suppose that \mathbb{S} is affine. Then $\nu_s \not\leq \mathbb{A}^2$ for all $s \in S$, and therefore statements (vi) and (vii) of Proposition 3.2.1 imply that $\mu_s \not\leq \mathbb{A}^2$ and $\nu_s^\tau \not\leq \mathbb{A}^2$ hold for all $s \in S$ and all fixed-point free permutations τ of $A \setminus S$. Assumption 2 (A), combined with statement (ix) of Proposition 3.2.1 shows that $\chi_{s,s} \not\leq \mathbb{A}^2$ and $\chi_{S,s} \not\leq \mathbb{A}^2$ hold for all $s \in S$. Therefore, by inspecting the possible subuniverses of \mathbb{A}^2 listed in Theorem 3.1.5 we conclude that all reduced subuniverses of \mathbb{A}^2 are θ -closed in their A -coordinates.

Thus $n \geq 3$. By permuting coordinates if necessary, we may assume that $i = n$, so B is a reduced subuniverse of \mathbb{A}^n such that $\text{pr}_n B = A$ and B is not θ -closed in its last coordinate. Hence, there exists some $\bar{a} \in B$ that satisfies $a_n \in S$, and $\{a_1\} \times \cdots \times \{a_{n-1}\} \times S \not\subseteq B$. We will contradict the assumption that such a subuniverse B exists by showing that the following three lemmas hold.

Lemma 5.4.2. *Suppose that \mathbb{A} satisfies Assumption 2, $n \geq 3$, and the reduced subuniverses of \mathbb{A}^{n-1} are θ -closed in their A -coordinates. Let B be a reduced subuniverse of \mathbb{A}^n , $\text{pr}_n B = A$, and $\bar{a} \in B$ where $a_n \in S$ and $\{a_1\} \times \cdots \times \{a_{n-1}\} \times S \not\subseteq B$. Then one of the following cases holds. Either,*

(I') *there exists $1 \leq j \leq n-1$ such that*

- *B_j is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, $\text{pr}_{j,n} B = A^2$, and $B(x_1, \dots, x_{j-1}, a_j, x_{j+1}, \dots, x_n)$ is an h.d.-automorphism of \mathbb{A} , and*
- *for all $1 \leq i \leq n-1$, $i \neq j$, B_i is an automorphism of \mathbb{S} or \mathbb{A} , $\text{pr}_{i,n} B \neq A^2$, and $B(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n)$ is reduced,*

or

(II') *for all $1 \leq i \leq n-1$, $B_i = \lambda_{S, a_n}$, $\text{pr}_{i,n} B = \chi_{S, S}$, and $B(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n)$ is reduced.*

Lemma 5.4.3. *Under the assumptions of Lemma 5.4.2, case (I') of Lemma 5.4.2 cannot occur.*

Lemma 5.4.4. *Under the assumptions of Lemma 5.4.2, case (II') of Lemma 5.4.2 cannot occur.*

The proof of these lemmas will be postponed. We begin with a sequence of claims.

The subuniverses of \mathbb{A}^2 that are defined by $B(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{n-1}, x_n)$, $i \in \overline{n-1}$, will play an important role in the proof of this theorem. Therefore, for $i \in \overline{n-1}$, we will denote the i^{th} such subuniverse by $B_i := B(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{n-1}, x_n)$. The following claim states important properties of B_i , for all $i \in \overline{n-1}$.

Claim 5.4.4.1. *Under the assumptions of Lemma 5.4.2, B_i satisfies the following properties for all $i \in \overline{n-1}$,*

- (1) $(a_i, a_n) \in B_i$, where $a_n \in S$ and $\{a_i\} \times S \not\subseteq B_i$,
- (2) $S \subseteq \text{pr}_n B_i$ and $\{a_i\} \times S \subseteq \text{pr}_{i,n} B$,
- (3) $S \subseteq \text{pr}_i B_i$,
- (4) $B_i = \begin{cases} \text{an automorphism of } \mathbb{A}, \\ \text{an automorphism of } \mathbb{S}, \\ \text{an isomorphism } \mathbb{A}/\theta \rightarrow \mathbb{S}, \text{ given that } \theta \in \text{Con}(\mathbb{A}), \\ \kappa_{b, a_n}, \\ \lambda_{S, a_n}. \end{cases}$

Proof of claim. WLOG, we will prove that properties (1)–(4) hold for $B_1 = B(x_1, a_2, \dots, a_{n-1}, x_n)$.

[(1)] This property clearly follows from the assumption that $(a_1, \dots, a_n) \in B$, $a_n \in S$ and $\{a_1\} \times \dots \times \{a_n - 1\} \times S \not\subseteq B$.

[(2)] Since B is reduced we get, from Lemma 5.1.4, that the projection, $\text{pr}_{\overline{n} \setminus \{1\}} B$, is a reduced subuniverse of \mathbb{A}^{n-1} . Furthermore, $\text{pr}_n(\text{pr}_{\overline{n} \setminus \{1\}} B) = \text{pr}_n B = A$. Therefore, by the minimality of n , $\text{pr}_{\overline{n} \setminus \{1\}} B$ is θ -closed in its last coordinate. Thus $(a_2, \dots, a_n) \in \text{pr}_{\overline{n} \setminus \{1\}} B$, where $a_n \in S$, implies $\{a_2\} \dots \{a_{n-1}\} \times S \subseteq \text{pr}_{\overline{n} \setminus \{1\}} B$, which means $S \subseteq \text{pr}_n B(x_1, a_2, \dots, a_{n-1}, x_n) = \text{pr}_n B_1$.

A similar proof shows that $S \subseteq \text{pr}_n B_2 = \text{pr}_n B(a_1, x_2, a_3, \dots, a_{n-1}, x_n)$. Therefore, for all $s \in S$, there exists some $c_s \in A$ such that $(c_s, s) \in B_2$, which means $(a_1, c_s, a_3, \dots, a_{n-1}, s) \in B$. Then $(a_1, s) \in \text{pr}_{1,n} B$, for all $s \in S$. Therefore $\{a_1\} \times S \subseteq \text{pr}_{1,n} B$.

[(3)] Recall that $|S| \geq 2$ and $a_n \in S$, therefore there exists an element $s \in S \setminus \{a_n\}$. From property (2), $S \subseteq \text{pr}_n B_1$ implies that there exists some $c_s \in A$ such that $(c_s, s) \in B_1$. We claim that $c_s \neq a_1$, otherwise if $c_s = a_1$, then $(a_1, a_n), (a_1, s) = (c_s, s) \in B_1$ and $a_n \neq s$ implies, by Lemma 3.1.7, that $\{a_1\} \times S \subseteq B_1$, which contradicts property (1). Therefore a_1 and c_s are distinct elements in $\text{pr}_1 B_1$. Thus $\text{pr}_1 B_1$ is a nontrivial subuniverse of \mathbb{A} , which means $S \subseteq \text{pr}_1 B_1$.

[(4)] By properties (2) and (3) the unary projections of B_1 are nontrivial. Thus if B_1 is not reduced then it must be an automorphism of \mathbb{A} or \mathbb{S} . Suppose that B_1 is reduced. Since the reduced subuniverses of \mathbb{A}^2 are θ -closed in their A -coordinates and property (1) implies that B_1 is not θ -closed in its last coordinate, it must be that $\text{pr}_n B_1 = S$. Property (1) also implies that $B_1 \neq S \times S$ and $B_1 \neq A \times S$. By Assumption 2, \mathbb{S} is quasiprimal or affine, therefore there is no (S, S) -cross among the subuniverses of \mathbb{A}^2 . Hence, by Theorem 3.1.5, if B_1 is reduced, then B_1 is either an isomorphism from \mathbb{A}/θ to \mathbb{S} , an (A, S) -cross, or a thick (A, S) -cross. This proves property (4) and completes the proof of the claim. \square

Property (4) of Claim 5.4.4.1 narrows the possibilities for each subuniverse B_i . We will now show that, in fact, there are only three possible cases that all of the subuniverses B_i , $1 \leq i \leq n-1$, may simultaneously satisfy. This is the first step in showing that Lemma 5.4.2 holds.

Claim 5.4.4.2. *Under the assumptions of Lemma 5.4.2, one of the following cases holds. Either*

(I) *for all $1 \leq i \leq n-1$,*

$$B_i = \begin{cases} \text{an automorphism of } \mathbb{S}, \\ \text{an automorphism of } \mathbb{A}, \\ \text{an isomorphism } \mathbb{A}/\theta \rightarrow \mathbb{S}, \end{cases}$$

and there exists some $j \in \overline{n-1}$ such that B_j is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, or

(II) *$B_i = \lambda_{S, a_n}$, for all $1 \leq i \leq n-1$, or*

(III) *there exists some $b \in A \setminus S$ such that $B_i = \kappa_{b, a_n}$, for all $1 \leq i \leq n-1$.*

Proof of claim. From property (4) of Claim 5.4.4.1 we know that, for each $i \in \overline{n-1}$, the subuniverse B_i is either an automorphism of \mathbb{S} , an automorphism of \mathbb{A} , an isomorphism from $\mathbb{A}/\theta \rightarrow \mathbb{S}$, an (A, S) -cross, or a thick (A, S) -cross.

First suppose that no subuniverse B_i is an automorphism of \mathbb{S} , an automorphism of \mathbb{A} , or an isomorphism from $\mathbb{A}/\theta \rightarrow \mathbb{S}$. Then by property (4) of Claim 5.4.4.1, each B_i is either an (A, S) -cross

or a thick (A, S) -cross. By property (ii) of Corollary 3.2.5, either B_i is a thick (A, S) -cross, for all $1 \leq i \leq n-1$, or B_i is an (A, S) -cross, for all $1 \leq i \leq n-1$. If B_i is a thick (A, S) -cross, for all $1 \leq i \leq n-1$, then for each $i \in \overline{n-1}$, there exists some $s_i \in S$ such that $B_i = \lambda_{S, s_i}$. In this case let $G = S$. Otherwise B_i is an (A, S) -cross, for all $1 \leq i \leq n-1$, and by statements (i) and (iv) of Corollary 3.2.5 we get that there exists $b \in A \setminus S$ such that, for each $i \in \overline{n-1}$, there exists some $s_i \in S$ such that $B_i = \kappa_{b, s_i}$. In this case let $G = \{b\}$. Then, in either case, $G \times S \subseteq B_i$ which means, by property (1) of Claim 5.4.4.1, that $a_i \notin G$, $1 \leq i \leq n-1$. Thus, $(a_i, a_n) \in B_i \in \{\lambda_{S, s_i}, \kappa_{b, s_i}\}$ and $a_i \notin G$ implies $s_i = a_n$ for all $1 \leq i \leq n-1$, therefore (II) or (III) holds. It remains to show that, in all other cases, (I) holds.

Suppose that there exists some $j \in \overline{n-1}$ such that B_j is either an automorphism of \mathbb{S} , or an automorphism of \mathbb{A} , or an isomorphism from $\mathbb{A}/\theta \rightarrow \mathbb{S}$. We will complete the proof of the claim by first showing some subclaims.

Subclaim 5.4.4.2.1. *There exists no $\{i, j\} \subseteq \overline{n-1}$, $i \neq j$, such that B_i is an automorphism of \mathbb{S} or an automorphism of \mathbb{A} and B_j is a (thick) (A, S) -cross.*

Proof of subclaim. Suppose, for contradiction, that $\{i, j\} \subseteq \overline{n-1}$, $i \neq j$, such that B_i is an automorphism of \mathbb{S} or an automorphism of \mathbb{A} and B_j is a (thick) (A, S) -cross. The later implies, by Assumption 2, that \mathbb{S} is quasiprimal. We will assume, WLOG, that $i = 1$, $j = 2$. Then $B(x_1, x_2, a_3, \dots, a_{n-1}, x_n)$ is a subuniverse of \mathbb{A}^3 and thus $B(x_1, x_2, a_3, \dots, a_{n-1}, x_n) \cap (S \times A \times S) \leq \mathbb{S} \times \mathbb{A} \times \mathbb{S}$. Let $C := B(x_1, x_2, a_3, \dots, a_{n-1}, x_n) \cap (S \times A \times S)$. Then $C(x_1, a_2, x_n) = B(x_1, a_2, a_3, \dots, a_{n-1}, x_n) \cap (S \times S) = B_1 \cap S^2$, therefore $C(x_1, a_2, x_n)$ is an automorphism of \mathbb{S} . Now $(a_1, a_n) \in C(x_1, a_2, x_n)$ and $a_n \in S$ imply that $a_1 \in S$. A similar argument shows that $C(a_1, x_2, x_3) = B_2 \cap (A \times S)$. Since B_2 is a (thick) (A, S) -cross, it follows that $C(x_1, a_2, x_3) = B_2$ is a (thick) (A, S) -cross. Therefore, we have shown that $C \leq \mathbb{S} \times \mathbb{A} \times \mathbb{S}$ and there exists a tuple $(a_1, a_2, a_n) \in C$ such that $C(x_1, a_2, x_n)$ is an automorphism of \mathbb{S} and $C(a_1, x_2, x_n)$ is a (thick) (A, S) -cross. Recall that \mathbb{S} is quasiprimal. Then applying Theorem 5.3.1 to C and the tuple (a_1, a_2, a_n) gives that either Λ or \mathcal{K}_b is a subuniverse of \mathbb{A}^3 for some $b \in A \setminus S$, which contradicts

Assumption 2 (Q). This completes the proof of the subclaim. \square

Subclaim 5.4.4.2.2. *There exists no $\{i, j\} \subseteq \overline{n-1}$, $i \neq j$, such that B_i is an isomorphism from \mathbb{A}/θ to \mathbb{S} and B_j is either an (A, S) -cross or a thick (A, S) -cross.*

Proof of subclaim. Suppose that B_i is an isomorphism from \mathbb{A}/θ to \mathbb{S} , for some $i \in \overline{n-1}$. We are assuming that \mathbb{S} is either quasiprimal or affine, thus it follows from properties (vi) and (vii) of Corollary 3.2.5, respectively, that there is no (A, S) -cross and no thick (A, S) -cross among the subuniverses of \mathbb{A}^2 . The subclaim follows. \square

Subclaim 5.4.4.2.3. *There exists some $1 \leq i \leq n-1$, such that B_i is neither an automorphism of \mathbb{S} nor an automorphism of \mathbb{A} .*

Proof of subclaim. Suppose, for contradiction, that B_i is an automorphism of \mathbb{S} or an automorphism of \mathbb{A} for all $1 \leq i \leq n-1$. Then $B_i \cap S^2 \in \text{Aut}(\mathbb{S})$, for all $1 \leq i \leq n-1$. By property (1) of Claim 5.4.4.1, we have that $(a_i, a_n) \in B_i$ and $a_n \in S$, therefore $a_i \in S$, for all $1 \leq i \leq n$, which means $(a_1, \dots, a_n) \in B \cap S^n$.

Let $\hat{B} = B \cap S^n$. Then \hat{B} is a subuniverse of \mathbb{S}^n that satisfies $(a_1, \dots, a_n) \in \hat{B}$ and $\hat{B}(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{n-1}, x_n)$ is an automorphism of \mathbb{S} , for all $1 \leq i \leq n-1$. Recall that $n \geq 3$ and \mathbb{S} is either quasiprimal or affine. Under these assumptions on \hat{B} we get that \hat{B} satisfies the assumptions of Proposition 5.2.4, therefore it follows from applying statements (i) and (ii) of Proposition 5.2.4 that \mathbb{S} must be affine and that $\hat{B} = B \cap S^n$ is an h.d.-automorphism of \mathbb{S} . Since $\text{pr}_n B = A$, we have that $B \not\leq B \cap S^n$, therefore it follows from Corollary 5.2.3 that B is an h.d.-automorphism of \mathbb{A} . This contradicts our assumption that B is reduced and completes the proof of the subclaim. \square

To sum up, recall that we are considering the case when there exists some $1 \leq j \leq n-1$ such that B_j is either an automorphism of \mathbb{S} , or an automorphism of \mathbb{A} , or an isomorphism from $\mathbb{A}/\theta \rightarrow \mathbb{S}$. Thus we get from Subclaims 5.4.4.2.1 and 5.4.4.2.2, that, in fact, for every $1 \leq i \leq n-1$, B_i is either an automorphism of \mathbb{S} , an automorphism of \mathbb{A} , or an isomorphism from $\mathbb{A}/\theta \rightarrow \mathbb{S}$. Furthermore, by

Subclaim 5.4.4.2.3, there exists some $1 \leq j \leq n - 1$ such that B_j is an isomorphism from $\mathbb{A}/\theta \rightarrow \mathbb{S}$.

Therefore (I) holds.

This completes the proof of Claim 5.4.4.2. \square

We must show three more claims before proving Lemma 5.4.2.

Claim 5.4.4.3. *Under the assumptions of Lemma 5.4.2, the following implications hold.*

(i) *If case (I) of Claim 5.4.4.2 holds and B_i is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, for some $i \in \overline{n-1}$, then $\text{pr}_{i,n} B = A^2$.*

(ii) *If case (II) of Claim 5.4.4.2 holds, then, for any $i \in \overline{n-1}$, $\text{pr}_{i,n} B = \begin{cases} \chi_{S,S} \\ A^2 \end{cases}$.*

(iii) *If case (III) of Claim 5.4.4.2 holds, then, for any $i \in \overline{n-1}$, $\text{pr}_{i,n} B = A^2$.*

Proof of claim. Since B is reduced and $\text{pr}_n B = A$, the binary projection $\text{pr}_{i,n} B$ is a reduced subuniverse of \mathbb{A}^2 that contains some tuple (w_i, w_n) , where $w_n \in A \setminus S$. Since $\text{pr}_n(\text{pr}_{i,n} B) = \text{pr}_n B = A$, it follows from the minimality of n and $n \geq 3$ that $\text{pr}_{i,n} B$ is θ -closed in its second coordinate.

Suppose that B satisfies the assumptions of (i), (ii), or (iii). Then $B_i \subseteq \text{pr}_{i,n} B$, where B_i is either an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, a thick (A, S) -cross, or an (A, S) -cross. We showed that $\text{pr}_{i,n} B$ is θ -closed in its second coordinate, therefore, in all cases we get that $A \times S \subseteq \text{pr}_{i,n} B$. Thus $(A \times S) \cup \{(w_i, w_n)\} \subseteq \text{pr}_{i,n} B$, where $w_n \in A \setminus S$, which implies, in particular, that both coordinates of $\text{pr}_{i,n} B$ are A -coordinates. Hence we get from Theorem 3.1.5 that $\text{pr}_{i,n} B = A^2$ or $\chi_{S,S}$.

It is clear that if case (II) of Claim 5.4.4.2 holds, then implication (ii) holds.

If case (III) of Claim 5.4.4.2 holds, then $\kappa_{b,a_3} \leq A^2$ for some $b \in A \setminus S$. By property (iii) of Corollary 3.2.5, $\chi_{S,S} \not\leq A^2$, therefore implication (iii) holds.

Finally, if case (I) of Claim 5.4.4.2 holds and B_i is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$, then our assumption that \mathbb{S} is either quasiprimal or affine and statement (v) of Corollary 3.2.5 imply that

$\chi_{S,S} \not\leq \mathbb{A}^2$. Hence implication (i) holds. This completes the proof of the claim. \square

Claim 5.4.4.4. *Under the assumptions of Lemma 5.4.2, if $n = 3$, then case (I) of Claim 5.4.4.2 does not hold.*

Proof of claim. Suppose, for contradiction, that $n = 3$ and case (I) of Claim 5.4.4.2 holds. WLOG, suppose that $B_1 = B(x_1, a_2, x_3)$ is an isomorphism from \mathbb{A}/θ to \mathbb{S} .

By Claim 5.4.4.3 we have that $\text{pr}_{1,3} B = A^2$. Let $a \in A$. Then $\{a\} \times A \subseteq A^2 = \text{pr}_{1,3} B$ implies that for each $c \in A$ there exists some $c' \in A$ such that $(a, c', c) \in B$. Therefore, $B(a, x_2, x_3)$ is a subuniverse of \mathbb{A}^2 and $\text{pr}_3 B(a, x_2, x_3) = A$.

Since $B_1 = B(x_1, a_2, x_3)$ is an isomorphism from \mathbb{A}/θ to \mathbb{S} , we get that for each $a \in A$ there exists some $a_s \in S$ such that $(a, a_s) \in B(x_1, a_2, x_2)$ and $\{a\} \times S \not\subseteq B(x_1, a_2, x_2)$. Thus $(a, a_2, a_s) \in B$, where $a_s \in S$ and $\{a\} \times \{a_2\} \times S \not\subseteq B$.

Under the assumptions of Claim 5.4.4.4 we have that the assumptions of Lemma 5.4.2 hold. Then replacing the tuple (a_1, a_2, a_3) with the tuple $(a, a_2, a_s) \in B$, we get that B and the tuple $(a, a_2, a_s) \in B$ satisfy the assumptions of Claim 5.4.4.1. Therefore when we apply Claim 5.4.4.1 to the tuple (a, a_2, a_s) in place of (a_1, a_2, a_3) , we get from property (4) of Claim 5.4.4.1 that $B(a, x_2, x_3)$ is either an automorphism of \mathbb{S} , an automorphism of \mathbb{A} , an isomorphism from \mathbb{A}/θ to \mathbb{S} , or a (thick) (A, S) -cross. Since we have that $\text{pr}_3 B(a, x_2, x_3) = A$, it must be that $B(a, x_2, x_3)$ is an automorphism of \mathbb{A} . The element a was an arbitrary element in A , thus, for all $a \in A$, $B(a, x_2, x_3) = \sigma_a$, where σ_a is an automorphism of \mathbb{A} . This implies that $\text{pr}_{2,3} B \subseteq S^2 \cup (A \setminus S)^2$ and both coordinates of $\text{pr}_{2,3} B$ are A -coordinates. Since B is reduced, we know from Proposition 5.1.4 that $\text{pr}_{2,3} B$ is reduced, therefore it follows from Theorem 3.1.5 that $\text{pr}_{2,3} B$ is an automorphism of \mathbb{A}/θ that fixes \bar{S} . Let $\text{pr}_{2,3} B := \sigma$. Since $\sigma_a = B(a, x_2, x_3) \subseteq \text{pr}_{2,3} B = \sigma$ for all $a \in A$, we get that $\sigma_a|_{A \setminus S} = \sigma|_{A \setminus S}$. Thus, by Lemma 3.0.6, $\sigma_a = \sigma_{a'}$, for all distinct $a, a' \in A$, which means $B(a, x_2, x_3) = \sigma_a = B(a', x_2, x_3)$. Therefore, $B = \{(y, x, \sigma_a(x)) : y, x \in A\}$, which implies $\text{pr}_{2,3} B = \sigma_a \in \text{Aut}(\mathbb{A})$, a contradiction to the fact that the binary projections of B are reduced. \square

We have one more claim to show before proving Lemma 5.4.2.

Claim 5.4.4.5. *Under the assumptions of Lemma 5.4.2, for all $1 \leq i \leq n - 1$, the subuniverse $B(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n) \leq \mathbb{A}^{n-1}$ is either reduced or it is an h.d.-automorphism of \mathbb{S} or \mathbb{A} .*

Proof of claim. WLOG, we will prove the claim for the subuniverse $B(a_1, x_2, \dots, x_n) \leq \mathbb{A}^{n-1}$.

Subclaim 5.4.4.5.1. *No unary projection of $B(a_1, x_2, \dots, x_n)$ is a singleton.*

Proof of subclaim. Let i be arbitrary, $2 \leq i \leq n - 1$. Then the unary projection

$$\text{pr}_i B(a_1, x_2, \dots, x_n) \supseteq \text{pr}_i B(a_1, a_2, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{n-1}, x_n) = \text{pr}_i B_i.$$

Property (3) of Claim 5.4.4.1 implies that $S \subseteq \text{pr}_i B_i$, therefore $S \subseteq \text{pr}_i B_i \subseteq \text{pr}_i B(a_1, x_2, \dots, x_n)$.

Furthermore, $\text{pr}_n B(a_1, x_2, \dots, x_n) \supseteq \text{pr}_n B(a_1, x_2, a_3, \dots, a_{n-1}, x_n) = \text{pr}_n B_2$ and from property (2) of Claim 5.4.4.1 we have that $S \subseteq \text{pr}_n B_2$, thus $S \subseteq \text{pr}_n B_2 \subseteq \text{pr}_n B(a_1, x_2, \dots, x_n)$. Hence no unary projection of $B(a_1, x_2, \dots, x_n)$ is a singleton. \square

Subclaim 5.4.4.5.2. *If $n = 3$, then $B(a_1, x_2, x_3)$ is not an automorphism of \mathbb{S} or \mathbb{A} .*

Proof of subclaim. Suppose, for contradiction, that $B(a_1, x_2, x_3)$ is an automorphism of \mathbb{S} or an automorphism of \mathbb{A} . Then cases (II) and (III) of Claim 5.4.4.2 cannot hold for B , therefore it must be that case (I) of Claim 5.4.4.2 holds, which contradicts Claim 5.4.4.4. This proves the subclaim. \square

Subclaim 5.4.4.5.3. *If $n \geq 4$, then for $2 \leq m \leq n - 2$, no m -ary projection of $B(a_1, x_2, \dots, x_n)$ is an automorphism of \mathbb{S} , an automorphism of \mathbb{A} , an h.d.-automorphism of \mathbb{S} or an h.d.-automorphism of \mathbb{A} .*

Proof of subclaim. Let $n \geq 4$ and $2 \leq m \leq n - 2$. Suppose, for contradiction, that there exists some $I \subseteq \bar{n} \setminus \{1\}$, where $|I| = m$, such that the projection $\text{pr}_I B(a_1, x_2, \dots, x_n)$ is either an automorphism of \mathbb{S} , an automorphism of \mathbb{A} , an h.d.-automorphism of \mathbb{S} or an h.d.-automorphism of \mathbb{A} . There are two cases to be considered: the case when $n \notin I$ and the case when $n \in I$.

First suppose that $n \notin I$. WLOG, permute the coordinates of B so that $I = \{2, \dots, m + 1\}$.

Recall that automorphisms and h.d.-automorphisms share the property that if one fixes all but one

of the coordinates of a tuple from the relation, then there is exactly one element that can satisfy the remaining coordinate. Since $\bar{a} \in B$ implies $(a_2, \dots, a_m, a_{m+1}) = \bar{a}_I \in \text{pr}_I B(a_1, x_2, \dots, x_n)$ and $\text{pr}_I B(a_1, x_2, \dots, x_n)$ is either an automorphism of \mathbb{S} , an automorphism of \mathbb{A} , an h.d.-automorphism of \mathbb{S} or an h.d.-automorphism of \mathbb{A} , it follows that

$$\text{pr}_{m+1} B(a_1, a_2, \dots, a_m, x_{m+1}, x_{m+2}, \dots, x_n) = \{a_{m+1}\}.$$

However, by property (3) of Claim 5.4.4.1, we have that

$$\begin{aligned} S &\subseteq \text{pr}_{m+1} B_{m+1} = \text{pr}_{m+1} B(a_1, a_2, \dots, a_m, x_{m+1}, a_{m+2}, \dots, a_{n-1}, x_n) \\ &\subseteq \text{pr}_{m+1} B(a_1, a_2, \dots, a_m, x_{m+1}, x_{m+2}, \dots, x_{n-1}, x_n) = \{a_{m+1}\}, \end{aligned}$$

which is a contradiction.

Suppose that $n \in I$. WLOG, permute the coordinates of B so that $I = \{2, \dots, m-1, n\}$. Then $\bar{a} \in B$ implies $(a_2, \dots, a_{m-1}, a_n) = \bar{a}_I \in \text{pr}_I B(a_1, x_2, \dots, x_n)$. Therefore, by our assumptions on $\text{pr}_I B(a_1, x_2, \dots, x_n)$, we get that

$$\text{pr}_n B(a_1, a_2, \dots, a_{m-1}, x_m, x_{m+1}, \dots, x_n) = \{a_n\}.$$

However, by property (2) of Claim 5.4.4.1, we have that

$$\begin{aligned} S &\subseteq \text{pr}_n B_m = \text{pr}_n B(a_1, a_2, \dots, a_{m-1}, x_m, a_{m+1}, \dots, a_{n-1}, x_n) \\ &\subseteq \text{pr}_n B(a_1, a_2, \dots, a_{m-1}, x_m, x_{m+1}, \dots, x_{n-1}, x_n) = \{a_n\}, \end{aligned}$$

therefore we have a contradiction. The proof of the subclaim is complete. \square

We have shown in Subclaims 5.4.4.5.1 and 5.4.4.5.3 that, for $1 \leq m \leq n-2$, no m -ary projection of $B(a_1, x_2, \dots, x_n)$ is a singleton, an automorphism of \mathbb{S} , an automorphism of \mathbb{A} , an h.d.-automorphism of \mathbb{S} , or an h.d.-automorphism of \mathbb{A} . Furthermore, we have shown in Subclaim 5.4.4.5.2 that for $n=3$, $B(a_1, x_2, x_3)$ itself is not an automorphism of \mathbb{S} or \mathbb{A} . Additionally, since $B(a_1, x_2, x_3) \leq \mathbb{A}^2$, it is clear from Definition 5.0.9 that $B(a_1, x_2, x_3) \leq \mathbb{A}^2$ cannot be an h.d.-automorphism of \mathbb{S} or \mathbb{A} . Therefore it follows from Definition 5.1.1 that $B(a_1, x_2, \dots, x_n)$ is either reduced or it is an h.d.-automorphism of \mathbb{S} or \mathbb{A} . This completes the proof of the claim. \square

We are now ready to prove Lemma 5.4.2.

Proof of Lemma 5.4.2. First we will prove an auxiliary result.

Subclaim 5.4.4.5.4. *Let $i \in \overline{n-1}$. If $B(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n)$ is reduced then $\text{pr}_{i,n} B \neq A^2$.*

Proof of subclaim. WLOG, we will show that the subclaim for $i = 1$. Suppose $B(a_1, x_2, \dots, x_n)$ is reduced. It follows from our assumptions on B , that $(a_2, \dots, a_n) \in B(a_1, x_2, \dots, x_n)$, where $a_n \in S$ and $\{a_2\} \times \dots \times \{a_{n-1}\} \times S \not\subseteq B(a_1, x_2, \dots, x_n)$. Thus $B(a_1, x_2, \dots, x_n)$ is a reduced subuniverse of \mathbb{A}^{n-1} that is not θ -closed in its last coordinate. By the minimality of n , we get that $\text{pr}_n B(a_1, x_2, \dots, x_n) = S$. This means, for any $c \in A \setminus S$ there exists no $\bar{u} \in A^{n-2}$ such that $(a_1, \bar{u}, c) \in B$. Hence $\{a_1\} \times A \not\subseteq \text{pr}_{1,n} B$, which means $\text{pr}_{1,n} B \neq A^2$. \square

Now we start proving Lemma 5.4.2. It follows from Claim 5.4.4.5 that, for each $i \in \overline{n-1}$, the subuniverse $B(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n)$ of \mathbb{A}^{n-1} is either reduced or it is an h.d.-automorphism of \mathbb{S} or \mathbb{A} .

Suppose first that there exists some $j \in \overline{n-1}$ such that $B(x_1, \dots, x_{j-1}, a_j, x_{j+1}, \dots, x_n)$ is not reduced. Then $B(x_1, \dots, x_{j-1}, a_j, x_{j+1}, \dots, x_n)$ is an h.d.-automorphism of \mathbb{S} or an h.d.-automorphism of \mathbb{A} . Thus, it follows from Definition 5.0.9 that $(a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n) \in B(x_1, \dots, x_{j-1}, a_j, x_{j+1}, \dots, x_n)$ implies B_i is the graph of a bijection $\text{pr}_i B_i \rightarrow \text{pr}_n B_i$ for all $i \in \overline{n-1} \setminus \{j\}$. We know from statements (2) and (3) of Claim 5.4.4.1 that $S \subseteq \text{pr}_n B_i$ and $S \subseteq \text{pr}_i B_i$, therefore B_i is an automorphism of \mathbb{S} or an automorphism of \mathbb{A} . Hence case (I) of Claim 5.4.4.2 holds and j is unique with respect to the property that B_j is an isomorphism from \mathbb{A}/θ to \mathbb{S} . Hence j is also unique with respect to the property that $B(x_1, \dots, x_{j-1}, a_j, x_{j+1}, \dots, x_n)$ is not reduced. Thus $B(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n)$ is reduced for all $i \in \overline{n-1} \setminus \{j\}$, which implies by Subclaim 5.4.4.5.4 that $\text{pr}_{i,n} B \neq A^2$ for all such i . For j , the equality $\text{pr}_{j,n} B = A^2$ was established in Claim 5.4.4.3.

To see that (I') holds, it remains to verify that $B(x_1, \dots, x_{j-1}, a_j, x_{j+1}, \dots, x_n)$ is an h.d.-automorphism of \mathbb{A} (rather than an h.d.-automorphism of \mathbb{S}). To simplify notation we will assume that $j = 1$. We have that $\text{pr}_{1,n} B = A^2$, which means $\{a_1\} \times A \subseteq \text{pr}_{1,n} B$. Then, for each $c \in A$, there exists some $\bar{u} \in \text{pr}_{2,\dots,n-1} B$ such that $(a_1, \bar{u}, c) \in B$. Thus $\text{pr}_n B(a_1, x_2, \dots, x_n) = A$. Hence $B(a_1, x_2, \dots, x_n)$ is not an h.d.-automorphism of \mathbb{S} , it must be an h.d.-automorphism of \mathbb{A} . This shows that if $B(x_1, \dots, x_{j-1}, a_j, x_{j+1}, \dots, x_n)$ is not reduced for some $1 \leq j \leq n-1$, then (I') holds.

Now suppose that $B(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n)$ is reduced, for all $1 \leq i \leq n-1$. Then it follows from Subclaim 5.4.4.5.4 that $\text{pr}_{i,n} B \neq A^2$, for all $1 \leq i \leq n-1$. Reviewing Claim 5.4.4.3, we see that if $\text{pr}_{i,n} B \neq A^2$, for all $1 \leq i \leq n-1$, then only case (II) of Claim 5.4.4.2 can hold with $\text{pr}_{i,n} B = \chi_{S,S}$, for all $1 \leq i \leq n-1$. This shows (II'). \square

To finish the proof of this theorem it remains to prove Lemmas 5.4.3 and 5.4.4, that is, we must show that cases (I') and (II') of Lemma 5.4.2 each cannot occur. We will first show that case (I') of Lemma 5.4.2 cannot occur, thus proving Lemma 5.4.3.

Proof of Lemma 5.4.3. Suppose, for contradiction, that case (I') of Lemma 5.4.2 holds. Then Claim 5.4.4.4 implies that $n > 3$. WLOG, suppose $j = 1$ is the unique element of $\overline{n-1}$ such that the subuniverse B_1 is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$. Then, by Lemma 5.4.2, $B(a_1, x_2, \dots, x_n)$ is an h.d.-automorphism of \mathbb{A} , so \mathbb{S} is affine.

Subclaim 5.4.4.5.5. *If B_1 is an isomorphism from \mathbb{A}/θ to \mathbb{S} , then for each $a \in A$, there exists some $s_a \in S$ such that the tuple $(a, a_2, \dots, a_{n-1}, s_a) \in B$ and $\{a\} \times \{a_2\} \times \dots \times \{a_{n-1}\} \times S \not\subseteq B$.*

Proof of subclaim. Suppose that B_1 is an isomorphism from \mathbb{A}/θ to \mathbb{S} . Then for all $a \in A$, there exists some $s_a \in S$ such that $(a, s_a) \in B_1$ and $\{a\} \times S \not\subseteq B_1$. Since $B_1 = B(x_1, a_2, \dots, a_{n-1}, x_n)$, this means the tuple $(a, a_2, \dots, a_{n-1}, s_a) \in B$, where $s_a \in S$ and $\{a\} \times \{a_2\} \times \dots \times \{a_{n-1}\} \times S \not\subseteq B$. \square

Subclaim 5.4.4.5.6. *If B_1 is an isomorphism from \mathbb{A}/θ to \mathbb{S} , then for each $a \in A$ the subuniverse $B(a, x_2, \dots, x_n)$ of $\mathbb{A}^{\overline{n} \setminus \{1\}}$ is an h.d.-automorphism of \mathbb{A} .*

Proof of subclaim. Let B_1 an isomorphism from \mathbb{A}/θ to \mathbb{S} and let $a \in A$. From Subclaim 5.4.4.5.5, there exists a tuple $(a, a_2, \dots, a_{n-1}, s_a) \in B$, where $s_a \in S$ and $\{a\} \times \{a_2\} \times \dots \times \{a_{n-1}\} \times S \not\subseteq B$. Under the assumptions of Lemma 5.4.3 we have that the assumptions of Lemma 5.4.2 hold. Then replacing the tuple (a_1, \dots, a_n) with the tuple $(a, a_2, \dots, a_{n-1}, s_a) \in B$, we get that B and the tuple $(a, a_2, \dots, a_{n-1}, s_a) \in B$ satisfy the assumptions of Lemma 5.4.2. Since the assumption that B_1 is an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$ implies, by Lemma 5.4.2, that $\text{pr}_{1,n} B = A^2$ and $\text{pr}_{i,n} B \neq A^2$ for $i = 2, \dots, n-1$, therefore when we apply Lemma 5.4.2 to the tuple $(a, a_2, \dots, a_{n-1}, s_a)$ in place of $(a_1, a_2, \dots, a_{n-1}, a_n)$, the only case possible for $(a, a_2, \dots, a_{n-1}, s_a)$ is again case (I') with $j = 1$. This shows that $B(a, x_2, \dots, x_n)$ is an h.d.-automorphism of \mathbb{A} , as claimed. \square

It follows from Corollary 5.2.3 that for each $a \in A$,

$$B(a, x_2, \dots, x_n) = (B(a, x_2, \dots, x_n) \cap S^{n-1}) \cup \sigma_a$$

where $\sigma_a = \{(x, \pi_3^a(x), \dots, \pi_n^a(x)) : x \in A \setminus S\}$ and π_i^a is an automorphism of \mathbb{A}/θ that fixes \bar{S} , for all $3 \leq i \leq n$.

Subclaim 5.4.4.5.7. *For all distinct $a, a' \in A$, $\sigma_a = \sigma_{a'}$.*

Proof of subclaim. Let $a, a' \in A$, $a \neq a'$. Suppose that $\sigma_a = \{(x, \pi_3(x), \dots, \pi_n(x)) : x \in A \setminus S\}$ and $\sigma_{a'} = \{(x, \gamma_3(x), \dots, \gamma_n(x)) : x \in A \setminus S\}$, where π_i and γ_i are automorphisms of \mathbb{A}/θ that fix \bar{S} , for all $3 \leq i \leq n$. Let $i \in \{3, \dots, n\}$. Then $\text{pr}_{2,i} B \supseteq \{(x, \pi_i(x)) : x \in A \setminus S\} \cup \{(x, \gamma_i(x)) : x \in A \setminus S\}$. Since $\text{pr}_{2,i} B = \text{pr}_{2,i}(\bigcup_{a \in A} B(a, x_2, \dots, x_n)) = \bigcup_{a \in A} (\text{pr}_{2,i} B(a, x_2, \dots, x_n)) \subseteq S^2 \cup (A \setminus S)^2$ and $\text{pr}_{2,i} B \leq \mathbb{A}^2$ we have that $\text{pr}_{2,i} B$ is an automorphism of \mathbb{A}/θ that fixes \bar{S} , therefore it must be that $\pi_i(x) = \gamma_i(x)$, for all $x \in A \setminus S$. The element $i \in \{3, \dots, n\}$ was arbitrary, hence $\sigma_a = \sigma_{a'}$. \square

It follows from Subclaim 5.4.4.5.7 that, for all $a \in A$,

$$B(a, x_2, \dots, x_n) = (B(a, x_2, \dots, x_n) \cap S^{n-1}) \cup \sigma,$$

where $\sigma = \{(x, \pi_3(x), \dots, \pi_n(x)) : x \in A \setminus S\}$ and π_i is an automorphism of \mathbb{A}/θ that fixes \overline{S} , for all $3 \leq i \leq n$.

Furthermore, for each $a \in A$, we have shown that $B(a, x_2, \dots, x_n)$ is an h.d.-automorphism of \mathbb{A} , therefore there exists a map

$$f_a : \text{pr}_{2, \dots, n-1} B(a, x_2, \dots, x_n) \rightarrow \text{pr}_n B(a, x_2, \dots, x_n) : (x_2, \dots, x_{n-1}) \mapsto x_n,$$

where x_n is the unique element of $\text{pr}_n B(a, x_2, \dots, x_n)$ such that $(x_2, \dots, x_{n-1}, x_n) \in B(a, x_2, \dots, x_n)$.

Subclaim 5.4.4.5.8. *Let $c \in A \setminus S$, $c' \in A$, $c \neq c'$. Then $f_c(s_2, \dots, s_{n-1}) \neq f_{c'}(s_2, \dots, s_{n-1})$ for all $(s_2, \dots, s_{n-1}) \in S^{n-2}$.*

Proof of subclaim. Suppose, for contradiction, that there exists some $(s_2, \dots, s_{n-1}) \in S^{n-2}$ such that $f_c(s_2, \dots, s_{n-1}) = f_{c'}(s_2, \dots, s_{n-1})$. Then there exists some $s_n \in S$ such that $f_c(s_2, \dots, s_{n-1}) = s_n = f_{c'}(s_2, \dots, s_{n-1})$.

We claim that $A \times \{s_2\} \times \dots \times \{s_n\} \subseteq B$. From $f_c(s_2, \dots, s_{n-1}) = s_n = f_{c'}(s_2, \dots, s_{n-1})$, we get that $(c, s_2, \dots, s_{n-1}, s_n), (c', s_2, \dots, s_{n-1}, s_n) \in B$, thus the subuniverse $B(x_1, s_2, \dots, s_{n-1}, x_2)$ of \mathbb{A}^2 contains $(c, s_n), (c', s_n)$. Since $c \neq c'$ and $c \in A \setminus S$, it follows from Lemma 3.1.8 that $A \times \{s_n\} \subseteq B(x_1, s_2, \dots, s_{n-1}, x_2)$, thus $A \times \{s_2\} \times \dots \times \{s_n\} \subseteq B$.

In particular, for arbitrary $a \in A$, the tuple $(a, s_2, \dots, s_{n-1}, s_n) \in B$. Since $B(a, x_2, \dots, x_n)$ is an h.d.automorphism of \mathbb{A} and $(s_2, \dots, s_{n-1}, s_n) \in B(a, x_2, \dots, x_n)$, it follows from Definition 5.0.9 that $B(a, s_2, \dots, s_{n-1}, x_n) = \{s_n\}$. Hence $B(a, s_2, \dots, s_{n-1}, x_n) = \{s_n\}$, for all $a \in A$.

Clearly, $B(a, s_2, \dots, s_{n-1}, x_n) = \{s_n\}$ implies $S \not\subseteq B(a, s_2, \dots, s_{n-1}, x_n)$, therefore we have that $(a, s_2, \dots, s_{n-1}, s_n) \in B$, where $s_n \in S$ and $\{a\} \times \{s_2\} \times \dots \times \{s_{n-1}\} \times S \not\subseteq B$. Under the assumptions of Lemma 5.4.3 we have that the assumptions of Lemma 5.4.2 hold. Then replacing the tuple (a_1, \dots, a_n) with the tuple $(a, s_2, \dots, s_{n-1}, s_n) \in B$, we get that B and the tuple $(a, s_2, \dots, s_{n-1}, s_n) \in B$ satisfy the assumptions of Claim 5.4.4.1. Therefore when we apply property (2) of Claim 5.4.4.1 to the tuple $(a, s_2, \dots, s_{n-1}, s_n)$ in place of (a_1, \dots, a_n) , we get that $S \subseteq \text{pr}_2 B(x_1, s_2, \dots, s_{n-1}, x_2)$. Thus, for each $s \in S$, there exists some $a_s \in A$ such that $(a_s, s) \in B(x_1, s_2, \dots, s_{n-1}, x_2)$.

Let $s \in S \setminus \{s_n\}$. Such an element exists since $|S| \geq 2$. Then $(a_s, s_2, \dots, s_{n-1}, s) \in B$. Furthermore, we found that $A \times \{s_1\} \times \dots \times \{s_n\} \subseteq B$, therefore, $(a_s, s_2, \dots, s_{n-1}, s_n) \in B$. Then $B(a_s, s_2, \dots, s_{n-1}, x_n)$ is a subuniverse of \mathbb{A} contains $\{s, s_n\}$, where $s \neq s_n$. However, this contradicts $B(a_s, s_2, \dots, s_{n-1}, x_n) = \{s_n\}$. This completes the proof of the subclaim. \square

Subclaim 5.4.4.5.9. $A \setminus S$ is a subuniverse of \mathbb{A} .

Proof of subclaim. Let $c \in A \setminus S$, $c' \in A$, $c \neq c'$. Let $\Delta = \{(x, x) : x \in A\}$. Let

$$D := \{(x, x') : \text{there exists } \bar{x} \in A^{n-2} \text{ such that } (c, \bar{x}, x), (c', \bar{x}, x') \in B\}.$$

Then D is a subuniverse of \mathbb{A}^2 and we claim that $D \cap \Delta = \{(x, x) : x \in A \setminus S\}$.

By definition of D , we have that D is a subuniverse of \mathbb{A}^2 . Suppose $(x, x') \in D$. Then there exists some $(a_2, \dots, a_{n-1}) \in A^{n-1}$ such that $(c, a_2, \dots, a_{n-1}, x), (c', a_2, \dots, a_{n-1}, x') \in B$, which means $(a_2, \dots, a_{n-1}, x) \in B(c, x_2, \dots, x_n)$ and $(a_2, \dots, a_{n-1}, x') \in B(c', x_1, \dots, x_{n-1})$. By Subclaims 5.4.4.5.6 and 5.4.4.5.7, $B(c, x_2, \dots, x_n)$ and $B(c', x_2, \dots, x_n)$ are h.d.-automorphisms of \mathbb{A} , where

$$B(c, x_2, \dots, x_n) = (B(c, x_2, \dots, x_n) \cap S^{n-1}) \cup \sigma,$$

$$B(c', x_2, \dots, x_n) = (B(c', x_2, \dots, x_n) \cap S^{n-1}) \cup \sigma,$$

and $\sigma \subseteq (A \setminus S)^{n-1}$. Hence $(a_2, \dots, a_{n-1}, x), (a_2, \dots, a_{n-1}, x') \in S^{n-1} \cup (A \setminus S)^{n-1}$.

If $(a_2, \dots, a_{n-1}, x) \in S^{n-1}$, then $(a_2, \dots, a_{n-1}) \in S^{n-2}$ and by Subclaim 5.4.4.5.8, we get that $x = f_c(a_2, \dots, a_{n-1}) \neq f_{c'}(a_2, \dots, a_{n-1}) = x'$. Therefore $x \neq x'$ and $(x, x') \in S^2$.

If $(a_2, \dots, a_{n-1}, x) \in (A \setminus S)^{n-1}$, then $(a_2, \dots, a_{n-1}, x), (a_2, \dots, a_{n-1}, x') \in \sigma$, hence $x = \pi_n(a_2) = x'$. Hence $x = x'$ and $(x, x') \in (A \setminus S)^2$.

By the definition of σ , we get that the tuple $(a, \pi_2(a), \dots, \pi_n(a)) \in \sigma \subseteq B(c, x_1, \dots, x_{n-1}) \cap B(c, x_1, \dots, x_{n-1})$ for any $a \in A \setminus S$. Thus $(c, a, \pi_2(a), \dots, \pi_n(a)), (c', a, \pi_2(a), \dots, \pi_n(a)) \in B$ for any $a \in A \setminus S$, which means $\{(x, x) : x \in A \setminus S\} \subseteq D$. Therefore $D \cap \Delta = \{(x, x) : x \in A \setminus S\}$. Since

D and Δ are subuniverses of \mathbb{A}^2 and relational clones are closed under intersections and projections, we get that $\text{pr}_1(D \cap \Delta) = A \setminus S$ is a subalgebra of \mathbb{A} . \square

Recall that \mathbb{S} is the unique nontrivial subalgebra of \mathbb{A} , therefore we have a contradiction to Subclaim 5.4.4.5.9. Hence, case (I') of Lemma 5.4.2 cannot occur. \square

Since case (I') of Lemma 5.4.2 cannot occur, it must be that case (II') of Lemma 5.4.2 holds. To complete the proof of this theorem, we will show that case (II') cannot occur, this is Lemma 5.4.4.

Recall from Definition 4.1.2 that if \mathbb{A} is a finite idempotent algebra, $\mathbb{A}' \leq \mathbb{A}$, $G \subsetneq A'$, and $n \geq 1$, then the n -dimensional cross on A' at G is

$$X_n^{A',G} := \{(a_1, \dots, a_n) \in (\mathbb{A}')^n : \text{there exists } i \text{ such that } a_i \in G\}.$$

In the case that $\mathbb{A}' = \mathbb{A}$, we will simply write X_n^G . Therefore, the n -dimensional cross on A at S is

$$X_n^S := \{(a_1, \dots, a_n) \in (\mathbb{A})^n : \text{there exists } i \text{ such that } a_i \in S\}.$$

Notice that $\chi_{S,S} = X_2^S$.

We must show three claims before proving Lemma 5.4.4.

Claim 5.4.4.6. *Let $m \geq 3$. Suppose \mathbb{S} is quasiprimal, D is a reduced subuniverse of \mathbb{A}^m , and $\chi_{S,S} \leq \mathbb{A}^2$. If there exists an element $(a_1, \dots, a_m) \in D$, where $a_m \in S$ and $\{a_1\} \times \dots \times \{a_{m-1}\} \times S \not\subseteq D$, and if the subuniverse $D(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{m-1}, x_m) = \lambda_{S,a_m}$, for all $1 \leq i \leq m-1$, then the following properties hold for D ,*

$$(i) \quad A^{m-1} \times \{a_m\} \subseteq D,$$

$$(ii) \quad X_{m-1}^S \times S \subseteq D.$$

Proof of claim. Let $D_i := D(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{m-1}, x_m)$. Then $D_i = \lambda_{S,a_m}$, for all $i \in \overline{m-1}$. Since $(a_1, \dots, a_m) \in D$, where $a_m \in S$ and $\{a_1\} \times \dots \times \{a_{m-1}\} \times S \not\subseteq D$, we get that $(a_i, a_m) \in D_i$ and $\{a_i\} \times S \not\subseteq D_i$, for all $1 \leq i \leq m-1$. Thus $D_i = \lambda_{S,a_m}$ implies $a_i \in A \setminus S$, for all $1 \leq i \leq m-1$.

We will prove the claim by inducting on m . First suppose that $m = 3$. Then for $i = 1, 2$ we have that $D_i = \lambda_{S, a_3}$ and $a_i \in A \setminus S$. Thus $A \times \{a_3\} \subseteq D_1 \cap D_2$, which means $(A \times \{a_2\} \times \{a_3\}) \cup (\{a_1\} \times A \times \{a_3\}) \subseteq D$. Therefore $D(x_1, x_2, a_3)$ is a subuniverse of \mathbb{A}^2 that contains $(A \times \{a_2\}) \cup (\{a_1\} \times A) = \mu_{a_1, a_2}$ where $a_1, a_2 \in A \setminus S$. We are assuming that $\chi_{S, S} \leq \mathbb{A}^2$, this implies by property (x) of Proposition 3.2.1 that $\mu_b \not\leq \mathbb{A}^2$ for any $b \in A \setminus S$. Therefore it follows from Theorem 3.1.5 that $D(x_1, x_2, a_3) = A^2$, hence $A^2 \times \{a_3\} \subseteq D$ which proves property (i) when $m = 3$. Now let $s \in S \setminus \{a_3\}$ and consider the subuniverse $D(x_1, x_2, s)$ of \mathbb{A}^2 . Since $S \times \{s\} \subseteq \lambda_{S, a_3} = D_i$ for $i = 1, 2$, we have that $(S \times \{a_2\} \times \{s\}) \cup (\{a_1\} \times S \times \{s\}) \subseteq D$, therefore $D(x_1, x_2, s) \supseteq (S \times \{a_2\}) \cup (\{a_1\} \times S)$ where $a_1, a_2 \in A \setminus S$. We are assuming that $\chi_{S, S} \leq \mathbb{A}^2$, thus it follows from property (vi) of Proposition 3.2.4 that every automorphism of \mathbb{A}/θ fixes \bar{S} . In particular, there is no automorphism among the subuniverses of \mathbb{A}^2 that fixes an element $\bar{b} \in A/\theta$ where $b \in A \setminus S$. Thus $(S \times \{a_2\}) \cup (\{a_1\} \times S) \subseteq D(x_1, x_2, s)$ implies, by Theorem 3.1.5, that $\chi_{S, S} \subseteq D(x_1, x_2, s)$. Hence $D \supseteq (A \times S \times \{s\}) \cup (S \times A \times \{s\}) = X_2^S \times \{s\}$. We chose $s \in S \setminus \{a_3\}$ arbitrarily, furthermore from property (i) we have that $D \supseteq A^2 \times \{a_3\} \supseteq X_2^S \times \{a_3\}$, therefore

$$D \supseteq (X_2^S \times \bigcup_{s \in A \setminus \{a_3\}} \{s\}) \cup (X_2^S \times \{a_3\}) = X_2^S \times S.$$

This proves property (ii) when $m = 3$.

We have shown that if the claim fails, then it must fail for some $m > 3$. To aid in our inductive step we will show that if $m > 3$, then $D(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m)$ is a subuniverse of $\mathbb{A}^{\overline{m} \setminus \{i\}}$ that satisfies the assumptions of the claim for all $1 \leq i \leq m-1$. Fix $i \in \overline{m-1}$. The assumptions of the claim clearly imply that $(a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m) \in D(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m)$, where $a_m \in S$ and $\{a_1\} \times \dots \times \{a_{i-1}\} \times \{a_{i+1}\} \times \dots \times \{a_{m-1}\} \times S \not\subseteq D(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m)$. Furthermore, for all $j \in \overline{m-1} \setminus \{i\}$, $D(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_{m-1}, x_m) = D_j = \lambda_{S, a_m}$. Lastly, to finish showing that $D(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m)$ satisfies the assumptions of this claim, we must show that $D(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m)$ is reduced.

Subclaim 5.4.4.6.1. *Let $i \in \overline{m-1}$. Then $D(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m)$ is a reduced subuniverse of \mathbb{A}^{m-1} .*

Proof of subclaim. By the assumptions of Claim 5.4.4.6, we have that \mathbb{S} is quasiprimal, therefore \mathbb{S} and \mathbb{A} have no h.d.-automorphisms, by definition. WLOG we will prove the subclaim for $D(a_1, x_2, \dots, x_m)$. We must show that $D(a_1, x_2, \dots, x_m)$ has no singleton unary projection and no binary projection that is an automorphism of \mathbb{S} or an automorphism of \mathbb{A} . We have that $D_i = \lambda_{S, a_i}$, for all $2 \leq i \leq m-1$. Thus, for any $2 \leq i \leq m-1$, the unary projection

$$\text{pr}_i D(a_1, x_2, \dots, x_m) \supseteq \text{pr}_i D(a_1, a_2, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{m-1}, x_m) = \text{pr}_i D_i = A,$$

and

$$\text{pr}_m D(a_1, x_2, \dots, x_m) \supseteq \text{pr}_m D(a_1, x_2, a_3, \dots, a_{m-1}, x_m) = \text{pr}_m D_2 = S.$$

Therefore, no unary projection of $D(a_1, x_2, \dots, x_m)$ is a singleton.

Furthermore, for any $2 \leq i \leq m-1$,

$$\text{pr}_{i,m} D(a_1, x_2, \dots, x_m) \supseteq D(a_1, a_2, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{m-1}, x_m) = D_i = \lambda_{S, a_i}.$$

Therefore, for all $2 \leq i \leq m-1$ the binary projection $\text{pr}_{i,m} D(a_1, x_2, \dots, x_m)$ is not an automorphism of \mathbb{S} or an automorphism of \mathbb{A} .

Finally suppose, for contradiction, that $\text{pr}_{i,j} D(a_1, x_2, \dots, x_m)$ is an automorphism of \mathbb{S} or an automorphism of \mathbb{A} , for some $2 \leq i < j < m$. Recall that $m > 3$ so, WLOG, suppose $i = 2, j = 3$. Then $(a_2, \dots, a_m) \in D(a_1, x_2, \dots, x_m)$ implies that $(a_2, a_3) \in \text{pr}_{2,3} D(a_1, x_2, \dots, x_m)$, which means $\text{pr}_3 D(a_1, a_2, x_3, \dots, x_m) = \{a_3\}$. However, since $D_3 = \lambda_{S, a_3}$, we get that

$$A = \text{pr}_3 D_3 = \text{pr}_3 D(a_1, a_2, x_3, a_4, \dots, a_{m-1}, x_m) \subseteq \text{pr}_3 D(a_1, a_2, x_3, \dots, x_m) = \{a_3\},$$

which is a contradiction. This completes the proof of the subclaim. \square

Since $i \in \overline{m-1}$ was arbitrary, we have shown that if $m > 3$, then for all $1 \leq i \leq m-1$, $D(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m)$ is a subuniverse of \mathbb{A}^{m-1} that satisfies the assumptions of Claim 5.4.4.6. We will now prove properties (i) and (ii).

[(i)] Let m be minimal such that property (i) fails for a subuniverse, D , of \mathbb{A}^m that satisfies the assumptions of the claim. We showed above that property (i) holds if $m = 3$. Thus, $m > 3$ and

by the minimality of m , we get that property (i) holds for $D(a_1, x_2, \dots, x_m), D(x_1, a_2, x_3, \dots, x_m) \leq \mathbb{A}^{m-1}$, which means $A^{m-2} \times \{a_m\} \subseteq D(a_1, x_2, \dots, x_m) \cap D(x_1, a_2, x_3, \dots, x_m)$ and thus, $(\{a_1\} \times A^{m-2} \times \{a_n\}) \cup (A \times \{a_2\} \times A^{m-3} \times \{a_n\}) \subseteq D$. Let $\bar{u} \in A^{m-3}$. Then $D(x_1, x_2, \bar{u}, a_n)$ is a subuniverse of \mathbb{A}^2 that contains $(\{a_1\} \times A) \cup (A \times \{a_2\}) = \mu_{a_1, a_2}$, where $a_1, a_2 \in A \setminus S$. Since $\chi_{S, S} \leq \mathbb{A}^2$, it follows from property (x) of Proposition 3.2.1 and Theorem 3.1.5 that $D(x_1, x_2, \bar{u}, a_n) = A^2$. Thus, $A^2 \times \{\bar{u}\} \times \{a_n\} \subseteq D$. Since \bar{u} was an arbitrary element of A^{m-3} , we get that

$$D \supseteq A^2 \times \bigcup_{\bar{u} \in A^{m-3}} \{\bar{u}\} \times \{a_n\} = A^{m-1} \times \{a_n\},$$

which contradicts the minimality of m . This completes the proof of property (i).

[(ii)] We have shown that property (ii) holds when $m = 3$. Let $m > 3$ and suppose that property (ii) holds for all subuniverses of \mathbb{A}^{m-1} that satisfy the assumptions of Claim 5.4.4.6. Let $D \leq \mathbb{A}^m$ that satisfies the assumptions of Claim 5.4.4.6. We will show that $D \supseteq A \times \dots \times A \times S \times A \times \dots \times A \times S$ where the first S appears in the i^{th} -coordinate for any $i \in \{1, \dots, m-1\}$, therefore $X_{m-1}^S \times S \subseteq D$.

WLOG, let $i = m-1$. We showed above that $D(a_1, x_2, \dots, x_m)$ and $D(x_1, a_2, x_3, \dots, x_m)$ are subuniverses of \mathbb{A}^{m-1} that satisfy the assumptions of Claim 5.4.4.6, therefore, by the induction hypothesis, we get that $D(a_1, x_2, \dots, x_m) \cap D(x_1, a_2, x_3, \dots, x_m) \supseteq X_{m-2}^S \times S \supseteq A^{m-3} \times S^2$. Thus, $(\{a_1\} \times A \times A^{m-4} \times S^2) \cup (A \times \{a_2\} \times A^{m-4} \times S^2) \subseteq D$. Let $\bar{u} \in A^{m-4} \times S^2$. Then $D(x_1, x_2, \bar{u})$ is a subuniverse of \mathbb{A}^2 that contains $(\{a_1\} \times A) \cup (A \times \{a_2\}) = \mu_{a_1, a_2}$ where $a_1, a_2 \in A \setminus S$. By assumption, $\chi_{S, S} \leq \mathbb{A}^2$, therefore it follows from property (x) of Proposition 3.2.1 and Theorem 3.1.5 that $D(x_1, x_2, \bar{u}) = A^2$. The tuple \bar{u} was chosen arbitrarily from $A^{m-4} \times S^2$, thus

$$D \supseteq A^2 \times \bigcup_{\bar{u} \in A^{m-4} \times S^2} \{\bar{u}\} = A^2 \times A^{m-4} \times S^2 = A^{m-2} \times S^2.$$

Since $i = m-1$ was an arbitrary choice of for $i \in \overline{m-1}$, we have shown that $D \supseteq A \times \dots \times A \times S \times A \times \dots \times A \times S$ where the first S appears in the i^{th} -coordinate for any $i \in \{1, \dots, m-1\}$. Hence, $D \supseteq X_{m-1}^S \times S$. This completes the proof of the claim. □

Claim 5.4.4.7. *Suppose that $\chi_{S,S} \leq \mathbb{A}^2$. Let $m \geq 3$, $D \leq \mathbb{A}^m$, and $(X_{m-1}^S \times S) \cup (A^{m-1} \times \{a_m\}) \subseteq D$ for some $a_m \in S$. If $D \ni \bar{u} = (u_1, \dots, u_m)$, where $u_i \in A \setminus S$ for all $1 \leq i \leq m-1$ and $u_m \neq a_m$, then $A^{m-1} \times S \subseteq D$.*

Proof. We will show the claim by inducting on m . First suppose that $m = 3$ and $D \leq \mathbb{A}^3$ satisfies the assumptions of the claim. Suppose that $(u_1, u_2, u_3) \in D$, where $u_1, u_2 \in A \setminus S$ and $u_3 \neq a_3$. Since we also have that $(u_1, u_2, a_3) \in A^2 \times \{a_3\} \subseteq D$, we get that $D(u_1, u_2, x_3)$ is a subuniverse of \mathbb{A} that contains $\{u_3, a_3\}$ where $u_3 \neq a_3$. Therefore $D(u_1, u_2, x_3) \supseteq S$, which means $\{u_1\} \times \{u_2\} \times S \subseteq D$.

Let $s \in S \setminus \{a_3\}$ and consider the subuniverse $D(x_1, x_2, s)$ of \mathbb{A}^2 . By assumption, $D \supseteq X_2^S \times S \supseteq (S \times A \times \{s\}) \cup (A \times S \times \{s\})$. Also, $(u_1, u_2, s) \in \{u_1\} \times \{u_2\} \times S \subseteq D$. Therefore $D(x_1, x_2, s) \supseteq (A \times S) \cup (S \times A) \cup \{(u_1, u_2)\} = \chi_{S,S} \cup \{(u_1, u_2)\}$, where $u_1, u_2 \in A \setminus S$. It follows from Theorem 3.1.5 that $D(x_1, x_2, s) = A^2$. Since $s \in S \setminus \{a_3\}$ was arbitrary and we are assuming that $D \supseteq A^2 \times \{a_3\}$, we get that

$$D \supseteq (A^2 \times \cup_{s \in S \setminus \{a_3\}} \{s\}) \cup (A^2 \times \{a_3\}) = A^2 \times S.$$

This completes the proof of the case when $m = 3$.

Now suppose that $m > 3$ and Claim 5.4.4.7 holds for all subuniverses of \mathbb{A}^{m-1} that satisfy the assumption of the claim. Let $D \leq \mathbb{A}^m$ where $(X_{m-1}^S \times S) \cup (A^{m-1} \times \{a_m\}) \subseteq D$ for some $a_m \in S$. Suppose that $D \ni \bar{u} = (u_1, \dots, u_m)$ where $u_i \in A \setminus S$, $1 \leq i \leq m-1$, and $u_m \neq a_m$. We will show that $D(u_1, x_2, \dots, x_m)$ is a subuniverse of \mathbb{A}^{m-1} that satisfies the assumptions of Claim 5.4.4.7, therefore, by the induction hypothesis, $A^{m-2} \times S \subseteq D(u_1, x_2, \dots, x_m)$.

First we will show that $X_{m-2}^S \times S \subseteq D(u_1, x_2, \dots, x_m)$. By assumption we have that

$$\begin{aligned} D &\supseteq X_{m-1}^S \times S \\ &\supseteq A \times X_{m-2}^S \times S \\ &\supseteq \{u_1\} \times X_{m-2}^S \times S \end{aligned}$$

Therefore $D(u_1, x_2, \dots, x_m) \supseteq X_{m-2}^S \times S$. Now, notice that since $D \supseteq A^{m-1} \times \{a_m\} \supseteq \{u_1\} \times A^{m-2} \times \{a_m\}$, we get that $D(u_1, x_2, \dots, x_m) \supseteq A^{m-2} \times \{a_m\}$. Finally, $\bar{u} \in D$ implies that

$D(u_1, x_2, \dots, x_m) \ni (u_2, \dots, u_m)$ where $u_2, \dots, u_{m-1} \in A \setminus S$ and $u_m \neq a_m$. Therefore, by the induction hypothesis, we get that $A^{m-2} \times S \subseteq D(u_1, x_2, \dots, x_m)$.

A similar argument shows that $A^{m-2} \times S \subseteq D(x_1, u_2, x_3, \dots, x_m)$. Hence $(\{u_1\} \times A \times A^{m-3} \times S) \cup (A \times \{u_2\} \times A^{m-3} \times S) \subseteq D$. Let $\bar{v} \in A^{m-3} \times S$. Then $D(x_1, x_2, \bar{v})$ is a subuniverse of \mathbb{A}^2 that contains $(\{u_1\} \times A) \cup (A \times \{u_2\}) = \mu_{u_1, u_2}$ where $u_1, u_2 \in A \setminus S$. We are assuming that $\chi_{S, S} \leq \mathbb{A}^2$, therefore it follows from property (x) of Proposition 3.2.1 that $\mu_b \not\leq \mathbb{A}^2$ for any $b \in A \setminus S$. Therefore, by Theorem 3.1.5, we have that $D(x_1, x_2, \bar{v}) = A^2$. Since \bar{v} was chosen arbitrarily from $A^{m-3} \times S$, we can conclude that

$$D \supseteq A^2 \times \bigcup_{\bar{v} \in A^{m-3} \times S} \{\bar{v}\} = A^2 \times A^{m-3} \times S = A^{m-1} \times S.$$

This completes the proof of the claim. \square

Claim 5.4.4.8. *If case (II') of Lemma 5.4.2 holds, then $S^{n-1} \times A \subseteq B$.*

Proof of claim. Suppose that case (II') of Lemma 5.4.2 holds. Then $\chi_{S, S}, \lambda_{S, a_n} \leq \mathbb{A}^2$. By the assumptions of Lemma 5.4.2, we have that (1) \mathbb{A} satisfies Assumption 2, therefore $\lambda_{S, a_3} \leq \mathbb{A}^2$ implies that \mathbb{S} is quasiprimal, and (2) B is a reduced subuniverse of \mathbb{A}^n , $n \geq 3$, that contains an element (a_1, \dots, a_n) where $a_n \in S$ and $\{a_1\} \times \dots \times \{a_{n-1}\} \times S \not\subseteq B$. Furthermore, from case (II') of Lemma 5.4.2 we have that $B(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{n-1}, x_n) = \lambda_{S, a_n}$, for all $1 \leq i \leq n-1$. Therefore B satisfies the assumptions of Claim 5.4.4.6, so we can apply Claim 5.4.4.6 to B in place of D

By property (ii) of Claim 5.4.4.6 we have that $B \supseteq X_{n-1}^S \times S \supseteq S^{n-1} \times S$. We will show that for all $c \in A \setminus S$ the subuniverse $B(x_1, \dots, x_{n-1}, c)$ is equal to S^{n-1} . This means that $S^{n-1} \times \{A \setminus S\} \subseteq B$. Therefore, the claim follows from the union $(S^{n-1} \times S) \cup (S^{n-1} \times \{A \setminus S\}) \subseteq B$.

Let $c \in A \setminus S$. Then $c \in A = \text{pr}_n B$, which means $B(x_1, \dots, x_{n-1}, c) \neq \emptyset$. Therefore we have that $B(x_1, \dots, x_{n-1}, c)$ is a subuniverse of A^{n-1} . We will first show that $B(x_1, \dots, x_{n-1}, c) \leq_{s,d} \mathbb{S}^{n-1}$. Let $i \in \overline{n-1}$. We claim that $\text{pr}_i B(x_1, \dots, x_{n-1}, c) = S$. WLOG, suppose that $i = 1$. From case (II'), we have that $\text{pr}_{1,n} B = \chi_{S, S}$, thus $S \times \{c\} \subseteq \chi_{S, S} = \text{pr}_{1,n} B$. This means, for each

$s \in S$ there exists some $\bar{u}_s \in A^{n-2}$ such that $(s, \bar{u}_s, c) \in B$. Thus $(s, \bar{u}_s) \in B(x_1, \dots, x_{n-1}, c)$, for each $s \in S$, hence $S \subseteq \text{pr}_1 B(x_1, \dots, x_{n-1}, c)$. Furthermore, equality must hold. Otherwise, if $d \in \text{pr}_1 B(x_1, \dots, x_{n-1}, c)$, for some $d \in A \setminus S$, then there exists some $\bar{u} \in A^{n-2}$ such that $(d, \bar{u}) \in B(x_1, \dots, x_{n-1}, c)$, which means $(d, \bar{u}, c) \in B$ and $(d, c) \in \text{pr}_{1,n} B = \chi_{S,S}$, where $d, c \in A \setminus S$, which is a contradiction. Hence $\text{pr}_1 B(x_1, \dots, x_{n-1}, c) = S$. Therefore, we have shown that $B(x_1, \dots, x_{n-1}, c) \leq_{s,d} \mathbb{S}^{n-1}$, which means $B(x_1, \dots, x_{n-1}, c)$ is a subuniverse of S^{n-1} and no unary projection of $B(x_1, \dots, x_{n-1}, c)$ is a singleton.

Since \mathbb{S} is quasiprimal, we know from Proposition 2.4.5 that a subuniverse of \mathbb{S}^n has either unary projections that are singletons, or binary projections that are automorphisms of \mathbb{S} , or it is equal to the full direct product S^n . Therefore, to show that $B(x_1, \dots, x_{n-1}, c) = S^{n-1}$, it remains to show that no binary projection of $B(x_1, \dots, x_{n-1}, c)$ is an automorphism of \mathbb{S} .

Suppose not. WLOG, suppose $\text{pr}_{1,2} B(x_1, \dots, x_{n-1}, c) \in \text{Aut}(\mathbb{S})$. Let $\bar{u} \in B(x_1, \dots, x_{n-1}, c)$, $\bar{u} = (u_1, \dots, u_{n-1})$. Then $B(x_1, \dots, x_{n-1}, c) \leq_{s,d} \mathbb{S}^{n-1}$ implies $\bar{u} \in S^{n-1}$. Also, $(u_1, u_2) \in \text{pr}_{1,2} B(x_1, \dots, x_{n-1}, c) \in \text{Aut}(\mathbb{S})$ implies $B(x_1, u_2, u_3, \dots, u_{n-1}, c) = \{u_1\}$. Consider the subuniverse $B(x_1, u_2, \dots, u_{n-1}, x_n)$ of A^2 . We claim that $B(x_1, u_2, \dots, u_{n-1}, x_n) = \chi_{S,S}$. First we will show that $\{(u_1, c)\} \cup (A \times \{a_n\}) \cup S^2 \subseteq B(x_1, u_2, \dots, u_{n-1}, x_n)$.

We are assuming that $\bar{u} \in B(x_1, \dots, x_{n-1}, c)$, this means that $(u_1, \dots, u_{n-1}, c) \in B$ and $(u_1, c) \in B(x_1, u_2, \dots, u_{n-1}, x_n)$. Recall that B satisfies the assumptions of Claim 5.4.4.6, therefore properties (i) and (ii) of Claim 5.4.4.6 hold for B . From property (i) of Claim 5.4.4.6 we get that $B \supseteq A^{n-1} \times \{a_n\} \supseteq A \times \{u_2\} \times \dots \times \{u_{n-1}\} \times \{a_n\}$. Therefore $B(x_1, u_2, \dots, u_{n-1}, x_n) \supseteq A \times \{a_n\}$. Finally, from property (ii) of Claim 5.4.4.6, we get that $B \supseteq X_{n-1}^S \times S \supseteq S^n$. Thus, $\bar{u} \in S^{n-1}$ implies $S \times \{u_2\} \times \dots \times \{u_{n-1}\} \times S \subseteq S^n \subseteq B$ which means $S^2 \subseteq B(x_1, u_2, \dots, u_{n-1}, x_n)$.

Therefore, we have shown that $B(x_1, u_2, \dots, u_{n-1}, x_n) \supseteq \{(u_1, c)\} \cup A \times \{a_n\} \cup S^2$, where $u_1, a_n \in S$ and $c \in A \setminus S$. By Theorem 3.1.5, it is clear that $\chi_{u_1, a_n} \subseteq B(x_1, u_2, \dots, u_{n-1}, x_n)$. Then $B(x_1, u_2, \dots, u_{n-1}, x_n)$ is a reduced subuniverse of A^2 , therefore, by the assumptions of the theorem, $B(x_1, u_2, \dots, u_{n-1}, x_n)$ is θ -closed in its A -coordinates, which means $\chi_{S,S} \subseteq B(x_1, u_2, \dots, u_{n-1}, x_n)$. Furthermore, $B(x_1, u_2, \dots, u_{n-1}, x_n) \subseteq \text{pr}_{1,n} B = \chi_{S,S}$, hence $B(x_1, u_2, \dots, u_{n-1}, x_n) = \chi_{S,S}$.

Since $B(x_1, u_2, \dots, u_{n-1}, x_n) = \chi_{S,S}$, we get that $S \times \{c\} \subseteq \chi_{S,S} = B(x_1, u_2, \dots, u_{n-1}, x_n)$ which means $S \subseteq B(x_1, u_2, \dots, u_{n-1}, c)$. However, we showed that $B(x_1, u_2, \dots, u_{n-1}, c) = \{u_1\}$, therefore we have a contradiction to the assumption that $\text{pr}_{1,2} B(x_1, x_2, \dots, x_{n-1}, c) \in \text{Aut}(\mathbb{S})$.

We have shown that $B(x_1, x_2, \dots, x_{n-1}, c)$ is a subdirect subuniverse of \mathbb{S}^{n-1} , no unary projection of $B(x_1, x_2, \dots, x_{n-1}, c)$ is a singleton, and no binary projection of $B(x_1, x_2, \dots, x_{n-1}, c)$ is an automorphism of \mathbb{S} . Since \mathbb{S} is quasiprimal, it follows that $B(x_1, x_2, \dots, x_{n-1}, c) = S^{n-1}$. Furthermore, $c \in A \setminus S$ was chosen arbitrarily, therefore $B(x_1, x_2, \dots, x_{n-1}, c) = S^{n-1}$, for all $c \in A \setminus S$. As we noted at the start of the proof, this completes the proof of the claim. \square

We will now show that case (II') of Lemma 5.4.2 cannot occur, thus proving Lemma 5.4.4.

Proof of Lemma 5.4.4. Suppose, for contradiction, that case (II') of Lemma 5.4.2 holds. Then $\chi_{S,S}, \lambda_{S,a_n} \leq \mathbb{A}^2$. By the assumptions of Lemma 5.4.2, we have that (1) \mathbb{A} satisfies Assumption 2, therefore $\lambda_{S,a_3} \leq \mathbb{A}^2$ implies that \mathbb{S} is quasiprimal, and (2) B is a reduced subuniverse of \mathbb{A}^n , $n \geq 3$, that contains an element (a_1, \dots, a_n) where $a_n \in S$ and $\{a_1\} \times \dots \times \{a_{n-1}\} \times S \not\subseteq B$. Furthermore, from case (II') of Lemma 5.4.2 we have that $B(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_{n-1}, x_n) = \lambda_{S,a_n}$, for all $1 \leq i \leq n-1$. Therefore B satisfies the assumptions of Claim 5.4.4.6 and applying Claim 5.4.4.6 to B in place of D gives that $(A^{n-1} \times \{a_n\}) \cup (X_{n-1}^S \times S) \subseteq B$.

Notice that $\{a_1\} \times \dots \times \{a_{n-1}\} \times S \not\subseteq B$ has the following two implications. The first is that $\{a_1\} \times S \not\subseteq B(x_1, a_2, \dots, a_{n-1}, x_n) = \lambda_{S,a_n}$, therefore $a_1 \in A \setminus S$. The second is that $A^{n-1} \times S \not\subseteq B$.

Therefore, we have that $\chi_{S,S} \leq \mathbb{A}^2$, $B \leq \mathbb{A}^n$ for some $n \geq 3$, and $(A^{n-1} \times \{a_n\}) \cup (X_{n-1}^S \times S) \subseteq B$. Since $A^{n-1} \times S \not\subseteq B$, it follows from applying Claim 5.4.4.7 to B in place of D that if there exists a tuple $\bar{u} \in B$ where $\bar{u} = (u_1, \dots, u_n)$ and $u_i \in A \setminus S$ for all $1 \leq i \leq n-1$, then $u_n = a_n$. To prove the lemma, we will show that if case (II') of Lemma 5.4.2 holds, then for any $s \in S \setminus \{a_n\}$ the tuple $(a_1, \dots, a_1, s) \in B$. However, as we just stated, $(a_1, \dots, a_1, s) \in B$ and $a_1 \in A \setminus S$ implies that $s = a_n$, which contradicts $s \neq a_n$.

Consider the subuniverse $C := B \cap \{(x, \dots, x, y) : x, y \in A\}$ of \mathbb{A}^n . Since $A^{n-1} \times \{a_n\} \subseteq B$, we get that $\{(x, \dots, x, a_n) : x \in A\} \subseteq B$. Furthermore, we have from Claim 5.4.4.8 that $S^{n-1} \times A \subseteq B$,

therefore $\{(x, \dots, x, y) : x \in S, y \in A\} \subseteq B$. Hence $\{(x, \dots, x, a_n) : x \in A\} \cup \{(x, \dots, x, y) : x \in S, y \in A\} \subseteq C$. Then $\text{pr}_{1,n} C$ is a subuniverse of \mathbb{A}^2 that contains $\{(x, a_n) : x \in A\} \cup \{(x, y) : x \in S, y \in A\} = \chi_{S, a_n}$. Under Assumption 2 (Q), the reduced subuniverses of \mathbb{A}^2 are θ -closed in their A -coordinates. Then $\text{pr}_{1,n} C \supseteq \chi_{S, a_n}$ implies that $\text{pr}_{1,n} C$ is a reduced subuniverses of \mathbb{A}^2 , therefore closing $\text{pr}_{1,n} C$ in its A -coordinates gives that $\text{pr}_{1,n} C \supseteq \chi_{S, S} \supseteq \{b\} \times S$ for all $b \in A \setminus S$. In particular, $\text{pr}_{1,n} C \supseteq \{a_1\} \times S$. However, $\{a_1\} \times S \subseteq \text{pr}_{1,n} C$ implies $\{a_1\}^{n-1} \times S \subseteq C \subseteq B$. By assumption, $|S| > 2$, so let $s \in S \setminus \{a_n\}$. Then $(a_1, \dots, a_1, s) \in \{a_1\}^{n-1} \times S \subseteq B$ where $a_1 \in A \setminus S$. As we explained in the previous paragraph, the existence of the tuple $(a_1, \dots, a_1, s) \in B$ where $a_1 \in A \setminus S$ and $s \neq a_n$ gives a contradiction. This completes the proof of the lemma. \square

We have shown that both case (I') and case (II') of Lemma 5.4.2 cannot occur, thus the assumptions of Lemma 5.4.2 are false. Therefore, if \mathbb{A} satisfies Assumption 2, then the reduced subuniverses of \mathbb{A}^n are θ -closed in their A -coordinates. This concludes the proof of this theorem. \square

5.5 The Clone of \mathbb{A}

Under Assumption 2, and the additional assumption that \mathbb{A} is not simple, we will now describe the clone of \mathbb{A} by determining a transparent generating set for the relational clone of \mathbb{A} .

Recall that we found in Proposition 5.1.3 that if $B \leq \mathbb{A}^n$, $n \geq 1$, then $B \in \langle \text{pr}_I B, \mathcal{T}_{\mathbb{A}} \rangle_{RC\text{clone}}$, where $\text{pr}_I B$ is a reduced subuniverse of \mathbb{A}^I , for some nonempty $I \subseteq \bar{n}$. This shows that the relational clone of \mathbb{A} is generated by $\mathcal{T}_{\mathbb{A}}$ and the reduced subuniverses of finite powers of \mathbb{A} . Therefore, to find a generating set for the relational clone of \mathbb{A} , we must find a description for the reduced subuniverses of finite powers of \mathbb{A} .

If B is a reduced subuniverse of \mathbb{A}^n , $B \leq_{s.d.} \prod_{i=1}^n \mathbb{A}_i$, where $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}\}$, then under Assumption 2, it follows from Theorem 5.4.1 that B is θ -closed in its A -coordinates. Let ρ be the natural homomorphism $\rho : \prod_{i=1}^n \mathbb{A}_i \rightarrow \prod_{i=1}^n \mathbb{A}_i / \Theta_i$, where Θ_i is the equality relation if $\mathbb{A}_i = \mathbb{S}$ and $\Theta_i = \theta$ if $\mathbb{A}_i = \mathbb{A}$. Let $B' = \rho(B)$. Then, by Proposition 2.2.8, B is the full inverse image of B' under ρ . Therefore, if we can describe B' , then we have a description for B . We found in Proposition 5.1.6

that $B' \in \langle \text{pr}_{I'} B', \mathcal{T}'_{\mathbb{A}} \rangle_{RC\text{clone}}$, where $\text{pr}_{I'} B'$ is a reduced subuniverse of $\prod_{i \in I'} \mathbb{A}_i$, for some nonempty $I' \in \bar{n}$. Therefore, to understand the reduced subuniverses of finite powers of \mathbb{A} we must find a description for the reduced subuniverses of $\prod_{i=1}^n \mathbb{A}_i$, where $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$, $n \geq 1$. This will be the focus of this section. At the end of the section we will give a complete description of the relational clone of \mathbb{A} .

Definition 5.5.1. Let $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$, $1 \leq i \leq n$. Let $\{I, J\}$ be the partition of \bar{n} such that $\mathbb{A}_i = \mathbb{S}$ whenever $i \in I$ and $\mathbb{A}_i = \mathbb{A}/\theta$ whenever $i \in J$. Let $s \in S$ and $\bar{a} \in A/\theta$. Then we will call the set

$$\mathcal{X}_{(I,J)}^{(s,\bar{a})} := \{(x_1, \dots, x_n) \in \prod_{i=1}^n \mathbb{A}_i : \text{there exists some } i \text{ such that } x_i = s \text{ if } i \in I \text{ and } x_i = \bar{a} \text{ if } i \in J\}$$

a cross on $\prod_{i=1}^n \mathbb{A}_i$. If $\mathbb{A}_1 = \dots = \mathbb{A}_n$, and hence $\{I, J\} = \{\emptyset, \bar{n}\}$, then we will simply denote a cross on $\prod_{i=1}^n \mathbb{A}_i$ by \mathcal{X}_n^g , where $g \in A_1$.

Under Assumption 2 we have that \mathbb{S} is either quasiprimal or affine, thus there is no (S, S) -cross among the subuniverses of \mathbb{A}^2 . This fact implies the following restriction on the size of I in the above definition.

Proposition 5.5.2. Suppose that \mathbb{A} satisfies Assumption 2. If $B' = \mathcal{X}_{(I,J)}^{(s,\bar{a})}$ is a cross on $\prod_{i=1}^n \mathbb{A}_i$, where $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$, $s \in S$, $\bar{a} \in A/\theta$, and $\{I, J\}$ is the partition of \bar{n} such that $\mathbb{A}_i = \mathbb{S}$ whenever $i \in I$ and $\mathbb{A}_i = \mathbb{A}/\theta$ whenever $i \in J$, then $|I| \leq 1$.

Proof. For contradiction, suppose that $|I| > 1$. By Assumption 2 we have that \mathbb{S} is either quasiprimal or affine which means there is no (S, S) -cross among the subuniverses of \mathbb{S}^2 , this means that $n > 2$, otherwise B' is an (S, S) -cross which gives a contradiction. WLOG, suppose that $1, 2 \in I$. Then $B' \leq \mathbb{S} \times \mathbb{S} \times \prod_{i=3}^n \mathbb{A}_i$. Let $\bar{u} \in \prod_{i=3}^n \mathbb{A}_i$, $\bar{u} = (u_3, \dots, u_n)$, where $u_i \neq s$ for all $i \in I \cap \{3, \dots, n\}$ and $u_i \neq \bar{a}$ for all $i \in J \cap \{3, \dots, n\}$. Then by Definition 5.5.1 we have that $\{s\} \times S \times \{\bar{u}\} \subseteq \mathcal{X}_{(I,J)}^{(s,\bar{a})} = B'$ and $S \times \{s\} \times \{\bar{u}\} \subseteq \mathcal{X}_{(I,J)}^{(s,\bar{a})} = B'$. Therefore $B'(x_1, x_2, \bar{u})$ is a subuniverse of \mathbb{S}^2 that contains $(\{s\} \times S) \cup (S \times \{s\})$. In fact, $B'(x_1, x_2, \bar{u}) = (\{s\} \times S) \cup (S \times \{s\})$, otherwise there exists some $(u_1, u_2) \in B'(x_1, x_2, \bar{u}) \setminus [(\{s\} \times S) \cup (S \times \{s\})]$, thus $(u_1, u_2, u_3, \dots, u_n) \in B' = \mathcal{X}_{(I,J)}^{(s,\bar{a})}$ where $u_1 \neq s$, $u_2 \neq s$, $u_i \neq s$ for all $i \in I \cap \{3, \dots, n\}$ and $u_i \neq \bar{a}$ for all $i \in J \cap \{3, \dots, n\}$ which

is a contradiction. Therefore $B'(x_1, x_2, \bar{u}) = (\{s\} \times S) \cup (S \times \{s\})$ is a subuniverse of \mathbb{S}^2 , however this contradicts the fact that there is no (S, S) -cross among the subuniverses of \mathbb{S}^2 . \square

To describe the reduced subuniverses of $\prod_{i=1}^n \mathbb{A}_i$, where $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$, we will distinguish three cases. In Proposition 5.5.4 we consider the case when the strictly simple algebra \mathbb{A}/θ is either quasiprimal or affine. In Propositions 5.5.5 and 5.5.6, \mathbb{A}/θ is assumed to be the third kind of strictly simple idempotent algebra, and the description splits into two cases according to whether there is a cross among the subuniverses of $\mathbb{A}/\theta \times \mathbb{S}$.

Lemma 5.5.3. *Suppose that \mathbb{A} satisfies Assumption 1, \mathbb{S} is quasiprimal, and θ is a congruence on \mathbb{A} . Suppose that $B' \leq \mathbb{A}/\theta \times \mathbb{S} \times \mathbb{A}/\theta$, where $(a_1, a_2, c_3) \in B'$, $B'(x_1, a_2, x_3) \in \text{Aut}(\mathbb{A}/\theta)$, $B'(a_1, x_2, x_3) = \varkappa_{c_2, c_3}$, for some $a_1, c_3 \in A/\theta$ and distinct $a_2, c_2 \in S$. Then either $\mathbb{S} \cong \mathbb{A}/\theta$ or $\lambda_{S, c_2}, \kappa_{d, c_2} \leq \mathbb{A} \times \mathbb{S}$ for all $d \in A \setminus S$.*

Proof. Under the assumptions of the lemma we have that $B'(x_1, a_2, x_3)$ is the graph of Φ for some $\Phi \in \text{Aut}(\mathbb{A}/\theta)$ and $B'(a_1, x_2, x_3) = \varkappa_{c_2, c_3}$. Since \mathbb{S} is quasiprimal, statement (ix) of Corollary 3.2.5, $\Phi \in \text{Aut}(\mathbb{A}/\theta)$, and $\varkappa_{c_2, c_3} \leq \mathbb{S} \times \mathbb{A}/\theta$ imply that Φ fixes c_3 . From $(a_1, a_2, c_3) \in B'$ we get that $(a_1, c_3) \in B'(x_1, a_2, x_3)$, thus $\Phi(a_1) = c_3$, since c_3 is fixed by Φ it follows that $a_1 = c_3$.

Claim 5.5.3.1. *For any $b \in (A/\theta) \setminus \{c_3\}$, either $B'(x_1, x_2, \Phi(b))$ is an isomorphism from \mathbb{A}/θ to \mathbb{S} or $B'(x_1, x_2, \Phi(b))$ is the $(A/\theta, S)$ -cross κ_{b, c_2} .*

Proof of claim. Let $b \in (A/\theta) \setminus \{c_3\}$. Notice that $(b, \Phi(b)) \in B'(x_1, a_2, x_3)$ which means, since $b \neq c_3$ and Φ fixes c_3 , that $\Phi(b) \neq c_3$.

From the assumption that $B'(a_1, x_2, x_3)$ is the graph of Φ we get that $(b, \Phi(b)) \in B'(x_1, a_2, x_3)$ and $A/\theta \times \{\Phi(b)\} \not\subseteq B'(x_1, a_2, x_3)$. Therefore the tuple $(b, a_2) \in B'(x_1, x_2, \Phi(b))$ and $A/\theta \times \{a_2\} \not\subseteq B'(x_1, x_2, \Phi(b))$. From the assumption that $B'(a_1, x_2, x_3) = \varkappa_{c_2, c_3}$ it follows that $(c_2, \Phi(b)) \in \{c_2\} \times A/\theta \subseteq B'(a_1, x_2, x_3)$. Also, $\Phi(b) \neq c_3$ implies that $S \times \{\Phi(b)\} \not\subseteq \varkappa_{c_2, c_3} = B'(a_1, x_2, x_3)$. Therefore, $(a_1, c_2) \in B'(x_1, x_2, \Phi(b))$ and $\{a_1\} \times S \not\subseteq B'(x_1, x_2, \Phi(b))$.

Then $(b, a_2), (a_1, c_2) \in B'(x_1, x_2, \Phi(b))$ implies $\{a_2, c_2\} \subseteq \text{pr}_2 B'(x_1, x_2, \Phi(b)) \leq \mathbb{S}$. By assumption, a_2 and c_2 are distinct elements of S , therefore $\text{pr}_2 B'(x_1, x_2, \Phi(b)) = S$. Furthermore, $\{b, a_1\} \subseteq \text{pr}_1 B'(x_1, x_2, \Phi(b)) \leq \mathbb{A}/\theta$. We claim that $a_1 \neq b$, otherwise $(a_1, a_2) = (b, a_2), (a_1, c_2) \in B'(x_1, x_2, \Phi(b))$, $a_2 \neq c_2$ implies by Lemma 3.1.7 that $\{a_1\} \times S \subseteq B'(x_1, x_2, \Phi(b))$, which contradicts $\{a_1\} \times S \not\subseteq B'(x_1, x_2, \Phi(b))$. Therefore $\text{pr}_1 B'(x_1, x_2, \Phi(b)) = \mathbb{A}/\theta$ and we have shown that $B'(x_1, x_2, \Phi(b)) \leq_{s.d} \mathbb{A}/\theta \times \mathbb{S}$. Since $\{a_1\} \times S \not\subseteq B'(x_1, x_2, \Phi(b))$ it is clear that $B'(x_1, x_2, \Phi(b)) \neq \mathbb{A}/\theta \times S$. Then it follows from Corollary 3.1.6 that $B'(x_1, x_2, \Phi(b))$ is either an isomorphism from \mathbb{A}/θ to \mathbb{S} or $B'(x_1, x_2, \Phi(b))$ is an $(\mathbb{A}/\theta, S)$ -cross.

Suppose that $B'(x_1, x_2, \Phi(b))$ is not an isomorphism $\mathbb{A}/\theta \rightarrow \mathbb{S}$. Then $(b, a_2), (a_1, c_2) \in B'(x_1, x_2, \Phi(b))$, $\mathbb{A}/\theta \times \{a_2\} \not\subseteq B'(x_1, x_2, \Phi(b))$, and $\{a_1\} \times S \not\subseteq B'(x_1, x_2, \Phi(b))$ implies that $B'(x_1, x_2, \Phi(b)) = \varkappa_{b, c_2}$. \square

It is clear that $B'(x_1, x_2, \Phi(b))$ is a subuniverse of $\mathbb{A}/\theta \times \mathbb{S}$ for all $b \in \mathbb{A}/\theta \setminus \{c_3\}$. Suppose that \mathbb{S} and \mathbb{A}/θ are not isomorphic. Then it follows from Claim 5.5.3.1 that $\varkappa_{b, c_2} \leq \mathbb{A}/\theta \times \mathbb{S}$ for all $b \in \mathbb{A}/\theta \setminus \{c_3\}$. Furthermore, we have by assumption that $B'(a_1, x_2, x_3) = \varkappa_{c_2, c_3}$ is a subuniverse of $\mathbb{S} \times \mathbb{A}/\theta$, thus $\varkappa_{c_3, c_2} = (\varkappa_{c_2, c_3})^{-1} \leq \mathbb{A}/\theta \times \mathbb{S}$. This means that $\varkappa_{\bar{S}, c_2}, \varkappa_{\bar{d}, c_2} \leq \mathbb{A}/\theta \times \mathbb{S}$ for all $d \in \mathbb{A} \setminus S$.

Let $\rho : \mathbb{A} \times \mathbb{S} \rightarrow \mathbb{A}/\theta \times \mathbb{S}$ be the natural homomorphism. Then $\lambda_{S, c_2} = \rho^{-1}(\varkappa_{\bar{S}, c_2}) \leq \mathbb{A} \times \mathbb{S}$ and $\kappa_{d, c_2} = \rho^{-1}(\varkappa_{\bar{d}, c_2}) \leq \mathbb{A} \times \mathbb{S}$ for all $d \in \mathbb{A} \setminus S$. This completes the proof of the lemma. \square

Proposition 5.5.4. *Suppose that \mathbb{A} satisfies Assumption 2, θ is a congruence on \mathbb{A} , and \mathbb{A}/θ is either quasiprimal or affine. Let $n \geq 2$ and $B' \leq_{s.d} \prod_{i=1}^n \mathbb{A}_i$, where $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$, for all $1 \leq i \leq n$. If B' is reduced, then $B' = \{(x_1, \dots, x_n) \in \prod_{i=1}^n \mathbb{A}_i : (x_i, x_j) \in \text{pr}_{i,j} B', 1 \leq i < j \leq n\}$.*

Proof. Suppose not. The proposition clearly holds if $n = 2$, therefore it must be that $n > 2$. Let n be minimal such that there exists a reduced subuniverse $B' \leq_{s.d} \prod_{i=1}^n \mathbb{A}_i$, $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$, where $B' \neq \{(x_1, \dots, x_n) \in \prod_{i=1}^n \mathbb{A}_i : (x_i, x_j) \in \text{pr}_{i,j} B', 1 \leq i < j \leq n\}$. It is clear that the containment (\subseteq) must hold. Therefore, there exists $(a_i, a_j) \in \text{pr}_{i,j} B'$ for all $1 \leq i < j \leq n$ such that $\bar{a} = (a_1, \dots, a_n) \notin B'$. Let $I \subseteq \bar{n}$, $|I| = n - 1$. Since B' is a reduced subuniverse of dimension n

we have that $\text{pr}_I B'$ is a reduced subuniverse of dimension $n - 1$, furthermore $(a_i, a_j) \in \text{pr}_{i,j}(\text{pr}_I B')$ for all $i, j \in I$. Then by the minimality of n we get that $\text{pr}_I \bar{a} \in \text{pr}_I B'$. Since I was an arbitrary subset of \bar{n} containing $n - 1$ elements, we get that for each $i \in \bar{n}$ there exists some $c_i \in A$ such that $(a_1, \dots, a_{i-1}, c_i, a_{i+1}, \dots, a_n) \in B'$. Clearly $c_i \neq a_i$ for all $1 \leq i \leq n$, otherwise we would have that $\bar{a} \in B'$.

Claim 5.5.4.1. *There exists some distinct $i, j \in \bar{n}$ such that $A_i = S$ and $A_j = A/\theta$, also $\mathbb{S} \not\cong \mathbb{A}/\theta$.*

Proof of claim. To prove this claim we will first show two subclaims.

Subclaim 5.5.4.1.1. *If $A_1 = \dots = A_n$, then $B' = \Pi_{i=1}^n A_i$.*

Proof of subclaim. Suppose that $\mathbb{A}_1 = \mathbb{A}_2 = \dots = \mathbb{A}_n$. Then $B' \in \text{RClo}(\mathbb{G})$ for some $G \in \{\mathbb{S}, \mathbb{A}/\theta\}$. We have that \mathbb{G} is simple algebra that is either quasiprimal or affine, moreover we have that B' is reduced, therefore it follows from the discription given in Propositions 2.4.5 and 2.4.6 that B' must be equal to the full direct product, \mathbb{G}^n . \square

Subclaim 5.5.4.1.2. *If $\mathbb{S} \cong \mathbb{A}/\theta$, then $B' = \Pi_{i=1}^n A_i$.*

Proof of subclaim. Suppose that \mathbb{S} is isomorphic to \mathbb{A}/θ . Let ι be an isomorphism $\iota : \mathbb{A}/\theta \rightarrow \mathbb{S}$ and define the map $\Pi_{i=1}^n \iota_i : \Pi \mathbb{A}_i \rightarrow \mathbb{S}^n$, by letting $\iota_i = \text{id}_S$, if $\mathbb{A}_i = \mathbb{S}$, and $\iota_i = \iota$, if $\mathbb{A}_i = \mathbb{A}/\theta$. Clearly $\Pi \iota_i$ is an isomorphism. Thus, applying $\Pi \iota_i$ to B' we get $\hat{B} := (\Pi \iota_i)(B') \leq \mathbb{S}^n$. Since $\Pi \iota_i$ is a product isomorphism, we have that $\Pi \iota_i$ and $(\Pi \iota_i)^{-1}$ must preserve the size of unary projection and projections that are defined by bijective maps. Since B' is reduced, it follows that $\hat{B} = (\Pi \iota_i)(B')$ is a reduced subuniverse of \mathbb{S}^n . Then Subclaim 5.5.4.1.1 implies $\hat{B} = \mathbb{S}^n$. If we now apply the inverse map, $(\Pi \iota_i)^{-1}$, to \hat{B} , we get that $B' = (\Pi \iota_i)^{-1}(\hat{B}) = (\Pi \iota_i)^{-1}(\mathbb{S}^n) = \Pi \mathbb{A}_i$. Hence B' is the full direct product. \square

Suppose for contradiction that either $A_1 = \dots = A_n$ or that $\mathbb{S} \cong \mathbb{A}/\theta$. Then by Subclaims 5.5.4.1.1 and 5.5.4.1.2 we get that $B' = \Pi_{i=1}^n A_i$. Thus $\bar{a} \in \Pi_{i=1}^n A_i = B'$, which contradicts $\bar{a} \notin B'$. \square

WLOG, suppose that $A_1 = S$ and $A_2 = A/\theta$. Let $\hat{B} := B'(x_1, x_2, x_3, a_4, \dots, a_n)$. Then \hat{B} is a subuniverse of $\mathbb{S} \times \mathbb{A}/\theta \times \mathbb{A}_3$ that contains the tuples $(c_1, a_2, a_3), (a_1, c_2, a_3), (a_1, a_2, c_3), (a_1, a_2, a_3) \notin \hat{B}$, and $c_i \neq a_i$ for $1 \leq i \leq 3$. We claim that \hat{B} is reduced. Since $\text{pr}_i \hat{B} \supseteq \{c_i, a_i\}$ for $i = 1, 2, 3$ and $c_i \neq a_i$ we have that no unary projection of \hat{B} is a singleton and, in fact, since $\mathbb{S}, \mathbb{A}/\theta$, and \mathbb{A}_3 are all simple algebras, this means that $\hat{B} \leq_{s.d.} \mathbb{S} \times \mathbb{A}/\theta \times \mathbb{A}_3$. Suppose, for contradiction, that $\text{pr}_{i,j} \hat{B}$ is an automorphism of \mathbb{S} or \mathbb{A}/θ for some $1 \leq i < j \leq 3$. Then $(a_i, c_j), (a_i, a_j) \in \text{pr}_{i,j} \hat{B}$ implies $c_j = a_j$, which contradicts and $c_i \neq a_i$ for $1 \leq i \leq 3$. Finally, by Claim 5.5.4.1 we have that $\mathbb{S} \not\cong \mathbb{A}/\theta$, thus \hat{B} cannot be an h.d.-automorphism of \mathbb{S} or of \mathbb{A}/θ . Therefore \hat{B} is reduced. We have shown that \hat{B} is a reduced subdirect subuniverse of $\mathbb{S} \times \mathbb{A}/\theta \times \mathbb{A}_3$, where $\mathbb{A}_3 \in \{\mathbb{S}, \mathbb{A}/\theta\}$, and $(a_i, a_j) \in \text{pr}_{i,j} \hat{B}$ for $1 \leq i < j \leq 3$, but the tuple $(a_1, a_2, a_3) \notin \hat{B}$. Thus, the proposition fails for $n = 3$.

WLOG, suppose that $B' \leq_{s.d.} \mathbb{G} \times \mathbb{H} \times \mathbb{G}$, where $\{\mathbb{G}, \mathbb{H}\} = \{\mathbb{S}, \mathbb{A}/\theta\}$. We showed above that there exist tuples $(c_1, a_2, a_3), (a_1, c_2, a_3), (a_1, a_2, c_3) \in B'$, $(a_1, a_2, a_3) \notin B'$, and $c_i \neq a_i$ for $1 \leq i \leq 3$. Consider the subuniverse $B'(a_1, x_2, x_3)$ of $\mathbb{H} \times \mathbb{G}$. We have that $(c_2, a_3), (a_2, c_3) \in B'(a_1, x_2, x_3)$ where $c_i \neq a_i$ for $i = 2, 3$, thus, $\text{pr}_i B'(a_1, x_2, x_3)$ is not a singleton for $i = 2, 3$. Since \mathbb{H} and \mathbb{G} are both strictly simple algebras and thus contain no proper subalgebras, this means that $B'(a_1, x_2, x_3) \leq_{s.d.} \mathbb{H} \times \mathbb{G}$. Additionally, we have that $(a_2, a_3) \notin B'(a_1, x_2, x_3)$, therefore $B'(a_1, x_2, x_3) \neq \mathbb{H} \times \mathbb{G}$. We showed in Claim 5.5.4.1 that $\mathbb{H} \not\cong \mathbb{G}$, hence it follows from Corollary 3.1.6 that $B'(a_1, x_2, x_3) = \kappa_{c_2, c_3}$.

We claim that $B'(a_1, x_2, x_3) = \kappa_{c_2, c_3} \leq \mathbb{H} \times \mathbb{G}$ implies that \mathbb{S} is quasiprimal. Let

$$C := \begin{cases} \kappa_{c_2, c_3}, & \text{if } H = A/\theta \text{ and } G = S, \\ \kappa_{c_2, c_3}^{-1}, & \text{if } H = S \text{ and } G = A/\theta. \end{cases}$$

Let ρ be the natural homomorphism $\rho : \mathbb{A} \times \mathbb{S} \rightarrow \mathbb{A}/\theta \times \mathbb{S}$. Then $\rho^{-1}(C)$ is a subuniverse of \mathbb{A}^2 , furthermore $\rho^{-1}(C)$ is a (thick) (A, S) -cross. By Assumption 2 we have that if there is a (thick) (A, S) -cross among the subuniverses of \mathbb{A}^2 , then \mathbb{S} is quasiprimal.

Now consider the subuniverse $B'(x_1, a_2, x_3)$ of \mathbb{G}^2 . We have that \mathbb{G} is a strictly simple

algebra and $(c_1, a_3), (a_1, c_3) \in B'(x_1, a_2, x_3)$ where $c_i \neq a_i$ for $i = 1, 3$. Then \mathbb{G} contains no proper subalgebras and $\text{pr}_i B'(x_1, a_2, x_3) \leq \mathbb{G}$ where $|\text{pr}_i B'(x_1, a_2, x_3)| > 1$ ($i = 1, 3$), therefore $B'(x_1, a_2, x_3) \leq_{s,d} \mathbb{G}^2$. Furthermore, $(a_1, a_3) \notin B'(x_1, a_2, x_3)$, so $B'(x_1, a_2, x_3) \neq \mathbb{G}^2$. Finally, we have shown that \mathbb{S} is quasiprimal and we are assuming that \mathbb{A}/θ is either quasiprimal or affine, therefore $G \in \{S, A/\theta\}$ implies that there is no cross among the subuniverses of \mathbb{G}^2 . So it follows from Corollary 3.1.6 that B' is an automorphism of \mathbb{G} .

We that we are assuming that θ is a congruence on \mathbb{A} and that Assumption 2 holds, therefore we have that Assumption 1 holds. We have shown that \mathbb{S} is quasiprimal. Additionally, we have shown that $B' \leq \mathbb{G} \times \mathbb{H} \times \mathbb{G}$ contains the tuple (a_1, a_2, c_3) , $B'(x_1, a_2, x_3)$ is an automorphism of \mathbb{G} , and $B'(a_1, x_2, x_3) = \varkappa_{c_2, c_3}$. There are two cases to consider: either $\mathbb{G} = \mathbb{S}$ and $\mathbb{H} = \mathbb{A}/\theta$ or $\mathbb{G} = \mathbb{A}/\theta$ and $\mathbb{H} = \mathbb{S}$.

If $\mathbb{G} = \mathbb{S}$ and $\mathbb{H} = \mathbb{A}/\theta$, then B' and the tuple (a_1, a_2, c_3) satisfy the assumptions of Corollary 5.3.2. Thus Corollary 5.3.2 implies that either $\Lambda \leq \mathbb{A}^3$ or $\mathcal{K}_b \leq \mathbb{A}^3$ where $c_2 = b/\theta$ for some $b \in A \setminus S$. However, both cases contradict Assumption 2 (Q).

Now suppose that $\mathbb{G} = \mathbb{A}/\theta$ and $\mathbb{H} = \mathbb{S}$. Since $c_2 \neq a_2$, we have that B' and the tuple (a_1, a_2, c_3) satisfy the assumptions of Lemma 5.5.3. Then Lemma 5.5.3 implies that either $\mathbb{S} \cong \mathbb{A}/\theta$ or $\lambda_{S, c_2}, \kappa_{d, c_2} \leq \mathbb{A} \times \mathbb{S}$ for all $d \in A \setminus S$. By Claim 5.5.4.1 we have that $\mathbb{S} \not\cong \mathbb{A}/\theta$, therefore $\lambda_{S, c_2}, \kappa_{d, c_2} \leq \mathbb{A} \times \mathbb{S}$ for all $d \in A \setminus S$. However, since \mathbb{S} is quasiprimal we have from property (ii) of Corollary 3.2.5 that $\kappa_{a, s} \leq \mathbb{A} \times \mathbb{S}$ implies $\lambda_{S, s'} \not\leq \mathbb{A} \times \mathbb{S}$ for all $a \in A$, $s, s' \in S$, therefore we have a contradiction. This completes the proof of the proposition. \square

Proposition 5.5.5. *Suppose that \mathbb{A} satisfies Assumption 2, θ is congruence on \mathbb{A} , and there exists an $(A/\theta, A/\theta)$ -cross among the subuniverses of $(\mathbb{A}/\theta)^2$, but there is no $(S, A/\theta)$ -cross among the subuniverses of $\mathbb{S} \times \mathbb{A}/\theta$. If B' is a reduced subuniverse of $\mathbb{S}^r \times (\mathbb{A}/\theta)^{n-r}$ for some $n \geq 2$, $0 \leq r \leq n$, then $B' = \text{pr}_{\bar{r}} B' \times \text{pr}_{\bar{n} \setminus \bar{r}} B'$.*

Proof. Suppose that $\eta_{\bar{a}} \leq (A/\theta)^2$, for some $\bar{a} \in A/\theta$ and there is no (thick) $(S, A/\theta)$ -cross among the subuniverses of $\mathbb{S} \times \mathbb{A}/\theta$. Let B' be a reduced subuniverse of $\mathbb{S}^r \times (\mathbb{A}/\theta)^{n-r}$, for some $n \geq 2$,

$0 \leq r \leq n$.

If $r = 0$, then $B' \leq (\mathbb{A}/\theta)^n$, thus $B' = \text{pr}_{\bar{n}\setminus\bar{r}} B'$. Alternatively, if $r = n$, then $B' \leq \mathbb{S}^r$ and $B' = \text{pr}_{\bar{r}} B'$. Therefore, it remains to show that the proposition holds when there exists some $i, j \in \bar{n}$ such that $\text{pr}_i B' = S$ and $\text{pr}_j B' = A/\theta$.

For contradiction, suppose the statement of the proposition does not hold. If $n = 2$, then $B' \leq \mathbb{S} \times \mathbb{A}/\theta$. Recall that there exists a $(A/\theta, A/\theta)$ -cross among the subuniverses of $(\mathbb{A}/\theta)^2$ and there does not exist an (S, S) -cross among the subuniverses of \mathbb{A}^2 , therefore \mathbb{S} and \mathbb{A}/θ are not isomorphic. Furthermore, we are assuming that there is no $(S, A/\theta)$ -cross among the subuniverses of $\mathbb{S} \times \mathbb{A}/\theta$. Therefore, it follows from Corollary 3.1.6 that $B' = S \times A/\theta = \text{pr}_1 B' \times \text{pr}_2 B'$. This means that if the proposition fails for some n -dimensional subuniverse, then it must be that $n \geq 3$.

Let n be minimal such that there exists a reduced subuniverse $B' \leq \mathbb{S}^r \times (\mathbb{A}/\theta)^{n-r}$, where $B \neq \text{pr}_{\bar{r}} B' \times \text{pr}_{\bar{n}\setminus\bar{r}} B'$. We showed that the proposition holds if $r \in \{0, n\}$, therefore it must be that $\text{pr}_1 B' = S$ and $\text{pr}_n B' = A/\theta$.

Clearly, $B' \subseteq \text{pr}_{\bar{r}} B' \times \text{pr}_{\bar{n}\setminus\bar{r}} B'$. The projection of a reduced subuniverse is reduced, thus $\text{pr}_{2, \dots, n} B' \leq \prod_{i \in \bar{n}\setminus\{1\}} \mathbb{A}_i$ and $\text{pr}_{1, \dots, n-1} B' \leq \prod_{i \in \bar{n}-1} \mathbb{A}_i$ are reduced subuniverses. Therefore, by the minimality of n , we get that $\text{pr}_{2, \dots, n} B' = \text{pr}_{\bar{r}\setminus\{1\}} B' \times \text{pr}_{\bar{n}\setminus\bar{r}} B'$ and $\text{pr}_{1, \dots, n-1} B' = \text{pr}_{\bar{r}} B' \times \text{pr}_{\bar{n}-1\setminus\bar{r}} B'$.

We are assuming that $B' \neq \text{pr}_{\bar{r}} B' \times \text{pr}_{\bar{n}\setminus\bar{r}} B'$, therefore there exists a tuple $(u_1, \dots, u_n) \in \text{pr}_{\bar{r}} B' \times \text{pr}_{\bar{n}\setminus\bar{r}} B'$ such that $(u_1, \dots, u_n) \notin B'$. However, $(u_1, \dots, u_n) \in \text{pr}_{\bar{r}} B' \times \text{pr}_{\bar{n}\setminus\bar{r}} B'$ implies that $(u_2, \dots, u_n) \in \text{pr}_{\bar{r}\setminus\{1\}} B' \times \text{pr}_{\bar{n}\setminus\bar{r}} B' = \text{pr}_{2, \dots, n} B'$ and $(u_1, \dots, u_{n-1}) \in \text{pr}_{\bar{r}} B' \times \text{pr}_{\bar{n}-1\setminus\bar{r}} B' = \text{pr}_{1, \dots, n-1} B'$. Therefore, there exists some $c_1 \in S$, $c_n \in A/\theta$ such that B' contains the tuples $(c_1, u_2, \dots, u_{n-1}, u_n)$ and $(u_1, u_2, \dots, u_{n-1}, c_n)$. Since $(u_1, u_2, \dots, u_{n-1}, u_n) \notin B'$, we get that $c_i \neq u_i$, for $i = 1, n$.

Then $B'(x_1, u_2, \dots, u_{n-1}, x_n)$ is a subuniverse of $\mathbb{S} \times \mathbb{A}/\theta$ that contains the tuples (c_1, u_n) and (u_1, c_n) . Furthermore, since $\{u_i, c_i\} \in \text{pr}_i B'(x_1, u_2, \dots, u_{n-1}, x_n)$ and $u_i \neq c_i$, for $i \in \{1, n\}$, we get that $B'(x_1, u_2, \dots, u_{n-1}, x_n)$ is a subdirect subuniverse of $\mathbb{S} \times \mathbb{A}/\theta$. We are assuming that \mathbb{S} and \mathbb{A}/θ are not isomorphic and there exists no $(S, A/\theta)$ -cross among the subuniverses of $\mathbb{S} \times \mathbb{A}/\theta$, therefore, by Corollary 3.1.6, we get that $B'(x_1, u_2, \dots, u_{n-1}, x_n) = \mathbb{S} \times \mathbb{A}/\theta$. Hence, $(u_1, u_n) \in \mathbb{S} \times \mathbb{A}/\theta =$

$B'(x_1, u_2, \dots, u_{n-1}, x_n)$, which means $(u_1, \dots, u_n) \in B'$, a contradiction to $(u_1, \dots, u_n) \notin B'$. This completes the proof of the proposition. \square

Proposition 5.5.6. *Suppose that \mathbb{A} satisfies Assumption 2, θ is a congruence on \mathbb{A} , and there exists an $(A/\theta, A/\theta)$ -cross among the subuniverses of $(\mathbb{A}/\theta)^2$ and there exists an $(A/\theta, S)$ -cross among the subuniverses of $\mathbb{A}/\theta \times \mathbb{S}$. Let B' be a reduced subuniverse of $\prod_{i=1}^n \mathbb{A}_i$, where $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$, $1 \leq i \leq n$, and $n \geq 2$. Let P be the family of subsets of \bar{n} such that $I \in P$ if and only if $\text{pr}_I B'$ is a cross on $\prod_{i \in I} \mathbb{A}_i$. Then $B' = \{\bar{a} \in \prod_{i=1}^n \mathbb{A}_i : \bar{a}_I \in \text{pr}_I B' \text{ for all } I \in P\}$.*

Proof. We are assuming that there exists an $(A/\theta, S)$ -cross among the subuniverses of $\mathbb{A}/\theta \times \mathbb{S}$, therefore its full inverse image under the natural homomorphism $\rho : A \times S \rightarrow A/\theta \times S$ is a subuniverse of $\mathbb{A} \times \mathbb{S}$. Hence there exists a (thick) (A, S) -cross among the subuniverses of \mathbb{A}^2 . By the Assumption 2, it must be that \mathbb{S} is quasiprimal. Furthermore, there is no (S, S) -cross that is a subuniverse of \mathbb{S}^2 .

We are also assuming that there is an $(A/\theta, A/\theta)$ -cross among the subuniverses of $(\mathbb{A}/\theta)^2$, therefore \mathbb{S} and \mathbb{A}/θ are not isomorphic. Let us suppose that $\eta_{\bar{a}} \leq (\mathbb{A}/\theta)^2$, for some $\bar{a} \in A/\theta$. Then by statement (iii) of Proposition 3.2.1, we get that $\eta_{\bar{a}}$ is the unique $(A/\theta, A/\theta)$ -cross that is a subuniverse of $(\mathbb{A}/\theta)^2$. Furthermore, if $\varkappa_{\bar{b}, s}$ is a subuniverse of $\mathbb{A}/\theta \times \mathbb{S}$, for some $\bar{b} \in A/\theta, s \in S$, then by statement (x) of Corollary 3.2.5, we get that $\eta_{\bar{a}} \leq (\mathbb{A}/\theta)^2$ and $\varkappa_{\bar{b}, s} \leq \mathbb{A}/\theta \times \mathbb{S}$ implies that $\bar{a} = \bar{b}$. Therefore, every $(A/\theta, S)$ -cross that is a subuniverse of $\mathbb{A}/\theta \times \mathbb{S}$ is of the form $\varkappa_{\bar{a}, s}$, for some $s \in S$.

Finally, we claim that $\eta_{\bar{a}} \leq (\mathbb{A}/\theta)^2$ implies that \bar{a} is the special element of A/θ such that $\mathbf{X}_{(I, J)}^{(s, \bar{a})}$ is a cross on $\prod_{i=1}^n \mathbb{A}_i$, for some $s \in S, \mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$, some partition $\{I, J\}$ of \bar{n} , and $n \geq 2$. For contradiction, suppose that $\bar{a} \neq \bar{b} \in A/\theta$ and $\mathbf{X}_{(I, J)}^{(s, \bar{b})}$ is a cross on $\prod_{i=1}^n \mathbb{A}_i$, for some $s \in S, \mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$, some partition $\{I, J\}$ of \bar{n} , and $n \geq 2$. Either $I = \emptyset$ or $I \neq \emptyset$. If $I = \emptyset$, then $J = \bar{n}$ and $C := \mathbf{X}_{(I, J)}^{(s, \bar{b})} = \mathbf{X}_{\bar{n}}^{\bar{b}} \leq (\mathbb{A}/\theta)^n$. Let $\bar{u} \in (\mathbb{A}/\theta)^{n-2}$. Then $C(x_1, x_2, \bar{u})$ is a subuniverse of $(\mathbb{A}/\theta)^2$ and, by definition of C , $C(x_1, x_2, \bar{u}) = \eta_{\bar{b}}$. However, $\eta_{\bar{b}} \leq (\mathbb{A}/\theta)^2$ and $\bar{a} \neq \bar{b}$ contradicts the above statement that $\eta_{\bar{a}}$ is the unique $(A/\theta, A/\theta)$ -cross that is a subuniverse of $(\mathbb{A}/\theta)^2$. Thus $I \neq \emptyset$. Let

$i \in I$ and $j \in J$. WLOG, suppose that $i = 2$ and $j = 1$. Let $C := \mathbf{X}_{(I,J)}^{(s,\bar{b})} \leq \mathbb{A}/\theta \times \mathbb{S} \times \prod_{i=3}^n \mathbb{A}_i$ and let $\bar{u} \in \prod_{i=3}^n \mathbb{A}_i$. Then $C(x_1, x_2, \bar{u})$ is a subuniverse of $\mathbb{A}/\theta \times \mathbb{S}$ and, by definition of C , $C(x_1, x_2, \bar{u}) = (\{\bar{b}\} \times \mathbb{S}) \cup (\mathbb{A}/\theta \times \{s\}) = \varkappa_{\bar{b},s}$. However $\varkappa_{\bar{b},s} \leq \mathbb{A}/\theta \times \mathbb{S}$ and $\bar{a} = \bar{b}$ contradicts the above statement that every $(A/\theta, S)$ -cross that is a subuniverse of $\mathbb{A}/\theta \times \mathbb{S}$ is of the form $\varkappa_{\bar{a},s}$, for some $s \in S$. Thus, we have shown that \bar{a} is the special element of A/θ such that $\mathbf{X}_{(I,J)}^{(s,\bar{a})}$ is a cross on $\prod_{i=1}^n \mathbb{A}_i$.

We will first show that the proposition holds when either $n = 2$, or $\mathbb{A}_1 = \cdots = \mathbb{A}_n$, or $\bar{n} \in P$. Suppose that $n = 2$ and B' is a reduced subuniverse of $\mathbb{A}_1 \times \mathbb{A}_2$. Since \mathbb{A}/θ and \mathbb{S} are not isomorphic, it follows from Corollary 3.1.6 that B' is either an $(A/\theta, S)$ -cross, an $(S, A/\theta)$ -cross, an $(A/\theta, A/\theta)$ -cross, $A/\theta \times S$, $S \times A/\theta$, S^2 , or $(A/\theta)^2$, therefore the proposition holds.

If $\mathbb{A}_1 = \cdots = \mathbb{A}_n$, then either $B' \leq \mathbb{S}^n$ or $B' \leq (\mathbb{A}/\theta)^n$. Suppose $B' \leq \mathbb{S}^n$. Since \mathbb{S} is quasiprimal and B' is reduced, it follows from Proposition 2.4.5 that $B' = \mathbb{S}^n$. Now suppose that $B' \leq (\mathbb{A}/\theta)^n$. Since $\eta_{\bar{a}} \leq (\mathbb{A}/\theta)^2$ and \mathbb{A}/θ is strictly simple, it follows from Proposition 2.4.7 that the proposition holds for B' .

Lastly, suppose that $\bar{n} \in P$. It is clear that $B' \subseteq \{\bar{a} \in \prod_{i=1}^n \mathbb{A}_i : \bar{a}_I \in \text{pr}_I B', I \in P\}$. Furthermore, since $\bar{n} \in P$, for any $\bar{a} \in \{\bar{a} \in \prod_{i=1}^n \mathbb{A}_i : \bar{a}_I \in \text{pr}_I B', I \in P\}$, we get that $\bar{a} = (\bar{a})_{\bar{n}} \in \text{pr}_{\bar{n}} B' = B'$, hence $B' \supseteq \{\bar{a} \in \prod_{i=1}^n \mathbb{A}_i : \bar{a}_I \in \text{pr}_I B', I \in P\}$. Therefore, we have shown that the proposition holds when either $n = 2$, or $\mathbb{A}_1 = \cdots = \mathbb{A}_n$, or $\bar{n} \in P$.

Now let us suppose, for contradiction, that the proposition fails. Let n be minimal such that there exists a reduced subuniverse B' of $\prod_{i=1}^n \mathbb{A}_i$ where $B' \neq \{\bar{a} \in \prod_{i=1}^n \mathbb{A}_i : \bar{a}_I \in \text{pr}_I B', I \in P\}$. Then $n \geq 3$, there exists some distinct $i, j \in \bar{n}$ such that $\text{pr}_i B' = S$ and $\text{pr}_j B' = A/\theta$, and $\bar{n} \notin P$.

Let $k \in \bar{n}$ and P_k be the family of subsets of $\bar{n} \setminus \{k\}$ such that $I \in P_k$ if and only if $\text{pr}_I(\text{pr}_{\bar{n} \setminus \{k\}} B')$ is a cross on $\prod_{i \in \bar{n} \setminus \{k\}} \mathbb{A}_i$. Recall that $\text{pr}_I(\text{pr}_{\bar{n} \setminus \{k\}} B') = \text{pr}_I B'$. Therefore, $P_k = \{I \subseteq \bar{n} : I \in P, k \notin I\}$. Since, for any $k \in \bar{n}$, $\text{pr}_{\bar{n} \setminus \{k\}} B'$ is a subuniverse of $\prod_{i \in \bar{n} \setminus \{k\}} \mathbb{A}_i$, it follows from the minimality of n that $\text{pr}_{\bar{n} \setminus \{k\}} B' = \{\bar{a} \in \prod_{i \in \bar{n} \setminus \{k\}} \mathbb{A}_i : \bar{a}_I \in \text{pr}_I B', I \in P_k\} = \{\bar{a} \in \prod_{i \in \bar{n} \setminus \{k\}} \mathbb{A}_i : \bar{a}_I \in \text{pr}_I B', I \in P, k \notin I\}$.

Clearly, $B' \subseteq \{\bar{a} \in \prod_{i=1}^n \mathbb{A}_i : \bar{a}_I \in \text{pr}_I B', I \in P\}$. Since the proposition fails for B' , it must

be that there exists some $\bar{u} = (u_1, \dots, u_n) \in \{\bar{a} \in \prod_{i=1}^n \mathbb{A}_i : \bar{a}_I \in \text{pr}_I B', I \in P\}$ such that $\bar{u} \notin B'$. Then $\bar{u}_{\bar{n} \setminus \{k\}} \in \{\bar{a} \in \prod_{i \in \bar{n} \setminus \{k\}} \mathbb{A}_i : \bar{a}_I \in \text{pr}_I B', I \in P, k \notin I\} = \text{pr}_{\bar{n} \setminus \{k\}} B'$. Thus, for all $1 \leq k \leq n$, there exists some $c_k \in A_k$ such that

$$(u_1, \dots, u_{k-1}, c_k, u_{k+1}, \dots, u_n) \in B'. \quad (5.3)$$

Claim 5.5.6.1. *If there exists some $\bar{u} \in \prod_{i=1}^n \mathbb{A}_i$, $c_k \in A_k$, for all $1 \leq k \leq n$, such that $\bar{u} \notin B'$ and $(u_1, \dots, u_{k-1}, c_k, u_{k+1}, \dots, u_n) \in B'$, for all $1 \leq k \leq n$, then, for each $1 \leq i < j \leq n$, $B'(u_1, \dots, u_{i-1}, x_i, u_{i+1}, \dots, u_{j-1}, x_j, u_{j+1}, \dots, u_n)$ is a subuniverse of $\mathbb{A}_i \times \mathbb{A}_j$ and*

$$B'(u_1, \dots, u_{i-1}, x_i, u_{i+1}, \dots, u_{j-1}, x_j, u_{j+1}, \dots, u_n) = \begin{cases} \text{an automorphism of } \mathbb{S}, \text{ if } A_i = S = A_j, \text{ or} \\ \text{the } (A/\theta, S)\text{-cross, } \varkappa_{\bar{a}, c_j}, \text{ and } c_i = \bar{a}, \text{ if } A_i = A/\theta \text{ and } A_j = S, \text{ or} \\ \text{the } (S, A/\theta)\text{-cross, } \varkappa_{c_i, \bar{a}}, \text{ and } c_j = \bar{a}, \text{ if } A_i = S \text{ and } A_j = A/\theta, \text{ or} \\ \text{the } (A/\theta, A/\theta)\text{-cross, } \eta_{\bar{a}}, \text{ and } c_i = \bar{a} = c_j, \text{ if } A_i = A/\theta = A_j. \end{cases}$$

Proof of claim. It is clear that $B'(u_1, \dots, u_{i-1}, x_i, u_{i+1}, \dots, u_{j-1}, x_j, u_{j+1}, \dots, u_n)$ is a subuniverse of $\mathbb{A}_i \times \mathbb{A}_j$ that contains the tuples $(c_i, u_j), (u_i, c_j)$ but does not contain (u_i, u_j) , thus $c_i \neq u_i$ and $c_j \neq u_j$. Then

$$B'(u_1, \dots, u_{i-1}, x_i, u_{i+1}, \dots, u_{j-1}, x_j, u_{j+1}, \dots, u_n) \leq_{s.d.} \mathbb{A}_i \times \mathbb{A}_j$$

and $B'(u_1, \dots, u_{i-1}, x_i, u_{i+1}, \dots, u_{j-1}, x_j, u_{j+1}, \dots, u_n) \neq \mathbb{A}_i \times \mathbb{A}_j$.

WLOG, we will suppose that $i = 1$ and $j = 2$. If $A_1 = S = A_2$, then $B'(x_1, x_2, u_3, \dots, u_n) \leq_{s.d.} \mathbb{S}^2$ and $B'(x_1, x_2, u_3, \dots, u_n) \neq \mathbb{S}^2$, implies, by Corollary 3.1.6, that $B'(x_1, x_2, u_3, \dots, u_n)$ is an automorphism of \mathbb{S} .

Suppose that $A_1 = A/\theta$ and $A_2 = S$. We saw that \mathbb{A}/θ and \mathbb{S} are not isomorphic and $B'(x_1, x_2, u_3, \dots, u_n) \neq A/\theta \times S$, therefore it follows from Corollary 3.1.6 that $B'(x_1, x_2, u_3, \dots, u_n)$ is an $(A/\theta, S)$ -cross. Furthermore, since $(u_1, c_2), (c_1, u_2) \in B'(x_1, x_2, u_3, \dots, u_n)$ and $(u_1, u_2) \notin$

$B'(x_1, x_2, u_3, \dots, u_n)$ we get that $B'(x_1, x_2, u_3, \dots, u_n) = \varkappa_{c_1, c_2}$, which means, as we discussed at the beginning of the proof, $c_1 = \bar{a}$. A symmetric proof shows the result when $A_1 = S$ and $A_2 = A/\theta$.

Finally, suppose that $A_1 = A/\theta = A_2$. We have that $(u_1, c_2), (c_1, u_2) \in B'(x_1, x_2, u_3, \dots, u_n)$. We claim that $c_1 = \bar{a} = c_2$. Recall the $A_i = S$ for some $i \in \bar{n}$. WLOG, suppose that $i = 3$. Then we have that $A_1 = A/\theta$ and $A_3 = S$ implies $B'(x_1, u_2, x_3, u_4, \dots, u_n)$ is the $(A/\theta, S)$ -cross, \varkappa_{c_1, c_3} and $c_1 = \bar{a}$. Similarly, $A_2 = A/\theta$ and $A_3 = S$ implies $B'(u_1, x_2, x_3, u_4, \dots, u_n)$ is the $(A/\theta, S)$ -cross, \varkappa_{c_2, c_3} and $c_2 = \bar{a}$. Thus $(u_1, c_2) = (\bar{a}, c_2), (c_1, u_2) = (\bar{a}, c_2)$ and $(\bar{a}, c_2), (\bar{a}, c_2) \in B'(x_1, x_2, u_3, \dots, u_n)$. Since $B'(x_1, x_2, u_3, \dots, u_n)$ is a subdirect subproduct of $(\mathbb{A}/\theta)^2$ and is not equal to the full direct product $(\mathbb{A}/\theta)^2$, it follows from Corollary 3.1.6 that $B'(x_1, x_2, u_3, \dots, u_n)$ is either an automorphism of \mathbb{A}/θ or an $(A/\theta, A/\theta)$ -cross. By statement (vii) of Proposition 3.2.4, we have that $\eta_{\bar{a}} \leq (\mathbb{A}/\theta)^2$ implies every automorphism of \mathbb{A}/θ must fix \bar{a} . Thus, if $B'(x_1, x_2, u_3, \dots, u_n)$ is an automorphism of \mathbb{A}/θ , then $(\bar{a}, u_2) \in B'(x_1, x_2, u_3, \dots, u_n)$ implies $u_2 = \bar{a}$, thus $u_2 = \bar{a} = c_2$, which is a contradiction. Hence, $B'(x_1, x_2, u_3, \dots, u_n)$ is an $(A/\theta, A/\theta)$ -cross and, since $\eta_{\bar{a}}$ is the unique $(A/\theta, A/\theta)$ -cross that is a subuniverse of $(\mathbb{A}/\theta)^2$, we get that $B'(x_1, x_2, u_3, \dots, u_n) = \eta_{\bar{a}}$. This completes the proof of the claim. \square

Claim 5.5.6.2. *There exists at most one $i \in \bar{n}$ such that $A_i = S$.*

Proof of claim. Suppose not. WLOG, suppose $A_1 = S = A_3$. We are assuming that there exists some $j \in \bar{n}$ such that $A_j = A/\theta$. WLOG, suppose that $j = 2$. Then by Claim 5.5.6.1 we have that $B'(x_1, u_2, x_3, u_4, \dots, u_n)$ is an automorphism of \mathbb{S} and $B'(u_1, x_2, x_3, u_4, \dots, u_n)$ is an $(A/\theta, S)$ -cross. Therefore $B'(x_1, x_2, x_3, u_4, \dots, u_n)$ is a subuniverse of $\mathbb{S} \times \mathbb{A}/\theta \times \mathbb{S}$ that satisfies the assumptions of Corollary 5.3.2. Then it follows from Corollary 5.3.2, that either $\Lambda \leq \mathbb{A}^2$ or $\mathcal{K}_b \leq \mathbb{A}^3$, for some $b \in A \setminus S$. However, since \mathbb{S} is quasiprimal, this is a contradiction to Assumption 2. This completes the proof of the claim. \square

Since we have that the proposition fails for $B' \leq \prod_{i=1}^n \mathbb{A}_i$ where $\mathbb{A}_i = \mathbb{S}$ for some $i \in \bar{n}$, it follows from Claim 5.5.6.2 that there exists a unique element $\iota \in \bar{n}$ such that $\mathbb{A}_\iota = S$.

Either $P \neq \emptyset$ or $P = \emptyset$. We will consider these two cases separately. First suppose that $P \neq \emptyset$ and let $I \in P$. WLOG, we may permute the coordinates of B' so that $I = \{1, \dots, m\}$. Recall that we showed that proposition holds if $\bar{n} \in P$, thus, since we are assuming that the proposition fails for B' , it must be that $m < n$. By Claim 5.5.6.2 there exists at most one $i \in n$ such that $A_i = S$. Therefore we may assume, WLOG, that $A_i = A/\theta$ for all $1 \leq i \leq m-1$. Since $m < n$ it follows from (5.3) that $(u_1, \dots, u_{m-1}, u_m), (u_1, \dots, u_{m-1}, c_m) \in \text{pr}_I B'$. We claim that $u_i = \bar{a}$ for some $1 \leq i \leq m-1$. Suppose not. Then $(u_1, \dots, u_{m-1}, u_m), (u_1, \dots, u_{m-1}, c_m) \in \text{pr}_I B'$ and $\text{pr}_I B'$ a cross on $\prod_{j \in I} \mathbb{A}_j$ with special element $\bar{a} \in A/\theta$ implies that $u_m = c_m$, which contradicts $u_i \neq c_i$, for all $1 \leq i \leq n$. Therefore $u_i = \bar{a}$ for some $1 \leq i \leq m-1$. WLOG, suppose that $u_1 = \bar{a}$.

We have made no assumption on A_m , thus either $A_m = S$ or $A_m = A/\theta$. We will consider each case separately. First suppose that $A_m = S$. Then $A_1 = A/\theta$, $A_m = S$ implies, by Claim 5.5.6.1, that $B'(x_1, u_2, \dots, u_{m-1}, x_m, u_{m+1}, \dots, u_n)$ is the $(A/\theta, S)$ -cross, $\varkappa_{\bar{a}, c_m}$, and $c_1 = \bar{a}$. Then $c_1 = \bar{a} = u_1$, a contradiction. Now suppose that $A_m = A/\theta$. Then $A_1 = A/\theta = A_m$ implies, by Claim 5.5.6.1, that $B'(x_1, u_2, \dots, u_{m-1}, x_m, u_{m+1}, \dots, u_n)$ is the $(A/\theta, A/\theta)$ -cross, $\eta_{\bar{a}}$, and $c_1 = \bar{a} = c_2$. Thus $c_1 = \bar{a} = u_1$, a contradiction. Therefore $P \neq \emptyset$ contradicts $u_i \neq c_i$, for all $1 \leq i \leq n$.

Now suppose that $P = \emptyset$. We will first consider the case when $n = 3$ and then consider the case when $n > 3$. Suppose that $n = 3$. Then B' is a reduced subuniverse of $\prod_{i=1}^3 \mathbb{A}_i$ such that no projection of \mathbb{A} is a cross and, since the proposition fails for B' , $B' \neq \prod_{i=1}^3 \mathbb{A}_i$. Finally, we showed that ι is the unique such element of $\{1, 2, 3\}$ such that $A_\iota = S$. WLOG, we will permute the coordinates of B' so that $B' \leq \mathbb{A}/\theta \times \mathbb{A}/\theta \times \mathbb{S}$.

Then by Claim 5.5.6.1 we get that $B'(u_1, x_2, x_3) = \varkappa_{\bar{a}, u_3} = B'(x_1, u_2, x_3)$, $B'(x_1, x_2, u_3) = \eta_{\bar{a}}$, and $c_1 = \bar{a} = c_2$. Then,

$$\begin{aligned} B' \supseteq & (\{u_1\} \times \{\bar{a}\} \times S) \cup (\{u_1\} \times A/\theta \times \{c_3\}) \\ & \cup (\{\bar{a}\} \times \{u_2\} \times S) \cup (A/\theta \times \{u_2\} \times \{c_3\}) \\ & \cup (A/\theta \times \{\bar{a}\} \times \{u_3\}) \cup (\{\bar{a}\} \times A/\theta \times \{u_3\}). \end{aligned}$$

Let $I = \{3\}$ and $J = \{1, 2\}$. We claim that $B' \supseteq X_{I,J}^{c_3, \bar{a}}$. Suppose not. Then there exists some

tuple $(v_1, v_2, v_3) \in \mathsf{X}_{I,J}^{c_3, \bar{a}}$ such that $(v_1, v_2, v_3) \notin B'$. By the definition of $\mathsf{X}_{I,J}^{c_3, \bar{a}}$ we know that either $v_1 = \bar{a}$, or $v_2 = \bar{a}$, or $v_3 = c_3$. Suppose that $v_1 = \bar{a}$. From the above list of subsets of B' , we get that $B'(\bar{a}, x_2, x_3)$ is a subuniverse of $\mathbb{A}/\theta \times \mathbb{S}$ that contains $(\{u_2\} \times S) \cup (A/\theta \times \{u_3\})$ and $u_2 \neq c_2 = \bar{a}$, therefore, it follows that $B'(\bar{a}, x_2, x_3)$ is not an $(A/\theta, S)$ -cross and thus, by Corollary 3.1.6, we have that $B'(\bar{a}, x_2, x_3) = A/\theta \times S$. Then $B' \supseteq \{\bar{a}\} \times A/\theta \times S \ni (\bar{a}, v_2, v_3) = (v_1, v_2, v_3)$, a contradiction. A symmetric argument shows that we get a contradiction if $v_2 = \bar{a}$. Suppose that $v_3 = c_3$. From above we have that $B'(x_1, x_2, c_3)$ is a subuniverse of $(\mathbb{A}/\theta)^2$ that contains $(\{u_1\} \times A/\theta) \cup (A/\theta \times \{u_2\})$ and $u_i \neq c_i = \bar{a}$, for $i = 1, 2$, thus, $B'(x_1, x_2, c_3)$ is not an $(A/\theta, A/\theta)$ -cross, therefore, by Corollary 3.1.6, $B'(x_1, x_2, c_3) = (\mathbb{A}/\theta)^2$. Then $B' \supseteq (\mathbb{A}/\theta)^2 \times \{c_3\} \ni (v_1, v_2, c_3) = (v_1, v_2, v_3)$, a contradiction. Therefore we have shown that $B' \supseteq \mathsf{X}_{I,J}^{c_3, \bar{a}}$.

Since $P = \emptyset$ we have that $B' \neq \mathsf{X}_{I,J}^{c_3, \bar{a}}$. Therefore there exists some tuple $(v_1, v_2, v_3) \in B'$ such that $(v_1, v_2, v_3) \notin \mathsf{X}_{I,J}^{c_3, \bar{a}}$, which means $v_1 \neq \bar{a} \neq v_2$ and $v_3 \neq c_3$. Then $v_1 \in A/\theta$ implies that

$$B' \supseteq \mathsf{X}_{I,J}^{c_3, \bar{a}} \supseteq ((A/\theta)^2 \times \{c_3\}) \cup (A/\theta \times \{\bar{a}\} \times S) \supseteq (\{v_1\} \times A/\theta \times \{c_3\}) \cup (\{v_1\} \times \{\bar{a}\} \times S).$$

Therefore $B(v_1, x_2, x_3)$ is a subuniverse of $\mathbb{A}/\theta \times \mathbb{S}$ that contains $(A/\theta \times \{c_3\}) \cup (\{\bar{a}\} \times S) \cup \{(v_2, v_3)\}$. Since $v_2 \neq \bar{a}$ and $v_3 \neq c_3$, it follows that $B(v_1, x_2, x_3) = A/\theta \times S$. Thus $(\{v_1\} \times A/\theta \times S) \subseteq B'$. Also, $(\{\bar{a}\} \times A/\theta \times S) \subseteq \mathsf{X}_{I,J}^{c_3, \bar{a}} \subseteq B'$, where $\bar{a} \neq v_1$. Let $\bar{w} \in \text{pr}_{2,3} B' = \mathbb{A}/\theta \times \mathbb{S}$. Then $B'(x_1, \bar{w})$ is a subuniverse of \mathbb{A}/θ that contains v_1, \bar{a} , where $v_1 \neq \bar{a}$, which means $B'(x_1, \bar{w}) = A/\theta$. Since \bar{w} was an arbitrary element of $\text{pr}_{2,3} B'$, we get that

$$B' = A/\theta \times \bigcup_{\bar{w} \in \text{pr}_{2,3} B'} \{\bar{w}\} = A/\theta \times A/\theta \times S,$$

which contradicts the assumption that $B' \neq \Pi_{i=1}^3 A_i$. This completes the proof when $n = 3$ and $P = \emptyset$.

We will now suppose that $P = \emptyset$ and $n \geq 4$.

Claim 5.5.6.3. *Let $j \in \bar{n}$. Then $B'(x_1, \dots, x_{j-1}, u_j, x_{j+1}, \dots, x_n)$ is a cross on $\Pi_{i \in \bar{n} \setminus \{j\}} \mathbb{A}_i$.*

Proof of claim. Let $j \in \bar{n}$. Then $B'(x_1, \dots, x_{j-1}, u_j, x_{j+1}, \dots, x_n)$ is a subuniverse of $\Pi_{i \in \bar{n} \setminus \{j\}} \mathbb{A}_i$. WLOG, suppose that $j = 1$. Clearly, $B'(u_1, x_2, \dots, x_n)$ is a subuniverse of $\Pi_{i \in \bar{n} \setminus \{1\}} \mathbb{A}_i$. We will

show that $B'(u_1, x_2, \dots, x_n)$ is reduced, no m -ary projection of $B'(u_1, x_2, \dots, x_n)$ is a cross, for $2 \leq m < n - 2$, and $B'(u_1, x_2, \dots, x_n) \neq \prod_{i \in \bar{n} \setminus \{j\}} \mathbb{A}_i$. Then the result of the claim will follow from the minimality of n .

First we will show that $B'(u_1, x_2, \dots, x_n)$ is reduced. Since $c_k, u_k \in \text{pr}_k B'(u_1, x_2, \dots, x_n)$ and $c_k \neq u_k$, for all $2 \leq k \leq n$, we get that no unary projection of $B'(u_1, x_2, \dots, x_n)$ is trivial. Furthermore, since $n \geq 4$, for all $2 \leq i < j \leq n$, we have that $(c_i, u_j), (u_i, u_j) \in \text{pr}_{i,j} B'(u_1, x_2, \dots, x_n)$. Thus, if $\text{pr}_{i,j} B'(u_1, x_2, \dots, x_n)$ is an automorphism, then $c_i = u_i$, which is a contradiction. Therefore, no binary projection of $B'(u_1, x_2, \dots, x_n)$ is an automorphism. Since \mathbb{S} is quasiprimal this shows that $B'(u_1, x_2, \dots, x_n)$ is reduced.

Now, we will show that no m -ary projection of $B'(u_1, x_2, \dots, x_n)$ is a cross, for $2 \leq m < n - 1$. Let $I' \subseteq \{2, \dots, n\}$, $|I'| = m$. Let $I = I' \cup \{1\}$. Then $|I| = m + 1 < n$ and $\text{pr}_I B'$ is a reduced subuniverse of $\prod_{i \in I} \mathbb{A}_i$ such that, since $P = \emptyset$, no projection of $\text{pr}_I B'$ is a cross. By the minimality of n we get that $\text{pr}_I B' = \prod_{i \in I} \mathbb{A}_i$. Then $u_1 \in \text{pr}_1 B'$ implies $\text{pr}_I B' \supseteq \{u_1\} \times \prod_{i \in I \setminus \{1\}} \mathbb{A}_i = \{u_1\} \times \prod_{i \in I'} \mathbb{A}_i$. Hence $\text{pr}_{I'} B'(u_1, x_2, \dots, x_n) \supseteq \prod_{i \in I'} \mathbb{A}_i$. Since I' was arbitrary, we have shown that no m -ary projection of $B'(u_1, x_2, \dots, x_n)$ is a cross, for $1 \leq m < n - 1$.

Then it follows from the minimality of n that $B'(u_1, x_2, \dots, x_n)$ is the full cross on $\prod_{i=2}^n \mathbb{A}_i$. This completes the proof of the claim. \square

Recall that ι is the unique element of \bar{n} such that $A_\iota = S$. Let $I = \{\iota\}$ and $J = \bar{n} \setminus \{\iota\}$.

Claim 5.5.6.4. $B' \supseteq \mathcal{X}_{I,J}^{u_\iota, \bar{a}}$

Proof of claim. To show this claim we will show that, for $j \in J$, $B'(x_1, \dots, x_{j-1}, \bar{a}, x_{j+1}, \dots, x_n) = \prod_{i \in \bar{n} \setminus \{j\}} \mathbb{A}_i$ and, for $j \in I$, $B'(x_1, \dots, x_{j-1}, u_j, x_{j+1}, \dots, x_n) = \prod_{i \in \bar{n} \setminus \{j\}} \mathbb{A}_i$. WLOG, we will show this result for $j = 1$.

First it will be useful to show that, for any $2 \leq k \leq n$,

$$\{a_1\} \times \prod_{i \in \bar{n} \setminus \{1,k\}} \mathbb{A}_i \subseteq B''(x_1, \dots, x_{k-1}, u_k, x_{k+1}, \dots, x_n),$$

where $a_1 = u_1$, if $A_1 = S$, and $a_1 = \bar{a}$, if $A_1 = A/\theta$. WLOG, suppose that $k = n$. Claim 5.5.6.3

implies that $B'(x_1, \dots, x_{n-1}, u_n)$ is a cross on $\Pi_{i=1}^{n-1} \mathbb{A}_i$. Thus there exists some $a \in A_1$ such that $\{a\} \times \Pi_{i=2}^{n-1} \mathbb{A}_i \subseteq B'(x_1, \dots, x_{n-1}, u_n)$. Since A_1 and A_2 are not both equal to S , it follows from Claim 5.5.6.1 that $B'(x_1, x_2, u_3, \dots, u_n) = (\{a_1\} \times A_2) \cup (A_1 \times \{a_2\})$, where $a_i = u_i$ if $A_i = S$ and $a_i = \bar{a}$ if $A_i = A/\theta$, for $i = 1, 2$. We claim that $a = a_1$. Suppose not. Note that $\{a\} \times \Pi_{i=2}^{n-1} \mathbb{A}_i \subseteq B'(x_1, \dots, x_{n-1}, u_n)$ implies $\{a\} \times A_2 \times \{(u_3, \dots, u_n)\} \subseteq B''$. Also $B'(x_1, x_2, u_3, \dots, u_n) = (\{a_1\} \times A_2) \cup (A_1 \times \{a_2\})$ implies $\{a_1\} \times A_2 \times \{(u_3, \dots, u_n)\} \subseteq B''$. Therefore $(\{a\} \times A_2) \cup (\{a_1\} \times A_2) \subseteq B'(x_1, x_2, u_3, \dots, u_n)$. Then for $v \in A_2$, we have that $B'(x_1, v, u_3, \dots, u_n)$ is a subuniverse of A_1 that contains the distinct elements a, a_1 . Since A_1 is strictly simple, it follows that $B'(x_1, v, u_3, \dots, u_n) = A_1$. We chose $v \in A_2$ arbitrarily, therefore $B'(x_1, x_2, u_3, \dots, u_n) = A_1 \times \bigcup_{v \in A_2} \{v\} = A_1 \times A_2$, which contradicts $B'(x_1, x_2, u_3, \dots, u_n) = (\{a_1\} \times A_2) \cup (A_2 \times \{a_2\})$. Thus $a = a_1$ and we have shown that $\{a_1\} \times \Pi_{i=2}^{n-1} \mathbb{A}_i \subseteq B'(x_1, \dots, x_{n-1}, u_n)$, where $a_1 = u_1$, if $A_1 = S$, and $a_1 = \bar{a}$, if $A_1 = A/\theta$.

Let,

$$g = \begin{cases} u_1, & \text{if } A_1 = S \\ \bar{a}, & \text{if } A_1 = A/\theta. \end{cases}$$

Then we have shown that $\{g\} \times \Pi_{i=2}^{k-1} \mathbb{A}_i \times \{u_k\} \times \Pi_{k+1}^n \mathbb{A}_i \subseteq B''$, for each $2 \leq k \leq n$. Thus

$$B'(g, x_2, \dots, x_n) \supseteq \bigcup_{k \in \{2, \dots, n\}} (\Pi_{i=2}^{k-1} \mathbb{A}_i \times \{u_k\} \times \Pi_{k+1}^n \mathbb{A}_i).$$

Clearly $B'(g, x_2, \dots, x_n)$ is reduced and no m -ary projection of $B'(g, x_2, \dots, x_n)$ is a cross, for $2 \leq m \leq n-2$. Furthermore, we claim that $B'(g, x_2, \dots, x_n)$ is not a cross on $\Pi_{i=2}^n \mathbb{A}_i$. Recall that $n \geq 4$, $A_i = S$, and $A_j = A/\theta$, for all $j \in J$. Then it follows from Claim 5.5.6.1 that $c_j = \bar{a}$ for all $j \in J$. If $B'(g, x_2, \dots, x_n)$ is a cross on $\Pi_{i=2}^n \mathbb{A}_i$, then we have that

$$B'(g, x_2, \dots, x_n) \supseteq \bigcup_{k \in \{2, \dots, n\}} (\Pi_{i=2}^{k-1} \mathbb{A}_i \times \{u_k\} \times \Pi_{k+1}^n \mathbb{A}_i)$$

implies $u_j = \bar{a}$, for all $j \in J$. Thus $u_i = \bar{a} = c_i$, which is a contradiction. Therefore $B'(g, x_2, \dots, x_n)$ is not a full cross on $\Pi_{i=2}^n \mathbb{A}_i$. Then it follows from the minimality of n that $B'(g, x_2, \dots, x_n) = \Pi_{i=2}^n \mathbb{A}_i$. This completes the proof of the claim. \square

From Claim 5.5.6.4 we have that B' contains the cross $X_{I,J}^{(u_i, \bar{a})}$. We are assuming that no projection of B' is a cross, therefore there must exist some tuple $(v_1, \dots, v_n) \in B'$ such that $(v_1, \dots, v_n) \notin X_{I,J}^{(u_i, \bar{a})}$. Then $v_i \neq u_i$ and $v_j \neq \bar{a}$, for all $j \in J$.

WLOG, we will permute the coordinates of B' so that $A_1 = A/\theta$. Clearly $B'(v_1, x_2, \dots, x_n)$ is a subuniverse of $\prod_{i=2}^n \mathbb{A}_i$. We claim that $B'(v_1, x_2, \dots, x_n)$ is reduced and no projection of $B'(v_1, x_2, \dots, x_n)$ is a cross. Since B' contains the full cross $X_{I,J}^{(u_i, \bar{a})}$, we have that $\{v_1\} \times X_{I, J \setminus \{1\}}^{(u_i, \bar{a})} \subseteq B'$, thus $B'(v_1, x_2, \dots, x_n) \supseteq X_{I, J \setminus \{1\}}^{(u_i, \bar{a})}$. Therefore it is clear that $B'(v_1, x_2, \dots, x_n)$ is reduced and no m -ary projection of $B'(v_1, x_2, \dots, x_n)$ is a cross, for $2 \leq m \leq n-2$. Furthermore, $X_{I, J \setminus \{1\}}^{(u_i, \bar{a})} \cup (v_2, \dots, v_n) \subseteq B'(v_1, x_2, \dots, x_n)$, where $v_j \neq \bar{a}$, for all $j \in J$, and $v_i \neq u_i$. Thus $B'(v_1, x_2, \dots, x_n) \neq X_{I, J \setminus \{1\}}^{(u_i, \bar{a})}$ and, since $B'(v_1, x_2, \dots, x_n) \supseteq X_{I, J \setminus \{1\}}^{(u_i, \bar{a})}$, it cannot be that $B'(v_1, x_2, \dots, x_n)$ is a cross on $\prod_{i=2}^n \mathbb{A}_i$, therefore by the minimality of n we get that $B'(v_1, x_2, \dots, x_n) = \prod_{i=2}^n \mathbb{A}_i$.

We have shown that $(\{v_1\} \times \prod_{i=2}^n \mathbb{A}_i) \subseteq B'$ and $\text{pr}_{2, \dots, n} B' = \prod_{i=2}^n \mathbb{A}_i$. Furthermore, since B' contains $X_{I,J}^{(u_i, \bar{a})}$, we have that $(\{\bar{a}\} \times \prod_{i=2}^n \mathbb{A}_i) \subseteq B'$. Thus $(\{\bar{a}\} \times \prod_{i=2}^n \mathbb{A}_i) \cup (\{v_1\} \times \prod_{i=2}^n \mathbb{A}_i) \subseteq B'$, where $v_1 \neq \bar{a}$. Let $\bar{w} \in \text{pr}_{2, \dots, n} B' = \prod_{i=2}^n \mathbb{A}_i$. Then $B'(x_1, \bar{w})$ is a subuniverse of \mathbb{A}/θ that contains v_1, \bar{a} , where $v_1 \neq \bar{a}$. Since \mathbb{A}/θ is strictly simple, we get that $B'(x_1, \bar{w}) = A/\theta$. The choice of \bar{w} was arbitrary in $\text{pr}_{2, \dots, n} B'$, therefore

$$B' = A/\theta \times \bigcup_{\bar{w} \in \text{pr}_{2, \dots, n} B'} \{\bar{w}\} = A_1 \times \prod_{i=2}^n \mathbb{A}_i = \prod_{i=1}^n \mathbb{A}_i,$$

which contradicts our assumption that $B' \neq \prod_{i=1}^n \mathbb{A}_i$. Therefore our assumption that there exists some reduced subuniverse for which the proposition fails is incorrect. This completes the proof of the proposition. \square

In Propositions 5.5.4, 5.5.5, and 5.5.6 we obtained similar descriptions for the reduced subuniverses of $\prod_{i=1}^n \mathbb{A}_i$, where $\mathbb{A}_i \in \{\mathbb{S}, \mathbb{A}/\theta\}$, under various assumptions on the existence of crosses among the subuniverses of $(\mathbb{A}/\theta)^2$ and $\mathbb{A}/\theta \times \mathbb{S}$. Next we will translate these results into descriptions of subuniverses of finite powers of \mathbb{A} .

The analog of Definition 5.5.1 is the following.

Definition 5.5.7. Let $\mathbb{B}_i \in \{\mathbb{S}, \mathbb{A}\}$, $1 \leq i \leq n$. Let $\{I, J\}$ be the partition of \bar{n} such that $\mathbb{B}_i = \mathbb{S}$ whenever $i \in I$ and $\mathbb{B}_i = \mathbb{A}$ whenever $i \in J$. Let $s \in S$ and let $G = S$ or $G = \{b\}$ for some $b \in A \setminus S$. Then we will call the set

$$\mathbf{X}_{(I,J)}^{(s,G)} := \{(x_1, \dots, x_n) \in \prod_{i=1}^n \mathbb{B}_i : \text{there exists some } i \text{ such that } x_i = s \text{ if } i \in I \text{ and } x_i \in G \text{ if } i \in J\}$$

a $(\mathbb{B}_i)_{i=1}^n$ -cross. If $\mathbb{B}_1 = \dots = \mathbb{B}_n \in \{\mathbb{S}, \mathbb{A}\}$, then we will simply denote an $(S)_{i=1}^n$ -cross by \mathbf{X}_n^s , and an $(A)_{i=1}^n$ -cross by \mathbf{X}_n^G with s and G as before.

It is easy to see that if ρ denotes the natural homomorphism $\prod_{i=1}^n \mathbb{B}_i \rightarrow \prod_{i=1}^n \mathbb{A}_i$ where $\mathbb{A}_i = \mathbb{A}/\theta$ if $\mathbb{B}_i = \mathbb{A}$ and $\mathbb{A}_i = \mathbb{S}$ if $\mathbb{B}_i = \mathbb{S}$, then the $(\mathbb{B}_i)_{i=1}^n$ -cross $\mathbf{X}_{(I,J)}^{s,G}$ is the full inverse image, under ρ , of the cross $\mathbf{X}_{(I,J)}^{s,\bar{a}}$ on $\prod_{i=1}^n \mathbb{A}_i$ where $\bar{a} = \bar{S}$ if $G = S$ and $\bar{a} = \bar{b}$ if $G = \{b\}$ for some $b \in A \setminus S$.

In particular, for $n = 2$, $s \in S$ and $b \in A \setminus S$ we have

$$\mathbf{X}_2^s = \nu_s, \quad \mathbf{X}_2^S = \chi_{S,S}, \quad \mathbf{X}_2^b = \mu_b, \quad \mathbf{X}_{(\{1\},\{2\})}^{(s,S)} = \lambda_{S,s}^{-1}, \quad \mathbf{X}_{(\{1\},\{2\})}^{(s,b)} = \kappa_{b,s}^{-1}.$$

Theorem 5.5.8. Suppose that \mathbb{A} satisfies Assumption 2 and θ is a congruence on \mathbb{A} . Let $B \leq_{s.d.} \prod_{i=1}^n \mathbb{B}_i$ ($n \geq 2$) where $\mathbb{B}_i \in \{\mathbb{A}, \mathbb{S}\}$ for all $1 \leq i \leq n$, and let ρ be the natural homomorphism $\prod_{i=1}^n \mathbb{B}_i \rightarrow \prod_{i=1}^n \mathbb{A}_i$ where $\mathbb{A}_i = \mathbb{A}/\theta$ if $\mathbb{B}_i = \mathbb{A}$ and $\mathbb{A}_i = \mathbb{S}$ if $\mathbb{B}_i = \mathbb{S}$. If $B' = \rho(B)$ is a reduced subuniverse of $\prod_{i=1}^n \mathbb{A}_i$, then

$$B = \{\bar{a} \in \prod_{i=1}^n \mathbb{B}_i : \bar{a}_I \in \text{pr}_I B \text{ for all } I \in P\}, \quad (5.4)$$

where P is the set of all subsets I of \bar{n} such that $\text{pr}_I B$ is a $(\mathbb{B}_i)_{i \in I}$ -cross.

Proof. Under the assumptions of the theorem, \mathbb{A}/θ is a finite idempotent strictly simple algebra. Recall that $|A \setminus S| > 1$, therefore, $|A/\theta| > 2$. Then by Corollary 2.4.9, \mathbb{A}/θ is either quasiprimal or affine or has an $(A/\theta, A/\theta)$ -cross among its subuniverses. Therefore one of Propositions 5.5.4, 5.5.5, or 5.5.6 applies to B' . In each case, since B' is reduced, therefore B is a reduced subuniverse of \mathbb{A}^n . Hence, by Theorem 5.4.1, B is θ -closed in its A -coordinates, which implies by Proposition 2.2.8 that $B = \rho^{-1}(B')$.

Therefore, if \mathbb{A}/θ is quasiprimal or affine, then the equality proved in Proposition 5.5.4 implies, by taking inverse images, that

$$B = \{(x_1, \dots, x_n) \in \prod_{i=1}^n B_i : (x_i, x_j) \in \text{pr}_{i,j} B, 1 \leq i < j \leq n\}.$$

As we noted at the beginning of the proof of Propositions 5.5.4, in this case a binary projection of B' is either an $(S, A/\theta)$ -cross, an $(A/\theta, S)$ -cross, or a direct product $S^2, (A/\theta)^2, S \times A/\theta$, or $A/\theta \times S$. Thus a binary projection of B is either an (S, A) -cross, an (A, S) -cross, or a direct product $S^2, A^2, S \times A$, or $A \times S$. If $\text{pr}_{i,j} B$ is a direct product, then $\text{pr}_{i,j} B = \text{pr}_i B \times \text{pr}_j B = B_i \times B_j$, so the condition $(x_i, x_j) \in \text{pr}_{i,j} B$ makes no restriction, and can be omitted. Thus we get that (5.4) is true in this case.

Suppose there exists an $(A/\theta, A/\theta)$ -cross among the subuniverses of $(\mathbb{A}/\theta)^2$, but there is no $(S, A/\theta)$ -cross among the subuniverses of $\mathbb{S} \times \mathbb{A}/\theta$, then by permuting coordinates if necessary, we get from Proposition 5.5.5 that the equality $B' = \text{pr}_J B' \times \text{pr}_K B'$ holds for B' where J is the set of all $i \in \bar{n}$ such that $\mathbb{B}_i = \mathbb{S}$, and K is the set of all $i \in \bar{n}$ such that $\mathbb{B}_i = \mathbb{A}$. Here $\text{pr}_K B'$ is a reduced subuniverse of a power of the strictly simple algebra \mathbb{A}/θ , therefore it follows from Proposition 2.4.7 that $\text{pr}_K B' = \{\bar{a} \in \prod_{i \in K} \mathbb{A}_i : \bar{a}_I \in \text{pr}_I B' \text{ for all } I \in P'\}$, where P' is the set of all $I \subseteq K$ such that $\text{pr}_I B'$ is an cross on $(\mathbb{A}/\theta)^I$. Hence $B' = \{\bar{a} \in \prod_{i=1}^n \mathbb{A}_i : \bar{a}_I \in \text{pr}_I B' \text{ for all } I \in P'\}$. By taking inverse images under ρ we get that (5.4) is true.

Finally, if there exists an $(A/\theta, A/\theta)$ -cross among the subuniverses of $(\mathbb{A}/\theta)^2$ and also an $(S, A/\theta)$ -cross among the subuniverses of $\mathbb{S} \times \mathbb{A}/\theta$, then (5.4) follows immediately from the equality proved in Proposition 5.5.6, by taking inverse images. \square

Theorem 5.5.9. *Suppose that \mathbb{A} satisfies Assumption 2 and that θ is a congruence on \mathbb{A} . Then the relational clone $\text{RClo}(\mathbb{A})$ of \mathbb{A} is generated by the following members of $\text{RClo}(\mathbb{A})$.*

- (i) All $\{a\}$ for $a \in A$.
- (ii) All automorphisms of \mathbb{A}, \mathbb{S} and \mathbb{A}/θ .
- (iii) All isomorphisms $\mathbb{S} \rightarrow \mathbb{A}/\theta$.

(iv) All h.d.-automorphisms of \mathbb{A} , \mathbb{S} , and \mathbb{A}/θ .

(v) All higher dimensional crosses $\mathbf{X}_{(I,J)}^{(s,G)}$ in $\text{RClo}(\mathbb{A})$ where $s \in S$, $G = S$ or $G = \{b\}$ for some $b \in A \setminus S$, and $\{I, J\}$ is a partition of \bar{n} , $n \geq 2$ with $|I| \leq 1$.

Proof. Let R denote the set of relations listed in (i)–(vi). It is clear that $R \subseteq \text{RClo}(\mathbb{A})$. To show that R generates $\text{RClo}(\mathbb{A})$ we will choose any subuniverse B of a finite power of \mathbb{A} , and want to show that B is contained in the relational clone $\langle R \rangle_{\text{RClo}}$ generated by R .

All members of

$$\mathcal{T}_{\mathbb{A}} = \{\{a\} : a \in A\} \cup \text{Aut}(\mathbb{S}) \cup \text{Aut}(\mathbb{A}) \cup \text{Aut}_{\text{h.d.}}(\mathbb{S}) \cup \text{Aut}_{\text{h.d.}}(\mathbb{A})$$

are listed in R , so it follows that $\mathcal{T}_{\mathbb{A}} \subseteq \langle R \rangle_{\text{RClo}}$. All remaining members of

$$\mathcal{T}'_{\mathbb{A}} = \text{Aut}(\mathbb{A}/\theta) \cup \text{Isom}(\mathbb{S}, \mathbb{A}/\theta) \cup \text{Aut}_{\text{h.d.}}(\mathbb{A}/\theta)$$

are listed in R , so we get as before that $\mathcal{T}'_{\mathbb{A}} \subseteq \langle R \rangle_{\text{RClo}}$.

Now let B be a subuniverse of \mathbb{A}^n . Then it follows from Proposition 5.1.3 that there exists a nonempty subset $I \subseteq \bar{n}$ such that $B \in \langle \text{pr}_I B, \mathcal{T}_{\mathbb{A}} \rangle_{\text{RClo}}$ and $\text{pr}_I B$ is a reduced subuniverse of $\mathbb{A}^{|I|}$. Thus, it will suffice to show that $\text{pr}_I B \in \langle R \rangle_{\text{RClo}}$.

Therefore, replacing B by $\text{pr}_I B$, we may assume that B is reduced. Let $B \leq_{s.d.} \prod_{i=1}^n \mathbb{B}_i$ where $\mathbb{B}_i \in \{\mathbb{S}, \mathbb{A}\}$ for all $1 \leq i \leq n$. With the same notation as in the preceding theorem, let $B' = \rho(B)$. We know from Theorem 5.4.1 that B is θ -closed in its A -coordinates, and therefore $B = \rho^{-1}(B')$. Also, it follows from Proposition 5.1.6 that there exists some nonempty $J \subseteq \bar{n}$ such that $B' \in \langle \text{pr}_J B', \mathcal{T}'_{\mathbb{A}} \rangle_{\text{RClo}}$ and $\text{pr}_J B'$ is reduced. By examining the proof, and using the fact that $B = \rho^{-1}(B')$, one can see that $B \in \langle \text{pr}_J B, \mathcal{T}'_{\mathbb{A}} \rangle_{\text{RClo}}$ also holds.

Therefore, replacing B by $\text{pr}_J B$, we may assume that B is such that B' is reduced. Now Theorem 5.5.8 shows that B is in the relational clone generated by the relations in (vi). This proves that $B \in \langle R \rangle_{\text{RClo}}$. \square

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