Nonanalyticities in the Time Evolution of Unconventional Superconductors

by

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Thesis directed by Professor Victor Gurarie

Dynamical Quantum Phase Transitions (DQPTs) are nonanalyticities in the Loschmidt echo, a quantity that characterizes the time evolution of quantum systems. These are analogous to thermal phase transitions, which are nonanalyticities in the thermal partition function. Over the past decade, DQPTs have been shown to have a close relationship with the equilibrium phases of quantum systems. DQPTs can serve as a probe of the underlying quantum critical points of a system.

In this thesis, we explore how DQPTs can occur in superconductors. We analytically show that DQPTs can occur in topological superconductors but not in the topologically trivial s-wave superconductor. We extend this analysis to study the spectral form factor, which is a related quantity that tells us about the time evolution of systems. We see how the spectral form factor of unconventional superconductors can have nonanalyticities, whereas that of the s-wave superconductor is featureless. These nonanalyticities arise due to the nontrivial structure of the superconducting gap function in momentum space, which in turn result from the symmetries of the Cooper pair wavefunction and the underlying lattice. We propose a method by which the spectral form factor can be measured if the superconducting Hamiltonian is simulated using qubits realized as Anderson pseudospins.

We find that the Schwinger-Keldysh mean field formalism is insufficient to evaluate these nonanalyticities in superconductors, and therefore develop a generalized mean field theory for this purpose. We present a simple model of a flat-band superconductor for which we can explicitly verify the validity of our generalized mean field theory.

Dedication

To my friends and family

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First and foremost, I would like to express my deep gratitude and appreciation for my advisor, Victor Gurarie. Ever since the first day I approached him as a directionless graduate student, he has devoted a considerable amount of time and effort helping and guiding me. He helped me build my knowledge of condensed matter physics from ground up. He constantly kept supporting and pushing me when I thought that my research was at a dead end. He always made time for me, even when he was overwhelmed with other commitments. I appreciate how I could discuss with Victor for long hours, and he would never get tired of it. These discussions not only helped me in my research, but significantly shaped the way I think and approach problems in general.

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Not all those who wander are lost. - J. R. R. Tolkien

Chapter 1

Introduction

Condensed matter physics is the study of the phases of matter which arise in systems with many atoms and molecules. The properties of these phases can be very different than those of the individual atoms and molecules comprising these phases [1]. These phases include classical states of matter like solids, liquids and gases, as well as quantum phases such as Fermi liquids, superconductors, superfluids, topological insulators, quantum spin liquids, neutron stars, and a myriad others [2–11]. The theoretical and experimental investigation of these phases of matter is at the heart of the field of condensed matter physics.

Over the previous decade, a new approach has emerged in this study of phases of matter. Studying the dynamics of a quantum system can reveal insights into its equilibrium properties. In particular, the phenomenon of dynamical quantum phase transitions (DQPTs) is closely related to quantum critical points and equilibrium phase transitions in such systems [12]. Also, DQPTs turn out to be an interesting phenomenon to study in itself.

Another area of highly active research is that of topological insulators and superconductors [9, 13]. These are systems which have associated topological invariants which cannot be changed in a continuous manner. Topological superconductors are unconventional superconductors whose gap function can vanish at points in its momentum space. The topological invariants for these materials cannot be changed by a continuous tuning of its parameters, but rather involve closing the energy gap in its bulk. These systems are of great interest to the quantum computing community, since they can be used to implement qubits that are topologically protected [14].

In this chapter, we will review some of the important concepts regarding DQPTs and their occurrence in topological insulators and superconductors.

1.1 Formulation of dynamical quantum phase transitions

We start by recalling the main quantity of interest in equilibrium statistical physics. The central quantity in statistical physics is the partition function, defined as [3]

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta \hat{H}} = \sum_{\nu} e^{-\beta E_\nu} \ , \tag{1.1}$$

where $\hat{H}$ is the Hamiltonian of our system with energy eigenvalues E_ν, and Tr represents the trace operator. β is the inverse temperature given by $\beta = \frac{1}{k_B T}$, where k_B is the Boltzmann constant. If we have the expression for the partition function, we can get the free energy as

$$F = -k_B T \ln \mathcal{Z} \ . \tag{1.2}$$

This is related to several physical observables. For example, the derivative of the free energy with volume yields pressure, and the second derivative of the free energy with temperature is related to the heat capacity. If the free energy as a function of temperature becomes non-analytic at some temperature, then an equilibrium phase transition is said to occur.

A dynamical quantum phase transition is defined in a similar way [12, 15]. We work with a Hamiltonian with some tunable parameter g. Let us have $g = g_0$ at the start of our process. We initialize the system in the ground state $|\psi_0\rangle$ of the Hamiltonian $\hat{H}_0 = \hat{H}(g_0)$. At time $t = 0$, the parameter g is changed from g_0 to some other value g_f. This is known as a quantum quench. The initial state $|\psi_0\rangle$ then evolves under the Hamiltonian $\hat{H} = \hat{H}(g_f)$. With this we can define the Loschmidt amplitude as

$$\mathcal{G}(t) = \langle \psi_0 | e^{-i\hat{H}t} | \psi_0 \rangle \ , \tag{1.3}$$

and the Loschmidt echo,

$$\mathcal{L}(t) = |\mathcal{G}(t)|^2 \ . \tag{1.4}$$

The Loschmidt amplitude is the quantity of interest here, analogous to the partition function in the study of thermal phase transitions. Analogous to the free energy, we can define the rate function

$$\lambda(t) = -\frac{1}{N} \ln \mathcal{L}(t) \ . \tag{1.5}$$

When there is a non-analyticity in the rate function $\lambda(t)$ at some time t, a dynamical quantum phase transition (DQPT) is said to occur.

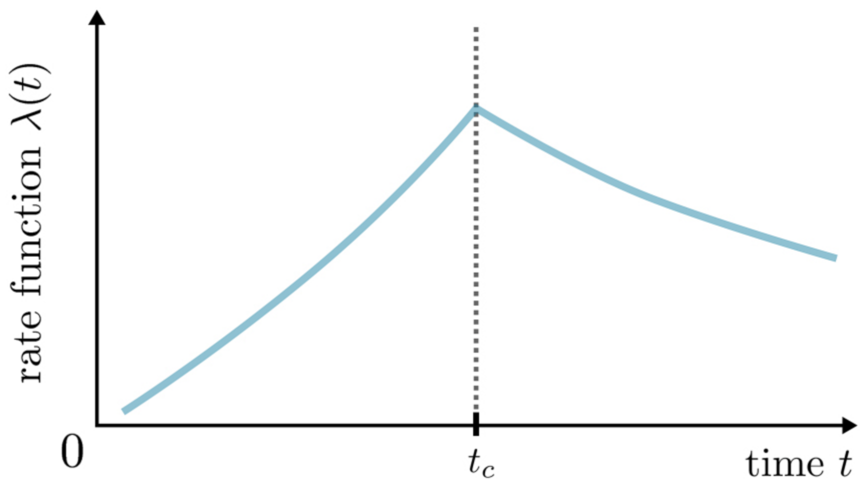

Figure 1.1: Schematic of a dynamical quantum phase transition (DQPT). Figure reproduced from Ref. [12].

1.2 Fisher zeros

We can further understand DQPTs using a concept called Fisher zeros, which was used by Michael Fisher in the study of equilibrium phase transitions [16]. We extend time t from the real axis to the complex plane, $t \to z = t + i\tau$, and analytically continue the Loschmidt amplitude to

$$\mathcal{G}(z) = \langle \psi_0 | e^{-i\hat{H}z} | \psi_0 \rangle \ . \tag{1.6}$$

If our system has a finite number of degrees of freedom, which is the case for fermionic or spin systems, this becomes

$$\mathcal{G}(z) = \sum_{\nu} |\langle E_\nu|\psi_0\rangle|^2 \, e^{-iE_\nu z} \; , \tag{1.7}$$

where $|E_\nu\rangle$ is an energy eigenstate with eigenvalue E_ν. Since $\mathcal{G}(z)$ is a sum over a finite number of analytic functions, $\mathcal{G}(z)$ is itself an analytic function in z. Now using the Weierstrass factorization theorem [17], we can write

$$\mathcal{G}(z) = e^{\mu(z)} \prod_j (z_j - z) \; , \tag{1.8}$$

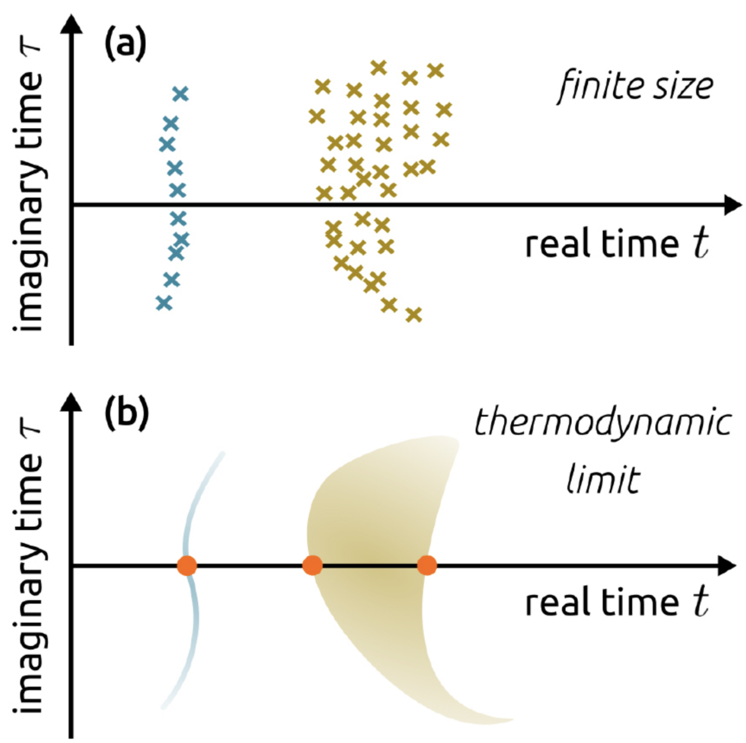

Figure 1.2: Schematic of Fisher zeros in the complex time plane. (a) For systems with a finite number of degrees of freedom, the Fisher zeros are isolated points in the complex plane. (b) In the thermodynamic limit, the Fisher zeros coalesce onto lines or areas. Figure reproduced from Ref. [12].

where z_j are the zeros of $\mathcal{G}(z)$ in the complex plane and $\mu(z)$ is an analytic function. $\mu(z)$ is not of interest here since the singular part of the rate function is given by

$$\lambda_s(z) = -\frac{2}{N}\sum_j \ln|z_j - z| \ . \tag{1.9}$$

For finite N, the zeros z_j are isolated points in the complex plane. But in the thermodynamic limit, these zeros coalesce onto lines or areas in the complex plane, depending on the dimensionality and symmetries of the system [18], see Fig 1.2.

DQPTs occur whenever a line or the boundary of an area of Fisher zeros intersects the real-time axis [16]. The singularity in the rate function is obtained as

$$\lambda_s(z) = \int_{\mathbf{C}} dz' \, \rho(z') \ln\left|z' - z\right| \ , \tag{1.10}$$

where $\rho(z')$ gives the density of Fisher zeros at point z' in the complex plane.

1.3 Example of DQPT: Transverse Field Ising Model

In this section, we will see how a DQPT can take place in a physical model. The first model in which DQPTs were found analytically is the Transverse Field Ising Model (TFIM) in one spatial dimension [15]. We shall briefly detail this calculation in order to show how DQPTs can arise.

We follow the calculation presented in Ref. [15] where the authors work with a TFIM in (1+1)D with nearest-neighbor coupling, as it has an exact analytical solution. We start with the Hamiltonian,

$$\hat{H}(g) = -\frac{1}{2}\sum_{i=1}^{N-1} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + \frac{g}{2}\sum_{i=1}^{N} \hat{\sigma}_i^x \ . \tag{1.11}$$

These are N quantum spin-$\frac{1}{2}$'s in 1D with periodic boundary conditions. The strength of the transverse field, g, is the tuning parameter here. This model has a quantum critical point at $g = 1$ [19]. For $g < 1$ the ground state is ferromagnetic, while for $g > 1$ the ground state is paramagnetic. The Hamiltonian can be mapped to a quadratic fermionic model using the Jordan-

Wigner transformation [20],

$$\hat{H}(g) = -\frac{1}{2}\sum_{i=1}^{N-1}(\hat{c}_i^\dagger \hat{c}_{i+1} + \hat{c}_i^\dagger \hat{c}_{i+1}^\dagger + \text{H.c.}) + g\sum_{i=1}^{N}\hat{c}_i^\dagger \hat{c}_i \ , \tag{1.12}$$

where $\hat{c}_i$ and $\hat{c}_i^\dagger$ respectively annihilate and create a fermion at site i. This is a translation invariant Hamiltonian that can be diagonalized in the momentum basis to yield the dispersion relation

$$\epsilon_k = \sqrt{(g - \cos k)^2 + \sin^2 k} \ . \tag{1.13}$$

Each momentum mode k has two energy eigenvalues, ϵ_k and $-\epsilon_k$.

We initialize the system in the ground state $|\psi_0\rangle$ of Hamiltonian $\hat{H}(g_0)$. We follow the standard procedure laid out in Section 1.1 and quench the Hamiltonian from $\hat{H}(g_0)$ to $\hat{H}(g_1)$ at time $t = 0$. We then evaluate the Loschmidt amplitude and the rate function,

$$\mathcal{Z}(z) = \langle\psi_0|e^{-z\hat{H}}|\psi_0\rangle \quad , \quad f(z) = -\lim_{N\to\infty}\frac{1}{N}\ln \mathcal{Z}(z) \ . \tag{1.14}$$

The rate function then evaluates to [15]

$$f(z) = -\int_0^\infty \frac{dk}{2\pi}\ln\left(\cos^2\phi_k + \sin^2\phi_k\, e^{-2z\epsilon_k(g_1)}\right) + (\text{analytic}) \ , \tag{1.15}$$

where $\phi_k = \theta_k(g_0) - \theta_k(g_1)$, and

$$\tan(2\theta_k(g)) = \frac{\sin k}{g - \cos k} \quad , \quad \theta_k(g) \in \left[0, \frac{\pi}{2}\right] \ . \tag{1.16}$$

From the rate function, we get the Fisher zeros as

$$z_n(k) = \frac{1}{2\epsilon_k(g_1)}\left[\ln\left(\tan^2\phi_k\right) + i\pi(2n+1)\right] \ . \tag{1.17}$$

Note from the definition (1.14) of the Loschmidt amplitude used by the authors of Ref. [15], a

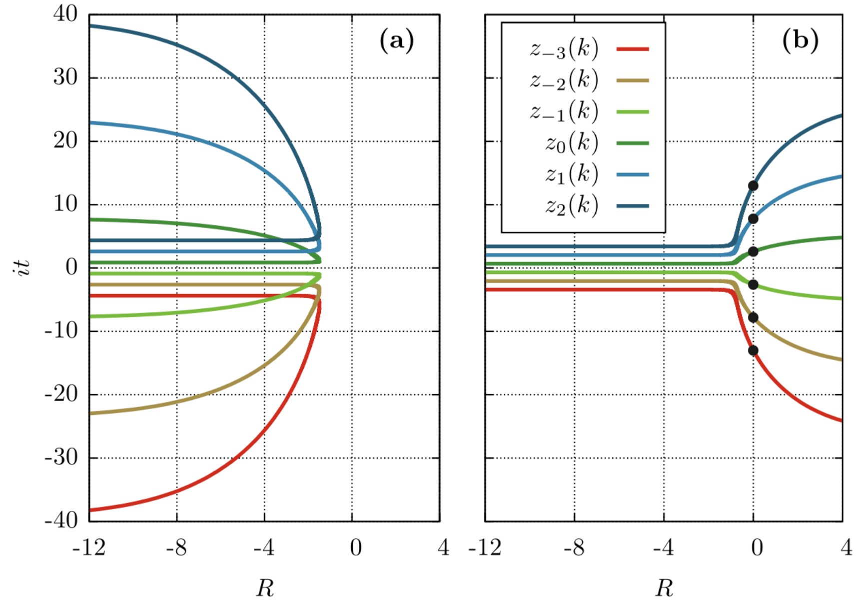

Figure 1.3: Lines of Fisher zeros $z_n(k) = R + it$ for the 1D TFIM (1.11). Left shows a quench within the same phase ($g_0 = 0.4 \to g_1 = 0.8$) whereas right shows a quench across the quantum critical point ($g_0 = 0.4 \to g_1 = 1.3$). Figure reproduced from Ref [15].

Fisher zero must lie on the positive imaginary axis for a DQPT to occur. From the above equation, we find that $z_n(k)$ can be purely imaginary if $\phi_k = \pm\pi/4$. From Eq. (1.16) we find

$$\phi_{k=0} = \begin{cases} 0 \ , \ \text{quench in same phase} \\ \pi/4 \ , \ \text{quench to or from quantum critical point} \\ \pi/2 \ , \ \text{quench across quantum critical point} \end{cases} \quad , \quad \phi_{k=\pi} = 0 \ . \tag{1.18}$$

From the above condition we find that for a quench across the quantum critical point we get the following limits,

$$\lim_{k\to 0} \mathrm{Re}[z_n(k)] = +\infty \quad , \quad \lim_{k\to\pi} \mathrm{Re}[z_n(k)] = -\infty \ . \tag{1.19}$$

Since θ_k is continuous, we see in this case that there must be some value of $k \in (0, \pi)$ for which $\mathrm{Re}[z_n(k)] = 0$ so that $z_n(k)$ lies on the imaginary axis.

Figure 1.3 shows the distribution of Fisher zeros as k is varied across the Brillouin zone. The left figure shows a quench within the ferromagnetic phase. Here none of the lines of Fisher zeros intersects the imaginary axis, so there is no DQPT. Contrast this with the figure on the right which represents a quench across the quantum critical point. The lines of Fisher zeros intersect the imaginary axis at equal intervals, giving DQPTs at times

$$t_n^* = t^* \left(n + \frac{1}{2} \right) \quad , \quad \text{where } t^* = \frac{\pi}{\epsilon_{k^*}(g_1)} \ . \tag{1.20}$$

k^* is the momentum for which $\phi_{k^*} = \pi/4$ (see Eq. (1.17)). t^* is purely a dynamical quantity which has no equilibrium counterpart. Thus we see theoretically how DQPTs can occur in the TFIM.

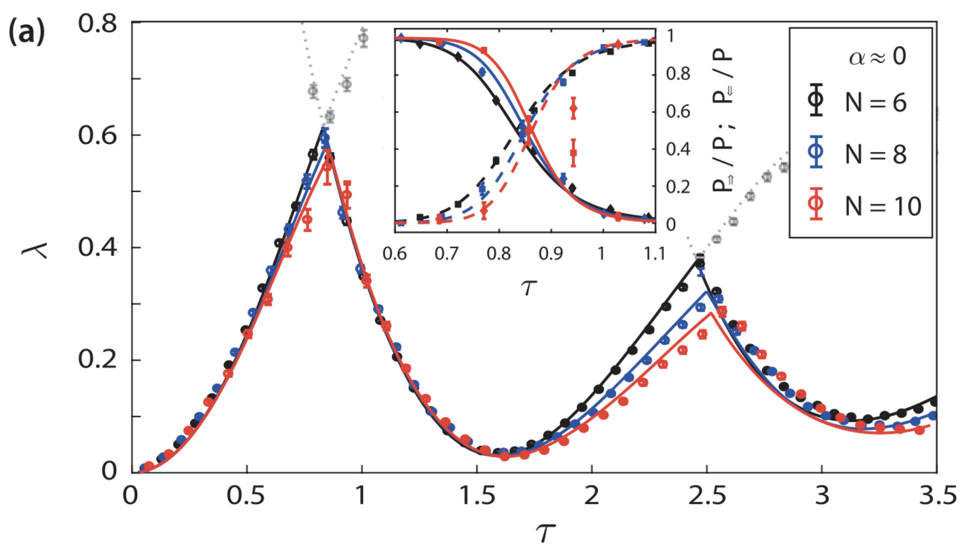

Figure 1.4: DQPT in a TFIM simulated using a string of trapped ions. Figure reproduced from Ref. [21].

The Transverse Field Ising Model is also the first model for which DQPTs were observed in experiment [21]. The authors of Ref. [21] simulated a spin chain using a string of trapped Ca^+ ions. Their observation is shown in Figure 1.4. In their experiment, the simulated TFIM did not have nearest-neighbor interactions as in Eq. (1.11), but had long-range interactions. The rate function λ is defined in a different way than (1.14). However the physics remains similar. The authors also show the effect of a DQPT on observables. In their case, the longitudinal magnetization flips at a DQPT.

1.4 DQPTs in two-band topological insulators

In this section we show how DQPTs can occur in two-band topological insulators with no inter-electron interactions. These are the simplest condensed matter systems that can show nontrivial topological properties. We follow the theoretical work first presented in Ref. [22].

We work with a quadratic fermionic Hamiltonian of the form

$$\hat{H} = \sum_{\mathbf{k}} \hat{\mathbf{c}}_{\mathbf{k}}^{\dagger} h_{\mathbf{k}} \hat{\mathbf{c}}_{\mathbf{k}} \quad , \quad \text{where } h_{\mathbf{k}} = d_{\mathbf{k}}^{x} \sigma^{x} + d_{\mathbf{k}}^{y} \sigma^{y} + d_{\mathbf{k}}^{z} \sigma^{z} \ . \tag{1.21}$$

Here $\hat{\mathbf{c}}_{\mathbf{k}} = (\hat{c}_{\mathbf{k},A}, \hat{c}_{\mathbf{k},B})$ represents the fermionic annihilation operators, where the indices A and B stand for two different sublattices. σ^x, σ^y and σ^z are the 2×2 Pauli matrices. This could represent the Hamiltonian of a topological superconductor rather than a topological insulator if we used the basis $\hat{\mathbf{c}}_{\mathbf{k}} = (\hat{c}_{\mathbf{k}}, \hat{c}_{-\mathbf{k}}^{\dagger})$ instead. The winding of the vector $\mathbf{d}_{\mathbf{k}}$ as $\mathbf{k}$ varies over the Brillouin zone determines the topological invariant [9]. This Hamiltonian has been realized experimentally in cold atomic and condensed matter systems [23–26].

The system is prepared in the ground state of the Hamiltonian with $\mathbf{d}_{\mathbf{k}}^{\mathbf{0}}$. At time $t = 0$, the Hamiltonian is quenched from $\mathbf{d}_{\mathbf{k}}^{\mathbf{0}}$ to $\mathbf{d}_{\mathbf{k}}^{\mathbf{1}}$. The Loschmidt amplitude (1.3) evaluates to

$$\mathcal{G}(t) = \prod_{\mathbf{k}} \left[\cos\left(\epsilon_{\mathbf{k}}^{1} t\right) + i \hat{\mathbf{d}}_{\mathbf{k}}^{0} \cdot \hat{\mathbf{d}}_{\mathbf{k}}^{1} \sin\left(\epsilon_{\mathbf{k}}^{1} t\right)\right] \ , \tag{1.22}$$

where the energy eigenvalues are $\epsilon_{\mathbf{k}}^{i} = \left|\mathbf{d}_{\mathbf{k}}^{\mathbf{i}}\right|$ and $\hat{\mathbf{d}}_{\mathbf{k}}^{i}$ is the unit vector pointing along the vector $\mathbf{d}_{\mathbf{k}}^{\mathbf{i}}$. This gives us the Fisher zeros,

$$z_n(\mathbf{k}) = \frac{i\pi}{\epsilon_{\mathbf{k}}^{1}} \left(n + \frac{1}{2}\right) - \frac{1}{\epsilon_{\mathbf{k}}^{1}} \tanh^{-1}(\hat{\mathbf{d}}_{\mathbf{k}}^{0} \cdot \hat{\mathbf{d}}_{\mathbf{k}}^{1}) \ . \tag{1.23}$$

The necessary condition for a DQPT to occur is that $\hat{\mathbf{d}}_{\mathbf{k}}^{0} \cdot \hat{\mathbf{d}}_{\mathbf{k}}^{1}$ vanishes for some $\mathbf{k}$ in the Brillouin zone. This happens when the vector $\mathbf{d}_{\mathbf{k}}$ for the post-quench Hamiltonian is orthogonal to that for the pre-quench Hamiltonian. This is a geometric condition that relates DQPTs to the topology of the initial and the final Hamiltonian.

In the next two subsections we demonstrate how DQPTs can occur in topological insulators in one and two spatial dimensions respectively.

1.4.1 DQPTs for topological insulators in 1D

Topological insulators in one spatial dimension belong to either the AIII or the BDI symmetry class of the tenfold symmetry classification [27]. They possess sublattice symmetry which constrains $\mathbf{d_k}$ to lie in a 2D plane.

Let us say that the plane on which $\mathbf{d_k}$ lies is the $x-y$ plane. As k sweeps across the Brillouin zone, we find the number of times $\mathbf{d_k}$ goes around the origin of the plane as [9]

$$\nu = \frac{1}{2\pi}\int dk\,(\hat{d}_k^x \partial_k \hat{d}_k^y - \hat{d}_k^y \partial_k \hat{d}_k^x)\ . \tag{1.24}$$

This calculation yields ν as an integer. The winding number, ν, is a topological invariant for the 1D topological insulator.

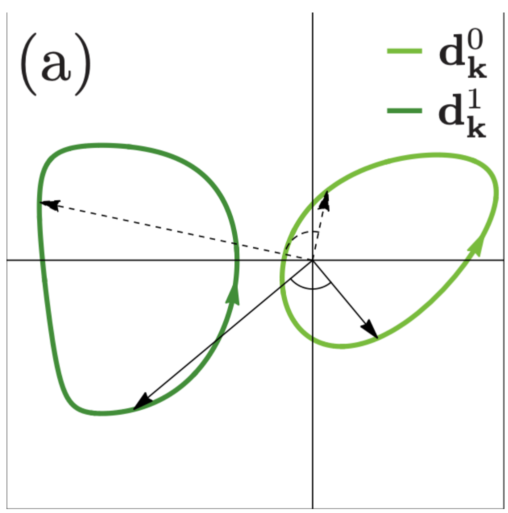

Figure 1.5: Illustration showing winding of vectors $\mathbf{d_k^0}$ and $\mathbf{d_k^1}$ in a 2D plane. If the winding numbers of the two Hamiltonians differ by $\Delta\nu$, then there are at least $2\Delta\nu$ values of k at which $\mathbf{d_k^0}$ and $\mathbf{d_k^1}$ become perpendicular. Figure reproduced from Ref. [22].

If the winding numbers of the pre-quench and the post-quench Hamiltonian differ by some positive integer value $\Delta\nu$, then the scalar product $\hat{\mathbf{d}}^0_{\mathbf{k}} \cdot \hat{\mathbf{d}}^1_{\mathbf{k}}$ covers the interval $[-1, 1]$ at least $2\Delta\nu$ times [22]. Figure 1.5 is an illustration of how this can occur. We can find at least $2\Delta\nu$ points in momentum space where $\mathbf{d^0_k}$ and $\mathbf{d^1_k}$ become orthogonal.

Thus we see that for topological insulators in 1D, a sufficient condition for DQPTs to occur is that the pre-quench and post-quench Hamiltonians have different winding numbers. However, this is not a necessary condition. It turns out that even when $\Delta\nu = 0$, there can be some k for which $\mathbf{d^0_k}$ and $\mathbf{d^1_k}$ are perpendicular. These DQPTs are not topologically protected. A physical example where such DQPTs are found is given in Ref. [28].

1.4.2 DQPTs for topological insulators in 2D

Just as for 1D, we can define a topological invariant for topological insulators in 2D. The Brillouin zone in 2D has the topology of a 2-torus (T^2). As $\mathbf{k}$ sweeps across the Brillouin zone, the number of times $\hat{\mathbf{d}}_{\mathbf{k}}$ covers the unit sphere is obtained as [29]

$$Q = \frac{1}{4\pi} \int_{\text{B.Z.}} dk_x \, dk_y \, \hat{\mathbf{d}}_{\mathbf{k}} \cdot (\partial_{k_x} \hat{\mathbf{d}}_{\mathbf{k}} \times \partial_{k_y} \hat{\mathbf{d}}_{\mathbf{k}}) \ . \tag{1.25}$$

Q is called the Chern number and is an integer topological invariant.

The result for topological insulators in 2D is as follows; for a quench from the Hamiltonian with $\mathbf{d^0_k}$ to the Hamiltonian with $\mathbf{d^1_k}$, if we have for the Chern numbers $|Q_0| \neq |Q_1|$, then the image of $\hat{\mathbf{d}}^0_{\mathbf{k}} \cdot \hat{\mathbf{d}}^1_{\mathbf{k}}$ is $[-1, 1]$ [22]. This can be shown with the following argument. If $\hat{\mathbf{d}}^0_{\mathbf{k}} \cdot \hat{\mathbf{d}}^1_{\mathbf{k}} > -1$ for all $\mathbf{k}$ then there exists the continuous mapping

$$\mathbf{f}_{\mathbf{k}}(\gamma) = (1-\gamma)\hat{\mathbf{d}}^0_{\mathbf{k}} + \gamma\hat{\mathbf{d}}^1_{\mathbf{k}} \quad , \quad \gamma \in [0, 1] \ , \tag{1.26}$$

so that

$$|\mathbf{f}_{\mathbf{k}}(\gamma)|^2 = 1 + 2\gamma(1-\gamma)(-1 + \hat{\mathbf{d}}^0_{\mathbf{k}} \cdot \hat{\mathbf{d}}^1_{\mathbf{k}}) > 0 \tag{1.27}$$

for all $\mathbf{k}$. This shows that if $\hat{\mathbf{d}}^0_{\mathbf{k}}$ and $\hat{\mathbf{d}}^1_{\mathbf{k}}$ are nowhere antiparallel, then $\hat{\mathbf{d}}^0_{\mathbf{k}}$ can be deformed continuously into $\hat{\mathbf{d}}^1_{\mathbf{k}}$ for all $\mathbf{k}$ in the Brillouin zone, resulting in the Chern numbers Q_0 and Q_1 being equal [30]. The converse of this is the result that if $Q_0 \neq Q_1$, then $\hat{\mathbf{d}}^0_{\mathbf{k}} \cdot \hat{\mathbf{d}}^1_{\mathbf{k}} = -1$ for some $\mathbf{k}$. Also, considering the additive inverse of $\hat{\mathbf{d}}^1_{\mathbf{k}}$, we get that if $Q_0 \neq -Q_1$, then $\hat{\mathbf{d}}^0_{\mathbf{k}} \cdot \hat{\mathbf{d}}^1_{\mathbf{k}} = 1$ for some (other) $\mathbf{k}$. Since both $\mathbf{d}^{\mathbf{0}}_{\mathbf{k}}$ and $\mathbf{d}^{\mathbf{1}}_{\mathbf{k}}$ are continuous and non-vanishing (in order for it to be a topological insulator [9]), we obtain the result that if $|Q_0| \neq |Q_1|$, then the image of $\hat{\mathbf{d}}^0_{\mathbf{k}} \cdot \hat{\mathbf{d}}^1_{\mathbf{k}}$ is $[-1, 1]$. Thus there exists some $\mathbf{k}$ for which $\hat{\mathbf{d}}^0_{k}$ and $\hat{\mathbf{d}}^1_{\mathbf{k}}$ are orthogonal, resulting in a DQPT.

Some 2D models where DQPTs can occur using the above formalism are the Haldane model [31], the half-BHZ model [32], and the chiral topological $p + ip$ superconductor [33]. DQPTs have been experimentally observed in a 2D Haldane model simulated using ultracold atoms with full-state tomography [34].

1.5 Some more results on DQPTs

There has been a lot of investigation into DQPTs in quantum systems over the past decade. In the previous two sections, we showed how DQPTs can occur in the transverse field Ising model in 1D, and in topological insulators in 1D and 2D. In this section, we list some more important results regarding DQPTs.

From the definition of the Loschmidt amplitude (1.3), one might think that DQPTs affect only the ground state of the pre-quench Hamiltonian. This is not the case in general. In Ref. [15], the authors showed that DQPTs affect not only the projection of the time-evolved state on the ground state of the pre-quench Hamiltonian, but their effect can also be seen in the projection of the time-evolved state on the higher energy eigenstates of the pre-quench Hamiltonian. DQPTs can also affect the behavior of observables. For example, in the Transverse Field Ising Model (TFIM) in 1D, a quench within the same phase results in an exponential decay of the longitudinal magnetization, whereas a quench across the quantum critical point causes an oscillatory decay in the longitudinal magnetization [15]. The periodicity of these oscillations coincides with that of the DQPTs. Ref. [35] further defines a dynamical order parameter for topological systems which is qualitatively different in periods of time separated by a DQPT.

In Ref. [36] the authors study the 1D TFIM with integrability-breaking perturbations. They study a TFIM with axial next-nearest-neighbor interactions, and also an Ising chain in a tilted magnetic field. They find that DQPTs do occur but are no longer periodic in time. Thus for the TFIM, DQPTs are stable against weak integrability-breaking perturbations. This is what is observed in experiment as well [21]. The 1D TFIM used in the experiment has long-range interactions, and is not integrable. The authors of Ref. [21] find that DQPTs do occur but are not periodic. The form of the long-range interaction and its strength could be tuned in the experiment. The DQPTs were found to be stable over a broad range of the interaction form and strength. Ref. [37] shows how the periodicity of DQPTs can be broken if we have multiple bands rather than changing the form of the coupling.

DQPTs have been found numerically in the 2D quantum Ising model with strong correlations in Ref. [38]. The model the authors consider is not integrable, so the Loschmidt echo cannot be analytically calculated. They calculate the Loschmidt echo numerically, using the method of Loschmidt cumulants discussed in Ref. [39].

The nonanalyticity at a DQPT can have different forms. If one looks for DQPTs using Fisher zeros (see Section 1.2), then depending on whether the Fisher zeros lie on lines or areas, there can be a discontinuity in the first time-derivative or the second time-derivative of the rate function [22].

DQPTs can also be looked at from a symmetry perspective. In the TFIM shown in Section 1.3, when there is a quench from the ferromagnetic to the paramagnetic phase, the broken symmetry in the order parameter is restored at a DQPT. This symmetry restoration at a DQPT holds not just for the ground state, but also extends to higher energy states [21]. In Ref. [40], it is shown that this restoration of symmetry works not just for the discrete Ising symmetry, but applies to continuous symmetries as well.

We saw in Section 1.4 that there exists a close relationship between DQPTs in topological systems and topological phase transitions. In general, quenches between phases with different topological numbers yields a DQPT. However, the converse does not hold [22].

In general, there exists a relationship between DQPTs and equilibrium phase transitions

(EPTs). But it has been shown that DQPTs and EPTs are not in one-to-one correspondence. The authors of Ref. [28] studied DQPTs in an XY chain with a transverse magnetic field. They found that a quench across critical lines of parameters in the XY model doesn't yield DQPTs. Thus DQPTs turn out to be an interesting critical phenomenon, distinct from equilibrium phase transitions. More nonequilibrium critical points can be found in Ising spin chains, that are not related to any underlying equilibrium quantum or thermal phase transition [41–43].

1.6 Outline of the thesis

In this thesis, we look at DQPTs and other nonanalyticities in the time evolution of superconductors.

In Chapter 2, we study how DQPTs that can occur in topological superconductors. This is different than previous research on this topic, such as Ref. [22], because we do not use the equilibrium superconducting gap function. Rather we calculate the superconducting mean field gap function self-consistently. We derive the required nonequilibrium mean field theory from first principles and show that the resulting gap function is very different than the equilibrium gap function. We find the form of singularities that can occur in the Loschmidt echo of topological superconductors, and see that no DQPT can occur in the topologically trivial s-wave superconductor. This work is published as Ref. [44].

In Chapter 3, we study the spectral form factor for superconductors. The spectral form factor is a quantity similar to the Loschmidt echo, that gives us information about the time evolution of the system. It can be considered a more general property of the Hamiltonian since unlike the Loschmidt echo, it does not depend on any quench or any particular choice of initial state. We use an appropriate mean field theory to calculate the spectral form factor for the superconductor. We show that unconventional superconductors can have singularities in their spectral form factor, whereas the trivial s-wave superconductor has a featureless spectral form factor. This is due to the difference in the structure of their superconducting gap functions, which comes from the different symmetries of the Cooper pair wavefunctions. We calculate the form of singularities in the spectral form factor for topological superconductors of different symmetry classes. We then discuss how the spectral form factor can be experimentally measured in such systems. This work is published as Ref. [45].

Chapter 4 looks at the mean field theory for the spectral form factor of superconductors in detail. The most general time-dependent mean field theory is derived in detail for a simple model which is that of a single-channel flat-band superconductor. This model is useful because its spectral form factor can also be calculated exactly, without the use of any mean field. We show that the exact result agrees with the result obtained using our mean field theory, thus providing a verification of our mean field formalism. In addition, this shows the existence of complex nontrivial saddle points in the path integral representation of the spectral form factors of superconductors. This is the first model for which such nontrivial saddle points have been explicitly calculated. This work is presented in Ref. [46].

Chapter 5 ends the thesis with a summary of our important results and a discussion of topics for further research.

Chapter 2

Dynamical Quantum Phase Transitions in Superconductors

2.1 Introduction

Superconducting Hamiltonians are considered to be highly non-trivial since they include electron-electron interaction terms [7, 47]. These Hamiltonians can be dealt with using a mean-field approximation [48]. The expectation value of the sum of electron-pair-annihilation operators can be approximated as a mean field. Using this mean field, the superconducting Hamiltonian reduces to a Hamiltonian with non-interacting electrons. This can then be diagonalized to solve for the mean field self-consistently. The mean-field approximation becomes exact in the thermodynamic limit. When the mean-field theory is used to calculate the ground state of the superconducting Hamiltonian, then this mean field is also known as the gap function. The energy gap between the ground state and the first excited state is twice the gap function.

Dynamics of superconducting Hamiltonians has been significantly studied in the literature [12, 22, 49–61]. Out of these, the authors in Refs. [12, 22, 57, 58] get results for dynamical quantum phase transitions (DQPT) in superconductors. However, these authors either take the mean field to be constant throughout the time-evolution, or evaluate it using an equilibrium formalism. Their mean-field formalism doesn't necessarily satisfy the time-dependent self-consistency equations, as we shall see in this chapter.

In this chapter, we derive the expression for the Loschmidt echo of a superconductor from first principles. We shall see how the Loschmidt echo can have nonanalyticities, leading to DQPTs. In particular, we will see how DQPTs can occur in topological superconductors, but not in the topologically trivial s-wave superconductor.

2.2 Path integral formulation of the Loschmidt echo

We start by calculating the Loschmidt echo for the Hamiltonian of an s-wave superconductor. This calculation can be easily generalized to that for other superconducting Hamiltonians.

The s-wave superconducting Hamiltonian takes the form [7]

$$\hat{H} = \sum_{\mathbf{p},\sigma} \xi_p \, \hat{a}^\dagger_{\mathbf{p}\sigma} \hat{a}_{\mathbf{p}\sigma} - \frac{g}{V} \sum_{\mathbf{p},\mathbf{q},\mathbf{k}} \hat{a}^\dagger_{\frac{\mathbf{k}}{2}+\mathbf{p}\uparrow} \hat{a}^\dagger_{\frac{\mathbf{k}}{2}-\mathbf{p}\downarrow} \hat{a}_{\frac{\mathbf{k}}{2}-\mathbf{q}\downarrow} \hat{a}_{\frac{\mathbf{k}}{2}+\mathbf{q}\uparrow} \, , \tag{2.1}$$

with

$$\xi_p = \frac{p^2}{2m} - \mu \, , \tag{2.2}$$

where $\hat{a}^\dagger_{\mathbf{p}\sigma}$ and $\hat{a}_{\mathbf{p}\sigma}$ are respectively the creation and annihilation operators for the fermions with momentum $\mathbf{p}$ and spin σ, g is the inter-electron interaction strength, μ is the chemical potential, and V is the volume of the system. In the applications to superconductivity, only the terms with $\mathbf{k} = 0$ play a substantial role [7]. From now on we will therefore remove summation over $\mathbf{k}$ and set $\mathbf{k} = 0$ with the result

$$\hat{H} = \sum_{\mathbf{p},\sigma} \xi_p \, \hat{a}^\dagger_{\mathbf{p}\sigma} \hat{a}_{\mathbf{p}\sigma} - \frac{g}{V} \sum_{\mathbf{p},\mathbf{q}} \hat{a}^\dagger_{\mathbf{p}\uparrow} \hat{a}^\dagger_{-\mathbf{p}\downarrow} \hat{a}_{-\mathbf{q}\downarrow} \hat{a}_{\mathbf{q}\uparrow} \, . \tag{2.3}$$

We are interested in calculating the Loschmidt echo, that is the quantity[1]

$$\mathcal{Z} = \langle \Psi_i | \, e^{-i\hat{H}t} \, | \Psi_i \rangle \, . \tag{2.4}$$

Here $|\Psi_i\rangle$ is some state which is not an eigenstate of $\hat{H}$. We would like to use mean field theory to do this calculation, as is common in the theory of superconductivity. It is argued that as long as the total number of fermions is large, mean field theory is a good approximation for the Hamiltonian (2.1) [7].

[1] The standard definition of the Loschmidt echo is $\left| \langle \Psi_i | \, e^{iH_2 t} e^{-iH_1 t} \, |\Psi\rangle \right|^2$. If Ψ is the eigenstate of H_1, up to taking its absolute value square it effectively reduces to Eq. (2.4).

Setting up mean field theory for purpose of calculating Loschmidt echo is subtle. Let us set up the appropriate formalism to do it.

We express the echo in terms of the coherent state functional integral [62]

$$\mathcal{Z} = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{i\int_0^t d\tau \left\{\sum_{\mathbf{p}\sigma}\left[\bar{\psi}_{\mathbf{p}\sigma}\left(i\frac{\partial}{\partial\tau}-\xi_p\right)\psi_{\mathbf{p}\sigma}\right]+\frac{g}{V}\sum_{\mathbf{p},\mathbf{q}}\bar{\psi}_{\mathbf{p}\uparrow}\bar{\psi}_{-\mathbf{p}\downarrow}\psi_{-\mathbf{q}\downarrow}\psi_{\mathbf{q}\uparrow}\right\}} . \tag{2.5}$$

In order to represent the required matrix element of the evolution operator, the fields $\psi_{\mathbf{p}\sigma}$ and $\bar{\psi}_{\mathbf{p}\sigma}$ must satisfy the appropriate boundary conditions at $\tau = 0$ and $\tau = t$ whose specific form will not be important here.

As common in the theory of superconductivity we introduce the Hubbard-Stratonovich field Δ [63], which results in

$$\mathcal{Z} = \int \mathcal{D}\Delta\mathcal{D}\bar{\Delta}\, e^{iW}, \tag{2.6}$$

where

$$\begin{aligned} e^{iW} &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{iS} , \\ S &= \int_0^t d\tau \left\{\sum_{\mathbf{p}\sigma}\left[\bar{\psi}_{\mathbf{p}\sigma}\left(i\frac{\partial}{\partial\tau}-\xi_p\right)\psi_{\mathbf{p}\sigma}\right] + \Delta\sum_{\mathbf{p}}\bar{\psi}_{\mathbf{p}\uparrow}\bar{\psi}_{-\mathbf{p}\downarrow} + \bar{\Delta}\sum_{\mathbf{p}}\psi_{-\mathbf{p}\downarrow}\psi_{\mathbf{p}\uparrow} - \frac{V}{g}\bar{\Delta}\Delta\right\} \end{aligned} \tag{2.7}$$

We calculate the integral over Δ and $\bar{\Delta}$ using the saddle point approximation. Varying W over $\bar{\Delta}(\tau)$ at some time τ, we find

$$\frac{1}{\mathcal{Z}}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\left(\sum_{\mathbf{p}}\psi_{-\mathbf{p}\downarrow}(\tau)\psi_{\mathbf{p}\uparrow}(\tau) - \frac{V}{g}\Delta(\tau)\right)e^{iS} = 0 . \tag{2.8}$$

Going back to the operator formalism and restoring the boundary conditions, this resulting functional integral can be reinterpreted as

$$\Delta(\tau) = \frac{g}{V\mathcal{Z}}\sum_{\mathbf{p}}\langle\Psi_i|\, e^{-i\hat{H}(t-\tau)}\hat{a}_{-\mathbf{p}\downarrow}\hat{a}_{\mathbf{p}\uparrow}e^{-i\hat{H}\tau}\,|\Psi_i\rangle\ . \tag{2.9}$$

Similarly, by varying W over Δ we find

$$\bar{\Delta}(\tau) = \frac{g}{V\mathcal{Z}} \sum_{\mathbf{p}} \langle \Psi_i | \, e^{-i\hat{H}(t-\tau)} \hat{a}^{\dagger}_{\mathbf{p}\uparrow} \hat{a}^{\dagger}_{-\mathbf{p}\downarrow} e^{-i\hat{H}\tau} \, |\Psi_i\rangle \ . \tag{2.10}$$

This is the version of the saddle point (or gap) equation appropriate for calculating the Loschmidt echo. We notice that Eq. (2.10) is not the complex conjugate of Eq. (2.9), unlike in the more common occurrences of the gap equation where Δ and $\bar{\Delta}$ are usually complex conjugates of each other.

The mean field Hamiltonian follows from Eq. (2.7) to be

$$\hat{H} = \sum_{\mathbf{p}\sigma} \xi_p \, \hat{a}^{\dagger}_{\mathbf{p}\sigma} \hat{a}_{\mathbf{p}\sigma} - \Delta \sum_{\mathbf{p}} \hat{a}^{\dagger}_{\mathbf{p}\uparrow} \hat{a}^{\dagger}_{-\mathbf{p}\downarrow} - \bar{\Delta} \sum_{\mathbf{p}} \hat{a}_{-\mathbf{p}\downarrow} \hat{a}_{\mathbf{p}\uparrow} + \frac{V}{g} \bar{\Delta}\Delta \ . \tag{2.11}$$

The initial state $|\Psi_i\rangle$ in our formalism is a coherent state and so does not have a fixed number of particles. To fix the particle number we can introduce the time-dependent chemical potential as a Lagrange multiplier. That generalizes the formalism to include

$$\mathcal{Z} = \int \mathcal{D}\Delta \mathcal{D}\bar{\Delta} \mathcal{D}\mu \, e^{iW - iN\int_0^t d\tau \, \mu(\tau)}, \tag{2.12}$$

where

$$e^{iW} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \, e^{iS[\mu]} \ , \tag{2.13}$$

$$S[\mu] = \int_0^t d\tau \left\{ \sum_{\mathbf{p}\sigma} \left[\bar{\psi}_{\mathbf{p}\sigma} \left(i\frac{\partial}{\partial\tau} - \frac{p^2}{2m} + \mu(\tau) \right) \psi_{\mathbf{p}\sigma} \right] + \Delta \sum_{\mathbf{p}} \bar{\psi}_{\mathbf{p}\uparrow} \bar{\psi}_{-\mathbf{p}\downarrow} + \bar{\Delta} \sum_{\mathbf{p}} \psi_{-\mathbf{p}\downarrow} \psi_{\mathbf{p}\uparrow} - \frac{V}{g} \bar{\Delta}\Delta \right\} .$$

We now apply the saddle point approximation not only over Δ and $\bar{\Delta}$, but also over μ. The saddle point over μ gives

$$N = \frac{1}{\mathcal{Z}} \sum_{\mathbf{p}\sigma} \langle \Psi_i | \, e^{-i\hat{H}(t-\tau)} \hat{a}^{\dagger}_{\mathbf{p}\sigma} \hat{a}_{\mathbf{p}\sigma} e^{-i\hat{H}\tau} \, |\Psi_i\rangle \ . \tag{2.14}$$

This equation should be solved together with the gap equation (2.9) and (2.10). It will fix the value of the chemical potential.

2.3 Anderson Pseudospins

It is customary when working with Hamiltonians such as (2.3) to define the effective spin operators, often called Anderson pseudospins [64],

$$\hat{s}^-_{\mathbf{p}} = \hat{a}_{-\mathbf{p}\downarrow}\hat{a}_{\mathbf{p}\uparrow} \quad , \quad \hat{s}^+_{\mathbf{p}} = \hat{a}^\dagger_{\mathbf{p}\uparrow}\hat{a}^\dagger_{-\mathbf{p}\downarrow} \quad , \quad \hat{s}^z_{\mathbf{p}} = \frac{1}{2}\left(\hat{a}^\dagger_{\mathbf{p}\uparrow}\hat{a}_{\mathbf{p}\uparrow} + \hat{a}^\dagger_{-\mathbf{p}\downarrow}\hat{a}_{-\mathbf{p}\downarrow} - 1\right) \ , \tag{2.15}$$

where

$$\hat{s}^-_{\mathbf{p}} = \hat{s}^x_{\mathbf{p}} - i\hat{s}^y_{\mathbf{p}} \quad , \quad \hat{s}^+_{\mathbf{p}} = \hat{s}^x_{\mathbf{p}} + i\hat{s}^y_{\mathbf{p}} \ . \tag{2.16}$$

These satisfy the usual SU(2) algebra of the angular momentum operators,

$$[\hat{s}^x_{\mathbf{p}}, \hat{s}^y_{\mathbf{p}}] = i\hat{s}^z_{\mathbf{p}} \quad , \quad [\hat{s}^y_{\mathbf{p}}, \hat{s}^z_{\mathbf{p}}] = i\hat{s}^x_{\mathbf{p}} \quad , \quad [\hat{s}^z_{\mathbf{p}}, \hat{s}^x_{\mathbf{p}}] = i\hat{s}^y_{\mathbf{p}} \ . \tag{2.17}$$

In terms of these, the Hamiltonian (2.3) can be rewritten as

$$\hat{H} = 2\sum_{\mathbf{p}} \xi_p \hat{s}^z_{\mathbf{p}} - \frac{g}{V}\sum_{\mathbf{p},\mathbf{q}} \hat{s}^+_{\mathbf{p}}\hat{s}^-_{\mathbf{q}} \ . \tag{2.18}$$

We use this Hamiltonian to derive equations of motion of the time-dependent Heisenberg spin operators, with the result

$$\dot{\hat{s}}^-_{\mathbf{p}} = -2i\xi_p\hat{s}^-_{\mathbf{p}} - 2i\hat{\Delta}\,\hat{s}^z_{\mathbf{p}} \ , \ \dot{\hat{s}}^+_{\mathbf{p}} = 2i\xi_p\hat{s}^+_{\mathbf{p}} + 2i\hat{\bar{\Delta}}\,\hat{s}^z_{\mathbf{p}} \ , \ \dot{\hat{s}}^z_{\mathbf{p}} = i\left(\hat{\Delta}\hat{s}^+_{\mathbf{p}} - \hat{\bar{\Delta}}\hat{s}^-_{\mathbf{p}}\right) \ , \tag{2.19}$$

where

$$\hat{\Delta} = \frac{g}{V}\sum_{\mathbf{p}} \hat{s}^-_{\mathbf{p}} \quad , \quad \hat{\bar{\Delta}} = \frac{g}{V}\sum_{\mathbf{p}} \hat{s}^+_{\mathbf{p}} \ . \tag{2.20}$$

Now the key is that operators $\hat{\Delta}$ and $\hat{\bar{\Delta}}$ are expressed as a sum over a large number of spins. Therefore, their quantum fluctuations can be neglected and they can be replaced by their expectation values with respect to the state of the system at the initial moment of time $|\Psi_i\rangle$,

$$\Delta = \frac{g}{V}\sum_{\mathbf{p}} \langle\Psi_i|\, \hat{s}^-_{\mathbf{p}}\, |\Psi_i\rangle \quad , \quad \bar{\Delta} = \frac{g}{V}\sum_{\mathbf{p}} \langle\Psi_i|\, \hat{s}^+_{\mathbf{p}}\, |\Psi_i\rangle \ . \tag{2.21}$$

Substituting this into the equations (2.19) makes the pseudospin equations of motion linear in operators. As a result, we can replace the remaining operators in these equations of motion by their expectation values, also over the initial state $|\Psi_i\rangle$, and obtain fully classical equations of motion

$$\dot{s}^-_{\mathbf{p}} = -2i\xi_p s^-_{\mathbf{p}} - 2i\Delta\, s^z_{\mathbf{p}} \ , \ \dot{s}^+_{\mathbf{p}} = 2i\xi_p s^+_{\mathbf{p}} + 2i\bar{\Delta}\, s^z_{\mathbf{p}} \ , \ \dot{s}^z_{\mathbf{p}} = i\left(\Delta s^+_{\mathbf{p}} - \bar{\Delta} s^-_{\mathbf{p}}\right) \ , \tag{2.22}$$

where

$$\Delta = \frac{g}{V}\sum_{\mathbf{p}} s^-_{\mathbf{p}} \quad , \quad \bar{\Delta} = \frac{g}{V}\sum_{\mathbf{p}} s^+_{\mathbf{p}} \ . \tag{2.23}$$

Equations (2.22) together with (2.23) represent nonlinear but exactly solvable (integrable) equations of motion in this problem. These can in principle be solved to find the classical pseudospins $s^-_{\mathbf{p}}$, $s^+_{\mathbf{p}}$, $s^z_{\mathbf{p}}$ evolving in time.

It is also instructive to rewrite these pseudospins in terms of the amplitudes appearing in the Bardeen-Cooper-Schrieffer (BCS) wavefunction [7]. We are interested in time evolution of the states which take the form

$$|\Psi(\tau)\rangle = \prod_{\mathbf{p}} \left(u_{\mathbf{p}}(\tau) + v_{\mathbf{p}}(\tau)\hat{s}^+_{\mathbf{p}}\right)|0\rangle \ , \tag{2.24}$$

where $|0\rangle$ is the fermion vacuum or the state where all the pseudospins point downward so that it is annihilated by all the operators $\hat{s}^-_{\mathbf{p}}$. For the initial state $|\Psi_i\rangle$, $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$ take the initial values $u_{i\mathbf{p}}$, $v_{i\mathbf{p}}$.

We can derive equations of motion for $u_{\mathbf{p}}(\tau)$, $v_{\mathbf{p}}(\tau)$ by applying the Hamiltonian (2.18) to the wave function (2.24), which gives

$$i\dot{u}_{\mathbf{p}} = -\xi_p u_{\mathbf{p}} - \bar{\Delta}\, v_{\mathbf{p}} \quad , \quad i\dot{v}_{\mathbf{p}} = \xi_p v_{\mathbf{p}} - \Delta\, u_{\mathbf{p}} \ , \tag{2.25}$$

where we apply the same quasiclassical approximation to produce Δ as given by Eq. (2.23). These amplitudes $u_{\mathbf{p}}$, $v_{\mathbf{p}}$ are related to the classical pseudospins as

$$s^+_{\mathbf{p}}(\tau) = \langle\Psi(\tau)|\,\hat{s}^+_{\mathbf{p}}\,|\Psi(\tau)\rangle = v^*_{\mathbf{p}}(\tau)u_{\mathbf{p}}(\tau)\ ,\ s^-_{\mathbf{p}}(\tau) = \langle\Psi(\tau)|\,\hat{s}^-_{\mathbf{p}}\,|\Psi(\tau)\rangle = u^*_{\mathbf{p}}(\tau)v_{\mathbf{p}}(\tau)\ , \tag{2.26}$$

$$s^z_{\mathbf{p}}(\tau) = \langle\Psi(\tau)|\,\hat{s}^z_{\mathbf{p}}\,|\Psi(\tau)\rangle = \frac{v^*_{\mathbf{p}}(\tau)v_{\mathbf{p}}(\tau) - u^*_{\mathbf{p}}(\tau)u_{\mathbf{p}}(\tau)}{2}\ . \tag{2.27}$$

We see that these classical pseudospins are constrained as

$$\sqrt{s^+_{\mathbf{p}}s^-_{\mathbf{p}} + \left(s^z_{\mathbf{p}}\right)^2} = \frac{1}{2}\ . \tag{2.28}$$

This allows us to rewrite the equation (2.23) as

$$\Delta(\tau) = \frac{g}{V}\sum_{\mathbf{p}} u^*_{\mathbf{p}}(\tau)v_{\mathbf{p}}(\tau) \quad , \quad \bar{\Delta}(\tau) = \frac{g}{V}\sum_{\mathbf{p}} v^*_{\mathbf{p}}(\tau)u_{\mathbf{p}}(\tau)\ . \tag{2.29}$$

Of course, under the definitions (2.26), (2.27), equations of motion (2.22) together with (2.23) map into (2.25) together with (2.29). Therefore, these two sets of equations represent two equivalent ways of representing the equations of motion in this problem.

This is the usual way in which dynamics of superconductors is calculated in the literature [49–56, 58–60]. But following the discussion in Section 2.2, we need to calculate Δ and $\bar{\Delta}$ using a different approach when trying to evaluate the Loschmidt echo. From Eqs. (2.9) and (2.10), we get

$$\Delta(\tau) = \frac{g}{V}\sum_{\mathbf{p}} \frac{\langle\Psi_i|\,e^{-i\hat{H}(t-\tau)}\hat{s}^-_{\mathbf{p}}e^{-i\hat{H}\tau}\,|\Psi_i\rangle}{\langle\Psi_i|\,e^{-i\hat{H}t}\,|\Psi_i\rangle}\ ,\ \bar{\Delta}(\tau) = \frac{g}{V}\sum_{\mathbf{p}} \frac{\langle\Psi_i|\,e^{-i\hat{H}(t-\tau)}\hat{s}^+_{\mathbf{p}}e^{-i\hat{H}t}\,|\Psi_i\rangle}{\langle\Psi_i|\,e^{-i\hat{H}t}\,|\Psi_i\rangle}\ . \tag{2.30}$$

These are similar to the generalized expectation values introduced in Ref. [65]. To work with these equations we define

$$|\Psi(\tau)\rangle = e^{-i\hat{H}\tau}\,|\Psi_i\rangle\ ,\ \left|\tilde{\Psi}(\tau)\right\rangle = e^{-i\hat{H}(\tau-t)}\,|\Psi_i\rangle\ . \tag{2.31}$$

These result in two sets of amplitudes, $u_{\mathbf{p}}$, $v_{\mathbf{p}}$ as well as $\tilde{u}_{\mathbf{p}}$ and $\tilde{v}_{\mathbf{p}}$. The latter are defined according to Eq. (2.24) but with Ψ replaced with $\tilde{\Psi}$, $u_{\mathbf{p}}$ with $\tilde{u}_{\mathbf{p}}$, and $v_{\mathbf{p}}$ with $\tilde{v}_{\mathbf{p}}$.

$\tilde{u}_{\mathbf{p}}$ and $\tilde{v}_{\mathbf{p}}$ satisfy the same equations of motion as their $u_{\mathbf{p}}$, $v_{\mathbf{p}}$ counterparts, Eq. (2.25). However, whereas we have the initial conditions

$$u_{\mathbf{p}}(0) = u_{i\mathbf{p}} \ , \ v_{\mathbf{p}}(0) = v_{i\mathbf{p}} \ , \tag{2.32}$$

we have the boundary conditions for the new amplitudes,

$$\tilde{u}_{\mathbf{p}}(t) = u_{i\mathbf{p}} \ , \ \tilde{v}_{\mathbf{p}}(t) = v_{i\mathbf{p}} \ . \tag{2.33}$$

At the same time, we can evaluate the expectation values in Eq. (2.30) to produce

$$\Delta(\tau) = \frac{g}{V}\sum_{\mathbf{p}} \frac{\tilde{u}^*_{\mathbf{p}}(\tau) v_{\mathbf{p}}(\tau)}{2S_{\mathbf{p}}} \quad , \quad \bar{\Delta}(\tau) = \frac{g}{V}\sum_{\mathbf{p}} \frac{\tilde{v}^*_{\mathbf{p}}(\tau) u_{\mathbf{p}}(\tau)}{2S_{\mathbf{p}}} \ . \tag{2.34}$$

Here we defined

$$2S_{\mathbf{p}} = \tilde{u}^*_{\mathbf{p}}(\tau) u_{\mathbf{p}}(\tau) + \tilde{v}^*_{\mathbf{p}}(\tau) v_{\mathbf{p}}(\tau) \ . \tag{2.35}$$

While the amplitudes on the right hand side of this equation are all dependent on τ, one can check with the equations of motion (2.25) that $S_{\mathbf{p}}$ are independent of τ, while of course they depend on t. Eq. (2.34) replaces the earlier equation (2.29) which is not adequate for the purpose of calculating the Loschmidt echo.

Therefore, we arrive at what appears to be a rather difficult problem. We need to solve the equations of motion (2.25) for $u_{\mathbf{p}}$, $v_{\mathbf{p}}$ with the initial conditions (2.32) as well as the same equations of motion for their counterparts $\tilde{u}_{\mathbf{p}}$, $\tilde{v}_{\mathbf{p}}$ but with the initial conditions (2.33), all the while keeping Δ to be given by the Eq. (2.34).

We can simplify the problem somewhat if we define a new set of variables by generalizing

Eqs. (2.26), (2.27) as

$$s^+_{\mathbf{p}}(\tau) = \frac{\tilde{v}^*_{\mathbf{p}}(\tau)u_{\mathbf{p}}(\tau)}{2S_{\mathbf{p}}} \ , \ s^-_{\mathbf{p}}(\tau) = \frac{\tilde{u}^*_{\mathbf{p}}(\tau)v_{\mathbf{p}}(\tau)}{2S_{\mathbf{p}}} \ , \ s^z_{\mathbf{p}}(\tau) = \frac{\tilde{v}^*_{\mathbf{p}}(\tau)v_{\mathbf{p}}(\tau) - \tilde{u}^*_{\mathbf{p}}(\tau)u_{\mathbf{p}}(\tau)}{4S_{\mathbf{p}}} \ . \tag{2.36}$$

The normalization condition (2.28) still holds for the new variables. In other words, these are new pseudospins – vectors of fixed length. Remarkably, the new pseudospins satisfy the same equations of motion (2.22) as the old ones and, moreover, the gap equations also retain their form (2.23). So this approach has the advantage of keeping the familiar equations of motion. The difficulty is now that the boundary conditions (2.32) and (2.33) must now be reworked in terms of these pseudospins. One can check that this gives

$$\frac{1+2s^z_{\mathbf{p}}(0)}{2s^+_{\mathbf{p}}(0)} = \frac{v_{i\mathbf{p}}}{u_{i\mathbf{p}}} \quad , \quad \frac{1+2s^z_{\mathbf{p}}(t)}{2s^-_{\mathbf{p}}(t)} = \frac{v^*_{i\mathbf{p}}}{u^*_{i\mathbf{p}}}. \tag{2.37}$$

Therefore, the goal is now to solve the equations of motion (2.22) with (2.23), while imposing the boundary conditions (2.37). One should also note that for these new pseudospins $s^+_{\mathbf{p}}$ is not equal to the complex conjugate of $s^-_{\mathbf{p}}$, and $s^z_{\mathbf{p}}$ is not necessarily real.

2.4 Evaluating the Loschmidt echo for a superconducting Hamiltonian

As shown in the previous section, in order to evaluate the Loschmidt echo for an s-wave superconductor, we need to solve the equations of motion (2.22) and (2.23), with the boundary condition (2.37). This is a hard problem since we have the boundary conditions at two different values of the time τ, i.e., at 0 and t. This is unlike the calculation of the dynamics of superconductors in literature, where the initial conditions directly specify the values of the pseudospins $\mathbf{s}_{\mathbf{p}}$ at $\tau = 0$.

One approach to solve this problem would be an iterative procedure. We would start with an initial guess for the pseudospins $\mathbf{s}_{\mathbf{p}}(\tau = 0)$, satisfying the first equation in (2.37). The pseudospins are evolved according to Eqs. (2.22) and (2.23). If at the end of the time-evolution, the boundary condition at $\tau = t$ from Eq. (2.37) is satisfied, then our initial guess for $\mathbf{s}_{\mathbf{p}}(\tau = 0)$ was correct and the Loschmidt echo can be evaluated. However, in general, the boundary condition at $\tau = t$ will not be satisfied. In that case, the initial guess should be modified so that the left-hand-side of the boundary condition at $\tau = t$ in Eq. (2.37) approaches the right-hand-side. This is usually done using methods like gradient descent. But this is not feasible for this problem where there are a large number of pseudospins. We shall also see later in this chapter that such a procedure does not reveal DQPTs in the time evolution.

However, it is worth exploring how these equations of motion (2.22) and (2.23) work. We look at a simple case of a superconducting Hamiltonian where it is relatively easy to calculate the Loschmidt echo.

2.4.1 Flat-band single channel model

We study a simple case here involving a flat-band superconducting Hamiltonian. For a flat band, the bare fermion dispersion relation is independent of the momentum of the fermion. We have $\xi_p = \xi$ for all $\mathbf{p}$. To avoid worrying about the underlying lattice which can produce this dispersion relation, we do away with the momentum label $\mathbf{p}$. Instead, we index the distinct modes of fermions (and pseudospins) with index n. We consider N species of spin-$\frac{1}{2}$ fermions with the Hamiltonian

$$\hat{H} = \xi \sum_{n=1}^{N} \left(\hat{a}_{n\uparrow}^{\dagger} \hat{a}_{n\uparrow} + \hat{a}_{n\downarrow}^{\dagger} \hat{a}_{n\downarrow} \right) - \frac{g}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} \hat{a}_{n\uparrow}^{\dagger} \hat{a}_{n\downarrow}^{\dagger} \hat{a}_{m\downarrow} \hat{a}_{m\uparrow} \ . \tag{2.38}$$

The corresponding pseudospin evolution equations (2.22) become

$$\dot{s}_n^- = -2i\xi s_n^- - 2i\Delta s_n^z \ , \ \dot{s}_n^+ = 2i\xi s_n^+ + 2i\bar{\Delta} s_n^z \ , \ \dot{s}_n^z = i\Delta s_n^+ - i\bar{\Delta} s_n^- \ , \tag{2.39}$$

and the self-consistency condition Eq. (2.23) becomes

$$\Delta = \frac{g}{N} \sum_n s_n^- \quad , \quad \bar{\Delta} = \frac{g}{N} \sum_n s_n^+ \ . \tag{2.40}$$

Let us introduce the notation,

$$s^+ = \frac{1}{N}\sum_n s_n^+ = \frac{\bar{\Delta}}{g} \ , \ s^- = \frac{1}{N}\sum_n s_n^- = \frac{\Delta}{g} \ , \ s^z = \frac{1}{N}\sum_n s_n^z \ . \tag{2.41}$$

These satisfy

$$\dot{s}^- = -2i\xi s^- - 2igs^-s^z \ , \ \dot{s}^+ = 2i\xi s^+ + 2igs^+s^z \ , \ \dot{s}^z = 0 \ . \tag{2.42}$$

s^z is constant in time τ, so we get

$$\dot{s}^- = -2i(\xi + gs^z)s^- \quad , \quad \dot{s}^+ = 2i(\xi + gs^z)s^+ \ . \tag{2.43}$$

This gives us the time-dependence of s^- and s^+ as

$$s^-(\tau) = s_0^- e^{-2i(\xi+gs^z)\tau} \quad , \quad s^+(\tau) = s_0^+ e^{2i(\xi+gs^z)\tau} \ . \tag{2.44}$$

At this stage s^z is an arbitrary constant. Using these, the equations of motion for individual spins (2.39) become

$$\begin{aligned} \dot{s}_n^- &= -2i\xi s_n^- - 2igs_0^- e^{-2i(\xi+gs^z)\tau} s_n^z \ , \\ \dot{s}_n^+ &= 2i\xi s_n^+ + 2igs_0^+ e^{2i(\xi+gs^z)\tau} s_n^z \ , \\ \dot{s}_n^z &= igs_0^- e^{-2i(\xi+gs^z)\tau} s_n^+ - igs_0^+ e^{2i(\xi+gs^z)\tau} s_n^- \ . \end{aligned} \tag{2.45}$$

Let's denote

$$s_n^- = c_n^- e^{-2i(\xi+gs^z)\tau} \quad , \quad s_n^+ = c_n^+ e^{2i(\xi+gs^z)\tau} \ , \tag{2.46}$$

so that the equations of motion for s_n^- and s_n^+ simplify to

$$\begin{aligned} \dot{c}_n^- &= -2igs_0^- s_n^z + 2igs^z c_n^- \ , \\ \dot{c}_n^+ &= 2igs_0^+ s_n^z - 2igs^z c_n^+ \ , \\ \dot{s}_n^z &= igs_0^- c_n^+ - igs_0^+ c_n^- \ . \end{aligned} \tag{2.47}$$

These transformations eliminate ξ from the time-evolution equations. We get a system of linear differential equations in (c_n^-, c_n^+, s_n^z) for each n. The eigenfrequencies of this system are 0 and $\pm\omega$, where

$$\omega = 2gs \quad , \quad s = \sqrt{(s^z)^2 + s_0^+ s_0^-} \ . \tag{2.48}$$

The pseudospin is not normalized here, in the sense that s doesn't equal $\frac{1}{2}$ in general. The general solution of Eq. (2.47) is

$$\begin{pmatrix} c_n^-(\tau) \\ c_n^+(\tau) \\ s_n^z(\tau) \end{pmatrix} = A_n \begin{pmatrix} s_0^- \\ s_0^+ \\ s^z \end{pmatrix} + B_n \begin{pmatrix} \frac{s-s^z}{s_0^+} \\ -\frac{s_0^+}{s-s^z} \\ 1 \end{pmatrix} e^{-2igs\tau} + C_n \begin{pmatrix} -\frac{s+s^z}{s_0^+} \\ \frac{s_0^+}{s+s^z} \\ 1 \end{pmatrix} e^{2igs\tau} \ . \tag{2.49}$$

Rewriting this, we get the simplified form,

$$\begin{aligned} s_n^-(\tau) &= s_0^- e^{-2i(\xi+gs^z)\tau} \left(A_n + \frac{B_n}{s+s^z} e^{-2igs\tau} - \frac{C_n}{s-s^z} e^{2igs\tau} \right) \ , \\ s_n^+(\tau) &= s_0^+ e^{2i(\xi+gs^z)\tau} \left(A_n - \frac{B_n}{s-s^z} e^{-2igs\tau} + \frac{C_n}{s+s^z} e^{2igs\tau} \right) \ , \\ s_n^z(\tau) &= A_n s^z + B_n e^{-2igs\tau} + C_n e^{2igs\tau} \ . \end{aligned} \tag{2.50}$$

The conditions (2.41) enforce on the coefficients,

$$\frac{1}{N} \sum_n A_n = 1 \quad , \quad \frac{1}{N} \sum_n B_n = 0 \quad , \quad \frac{1}{N} \sum_n C_n = 0 \ . \tag{2.51}$$

We can solve for the coefficients A_n, B_n, and C_n, from Eq. (2.50) along with the boundary condition (2.37). Let us write the initial condition as $\alpha_n = v_{in}/u_{in}$, where u_{in} and v_{in} are the initial amplitudes defined in Eq. (2.32). Also let's write $\phi = 2(\xi + gs^z)t$ and $\theta = 2gst$. Using these we obtain the solution,

$$A_n = \frac{1}{2s}\frac{e^{i\theta}((s+s^z)\alpha_n^* + s_0^+ e^{i\phi})((s-s^z)+s_0^+\alpha_n) + ((s-s^z)\alpha_n^* - s_0^+ e^{i\phi})((s+s^z)-s_0^+\alpha_n)}{e^{i\theta}((s+s^z)\alpha_n^* + s_0^+ e^{i\phi})((s-s^z)+s_0^+\alpha_n) - ((s-s^z)\alpha_n^* - s_0^+ e^{i\phi})((s+s^z)-s_0^+\alpha_n)},$$
$$B_n = \frac{-e^{i\theta}}{2s}\frac{(s-s^z)((s+s^z)\alpha_n^* + s_0^+ e^{i\phi})((s+s^z)-s_0^+\alpha_n)}{e^{i\theta}((s+s^z)\alpha_n^* + s_0^+ e^{i\phi})((s-s^z)+s_0^+\alpha_n) - ((s-s^z)\alpha_n^* - s_0^+ e^{i\phi})((s+s^z)-s_0^+\alpha_n)},$$
$$C_n = \frac{1}{2s}\frac{(s+s^z)((s-s^z)\alpha_n^* - s_0^+ e^{i\phi})((s-s^z)+s_0^+\alpha_n)}{e^{i\theta}((s+s^z)\alpha_n^* + s_0^+ e^{i\phi})((s-s^z)+s_0^+\alpha_n) - ((s-s^z)\alpha_n^* - s_0^+ e^{i\phi})((s+s^z)-s_0^+\alpha_n)}. \tag{2.52}$$

Substituting these expressions in the condition (2.51), we can solve for s_0^+, s_0^- and s^z in terms of the initial conditions α_n.

In a similar way, we can calculate the time-dependence of the amplitudes u_n and v_n. Evaluating the time-evolution equation (2.25), we get

$$\begin{aligned} u_n(\tau) &= e^{i(\epsilon+gs^z)\tau}\left[\frac{(s-s^z)u_{in}+s_0^+v_{in}}{2s}e^{igs\tau} + \frac{(s+s^z)u_{in}-s_0^+v_{in}}{2s}e^{-igs\tau}\right], \\ v_n(\tau) &= e^{-i(\epsilon+gs^z)\tau}\left[\frac{s_0^-u_{in}+(s+s^z)v_{in}}{2s}e^{igs\tau} - \frac{s_0^-u_{in}-(s-s^z)v_{in}}{2s}e^{-igs\tau}\right]. \end{aligned} \tag{2.53}$$

In terms of these amplitudes, the Loschmidt echo (2.4) can be written as

$$\mathcal{Z}(t) = e^{-iN\frac{\bar{\Delta}\Delta}{g}t}\prod_n \left(u_{in}^* u_n(t) + v_{in}^* v_n(t)\right). \tag{2.54}$$

Therefore the Loschmidt echo for the single-channel flat-band superconducting Hamiltonian evaluates to

$$\begin{aligned} \mathcal{Z}(t) = e^{-iNgs_0^+s_0^-t}\prod_n \Bigg[& u_{in}^* e^{i(\epsilon+gs^z)t}\left[\frac{(s-s^z)u_{in}+s_0^+v_{in}}{2s}e^{igst} + \frac{(s+s^z)u_{in}-s_0^+v_{in}}{2s}e^{-igst}\right] + \\ & v_{in}^* e^{-i(\epsilon+gs^z)t}\left[\frac{s_0^-u_{in}+(s+s^z)v_{in}}{2s}e^{igst} - \frac{s_0^-u_{in}-(s-s^z)v_{in}}{2s}e^{-igst}\right]\Bigg]. \end{aligned} \tag{2.55}$$

We evaluate the Loschmidt echo by substituting for s_0^+, s_0^-, s^z, and $s = \sqrt{(s^z)^2 + s_0^+s_0^-}$, from the self-consistency condition (2.51). Thus we get the explicit expression of the Loschmidt echo for the superconducting Hamiltonian in (2.38).

2.5 Dynamical Quantum Phase Transitions in Quenched Topological Superconductors

This section is based on the results presented in our paper [44]. In the previous section, we saw how to evaluate the Loschmidt echo of a superconductor. We were able to find the analytic expression for the Loschmidt echo of a flat-band single-channel superconductor in Eq. (2.55). This is not the case for a general superconductor. Solving the equations of motion (2.22) with self-consistency conditions (2.30) and boundary conditions (2.37) remains an open problem in general. The reason for this is the complicated form of the mean field equation (2.30). In order to calculate $\Delta(\tau)$ and $\bar{\Delta}(\tau)$ at some time τ between 0 and t, we need to know the mean-field Hamiltonian $\hat{H}$ at all times τ' between 0 and t. This implies from Eq. (2.11) that to calculate $\Delta(\tau)$ and $\bar{\Delta}(\tau)$, we need to know the values of $\Delta(\tau')$ and $\bar{\Delta}(\tau')$ for all times τ' between 0 and t.

It turns out that even though we cannot explicitly calculate the expression for the Loschmidt echo, it is possible in certain cases to look for singularities in the Loschmidt echo. In the seminal paper on dynamical quantum phase transitions (DQPTs) in quantum systems [15], the authors define a DQPT to be a singularity in $\ln \mathcal{Z}(t)$. A singularity occurs in $\ln \mathcal{Z}(t)$ at a critical time $t = t_n$ where $\mathcal{Z}(t_n) = 0$. The mean-fields given by Eqs. (2.9) and (2.10) diverge since the denominator on the right of these equations vanishes when $\mathcal{Z}(t) = 0$. In this section we shall see that it is possible to calculate the divergent term in $\ln \mathcal{Z}(t)$ near the critical time $t = t_n$, without explicitly calculating $\mathcal{Z}(t)$.

In this section, we first look for DQPTs in the 2D chiral p-wave superconductor. This turns out to be the simplest superconducting system which shows DQPTs. A 2D chiral p-wave superconducting Hamiltonian can be realized using a 2D crystal of ions in a Penning trap [60]. Such a Hamiltonian involves spinless or spin-polarized fermions that cannot interact in the s-wave channel [33, 53, 66]. Their strongest interactions are in the p-wave channel. In 2D space, their Hamiltonian takes the following form

$$\hat{H} = \sum_{\mathbf{p}} \xi_p \, \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} - \frac{2g}{V} \sum_{\mathbf{p},\mathbf{q},\mathbf{k}} \mathbf{p} \cdot \mathbf{q} \; \hat{a}^\dagger_{\frac{\mathbf{k}}{2}+\mathbf{p}} \hat{a}^\dagger_{\frac{\mathbf{k}}{2}-\mathbf{p}} \hat{a}_{\frac{\mathbf{k}}{2}-\mathbf{q}} \hat{a}_{\frac{\mathbf{k}}{2}+\mathbf{q}} \, . \tag{2.56}$$

Here

$$\xi_p = \frac{p^2}{2m} - \mu, \tag{2.57}$$

$\hat{a}^\dagger_{\mathbf{p}}$ and $\hat{a}_{\mathbf{p}}$ are the creation and annihilation operators for the spinless fermions, g is the coupling constant and V is the volume of the system. When describing superconductivity, only the terms with $\mathbf{k} = 0$ play a substantial role. So we drop the summation over $\mathbf{k}$ and set $\mathbf{k} = 0$ to get

$$\hat{H} = \sum_{\mathbf{p}} \xi_p \, \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} - \frac{2g}{V} \sum_{\mathbf{p},\mathbf{q}} \mathbf{p} \cdot \mathbf{q} \; \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{-\mathbf{p}} \hat{a}_{-\mathbf{q}} \hat{a}_{\mathbf{q}} \, . \tag{2.58}$$

The most interesting superconducting phase described by this Hamiltonian is the chiral phase [33] where

$$\langle \hat{a}_{-\mathbf{p}} \hat{a}_{\mathbf{p}} \rangle \sim e^{i\phi_{\mathbf{p}}} \, , \tag{2.59}$$

where $\phi_{\mathbf{p}}$ is the polar angle of the vector $\mathbf{p}$ in 2D space. It is also often referred to as $p_x + ip_y$ phase of the superconductor. In this phase, it is advantageous to split the interaction according to

$$\mathbf{p} \cdot \mathbf{q} = \frac{1}{2} \left[(p_x + ip_y)(q_x - iq_y) + (p_x - ip_y)(q_x + iq_y) \right] \, . \tag{2.60}$$

The right-most term in (2.60), when substituted into Eq. (2.58), produces zero in the $p_x + ip_y$ phase and therefore can be removed altogether, with the result

$$\hat{H} = \sum_{\mathbf{p}} \xi_p \, \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} - \frac{g}{V} \sum_{\mathbf{p},\mathbf{q}} p \, q \, e^{i(\phi_{\mathbf{p}} - \phi_{\mathbf{q}})} \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{-\mathbf{p}} \hat{a}_{-\mathbf{q}} \hat{a}_{\mathbf{q}} \, , \tag{2.61}$$

Importantly, this nullification happens in any state, ground state or excited state, where Eq. (2.59) holds. Here we restrict ourselves only to such states. In particular, we are going to consider time evolution of some initial state. All such initial states will be chosen to satisfy Eq. (2.59). Then we can fully rely on Eq. (2.61) as the Hamiltonian describing the time evolution.

The Hamiltonian (2.61) can be rewritten in a simpler form by absorbing the phases $\phi_{\mathbf{p}}$ and $\phi_{\mathbf{q}}$ in the creation and annihilation operators

$$e^{i\frac{\phi_{\mathbf{p}}}{2}}\hat{a}^{\dagger}_{\mathbf{p}} \to \hat{a}^{\dagger}_{\mathbf{p}} \quad , \qquad e^{-i\frac{\phi_{\mathbf{p}}}{2}}\hat{a}_{\mathbf{p}} \to \hat{a}_{\mathbf{p}} \; , \tag{2.62}$$

with the result

$$\hat{H} = \sum_{\mathbf{p}} \xi_p \, \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{p}} - \frac{g}{V}\sum_{\mathbf{p},\mathbf{q}} p\, q\, \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}^{\dagger}_{-\mathbf{p}}\hat{a}_{-\mathbf{q}}\hat{a}_{\mathbf{q}}. \tag{2.63}$$

Using mean-field theory, this p-wave Hamiltonian becomes

$$\hat{H} = \sum_{\mathbf{p}} \xi_p \, \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}_{\mathbf{p}} - \Delta \sum_{\mathbf{p}} p\, \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}^{\dagger}_{-\mathbf{p}} - \bar{\Delta}\sum_{\mathbf{p}} p\, \hat{a}_{-\mathbf{p}}\hat{a}_{\mathbf{p}} \; . \tag{2.64}$$

As shown in the previous sections, the mean fields Δ and $\bar{\Delta}$ are in general time-dependent, and are evaluated using the self-consistency equations. We evaluate these mean fields later in this section. We shall see that for the chiral p-wave superconductor, Δ and $\bar{\Delta}$ are continuous functions of time and do not diverge at a DQPT. For now we treat Δ and $\bar{\Delta}$ as some given time-dependent functions, and show how one can obtain singularities in the Loschmidt echo.

We start by writing a BCS (Bardeen-Cooper-Schrieffer) eigenstate of the Hamiltonian (2.64) as

$$|\Psi\rangle = \prod_{\mathbf{p}} \left(u_{\mathbf{p}} + v_{\mathbf{p}}\, \hat{a}^{\dagger}_{\mathbf{p}}\hat{a}^{\dagger}_{-\mathbf{p}}\right)|0\rangle \;\; , \tag{2.65}$$

where $|0\rangle$ is a vacuum. Normalization of this wave function requires that

$$|u_{\mathbf{p}}|^2 + |v_{\mathbf{p}}|^2 = 1 \; . \tag{2.66}$$

We consider the standard quench problem. We initialize the system in the ground state $|\Psi_i\rangle$ of the form (2.65) parametrized by $u_{i\mathbf{p}}$ and $v_{i\mathbf{p}}$. The chemical potential μ of the Hamiltonian is suddenly changed to a new value so that $|\Psi_i\rangle$ is no longer its eigenstate. The state now evolves forward with the new Hamiltonian. Knowing the time-dependent state $|\Psi(\tau)\rangle$, we can calculate the Loschmidt echo according to Eq. (2.4), or with $\mathcal{Z}(t) = \left\langle \Psi_i \middle| \Psi(t) \right\rangle$. We would like to see if there are values of "Loschmidt time" t at which $\mathcal{Z}(t)$ is singular.

Let the initial values of the amplitudes be $u_{\mathbf{p}}(0) = u_{i\mathbf{p}}$ and $v_{\mathbf{p}}(0) = v_{i\mathbf{p}}$. We get the time-evolution equations, analogous to Eq. (2.25), as

$$i\dot{u}_{\mathbf{p}} = -\xi_p u_{\mathbf{p}} - \bar{\Delta}\, p\, v_{\mathbf{p}} \quad , \quad i\dot{v}_{\mathbf{p}} = \xi_p v_{\mathbf{p}} - \Delta\, p\, u_{\mathbf{p}} \ . \tag{2.67}$$

To calculate the Loschmidt echo, we evolve $|\Psi\rangle$ to the Loschmidt time t and project it back onto itself, with the result

$$\mathcal{Z} = \prod_{\mathbf{p}} 2S_{\mathbf{p}} \quad , \quad \ln \mathcal{Z} = V \int d^2p \, \ln\left(2S_{\mathbf{p}}\right), \tag{2.68}$$

where

$$2S_{\mathbf{p}}(t) = u^*_{i\mathbf{p}} u_{\mathbf{p}}(t) + v^*_{i\mathbf{p}} v_{\mathbf{p}}(t). \tag{2.69}$$

A typical mechanism for $\ln \mathcal{Z}$ to become singular is for $S_{\mathbf{p}}$ to vanish at some critical value $t = t_n$, at some value p_c. Let us show that the $S_0 \equiv \lim_{\mathbf{p}\to 0} S_{\mathbf{p}}$ can vanish in a particularly simple way. Indeed, the equations of motion of u_0, v_0 (also understood as limits when $\mathbf{p}$ is taken to zero) can be solved directly, as they decouple from the functions $\Delta(\tau)$, $\bar{\Delta}(\tau)$, with the result

$$u_0 = u_{i0}\, e^{i\xi_0 \tau}, \; v_0 = v_{i0}\, e^{-i\xi_0 \tau} \ . \tag{2.70}$$

Substituting this into Eq. (2.69) and taking into account Eq. (2.66) we find

$$2S_0(t) = \cos(\xi_0 t) + i \left(u^*_{i0} u_{i0} - v^*_{i0} v_{i0}\right) \sin(\xi_0 t) \ . \tag{2.71}$$

S_0 vanishes as a function of t only if

$$|u_{i0}|^2 = |v_{i0}|^2 \ . \tag{2.72}$$

Therefore let us restrict our attention to this case. It is well-known that the p-wave superconductor we study here can be in the weakly coupled phase or strongly coupled phase depending on the sign of $\xi_0 = -\mu$. The condition (2.72) holds true in the ground state at the critical point between the phases only. Therefore, from now on we consider $|\Psi_i\rangle$ to be the ground state of a critical p-wave superconductor, while the Hamiltonian after the quench will describe the strongly coupled $\xi_0 > 0$ or weakly coupled $\xi_0 < 0$ phase. In other words, the chemical potential effectively changes from $\mu_i = 0$ to a nonzero μ as a result of the quench. It is necessary to keep μ in the mean-field Hamiltonian (2.64) to make sure we have the correct average fermion number [58].

With S_0 turning to zero at times $t_n = \pi(n + 1/2)/|\xi_0|$ with integer n, $\mathcal{Z}$ can now be singular at $t = t_n$. However, S_0 becoming zero at these times is not by itself a sufficient condition for $\mathcal{Z}$ to be singular. To understand if it becomes singular at these times, we need to examine not just the point $\mathbf{p} = 0$ in momentum space but also its vicinity. Fortunately in this region we can solve the equations of motion (2.67) perturbatively, using p as a small parameter. The solution reads

$$\begin{aligned} u_{\mathbf{p}}(\tau) &= \left(u_{i\mathbf{p}} + i v_{i\mathbf{p}}\, p \int_0^\tau d\tau'\, \bar{\Delta}(\tau')\, e^{-2i\xi_p \tau'} \right) e^{i\xi_p \tau} \ , \\ v_{\mathbf{p}}(\tau) &= \left(v_{i\mathbf{p}} + i u_{i\mathbf{p}}\, p \int_0^\tau d\tau'\, \Delta(\tau')\, e^{2i\xi_p \tau'} \right) e^{-i\xi_p \tau}. \end{aligned} \tag{2.73}$$

This allows us to calculate, from Eq. (2.69),

$$2S_{\mathbf{p}}(t) \approx u^*_{i\mathbf{p}} u_{i\mathbf{p}}\, e^{i\xi_p t} + v^*_{i\mathbf{p}} v_{i\mathbf{p}}\, e^{-i\xi_p t} + i v^*_{i\mathbf{p}} u_{i\mathbf{p}}\, p\, f_{\mathbf{p}} e^{-i\xi_p t} + i u^*_{i\mathbf{p}} v_{i\mathbf{p}}\, p\, \bar{f}_{\mathbf{p}} e^{i\xi_p t} \ , \tag{2.74}$$

where

$$f_{\mathbf{p}} = \int_0^t d\tau \Delta(\tau)\, e^{2i\xi_p \tau} \quad , \quad \bar{f}_{\mathbf{p}} = \int_0^t d\tau \bar{\Delta}(\tau)\, e^{-2i\xi_p \tau} \ . \tag{2.75}$$

Eq. (2.74) is an expansion in powers of p, therefore $u_{i\mathbf{p}}$, $v_{i\mathbf{p}}$ and ξ_p themselves need to be expanded in powers of p. Generally, taking into account Eq. (2.72), this expansion has the form

$$2\,|u_{i\mathbf{p}}|^2 = 1 + \alpha p + \ldots \quad , \quad 2\,|v_{i\mathbf{p}}|^2 = 1 - \alpha p + \ldots , \tag{2.76}$$

where α is some (real) constant. This leads to

$$2S_p \approx \cos(\xi_0 t) + \frac{1}{2}\left(i\bar{f}_0 + \alpha\right) p\, e^{i\xi_0 t} + \frac{1}{2}\left(if_0 - \alpha\right) p\, e^{-i\xi_0 t} \; . \tag{2.77}$$

We examine the vicinity of the point in time where S_0 vanishes. Expanding in powers of $t - t_n$ and p we find

$$2S_p \approx (-1)^n \left[|\xi_0|(t_n - t) + \text{sign}(\xi_0)\beta p\right] \quad , \quad \beta = i\alpha + \left(f_0 - \bar{f}_0\right)/2 \; . \tag{2.78}$$

We now know, quite generally, the behavior of $S_{\mathbf{p}}$ in the vicinity of the point $p = 0$ and time t_n where a singularity of $\mathcal{Z}$ can occur. We can estimate the contribution of small $\mathbf{p}$ to the Loschmidt echo (2.68) by writing

$$\frac{1}{\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial t} = 2\pi V \int_0^{p_0} \frac{p\, dp}{t - t_n - \beta p/\xi_0} \; , \tag{2.79}$$

where p_0 is some momentum beyond which the expansion (2.78) no longer holds. The integral is easy to evaluate and produces

$$\frac{1}{V\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial t} = \frac{2\pi\xi_0^2\,(t - t_n)}{\beta^2} \ln\left[\frac{\xi_0(t - t_n)}{\beta p_0}\right] \tag{2.80}$$

as the singular contribution to the Loschmidt echo (with the second derivative of $\ln \mathcal{Z}$ over time t and therefore also with $\partial^2\mathcal{Z}/\partial t^2$ having a logarithmic singularity).

Equation (2.80) is an important result here. We have obtained the form of singularities in the Loschmidt echo of a 2D p-wave superconductor, without explicitly calculating the expression for the Loschmidt echo at all times. A 2D p-wave superconductor when quenched out of its critical point into either its weak (topological) or strong (non-topological) phase has periodic singularities in its Loschmidt echo where its second derivative over time diverges logarithmically. The singularity is driven by the behavior of small momentum fermions. The parameter α which appears in the calculations above is controlled by the initial amplitudes $u_{i\mathbf{p}}$, $v_{i\mathbf{p}}$. Those can be found as the ground state of the Hamiltonian before quench is known. Being at the critical point where $\xi_0 = -\mu_i = 0$, it must have the form

$$\hat{H}_i = \sum_{\mathbf{p}} \frac{p^2}{2m} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} - \Delta_i \sum_{\mathbf{p}} p \, \hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{-\mathbf{p}} - \bar{\Delta}_i \sum_{\mathbf{p}} p \, \hat{a}_{-\mathbf{p}} \hat{a}_{\mathbf{p}} \ , \tag{2.81}$$

where Δ_i and $\bar{\Delta}_i$ are the equilibrium gap functions. Calculating its ground state is a standard exercise [7]. Evaluating $u_{i\mathbf{p}}$ and $v_{i\mathbf{p}}$ leads to $\alpha = 1/\left(4m\sqrt{\bar{\Delta}_i \Delta_i}\right)$.

We would like to point out that the ratio of the amplitudes $v_{i\mathbf{p}}/u_{i\mathbf{p}}$ goes either to infinity or to zero in the two phases of the 2D chiral p-wave superconductors as $p \to 0$, determining the topological properties of these phases [67]. The only exception is the critical point between the phases where this ratio is 1. Therefore, the criticality in the Loschmidt echo directly reflects the fact that before the quench the system is at the critical point between the two phases with different topology.

The question still remains whether $S_{\mathbf{p}}$ can also vanish at other values of momenta, perhaps leading to other singularities in $\mathcal{Z}$ which we also need to explore. We would like to argue that this does not happen. To do this, we need to understand further how Δ and $\bar{\Delta}$ are related to the fermions. Usually in the problem of quantum quench where the goal is to calculate $|\Psi(\tau)\rangle$ after the quench, the following equation is used for Δ (analogous to Eq. (2.29)) for the s-wave superconductor),

$$\Delta(\tau) = \frac{g}{V} \sum_{\mathbf{p}} p \, \langle \Psi(\tau) | \, \hat{a}_{-\mathbf{p}} \hat{a}_{\mathbf{p}} \, |\Psi(\tau)\rangle = \frac{g}{V} \sum_{\mathbf{p}} p \, u^*_{\mathbf{p}}(\tau) v_{\mathbf{p}}(\tau) \ , \tag{2.82}$$

and its complex conjugate for $\bar{\Delta}(\tau)$. Substituting this into Eq. (2.67) produces the equations of motion for $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$. They are nonlinear but known to be integrable. Their solution can be found for a variety of initial conditions and allows to calculate $|\Psi(t)\rangle$ in many interesting cases [53].

However, we showed in Section 2.3 that these equations are not suitable for calculating the Loschmidt echo (2.4). The resulting wave function $|\Psi(t)\rangle$, while well suited for calculating the expectation values of local observables in the problem and finding their time-dependence, produces wrong results if used to evaluate overlaps of $|\Psi_i\rangle$ and $|\Psi(t)\rangle$ occurring in the calculation of $\mathcal{Z}$.

We note that the Loschmidt echo is simply a matrix element of the evolution operator. These can be calculated using the conventional Feynman functional integral, avoiding the intricacies involved in the Schwinger-Keldysh functional integral construction. Analogous to Eq. (2.9) for the s-wave superconductor, the saddle point approximation with respect to Δ and $\bar{\Delta}$ calculated in the framework of the conventional Feynman functional integral produces (see also Ref. [65])

$$\Delta(\tau) = \frac{g}{V\mathcal{Z}} \sum_{\mathbf{p}} p \left\langle \Psi_i \right| e^{-i\hat{H}(t-\tau)} \hat{a}_{-\mathbf{p}} \hat{a}_{\mathbf{p}} e^{-i\hat{H}\tau} \left| \Psi_i \right\rangle \ , \tag{2.83}$$

and similarly for $\bar{\Delta}(\tau)$. This equation replaces the equation (2.82) for the purpose of determining Δ and $\bar{\Delta}$.

Just like in Section 2.3, we introduce the wavefunction

$$\left| \tilde{\Psi}(\tau) \right\rangle = e^{i\hat{H}(t-\tau)} \left| \Psi_i \right\rangle \ . \tag{2.84}$$

Similar to Eq. (2.24), this wavefunction is characterized by amplitudes $\tilde{u}_{\mathbf{p}}(\tau)$ and $\tilde{v}_{\mathbf{p}}(\tau)$ as

$$\left| \tilde{\Psi} \right\rangle = \prod_{\mathbf{p}} \left(\tilde{u}_{\mathbf{p}} + \tilde{v}_{\mathbf{p}} \, \hat{a}^{\dagger}_{\mathbf{p}} \hat{a}^{\dagger}_{-\mathbf{p}} \right) \left| 0 \right\rangle \ . \tag{2.85}$$

$\tilde{u}_{\mathbf{p}}(\tau)$ and $\tilde{v}_{\mathbf{p}}(\tau)$ satisfy the same equations of motion as (2.67), but with boundary conditions $\tilde{u}_{\mathbf{p}}(t) = u_{i\mathbf{p}}$ and $\tilde{v}_{\mathbf{p}}(t) = v_{i\mathbf{p}}$. In terms of these we find

$$\frac{1}{\mathcal{Z}} \left\langle \Psi_i \right| e^{-i\hat{H}(t-\tau)} \hat{a}_{-\mathbf{p}} \hat{a}_{\mathbf{p}} e^{-i\hat{H}\tau} \left| \Psi_i \right\rangle = \frac{\tilde{u}^*_{\mathbf{p}}(\tau) v_{\mathbf{p}}(\tau)}{2S_{\mathbf{p}}} \ , \tag{2.86}$$

where $S_{\mathbf{p}}$ can be expressed in terms of these amplitudes as

$$2S_{\mathbf{p}}(\tau) = \tilde{u}^*_{\mathbf{p}}(\tau) u_{\mathbf{p}}(\tau) + \tilde{v}^*_{\mathbf{p}}(\tau) v_{\mathbf{p}}(\tau) \ . \tag{2.87}$$

Using the equations of motion, it can be shown that $S_{\mathbf{p}}$ does not depend on τ. In particular, substituting $\tau = t$ we see that the definitions (2.87) and (2.69) coincide. With the help of these relations we find

$$\Delta(\tau)=\frac{g}{V}\sum_{\mathbf{p}}\frac{p\,\tilde{u}^*_{\mathbf{p}}(\tau)v_{\mathbf{p}}(\tau)}{2S_{\mathbf{p}}}\quad,\quad\bar{\Delta}(\tau)=\frac{g}{V}\sum_{\mathbf{p}}\frac{p\,\tilde{v}^*_{\mathbf{p}}(\tau)u_{\mathbf{p}}(\tau)}{2S_{\mathbf{p}}}\ , \tag{2.88}$$

analogous to Eq. (2.34). In order to calculate the Loschmidt echo, we need to solve Eq. (2.67) along with Eq. (2.88). Unlike in Eq. (2.82), here $\bar{\Delta}$ is not equal to the complex conjugate of Δ.

Solving these new equations of motion is an interesting problem by itself, which will be left as a subject for future work. Here we would just like to see whether they are compatible with the singularities in $\mathcal{Z}$ that we found earlier. In the quench scenario considered earlier, we had $S_{\mathbf{p}}$ vanish for small $\mathbf{p}$ according to Eq. (2.78). Substituting this into Eqs. (2.88) we find

$$\Delta\sim\int\frac{\tilde{u}^*_{\mathbf{p}}(\tau)v_{\mathbf{p}}(\tau)\,p^2dp}{t-t_n+\beta p}\ . \tag{2.89}$$

This shows that as a function of t, Δ will have a divergent second derivative. This by itself does not affect the earlier established fact of the divergent second derivative of $\mathcal{Z}$.

However, up until now we only analyzed vanishing of S_0. What if $S_{\mathbf{p}}$ vanishes for some nonzero p_c? If this were to happen, then the gap equation should be expected to read

$$\Delta\sim\int\frac{\tilde{u}^*_{\mathbf{p}}(\tau)v_{\mathbf{p}}(\tau)\,p^2dp}{t-t_c+\beta(p-p_c)}\ , \tag{2.90}$$

where the expression to be integrated is an approximation valid in the vicinity of $p\sim p_c$. Taking into account that β is generally complex and recalling the standard formula $\mathrm{Im}\,[1/(x\pm i\epsilon)]=\mp i\pi\delta(x)$, we see that Δ will then generally be a discontinuous function of t. If Δ is discontinuous, so will be $\mathcal{Z}$.[2] On the other hand, $\mathcal{Z}$ is closely related to the partition functions of quantum systems [12]. Partition functions of thermal systems cannot be discontinuous functions of temperature. Likewise we expect that the Loschmidt echo cannot be a discontinuous function of time t, thus we conclude

[2]Indeed, if Δ is a discontinuous function of t, so are $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$ as they obey their equations of motion (2.67) which involve Δ and $\bar{\Delta}$ explicitly. Therefore $\mathcal{Z}$, constructed from $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$, will also be a discontinuous function of t

that the scenario where $S_{\mathbf{p}}$ vanishes for some nonzero p cannot be realized.

Now that we demonstrated the existence of singularities in the 2D chiral p-wave superconductor, let us return to the simple model we started with – that of the s-wave superconductor. Let us test if an s-wave superconductor can have a vanishing $S_{\mathbf{p}}$ at some critical p_c. Putting this superconductor in d dimensional space for generality, the gap equation and the expression for the Loschmidt echo now read

$$\Delta \sim \int \frac{\tilde{u}^*_{\mathbf{p}}(\tau) v_{\mathbf{p}}(\tau) p^{d-1} dp}{t - t_c + \beta(p - p_c)} \quad , \quad \frac{\partial \mathcal{Z}}{\partial t} \sim \int \frac{p^{d-1} dp}{t - t_c + \beta(p - p_c)} \ . \tag{2.91}$$

Again, unless $p_c = 0$, the equations above lead to the discontinuity of Δ as a function of t and therefore the discontinuity in $\mathcal{Z}$, which we expect cannot happen. The only exception would be $p_c = 0$. However, here $\mathbf{p} = 0$ is not special in the same way as in p-wave superconductors. The spin-triplet pairing in p-wave superconductors ensures that the pseudospin at $\mathbf{p} = 0$ is decoupled from the rest of the pseudospins. Therefore we are unable to identify any initial conditions for which S_0 can vanish for the s-wave superconductor. We are also not aware of any alternative calculation showing vanishing of S_0 in s-wave superconductors. Overall, this leads to our conclusion that the Loschmidt echo in s-wave superconductors lacks any singularities.

On the contrary, we expect that the criticality in Loschmidt echo due to the $\mathbf{p} = 0$ mode also manifests itself in some other topological superconductors. Appendix A shows the calculation for the Loschmidt echo in the B-phase of superfluid He-3 in three spatial dimensions [8], which is an example of a class DIII superconductor [68]. Here we see a singularity of the form

$$\frac{1}{V\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial t} \sim \xi_0^3 (t - t_n)^2 \ln\left(\frac{\xi_0 (t - t_n)}{p_0}\right) \ . \tag{2.92}$$

In this case, it is the third derivative of $\ln \mathcal{Z}$ that is discontinuous at the DQPT.

Appendix B shows the calculation for the Loschmidt echo of a class CI superconductor. This is a model defined on a diamond lattice in three spatial dimensions, with spin singlet d-wave pairing [69]. In this case the first derivative of $\ln \mathcal{Z}$ with t shows discontinuous jumps periodically at times t_n.

2.6 Conclusions

In this chapter, we showed how to calculate the Loschmidt echo of quenched superconductors from first principles. The self-consistent mean-field theory for out-of-equilibrium superconductors yields the expressions for the mean fields (2.9) and (2.10), which are significantly different than their equilibrium counterparts such as those found in Ref. [7]. This leads to the existence of dynamical quantum phase transitions (DQPTs) in topological superconductors and their absence in the topologically trivial s-wave superconductor, as demonstrated in Section 2.5. The particular form of the singularity in the Loschmidt echo depends on the dimensionality of the momentum space, and symmetries of the underlying lattice and Cooper pair wavefunction, as evidenced by Eqs. (2.80) and (2.92), and the result in Appendix B.

Given the wave function $|\Psi(t)\rangle$ calculated using the standard approach of Eqs. (2.67) together with (2.82), one can ask whether its overlap with the initial wave function $|\Psi_i\rangle$ is still meaningful. Let us argue that

$$\mathcal{L} = \left|\left\langle \Psi_i \middle| \Psi(t) \right\rangle\right|^2 \tag{2.93}$$

coincides with the classical echo defined as [70, 71]

$$\mathcal{L} = \int d\mathbf{x}\, \rho(\mathbf{x}, 0)\rho(\mathbf{x}, t). \tag{2.94}$$

Here $\mathbf{x}$ are the coordinates parametrizing the phase space of a classical system and $\rho(\mathbf{x}, t)$ is the classical distribution function, which is in general time-dependent.

Indeed, equations of evolution of $|\Psi(t)\rangle$ (2.67) together with (2.82) are quasiclassical and should be equivalent to evolving the classical distribution function ρ. Therefore, it should not be surprising that Eqs. (2.93) and (2.94) coincide. Formal proof of that consists of identifying ρ for the interacting fermions system that we study here with the Wigner function computed from the quantum state of our system and showing formally that Eq. (2.93) reduces to (2.94); see Appendix C to see how this calculation can be carried out. The quantity $\mathcal{L}$ was calculated for s- and p-wave superconductors in Ref. [58] and found to have many singularities as a function of t. Thus we arrive at a striking conclusion: the classical echo (2.94) can be singular, even when the full quantum echo calculated here is not. If the Hamiltonian (2.56) is realized in experiment and its Loschmidt echo is measured, the result is expected to coincide with our finding of the quantum Loschmidt echo and not the classical Loschmidt echo.

Measurement of the Loschmidt echo has been reported in a trapped ion setup in Ref. [21]. Simulating the Anderson pseudospins using trapped ions provides a way to measure the Loschmidt echo for a system with a superconducting Hamiltonian. If the echo is measured experimentally, we need to account for the possibility that the initial state might be at a finite temperature T. This implies that a certain number of Bogoliubov excitations might be present in the initial state. The Bogoliubov excitations do not contribute to the echo; however they reduce the number of Cooper pairs resulting in the same singularity as given in Eq. (2.80) but now suppressed by a weight factor $0 < w < 1$ multiplying Eq. (2.80). This is consistent with other studies of Loschmidt echo in systems kept initially at finite temperature [72].

An interesting remaining question is the role of the chemical potential μ we introduced into the post-quench Hamiltonian. Normally in dynamical problems with conserved total particle number μ is arbitrary as changing μ simply changes the phase of the time-dependent wave function. However, our initial wave function is not an eigenstate of the total particle number operator $\hat{N} = \sum_{\mathbf{p}} \hat{a}^{\dagger}_{\mathbf{p}} \hat{a}_{\mathbf{p}}$. Note that $\langle \Psi(\tau) | \hat{N} | \Psi(\tau) \rangle$ can be expressed entirely in terms of $u_{i\mathbf{p}}$, $v_{i\mathbf{p}}$ characterizing the initial wave function which contains μ_i but is independent of μ. A more relevant expectation value which arises in the process of evaluating the Loschmidt echo and depends on μ is $\langle \Psi_i | e^{-i\hat{H}(t-\tau)} \hat{N} e^{-i\hat{H}\tau} | \Psi_i \rangle / \mathcal{Z}$, as shown in Eq. (2.14). Therefore, μ must be chosen in such a way as to make this expectation equal to the desired fermion number, i.e., we need to introduce μ to ensure that we describe the time evolution with the correct number of fermions.

Finally let us recall that in this chapter, we were able to find nonanalyticities in the Loschmidt echo of superconductors, without explicitly calculating the general expression of the Loschmidt echo. Evaluating the Loschmidt echo would require solving the mean field gap equations (2.9) and

(2.10) self-consistently with the Hamiltonian (2.11). These self-consistency equations are significantly harder to evaluate than those encountered in the case of an equilibrium mean field [7, 8]. At any time τ, the mean fields $\Delta(\tau)$ and $\bar{\Delta}(\tau)$ are self-consistently related to mean fields $\Delta(\tau')$ and $\bar{\Delta}(\tau')$ at all other times τ'. We have not found such a system of equations in the literature. Even when trying to solve these self-consistency equations numerically, it turns out that the numerical procedure does not converge if the initial guess is not appropriately chosen. We have not been able to find a suitable initial guess for the numerical method. Therefore finding the Loschmidt echo of a superconducting system, where the superconducting mean field is calculated self-consistently, remains an open problem.

Chapter 3

Spectral form factors of unconventional superconductors

3.1 Introduction

The previous chapter looked at the dynamics of quenched superconductors using the Loschmidt echo. The spectral form factor is a related quantity which can be used to describe the time-evolution of a system. It is defined as the trace of the time-evolution operator,

$$\mathcal{Z} = \mathrm{Tr}\, e^{-i\hat{H}t} = \sum_n e^{-iE_n t} \,, \tag{3.1}$$

and can be written as a sum over all the energy levels E_n of a quantum system. It is closely related to the thermal partition function of a quantum system, coinciding with its formal analytic continuation to the complex values of temperature. In a way, the spectral form factor is a more general quantity than the Loschmidt echo because unlike the Loschmidt echo, it does not involve any particular initial state or quench.

The spectral form factor is a way to characterize the energy spectrum of systems. It is a common tool in the study of quantum chaos and random-matrix theories [73–99]. Fourier transform of the absolute value square of the spectral form factor produces the correlation between energy levels of the system

$$\int dt\, e^{i\omega t} \left|\mathcal{Z}\right|^2 = 2\pi \sum_{nm} \delta\left(\omega - E_n + E_m\right). \tag{3.2}$$

In this chapter, we look at the spectral form factors of superconductors to see what properties of the underlying Hamiltonians they can reveal. This chapter follows the work presented in Ref. [45].

We show that spectral form factors in unconventional gapped superconductors have singularities which occur periodically in time, while their conventional counterparts have featureless spectral form factors. It follows that spectral form factors could be used as a test of the structure of the superconducting gap functions. Combined with the new proposals which make it possible to measure the spectral form factor in some atomic systems [95], this makes the spectral form factor an interesting observable to study.

3.2 Spectral form factor of the 2D chiral p-wave superconductor

We start by looking at the chiral p-wave superconductor in two spatial dimensions, which turns out to be the simplest superconducting Hamiltonian that exhibits singularities in its spectral form factor. This is the same model whose Loschmidt echo was explicitly calculated and studied in Section 2.5. The required 2D Hamiltonian for identical fermions with attractive interaction is given by [33, 66]

$$H = \sum_{\mathbf{p}} \xi_p \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}} - \frac{\lambda}{V} \sum_{\mathbf{p},\mathbf{k},\mathbf{q}} \mathbf{k}\cdot\mathbf{q}\, \hat{a}^\dagger_{\frac{\mathbf{p}}{2}+\mathbf{k}} \hat{a}^\dagger_{\frac{\mathbf{p}}{2}-\mathbf{k}} \hat{a}_{\frac{\mathbf{p}}{2}-\mathbf{q}} \hat{a}_{\frac{\mathbf{p}}{2}+\mathbf{q}}\ . \tag{3.3}$$

Here

$$\xi_p = \frac{p^2}{2m} - \mu \tag{3.4}$$

is the kinetic energy of these interacting spinless fermions, λ is the interaction constant and V is the volume of the system. These fermions are known to form a $p_x + ip_y$ paired fermionic superfluid, which for brevity we will refer to as a p-wave superconductor. It is a class D superconductor [27] which is topological if $\mu > 0$ and has a gap as long as $\mu \neq 0$. The Bogoliubov-de-Gennes (BdG) Hamiltonian of this superconductor takes the following standard form

$$\hat{H} = \sum_{\mathbf{p},\ p_y>0} \begin{pmatrix} \hat{a}^\dagger_{\mathbf{p}} & \hat{a}_{-\mathbf{p}} \end{pmatrix} \begin{pmatrix} \xi_p & \Delta(\mathbf{p}) \\ \bar{\Delta}(\mathbf{p}) & -\xi_p \end{pmatrix} \begin{pmatrix} \hat{a}_{\mathbf{p}} \\ \hat{a}^\dagger_{-\mathbf{p}} \end{pmatrix}\ . \tag{3.5}$$

Here $\Delta(\mathbf{p}) = (p_x + ip_y)\,\Delta_p$ and $\bar{\Delta}(\mathbf{p}) = (p_x - ip_y)\,\bar{\Delta}_p$ are the gap functions. Δ_p and $\bar{\Delta}_p$ are the magnitudes of the gap functions (the subscript p emphasizes that these are p-wave gap functions).

To avoid double counting, the summation over $\mathbf{p}$ is restricted to $p_y > 0$. Below all the sums over $\mathbf{p}$ for p-wave superconductors will be restricted in this way.

We use the BdG Hamiltonian to calculate the spectral form factor. To do that, we diagonalize the BdG Hamiltonian for each $\mathbf{p}$. Its eigenvalues $\omega_\pm(p)$ are

$$\omega_\pm(p) = \pm E(p) \ , \tag{3.6}$$

where

$$E(p) = \sqrt{\xi_p^2 + p^2 \bar{\Delta}_p \Delta_p} \ . \tag{3.7}$$

Therefore the trace of its evolution operator is

$$\mathcal{Z} = \prod_{\mathbf{p}} S_{\mathbf{p}} \quad , \quad S_{\mathbf{p}} = e^{-itE(p)} + e^{itE(p)} = 2\cos\left(tE(p)\right) \ . \tag{3.8}$$

Before proceeding to study $\mathcal{Z}$, let us briefly discuss its analytic properties. Each factor $S_{\mathbf{p}}$ is obviously an analytic function of time t. However, if $S_{\mathbf{p}}$ vanishes for some values of $\mathbf{p}$ at some critical time $t = t_c$ with all $S_{\mathbf{p}}$ remaining nonzero if t deviates from t_c, this could make $\mathcal{Z}$ nonanalytic at t_c (we postpone the discussion whether $S_{\mathbf{p}}$ can indeed behave in this way until later). Indeed, suppose $S_{\mathbf{p}}$ vanishes at $t = t_c$ if $\mathbf{p} = \mathbf{p}_c$. Quite generally we should expect that in the vicinity of $\mathbf{p} = \mathbf{p}_c$ and $t = t_c$, $S_{\mathbf{p}}$ has the following expansion

$$S_{\mathbf{p}} \approx C \left(t - t_c + \alpha \left| \mathbf{p} - \mathbf{p}_c \right|^2 \right) \ , \tag{3.9}$$

where α and C are some complex constants (we will see later that, even though it may not be obvious right now, the factors $S_{\mathbf{p}}$ are generally complex-valued). This immediately leads to

$$\frac{\partial \ln \mathcal{Z}}{\partial t} = \sum_{\mathbf{p}} \frac{\partial \ln S_{\mathbf{p}}}{\partial t} \approx \sum_{\mathbf{p}} \frac{1}{t - t_c + \alpha \left| \mathbf{p} - \mathbf{p}_c \right|^2}. \tag{3.10}$$

On the right hand side above the approximate expression for $S_{\mathbf{p}}$ valid with $\mathbf{p}$ in the vicinity of $\mathbf{p}_c$

and $t - t_c$ small is substituted. The sum above is a singular function of time at $t = t_c$, with the details of the singularity dependent on the dimensionality of space and on whether $\mathbf{p}_c$ is zero or nonzero. This makes $\ln \mathcal{Z}$ as well as $\mathcal{Z}$ itself a nonanalytic function of time at $t = t_c$ (we note an obvious similarity between the thermal free energy and $\ln \mathcal{Z}$ introduced above).

We now go back to Eq. (3.8). For a Hamiltonian (3.5) with given Δ_p, $\bar{\Delta}_p$, and ξ_p, Eq. (3.8) gives the answer for its spectral form factor. However, in a superconductor, Δ_p and $\bar{\Delta}_p$ are not fixed beforehand but must be determined self-consistently, by matching the Hamiltonian (3.3) with the BdG Hamiltonian (3.5). To understand how to do it, let us recall that to calculate thermal partition function $\mathrm{Tr}\exp\left(-\hat{H}/(k_B T)\right)$, we must determine Δ_p and $\bar{\Delta}_p$ by solving the thermal gap equation [8]. In a p-wave superconductor, this takes the form

$$\frac{1}{V}\sum_{\mathbf{p}} \frac{p^2 \tanh\left[\frac{E(p)}{2k_B T}\right]}{E(p)} = \frac{1}{\lambda} \,, \tag{3.11}$$

where T is the temperature, k_B is the Boltzmann constant, and $E(p)$ is the energy eigenvalue given by Eq. (3.7). This equation is solved for the product $\bar{\Delta}_p \Delta_p$ which enters $E(p)$. The solution to this equation can be used for example to calculate the thermal partition function of the superconductor. In order to adapt this to calculating the spectral form factor, we replace $1/(k_B T) \to it$, with the result

$$\frac{i}{V}\sum_{\mathbf{p}} \frac{p^2 \tan\left[\frac{tE(p)}{2}\right]}{E(p)} = \frac{1}{\lambda} \,. \tag{3.12}$$

This should be understood as an equation to determine $\bar{\Delta}_p \Delta_p$, which should then be substituted into Eqs. (3.7) and (3.8). See Appendix D for the steps necessary for a formal derivation of Eq. (3.12) from the Hamiltonian (3.5).

In principle, there could be many solutions of the equation (3.12). To find the one we should use we should identify the solution which gives the largest contribution to the spectral form factor. One strategy to do it could consist of first finding the solution of equation (3.11) for the temperatures T where the solution $\bar{\Delta}_p \Delta_p$ is nonzero, and then analytically continuing to the imaginary values of

T. We will leave the detailed study of the solutions of Eq. (3.11) for future work.

We observe that the self-consistency equation (3.12) predicts that the product $\bar{\Delta}_p\Delta_p$ must not be real. Indeed, if it is real, the left hand side of this equation is necessarily imaginary, while the right hand side is real. A similar situation occurs in evaluation of the Loschmidt echo where one also finds [44] that $\bar{\Delta}$ and Δ are not complex conjugates of each other. With $\bar{\Delta}_p\Delta_p$ being complex, $E(p)$ is also generally complex.

As a result, the factors $\cos(tE(p))$, which appear in the spectral form factor (3.8), generally do not vanish at any t. The exception to that is $p = 0$ where $E(0) = |\xi_0| = |\mu|$, and is independent of $\bar{\Delta}_p\Delta_p$. Quite remarkably, this takes us to the previously discussed scenario given by the Eq. (3.9) with $\mathbf{p}_c = 0$. Specifically, for t close to any of the values t_n given by

$$t_n = \frac{\pi}{2\,|\mu|}\,(1+2n)\ , \tag{3.13}$$

with an arbitrary integer n, we can write

$$S_{\mathbf{p}} = 2\cos\left(tE(p)\right) \approx C\left(t - t_n + \alpha p^2\right)\ . \tag{3.14}$$

Here

$$C = 2(-1)^{n+1}\,|\mu|\ , \tag{3.15}$$

and

$$\alpha = \pi\left(\frac{1}{2}+n\right)\frac{\bar{\Delta}_p\Delta_p m - \mu}{2\mu^2|\mu|m}\ . \tag{3.16}$$

are obtained by doing a simple Taylor expansion about $t = t_n$ and $p = 0$. Importantly, α is complex due to $\bar{\Delta}_p\Delta_p$ being complex. Note that this matches the conjectured form (3.9).

Working in the limit of large V and replacing summation over $\mathbf{p}$ with integration we find

$$\frac{1}{V}\frac{\partial \ln \mathcal{Z}}{\partial t} = \frac{1}{4\pi}\int \frac{p\,dp}{t - t_n + \alpha p^2}\ . \tag{3.17}$$

The integral above is taken over p varying from 0 to infinity, although we must remember that only the approximate value for the expression being integrated is written above, valid for small p only. In particular, that means that the integral above can be cut off at some momentum scale, avoiding any divergences at large p. We are interested in the singularity that occurs due to the integration over p near zero. It is then straightforward to see that the leading singularity is

$$\frac{1}{V}\frac{\partial \ln \mathcal{Z}}{\partial t} \approx -\frac{1}{8\pi\alpha} \ln |t - t_n| \ . \tag{3.18}$$

The expression here is approximate, valid when t is in the vicinity of t_n.

Therefore we arrive at the conclusion advertised earlier. The spectral form factor for the 2D p-wave chiral superconductor has periodic logarithmic singularities which occur at times t_n, defined above in Eq. (3.13).

It is important for this argument that α is complex and is not real, which in turn is related to $\bar{\Delta}_p \Delta_p$ being complex.

3.3 Spectral form factor of the s-wave superconductor

We contrast the behavior of the spinless fermions with attractive interactions studied in the previous section, with that of attractively interacting spin-$\frac{1}{2}$ fermions described by the Hamiltonian,

$$\hat{H} = \sum_{\mathbf{p}} \sum_{\sigma=\uparrow,\downarrow} \xi_p \, \hat{a}^\dagger_{\mathbf{p},\sigma} \hat{a}_{\mathbf{p},\sigma} - \frac{\lambda}{V} \sum_{\mathbf{p},\mathbf{q},\mathbf{k}} \hat{a}^\dagger_{\frac{\mathbf{k}}{2}+\mathbf{p},\uparrow} \hat{a}^\dagger_{\frac{\mathbf{k}}{2}-\mathbf{p},\downarrow} \hat{a}_{\frac{\mathbf{k}}{2}-\mathbf{q},\downarrow} \hat{a}_{\frac{\mathbf{k}}{2}+\mathbf{q},\uparrow} \ . \tag{3.19}$$

This is the Hamiltonian of an s-wave superconductor, also seen in Eq. (2.1). Its Bogoliubov-de-Gennes Hamiltonian takes the form

$$\hat{H} = \sum_{\mathbf{p}} \begin{pmatrix} \hat{a}^\dagger_{\mathbf{p}\uparrow} & \hat{a}_{-\mathbf{p}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_p & \Delta_s \\ \bar{\Delta}_s & -\xi_p \end{pmatrix} \begin{pmatrix} \hat{a}_{\mathbf{p}\uparrow} \\ \hat{a}^\dagger_{-\mathbf{p}\downarrow} \end{pmatrix} \ . \tag{3.20}$$

Here Δ_s, $\bar{\Delta}_s$ are momentum-independent s-wave gap functions. The spectral form factor takes the same form (3.8) but with the spectrum

$$E_s(p) = \sqrt{\xi_p^2 + \bar{\Delta}_s \Delta_s} \; . \tag{3.21}$$

The self-consistent solution for $\bar{\Delta}_s \Delta_s$ is given by the gap following equation, which is almost identical to the one for the p-wave superconductor,

$$\frac{i}{2V} \sum_{\mathbf{p}} \frac{\tan\left[\frac{tE_s(p)}{2}\right]}{E_s(p)} = \frac{1}{\lambda} \; . \tag{3.22}$$

This equation is explicitly derived in Appendix D as Eq. (D.15). The primary point is that, just like in case of Eq. (3.12), the solution of this equation necessarily corresponds to $\bar{\Delta}_s \Delta_s$ being complex-valued. As a result, $E_s(p)$ is complex-valued. Unlike in case of the p-wave superconductor, $E_s(p)$ is complex-valued for all p without exceptions. As a result, none of the factors $S_{\mathbf{p}}$ defined in Eq. (3.8) vanish for any time t, and the spectral form factor $\mathcal{Z}$ is analytic at all times. This is in contrast to the spectral form factor of the chiral p-wave superconductor (3.18), which has singularities periodic in time.

3.4 Spectral form factors of higher order unconventional superconductors

The key distinction in the behavior of the spectral form factors of the s-wave and chiral p-wave superconductors is due to the difference in their superconducting gap functions. In the p-wave case, there is the point $\mathbf{p} = 0$ in momentum space where the gap function $\Delta(\mathbf{p}) = (p_x + ip_y)\Delta_p$ vanishes. Furthermore, despite having to analytically continue the solution of the gap equation (3.11) to imaginary temperature $1/T \to it$, we expect that the analytically continued gap function must also vanish as $\mathbf{p} \to 0$. Indeed, from the structure of the Hamiltonian (3.3) the p-wave gap function must satisfy

$$\Delta(\mathbf{p}) = -\Delta(-\mathbf{p}). \tag{3.23}$$

This comes from the spin-triplet pairing, which forces the Cooper pair wavefunction to be antisymmetric under the exchange of the positions of the two fermions forming the pair. The above constraint enforces that the gap function must always vanish at $\mathbf{p} = 0$ even if it is a solution of the analytically continued gap equation (3.12). More generally, the key necessary condition for a nonanalytic spectral form factor is that the gap function $\Delta(\mathbf{p})$ vanishes at certain values of $\mathbf{p}$, not only at finite temperature, but also when analytically continued to imaginary values of temperature.

A good second example of a p-wave superconductor is a class DIII 3D topological superconductor [8, 100] (Helium III B phase) with the Bogoliubov-de-Gennes Hamiltonian

$$\hat{H} = \sum_{\mathbf{p}} \begin{pmatrix} \hat{a}^\dagger_{\mathbf{p}} & \hat{a}_{-\mathbf{p}} \end{pmatrix} \begin{pmatrix} \xi_p & ip_\mu \sigma^y \sigma^\mu \Delta_p \\ -ip_\mu \sigma^\mu \sigma^y \bar{\Delta}_p & -\xi_p \end{pmatrix} \begin{pmatrix} \hat{a}_{\mathbf{p}} \\ \hat{a}^\dagger_{-\mathbf{p}} \end{pmatrix} , \tag{3.24}$$

where σ^y and σ^μ are Pauli matrices acting on the spin indices of the operators $\hat{a}_{\mathbf{p}}$ and $\hat{a}^\dagger_{\mathbf{p}}$. Its spectrum is also given by the equation (3.7), but with $\mathbf{p}$ being the 3D vector. By analogy with the previous analysis leading to equation (3.17), we immediately find

$$\frac{1}{V}\frac{\partial \ln \mathcal{Z}}{\partial t} = \frac{1}{2\pi^2}\int \frac{p^2\, dp}{t - t_n + \alpha p^2} \sim \sqrt{|t - t_n|} \ . \tag{3.25}$$

On the other hand, let us examine 2D spin-singlet chiral d-wave superconductor, which belongs to the symmetry class C [68]. The corresponding Bogoliubov-de-Gennes Hamiltonian is

$$\hat{H} = \sum_{\mathbf{p}} \begin{pmatrix} \hat{a}^\dagger_{\mathbf{p}\uparrow} & \hat{a}_{-\mathbf{p}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_p & \Delta(\mathbf{p}) \\ \bar{\Delta}(\mathbf{p}) & -\xi_p \end{pmatrix} \begin{pmatrix} \hat{a}_{\mathbf{p}\uparrow} \\ \hat{a}^\dagger_{-\mathbf{p}\downarrow} \end{pmatrix} , \tag{3.26}$$

with $\Delta(\mathbf{p}) = (p_x + ip_y)^2 \Delta_d$, $\bar{\Delta}(\mathbf{p}) = (p_x - ip_y)^2 \bar{\Delta}_d$. What sets this example apart from others is that while the gap function vanishes at $\mathbf{p} = 0$, it is not automatically obvious that the gap function analytically continued to imaginary temperature would still vanish in this limit. To elucidate this further, we suppose that the gap function contains contributions from both d-wave and s-wave terms, $\Delta(\mathbf{p}) = \Delta_s + (p_x + ip_y)^2 \Delta_d$, $\bar{\Delta}(\mathbf{p}) = \bar{\Delta}_s + (p_x - ip_y)^2 \bar{\Delta}_d$. With rotationally invariant interactions, the gap equation should decouple into two separate equations for Δ_s, $\bar{\Delta}_s$ and for Δ_d, $\bar{\Delta}_d$. If Δ_s is equal to zero for any temperature T, its analytic continuation to imaginary values of T should also be zero. At the same time, just as earlier, $\bar{\Delta}_d \Delta_d$ becomes a complex number, with the spectrum given by $E(p) = \sqrt{\epsilon^2(p) + p^4 \bar{\Delta}_d \Delta_d /2}$. leading to the following singularity in the spectral form factor (it can be shown that β is real, while α is complex in the following equation)

$$\frac{1}{V} \frac{\partial \ln \mathcal{Z}}{\partial t} = \frac{1}{2\pi} \int \frac{p\, dp}{t - t_n + \beta p^2 + \alpha p^4} \sim \ln |t - t_n| \,. \tag{3.27}$$

However, if Δ_s is nonzero at some range of temperature, then it may still be nonzero after the analytic continuation $1/(k_B T) \to it$. In that case the superconductor will have a non-singular spectral form factor. To decide whether a particular superconductor of this form will have singularities in its spectral form factor we need to examine the original Hamiltonian of the interacting fermions which led to this superconductor and see if any s-wave pairing is possible in addition to the d-wave pairing. Therefore, the singularities in this case are not as ubiquitous as in the p-wave case.

All superconductors that we looked at so far were gapped to fermionic excitations. Let us now look at an example of a gapless superconductor. As an example, consider a 3D p-wave spin-triplet superconductor which has the Bogoliubov-de-Gennes equation (3.5) with the gap function which behaves as

$$\Delta(\mathbf{p}) = (p_x + i p_y)\, \Delta_p \ . \tag{3.28}$$

This gap function vanishes if $p_x = p_y = 0$, for all p_z. Furthermore, given $\xi_p = p^2/(2m) - \mu$ with $\mu > 0$, the excitation spectrum

$$E(\mathbf{p}) = \sqrt{\left(\frac{p^2}{2m} - \mu\right)^2 + \left(p_x^2 + p_y^2\right) \bar{\Delta}_p \Delta_p} \tag{3.29}$$

vanishes at $p_x = p_y = 0$, $p_z = \sqrt{2m\mu}$. Suppose just as in the previous examples, once the temperature is made imaginary, $\bar{\Delta}_p \Delta_p$ becomes complex, but otherwise no other terms appear in the gap function. However, unlike the previous examples of gapful superconductors, setting $p_x = p_y = 0$, we find that

$E(p_z)$ now ranges from zero to infinity. As a result, the spectral form factor $\mathcal{Z}(t)$ is now an analytic function of time t.

Now it is further possible to imagine that the fermions which formed this superconductor move on a lattice, as opposed to a continuous space. If so, then $E(p_z)$, at $p_x = p_y = 0$ now has a maximum somewhere as p_z is varied. Denoting this maximum as E_+, it is straightforward to see that this would lead to a singularities in the spectral form factor occurring at times $t_n = \pi(2n+1)/(2E_+)$. These arguments show that singularities are possible even in gapless superconductors, but they are not as ubiquitous and their existence requires some assumptions. In this case, the periodicity of the singularities would depend on the underlying lattice spacing.

Note however that if $\mu < 0$ in Eq. (3.29), then the resulting superconductor is gapful, although not topological [68]. It will still have singularities controlled by $E_- = |\mu|$.

Coming back to the gapful (topological) superconductors, we can rely on the classification of the topological superconductors [100] to see that there are five distinct classes of topological superconductors of interest, three in the two dimensional space and two more in the three dimensional space. We can summarize the behavior of their spectral form factors in the following table.

Class	Gap function	Spectral Form Factor
D, $2D$	$(p_x + ip_y)\,\Delta_p$	$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \ln\lvert t - t_n\rvert$
C, $2D$	$(p_x + ip_y)^2\,\Delta_d$	$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \ln\lvert t - t_n\rvert$
DIII, $2D$	$(\sigma^z p_x + ip_y)\,\Delta_p$	$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \ln\lvert t - t_n\rvert$
DIII, $3D$	$ip_\mu \sigma^y \sigma^\mu \Delta_p$	$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \sqrt{\lvert t - t_n\rvert}$
CI, $3D$	vanishes on surfaces	$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \sqrt{\lvert t - t_n\rvert}$

Table 3.1: Form of singularities in the spectral form factor for different classes of topological superconductors

The first two entries as well as the fourth entry of the above table have already been calculated in this chapter. In particular, class D and class DIII superconductors are p-wave and the singularities in their spectral form factor are ubiquitous. The class C superconductor may have singularities in their spectral form factor if its gap equation excludes the possibility of an additional s-wave gap function, as discussed above Eq. (3.27). The last entry refers to the class CI topological spin-singlet superconductor in three dimensions [69]. It is in the same class as the conventional s-wave spin-singlet superconductor and therefore will have singularities in the spectral form factor only if its gap equation excludes the possibility of an additional s-wave gap function. If this is excluded, then working out its singularities relies on the understanding that its gap function vanishes on 2D surfaces in its 3D Brillouin zone. Starting from the point on the surface where $E(p)$ has its minimum, following the arguments given here we obtain for class CI,

$$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \int \frac{d^2 q_1 dq_2}{t - t_n - \alpha q_1^2 - \beta q_2^2} \ . \tag{3.30}$$

Here q_1 is the coordinate parametrizing the surface and q_2 is the direction perpendicular to the surface, α is real while β is complex. By analogy with (3.25) we find

$$\frac{\partial \ln \mathcal{Z}}{\partial t} \sim \sqrt{|t - t_n|} \ , \tag{3.31}$$

just as stated in the table on the previous page.

3.5 Measuring the spectral form factor

Spectral form factors are new types of observables which only recently came within reach of experiment. It may not be obvious that they can be measured experimentally. We would like to present here a brief overview of the measurement techniques which were recently suggested in the literature [95].

The simplest object to measure would be the Loschmidt echo. That could be defined as

$$\mathcal{E} = \left| \langle \psi | \, e^{-i\hat{H}t} \, | \psi \rangle \right|^2 . \tag{3.32}$$

If the system under study is equivalent to a number of interacting spins, or qubits, and if there is experimental control over each of these spins, one could prepare the initial state $|\psi\rangle$ (assuming it is a product state), evolve it in time, and find the probability that after that evolution it is still the same state $|\psi\rangle$ as initially. This procedure has been carried out in a system of cold ions [21], where the state of each ion can be addressed independently.

The spectral form factor cannot be measured using this approach as it is given by

$$|\mathcal{Z}|^2 = \left| \sum_n \langle n| e^{-i\hat{H}t} |n\rangle \right|^2 \; . \tag{3.33}$$

Here $|n\rangle$ could be the eigenstates of $\hat{H}$ or any other complete set of orthonormal states. Instead, an alternative approach was proposed in Ref. [95] which allows to measure it. Just as in the example above, this approach still requires that the system under study consists of interacting spins or qubits.

In this work we study interacting fermions. However, all the relevant Hamiltonians presented here can be mapped into a system of interacting spin. The mapping which has extensively been discussed in the literature, and also in Section 2.3, consists of defining the Anderson pseudospin operators. For the s-wave Hamiltonian (3.19) the Anderson pseudospin operators are defined as

$$\hat{S}^+_{\mathbf{p}} = \hat{a}^\dagger_{\mathbf{p},\uparrow}\hat{a}^\dagger_{-\mathbf{p},\downarrow} \quad , \quad \hat{S}^-_{\mathbf{p}} = \hat{a}_{-\mathbf{p},\downarrow}\hat{a}_{\mathbf{p},\uparrow} \; , \tag{3.34}$$

$$\hat{S}^z_{\mathbf{p}} = \frac{1}{2}\left(\hat{a}^\dagger_{\mathbf{p},\uparrow}\hat{a}_{\mathbf{p},\uparrow} + \hat{a}^\dagger_{-\mathbf{p},\downarrow}\hat{a}_{-\mathbf{p},\downarrow} - 1\right) \; . \tag{3.35}$$

It is straightforward to check that they satisfy SU(2) algebra, as required for spins. In terms of these, the Hamiltonian becomes

$$\hat{H} = 2\sum_{\mathbf{p}} \xi_p \hat{S}^z_{\mathbf{p}} - \frac{\lambda}{V}\sum_{\mathbf{k},\mathbf{q}} \hat{S}^+_{\mathbf{k}}\hat{S}^-_{\mathbf{q}} . \tag{3.36}$$

Within mean field theory, this Hamiltonian reduces to

$$\hat{H} = 2\sum_{\mathbf{p}} \xi_p \hat{S}^z_{\mathbf{p}} - \Delta\sum_{\mathbf{p}} \hat{S}^+_{\mathbf{p}} - \bar{\Delta}\sum_{\mathbf{p}} \hat{S}^-_{\mathbf{p}} \; , \tag{3.37}$$

where Δ satisfies a gap equation almost identical to Eq. (3.22),

$$\frac{i}{2V}\sum_{\mathbf{p}}\frac{\tan\left[tE_s(p)\right]}{E_s(p)}=\frac{1}{\lambda}\ . \tag{3.38}$$

The absence of a factor of 2 in Eq. (3.38) when compared to Eq. (3.22) is due to a small difference between the spin system and the original interacting fermions. As can be readily seen, a spin flip excitation corresponds to exciting two Bogoliubov quasiparticles (with the opposite spin and the same excitation energy) in the superconductor. This does not affect the qualitative features of the spectral form factor.

We can now aim at creating a spin system obeying (3.36). A version of the spin system equivalent to a p-wave superconductor (3.3) obeys the Hamiltonian

$$\hat{H}=\sum_{\mathbf{p}}\xi_p\hat{S}^z_{\mathbf{p}}-\frac{\lambda}{V}\sum_{\mathbf{k},\mathbf{q}}\mathbf{k}\cdot\mathbf{q}\,\hat{S}^+_{\mathbf{k}}\hat{S}^-_{\mathbf{q}} \tag{3.39}$$

with the identification

$$\hat{S}^+_{\mathbf{p}}=\hat{a}^\dagger_{\mathbf{p}}\hat{a}^\dagger_{-\mathbf{p}}\quad,\quad\hat{S}^-_{\mathbf{p}}=\hat{a}_{-\mathbf{p}}\hat{a}_{\mathbf{p}}\ , \tag{3.40}$$

$$\hat{S}^z_{\mathbf{p}}=\frac{1}{2}\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}}+\hat{a}^\dagger_{-\mathbf{p}}\hat{a}_{-\mathbf{p}}-1\right)\ . \tag{3.41}$$

Ref. [60] addressed the question of how the Hamiltonian (3.39) can be created in an cold ion system where each spin (qubit) can be independently controlled and measured, at least in principle. The question remains how this addressability can be used to measure spectral form factors.

This question was resolved in another work Ref. [95]. That work proposed a protocol towards measuring the spectral form factor. The protocol itself is not elementary. It consists of the following steps. If the Hilbert space of a quantum many body system can be represented by a collection of N qubits, remarkably the square of the absolute value of the spectral form factor can be measured in terms of the probabilities $\left|\langle\mathbf{s}|\,U^\dagger e^{-i\hat{H}t}U\,|0\rangle\right|^2$. Here $|0\rangle$ is the state where all qubits are initialized in the "all spin-up" state, $\langle\mathbf{s}|$ is the state where the $j-th$ qubit points up if $s_j=0$ or down if $s_j=1$.

$U = \prod_{j=1}^{N} u_j$ and u_j is a unitary SU(2) rotation of the j-th qubit. It can be shown that

$$\left|\mathrm{Tr}\, e^{-i\hat{H}t}\right|^2 = \int \left[\prod_j du_j\right] \sum_{s_j=0,1} (-2)^{-\sum_{j=1}^{N} s_j} \left|\langle \mathbf{s}|\, U^\dagger e^{-i\hat{H}t} U \,|0\rangle\right|^2 \ . \tag{3.42}$$

Here the integrals du_j are over the SU(2) group's Haar measure.

To implement this proposal, it is envisioned that a system of spins is initialized in the "all spin-up" state. Subsequently it is rotated by a random rotation U, evolved in time, rotated again by $U^\dagger$, and the spins are measured producing the data of s_j. This is repeated many times and $2^{-\sum_j s_j}$ is averaged over many realizations of U as well as many repetitions of the same experiment. Since the quantum mechanical probability of observing an outcome of a set of s_j is given by $\left|\langle \mathbf{s}|\, U^\dagger e^{-i\hat{H}t} U \,|0\rangle\right|^2$, it should be clear that this procedure will, upon averaging over many measurements, produce the spectral form factor as long as Eq. (3.42) is correct.

The derivation of (3.42) involves Weingarten calculus and is given in Ref. [95].

3.6 Conclusions

In this chapter, we saw how the spectral form factor is an interesting observable in the study of superconductors. We studied the existence of a particular kind of singularities in the spectral form factor. The simplest superconductor, which has Cooper pairs with s-wave coupling, has a non-singular spectral form factor as shown in Section 3.3. Meanwhile, the simplest topologically non-trivial superconductor, which is a 2D chiral p-wave superconductor, shows periodic singularities in its spectral form factor as given in Eq. (3.18).

Singularities in the spectral form factor of a superconductor can arise when there are points in the momentum space where the superconducting gap function vanishes. This occurs when the superconducting Hamiltonian and the Cooper pair wavefunction obey certain symmetries. The singularities that occur in the different classes of topological superconductors are summarized in Table 3.1. If the singularities exist, then the type of the singularity depends only on the dimension of space.

Finally, as we saw in Section 3.5, spectral form factors are nowadays accessible to measurement in experiments, using the techniques of atomic physics. For example, if the superconductor is realized by means of cold ions [60], its spectral form factor could in principle be measured by directly evolving a random initial product-state up to some time t and measuring the distribution of Cooper pairs in the resulting state via the protocol proposed and developed in Ref. [95]. Thus the spectral form factor can serve as a probe of the structure of the superconducting order parameter, and hence reveal its topological characteristics.

Chapter 4

Nontrivial saddle points in the spectral form factor of a flat-band superconductor

4.1 Introduction

In Chapter 3, we talked about how the spectral form factor can be an interesting observable in the study of superconductors. Superconducting Hamiltonians can be realized in the laboratory using cold ions [60]. The spectral form factor of such a system can be measured using the protocol outlined in Ref. [95]. The type of singularities that can occur in the spectral form factor of a superconductor is determined by the structure of the superconducting order parameter in momentum space.

The key step in finding these singularities is to calculate the superconducting order parameter using the self-consistent mean field theory. We saw from the superconducting gap equations (3.12) and (3.22) in the previous chapter that the mean fields, Δ and $\bar{\Delta}$ are not complex conjugates of each other. We also observed this when calculating the Loschmidt echo in chapter 2, where the mean fields given by equations (2.9) and (2.10) are not complex conjugates of each other.

In contrast, there are no examples in the literature where the superconducting mean fields, Δ and $\bar{\Delta}$ are not complex conjugates. When the mean field theory is used to calculate the superconducting gap function at any finite temperature, then the real-temperature gap equation enforces $\bar{\Delta}\Delta$ to be real-valued [7, 8]. Even when looking at the time evolution of superconducting systems, authors use Δ and $\bar{\Delta}$ which are complex conjugates [12, 22, 49–61].

This can lead us to question the mean field theory we have been using in this text. In particular, we wish to ascertain whether it is possible to have the mean fields Δ and $\bar{\Delta}$ that are not complex conjugates, or whether there is a flaw in the mean field formalism developed in chapters 2 and 3.

In this chapter, we work with a model of a superconductor for which it is possible to calculate the spectral form factor without using mean fields. We compare the result to the one calculated using mean field theory and see that they match exactly. This model demonstrates explicitly that it is possible to have superconducting mean fields Δ and $\bar{\Delta}$ which are not complex conjugates. This work has been presented in Ref. [46].

The model we consider here is that of a single-channel flat-band superconductor. Flat-band superconductivity has recently been theoretically studied in the literature [101, 102]. Here we consider a toy model of a flat-band superconductor without worrying about the explicit material or platform where it can be implemented. We consider the Hamiltonian

$$\hat{H} = \epsilon \sum_{n=1}^{N} \left(\hat{a}_{n\uparrow}^{\dagger} \hat{a}_{n\uparrow} + \hat{a}_{n\downarrow}^{\dagger} \hat{a}_{n\downarrow} \right) - \frac{g}{N} \sum_{n=1}^{N} \sum_{m=1}^{N} \hat{a}_{n\uparrow}^{\dagger} \hat{a}_{n\downarrow}^{\dagger} \hat{a}_{m\downarrow} \hat{a}_{m\uparrow} \ , \tag{4.1}$$

where $\hat{a}$ and $\hat{a}^{\dagger}$ are respectively the annihilation and creation operators for attractively interacting identical fermions. We let the index n represent momentum modes. The bare fermion energy is given by ϵ, which is independent of n. Thus this represents a dispersionless or flat-band superconductor. We wish to calculate the spectral form factor,

$$\mathcal{Z}(t) = \mathrm{Tr}\, e^{-i\hat{H}t} = \sum_{n} e^{-iE_n t} \ , \tag{4.2}$$

where E_n are the energy eigenvalues of the Hamiltonian (4.1). In this calculation, we treat n just as an index, without worrying about the distribution of momenta in the Brillouin Zone.

4.2 Mean field approximation

The standard way to work with a superconducting Hamiltonian is to introduce a mean field. Here we shall set up the mean field to calculate the spectral form factor. From Eqs. (4.2) and (4.1), we can write $\mathcal{Z}(t)$ as a coherent state path integral [62],

$$\mathcal{Z}(t) = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\ \exp\left[i\int_0^t d\tau \left(\sum_{n,\sigma}\left(i\bar{\psi}_{n\sigma}\dot{\psi}_{n\sigma} - \epsilon\bar{\psi}_{n\sigma}\psi_{n\sigma}\right) + \frac{g}{N}\sum_{n,m}\bar{\psi}_{n\uparrow}\bar{\psi}_{n\downarrow}\psi_{m\downarrow}\psi_{m\uparrow}\right)\right] \ , \tag{4.3}$$

where $\psi_{n\sigma}$ and $\bar{\psi}_{n\sigma}$ are fermionic fields. In order for this to represent the trace, these fields must satisfy the boundary conditions,

$$\psi_{n\sigma}(t) = -\psi_{n\sigma}(0) \quad , \quad \bar{\psi}_{n\sigma}(t) = -\bar{\psi}_{n\sigma}(0) \ . \tag{4.4}$$

We have neglected subexponential prefactors in Eq. (4.3) as we are only trying to find the mean field here. Applying the Hubbard-Stratonovich transformation [63], Eq. (4.3) becomes

$$\begin{aligned} \mathcal{Z}(t) &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\Delta\mathcal{D}\bar{\Delta}\, e^{iS(t)} \quad , \\ S(t) &= \int_0^t d\tau \left(\sum_{n,\sigma}\left(i\bar{\psi}_{n\sigma}\dot{\psi}_{n\sigma} - \epsilon\bar{\psi}_{n\sigma}\psi_{n\sigma}\right) + \Delta\sum_n \bar{\psi}_{n\uparrow}\bar{\psi}_{n\downarrow} + \bar{\Delta}\sum_n \psi_{n\downarrow}\psi_{n\uparrow} - \frac{N}{g}\bar{\Delta}\Delta\right) \ . \end{aligned} \tag{4.5}$$

We approximate the functional integral over $\Delta(\tau)$ and $\bar{\Delta}(\tau)$ using only their saddle points. In this section, we calculate the mean fields from first principles. The mean fields, Δ and $\bar{\Delta}$ are dependent on time (τ) in general.

Sections 4.2.1, 4.2.2 and 4.2.3 detail the calculation of the saddle point. These sections can be skipped by the reader unconcerned with the specifics of the calculation. Section 4.2.4 discusses the result obtained using this mean-field theory.

4.2.1 Saddle Point Equations

The Hubbard-Stratonovich (HS) mean field Hamiltonian can be written as

$$\begin{aligned} \hat{H}(\tau) &= \sum_{n=1}^{N}\hat{H}_n(\tau) \ , \\ \hat{H}_n(\tau) &= \epsilon\left(\hat{a}^\dagger_{n\uparrow}\hat{a}_{n\uparrow} + \hat{a}^\dagger_{n\downarrow}\hat{a}_{n\downarrow}\right) - \Delta(\tau)\hat{a}^\dagger_{n\uparrow}\hat{a}^\dagger_{n\downarrow} - \bar{\Delta}(\tau)\hat{a}_{n\downarrow}\hat{a}_{n\uparrow} + \frac{1}{g}\bar{\Delta}(\tau)\Delta(\tau) \ , \end{aligned} \tag{4.6}$$

where $\Delta(\tau)$ and $\bar{\Delta}(\tau)$ are the HS mean fields at time τ. This is a time(τ)-dependent Hamiltonian quadratic in the fermion operators where the different $n-$modes are effectively decoupled. The spectral form factor can be written as a product,

$$\mathcal{Z}(t) = \left(\mathcal{Z}_0(t)\right)^N \ , \tag{4.7}$$

where

$$\mathcal{Z}_0(t) = \sum_{\{\alpha_n\}} \langle \alpha_n | T \exp\left(-i \int_0^t d\tau' \hat{H}_n(\tau')\right) | \alpha_n \rangle \ . \tag{4.8}$$

Here $\{|\alpha_n\rangle\}$ is a set of basis states for the fermion mode indexed by n, and T denotes the time-ordering operator. One choice of basis here is the BCS basis, for which $\{|\alpha_n\rangle\} = \{|11\rangle_n, |00\rangle_n\}$. Here the two basis states correspond respectively to the presence or absence of a Cooper pair in the $n-$fermionic mode. $\mathcal{Z}_0(t)$ is the same for all n since in this basis the elements of the Hamiltonian matrix $\hat{H}_n(\tau)$ (Eq. (4.6)) are independent of n.

The saddle point equation for $\Delta(\tau)$ is obtained by taking the functional derivative of the action in Eq. (4.5) with respect to $\bar{\Delta}(\tau)$ and setting it to zero. This yields the saddle point,

$$\Delta(\tau) = g \frac{\mathcal{Z}_1(\tau; t)}{\mathcal{Z}_0(t)} \ , \tag{4.9}$$

where

$$\mathcal{Z}_1(\tau; t) = \sum_{\{\alpha_n\}} \langle \alpha_n | \left(T \exp\left(-i \int_\tau^t d\tau' \hat{H}_n(\tau')\right)\right) \hat{a}_{n\downarrow} \hat{a}_{n\uparrow} \left(T \exp\left(-i \int_0^\tau d\tau' \hat{H}_n(\tau')\right)\right) | \alpha_n \rangle \ , \tag{4.10}$$

and $\mathcal{Z}_0(t)$ is given by Eq. (4.8). In a similar way, we obtain the saddle point equation for $\bar{\Delta}(\tau)$ as

$$\bar{\Delta}(\tau) = g \frac{\bar{\mathcal{Z}}_1(\tau; t)}{\mathcal{Z}_0(t)} \ , \tag{4.11}$$

where

$$\bar{\mathcal{Z}}_1(\tau;t) = \sum_{\{\alpha_n\}} \langle\alpha_n| \left(T \exp\left(-i\int_\tau^t d\tau' \hat{H}_n(\tau')\right)\right) \hat{a}^\dagger_{n\uparrow}\hat{a}^\dagger_{n\downarrow} \left(T \exp\left(-i\int_0^\tau d\tau' \hat{H}_n(\tau')\right)\right)|\alpha_n\rangle \ . \quad (4.12)$$

We see that $\Delta(\tau)$ and $\bar{\Delta}(\tau)$ are not complex conjugates in general. This is what we saw in Chapters 2 and 3 as well. These saddle points have explicit dependence on τ. Hence, the mean-field Hamiltonian $\hat{H}_n$ defined in Eq. (4.6) is non-Hermitian and time(τ)-dependent.

We note that there are two different times involved in this problem: t is the time at which we wish to evaluate the spectral form factor, whereas τ represents any time between 0 and t at which we are evaluating the mean fields Δ and $\bar{\Delta}$. In order to calculate $\mathcal{Z}_0(t)$, $\mathcal{Z}_1(\tau;t)$ and $\bar{\mathcal{Z}}_1(\tau;t)$, we need to know $\hat{H}_n(\tau')$ from (4.6), and hence the fields $\Delta(\tau')$ and $\bar{\Delta}(\tau')$, at all times τ' between 0 and t. Thus the self-consistency condition for these time-dependent mean fields is more complicated than that which occurs in the standard calculation for the superconducting partition function at finite temperature [7].

4.2.2 Time dependence of mean fields

The standard way to solve these saddle point equations would be an iterative procedure. We would begin by choosing arbitrary fields $\Delta(\tau)$ and $\bar{\Delta}(\tau)$ for all $\tau \in (0,t)$. We would calculate $\mathcal{Z}_0(t)$, $\mathcal{Z}_1(\tau;t)$ and $\bar{\mathcal{Z}}_1(\tau;t)$ with our chosen fields $\Delta(\tau)$ and $\bar{\Delta}(\tau)$. Then we use these values of $\mathcal{Z}_0(t)$, $\mathcal{Z}_1(\tau;t)$ and $\bar{\mathcal{Z}}_1(\tau;t)$ to calculate new fields $\Delta(\tau)$ and $\bar{\Delta}(\tau)$ according to Eqs. (4.9) and (4.11). We use these new fields $\Delta(\tau)$ and $\bar{\Delta}(\tau)$ to again calculate $\mathcal{Z}_0$, $\mathcal{Z}_1$ and $\bar{\mathcal{Z}}_1$. We get a self-consistent solution for the saddle point equations if the sequence of fields $(\Delta(\tau), \bar{\Delta}(\tau))$ generated iteratively, converges.

This standard procedure is computationally cumbersome. Also it is not certain that such an iteration would converge. For example, if $\mathcal{Z}_0(t)$ becomes sufficiently close to 0 at some step of the iteration, then the values for $\Delta(\tau)$ and $\bar{\Delta}(\tau)$ diverge, and the iterative procedure breaks down.

However, it turns out that we can do better for this flat-band model. We take the derivative of $\Delta(\tau)$ with τ. From Eq. (4.9),

$$\frac{\partial\Delta(\tau)}{\partial\tau} = \frac{g}{\mathcal{Z}_0(t)} \lim_{h\to 0} \frac{\mathcal{Z}_1(\tau+h;t) - \mathcal{Z}_1(\tau;t)}{h} \ . \tag{4.13}$$

Using the expression for $\mathcal{Z}_1(\tau;t)$ from Eq. (4.10), this becomes

$$\begin{aligned}\frac{\partial\Delta(\tau)}{\partial\tau} =& \frac{g}{\mathcal{Z}_0(t)} \lim_{h\to 0} \frac{1}{h} \sum_{\{\alpha_n\}} \langle\alpha_n| \left(T\exp\left(-i\int_{\tau+h}^{t} d\tau'\hat{H}_n(\tau')\right)\right) \hat{a}_{n\downarrow}\hat{a}_{n\uparrow} \left(T\exp\left(-i\int_{0}^{\tau+h} d\tau'\hat{H}_n(\tau')\right)\right) \\ &- \left(T\exp\left(-i\int_{\tau}^{t} d\tau'\hat{H}_n(\tau')\right)\right) \hat{a}_{n\downarrow}\hat{a}_{n\uparrow} \left(T\exp\left(-i\int_{0}^{\tau} d\tau'\hat{H}_n(\tau')\right)\right) |\alpha_n\rangle \ .\end{aligned} \tag{4.14}$$

This evaluates to

$$\begin{aligned}\frac{\partial\Delta(\tau)}{\partial\tau} =& \frac{ig}{\mathcal{Z}_0(t)} \sum_{\{\alpha_n\}} \langle\alpha_n| \left(T\exp\left(-i\int_{\tau}^{t} d\tau'\hat{H}_n(\tau')\right)\right) \\ & [\hat{H}_n(\tau), \hat{a}_{n\downarrow}\hat{a}_{n\uparrow}] \left(T\exp\left(-i\int_{0}^{\tau} d\tau'\hat{H}_n(\tau')\right)\right) |\alpha_n\rangle \ .\end{aligned} \tag{4.15}$$

Referring to the mean-field Hamiltonian in Eq. (4.6), we find that the commutator appearing above becomes

$$[\hat{H}_n(\tau), \hat{a}_{n\downarrow}\hat{a}_{n\uparrow}] = -2\epsilon\hat{a}_{n\downarrow}\hat{a}_{n\uparrow} - \Delta(\tau)(\hat{a}^\dagger_{n\uparrow}\hat{a}_{n\uparrow} + \hat{a}^\dagger_{n\downarrow}\hat{a}_{n\downarrow} - 1) \ . \tag{4.16}$$

Now let us define the generalized expectation value of the Cooper pair 'number' operator (analogous to $\Delta(\tau)$ in Eq. (4.9)) as

$$n(\tau) = \frac{\mathcal{Z}_2(\tau;t)}{\mathcal{Z}_0(t)} \ , \tag{4.17}$$

where $\mathcal{Z}_2(\tau;t)$ is defined as

$$\begin{aligned}\mathcal{Z}_2(\tau;t) = \sum_{\{\alpha_n\}} \langle\alpha_n| \left(T\exp\left(-i\int_{\tau}^{t} d\tau'\hat{H}_n(\tau')\right)\right) \cdot \\ (\hat{a}^\dagger_{n\uparrow}\hat{a}_{n\uparrow} + \hat{a}^\dagger_{n\downarrow}\hat{a}_{n\downarrow} - 1) \left(T\exp\left(-i\int_{0}^{\tau} d\tau'\hat{H}_n(\tau')\right)\right) |\alpha_n\rangle \ ,\end{aligned} \tag{4.18}$$

and $\mathcal{Z}_0(t)$ is given by Eq. (4.8). Using this definition of $n(\tau)$ and the commutator (4.16), Eq. (4.15) reduces to

$$\frac{\partial \Delta(\tau)}{\partial \tau} = -i\Delta(\tau)(2\epsilon + g\, n(\tau)) \ . \tag{4.19}$$

Thus we see that in the self-consistent mean field formalism, the time(τ)-evolution equation for $\Delta(\tau)$ depends only on the concurrent values of the fields $\Delta(\tau)$ and $n(\tau)$. This is a consequence of the flat-band property of the Hamiltonian.

Following the same procedure, we get the time-evolution equation for $\bar{\Delta}(\tau)$ as

$$\frac{\partial \bar{\Delta}(\tau)}{\partial \tau} = i\bar{\Delta}(\tau)(2\epsilon + g\, n(\tau)) \ . \tag{4.20}$$

Finally, we can find the time-evolution equation for $n(\tau)$ by differentiating $\mathcal{Z}_2(\tau;t)$ with τ from Eq. (4.18). We get

$$\begin{aligned}\frac{\partial n(\tau)}{\partial \tau} =& \frac{i}{\mathcal{Z}_0(t)} \sum_{\{\alpha_n\}} \langle \alpha_n | \left(T \exp\left(-i \int_\tau^t d\tau' \hat{H}_n(\tau') \right) \right) \\ & [\hat{H}_n(\tau), \hat{a}^\dagger_{n\uparrow}\hat{a}_{n\uparrow} + \hat{a}^\dagger_{n\downarrow}\hat{a}_{n\downarrow} - 1] \left(T \exp\left(-i \int_0^\tau d\tau' \hat{H}_n(\tau') \right) \right) |\alpha_n\rangle \ .\end{aligned} \tag{4.21}$$

Using Eq. (4.6) the commutator above yields

$$[\hat{H}_n(\tau), \hat{a}^\dagger_{n\uparrow}\hat{a}_{n\uparrow} + \hat{a}^\dagger_{n\downarrow}\hat{a}_{n\downarrow} - 1] = 2\Delta(\tau)\hat{a}^\dagger_{n\uparrow}\hat{a}^\dagger_{n\downarrow} - 2\bar{\Delta}(\tau)\hat{a}_{n\downarrow}\hat{a}_{n\uparrow} \ . \tag{4.22}$$

Substituting this into Eq. (4.21), we get

$$\frac{\partial n(\tau)}{\partial \tau} = \frac{2i}{g}(\Delta(\tau)\bar{\Delta}(\tau) - \bar{\Delta}(\tau)\Delta(\tau)) = 0 \ . \tag{4.23}$$

We find that $n(\tau)$ is constant as a function of τ. Let us call this constant n.

$$n = n(\tau) = n(0) = \frac{\mathcal{Z}_2(0;t)}{\mathcal{Z}_0(t)} \quad \text{[from Eq. (4.17)]}. \tag{4.24}$$

Using this in Eqs. (4.19) and (4.20), we obtain

$$\Delta(\tau) = \Delta_0 e^{-i(2\epsilon + g\,n)\tau} \quad , \quad \bar{\Delta}(\tau) = \bar{\Delta}_0 e^{i(2\epsilon + g\,n)\tau} \ , \tag{4.25}$$

where $\Delta_0 = \Delta(\tau = 0)$ and $\bar{\Delta}_0 = \bar{\Delta}(\tau = 0)$ are the initial values of the fields. Thus we find that in this simple flat-band model, the time(τ)-dependence of the mean fields can be expressed exactly.

4.2.3 Self-consistent solution of saddle point equations

To calculate the spectral form factor, we need to determine the values of n, Δ_0 and $\bar{\Delta}_0$ self-consistently. Using Eq. (4.25), the mean-field Hamiltonian (4.6) can be written in the BCS basis as

$$\hat{H}_n(\tau) = \begin{pmatrix} \epsilon & -\Delta_0 e^{-i(2\epsilon + g\,n)\tau} \\ -\bar{\Delta}_0 e^{i(2\epsilon + g\,n)\tau} & -\epsilon \end{pmatrix} + \frac{\bar{\Delta}_0 \Delta_0}{g} \mathbf{1} \ . \tag{4.26}$$

The time(τ)-dependence of this Hamiltonian can be factored by going into a ‘rotating’ basis,

$$\hat{H}_n(\tau) = \hat{U}_n^{-1}(\tau) \hat{H}_n \hat{U}_n(\tau) \ , \tag{4.27}$$

where

$$\hat{H}_n = \begin{pmatrix} \epsilon & -\Delta_0 \\ -\bar{\Delta}_0 & -\epsilon \end{pmatrix} + \frac{\bar{\Delta}_0 \Delta_0}{g} \mathbf{1} \quad , \quad \hat{U}_n(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i(2\epsilon + g\,n)\tau} \end{pmatrix} \ . \tag{4.28}$$

Note that $\hat{U}_n(\tau)$ is not in general a unitary matrix, since n as defined in Eq. (4.24) is not real-valued.

The time-evolution operator can be written as

$$T \exp\left(-i \int_0^t d\tau' \hat{H}_n(\tau')\right) = \lim_{\delta\tau' \to 0} T \prod_{m=1}^{t/\delta\tau'} \exp\Big(-i \hat{H}_n(m\delta\tau')\delta\tau'\Big) \ . \tag{4.29}$$

Using Eq. (4.27) this becomes

$$T \exp\left(-i \int_0^t d\tau' \hat{H}_n(\tau')\right) = \lim_{\delta\tau' \to 0} T \prod_{m=1}^{t/\delta\tau'} \exp\Big(-i\delta\tau' \hat{U}_n^{-1}(m\delta\tau') \hat{H}_n \hat{U}_n(m\delta\tau')\Big) \ , \tag{4.30}$$

giving

$$T\exp\left(-i\int_0^t d\tau' \hat{H}_n(\tau')\right) = \lim_{\delta\tau'\to 0} T\prod_{m=1}^{t/\delta\tau'} \hat{U}_n^{-1}(m\delta\tau')\exp\Big(-i\hat{H}_n\delta\tau'\Big)\hat{U}_n(m\delta\tau')\ . \tag{4.31}$$

To evaluate this product we note that

$$\hat{U}_n((m+1)\delta\tau')\,\hat{U}_n^{-1}(m\delta\tau') = \begin{pmatrix} 1 & 0 \\ 0 & e^{-i(2\epsilon+g\,n)\delta\tau'} \end{pmatrix} = \hat{U}_n(\delta\tau')\ . \tag{4.32}$$

Eq. (4.31) reduces to

$$T\exp\left(-i\int_0^t d\tau' \hat{H}_n(\tau')\right) = \lim_{\delta\tau'\to 0} \hat{U}_n^{-1}(t)e^{-i\hat{H}_n\delta\tau'}\left(\hat{U}_n(\delta\tau')e^{-i\hat{H}_n\delta\tau'}\right)^{t/\delta\tau'-1}\hat{U}_n(0)\ . \tag{4.33}$$

We can write

$$\hat{U}_n(\delta\tau')e^{-i\hat{H}_n\delta\tau'} = \mathbf{1} + iA\delta\tau'\ , \tag{4.34}$$

where

$$A = \begin{pmatrix} -\epsilon & \Delta_0 \\ \bar{\Delta}_0 & -(\epsilon+g\,n) \end{pmatrix} - \frac{\bar{\Delta}_0\Delta_0}{g}\mathbf{1}\ , \tag{4.35}$$

so that the time evolution operator becomes

$$T\exp\left(-i\int_0^t d\tau' \hat{H}_n(\tau')\right) = \hat{U}_n^{-1}(t)\exp(iAt)\hat{U}_n(0)\ . \tag{4.36}$$

To calculate the exponent of A, we diagonalize it. The eigenvalues of A are

$$\begin{aligned} \lambda_\pm &= -\left(\epsilon + \frac{g\,n}{2} + \frac{\bar{\Delta}_0\Delta_0}{g}\right) \pm D \quad , \quad \text{where} \\ D &= \sqrt{\bar{\Delta}_0\Delta_0 + \frac{g^2\,n^2}{4}}\ . \end{aligned} \tag{4.37}$$

We work with the case $D \neq 0$, in which case the eigenvalues are distinct and the matrix A is diagonalizable. The case $D = 0$ can be evaluated in a separate calculation not presented here; doing that calculation, we see that the result for $D = 0$ coincides with that for $D \neq 0$ in the limit $D \to 0$. Hence let us proceed with our calculation for $D \neq 0$. The matrix of eigenvectors corresponding to the eigenvalues λ_+ and λ_- is

$$V = \begin{pmatrix} \frac{g\,n}{2} + D & -\Delta_0 \\ \bar{\Delta}_0 & \frac{gn}{2} + D \end{pmatrix} . \tag{4.38}$$

The exponential of A (4.35) can now be written as

$$\exp(iA\tau) = V \begin{pmatrix} e^{i\lambda_+ \tau} & 0 \\ 0 & e^{i\lambda_- \tau} \end{pmatrix} V^{-1} . \tag{4.39}$$

Substituting this in Eq. (4.36) and using Eq. (4.37), we get

$$\begin{gathered} T\exp\left(-i\int_0^t d\tau' \hat{H}_n(\tau')\right) = e^{-i\frac{\bar{\Delta}_0\Delta_0}{g}t} \cdot \\ \begin{pmatrix} e^{-i\left(\epsilon+\frac{g\,n}{2}\right)t}\left(\cos(Dt) + i\frac{g\,n}{2D}\sin(Dt)\right) & e^{-i\left(\epsilon+\frac{g\,n}{2}\right)t}\,\frac{i\Delta_0}{D}\sin(Dt) \\ e^{i\left(\epsilon+\frac{g\,n}{2}\right)t}\,\frac{i\bar{\Delta}_0}{D}\sin(Dt) & e^{i\left(\epsilon+\frac{g\,n}{2}\right)t}\left(\cos(Dt) - i\frac{g\,n}{2D}\sin(Dt)\right) \end{pmatrix} . \end{gathered} \tag{4.40}$$

The spectral form factor (for a single fermionic mode) is just the trace of this time evolution operator as in Eq. (4.8). Thus we get

$$\mathcal{Z}_0(t) = e^{-i\frac{\bar{\Delta}_0\Delta_0}{g}t}\left[2\cos(Dt)\cos\left(\left(\epsilon + \frac{g\,n}{2}\right)t\right) + \frac{g\,n}{D}\sin(Dt)\sin\left(\left(\epsilon + \frac{g\,n}{2}\right)t\right)\right] . \tag{4.41}$$

Doing similar calculations as those from Eqs. (4.36)-(4.40), we get the remaining time-evolution operators as

$$\begin{gathered} T\exp\left(-i\int_0^\tau d\tau' \hat{H}_n(\tau')\right) = \hat{U}_n^{-1}(\tau)\exp(iA\tau)\hat{U}_n(0) = e^{-i\frac{\bar{\Delta}_0\Delta_0}{g}\tau} \cdot \\ \begin{pmatrix} e^{-i\left(\epsilon+\frac{g\,n}{2}\right)\tau}\left(\cos(D\tau) + i\frac{g\,n}{2D}\sin(D\tau)\right) & e^{-i\left(\epsilon+\frac{g\,n}{2}\right)\tau}\,\frac{i\Delta_0}{D}\sin(D\tau) \\ e^{i\left(\epsilon+\frac{g\,n}{2}\right)\tau}\,\frac{i\bar{\Delta}_0}{D}\sin(D\tau) & e^{i\left(\epsilon+\frac{g\,n}{2}\right)\tau}\left(\cos(D\tau) - i\frac{g\,n}{2D}\sin(D\tau)\right) \end{pmatrix} , \end{gathered} \tag{4.42}$$

and

$$T\exp\left(-i\int_\tau^t d\tau'\hat{H}_n(\tau')\right)=\hat{U}_n^{-1}(t)\exp(iA(t-\tau))\hat{U}_n(\tau)=e^{-i\frac{\bar{\Delta}_0\Delta_0}{g}(t-\tau)}\cdot$$

$$\begin{pmatrix} e^{-i\left(\epsilon+\frac{g\,n}{2}\right)(t-\tau)}\left(\cos(D(t-\tau))+i\frac{g\,n}{2D}\sin(D(t-\tau))\right) & e^{-i\left(\epsilon+\frac{g\,n}{2}\right)(t+\tau)}\frac{i\Delta_0}{D}\sin(D(t-\tau)) \\ e^{i\left(\epsilon+\frac{g\,n}{2}\right)(t+\tau)}\frac{i\bar{\Delta}_0}{D}\sin(D(t-\tau)) & e^{i\left(\epsilon+\frac{g\,n}{2}\right)(t-\tau)}\left(\cos(D(t-\tau))-i\frac{g\,n}{2D}\sin(D(t-\tau))\right) \end{pmatrix} \tag{4.43}$$

Substituting these time-evolution operators in Eq. (4.10), we get

$$\begin{aligned} \mathcal{Z}_1(\tau;t)&=e^{-i\frac{\bar{\Delta}_0\Delta_0}{g}t}e^{-i(2\epsilon+g\,n)\tau}\,\frac{i\Delta_0}{D}\left[\cos\left(\left(\epsilon+\frac{g\,n}{2}\right)t\right)\sin(Dt)\right. \\ &\left.+\,i\sin\left(\left(\epsilon+\frac{g\,n}{2}\right)t\right)\sin(D(2\tau-t))+\frac{g\,n}{D}\sin(D\tau)\sin(D(t-\tau))\sin\left(\left(\epsilon+\frac{g\,n}{2}\right)t\right)\right]\,. \end{aligned} \tag{4.44}$$

Also, Eq. (4.18) gives

$$\mathcal{Z}_2(\tau;t)=e^{-i\frac{\bar{\Delta}_0\Delta_0}{g}t}\left[-2i\cos(Dt)\sin\left(\left(\epsilon+\frac{g\,n}{2}\right)t\right)+\frac{ig\,n}{D}\sin(Dt)\cos\left(\left(\epsilon+\frac{g\,n}{2}\right)t\right)\right]\,. \tag{4.45}$$

To get the self-consistency condition for the mean-field we substitute the expressions for $\mathcal{Z}_0(t)$, $\mathcal{Z}_1(\tau;t)$ from Eqs. (4.41) and (4.44) into Eq. (4.9), giving

$$\begin{aligned} &\left[\cos\left(\left(\epsilon+\frac{g\,n}{2}\right)t\right)\left(2\cos(Dt)-\frac{ig}{D}\sin(Dt)\right)+\sin\left(\left(\epsilon+\frac{g\,n}{2}\right)t\right)\frac{g\,n}{D}\left(\sin(Dt)+\frac{ig}{D}\cos(Dt)\right)\right]\Delta_0 \\ &=-\frac{g}{D}\sin\left(\left(\epsilon+\frac{g\,n}{2}\right)t\right)\left[\sin(D(2\tau-t))-i\frac{g\,n}{D}\cos(D(2\tau-t))\right]\Delta_0\,. \end{aligned} \tag{4.46}$$

The left hand side of the above equation is independent of the intermediate time τ, whereas the right hand side explicitly depends on it. Thus, we must have

$$\Delta_0=0\quad\text{or}\quad\sin\left(\left(\epsilon+\frac{g\,n}{2}\right)t\right)=0\,. \tag{4.47}$$

It is also possible to have $D=0$. But then the eigenvalues in Eq. (4.37) are not distinct and the matrix A cannot be diagonalized. As mentioned after Eq. (4.37), we looked at this case separately and found that it leads to the same conditions as above.

In Eq. (4.44) we calculated $\mathcal{Z}_1(\tau;t)$ and used it in the self-consistency equation for $\Delta(\tau)$. In a similar manner we calculate $\bar{\mathcal{Z}}_1(\tau;t)$ and use it to get the self-consistency condition for $\bar{\Delta}(\tau)$. It turns out that analogous to Eq. (4.47), this gives

$$\bar{\Delta}_0 = 0 \quad \text{or} \quad \sin\left(\left(\epsilon + \frac{g\,n}{2}\right)t\right) = 0\ . \tag{4.48}$$

The final self-consistency condition is that for $n(\tau)$ in Eq. (4.17). Substituting from Eqs. (4.41) and (4.45), we get

$$\begin{aligned} &2\cos(Dt)\left[n\cos\left(\left(\epsilon + \frac{g\,n}{2}\right)t\right) + i\sin\left(\left(\epsilon + \frac{g\,n}{2}\right)t\right)\right] \\ &= \frac{g\,n}{D}\sin(Dt)\left[i\cos\left(\left(\epsilon + \frac{g\,n}{2}\right)t\right) - n\sin\left(\left(\epsilon + \frac{g\,n}{2}\right)t\right)\right]\ . \end{aligned} \tag{4.49}$$

We want to see if Eq. (4.47) can satisfy this self-consistency condition for $n(\tau)$. If we have $\Delta_0 = 0$ then from Eq. (4.37), we have $D = \pm g\,n/2$. Eq. (4.49) becomes

$$\begin{aligned} &2n\left[\cos\left(\left(\epsilon + \frac{g\,n}{2}\right)t\right)\cos\left(\frac{g\,n}{2}t\right) + \sin\left(\left(\epsilon + \frac{g\,n}{2}\right)t\right)\sin\left(\frac{g\,n}{2}t\right)\right] \\ &= 2i\left[\cos\left(\left(\epsilon + \frac{g\,n}{2}\right)t\right)\sin\left(\frac{g\,n}{2}t\right) - \sin\left(\left(\epsilon + \frac{g\,n}{2}\right)t\right)\cos\left(\frac{g\,n}{2}t\right)\right]\ . \end{aligned} \tag{4.50}$$

This reduces to

$$n = -i\tan(\epsilon t)\ . \tag{4.51}$$

Thus we have

$$\Delta_0 = \bar{\Delta}_0 = 0\ ,\ n = -i\tan(\epsilon t) \tag{4.52}$$

as a self-consistent saddle point.

With the other solution in Eq. (4.47), $\sin((\epsilon + g\,n/2)t) = 0$, the self-consistency equation

(4.46) for Δ and equation (4.49) for n, both reduce to

$$\frac{i}{2}\frac{\tan(Dt)}{D} = \frac{1}{g} \ . \tag{4.53}$$

Substituting for D from Eq. (4.37), we find that

$$n = \frac{2}{g}\left(\frac{k\pi}{t} - \epsilon\right) \ , \ \frac{i}{2\sqrt{\bar{\Delta}_0\Delta_0 + \left(\frac{k\pi}{t} - \epsilon\right)^2}} \tan\left(t\sqrt{\bar{\Delta}_0\Delta_0 + \left(\frac{k\pi}{t} - \epsilon\right)^2}\right) = \frac{1}{g} \tag{4.54}$$

is a valid saddle point for any integer k. The time-evolution equation for Δ and $\bar{\Delta}$, Eq. (4.25), becomes

$$\Delta(\tau) = \Delta_0 \exp\left(-i\frac{2\pi k\tau}{t}\right) \ , \ \bar{\Delta}(\tau) = \bar{\Delta}_0 \exp\left(i\frac{2\pi k\tau}{t}\right) \ . \tag{4.55}$$

We see that the self-consistency equation enforces $\Delta(t) = \Delta(0)$ and $\bar{\Delta}(t) = \bar{\Delta}(0)$. The choice $k = 0$ gives a time(τ)-independent saddle point.

4.2.4 Result

We find that there are two classes of time-dependent saddle point solutions, given by Eqs. (4.52) and (4.54). We calculate the contribution to the spectral form factor due to each. Substituting (4.52) in Eq. (4.41) gives

$$\mathcal{Z}_0(t) = 2\cos(\epsilon t) \tag{4.56}$$

whereas substituting the solution (4.54) into (4.41) gives

$$\mathcal{Z}_0(t) = 2(-1)^k \cos(t z_l(t)) \exp\left[-\frac{it}{g}\left(z_l(t)^2 - \left(\frac{k\pi}{t} - \epsilon\right)^2\right)\right] \ , \tag{4.57}$$

where $\mathcal{Z}_0(t) = (\mathcal{Z}(t))^{\frac{1}{N}}$, as defined in Eqs. (4.7) and (4.8), is the spectral form factor for a single fermionic mode of the mean field Hamiltonian, and $z_l(t)$ is a solution for D in the equation

$$\frac{i}{2}\frac{\tan(Dt)}{D} = \frac{1}{g} \ . \tag{4.58}$$

Eq. (4.58) has infinitely many solutions in the complex plane, which we index with subscript l. Note that these saddle points $z_l(t)$ depend on the interaction strength (g) and time (t), but not on the bare fermion energy (ϵ).

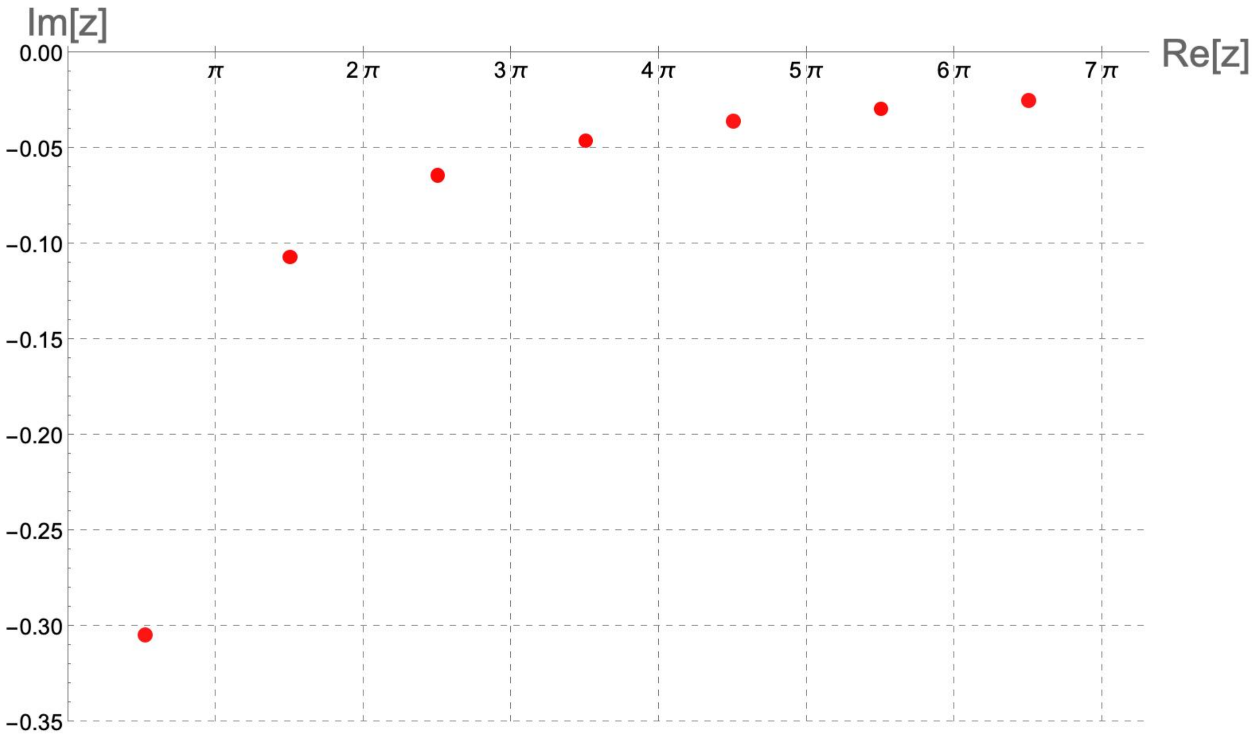

Figure 4.1: Saddle points $z_l(t)$ for g = 1 and t = 1. Shown here are the seven saddle points closest to the imaginary axis in the lower right quadrant of the complex plane.

Figure 4.1 shows the saddle points $z_l(t)$ which are the solutions to Eq. (4.58) for $g = 1$ and $t = 1$. The seven saddle points closest to the imaginary axis in the lower right quadrant of the complex plane are plotted. For each saddle point $z_l(t)$, there is a corresponding saddle point $-z_l(t)$ in the upper left quadrant of the complex plane not shown in the figure.

The infinitely many saddle point solutions to Eq. (4.58), each contribute to the spectral form factor according to Eq. (4.57). Let us define

$$b(t, z_l(t)) = \ln(\mathcal{Z}_0(t, z_l(t))) \ , \tag{4.59}$$

where the expression for $\mathcal{Z}_0(t, z_l(t))$ is given in Eq. (4.57). The magnitude of the contribution of saddle point $z_l(t)$ to the spectral form factor is determined by the real part of $b(t, z_l(t))$,

$$\mathrm{Re}[b(t, z_l(t))] = \ln 2 + \ln|\cos(tz_l(t))| + \frac{2t}{g}\mathrm{Re}[z_l(t)]\mathrm{Im}[z_l(t)] \ , \tag{4.60}$$

whereas the imaginary part determines its complex phase. Denoting the saddle points in the lower right quadrant of the complex plane as $z_1(t),\ z_2(t),\ z_3(t)$, and so on in order of increasing distance from the imaginary axis, the real part of $b(t, z_l(t))$ is shown as a function of t for the first three saddle points in Figure 4.2. The interaction strength g has been set to 1 for this figure. We see that $z_1(t)$, the saddle point closest to the imaginary axis, has the largest value of $\mathrm{Re}[b(t, z_l(t)]$ among all the saddle points.

Summing over all the saddle points the spectral form factor is obtained from Eqs. (4.7) and

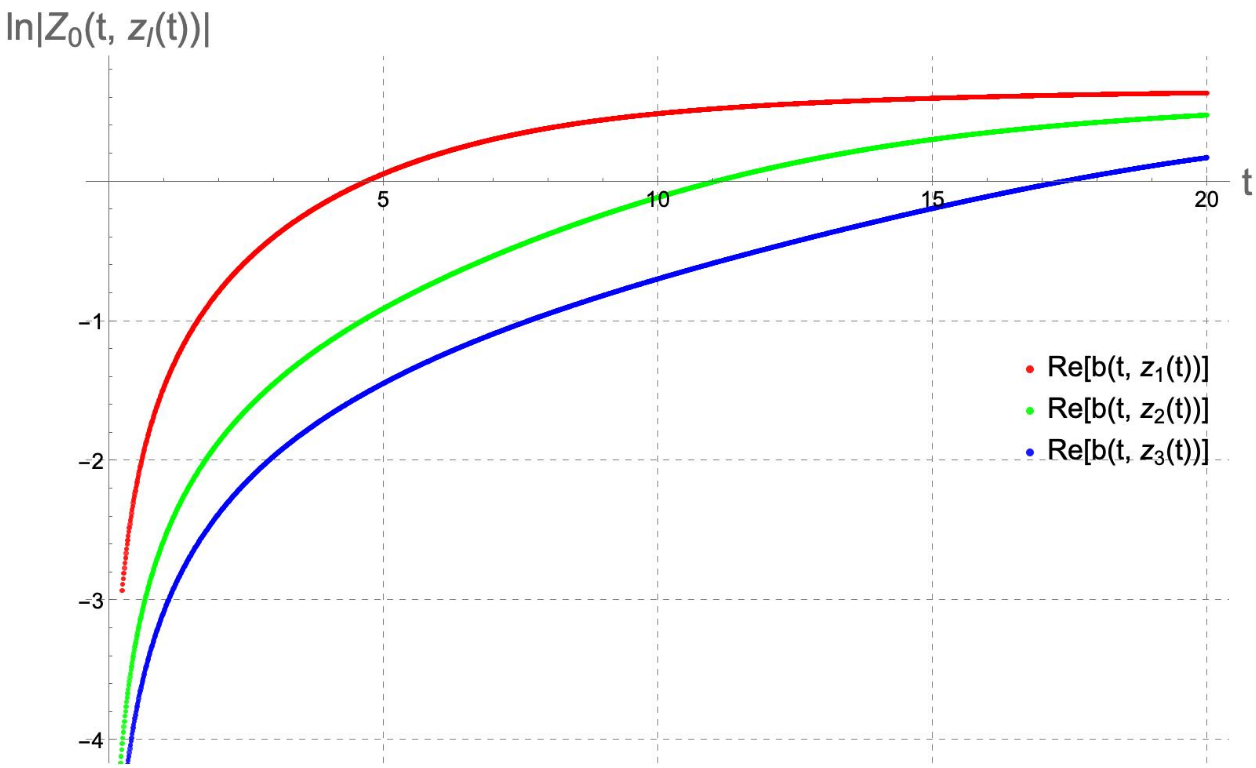

Figure 4.2: Contribution of saddle points z_1, z_2 and z_3 to Z_0 as a function of time t for $g = 1$

(4.59) as

$$\mathcal{Z}(t) = \sum_l c_l(t)\, e^{Nb(t,z_l(t))} \ , \tag{4.61}$$

where $c_l(t)$ is a prefactor that is subexponential in N. $c_l(t)$ comes from the omitted subexponential prefactor in Eq. (4.3) and the Gaussian integral around the saddle point. Thus from Eq. (4.61) and Figure 4.2 we see that saddle points $z_l(t)$ with $l > 1$ are exponentially suppressed as compared to the saddle point $z_1(t)$. To get the spectral form factor from Eq. (4.57) in the large-N limit, we only consider $z_l(t)$ with $l = 1$ at the leading order.

Among Eqs. (4.56) and (4.57), at leading order in N, we only consider the saddle point whose contribution to $\mathcal{Z}_0(t)$ has the larger absolute value. Using Eqs. (4.56), (4.57), (4.59), and (4.60), we obtain

$$\lim_{N\to\infty} \frac{\ln|\mathcal{Z}(t)|}{N} = \ln 2 + \max\left\{\ln(|\cos(\epsilon t)|), \left(\ln|\cos(tz_1(t))| + \frac{2t}{g}\,\mathrm{Re}[z_1(t)]\,\mathrm{Im}[z_1(t)]\right)\right\} \ . \tag{4.62}$$

This is an exact expression for the spectral form factor of the Hamiltonian in Eq. (4.1).

For saddle point $z_1(t)$, the Hubbard-Stratonovich mean fields satisfy, from Eq. (4.54), $\bar{\Delta}\Delta = z_1(t)^2 - \left(\frac{k\pi}{t} - \epsilon\right)^2$ for some integer k. $z_1(t)$ is neither purely real nor purely imaginary for any $t > 0$ since it is a solution for D in Eq. (4.58). Thus $\bar{\Delta}\Delta$ is complex-valued in general.

Singularities occur in the spectral form factor expression of Eq. (4.62) when the two arguments of the $\max\{\cdot\}$ function become equal. These singularities are of a different form than the ones studied in chapter 3, and are not related to Fisher zeros. For the superconducting Hamiltonians considered in chapter 3, looking for these singularities would involve solving the self-consistency equations for time-dependent mean fields. This is a hard problem, which we leave for future research.

4.3 Exact solution: sum over Anderson pseudospins

For the single-channel flat band superconductor considered here, it turns out that it is possible to evaluate the spectral form factor without using a mean field. We follow the standard mapping from fermionic operators to spin operators,

$$\hat{a}^\dagger_{n\uparrow}\hat{a}^\dagger_{n\downarrow} = \hat{s}^+_n \quad , \quad \hat{a}_{n\downarrow}\hat{a}_{n\uparrow} = \hat{s}^-_n \quad , \quad \frac{1}{2}(\hat{a}^\dagger_{n\uparrow}\hat{a}_{n\uparrow} + \hat{a}^\dagger_{n\downarrow}\hat{a}_{n\downarrow} - 1) = \hat{s}^z_n \ , \tag{4.63}$$

where

$$\hat{s}^+_n = \hat{s}^x_n + i\hat{s}^y_n \quad , \quad \hat{s}^-_n = \hat{s}^x_n - i\hat{s}^y_n \ . \tag{4.64}$$

These operators follow the commutation relations,

$$[\hat{s}^x_n, \hat{s}^y_n] = i\hat{s}^z_n \quad , \quad [\hat{s}^y_n, \hat{s}^z_n] = i\hat{s}^x_n \quad , \quad [\hat{s}^z_n, \hat{s}^x_n] = i\hat{s}^y_n \ , \tag{4.65}$$

and are called Anderson pseudospins [64].

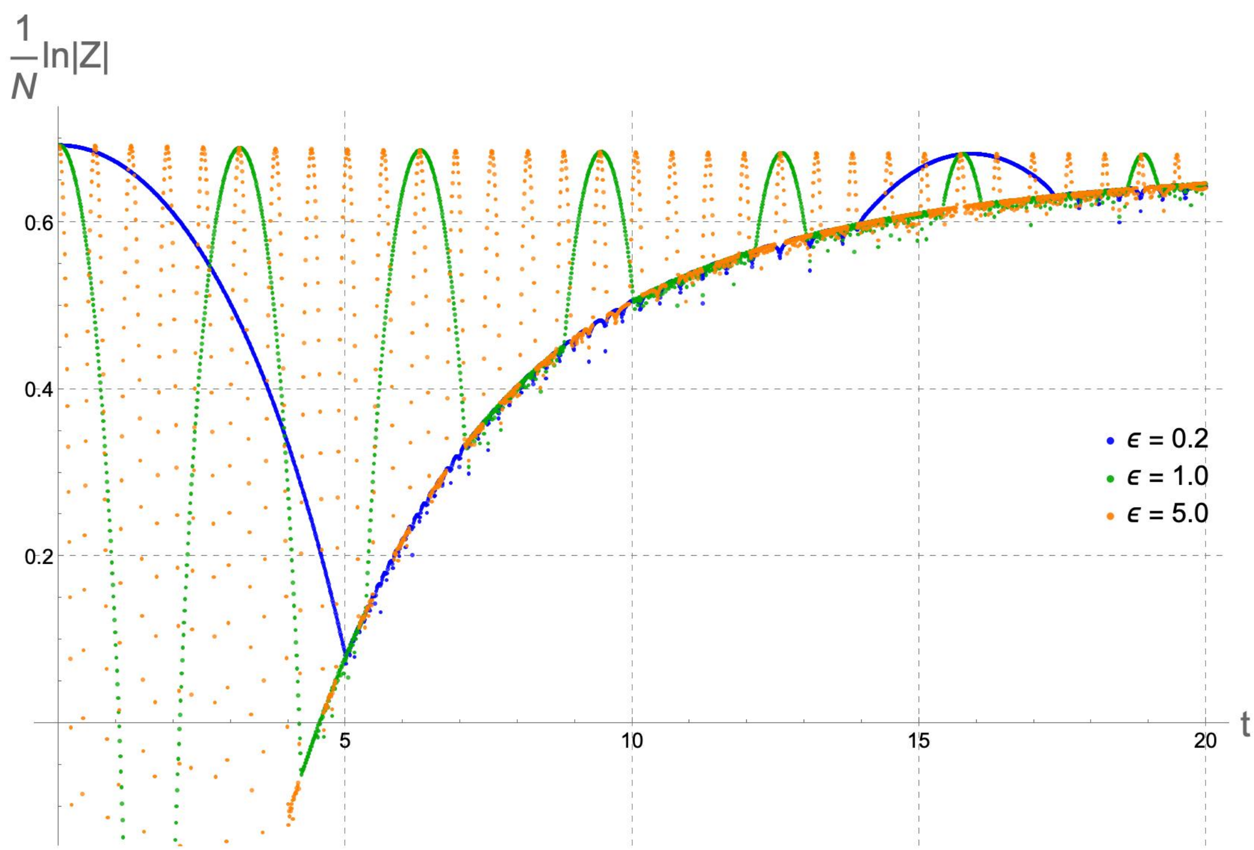

Figure 4.3: Exact spectral form factor, obtained by summing over $N = 200$ Anderson pseudospins, for three different values of ϵ, with $g = 1$

In terms of these Anderson pseudospins, the Hamiltonian Eq. (4.1) becomes

$$\hat{H} = 2\epsilon \sum_{n=1}^{N} \left(\hat{s}_n^z + \frac{1}{2} \right) - \frac{g}{N} \sum_{k,l=1}^{N} \hat{s}_k^+ \hat{s}_l^- \ . \tag{4.66}$$

Denoting the sum over all pseudospins as $\hat{\mathbf{S}} = \sum_{n=1}^{N} \hat{\mathbf{s}}_n$, the Hamiltonian becomes

$$\hat{H} = N\epsilon + 2\epsilon \hat{S}^z - \frac{g}{N} \hat{S}^+ \hat{S}^- \ . \tag{4.67}$$

From the pseudospin commutation relations, we have $\hat{S}^+\hat{S}^- = \hat{\mathbf{S}}^2 - (\hat{S}^z)^2 + \hat{S}^z$ and $[\hat{\mathbf{S}}^2, \hat{S}_z] = 0$. Thus, eigenstates of the Hamiltonian (4.67) are simultaneous eigenstates of $\hat{\mathbf{S}}^2$ and $\hat{S}_z$ operators. Let us denote these eigenstates as $|KM\rangle$, where

$$\hat{\mathbf{S}}^2 \,|KM\rangle = K(K+1)\,|KM\rangle \quad , \quad \hat{S}^z \,|KM\rangle = M\,|KM\rangle \ . \tag{4.68}$$

If we choose N even for convenience, then the value of K can be any integer from 0 to $N/2$. For a particular K, M can have integer values from $-K$ to K. The possible eigenvalues of the Hamiltonian (4.67) are hence,

$$E_{KM} = N\epsilon + 2\epsilon M - \frac{g}{N}[K(K+1) - M(M-1)] \ , \tag{4.69}$$

each with degeneracy [103],

$$D_K = \frac{N!(2K+1)}{\left(\frac{N}{2}+K+1\right)!\left(\frac{N}{2}-K\right)!} \ . \tag{4.70}$$

This gives us an expression for the spectral form factor (4.2) as

$$\mathcal{Z}(t) = e^{-iN\epsilon t} \sum_{K=0}^{\frac{N}{2}} \frac{N!(2K+1)}{\left(\frac{N}{2}+K+1\right)!\left(\frac{N}{2}-K\right)!} e^{it\frac{g}{N}K(K+1)} \sum_{M=-K}^{K} e^{-it\left(2\epsilon M + \frac{g}{N}M(M-1)\right)} \ . \tag{4.71}$$

Figure 4.3 shows the plot of the absolute value of the spectral form factor from Eq. (4.71) with $g = 1$ and $N = 200$, for three different values of the bare fermion energy ϵ.

4.4 Comparison of mean-field approximation with exact solution

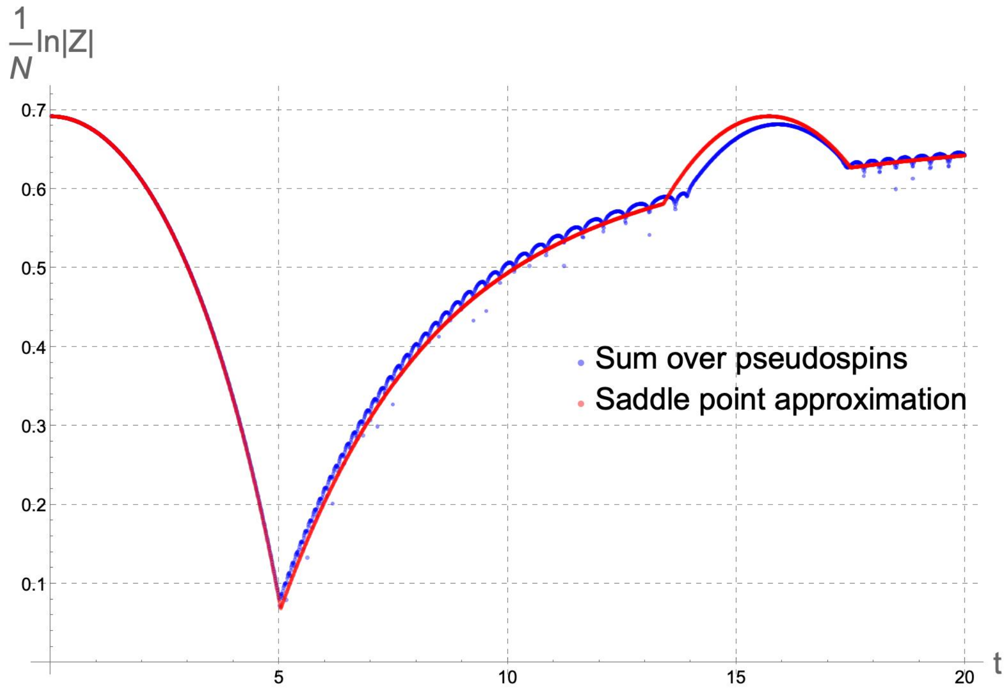

Figure 4.4: Comparison of the exact spectral form factor with the saddle point approximation, for $\epsilon = 0.2$, $g = 1$ and $N = 200$

We now compare the exact value of the spectral form factor obtained from Eq. (4.71), with the saddle point approximation (4.62) from mean field theory. Figures 4.4, 4.5 and 4.6 plot the results obtained from the two different methods for $\epsilon = 0.2$, 1 and 5 respectively. The interaction strength g has been set to 1 for convenience. The summation in Eq. (4.71) has been carried out over $N = 200$ pseudospins in all these plots. We see that the saddle point approximation matches the exact spectral form factor to leading order in N. The subleading terms are $\mathcal{O}\left(\frac{\ln N}{N}\right)$. For $N = 200$, we have $\frac{\ln N}{N} \approx 0.03$.

4.5 Conclusions

We see in this simple single-channel model for a flat-band superconductor that our time-dependent mean field approximation agrees with the exact result for the spectral form factor. We saw in Section 4.2 that for the mean fields, $\bar{\Delta}\Delta$ is not real-valued. Hence we have found a model

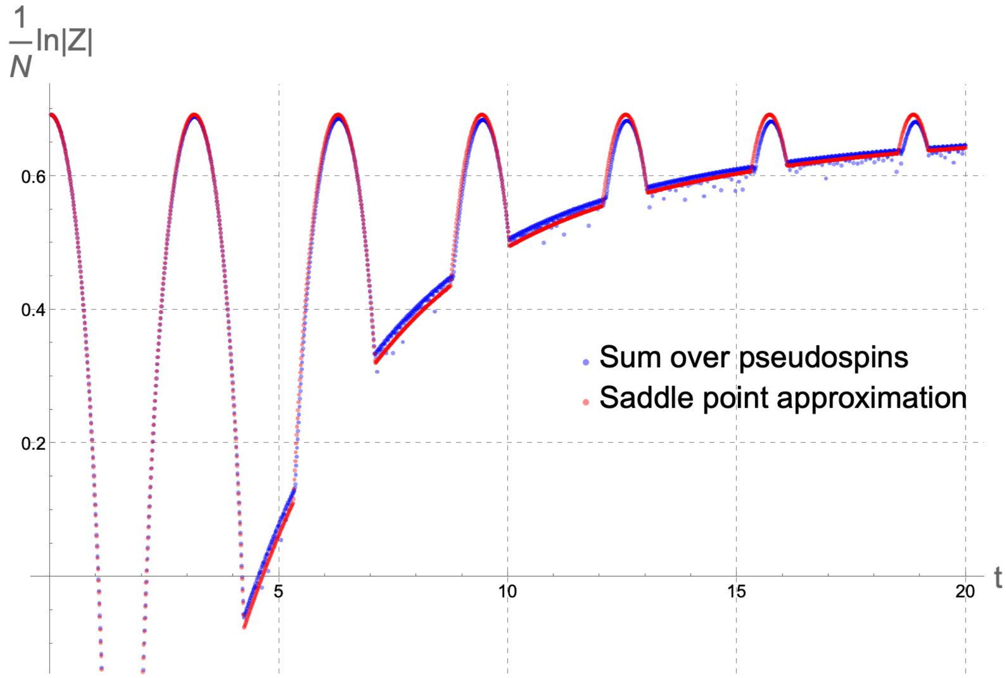

Figure 4.5: Comparison of the exact spectral form factor with the saddle point approximation, for $\epsilon = 1$, $g = 1$ and $N = 200$

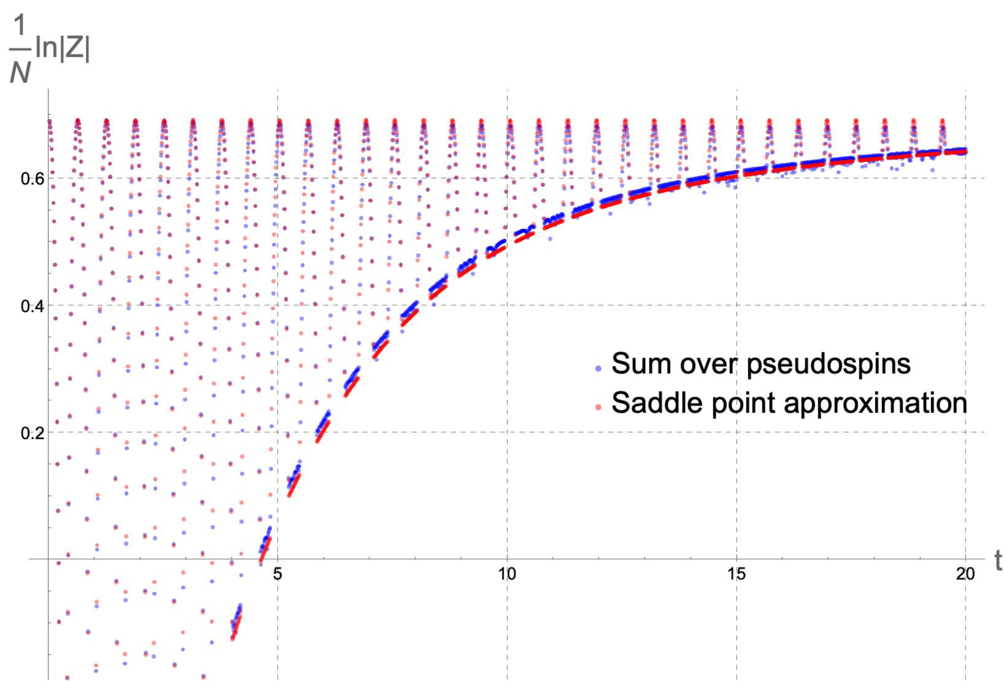

Figure 4.6: Comparison of the exact spectral form factor with the saddle point approximation, for $\epsilon = 5$, $g = 1$ and $N = 200$

where we explicitly showed the existence of mean fields Δ and $\bar{\Delta}$, that are not complex conjugates of each other. This answers the question posed in the beginning of the chapter. The reason we find $\bar{\Delta}\Delta$ to be complex-valued in chapters 2 and 3, is not because there is a flaw in our formalism. It is because this is the nature of the mean fields which are required to calculate the time evolution of superconductors. This agrees with our results published in Refs. [44] and [45].

The result here comes from the important fact that though the Hubbard-Stratonovich fields defined in Eq. (4.5) are complex conjugates of each other, their saddle points are not. This is also what is observed when a complex function is integrated over a segment of the real axis. If the integrand is complex-valued, then it can have saddle points anywhere in the complex plane, and not necessarily on the real axis [104].

This shows that one needs to be careful when using the saddle point approximation in the calculation of the time evolution of a quantum system. There are quite a few research articles involving the time evolution of superconducting systems, in which Δ and $\bar{\Delta}$ are taken to be complex conjugates. This should not be the case in general. In such context, Δ is not the superconducting gap function, but the saddle point of the mean field used in the Hubbard-Stratonovich path integral. Thus, one should not use the value of the equilibrium gap function for the mean field, but rather calculate the mean field self-consistently.

Ref. [95] provides a protocol to measure the spectral form factor of a system of quantum spins. Section 3.5 outlines how this protocol can be used to measure the spectral form factor of a superconducting Hamiltonian, where the Anderson pseudospins are realized using a trapped ion framework. If we wish to experimentally verify the mean-field result presented in this chapter, we need to recall that the expression in Eq. (4.62) is only valid in the large-N limit. Quantum simulators which measure the spectral form factor are limited by the quantum measurement technology of the time. Currently, the spectral form factor can be measured accurately only for systems with the number of spins N on the order of 10 [95]. In order to compare the theoretical prediction with experiment, Eq. (4.62) must be extended beyond the large-N limit. This involves including the sub-dominant saddle points in the calculation, the subexponential prefactor to the path integral in

Eq. (4.5), and doing a Taylor expansion of the integrand around each saddle point to include the higher order terms.

Chapter 5

Conclusions

In this thesis we studied nonanalyticities in the Loschmidt echo and the spectral form factor of superconductors. We found that these nonanalyticities can occur in unconventional superconductors, whose superconducting gap function has a nontrivial structure in momentum space. On the other hand, the conventional s-wave superconductor does not show any such nonanalyticity. The form of these nonanalyticities depends in general on the symmetries of the Cooper pair wavefunction and the dimensionality of space on which the system is defined.

A crucial tool in our study of the time evolution of superconductors was the generalized mean field theory. The superconducting mean field here is derived as the saddle point of the appropriate path integral involved in our calculation. We found that this path integral can have complex nontrivial saddle points which are different than the ones present in a static mean field theory. We explicitly calculated the most general time-dependent saddle points for the single-channel flat band superconductor and verified that our result agree with the exact result calculated without using mean field theory.

There is one important question which arises when using our generalized mean field theory. In Chapter 4, we found saddle points which have complex $\bar{\Delta}\Delta$, even though the Hubbard-Stratonovich fields Δ and $\bar{\Delta}$ are defined to be complex conjugates in the Gaussian path integral. Even if these complex saddle points exist, how can we be sure that they can be accessed by the path integral? In our case, we could match our mean field results with the exact solution obtained without using mean field. This showed that these complex saddle points do in fact contribute to the path integral.

But complex saddle points also exist for other systems which have been studied in literature. One prominent example is the thermal partition function of a superconductor. For the chiral p-wave superconductor Eq. (3.11), we can have saddle points for which $E(p)$ is purely imaginary. But these saddle points do not contribute to the path integral for the partition function, since such partition functions have been calculated in literature and found to be real-valued [105]. Also, one can show that if these complex saddle points did contribute to the thermal partition function, it is possible for their contribution to be larger than that of their real counterpart. Thus there needs to be further study on what conditions determine whether certain saddle points do or do not contribute to the path integral.

In Chapter 4, we found that the spectral form factor of the single-channel flat-band superconductor can have singularities, as observed in figures 4.4, 4.5, and 4.6. These singularities cannot be found in the formalism used in Chapters 2 and 3. The nonanalyticities in Chapter 4 occur due to the dominant saddle point switching between two different types of saddle points, and not due to Fisher zeros being encountered. Such nonanalyticities can also occur in the Loschmidt echo studied in Chapter 2. Finding such nonanalyticities requires calculating the Loschmidt echo at any general time, and not just near a DQPT. Therefore, using the nonequilibrium mean-field theory to explicitly calculate the Loschmidt echo for superconductors, and using it to find such nonanalyticities remains an open problem.

We can also use the nonequilibrium mean field theory developed here to look for mixed state dynamical quantum phase transitions (DQPTs) in superconductors. These mixed state DQPTs have been studied for the transverse field Ising model in Ref. [72], and for Gaussian models in Ref. [106]. This requires starting with a mixed state, such as an equilibrium thermal distribution, rather than the ground state of the pre-quench Hamiltonian. Such a calculation should match the results obtained in Chapters 2 and 3 in the two following limits. If we start with the equilibrium thermal distribution with temperature approaching zero, then it should approximate the standard DQPT result obtained in Chapter 2. On the other hand, if we start with the equilibrium thermal distribution with temperature approaching infinity, it should approximate the spectral form factor obtained in Chapter 3. Any deviation of the mixed state DQPTs from these expected limits would be an interesting phenomenon to study in itself. It would suggest that the order of operations of using the Hubbard-Stratonovich mean field approximation and taking the limit of the temperature to zero or infinity, do not commute.

References

[1] P. W. Anderson, “More is different”, Science **177**, 393 (1972).

[2] N. Ashcroft and N. Mermin, *Solid state physics*, HRW international editions (Holt, Rinehart and Winston, 1976).

[3] R. Pathria and P. Beale, *Statistical mechanics* (Academic Press, 2021).

[4] L. Landau and E. Lifshitz, *Statistical physics: volume 5*, v. 5 (Butterworth-Heinemann, 2013).

[5] G. Giuliani and G. Vignale, *Quantum theory of the electron liquid* (Cambridge University Press, 2008).

[6] J. F. Allen and A. Misener, “Flow phenomena in liquid helium ii”, Nature **142**, 643 (1938).

[7] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, “Theory of superconductivity”, Phys. Rev. **108**, 1175 (1957).

[8] A. J. Leggett, “A theoretical description of the new phases of liquid ^{3}He”, Rev. Mod. Phys. **47**, 331 (1975).

[9] M. Z. Hasan and C. L. Kane, “Colloquium: topological insulators”, Rev. Mod. Phys. **82**, 3045 (2010).

[10] L. Savary and L. Balents, “Quantum spin liquids: a review”, Reports on Progress in Physics **80**, 016502 (2016).

[11] D. Pines and M. A. Alpar, “Superfluidity in neutron stars”, Nature **316**, 27 (1985).

[12] M. Heyl, “Dynamical quantum phase transitions: a review”, Reports on Progress in Physics **81**, 054001 (2018).

[13] M. Sato and Y. Ando, “Topological superconductors: a review”, Reports on Progress in Physics **80**, 076501 (2017).

[14] L. Jiang, C. L. Kane, and J. Preskill, “Interface between topological and superconducting qubits”, Phys. Rev. Lett. **106**, 130504 (2011).

[15] M. Heyl, A. Polkovnikov, and S. Kehrein, “Dynamical quantum phase transitions in the transverse-field ising model”, Phys. Rev. Lett. **110**, 135704 (2013).

[16] M. Fisher and W. Brittin, “Statistical physics, weak interactions, field theory”, Lectures in Theoretical Physics (Boulder: University of Colorado Press) vol VIIC (1965).

[17] J. Conway, *Functions of one complex variable ii*, Graduate Texts in Mathematics (Springer New York, 2012).

[18] W. van Saarloos and D. A. Kurtze, “Location of zeros in the complex temperature plane: absence of lee-yang theorem”, J. Phys. A **17**, 1301 (1984).

[19] S. Sachdev, *Quantum phase transitions*, 2nd ed. (Cambridge University Press, 2011).

[20] E. Lieb, T. Schultz, and D. Mattis, “Two soluble models of an antiferromagnetic chain”, Annals of Physics **16**, 407 (1961).

[21] P. Jurcevic, H. Shen, P. Hauke, C. Maier, T. Brydges, C. Hempel, B. P. Lanyon, M. Heyl, R. Blatt, and C. F. Roos, “Direct observation of dynamical quantum phase transitions in an interacting many-body system”, Phys. Rev. Lett. **119**, 080501 (2017).

[22] S. Vajna and B. Dóra, “Topological classification of dynamical phase transitions”, Phys. Rev. B **91**, 155127 (2015).

[23] V. Mourik, K. Zuo, S. M. Frolov, S. R. Plissard, E. P. A. M. Bakkers, and L. P. Kouwenhoven, “Signatures of majorana fermions in hybrid superconductor-semiconductor nanowire devices”, Science **336**, 1003 (2012).

[24] M. Atala, M. Aidelsburger, J. T. Barreiro, D. Abanin, T. Kitagawa, E. Demler, and I. Bloch, “Direct measurement of the zak phase in topological bloch bands”, Nature Physics **9**, 795 (2013).

[25] G. Jotzu, M. Messer, R. Desbuquois, M. Lebrat, T. Uehlinger, D. Greif, and T. Esslinger, “Experimental realization of the topological haldane model with ultracold fermions”, Nature **515**, 237 (2014).

[26] S. Nadj-Perge, I. K. Drozdov, J. Li, H. Chen, S. Jeon, J. Seo, A. H. MacDonald, B. A. Bernevig, and A. Yazdani, “Observation of majorana fermions in ferromagnetic atomic chains on a superconductor”, Science **346**, 602 (2014).

[27] S. Ryu, A. P. Schnyder, A. Furusaki, and A. W. W. Ludwig, “Topological insulators and superconductors: tenfold way and dimensional hierarchy”, New Journal of Physics **12**, 065010 (2010).

[28] S. Vajna and B. Dóra, “Disentangling dynamical phase transitions from equilibrium phase transitions”, Phys. Rev. B **89**, 161105 (2014).

[29] X.-L. Qi, Y.-S. Wu, and S.-C. Zhang, “Topological quantization of the spin hall effect in two-dimensional paramagnetic semiconductors”, Phys. Rev. B **74**, 085308 (2006).

[30] M. Nakahara, *Geometry, topology and physics, second edition*, Graduate student series in physics (Taylor & Francis, 2003).

[31] F. D. M. Haldane, “Model for a quantum hall effect without landau levels: condensed-matter realization of the "parity anomaly"”, Phys. Rev. Lett. **61**, 2015 (1988).

[32] B. A. Bernevig, T. L. Hughes, and S.-C. Zhang, “Quantum spin hall effect and topological phase transition in hgte quantum wells”, Science **314**, 1757 (2006).

[33] V. Gurarie, L. Radzihovsky, and A. V. Andreev, “Quantum phase transitions across a p-wave feshbach resonance”, Phys. Rev. Lett. **94**, 230403 (2005).

[34] N. Fläschner, D. Vogel, M. Tarnowski, B.-S. Rem, D.-S. Lühmann, M. Heyl, J. C. Budich, L. Mathey, K. Sengstock, and C. Weitenberg, “Observation of dynamical vortices after quenches in a system with topology”, Nat. Phys. **14**, 265 (2017).

[35] J. C. Budich and M. Heyl, “Dynamical topological order parameters far from equilibrium”, Phys. Rev. B **93**, 085416 (2016).

[36] C. Karrasch and D. Schuricht, “Dynamical phase transitions after quenches in nonintegrable models”, Phys. Rev. B **87**, 195104 (2013).

[37] K. Cao, H. Guo, and G. Yang, "Aperiodic dynamical quantum phase transition in multi-band bloch hamiltonian and its origin", Journal of Physics: Condensed Matter **36**, 155401 (2024).

[38] F. Brange, S. Peotta, C. Flindt, and T. Ojanen, "Dynamical quantum phase transitions in strongly correlated two-dimensional spin lattices following a quench", Phys. Rev. Res. **4**, 033032 (2022).

[39] S. Peotta, F. Brange, A. Deger, T. Ojanen, and C. Flindt, "Determination of dynamical quantum phase transitions in strongly correlated many-body systems using loschmidt cumulants", Phys. Rev. X **11**, 041018 (2021).

[40] S. A. Weidinger, M. Heyl, A. Silva, and M. Knap, "Dynamical quantum phase transitions in systems with continuous symmetry breaking", Phys. Rev. B **96**, 134313 (2017).

[41] B. Zunkovic, M. Heyl, M. Knap, and A. Silva, "Dynamical quantum phase transitions in spin chains with long-range interactions: merging different concepts of nonequilibrium criticality", Phys. Rev. Lett. **120**, 130601 (2018).

[42] J. C. Halimeh and V. Zauner-Stauber, "Dynamical phase diagram of quantum spin chains with long-range interactions", Phys. Rev. B **96**, 134427 (2017).

[43] V. Zauner-Stauber and J. C. Halimeh, "Probing the anomalous dynamical phase in long-range quantum spin chains through fisher-zero lines", Phys. Rev. E **96**, 062118 (2017).

[44] S. Gaur, V. Gurarie, and E. A. Yuzbashyan, "Singularities in the loschmidt echo of quenched topological superconductors", Phys. Rev. B **106**, L220506 (2022).

[45] S. Gaur and V. Gurarie, "Spectral form factors of unconventional superconductors", Phys. Rev. B **109**, 144514 (2024).

[46] S. Gaur, E. A. Yuzbashyan, and V. Gurarie, "Nontrivial saddle points for spectral form factors of flat band superconductors", arXiv:2504.04057 (2025).

[47] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, "Microscopic theory of superconductivity", Phys. Rev. **106**, 162 (1957).

[48] N. N. Bogoliubov, V. V. Tolmachov, and D. V. Sirkov, "A new method in the theory of superconductivity", Fortschritte der Physik **6**, 605 (1958).

[49] R. A. Barankov, L. S. Levitov, and B. Z. Spivak, "Collective rabi oscillations and solitons in a time-dependent bcs pairing problem", Phys. Rev. Lett. **93**, 160401 (2004).

[50] E. A. Yuzbashyan, B. L. Altshuler, V. B. Kuznetsov, and V. Z. Enolskii, "Nonequilibrium cooper pairing in the nonadiabatic regime", Phys. Rev. B **72**, 220503 (2005).

[51] R. A. Barankov and L. S. Levitov, "Synchronization in the bcs pairing dynamics as a critical phenomenon", Phys. Rev. Lett. **96**, 230403 (2006).

[52] V. Gurarie, "Nonequilibrium dynamics of weakly and strongly paired superconductors", Phys. Rev. Lett. **103**, 075301 (2009).

[53] M. S. Foster, M. Dzero, V. Gurarie, and E. A. Yuzbashyan, "Quantum quench in a $p + ip$ superfluid: winding numbers and topological states far from equilibrium", Phys. Rev. B **88**, 104511 (2013).

[54] M. S. Foster, V. Gurarie, M. Dzero, and E. A. Yuzbashyan, "Quench-induced floquet topological p-wave superfluids", Phys. Rev. Lett. **113**, 076403 (2014).

[55] E. A. Yuzbashyan, M. Dzero, V. Gurarie, and M. S. Foster, "Quantum quench phase diagrams of an s-wave bcs-bec condensate", Phys. Rev. A **91**, 033628 (2015).

[56] F. Peronaci, M. Schiró, and M. Capone, "Transient dynamics of d-wave superconductors after a sudden excitation", Phys. Rev. Lett. **115**, 257001 (2015).

[57] A. A. Zvyagin, "Dynamical quantum phase transitions (review article)", Low Temperature Physics **42**, 971 (2016).

[58] C. C. Rylands, E. A. Yuzbashyan, V. Gurarie, A. Zabalo, and V. Galitski, "Loschmidt echo of far-from-equilibrium fermionic superfluids", Ann. Phys. **435**, 168554 (2021).

[59] R. J. Lewis-Swan, D. Barberena, J. R. K. Cline, D. J. Young, J. K. Thompson, and A. M. Rey, "Cavity-qed quantum simulator of dynamical phases of a bardeen-cooper-schrieffer superconductor", Phys. Rev. Lett. **126**, 173601 (2021).

[60] A. Shankar, E. A. Yuzbashyan, V. Gurarie, P. Zoller, J. J. Bollinger, and A. M. Rey, "Simulating dynamical phases of chiral $p + ip$ superconductors with a trapped ion magnet", PRX Quantum **3**, 040324 (2022).

[61] A. Nava, C. A. Perroni, R. Egger, L. Lepori, and D. Giuliano, "Dissipation-driven dynamical topological phase transitions in two-dimensional superconductors", Phys. Rev. B **109**, L041107 (2024).

[62] J. Martin, "The feynman principle for a fermi system", Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences **251**, 543 (1959).

[63] J. Hubbard, "Calculation of partition functions", Phys. Rev. Lett. **3**, 77 (1959).

[64] P. W. Anderson, "Random-phase approximation in the theory of superconductivity", Phys. Rev. **112**, 1900 (1958).

[65] E. Canovi, P. Werner, and M. Eckstein, "First-order dynamical phase transitions", Phys. Rev. Lett. **113**, 265702 (2014).

[66] V. Gurarie and L. Radzihovsky, "Resonantly paired fermionic superfluids", Ann. Phys. **322**, 2 (2007).

[67] N. Read and D. Green, "Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries and the fractional quantum hall effect", Phys. Rev. B **61**, 10267 (2000).

[68] A. P. Schnyder, S. Ryu, A. Furusaki, and A. W. W. Ludwig, "Classification of topological insulators and superconductors in three spatial dimensions", Phys. Rev. B **78**, 195125 (2008).

[69] A. P. Schnyder, S. Ryu, and A. W. W. Ludwig, "Lattice model of a three-dimensional topological singlet superconductor with time-reversal symmetry", Phys. Rev. Lett. **102**, 196804 (2009).

[70] B. Eckhardt, "Echoes in classical dynamical systems", Journal of Physics A: Mathematical and General **36**, 371 (2002).

[71] G. Veble and T. Prosen, "Faster than lyapunov decays of the classical loschmidt echo", Phys. Rev. Lett. **92**, 034101 (2004).

[72] U. Bhattacharya, S. Bandyopadhyay, and A. Dutta, "Mixed state dynamical quantum phase transitions", Phys. Rev. B **96**, 180303 (2017).

[73] M. L. Mehta, *Random matrices* (Elsevier Science & Technology Books, 1967).

[74] M. V. Berry, "Semiclassical theory of spectral rigidity", Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences **400**, 229 (1985).

[75] L. Leviandier, M. Lombardi, R. Jost, and J. P. Pique, “Fourier transform: a tool to measure statistical level properties in very complex spectra”, Phys. Rev. Lett. **56**, 2449 (1986).

[76] T. Guhr and H. Weidenmuller, “Correlations in anticrossing spectra and scattering theory. analytical aspects”, Chemical Physics **146**, 21 (1990).

[77] A. Delon, R. Jost, and M. Lombardi, “No2 jet cooled visible excitation spectrum: vibronic chaos induced by the x2a1–ã2b2 interaction”, The Journal of Chemical Physics **95**, 5701 (1991).

[78] U. Hartmann, H. Weidenmüller, and T. Guhr, “Correlations in anticrossing spectra and scattering theory: numerical simulations”, Chemical Physics **150**, 311 (1991).

[79] M. Lombardi and T. H. Seligman, “Universal and nonuniversal statistical properties of levels and intensities for chaotic rydberg molecules”, Phys. Rev. A **47**, 3571 (1993).

[80] E. Brézin and A. Zee, “Universality of the correlations between eigenvalues of large random matrices”, Nuclear Physics B **402**, 613 (1993).

[81] N. Argaman, Y. Imry, and U. Smilansky, “Semiclassical analysis of spectral correlations in mesoscopic systems”, Phys. Rev. B **47**, 4440 (1993).

[82] Y. Alhassid and N. Whelan, “Onset of chaos and its signature in the spectral autocorrelation function”, Phys. Rev. Lett. **70**, 572 (1993).

[83] A. G. Aronov, V. E. Kravtsov, and I. V. Lerner, “Spectral correlations in disordered electronic systems: crossover from metal to insulator regime”, Phys. Rev. Lett. **74**, 1174 (1995).

[84] H. Alt, H.-D. Gräf, T. Guhr, H. L. Harney, R. Hofferbert, H. Rehfeld, A. Richter, and P. Schardt, “Correlation-hole method for the spectra of superconducting microwave billiards”, Phys. Rev. E **55**, 6674 (1997).

[85] E. Brézin and S. Hikami, “Spectral form factor in a random matrix theory”, Phys. Rev. E **55**, 4067 (1997).

[86] R. E. Prange, “The spectral form factor is not self-averaging”, Phys. Rev. Lett. **78**, 2280 (1997).

[87] T. Guhr, A. Müller–Groeling, and H. A. Weidenmüller, “Random-matrix theories in quantum physics: common concepts”, Physics Reports **299**, 189 (1998).

[88] A. del Campo, J. Molina-Vilaplana, and J. Sonner, “Scrambling the spectral form factor: unitarity constraints and exact results”, Phys. Rev. D **95**, 126008 (2017).

[89] J. S. Cotler, G. Gur-Ari, M. Hanada, J. Polchinski, P. Saad, S. H. Shenker, D. Stanford, A. Streicher, and M. Tezuka, “Black holes and random matrices”, J. High Energ. Phys., 118 (2017).

[90] J. Cotler, N. Hunter-Jones, J. Liu, and B. Yoshida, “Chaos, complexity, and random matrices”, Journal of High Energy Physics **2017**, 1 (2017).

[91] J. Liu, “Spectral form factors and late time quantum chaos”, Phys. Rev. D **98**, 086026 (2018).

[92] M. Winer, S.-K. Jian, and B. Swingle, “Exponential ramp in the quadratic sachdev-ye-kitaev model”, Phys. Rev. Lett. **125**, 250602 (2020).

[93] Y. Liao, A. Vikram, and V. Galitski, “Many-body level statistics of single-particle quantum chaos”, Phys. Rev. Lett. **125**, 250601 (2020).

[94] P. Sierant, D. Delande, and J. Zakrzewski, "Thouless time analysis of anderson and many-body localization transitions", Phys. Rev. Lett. **124**, 186601 (2020).

[95] L. K. Joshi, A. Elben, A. Vikram, B. Vermersch, V. Galitski, and P. Zoller, "Probing many-body quantum chaos with quantum simulators", Phys. Rev. X **12**, 011018 (2022).

[96] G. Cipolloni, L. ErdHos, and D. Schröder, "On the spectral form factor for random matrices", Communications in Mathematical Physics **401**, 1665 (2023).

[97] B. Bertini, P. Kos, and T. Prosen, "Exact spectral form factor in a minimal model of many-body quantum chaos", Phys. Rev. Lett. **121**, 264101 (2018).

[98] A. Chan, A. De Luca, and J. T. Chalker, "Spectral statistics in spatially extended chaotic quantum many-body systems", Phys. Rev. Lett. **121**, 060601 (2018).

[99] A. Sarkar, S. Pachhal, A. Agarwala, and D. Das, "Spectral form factors of topological phases", Phys. Rev. B **109**, 155126 (2024).

[100] C.-K. Chiu, J. C. Y. Teo, A. P. Schnyder, and S. Ryu, "Classification of topological quantum matter with symmetries", Rev. Mod. Phys. **88**, 035005 (2016).

[101] M. Tovmasyan, S. Peotta, P. Törmä, and S. D. Huber, "Effective theory and emergent SU(2) symmetry in the flat bands of attractive hubbard models", Phys. Rev. B **94**, 245149 (2016).

[102] P. Törmä, L. Liang, and S. Peotta, "Quantum metric and effective mass of a two-body bound state in a flat band", Phys. Rev. B **98**, 220511 (2018).

[103] L. Landau and E. Lifshitz, "Chapter ix - identity of particles", in *Quantum mechanics (third edition)* (Pergamon, 1977), pp. 225–248.

[104] C. M. Bender and S. A. Orszag, "Asymptotic expansion of integrals", in *Advanced mathematical methods for scientists and engineers i: asymptotic methods and perturbation theory* (Springer New York, 1999), pp. 247–316.

[105] P. Coleman, "Finite-temperature many-body physics", in *Introduction to many-body physics* (Cambridge University Press, 2015), pp. 234–291.

[106] N. Sedlmayr, M. Fleischhauer, and J. Sirker, "Fate of dynamical phase transitions at finite temperatures and in open systems", Phys. Rev. B **97**, 045147 (2018).

[107] M. Sigrist, "Introduction to unconventional superconductivity", AIP Conference Proceedings **789**, 165 (2005).

[108] J. Davis, M. Kumari, R. B. Mann, and S. Ghose, "Wigner negativity in spin-j systems", Phys. Rev. Research **3**, 033134 (2021).

Appendix A

DQPTs in a class DIII superconductor

A.1 Hamiltonian and equations of motion

We start with the BdG (Bogoliubov-de Gennes) Hamiltonian for the B phase of superfluid He-3 in three spatial dimensions [8]. This is an example of a class DIII superconductor [68]. This is a translation-invariant Hamiltonian, therefore the momentum $\mathbf{k}$ is a good quantum number. For each momentum mode $\mathbf{k}$, we use the basis $\hat{C}_{\mathbf{k}} = (\hat{c}_{\mathbf{k}\uparrow}, \hat{c}_{\mathbf{k}\downarrow}, \hat{c}^\dagger_{-\mathbf{k}\uparrow}, \hat{c}^\dagger_{-\mathbf{k}\downarrow})^T$, where $\hat{c}_{\mathbf{k}\sigma}$ and $\hat{c}^\dagger_{\mathbf{k}\sigma}$ are respectively the annihilation and creation operators of the fermion with momentum $\mathbf{k}$ and spin σ. In this basis, the Hamiltonian is written as

$$\hat{H} = \sum_{\mathbf{k}} \hat{C}^\dagger_{\mathbf{k}} H_{\mathbf{k}} \hat{C}_{\mathbf{k}} \ , \tag{A.1}$$

$$H_{\mathbf{k}} = \begin{pmatrix} \xi_k \sigma_0 & \Delta_{\mathbf{k}} \\ \Delta^\dagger_{\mathbf{k}} & -\xi_k \sigma_0 \end{pmatrix} \ , \tag{A.2}$$

and

$$\xi_k = \frac{k^2}{2m} - \mu \ , \ \Delta_{\mathbf{k}} = i\Delta\sigma_2 \mathbf{k}\cdot\boldsymbol{\sigma} \ . \tag{A.3}$$

Here, Δ is the gap function. For later convenience, we write the Hamiltonian in expanded form,

$$\begin{aligned} \hat{H} = &\sum_{\mathbf{k}} \xi_k [\hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}_{\mathbf{k}\uparrow} + \hat{c}^\dagger_{\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\downarrow} - \hat{c}_{-\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\uparrow} - \hat{c}_{-\mathbf{k}\downarrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow}] \\ &+ \Delta \sum_{\mathbf{k}} [(k_x + ik_y)\hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\uparrow} - k_z \hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow} - k_z \hat{c}^\dagger_{\mathbf{k}\downarrow}\hat{c}^\dagger_{-\mathbf{k}\uparrow} - (k_x - ik_y)\hat{c}^\dagger_{\mathbf{k}\downarrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow}] \\ &+ \bar{\Delta} \sum_{\mathbf{k}} [(k_x - ik_y)\hat{c}_{-\mathbf{k}\uparrow}\hat{c}_{\mathbf{k}\uparrow} - k_z \hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\uparrow} - k_z \hat{c}_{-\mathbf{k}\uparrow}\hat{c}_{\mathbf{k}\downarrow} - (k_x + ik_y)\hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\downarrow}] \ . \end{aligned} \tag{A.4}$$

The summation is only over momentum modes with $k_z > 0$ so as to avoid double-counting.

We write a BCS state as

$$|\Psi\rangle = \prod_{\mathbf{k}} |\Psi_{\mathbf{k}}\rangle \ , \quad \text{where}$$
$$|\Psi_{\mathbf{k}}\rangle = (u_{\mathbf{k}} + v_{\mathbf{k}\uparrow\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow} + v_{\mathbf{k}\uparrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow} + v_{\mathbf{k}\downarrow\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow} + v_{\mathbf{k}\downarrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow} + v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow})\,|0\rangle \ . \tag{A.5}$$

The coefficients obey the normalization,

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}\uparrow\uparrow}|^2 + |v_{\mathbf{k}\uparrow\downarrow}|^2 + |v_{\mathbf{k}\downarrow\uparrow}|^2 + |v_{\mathbf{k}\downarrow\downarrow}|^2 + |v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}|^2 = 1 \ . \tag{A.6}$$

We wish to compute the action of the Hamiltonian on $|\Psi_{\mathbf{k}}\rangle$. To do this, let us list the action of each term of the Hamiltonian on $|\Psi_{\mathbf{k}}\rangle$.

$$\begin{aligned}
\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}_{\mathbf{k}\uparrow}|\Psi_{\mathbf{k}}\rangle &= (v_{\mathbf{k}\uparrow\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow} + v_{\mathbf{k}\uparrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow} + v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow})\,|0\rangle \\
\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\downarrow}|\Psi_{\mathbf{k}}\rangle &= (v_{\mathbf{k}\downarrow\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow} + v_{\mathbf{k}\downarrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow} + v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow})\,|0\rangle \\
\hat{c}_{-\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow}|\Psi_{\mathbf{k}}\rangle &= (u_{\mathbf{k}} + v_{\mathbf{k}\uparrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow} + v_{\mathbf{k}\downarrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow})\,|0\rangle \\
\hat{c}_{-\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow}|\Psi_{\mathbf{k}}\rangle &= (u_{\mathbf{k}} + v_{\mathbf{k}\uparrow\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow} + v_{\mathbf{k}\downarrow\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow})\,|0\rangle \\
\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow}|\Psi_{\mathbf{k}}\rangle &= (u_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow} - v_{\mathbf{k}\downarrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow})\,|0\rangle \\
\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow}|\Psi_{\mathbf{k}}\rangle &= (u_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow} + v_{\mathbf{k}\downarrow\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow})\,|0\rangle \\
\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow}|\Psi_{\mathbf{k}}\rangle &= (u_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow} + v_{\mathbf{k}\uparrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow})\,|0\rangle \\
\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow}|\Psi_{\mathbf{k}}\rangle &= (u_{\mathbf{k}}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow} - v_{\mathbf{k}\uparrow\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow})\,|0\rangle \\
\hat{c}_{-\mathbf{k}\uparrow}\hat{c}_{\mathbf{k}\uparrow}|\Psi_{\mathbf{k}}\rangle &= (v_{\mathbf{k}\uparrow\uparrow} - v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow})\,|0\rangle \\
\hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\uparrow}|\Psi_{\mathbf{k}}\rangle &= (v_{\mathbf{k}\uparrow\downarrow} + v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\downarrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow})\,|0\rangle \\
\hat{c}_{-\mathbf{k}\uparrow}\hat{c}_{\mathbf{k}\downarrow}|\Psi_{\mathbf{k}}\rangle &= (v_{\mathbf{k}\downarrow\uparrow} + v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\downarrow})\,|0\rangle \\
\hat{c}_{-\mathbf{k}\downarrow}\hat{c}_{\mathbf{k}\downarrow}|\Psi_{\mathbf{k}}\rangle &= (v_{\mathbf{k}\downarrow\downarrow} - v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\hat{c}^{\dagger}_{\mathbf{k}\uparrow}\hat{c}^{\dagger}_{-\mathbf{k}\uparrow})\,|0\rangle \ .
\end{aligned} \tag{A.7}$$

Using these, we get

$$\begin{aligned}
\hat{H}_{\mathbf{k}}\left|\Psi_{\mathbf{k}}\right\rangle = & [2\xi_k(-u_{\mathbf{k}} + v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{\mathbf{k}\downarrow}\hat{c}^\dagger_{-\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow}) \\
& + \Delta[u_{\mathbf{k}}((k_x+ik_y)\hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\uparrow} - k_z\hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow} - k_z\hat{c}^\dagger_{\mathbf{k}\downarrow}\hat{c}^\dagger_{-\mathbf{k}\uparrow} - (k_x-ik_y)\hat{c}^\dagger_{\mathbf{k}\downarrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow}) \\
& + (-(k_x+ik_y)v_{\mathbf{k}\downarrow\downarrow} - k_z v_{\mathbf{k}\downarrow\uparrow} - k_z v_{\mathbf{k}\uparrow\downarrow} + (k_x-ik_y)v_{\mathbf{k}\uparrow\uparrow})\hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{\mathbf{k}\downarrow}\hat{c}^\dagger_{-\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow}] \\
& + \bar{\Delta}[((k_x-ik_y)v_{\mathbf{k}\uparrow\uparrow} - k_z v_{\mathbf{k}\uparrow\downarrow} - k_z v_{\mathbf{k}\downarrow\uparrow} - (k_x+ik_y)v_{\mathbf{k}\downarrow\downarrow}) \\
& + v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}((k_x+ik_y)\hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\uparrow} - k_z\hat{c}^\dagger_{\mathbf{k}\uparrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow} - k_z\hat{c}^\dagger_{\mathbf{k}\downarrow}\hat{c}^\dagger_{-\mathbf{k}\uparrow} - (k_x-ik_y)\hat{c}^\dagger_{\mathbf{k}\downarrow}\hat{c}^\dagger_{-\mathbf{k}\downarrow})]]\left|0\right\rangle \ .
\end{aligned} \tag{A.8}$$

Now we obtain the equations of motion for the u and v amplitudes using $i\partial\left|\Psi_{\mathbf{k}}\right\rangle/\partial t = \hat{H}_{\mathbf{k}}\left|\Psi_{\mathbf{k}}\right\rangle$ as

$$\begin{aligned}
i\dot{u}_{\mathbf{k}} &= -2\xi_{\mathbf{k}}u_{\mathbf{k}} + \bar{\Delta}((k_x-ik_y)v_{\mathbf{k}\uparrow\uparrow} - k_z v_{\mathbf{k}\uparrow\downarrow} - k_z v_{\mathbf{k}\downarrow\uparrow} - (k_x+ik_y)v_{\mathbf{k}\downarrow\downarrow}) \\
i\dot{v}_{\mathbf{k}\uparrow\uparrow} &= \Delta(k_x+ik_y)u_{\mathbf{k}} + \bar{\Delta}(k_x+ik_y)v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow} \\
i\dot{v}_{\mathbf{k}\uparrow\downarrow} &= -\Delta k_z u_{\mathbf{k}} - \bar{\Delta}k_z v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow} \\
i\dot{v}_{\mathbf{k}\downarrow\uparrow} &= -\Delta k_z u_{\mathbf{k}} - \bar{\Delta}k_z v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow} \\
i\dot{v}_{\mathbf{k}\downarrow\downarrow} &= -\Delta(k_x-ik_y)u_{\mathbf{k}} - \bar{\Delta}(k_x-ik_y)v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow} \\
i\dot{v}_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow} &= 2\xi_{\mathbf{k}}v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow} + \Delta((k_x-ik_y)v_{\mathbf{k}\uparrow\uparrow} - k_z v_{\mathbf{k}\uparrow\downarrow} - k_z v_{\mathbf{k}\downarrow\uparrow} - (k_x+ik_y)v_{\mathbf{k}\downarrow\downarrow}) \ .
\end{aligned} \tag{A.9}$$

As $\mathbf{k} \to 0$, $\Delta_{\mathbf{k}}$ and $\bar{\Delta}_{\mathbf{k}}$ vanish (A.3). The above equations simplify to

$$i\dot{u}_0 = -2\xi_0 u_0 \ , \ \dot{v}_{0\uparrow\uparrow} = \dot{v}_{0\uparrow\downarrow} = \dot{v}_{0\downarrow\uparrow} = \dot{v}_{0\downarrow\downarrow} = 0 \ , \ i\dot{v}_{0\uparrow\downarrow\uparrow\downarrow} = 2\xi_0 v_{0\uparrow\downarrow\uparrow\downarrow}. \tag{A.10}$$

A.2 Condition for DQPT

We write $u(\tau)$, $v(\tau)$ as the coefficients at some time τ. The initial value of the coefficients are denoted as $v_{i\mathbf{k}} = v_{\mathbf{k}}(0)$. Using these, the Loschmidt echo is written as

$$\mathcal{Z} = \prod_{\mathbf{k}} 2S_{\mathbf{k}} \quad , \quad \ln\mathcal{Z} = V\int d^3k \ln(2S_{\mathbf{k}}) \ , \tag{A.11}$$

where

$$2S_{\mathbf{k}} = u^*_{i\mathbf{k}} u_{\mathbf{k}}(t) + v^*_{i\mathbf{k}\uparrow\uparrow} v_{\mathbf{k}\uparrow\uparrow}(t) + v^*_{i\mathbf{k}\uparrow\downarrow} v_{\mathbf{k}\uparrow\downarrow}(t) + v^*_{i\mathbf{k}\downarrow\uparrow} v_{\mathbf{k}\downarrow\uparrow}(t) + v^*_{i\mathbf{k}\downarrow\downarrow} v_{\mathbf{k}\downarrow\downarrow}(t) + v^*_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow} v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}(t) \ . \quad (A.12)$$

From the time evolution equations for $\mathbf{k} \to 0$ given in Eq. (A.10), we get for $2S_0$,

$$2S_0(t) = |u_{i0}|^2 e^{2i\xi_0 t} + |v_{i0\uparrow\uparrow}|^2 + |v_{i0\uparrow\downarrow}|^2 + |v_{i0\downarrow\uparrow}|^2 + |v_{i0\downarrow\downarrow}|^2 + |v_{i0\uparrow\downarrow\uparrow\downarrow}|^2 e^{-2i\xi_0 t} \ . \quad (A.13)$$

This can be simplified using the normalization condition Eq. (A.6) to

$$2S_0(t) = 1 + (|u_{i0}|^2 + |v_{i0\uparrow\downarrow\uparrow\downarrow}|^2)(\cos(2\xi_0 t) - 1) + i(|u_{i0}|^2 - |v_{i0\uparrow\downarrow\uparrow\downarrow}|^2)\sin(2\xi_0 t) \ . \quad (A.14)$$

There are two cases in which $S_0(t)$ can vanish.

Case 1: S_0 (t) vanishes at time

$$t_n = \frac{(2n+1)\pi}{2|\xi_0|} \ , \ \text{if } 1 - 2(|u_{i0}|^2 + |v_{i0\uparrow\downarrow\uparrow\downarrow}|^2) = 0 \ . \quad (A.15)$$

Case 2: If we have

$$|u_{i0}| = |v_{i0\uparrow\downarrow\uparrow\downarrow}| \ , \ \text{then } S_0(t) = 1 + 2|u_{i0}|^2(\cos(2\xi_0 t) - 1) \ . \quad (A.16)$$

$S_0(t)$ can vanish in (A.16) if we have $|u_{i0}|^2 > 1/4$.

A.3 Initial values of amplitudes

Now we wish to compute the coefficients at time $\tau = 0$. At $\tau = 0$, the state $|\Psi_{\mathbf{k}}\rangle$ is the ground state of the Hamiltonian (A.2) with a different value of μ, the value before the quench. For convenience, we follow the derivation in Ref. [107]. $\hat{H}$ can be diagonalized as

$$\hat{H} = \sum_{\mathbf{k}} \hat{A}^\dagger_{\mathbf{k}} \mathcal{E}_{\mathbf{k}} \hat{A}^\dagger_{\mathbf{k}} \ , \tag{A.17}$$

where

$$\hat{A}_{\mathbf{k}} = (\hat{a}_{\mathbf{k}\uparrow}, \hat{a}_{\mathbf{k}\downarrow}, \hat{a}^\dagger_{\mathbf{k}\uparrow}, \hat{a}^\dagger_{-\mathbf{k}\downarrow})^T \quad , \quad \mathcal{E}_{\mathbf{k}} = \mathrm{diag}(E_{\mathbf{k}}, E_{\mathbf{k}}, -E_{-\mathbf{k}}, -E_{-\mathbf{k}}) \ . \tag{A.18}$$

The eigenvalue $E_{\mathbf{k}}$ is given by

$$E_{\mathbf{k}} = \sqrt{\xi_k^2 + |\Delta_{\mathbf{k}}|^2} \quad , \quad \text{where } |\Delta_{\mathbf{k}}|^2 = \frac{1}{2}\mathrm{Tr}(\Delta^\dagger_{\mathbf{k}}\Delta_{\mathbf{k}}) \ . \tag{A.19}$$

In the ground state we have $\bar{\Delta} = \Delta^*$. In this problem, for $\Delta_{\mathbf{k}}$ given by Eq.(A.3), $E_{\mathbf{k}}$ becomes

$$E_{\mathbf{k}} = \sqrt{\xi_k^2 + |\Delta|^2 k^2} = E_k \ . \tag{A.20}$$

We get the eigenvector $\hat{A}_{\mathbf{k}}$ as $\hat{C}_{\mathbf{k}} = \mathcal{U}_{\mathbf{k}}\hat{A}_{\mathbf{k}}$ or $\hat{A}_{\mathbf{k}} = \mathcal{U}^\dagger_{\mathbf{k}}\hat{C}_{\mathbf{k}}$, where the unitary matrix $\mathcal{U}_{\mathbf{k}}$ is given by

$$\mathcal{U}_{\mathbf{k}} = \begin{pmatrix} U_{\mathbf{k}} & V_{\mathbf{k}} \\ V^*_{-\mathbf{k}} & U^*_{-\mathbf{k}} \end{pmatrix} \ , \text{ where } \quad U_{\mathbf{k}} = \frac{(E_k + \xi_k)\sigma_0}{\sqrt{2E_k(E_k+\xi_k)}} \ , \ V_{\mathbf{k}} = \frac{-\Delta_{\mathbf{k}}}{\sqrt{2E_k(E_k+\xi_k)}} \ . \tag{A.21}$$

Thus we get

$$\begin{pmatrix} \hat{a}_{\mathbf{k}\uparrow} \\ \hat{a}_{\mathbf{k}\downarrow} \\ \hat{a}^\dagger_{-\mathbf{k}\uparrow} \\ \hat{a}^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} = \frac{1}{\sqrt{2E_k(E_k+\xi_k)}} \begin{pmatrix} E_k+\xi_k & 0 & \Delta(k_x+ik_y) & -\Delta k_z \\ 0 & E_k+\xi_k & -\Delta k_z & -\Delta(k_x-ik_y) \\ -\bar{\Delta}(k_x-ik_y) & \bar{\Delta}k_z & E_k-\xi_k & 0 \\ \bar{\Delta}k_z & \bar{\Delta}(k_x+ik_y) & 0 & E_k-\xi_k \end{pmatrix} \begin{pmatrix} \hat{c}_{\mathbf{k}\uparrow} \\ \hat{c}_{\mathbf{k}\downarrow} \\ \hat{c}^\dagger_{-\mathbf{k}\uparrow} \\ \hat{c}^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} \tag{A.22}$$

In the ground state of the Hamiltonian, $|\Psi_i\rangle$, which is the state at $\tau = 0$, we have

$$\hat{a}_{\mathbf{k}\uparrow}|\Psi_{i\mathbf{k}}\rangle = 0 \quad , \quad \hat{a}_{\mathbf{k}\downarrow}|\Psi_{i\mathbf{k}}\rangle = 0 \ . \tag{A.23}$$

This leads to the conditions,

$$
\begin{aligned}
&((E_{ik}+\xi_{ik})\hat{c}_{\mathbf{k}\uparrow}+\Delta_i(k_x+ik_y)\hat{c}^\dagger_{-\mathbf{k}\uparrow}-\Delta_i k_z\hat{c}^\dagger_{-\mathbf{k}\downarrow})\,|\Psi_{i\mathbf{k}}\rangle=0\\
&((E_{ik}+\xi_{ik})\hat{c}_{\mathbf{k}\downarrow}-\Delta_i k_z\hat{c}^\dagger_{-\mathbf{k}\uparrow}-\Delta_i(k_x-ik_y)\hat{c}^\dagger_{-\mathbf{k}\downarrow})\,|\Psi_{i\mathbf{k}}\rangle=0\ .
\end{aligned}
\tag{A.24}
$$

With the form of $|\Psi_{ik}\rangle$ as given by Eq. (A.5), we get the following equations for the BCS coefficients,

$$
\begin{aligned}
&(E_{ik}+\xi_{ik})v_{i\mathbf{k}\uparrow\uparrow}+\Delta_i(k_x+ik_y)u_{i\mathbf{k}}=0\\
&(E_{ik}+\xi_{ik})v_{i\mathbf{k}\uparrow\downarrow}-\Delta_i k_z u_{i\mathbf{k}}=0\\
&(E_{ik}+\xi_{ik})v_{i\mathbf{k}\downarrow\uparrow}-\Delta_i k_z u_{i\mathbf{k}}=0\\
&(E_{ik}+\xi_{ik})v_{i\mathbf{k}\downarrow\downarrow}-\Delta_i(k_x-ik_y)u_{i\mathbf{k}}=0\\
&\Delta_i[(k_x+ik_y)v_{i\mathbf{k}\uparrow\downarrow}+k_z v_{i\mathbf{k}\uparrow\uparrow}]=0\\
&\Delta_i[-k_z v_{i\mathbf{k}\downarrow\downarrow}+(k_x-ik_y)v_{i\mathbf{k}\downarrow\uparrow}]=0\\
&(E_{ik}+\xi_{ik})v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}-\Delta_i(k_x+ik_y)v_{i\mathbf{k}\downarrow\downarrow}-\Delta_i k_z v_{i\mathbf{k}\downarrow\uparrow}=0\\
&(E_{ik}+\xi_{ik})v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}+\Delta_i(k_x-ik_y)v_{i\mathbf{k}\uparrow\uparrow}-\Delta_i k_z v_{i\mathbf{k}\uparrow\downarrow}=0\ .
\end{aligned}
\tag{A.25}
$$

Solving these equations with the normalization condition (A.6), we obtain

$$
\begin{aligned}
u_{i\mathbf{k}}&=\frac{E_{ik}+\xi_{ik}}{2E_{ik}}\quad,\quad v_{i\mathbf{k}\uparrow\uparrow}=-\frac{\Delta_i(k_x+ik_y)}{2E_{ik}}\quad,\quad v_{i\mathbf{k}\uparrow\downarrow}=v_{i\mathbf{k}\downarrow\uparrow}=\frac{\Delta_i k_z}{2E_{ik}}\ ,\\
v_{i\mathbf{k}\downarrow\downarrow}&=\frac{\Delta_i(k_x-ik_y)}{2E_{ik}}\quad,\quad v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}=\frac{\Delta_i^2k^2}{2E_{ik}(E_{ik}+\xi_{ik})}\ .
\end{aligned}
\tag{A.26}
$$

To evaluate the cases (A.15) and (A.16), we need to see what happens to the above expressions in the limit $k\to 0$.

$$
\text{If }\mu>0,\ k\to 0\ :\ \text{then }u_{i\mathbf{k}}=\frac{k^2|\Delta_i|^2}{4\mu^2}\ ,\ v_{i\mathbf{k}\sigma\sigma'}\sim\frac{k\Delta_i}{\mu}\ ,\ v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}=1\ . \tag{A.27}
$$

$$
\text{If }\mu<0,\ k\to 0\ :\ \text{then }u_{i\mathbf{k}}=1\ ,\ v_{i\mathbf{k}\sigma\sigma'}\sim\frac{k\Delta_i}{\mu}\ ,\ v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}=\frac{k^2\Delta_i^2}{4\mu^2}\ . \tag{A.28}
$$

$$\text{If } \mu = 0,\ k \to 0 \ :\ \text{then } u_{i\mathbf{k}} = \frac{1}{2} + \frac{k}{4m|\Delta_i|} \ , \ v_{i\mathbf{k}\uparrow\uparrow} = -\frac{(k_x + ik_y)\Delta_i}{2k|\Delta_i|} \ ,$$
$$v_{i\mathbf{k}\uparrow\downarrow} = v_{i\mathbf{k}\downarrow\uparrow} = \frac{k_z \Delta_i}{2k|\Delta_i|} \ , \ v_{i\mathbf{k}\downarrow\downarrow} = \frac{(k_x - ik_y)\Delta_i}{2k|\Delta_i|} \ , \ v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow} = \frac{\Delta_i^2}{|\Delta_i|^2}\left(\frac{1}{2} - \frac{k}{4m|\Delta_i|}\right) \ . \tag{A.29}$$

From Eqs. (A.15) and (A.16), we see that only the $\mu = 0$ case can have $S_0(t) = 0$ for some t.

We need to look at the singularity in $\ln \mathcal{Z}$ as we integrate around the $k = 0$ point in momentum space. To do this, let us expand the initial coefficients (A.29) to one further order in k.

$$\begin{aligned} u_{i\mathbf{k}} &= \frac{1}{2} + \frac{k}{4m|\Delta_i|} + \mathcal{O}(k^3) \ , \\ v_{i\mathbf{k}\uparrow\uparrow} &= -\frac{(k_x + ik_y)\Delta_i}{2k|\Delta_i|}\left(1 - \frac{k^2}{8m^2|\Delta_i|^2}\right) \ , \\ v_{i\mathbf{k}\uparrow\downarrow} = v_{i\mathbf{k}\downarrow\uparrow} &= \frac{k_z\Delta_i}{2k|\Delta_i|}\left(1 - \frac{k^2}{8m^2|\Delta_i|^2}\right) \ , \\ v_{i\mathbf{k}\downarrow\downarrow} &= \frac{(k_x - ik_y)\Delta_i}{2k|\Delta_i|}\left(1 - \frac{k^2}{8m^2|\Delta_i|^2}\right) \ , \\ v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow} &= \frac{\Delta_i^2}{|\Delta_i|^2}\left(\frac{1}{2} - \frac{k}{4m|\Delta_i|} + \mathcal{O}(k^3)\right) \ . \end{aligned} \tag{A.30}$$

A.4 Search for DQPTs

Now we use the time evolution equations (A.9) to get the coefficients at time t upto first order in k. Let us define

$$f_{\mathbf{k}} = \int_0^t d\tau\, \Delta(\tau)\, e^{2i\xi_k\tau} \quad , \quad \bar{f}_{\mathbf{k}} = \int_0^t d\tau\, \bar{\Delta}(\tau)\, e^{-2i\xi_k\tau} \ . \tag{A.31}$$

Up to first order in k, we get

$$
\begin{aligned}
u_{\mathbf{k}}(t) &= (u_{i\mathbf{k}} + i(-(k_x - ik_y)v_{i\mathbf{k}\uparrow\uparrow} + k_z(v_{i\mathbf{k}\uparrow\downarrow} + v_{i\mathbf{k}\downarrow\uparrow}) + (k_x + ik_y)v_{i\mathbf{k}\downarrow\downarrow})\bar{f}_{\mathbf{k}})\, e^{2i\xi_k t} \ , \\
v_{\mathbf{k}\uparrow\uparrow}(t) &= v_{i\mathbf{k}\uparrow\uparrow} - i(k_x + ik_y)(u_{i\mathbf{k}} f_{\mathbf{k}} + v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\bar{f}_{\mathbf{k}}) \ , \\
v_{\mathbf{k}\uparrow\downarrow}(t) &= v_{i\mathbf{k}\uparrow\downarrow} + ik_z(u_{i\mathbf{k}} f_{\mathbf{k}} + v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\bar{f}_{\mathbf{k}}) \ , \\
v_{\mathbf{k}\downarrow\uparrow}(t) &= v_{i\mathbf{k}\downarrow\uparrow} + ik_z(u_{i\mathbf{k}} f_{\mathbf{k}} + v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\bar{f}_{\mathbf{k}}) \ , \\
v_{\mathbf{k}\downarrow\downarrow}(t) &= v_{i\mathbf{k}\downarrow\downarrow} + i(k_x - ik_y)(u_{i\mathbf{k}} f_{\mathbf{k}} + v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\bar{f}_{\mathbf{k}}) \ , \\
v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}(t) &= (v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow} i(-(k_x - ik_y)v_{i\mathbf{k}\uparrow\uparrow} + k_z(v_{i\mathbf{k}\uparrow\downarrow} + v_{i\mathbf{k}\downarrow\uparrow}) + (k_x + ik_y)v_{i\mathbf{k}\downarrow\downarrow})f_{\mathbf{k}})\, e^{-2i\xi_k t} \ .
\end{aligned}
\tag{A.32}
$$

Substituting these in Eq. (A.12) and using Eq. (A.30) for the initial values of the coefficients, we get $S_{\mathbf{k}}(t)$ for small k upto first order in k as

$$
\begin{aligned}
S_{\mathbf{k}}(t) = &\left(\frac{1}{4} + \frac{k}{4m|\Delta_i|} + \frac{ik}{2}\frac{\Delta_i}{|\Delta_i|}\bar{f}_{\mathbf{k}}\right) e^{2i\xi_0 t} + \frac{1}{2} + \\
&\left(\frac{1}{4} - \frac{k}{4m|\Delta_i|} + \frac{ik}{2}\frac{\Delta_i}{|\Delta_i|} f_{\mathbf{k}}\right) e^{-2i\xi_0 t} + ik\frac{\bar{\Delta}_i}{|\Delta_i|}\frac{f_{\mathbf{k}} + \bar{f}_{\mathbf{k}}}{2} \ .
\end{aligned}
\tag{A.33}
$$

Now for t close to a DQPT at t_n given by Eq. (A.15) (Eq. (A.16) also gives the same value for t_n), we can expand in powers of $(t - t_n)$:

$$
2i\xi_0 t = 2i\xi_0 t_n + 2i\xi_0(t - t_n) = i(2n+1)\pi\frac{\xi_0}{|\xi_0|} + 2i\xi_0(t - t_n) \ . \tag{A.34}
$$

To first order in k and $(t - t_n)$, Eq. (A.12) becomes

$$
2S_k(t) = -\left(\frac{1}{2} + ik\frac{\Delta_i}{|\Delta_i|}\frac{f_{\mathbf{k}} + \bar{f}_{\mathbf{k}}}{2}\right) + \frac{1}{2} + ik\frac{\bar{\Delta}_i}{|\Delta_i|}\frac{f_{\mathbf{k}} + \bar{f}_{\mathbf{k}}}{2} + \mathcal{O}((t - t_n)^2) \ . \tag{A.35}
$$

Since in the ground state the gap function Δ_i is real and positive (this can be shown using the self-consistency condition), and $\bar{\Delta}_i$ is just the complex conjugate of Δ_i, we get

$$
2S_{\mathbf{k}}(t) = 0 \ , \tag{A.36}
$$

up to first order in k and $(t-t_n)$. Thus to observe the behavior of $S_{\mathbf{k}}(t)$ near t_n, we should expand it to second order in k and $(t-t_n)$.

Let us generalize Eq. (A.31) to

$$f_{\mathbf{k}}(\tau)=\int_0^\tau d\tau'\Delta(\tau')\,e^{2i\xi_k\tau'} \quad , \quad \bar{f}_{\mathbf{k}}(\tau')=\int_0^\tau d\tau'\bar{\Delta}(\tau')\,e^{-2i\xi_k\tau'}\ . \tag{A.37}$$

In addition to these, to go up to second order in k we also need to define the following,

$$\begin{aligned}
f_{\Delta\mathbf{k}}(t)&=\int_0^t d\tau\Delta(\tau)f_{\mathbf{k}}(\tau)\,e^{2i\xi_k\tau} \quad , \quad f_{\bar{\Delta}\mathbf{k}}(t)=\int_0^t d\tau\bar{\Delta}(\tau)f_{\mathbf{k}}(\tau)\,e^{-2i\xi_k\tau}\ ,\\
\bar{f}_{\Delta\mathbf{k}}(t)&=\int_0^t d\tau\Delta(\tau)\bar{f}_{\mathbf{k}}(\tau)\,e^{2i\xi_k\tau} \quad , \quad \bar{f}_{\bar{\Delta}\mathbf{k}}(t)=\int_0^t d\tau\bar{\Delta}(\tau)\bar{f}_{\mathbf{k}}(\tau)\,e^{-2i\xi_{\mathbf{k}}\tau}\ ,\\
\mathcal{V}_{i\mathbf{k}}&=-(k_x-ik_y)v_{i\mathbf{k}\uparrow\uparrow}+k_z(v_{i\mathbf{k}\uparrow\downarrow}+v_{i\mathbf{k}\downarrow\uparrow})+(k_x+ik_y)v_{i\mathbf{k}\downarrow\downarrow}\ .
\end{aligned} \tag{A.38}$$

Note that since $v_{i\mathbf{k}\sigma\sigma'}$ are $\mathcal{O}(1)$ in k (from Eq. (A.29)), $\mathcal{V}_{i\mathbf{k}}$ is $\mathcal{O}(k)$. Using these we can extend Eq. (A.32) to second order in k as

$$\begin{aligned}
u_{\mathbf{k}}(t)&=(u_{i\mathbf{k}}+i\mathcal{V}_{i\mathbf{k}}\bar{f}_{\mathbf{k}}(t)-2k^2(u_{i\mathbf{k}}f_{\bar{\Delta}\mathbf{k}}(t)+v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\bar{f}_{\bar{\Delta}\mathbf{k}}(t)))\,e^{2i\xi_kt}\ ,\\
v_{\mathbf{k}\uparrow\uparrow}(t)&=v_{i\mathbf{k}\uparrow\uparrow}-i(k_x+ik_y)(u_{i\mathbf{k}}f_{\mathbf{k}}(t)+v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\bar{f}_{\mathbf{k}}(t))+(k_x+ik_y)\mathcal{V}_{i\mathbf{k}}(\bar{f}_{\Delta\mathbf{k}}(t)+f_{\bar{\Delta}\mathbf{k}}(t))\ ,\\
&\qquad[\text{Now since } v_{i\mathbf{k}\downarrow\uparrow}=v_{i\mathbf{k}\uparrow\downarrow}]\\
v_{\mathbf{k}\uparrow\downarrow}(t)&=v_{\mathbf{k}\downarrow\uparrow}(t)=v_{i\mathbf{k}\uparrow\downarrow}+ik_z(u_{i\mathbf{k}}f_{\mathbf{k}}(t)+v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\bar{f}_{\mathbf{k}}(t))-k_z\mathcal{V}_{i\mathbf{k}}(\bar{f}_{\Delta\mathbf{k}}(t)+f_{\bar{\Delta}\mathbf{k}}(t))\ ,\\
v_{\mathbf{k}\downarrow\downarrow}(t)&=v_{i\mathbf{k}\downarrow\downarrow}+i(k_x-ik_y)(u_{i\mathbf{k}}f_{\mathbf{k}}(t)+v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\bar{f}_{\mathbf{k}}(t))-(k_x-ik_y)\mathcal{V}_{i\mathbf{k}}(\bar{f}_{\Delta\mathbf{k}}(t)+f_{\bar{\Delta}\mathbf{k}}(t))\ ,\\
v_{\mathbf{k}\uparrow\downarrow\uparrow\downarrow}(t)&=(v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}+i\mathcal{V}_{i\mathbf{k}}f_{\mathbf{k}}(t)-2k^2(u_{i\mathbf{k}}f_{\Delta\mathbf{k}}(t)+v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\bar{f}_{\Delta\mathbf{k}}(t))\,e^{-2i\xi_kt}\ .
\end{aligned} \tag{A.39}$$

Using these for $S_{\mathbf{k}}(t)$ in Eq. (A.12), we obtain

$$\begin{aligned}2S_{\mathbf{k}}(t) \approx\ & 1 + (|u_{i\mathbf{k}}|^2 + |v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}|^2)(\cos(2\xi_k t) - 1) + i(|u_{i\mathbf{k}}|^2 - |v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}|^2)\sin(2\xi_k t) \\ & + i(u^*_{i\mathbf{k}}\bar{f}_{\mathbf{k}}(t)\, e^{2i\xi_k t} + v^*_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow} f_{\mathbf{k}}(t)\, e^{-2i\xi_k t}) \\ & - 2k^2[u^*_{i\mathbf{k}}(u_{i\mathbf{k}} f_{\bar{\Delta}\mathbf{k}}(t) + v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\bar{f}_{\bar{\Delta}\mathbf{k}}(t))\, e^{2i\xi_k t} + v^*_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}(u_{i\mathbf{k}} f_{\Delta\mathbf{k}}(t) + v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\bar{f}_{\Delta\mathbf{k}}(t))\, e^{-2i\xi_k t}] \\ & + i\mathcal{V}^*_{i\mathbf{k}}(u_{i\mathbf{k}} f_{\mathbf{k}}(t) + v_{i\mathbf{k}\uparrow\downarrow\uparrow\downarrow}\bar{f}_{\mathbf{k}}(t)) - |\mathcal{V}_{i\mathbf{k}}|^2(\bar{f}_{\Delta\mathbf{k}}(t) + f_{\bar{\Delta}\mathbf{k}}(t))\ .\end{aligned} \tag{A.40}$$

Now up to $\mathcal{O}(k^2)$, we have

$$\begin{aligned} e^{2i\xi_k t} &= e^{2i\xi_0 t + ik^2 t/m} \approx e^{2i\xi_0 t}\left(1 + i\frac{k^2}{m}t\right)\ , \\ f_{\mathbf{k}}(t) &= \int_0^t d\tau \Delta(\tau)\, e^{2i\xi_0\tau}\left(1 + i\frac{k^2}{m}\tau\right) \end{aligned} \tag{A.41}$$

In Eq. (A.40), since $f_{\mathbf{k}}$ occurs only at $\mathcal{O}(k)$ and $f_{\Delta\mathbf{k}}$ at $\mathcal{O}(k^2)$, we can approximate $f_{\mathbf{k}} \approx f_0$ and $f_{\Delta\mathbf{k}} \approx f_{\Delta 0}$. The same approximation holds for the other quantities in Eqs. (A.37) and (A.38).

Using the initial values of the coefficients (A.30), we see from Eq. (A.38) that

$$\mathcal{V}_{i\mathbf{k}} = k\frac{\Delta_i}{|\Delta_i|}\ . \tag{A.42}$$

Substituting this along with Eq. (A.30) in Eq. (A.40), we get for $S_{\mathbf{k}}(t)$,

$$\begin{aligned} 2S_{\mathbf{k}}(t) \approx\ & 1 + \frac{1}{2}\left(1 + \frac{k^2}{4m^2|\Delta_i|^2}\right)(\cos(2\xi_0 t) - 1) - \frac{k^2}{2m}t\sin(2\xi_0 t) + i\frac{k}{2m|\Delta_i|}\sin(2\xi_0 t) \\ & + \frac{i}{2}\left(\frac{\Delta_i}{|\Delta_i|}\left(k + \frac{k^2}{2m|\Delta_i|}\right)\bar{f}_0(t)\, e^{2i\xi_0 t} + \frac{\bar{\Delta}_i}{|\Delta_i|}\left(k - \frac{k^2}{2m|\Delta_i|}\right) f_0(t)\, e^{-2i\xi_0 t}\right) \\ & - \frac{k^2}{2}\left(f_{\bar{\Delta}0}(t) + \frac{\Delta_i^2}{|\Delta_i|^2}\bar{f}_{\bar{\Delta}0}(t) + \frac{\bar{\Delta}_i^2}{|\Delta_i|^2} f_{\Delta 0}(t) + \bar{f}_{\Delta 0}(t)\right) - k^2(\bar{f}_{\Delta 0}(t) + f_{\bar{\Delta}0}(t)) \\ & + \frac{i}{2}\left(\frac{\bar{\Delta}_i}{|\Delta_i|}\left(k + \frac{k^2}{2m|\Delta_i|}\right) f_0(t) + \frac{\Delta_i}{|\Delta_i|}\left(k - \frac{k^2}{2m|\Delta_i|}\right)\bar{f}_0(t)\right) + \mathcal{O}(k^3)\ . \end{aligned} \tag{A.43}$$

Now we expand the terms in t to second order in $(t - t_n)$,

$$\begin{aligned} \cos(2\xi_0 t) &\approx -1 + 2\xi_0^2 (t-t_n)^2 \ , \ \sin(2\xi_0 t) \approx -2\xi_0 (t-t_n) \ , \\ e^{2i\xi_0 t} \approx -1 - 2i\xi_0(t-t_n) + 2\xi_0^2(t-t_n)^2 \ &, \ e^{-2i\xi_0 t} \approx -1 + 2i\xi_0(t-t_n) + 2\xi_0^2(t-t_n)^2 \ . \end{aligned} \tag{A.44}$$

Using these, $S_{\mathbf{k}}(t)$ becomes

$$\begin{aligned} 2S_{\mathbf{k}}(t) \approx \xi_0^2(t-t_n)^2 &+ ik\xi_0(t-t_n)\left[-\frac{1}{m|\Delta_i|} + \frac{i}{|\Delta_i|}(f_0(t)\bar{\Delta}_i - \bar{f}_0(t)\Delta_i)\right] \\ &+ k^2\Bigg[-\frac{1}{4m^2|\Delta_i|^2} - \frac{1}{2}\left(3f_{\bar{\Delta}0}(t) + \frac{\Delta_i^2}{|\Delta|_i^2}\bar{f}_{\bar{\Delta}0}(t) + \frac{\bar{\Delta}_i^2}{|\Delta_i|^2}f_{\Delta 0}(t) + 3\bar{f}_{\Delta 0}(t)\right) \\ &+ \frac{i}{2m|\Delta_i|}\left(f_0(t)\frac{\bar{\Delta}_i}{|\Delta_i|} - \bar{f}_0(t)\frac{\Delta_i}{|\Delta_i|}\right)\Bigg] \ . \end{aligned} \tag{A.45}$$

In the ground state of the Hamiltonian before quench, Δ_i is real and positive, so that $\Delta_i = \bar{\Delta}_i = |\Delta_i|$. Then up to $\mathcal{O}(k^2)$, $\mathcal{O}(k(t-t_n))$ and $\mathcal{O}((t-t_n)^2)$, $S_{\mathbf{k}}(t)$ simplifies to

$$2S_{\mathbf{k}}(t) \approx \xi_0^2(t-t_n)^2 - \beta k\,\xi_0(t-t_n) - \gamma k^2 \ , \tag{A.46}$$

where

$$\begin{aligned} \beta &= \frac{i}{m\Delta_i} + (f_0(t) - \bar{f}_0(t)) \ , \\ \gamma &= \frac{1}{4m^2\Delta_i^2} + \frac{1}{2}(3f_{\bar{\Delta}0}(t) + \bar{f}_{\bar{\Delta}0}(t) + f_{\Delta 0}(t) + 3\bar{f}_{\Delta 0}(t)) - \frac{i}{2m\Delta_i}(f_0(t) - \bar{f}_0(t)) \ . \end{aligned} \tag{A.47}$$

Differentiating the equation for $\ln \mathcal{Z}$ with time, we get

$$\begin{aligned} \frac{1}{V\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial t} &= \int d^3k \, \frac{1}{S_{\mathbf{k}}}\frac{\partial S_{\mathbf{k}}}{\partial t} \\ &= 4\pi \int_0^{k_0} dk \, \frac{2k^2\xi_0^2(t-t_n)/\gamma - \beta k^3 \xi_0/\gamma}{\xi_0^2(t-t_n)^2/\gamma - \beta k\,\xi_0(t-t_n)/\gamma - k^2} \ . \end{aligned} \tag{A.48}$$

Here k_0 is some momentum above which the expansion to $\mathcal{O}(k^2)$ is no longer sufficient. We are interested in the nonanalyticity that occurs due to the integration near $k = 0$. The poles of the integrand above are

$$k_{p_1/p_2} = \xi_0(t-t_n)\left[-\frac{\beta}{2\gamma} \pm \left(\frac{\beta^2}{4\gamma^2}+\frac{1}{\gamma}\right)^{1/2}\right] \, . \tag{A.49}$$

The integral in (A.48) can be written as

$$\frac{1}{V\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial t} = 4\pi \int_0^{k_0} dk \left(\frac{\beta\xi_0 k}{\gamma} - \xi_0^2(t-t_n)\left(\frac{\beta^2}{\gamma^2}+\frac{2}{\gamma}\right) + \frac{\rho_{p_1}}{k-k_{p_1}} + \frac{\rho_{p_2}}{k-k_{p_2}}\right) \, , \tag{A.50}$$

where

$$\rho_{p_1/p_2} = \pm\frac{\xi_0^3(t-t_n)^2}{k_{p_1}-k_{p_2}}\left[\frac{\beta k_{p_1/p_2}}{\gamma}\left(\frac{\beta^2}{\gamma^2}+\frac{3}{\gamma}\right) - \frac{\xi_0(t-t_n)}{\gamma}\left(\frac{\beta^2}{\gamma^2}+\frac{2}{\gamma}\right)\right] \, . \tag{A.51}$$

From Eqs. (A.49) and (A.51), we see that k_{p_1/p_2} is $\mathcal{O}(t-t_n)$ and ρ_{p_1/p_2} is $\mathcal{O}((t-t_n)^2)$. The singular part of the integral is given by

$$\frac{1}{V\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial t} = -4\pi\left(\rho_{p_1}\ln\left(\frac{k_{p_1}}{k_0}\right) + \rho_{p_2}\ln\left(\frac{k_{p_2}}{k_0}\right)\right) \tag{A.52}$$

Substituting in the appropriate expressions, we obtain

$$\frac{1}{V\mathcal{Z}}\frac{\partial \mathcal{Z}}{\partial t} \sim \xi_0^3(t-t_n)^2\ln\left(\frac{\xi_0(t-t_n)}{k_0}\right) \, . \tag{A.53}$$

This is the form of the singularities in the Loschmidt echo of the B-phase of superfluid He-3 in three spatial dimensions.

Appendix B

DQPTs in a class CI superconductor

B.1 Hamiltonian and equations of motion

We consider a diamond lattice in 3D with spin singlet d-wave pairing [69]. This is an example of a class CI topological superconductor. The Hamiltonian is given as $\hat{H} = \sum_{\mathbf{k}} \hat{\Psi}^{\dagger}_{\mathbf{k}} \mathcal{H}_{\mathbf{k}} \hat{\Psi}_{\mathbf{k}}$, with $\hat{\Psi}_{\mathbf{k}} = (\hat{a}_{\mathbf{k}\uparrow}, \hat{b}_{\mathbf{k}\uparrow}, \hat{a}^{\dagger}_{-\mathbf{k}\downarrow}, \hat{b}^{\dagger}_{-\mathbf{k}\downarrow})^T$, where $\hat{a}_{\mathbf{k}\alpha}$ and $\hat{b}_{\mathbf{k}\alpha}$ are the electron annihilation operators with spin α and momentum $\mathbf{k}$ on sublattice A and B of the diamond lattice, respectively. $\mathcal{H}_{\mathbf{k}}$ is given by

$$\mathcal{H}_{\mathbf{k}} = \begin{pmatrix} \Theta_{\mathbf{k}} & \Phi_{\mathbf{k}} & \Delta_{\mathbf{k}} & 0 \\ \Phi^*_{\mathbf{k}} & -\Theta_{\mathbf{k}} & 0 & \Delta_{\mathbf{k}} \\ \bar{\Delta}_{\mathbf{k}} & 0 & -\Theta_{\mathbf{k}} & -\Phi^*_{-\mathbf{k}} \\ 0 & \bar{\Delta}_{\mathbf{k}} & -\Phi_{-\mathbf{k}} & \Theta_{\mathbf{k}} \end{pmatrix} , \tag{B.1}$$

where $\Phi_{\mathbf{k}}$ is the nearest neighbor (NN) hopping term, $\Theta_{\mathbf{k}}$ is the next nearest neighbour (NNN) hopping term and $\Delta_{\mathbf{k}}$ is the pairing potential. A specific form of these which we use in our calculation is as follows [69]:

$$\Phi_{\mathbf{k}} = 2w \left[e^{ik_z/4} \cos\frac{k_x + k_y}{4} + e^{-ik_z/4} \cos\frac{k_x - k_y}{4} \right] , \tag{B.2}$$

$$\Delta_{\mathbf{k}} = 4\Delta \left[\cos\frac{k_x}{2} \cos\frac{k_y}{2} - \cos\frac{k_y}{2} \cos\frac{k_z}{2} - \cos\frac{k_z}{2} \cos\frac{k_x}{2} \right] , \tag{B.3}$$

$$\Theta_{\mathbf{k}} = 4w' \cos\frac{k_z}{2} \left(\cos\frac{k_y}{2} - \cos\frac{k_x}{2} \right) + \mu_s \ . \tag{B.4}$$

The energy eigenvalues of (B.1) are

$$E^{\pm}_{\mathbf{k}} = \pm\sqrt{|\Phi_{\mathbf{k}}|^2 + |\Theta_{\mathbf{k}}|^2 + |\Delta_{\mathbf{k}}|^2}\ , \tag{B.5}$$

exhibiting a twofold degeneracy for each $\mathbf{k}$. The system has time-reversal symmetry (TRS) giving a constraint on $\mathcal{H}_{\mathbf{k}}$,

$$\mathcal{H}^*_{-\mathbf{k}} = \mathcal{H}_{\mathbf{k}}\ . \tag{B.6}$$

With this we see that $\Theta_{\mathbf{k}}$ and $\Delta_{\mathbf{k}}$ are real-valued, and $\Phi_{\mathbf{k}}$ obeys $\Phi^*_{-\mathbf{k}} = \Phi_{\mathbf{k}}$. This simplifies $\mathcal{H}_{\mathbf{k}}$ to

$$\mathcal{H}_{\mathbf{k}} = \begin{pmatrix} \Theta_{\mathbf{k}} & \Phi_{\mathbf{k}} & \Delta_{\mathbf{k}} & 0 \\ \Phi^*_{\mathbf{k}} & -\Theta_{\mathbf{k}} & 0 & \Delta_{\mathbf{k}} \\ \bar{\Delta}_{\mathbf{k}} & 0 & -\Theta_{\mathbf{k}} & -\Phi_{\mathbf{k}} \\ 0 & \bar{\Delta}_{\mathbf{k}} & -\Phi^*_{\mathbf{k}} & \Theta_{\mathbf{k}} \end{pmatrix}\ , \tag{B.7}$$

We retain $\bar{\Delta}_{\mathbf{k}}$ instead of just $\Delta_{\mathbf{k}}$ because when we study the Loschmidt echo, $\bar{\Delta}_{\mathbf{k}}$ can turn out to be different from $\Delta_{\mathbf{k}}$ by the self-consistency condition.

When $\Theta_{\mathbf{k}} = 0$, the BCS dispersion given by $\Phi_{\mathbf{k}}$ and $\Delta_{\mathbf{k}}$ has four Dirac points where the bands are fourfold degenerate, $\mathbf{K}_{1,\pm} = 2\pi(\pm 1/3, 1, 0)$ and $\mathbf{K}_{2,\pm} = 2\pi(1, \pm 1/3, 0)$.

We can write a BCS state on this lattice as

$$|\Psi_{\rm BCS}\rangle = \prod_{\mathbf{k}} |\Psi_{\mathbf{k}}\rangle \text{ where}$$

$$|\Psi_{\mathbf{k}}\rangle = (u_{\mathbf{k}} + v_{\mathbf{k}aa}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}bb}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}ab}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}ba}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}aabb}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow})\,|0\rangle\ , \tag{B.8}$$

with the normalization,

$$|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}aa}|^2 + |v_{\mathbf{k}bb}|^2 + |v_{\mathbf{k}ab}|^2 + |v_{\mathbf{k}ba}|^2 + |v_{\mathbf{k}aabb}|^2 = 1\ . \tag{B.9}$$

We now calculate the action of each term of the Hamiltonian on the BCS state.

$$
\begin{aligned}
\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}_{\mathbf{k}\uparrow}|\Psi_\mathbf{k}\rangle &= (v_{\mathbf{k}aa}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}ab}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}aabb}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow})|0\rangle \ , \\
\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{b}_{\mathbf{k}\uparrow}|\Psi_\mathbf{k}\rangle &= (v_{\mathbf{k}bb}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}ba}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow})|0\rangle \ , \\
\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow}|\Psi_\mathbf{k}\rangle &= (u_\mathbf{k}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}bb}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow})|0\rangle \ , \\
\hat{a}_{-\mathbf{k}\downarrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow}|\Psi_\mathbf{k}\rangle &= (u_\mathbf{k} + v_{\mathbf{k}bb}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}ab}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow}) \ , \\
\hat{a}_{-\mathbf{k}\downarrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow}|\Psi_\mathbf{k}\rangle &= (-v_{\mathbf{k}aa}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow} - v_{\mathbf{k}ba}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow})|0\rangle \ , \\
\hat{a}_{-\mathbf{k}\downarrow}\hat{a}_{\mathbf{k}\uparrow}|\Psi_\mathbf{k}\rangle &= (v_{\mathbf{k}aa} + v_{\mathbf{k}aabb}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow})|0\rangle \ , \\
\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}_{\mathbf{k}\uparrow}|\Psi_\mathbf{k}\rangle &= (v_{\mathbf{k}bb}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}ba}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}aabb}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow})|0\rangle \ , \\
\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{a}_{\mathbf{k}\uparrow}|\Psi_\mathbf{k}\rangle &= (v_{\mathbf{k}aa}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}ab}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow})|0\rangle \ , \\
\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow}|\Psi_\mathbf{k}\rangle &= (u_\mathbf{k}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}aa}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow})|0\rangle \ , \\
\hat{b}_{-\mathbf{k}\downarrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow}|\Psi_\mathbf{k}\rangle &= (u_\mathbf{k} + v_{\mathbf{k}aa}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow} + v_{\mathbf{k}ba}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow})|0\rangle \ , \\
\hat{b}_{-\mathbf{k}\downarrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow}|\Psi_\mathbf{k}\rangle &= (-v_{\mathbf{k}bb}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow} - v_{\mathbf{k}ab}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow})|0\rangle \ , \\
\hat{b}_{-\mathbf{k}\downarrow}\hat{b}_{\mathbf{k}\uparrow}|\Psi_\mathbf{k}\rangle &= (v_{\mathbf{k}bb} + v_{\mathbf{k}aabb}\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow})|0\rangle \ .
\end{aligned}
\tag{B.10}
$$

Using these, we get

$$
\begin{aligned}
&\mathcal{H}_\mathbf{k}|\Psi_\mathbf{k}\rangle = [\bar{\Delta}_\mathbf{k}(v_{\mathbf{k}aa} + v_{\mathbf{k}bb}) \\
&+ (2\Theta_\mathbf{k}v_{\mathbf{k}aa} + (\Phi_\mathbf{k}v_{\mathbf{k}ba} + \Phi^*_\mathbf{k}v_{\mathbf{k}ab}) + (\Delta_\mathbf{k}u_\mathbf{k} + \bar{\Delta}_\mathbf{k}v_{\mathbf{k}aabb}))\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow} \\
&+ (-2\Theta_\mathbf{k}v_{\mathbf{k}bb} + (\Phi_\mathbf{k}v_{\mathbf{k}ba} + \Phi^*_\mathbf{k}v_{\mathbf{k}ab}) + (\Delta_\mathbf{k}u_\mathbf{k} + \bar{\Delta}_\mathbf{k}v_{\mathbf{k}aabb}))\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow} \\
&+ \Phi_\mathbf{k}(v_{\mathbf{k}aa} + v_{\mathbf{k}bb})\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow} + \Phi^*_\mathbf{k}(v_{\mathbf{k}aa} + v_{\mathbf{k}bb})\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow} \\
&+ \Delta_\mathbf{k}(v_{\mathbf{k}aa} + v_{\mathbf{k}bb})\hat{a}^\dagger_{\mathbf{k}\uparrow}\hat{a}^\dagger_{-\mathbf{k}\downarrow}\hat{b}^\dagger_{\mathbf{k}\uparrow}\hat{b}^\dagger_{-\mathbf{k}\downarrow}]|0\rangle \ .
\end{aligned}
\tag{B.11}
$$

The Schrödinger equation, $i\partial|\Psi_\mathbf{k}\rangle/\partial t = \mathcal{H}_\mathbf{k}|\Psi\rangle_\mathbf{k}$, gives us the time evolution equations for the amplitudes as

$$
\begin{aligned}
i\dot{u}_{\mathbf{k}} &= \bar{\Delta}_{\mathbf{k}}(v_{\mathbf{k}aa} + v_{\mathbf{k}bb}) \ , \\
i\dot{v}_{\mathbf{k}aa} &= 2\Theta_{\mathbf{k}} v_{\mathbf{k}aa} + (\Phi_{\mathbf{k}} v_{\mathbf{k}ba} + \Phi^*_{\mathbf{k}} v_{\mathbf{k}ab}) + (\Delta_{\mathbf{k}} u_{\mathbf{k}} + \bar{\Delta}_{\mathbf{k}} v_{\mathbf{k}aabb}) \ , \\
i\dot{v}_{\mathbf{k}bb} &= -2\Theta_{\mathbf{k}} v_{\mathbf{k}bb} + (\Phi_{\mathbf{k}} v_{\mathbf{k}ba} + \Phi^*_{\mathbf{k}} v_{\mathbf{k}ab}) + (\Delta_{\mathbf{k}} u_{\mathbf{k}} + \bar{\Delta}_{\mathbf{k}} v_{\mathbf{k}aabb}) \ , \\
i\dot{v}_{\mathbf{k}ab} &= \Phi_{\mathbf{k}}(v_{\mathbf{k}aa} + v_{\mathbf{k}bb}) \ , \\
i\dot{v}_{\mathbf{k}ba} &= \Phi^*_{\mathbf{k}}(v_{\mathbf{k}aa} + v_{\mathbf{k}bb}) \ , \\
i\dot{v}_{\mathbf{k}aabb} &= \Delta_{\mathbf{k}}(v_{\mathbf{k}aa} + v_{\mathbf{k}bb}) \ .
\end{aligned}
\tag{B.12}
$$

B.2 Condition for $2S_{\mathbf{k}}(t) = 0$

We denote the initial value of the amplitudes as $u_{i\mathbf{k}}$ and $v_{i\mathbf{k}}$. Since for $\mathbf{k} = \mathbf{K}_{1,\pm}$ or $\mathbf{K}_{2,\pm}$, we have $\Delta_{\mathbf{k}} = \bar{\Delta}_{\mathbf{k}} = \Phi_{\mathbf{k}} = 0$, the time evolution equations simplify to give

$$
\begin{aligned}
& u_{\mathbf{k}}(t) = u_{i\mathbf{k}} \ , \ v_{\mathbf{k}aa}(t) = v_{i\mathbf{k}aa} e^{-2i\Theta_{\mathbf{k}} t} \ , \ v_{\mathbf{k}bb} = v_{i\mathbf{k}bb} e^{2i\Theta_{\mathbf{k}} t} \ , \\
& v_{\mathbf{k}ab}(t) = v_{i\mathbf{k}ab} \ , \ v_{\mathbf{k}ba}(t) = v_{i\mathbf{k}ba} \ , \ v_{\mathbf{k}aabb}(t) = v_{i\mathbf{k}aabb} \ .
\end{aligned}
\tag{B.13}
$$

For these values of $\mathbf{k}$, we get

$$
\begin{aligned}
2S_{\mathbf{k}}(t) = \langle \Psi_{i\mathbf{k}} | \Psi_{\mathbf{k}}(t) \rangle &= |u_{i\mathbf{k}}|^2 + |v_{i\mathbf{k}aa}|^2 e^{-2i\Theta_{\mathbf{k}} t} + |v_{i\mathbf{k}bb}|^2 e^{2i\Theta_{\mathbf{k}} t} + |v_{i\mathbf{k}ab}|^2 + |v_{i\mathbf{k}ba}|^2 + |v_{i\mathbf{k}aabb}|^2 \\
&= 1 - (|v_{i\mathbf{k}aa}|^2 + |v_{i\mathbf{k}bb}|^2)(1 - \cos(2\Theta_{\mathbf{k}} t)) + i(|v_{i\mathbf{k}bb}|^2 - |v_{i\mathbf{k}aa}|^2) \sin(2\Theta_{\mathbf{k}} t) \ .
\end{aligned}
\tag{B.14}
$$

This can go to zero in two cases:

$$
\begin{aligned}
& \textbf{Case 1: } |v_{i\mathbf{k}aa}| = |v_{i\mathbf{k}bb}| \ , \ \sin^2(\Theta_{\mathbf{k}} t) = \frac{1}{4|v_{i\mathbf{k}aa}|^2} \ \Rightarrow \Theta_{\mathbf{k}} t = n\pi \pm \sin^{-1}\left(\frac{1}{2|v_{i\mathbf{k}aa}|}\right) \ , \\
& \textbf{Case 2: } \sin(2\Theta_{\mathbf{k}} t) = 0 \ , \ |v_{i\mathbf{k}aa}|^2 + |v_{i\mathbf{k}bb}|^2 = \frac{1}{2} \ \Rightarrow \Theta_{\mathbf{k}} t = \left(m + \frac{1}{2}\right)\pi \ .
\end{aligned}
\tag{B.15}
$$

B.3 Initial values of amplitudes

We now diagonalize the Hamiltonian in (B.7) to get the initial values of the amplitudes. Let $E_{\mathbf{k}} = (|\Phi_{\mathbf{k}}|^2 + |\Theta_{\mathbf{k}}|^2 + |\Phi_{\mathbf{k}}|^2)^{1/2}$ as in Eq. (B.5). Let us define s_x, s_y, s_z to be Pauli matrices acting on the $a - b$ sublattice space. We decompose $\Phi_{\mathbf{k}}$ into its real and imaginary parts, $\Phi_{\mathbf{k}} = \Phi_{\mathbf{k}x} + i\Phi_{\mathbf{k}y}$. The eigenvectors with energy $-E_{\mathbf{k}}$ will be the null eigenvectors of the matrix,

$$\begin{pmatrix} E_{\mathbf{k}}\mathbf{1} + \Phi_{\mathbf{k}x}s_x - \Phi_{\mathbf{k}y}s_y + \Theta_{\mathbf{k}}s_z & \Delta_{\mathbf{k}}\mathbf{1} \\ \Delta_{\mathbf{k}}\mathbf{1} & E_{\mathbf{k}}\mathbf{1} - \Phi_{\mathbf{k}x}s_x + \Phi_{\mathbf{k}y}s_y - \Theta_{\mathbf{k}}s_z \end{pmatrix} \tag{B.16}$$

Let us take the initial state to be the ground state of the Hamiltonian with the values of $\Theta_{\mathbf{k}}$, $\Phi_{\mathbf{k}}$, $\Delta_{\mathbf{k}}$ so that $E_{\mathbf{k}} = 0$ at $\mathbf{k} = \mathbf{K}_{1,\pm}$. From the time reversal symmetry (B.6) and the form of $\Delta_{\mathbf{k}}$ (B.3), in the initial state, we have $\bar{\Delta}_{\mathbf{k}} = \Delta_{\mathbf{k}}$. Let the null eigenvector of the above matrix be $(A, B)^T$. We get from the first row,

$$(E_{\mathbf{k}}\mathbf{1} + \Phi_{\mathbf{k}x}s_x - \Phi_{\mathbf{k}y}s_y + \Theta_{\mathbf{k}}s_z)A + \Delta_{\mathbf{k}}\mathbf{1}B = 0 \ , \tag{B.17}$$

so that

$$B = -\frac{1}{\Delta_{\mathbf{k}}}(E_{\mathbf{k}}\mathbf{1} + \Phi_{\mathbf{k}x}s_x - \Phi_{\mathbf{k}y}s_y + \Theta_{\mathbf{k}}s_z)A \ . \tag{B.18}$$

The second row yields the same equation. Now since the eigenvalue $-E_{\mathbf{k}}$ is doubly degenerate, we can find B for two linearly independent values of A.

$$\begin{aligned} &\text{For } A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ , \ B = -\frac{1}{\Delta_{\mathbf{k}}}\begin{pmatrix} E_{\mathbf{k}} + \Theta_{\mathbf{k}} \\ \Phi^*_{\mathbf{k}} \end{pmatrix} \ , \\ &\text{For } A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \ , \ B = -\frac{1}{\Delta_{\mathbf{k}}}\begin{pmatrix} \Phi_{\mathbf{k}} \\ E_{\mathbf{k}} - \Theta_{\mathbf{k}} \end{pmatrix} \ . \end{aligned} \tag{B.19}$$

The normalized eigenvectors for eigenvalue $-E_{\mathbf{k}}$ become

$$\begin{aligned} &\frac{1}{(2E_{\mathbf{k}}(E_{\mathbf{k}} + \Theta_{\mathbf{k}}))^{1/2}}(\Delta_{\mathbf{k}}, 0, -(E_{\mathbf{k}} + \Theta_{\mathbf{k}}), -\Phi^*_{\mathbf{k}})^T \text{ and} \\ &\frac{1}{(2E_{\mathbf{k}}(E_{\mathbf{k}} - \Theta_{\mathbf{k}}))^{1/2}}(0, \Delta_{\mathbf{k}}, -\Phi_{\mathbf{k}}, -(E_{\mathbf{k}} - \Theta_{\mathbf{k}}))^T \ . \end{aligned} \tag{B.20}$$

In the above, if we replace $E_{\mathbf{k}} \to -E_{\mathbf{k}}$, we get the eigenvectors with eigenvalue $E_{\mathbf{k}}$ as

$$\begin{aligned}&\frac{1}{(2E_{\mathbf{k}}(E_{\mathbf{k}}-\Theta_{\mathbf{k}}))^{1/2}}(\Delta_{\mathbf{k}}, 0, E_{\mathbf{k}}-\Theta_{\mathbf{k}}, -\Phi^*_{\mathbf{k}})^T \text{ and}\\ &\frac{1}{(2E_{\mathbf{k}}(E_{\mathbf{k}}+\Theta_{\mathbf{k}}))^{1/2}}(0, \Delta_{\mathbf{k}}, -\Phi_{\mathbf{k}}, E_{\mathbf{k}}+\Theta_{\mathbf{k}})^T \ .\end{aligned} \tag{B.21}$$

Let $A_{\mathbf{k}}$ be the basis in which the Hamiltonian is diagonal. we can write $\hat{\Psi}_{\mathbf{k}} = (\hat{a}_{\mathbf{k}\uparrow}, \hat{b}_{\mathbf{k}\uparrow}, \hat{a}^\dagger_{-\mathbf{k}\downarrow}, \hat{b}^\dagger_{-\mathbf{k}\downarrow})^T = U_{\mathbf{k}}\hat{A}_{\mathbf{k}}$, where the unitary transformation $U_{\mathbf{k}}$, can be obtained from the eigenvectors above. To simplify our notation, let us write $\varepsilon_+ = (2E_{\mathbf{k}}(E_{\mathbf{k}}+\Theta_{\mathbf{k}}))^{1/2}$, $\varepsilon_- = (2E_{\mathbf{k}}(E_{\mathbf{k}}-\Theta_{\mathbf{k}}))^{1/2}$ and drop the subscript $\mathbf{k}$ in our expression for $U_{\mathbf{k}}$. We get U as

$$U = \begin{pmatrix} \Delta/\varepsilon_+ & 0 & \Delta/\varepsilon_- & 0 \\ 0 & \Delta/\varepsilon_- & 0 & \Delta/\varepsilon_+ \\ -(E+\Theta)/\varepsilon_+ & -\Phi/\varepsilon_- & (E-\Theta)/\varepsilon_- & -\Phi/\varepsilon_+ \\ -\Phi^*/\varepsilon_+ & -(E-\Theta)/\varepsilon_- & -\Phi^*/\varepsilon_- & (E+\Theta)/\varepsilon_+ \end{pmatrix} . \tag{B.22}$$

We get $\hat{A}_{\mathbf{k}} = U^\dagger_{\mathbf{k}}\hat{\Psi}_{\mathbf{k}}$,

$$\hat{A}_{\mathbf{k}} = \begin{pmatrix} \Delta/\varepsilon_+ & 0 & -(E+\Theta)/\varepsilon_+ & -\Phi/\varepsilon_+ \\ 0 & \Delta/\varepsilon_- & -\Phi^*/\varepsilon_- & -(E-\Theta)/\varepsilon_+ \\ \Delta/\varepsilon_- & 0 & (E-\Theta)/\varepsilon_- & -\Phi/\varepsilon_- \\ 0 & \Delta/\varepsilon_+ & -\Phi^*/\varepsilon_+ & (E+\Theta)/\varepsilon_+ \end{pmatrix} \begin{pmatrix} \hat{a}_{\mathbf{k}\uparrow} \\ \hat{b}_{\mathbf{k}\uparrow} \\ \hat{a}^\dagger_{-\mathbf{k}\downarrow} \\ \hat{b}^\dagger_{-\mathbf{k}\downarrow} \end{pmatrix} . \tag{B.23}$$

In the ground state, we require $(\hat{A}_{\mathbf{k}})_1 |\Psi_{i\mathbf{k}}\rangle = 0$ and $(\hat{A}_{\mathbf{k}})_2 |\Psi_{i\mathbf{k}}\rangle = 0$, i.e., the annihilation operators in the diagonalized basis acting on the ground state yield zero. This gives the conditions,

$$\begin{aligned}&(\Delta_{i\mathbf{k}}\hat{a}_{\mathbf{k}\uparrow} - (E_{i\,\mathbf{k}} + \Theta_{i\mathbf{k}})\hat{a}^\dagger_{-\mathbf{k}\downarrow} - \Phi_{i\mathbf{k}}\hat{b}^\dagger_{-\mathbf{k}\downarrow}) |\Psi_{i\mathbf{k}}\rangle = 0 \ ,\\ &(\Delta_{i\mathbf{k}}\hat{b}_{\mathbf{k}\uparrow} - \Phi^*_{i\mathbf{k}}\hat{a}^\dagger_{-\mathbf{k}\downarrow} - (E_{i\mathbf{k}} - \Theta_{i\mathbf{k}})\hat{b}^\dagger_{-\mathbf{k}\downarrow}) |\Psi_{i\mathbf{k}}\rangle = 0 \ .\end{aligned} \tag{B.24}$$

From the form of $|\Psi_{\mathbf{k}}\rangle$ in Eq. (B.8), we get the following conditions for the initial amplitudes,

$$\begin{aligned}
&\Delta_{i\mathbf{k}} v_{i\mathbf{k}aa} - (E_{i\mathbf{k}} + \Theta_{i\mathbf{k}}) u_{i\mathbf{k}} = 0 \\
&\Delta_{i\mathbf{k}} v_{i\mathbf{k}ab} - \Phi_{i\mathbf{k}} u_{i\mathbf{k}} = 0 \\
&(E_{i\mathbf{k}} + \Theta_{i\mathbf{k}}) v_{i\mathbf{k}ab} - \Phi_{i\mathbf{k}} v_{i\mathbf{k}aa} = 0 \\
&\Delta_{i\mathbf{k}} v_{i\mathbf{k}aabb} - (E_{i\mathbf{k}} + \Theta_{i\mathbf{k}}) v_{i\mathbf{k}bb} + \Phi_{i\mathbf{k}} v_{i\mathbf{k}ba} = 0 \\
&\Delta_{i\mathbf{k}} v_{i\mathbf{k}ba} - \Phi^*_{i\mathbf{k}} u_{i\mathbf{k}} = 0 \\
&\Delta_{i\mathbf{k}} v_{i\mathbf{k}bb} - (E_{i\mathbf{k}} - \Theta_{i\mathbf{k}}) u_{i\mathbf{k}} = 0 \\
&-\Phi^*_{i\mathbf{k}} v_{i\mathbf{k}bb} + (E_{i\mathbf{k}} - \Theta_{i\mathbf{k}}) v_{i\mathbf{k}ba} = 0 \\
&\Delta_{i\mathbf{k}} v_{i\mathbf{k}aabb} + \Phi^*_{i\mathbf{k}} v_{i\mathbf{k}ab} - (E_{i\mathbf{k}} - \Theta_{i\mathbf{k}}) v_{i\mathbf{k}aa} = 0 \ .
\end{aligned} \tag{B.25}$$

Solving these equations along with the normalization (B.9), we get the initial amplitudes as

$$\begin{aligned}
v_{i\mathbf{k}aa} &= \frac{E_{i\mathbf{k}} + \Theta_{i\mathbf{k}}}{2E_{i\mathbf{k}}} \ , \ v_{i\mathbf{k}ab} = \frac{\Phi_{i\mathbf{k}}}{2E_{i\mathbf{k}}} \ , \ v_{i\mathbf{k}ba} = \frac{\Phi^*_{i\mathbf{k}}}{2E_{i\mathbf{k}}} \ , \\
v_{i\mathbf{k}bb} &= \frac{E_{i\mathbf{k}} - \Theta_{i\mathbf{k}}}{2E_{i\mathbf{k}}} \ , \ v_{i\mathbf{k}aabb} = \frac{\Delta_{i\mathbf{k}}}{2E_{i\mathbf{k}}} = u_{i\mathbf{k}} \ .
\end{aligned} \tag{B.26}$$

Here it makes sense that $v_{i\mathbf{k}aabb} = u_{i\mathbf{k}}$ since the bare single-particle dispersion energies of the A and B sublattices are equal in magnitude and opposite in sign. Thus simultaneously creating a particle on both either sublattice costs the same energy as not creating any particle at all.

B.4 Search for DQPTs

Let us now evaluate the condition (B.15) to see when we can have $2S_{\mathbf{k}}(t) = 0$. With $\Phi_{i\mathbf{k}} = \Delta_{i\mathbf{k}} = 0$, Case 1 in (B.15) gives

$$|v_{i\mathbf{k}aa}| = |v_{i\mathbf{k}bb}| \ \Rightarrow \ \Theta_{i\mathbf{k}} = 0 \ , \ |v_{i\mathbf{k}aa}| = |v_{i\mathbf{k}bb}| = \frac{1}{2} \ . \tag{B.27}$$

This happens at times t_n where

$$\sin^2(\Theta_{\mathbf{k}} t_n) = 1 \;\Rightarrow\; t_n = \left(n + \frac{1}{2}\right)\frac{\pi}{\Theta_{\mathbf{k}}} \ , \tag{B.28}$$

where n is any integer which gives a positive value for t_n (according to the sign of $\Theta_{\mathbf{k}}$). Case 2 of (B.15) gives the same result.

Since $\Theta_{i\mathbf{k}} = 0$, the system is gapless before the quench. $\mathbf{K}_{1,\pm}$ and $\mathbf{K}_{2,\pm}$ are Dirac points. After the quench to $\Theta_{\mathbf{k}} \neq 0$ for the above points in $k-$space, the system is gapped. Thus $2S_{\mathbf{k}}(t) = 0$ is achieved in a quench from a gapless to a gapped phase.

For $\mathbf{k} = \mathbf{K}_{1,\pm}$, $\Theta_{\mathbf{k}} = 0$ when $\mu_s = 6w'$, whereas for $\mathbf{k} = K_{2,\pm}$, $\Theta_{\mathbf{k}} = 0$ when $\mu_s = -6w'$. Depending on which initial condition we have, $\mu_s = 6w'$ or $\mu_s = -6w'$, $S_{\mathbf{k}}(t_c) = 0$ will occur for $\mathbf{k} = \mathbf{K}_{1,\pm}$ or $\mathbf{k} = \mathbf{K}_{2,\pm}$ respectively.

For convenience, let us choose the initial condition $\mu_s = 6w'$ and focus on $\mathbf{k} = \mathbf{K}_{1,+}$. Let us expand $S_{\mathbf{k}}(t)$ to lowest non-zero order in $(\mathbf{k} - \mathbf{K}_{1,+})$ and $(t - t_n)$. Let

$$\mathbf{k} = \mathbf{K}_{1,+} + \mathbf{q} = \left(\frac{2\pi}{3} + q_x, 2\pi + q_y, q_z\right) \ . \tag{B.29}$$

From Eq. (B.2),

$$\begin{aligned} \Phi_{\mathbf{K}_{1,+}+\mathbf{q}} &= 2w\left[e^{iq_z/4}\cos\left(\frac{2\pi}{3} + \frac{q_x + q_y}{4}\right) + e^{-iq_z/4}\cos\left(-\frac{\pi}{3} + \frac{q_x - q_y}{4}\right)\right] \\ &= 2w\left(-\frac{\sqrt{3}}{4}q_y - i\frac{q_z}{4}\right) \ . \end{aligned} \tag{B.30}$$

From Eq. (B.3), we have

$$\begin{aligned} \Delta_{\mathbf{K}_{1,+}+\mathbf{q}} &= 4\Delta\left(\cos\left(\frac{\pi}{3} + \frac{q_x}{2}\right)\cos\left(\pi + \frac{q_y}{2}\right) - \cos\left(\pi + \frac{q_y}{2}\right)\cos\left(\frac{q_z}{2}\right) - \cos\left(\frac{\pi}{3} + \frac{q_x}{2}\right)\cos\left(\frac{q_z}{2}\right)\right) \\ &= 2\sqrt{3}\Delta q_x \ . \end{aligned} \tag{B.31}$$

From Eq. (B.4),

$$\begin{aligned}\Theta_{\mathbf{K}_{1,+}+\mathbf{q}} &= 4w' \cos\left(\frac{q_z}{2}\right)\left(\cos\left(\pi + \frac{q_y}{2}\right) - \cos\left(\frac{\pi}{3} + \frac{q_x}{2}\right)\right) + 6w' \\ &= \sqrt{3} w' q_x \ .\end{aligned} \tag{B.32}$$

Using these, we get for $E_{i\mathbf{k}}$,

$$E_{i\mathbf{k}} = \left(3w'^2 q_x^2 + w^2 \frac{3q_y^2 + q_z^2}{4} + 12\Delta^2 q_x^2\right)^{1/2} \quad . \tag{B.33}$$

Now we wish to calculate the amplitudes at time t. To zeroth order in q, we have Eq. (B.13). To get the expressions to first order in q, let us define the following,

$$\begin{aligned} f_{\Delta\mathbf{k}} &= \int_0^t d\tau\, \Delta_{\mathbf{k}}(\tau)\, e^{2i\Theta_{\mathbf{k}}\tau} \ , \ f_{\bar{\Delta}\mathbf{k}} = \int_0^t d\tau\, \bar{\Delta}_{\mathbf{k}}(\tau)\, e^{2i\Theta_{\mathbf{k}}\tau} \ , \\ \bar{f}_{\Delta\mathbf{k}} &= \int_0^t d\tau\, \Delta_{\mathbf{k}}(\tau)\, e^{-2i\Theta_{\mathbf{k}}\tau} \ , \ \bar{f}_{\bar{\Delta}\mathbf{k}} = \int_0^t d\tau\, \bar{\Delta}_{\mathbf{k}}(\tau)\, e^{-2i\Theta_{\mathbf{k}}\tau} \ . \end{aligned} \tag{B.34}$$

These expressions are all $\mathcal{O}(q)$ from Eq. (B.31). We integrate Eq. (B.12) with time to first order in q giving

$$\begin{aligned} u_{\mathbf{k}}(t) &= u_{i\mathbf{k}} - i v_{i\mathbf{k}aa} \bar{f}_{\bar{\Delta}\mathbf{k}} - i v_{i\mathbf{k}bb} f_{\bar{\Delta}\mathbf{k}} \ , \\ v_{\mathbf{k}aa}(t) &= v_{i\mathbf{k}aa} e^{-2i\Theta_{\mathbf{k}}t} - \frac{1}{2\Theta_{\mathbf{k}}}(\Phi_{\mathbf{k}} v_{i\mathbf{k}ba} + \Phi^*_{\mathbf{k}} v_{i\mathbf{k}ab})(1 - e^{-2i\Theta_{\mathbf{k}}t}) - i(u_{i\mathbf{k}} f_{\Delta\mathbf{k}} + v_{i\mathbf{k}aabb} f_{\bar{\Delta}\mathbf{k}}) e^{-2i\Theta_{\mathbf{k}}t} \ , \\ v_{\mathbf{k}bb}(t) &= v_{i\mathbf{k}bb} e^{2i\Theta_{\mathbf{k}}t} - \frac{1}{2\Theta_{\mathbf{k}}}(\Phi_{\mathbf{k}} v_{i\mathbf{k}ba} + \Phi^*_{\mathbf{k}} v_{i\mathbf{k}ab})(e^{2i\Theta_{\mathbf{k}}t} - 1) - i(u_{i\mathbf{k}} \bar{f}_{\Delta\mathbf{k}} + v_{i\mathbf{k}aabb} \bar{f}_{\bar{\Delta}\mathbf{k}}) e^{2i\Theta_{\mathbf{k}}t} \ , \\ v_{\mathbf{k}ab}(t) &= v_{i\mathbf{k}ab} - \frac{\Phi_{\mathbf{k}}}{2\Theta_{\mathbf{k}}}(1 - e^{-2i\Theta_{\mathbf{k}}t})(v_{i\mathbf{k}aa} + v_{i\mathbf{k}bb} e^{2i\Theta_{\mathbf{k}}t}) \ , \\ v_{\mathbf{k}ba}(t) &= v_{i\mathbf{k}ba} - \frac{\Phi^*_{\mathbf{k}}}{2\Theta_{\mathbf{k}}}(1 - e^{-2i\Theta_{\mathbf{k}}t})(v_{i\mathbf{k}aa} + v_{i\mathbf{k}bb} e^{2i\Theta_{\mathbf{k}}t}) \ , \\ v_{\mathbf{k}aabb}(t) &= v_{i\mathbf{k}aabb} - i v_{i\mathbf{k}aa} \bar{f}_{\Delta\mathbf{k}} - i v_{i\mathbf{k}bb} f_{\Delta\mathbf{k}} \ . \end{aligned} \tag{B.35}$$

Therefore we have

$$2S_{\mathbf{k}}(t) = u^*_{i\mathbf{k}}u_{\mathbf{k}}(t) + v^*_{i\mathbf{k}aa}v_{\mathbf{k}aa}(t) + v^*_{i\mathbf{k}bb}v_{\mathbf{k}bb}(t) \\ + v^*_{i\mathbf{k}ab}v_{\mathbf{k}ab}(t) + v^*_{i\mathbf{k}ba}v_{\mathbf{k}ba}(t) + v^*_{i\mathbf{k}aabb}v_{\mathbf{k}aabb}(t) \ . \tag{B.36}$$

In this problem, we are just quenching μ_s, so that $\Phi_{\mathbf{k}}(\tau) = \Phi_{i\mathbf{k}}$ for any time τ. Also from Eq. (B.26), we see that $u_{i\mathbf{k}},\ v_{i\mathbf{k}aa},\ v_{i\mathbf{k}bb},\ v_{i\mathbf{k}aabb}$ are all real whereas $v_{i\mathbf{k}ab}$ and $v_{i\mathbf{k}ba}$ are complex. Substituting Eq. (B.35) in the equation for $2S_{\mathbf{k}}(t)$, we get

$$\begin{aligned} 2S_{\mathbf{k}}(t) = {} & u^2_{i\mathbf{k}} + v^2_{i\mathbf{k}aa}e^{-2i\Theta_{\mathbf{k}}t} + v^2_{i\mathbf{k}bb}e^{2i\Theta_{\mathbf{k}}t} \\ & + |v_{i\mathbf{k}ab}|^2 + |v_{i\mathbf{k}ba}|^2 + v^2_{i\mathbf{k}aabb} \\ & - (v_{i\mathbf{k}aa} + v_{i\mathbf{k}bb}e^{2i\Theta_{\mathbf{k}}t})\Big(iu_{i\mathbf{k}}(\bar{f}_{\Delta\mathbf{k}} + \bar{f}_{\bar{\Delta}\mathbf{k}}) + iu_{i\mathbf{k}}e^{-2i\Theta_{\mathbf{k}}t}(f_{\Delta\mathbf{k}} + f_{\bar{\Delta}\mathbf{k}}) \\ & + (1 - e^{-2i\Theta_{\mathbf{k}}t})\frac{|\Phi_{i\mathbf{k}}|^2}{\Theta_{\mathbf{k}}E_{i\mathbf{k}}}\Big) \ . \end{aligned} \tag{B.37}$$

Substituting (B.26) for the initial amplitudes, we get

$$\begin{aligned} 2S_{\mathbf{k}}(t) = {} & 1 + \frac{1}{4E^2_{i\mathbf{k}}}\Big[(E_{i\mathbf{k}} + \Theta_{i\mathbf{k}})^2(-1 + e^{-2i\Theta_{\mathbf{k}}t}) + (E_{i\mathbf{k}} - \Theta_{i\mathbf{k}})^2(-1 + e^{2i\Theta_{\mathbf{k}}t}) \\ & - ((E_{i\mathbf{k}} + \Theta_{i\mathbf{k}}) + (E_{i\mathbf{k}} - \Theta_{i\mathbf{k}})e^{2i\Theta_{\mathbf{k}}t})\Big[i\Delta_{i\mathbf{k}}((\bar{f}_{\Delta\mathbf{k}} + \bar{f}_{\bar{\Delta}\mathbf{k}}) \\ & + (f_{\Delta\mathbf{k}} + f_{\bar{\Delta}\mathbf{k}})e^{-2i\Theta_{\mathbf{k}}t}) + 2(1 - e^{-2i\Theta_{\mathbf{k}}t})\frac{|\Phi_{i\mathbf{k}}|^2}{\Theta_{\mathbf{k}}}\Big]\Big] \ . \end{aligned} \tag{B.38}$$

Now let us expand this to first order in $(t - t_n)$. We expand around $t_n = (n + 1/2)\pi/\Theta_{\mathbf{k}}$ where $2\Theta_{\mathbf{k}}t_n = (2n + 1)\pi$,

$$e^{2i\Theta_{\mathbf{k}}t} \approx -(1 + 2i\Theta_{\mathbf{k}}(t - t_n)) \ , \ e^{-2i\Theta_{\mathbf{k}}t} \approx -(1 - 2i\Theta_{\mathbf{k}}(t - t_n)) \ . \tag{B.39}$$

Eq. (B.38) simplifies to

$$2S_{\mathbf{k}}(t) \approx \frac{1}{4E_{i\mathbf{k}}^2}\Bigg[-4\Theta_{i\mathbf{k}}^2 + 8iE_{i\mathbf{k}}\Theta_{i\mathbf{k}}\Theta_{\mathbf{k}}(t-t_n) \\ -2\left[i\Theta_{i\mathbf{k}}\Delta_{i\mathbf{k}}(f_{\Delta\mathbf{k}} + \bar{f}_{\Delta\mathbf{k}} + f_{\bar{\Delta}\mathbf{k}} + \bar{f}_{\bar{\Delta}\mathbf{k}}) + \frac{4\Theta_{i\mathbf{k}}|\Phi_{i\mathbf{k}}|^2}{\Theta_{\mathbf{k}}}\right]\Bigg] , \tag{B.40}$$

where we have neglected terms of $\mathcal{O}(q(t-t_n))$ for small q, since the expression contains a term of $\mathcal{O}(t-t_n)$. Let us rank the terms in the above equation in orders of q. From Eqs. (B.31) and (B.34), we can write

$$f_{\Delta\mathbf{k}} = 2\sqrt{3}q_x\tilde{f}_{\Delta\mathbf{k}} , \tag{B.41}$$

so that $f_{\Delta\mathbf{k}}$ is $\mathcal{O}(q)$. The same holds for all the quantities in Eq. (B.34). Using Eqs. (B.30), (B.31) and (B.32), $2S_{\mathbf{k}}(t)$ becomes

$$2S_{\mathbf{k}}(t) \approx -\frac{3w'^2q_x^2}{E_{i\mathbf{k}}^2} + \frac{2\sqrt{3}iw'q_x\delta\mu}{E_{i\mathbf{k}}}(t-t_n) \\ -\frac{1}{2E_{i\mathbf{k}}^2}\left[12\sqrt{3}iw'\Delta_i q_x^3(\tilde{f}_{\Delta\mathbf{k}} + \tilde{\bar{f}}_{\Delta\mathbf{k}} + \tilde{f}_{\bar{\Delta}\mathbf{k}} + \tilde{\bar{f}}_{\bar{\Delta}\mathbf{k}}) + 4\sqrt{3}ww'\frac{q_x(3q_y^2/4 + q_z^2/4)}{\sqrt{3}w'q_x + \delta\mu}\right] , \tag{B.42}$$

where $\delta\mu = \mu_s - 6w'$ from Eq. (B.4) and $E_{i\mathbf{k}} = (3w'^2q_x^2 + w^2(3q_y^2 + q_z^2)/4 + 12\Delta^2q_x^2)^{1/2}$ from Eq. (B.33). The first term in the above equation is $\mathcal{O}(q^0)$, the second term is $\mathcal{O}(t-t_n)$, and the third and fourth terms are $\mathcal{O}(q)$. Let us ignore these latter two terms for now.

We want to calculate the singular part of the Loschmidt echo. The contribution to this from the region around $\mathbf{k} = \mathbf{K}_{1,+}$ is given by $\mathcal{Z} = V\iiint dq_xdq_ydq_z\ \ln(2S_{\mathbf{k}}(t))$, so that

$$\frac{1}{V\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial t} = \frac{1}{V}\iiint dq_xdq_ydq_z\frac{1}{S_{\mathbf{k}}(t)}\frac{\partial S_{\mathbf{k}}(t)}{\partial t} \approx \iiint dq_xdq_ydq_z\left((t-t_n) + i\frac{\sqrt{3}w'q_x}{2E_{i\mathbf{k}}\delta\mu}\right)^{-1} , \tag{B.43}$$

where V is the volume of the system.

Since in the initial condition, we chose μ_s as $6w'$, there is also a node at $\mathbf{k} = \mathbf{K}_{1,-}$. A similar calculation gives the contribution to $\mathcal{Z}$ from the region around $\mathbf{K}_{1,-}$ as

$$\frac{1}{V\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial t} \approx \iiint dq_x dq_y dq_z \left((t-t_n) - i\frac{\sqrt{3}w' q_x}{2E_{i\mathbf{k}}\delta\mu}\right)^{-1} . \tag{B.44}$$

Adding up these two contributions gives us

$$\frac{1}{V\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial t} \approx \iiint dq_x dq_y dq_z 2(t-t_n)\left((t-t_n)^2 + \frac{3}{4}\frac{w'^2 q_x^2}{E_{i\mathbf{k}}^2\delta\mu^2}\right)^{-1} . \tag{B.45}$$

Let us see if we can approximate this integral. The integrand has inversion symmetry in $\mathbf{q}$, it is invariant under the transformation, $\mathbf{q} \to -\mathbf{q}$. We double the integrand and integrate q_x from 0 to q_0, where q_0 is large enough to not interfere with the singularity at $q = 0$.

We split the integral over q_x into two parts. For some q_r which is of the order $\sqrt{3q_y^2 + q_z^2}/2$, we approximate $E_{i\mathbf{k}}$ from Eq. (B.33) as,

$$\begin{aligned} &\text{For } 0 < q_x < q_r \ \rightarrow\ E_{i\mathbf{k}} \approx \frac{w}{2}(3q_y^2 + q_z^2)^{1/2} , \\ &\text{For } q_r < q_x < q_0 \ \rightarrow\ E_{i\mathbf{k}} \approx (3w'^2 + 12\Delta_i^2)^{1/2} q_x . \end{aligned} \tag{B.46}$$

The integral in Eq. (B.45) is split as

$$\begin{aligned} \frac{1}{V\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial t} \approx \iint dq_y dq_z \Bigg[&\frac{w^2(3q_y^2+q_z^2)\delta\mu^2}{3w'^2}\int_0^{q_r} dq_x \frac{4(t-t_n)}{\frac{w^2}{3w'^2}(3q_y^2+q_z^2)\delta\mu^2(t-t_n)^2 + q_x^2} \\ &+ \int_{q_r}^{q_0} dq_x \frac{4(t-t_n)}{(t-t_n)^2 + \left(4\delta\mu^2(1+4\Delta_i^2/w'^2)\right)^{-1}} \Bigg] . \end{aligned} \tag{B.47}$$

The integrand of the second integral is independent of q, so it doesn't contribute to the singularity. Evaluating the first integral, we get

$$\frac{1}{V\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial t} \approx \iint dq_y dq_z \frac{4w\delta\mu}{\sqrt{3}w'}(3q_y^2+q_z^2)^{1/2}\tan^{-1}\left(\frac{\sqrt{3}w' q_r}{w(3q_y^2+q_z^2)^{1/2}\delta\mu(t-t_n)}\right) . \tag{B.48}$$

When we have $(t-t_n) \ll w' q_r/((3q_y^2+q_z^2)^{1/2} w\delta\mu)$, then we can approximate the $\tan^{-1}(\cdot)$ term as

$$\tan^{-1}\left(\frac{\sqrt{3}w' q_r}{w(3q_y^2+q_z^2)^{1/2}\delta\mu(t-t_n)}\right) \approx \frac{\pi}{2}\,\mathrm{sgn}[t-t_n]\ . \tag{B.49}$$

We see that the integrand is discontinuous as t crosses t_n. Thus we predict that $d\ln\mathcal{Z}/dt$ will have jump discontinuities periodically at times $t=t_n$. This gives us the form of DQPTs for the class CI superconductor described by Eqs. (B.1)-(B.4).

We made several approximations to get from Eq. (B.45) to Eq. (B.49). Instead, we can evaluate the integral in Eq. (B.45) numerically, to see if the jump discontinuity appears even without those approximations. Figures B.1 and B.2 show the numerical result for certain values of the parameters w, w', Δ_i, and $\delta\mu$. We indeed see that there is a jump discontinuity in $\frac{d\ln\mathcal{Z}}{dt}$ at time $t=t_n$. Thus this class CI superconductor shows DQPTs where the first derivative of $\ln\mathcal{Z}$ with time is discontinuous.

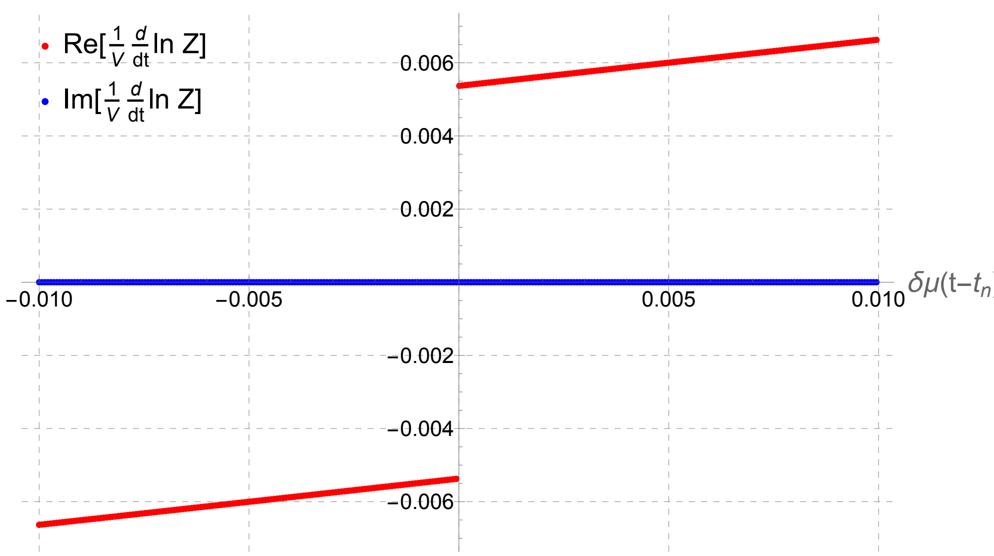

Figure B.1: Real and imaginary parts of $\frac{1}{V}\frac{d\ln\mathcal{Z}}{dt}$ as a function of $\delta\mu(t-t_n)$ for $w'=\frac{1}{2\sqrt{3}}, w=\frac{1}{10}, \Delta_i=\frac{1}{4\sqrt{3}}$, and $\delta\mu=1$.

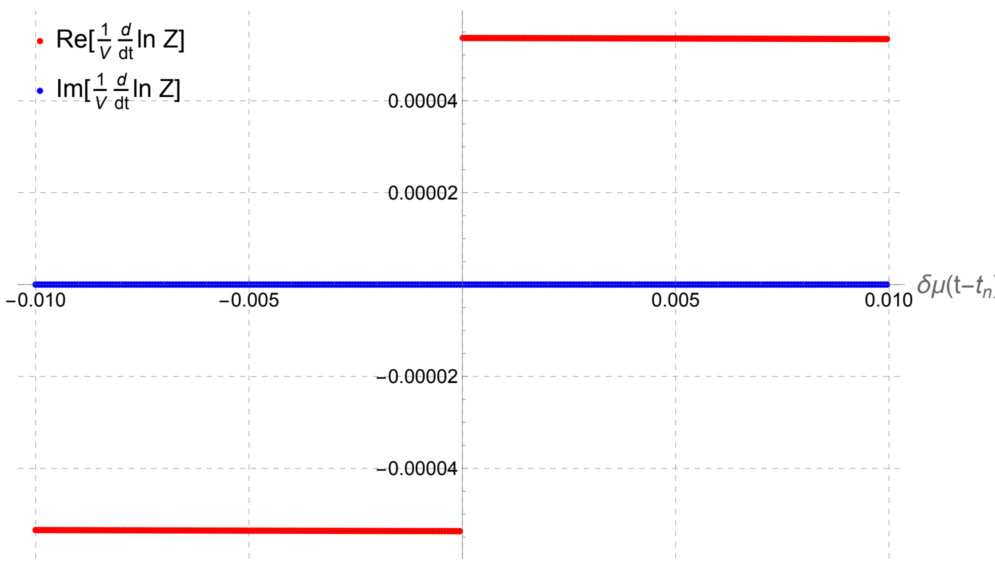

Figure B.2: Real and imaginary parts of $\frac{1}{V}\frac{d\ln\mathcal{Z}}{dt}$ as a function of $\delta\mu(t-t_n)$ for $w' = \frac{1}{2\sqrt{3}}$, $w = 1$, $\Delta_i = \frac{1}{40\sqrt{3}}$, and $\delta\mu = 1$.

Appendix C

Classical Loschmidt echo

In calculating the Loschmidt echo we need to compute

$$\mathcal{Z} = \langle \Psi_i | e^{-i\hat{H}t} | \Psi_i \rangle = \prod_{\mathbf{p}} 2S_{\mathbf{p}} \ . \tag{C.1}$$

Here $S_{\mathbf{p}}$ are defined in Eq. (2.35).

However, if we were not careful, we could have instead computed

$$|\Psi(t)\rangle = e^{-i\hat{H}t} |\Psi_i\rangle \tag{C.2}$$

using the conventional approach of solving equations (2.25) together with (2.29). We could then calculate

$$\left\langle \Psi_i \middle| \Psi(t) \right\rangle = \prod_{\mathbf{p}} \left(u^*_{i\mathbf{p}} u_{\mathbf{p}}(t) + v^*_{i\mathbf{p}} v_{\mathbf{p}}(t) \right) \ , \tag{C.3}$$

The right hand side of this equation appears to be superficially similar to the right hand side of the equation (C.1) if $\tau = t$ is used in the definition of $S_{\mathbf{p}}$, Eq. (2.35). However, importantly in evaluating the amplitudes $u_{\mathbf{p}}$, $v_{\mathbf{p}}$, Eq. (2.29) is used instead of Eq. (2.34). This in fact has been carried out in the literature [58].

One could ask about the meaning of the resulting quantity. We would like to present arguments that it is equivalent to the *classical echo* introduced in Refs. [70, 71].

Consider the quantity

$$\mathcal{L}(t) = \left|\left\langle \Psi_i \middle| \Psi(t) \right\rangle\right|^2 \ . \tag{C.4}$$

Using simple algebra, one can show that [58]

$$\mathcal{L}(t) = \prod_{\mathbf{p}} \mathcal{L}_{\mathbf{p}}(t) \ , \tag{C.5}$$

where

$$\mathcal{L}_{\mathbf{p}}(t) = \frac{1}{2} + 2\mathbf{s}_{\mathbf{p}}(0) \cdot \mathbf{s}_{\mathbf{p}}(t) \ , \tag{C.6}$$

with the spins defined according to Eq. (2.26), (2.27).

Classical Loschmidt echo is defined as [70, 71]

$$\mathcal{L}_{\mathrm{cl}}(t) = \int d\mathbf{x}\, \rho(\mathbf{x}, 0)\rho(\mathbf{x}, t), \tag{C.7}$$

where the integral is over the phase space $\mathbf{x}$ and $\rho(\mathbf{x}, t)$ is the phase space density of a classical system. The classical counterpart of spin is angular momentum (classical spin) and the corresponding phase space density is the Wigner function of spin-$\frac{1}{2}$ [108],

$$\rho(\mathbf{k}, t = 0) = \frac{1}{2} + \sqrt{3}\,\mathbf{k} \cdot \mathbf{s} \ , \tag{C.8}$$

where $\mathbf{k}$ is a unit vector representing the phase space of the spin $\mathbf{s}$. We choose to parametrize $\mathbf{k}$ by its spherical angles (θ, ϕ), i.e.,

$$\mathbf{k} = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta] \ . \tag{C.9}$$

Eq. (C.8) is a distribution of initial conditions for a classical spin that rotates according to classical equations of motion (2.22). Each initial condition [initial direction $\mathbf{k}$ of the classical spin] moves to a new point $\mathbf{k}'$ in the phase space in time t thus generating the evolution of the Wigner function in time.

It is sufficient to do the calculation for one spin $\mathbf{s_p}$ due to the product nature of (C.5). $\mathbf{s_p}(0)$ and $\mathbf{s_p}(t)$ are related by a rotation. Since we did not pick any coordinate axis until now, we can choose them arbitrarily without loss of generality. Let the axis of rotation be the z-axis and $\mathbf{s_p}(0)$ be in the $x-z$ plane with spherical coordinates $\theta=\theta_0$ and $\phi=0$. In this coordinate system we also have

$$\mathbf{s_p}(0)\cdot\mathbf{s_p}(t)=\frac{1}{4}\cos^2\theta_0+\frac{1}{4}\sin^2\theta_0\cos\alpha\ , \tag{C.10}$$

and

$$\mathbf{s_p}(0)=[\sin\theta_0/2,0,\cos\theta_0/2]\ . \tag{C.11}$$

The phase space point (θ,ϕ) evolves to $(\theta,\phi+\alpha)$ under this rotation, where α is the angle of rotation and the vector $\mathbf{k}$ evolves into

$$\mathbf{k}'=[\sin\theta\cos(\phi+\alpha),\sin\theta\sin(\phi+\alpha),\cos\theta]\ , \tag{C.12}$$

and

$$\rho(\theta,\phi,t)=\frac{1}{2}+\sqrt{3}\,\mathbf{k}'\cdot\mathbf{s}(0)\ . \tag{C.13}$$

The integration in Eq. (C.7) should be understood as integration (averaging) over the sphere, i.e.,

$$\int d\mathbf{x}=\frac{1}{4\pi}\int_{-1}^{1}d\cos\theta\int_0^{2\pi}d\phi\ . \tag{C.14}$$

Note that Wigner function (C.8) is normalized to one with this integration measure. Using this definition of $\int d\mathbf{x}$ in Eq. (C.7) with $\rho(\mathbf{x},0)$ and $\rho(\mathbf{x},t)$ from Eqs. (C.8) and (C.13) and the above expressions for $\mathbf{s}$, $\mathbf{k}$, and $\mathbf{k}'$, we obtain

$$\mathcal{L}_{\rm cl}=\frac{1}{4}+\mathbf{s_p}(0)\cdot\mathbf{s_p}(t)\ . \tag{C.15}$$

This coincides with $\mathcal{L}_{\mathbf{p}}$ in Eq. (C.6) up to a nonessential factor of 2.

Appendix D

Gap equation for the spectral form factor of s-wave superconductors

Here we present the derivation of the real-time gap equation for the s-wave spin singlet superconductor. The gap equations for other types of superconductors can be derived similarly.

We begin with the Hamiltonian (3.19) for the spin-1/2 attractively interacting fermions. We are interested in calculating the spectral form factor, that is the quantity

$$\mathcal{Z} = \mathrm{Tr}\, e^{-i\hat{H}t} \; . \tag{D.1}$$

Let us set up the coherent state path integral for the purpose of this calculation [62].

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi}\, \exp\left(i \int_0^t d\tau \int d^3x \left\{ \sum_{\sigma=\uparrow,\downarrow} \left(i\bar{\psi}_\sigma \dot{\psi}_\sigma - \frac{\nabla\bar{\psi}_\sigma \nabla\psi_\sigma}{2m} + \mu\bar{\psi}_\sigma\psi_\sigma \right) + \lambda\bar{\psi}_\uparrow\bar{\psi}_\downarrow\psi_\downarrow\psi_\uparrow \right\} \right) \; . \tag{D.2}$$

In order to represent the spectral form factor, which involves taking a trace, the fermionic fields ψ and $\bar{\psi}$ must satisfy the antiperiodic boundary conditions

$$\psi_\sigma(t) = -\psi_\sigma(0) \quad , \quad \bar{\psi}_\sigma(t) = -\bar{\psi}_\sigma(0) \; . \tag{D.3}$$

As standard in the theory of superconductivity we introduce the Hubbard-Stratonovich field Δ_s [63], which results in

$$\mathcal{Z} = \int \mathcal{D}\Delta_s \mathcal{D}\bar{\Delta}_s\, e^{iW} \; , \tag{D.4}$$

where

$$
\begin{aligned}
e^{iW} &= \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, e^{iS} \ , \\
S &= \int_0^t d\tau \int d^3x \left\{ \sum_{\sigma=\uparrow,\downarrow} \left(i\bar{\psi}_\sigma\dot{\psi}_\sigma - \frac{\nabla\bar{\psi}_\sigma\nabla\psi_\sigma}{2m} + \mu\bar{\psi}_\sigma\psi_\sigma \right) - \bar{\Delta}_s\psi_\downarrow\psi_\uparrow - \Delta_s\bar{\psi}_\uparrow\bar{\psi}_\downarrow - \frac{\bar{\Delta}_s\Delta_s}{\lambda} \right\} \ .
\end{aligned}
\tag{D.5}
$$

We calculate the integral over Δ_s and $\bar{\Delta}_s$ in the saddle point approximation. Varying W over $\bar{\Delta}_s(\mathbf{r},\tau)$ at some time τ and at some position $\mathbf{r}$, we find

$$
\frac{1}{\mathcal{Z}} \int \mathcal{D}\psi\mathcal{D}\bar{\psi} \left(\psi_\downarrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau) + \frac{1}{\lambda}\Delta_s(\mathbf{r},\tau) \right) e^{iS} = 0 \ . \tag{D.6}
$$

We will look for the solution of this equation in terms of $\Delta_s(\mathbf{r},\tau)$ and $\bar{\Delta}_s(\mathbf{r},\tau)$ which are constant in space and time and so, from now on, denote them simply as Δ_s and $\bar{\Delta}_s$. This gives

$$
\Delta_s = -\frac{\lambda}{\mathcal{Z}} \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\, \psi_\downarrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau)\, e^{iS} \ . \tag{D.7}
$$

We note that recasting this equation, and the corresponding equation for $\bar{\Delta}_s$, in the operator formalism gives

$$
\begin{aligned}
\Delta_s &= -\lambda\, \mathrm{Tr}\left(e^{-i\hat{H}_{\mathrm{BdG}}(t-\tau)} \hat{\psi}_\downarrow(\mathbf{r})\hat{\psi}_\uparrow(\mathbf{r}) e^{-i\hat{H}_{\mathrm{BdG}}\tau} \right) \ , \\
\bar{\Delta}_s &= -\lambda\, \mathrm{Tr}\left(e^{-i\hat{H}_{\mathrm{BdG}}(t-\tau)} \hat{\psi}^\dagger_\uparrow(\mathbf{r})\hat{\psi}^\dagger_\downarrow(\mathbf{r}) e^{-i\hat{H}_{\mathrm{BdG}}\tau} \right) \ .
\end{aligned}
\tag{D.8}
$$

Here $\hat{H}_{\mathrm{BdG}}$ is the Bogoliubov-de-Gennes Hamiltonian which follows from the action S in Eq. (D.5). Note that, the equation for $\bar{\Delta}_s$ is not the complex conjugate of the equation for Δ_s, therefore as pointed out in the Chapter 3, $\bar{\Delta}_s\Delta_s$ does not have to be real.

To proceed further, we need to calculate the anomalous Green's function which appears on the right hand side of the equation (D.7). This is computed by taking advantage of the functional integral over ψ, $\bar{\psi}$ being Gaussian. We rewrite the action S by using the Nambu notations in the frequency and momentum space. The frequencies as always are discrete and have the fermionic

Matsubara form, to ensure the antiperiodic boundary conditions (D.3),

$$\omega_n = \frac{\pi}{t}\left(1+2n\right). \tag{D.9}$$

We define

$$\begin{aligned} \psi_{\mathbf{p},\omega_n} &= \frac{1}{\sqrt{V}}\int_0^t d\tau \int d^3x\, \psi(\mathbf{r},\tau)\, e^{i\omega_n\tau - i\mathbf{p}\cdot\mathbf{r}}\ , \\ \psi(\mathbf{r},\tau) &= \frac{1}{t\sqrt{V}}\sum_{\omega_n,\mathbf{p}} \psi_{\mathbf{p},\omega_n}\, e^{-i\omega_n\tau + i\mathbf{p}\cdot\mathbf{r}}\ . \end{aligned} \tag{D.10}$$

The action in Eq. (D.5) becomes

$$S = \frac{1}{t}\sum_{\mathbf{p},\omega_n}\begin{pmatrix}\bar{\psi}_{\mathbf{p},\omega_n,\uparrow} & \psi_{-\mathbf{p},-\omega_n,\downarrow}\end{pmatrix}\begin{pmatrix}\omega_n - \frac{p^2}{2m}+\mu & -\Delta_s \\ -\bar{\Delta}_s & \omega_n + \frac{p^2}{2m}-\mu\end{pmatrix}\begin{pmatrix}\psi_{\mathbf{p},\omega_n,\uparrow} \\ \bar{\psi}_{-\mathbf{p},-\omega_n,\downarrow}\end{pmatrix} - \frac{tV}{\lambda}\bar{\Delta}_s\Delta_s\ . \tag{D.11}$$

To calculate the anomalous Green's function we invert the matrix

$$\begin{pmatrix}\omega_n - \frac{p^2}{2m}+\mu & -\Delta_s \\ -\bar{\Delta}_s & \omega_n + \frac{p^2}{2m}-\mu\end{pmatrix}^{-1} = \frac{1}{\omega_n^2 - \left(\frac{p^2}{2m}-\mu\right)^2 - \bar{\Delta}_s\Delta_s}\begin{pmatrix}\omega_n + \frac{p^2}{2m}-\mu & \Delta_s \\ \bar{\Delta}_s & \omega_n - \frac{p^2}{2m}+\mu\end{pmatrix}\ , \tag{D.12}$$

and read off the anomalous Green's function from the upper right corner of this matrix. We find

$$\frac{1}{\mathcal{Z}}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\psi_\downarrow(\mathbf{r},\tau)\psi_\uparrow(\mathbf{r},\tau)e^{iS} = -\frac{i}{tV}\sum_n\sum_{\mathbf{p}}\frac{\Delta_s}{\omega_n^2 - \left(\frac{p^2}{2m}-\mu\right)^2 - \bar{\Delta}_s\Delta_s}\ . \tag{D.13}$$

The saddle point equation (D.6) thus becomes

$$\frac{i}{tV}\sum_n\sum_{\mathbf{p}}\frac{1}{\omega_n^2 - \left(\frac{p^2}{2m}-\mu\right)^2 - \bar{\Delta}_s\Delta_s} = \frac{1}{\lambda}\ . \tag{D.14}$$

Summation over the Matsubara frequencies can now be carried out explicitly (see for example

Ref. [105]), with the result

$$\frac{i}{2V}\sum_{\mathbf{p}}\frac{\tan\left[\frac{tE_s(p)}{2}\right]}{E_s(p)}=\frac{1}{\lambda}\ , \tag{D.15}$$

where

$$E_s(p)=\sqrt{\left(\frac{p^2}{2m}-\mu\right)^2+\bar{\Delta}_s\Delta_s}\ . \tag{D.16}$$

This is the real-time gap equation which appears as Eq. (3.22) in Chapter 3. Given the solution $\bar{\Delta}_s\Delta_s$ of this equation, we can calculate the spectral form factor by following Eq. (3.8). Following a very similar blueprint we can also derive Eq. (3.12) from the Hamiltonian (3.5).

Note that if instead we had been interested in computing the thermal partition function $\mathrm{Tr}\, e^{-\hat{H}/(k_BT)}$, we would have employed the imaginary time formalism. It largely coincides with the formalism described here, and differs only by the replacement $it \to 1/(k_BT)$ where T is temperature and k_B Boltzmann constant, and by the analytic continuation of the time τ used above to imaginary values. It would have resulted in

$$\frac{k_BT}{V}\sum_n\sum_{\mathbf{p}}\frac{1}{\omega_n^2+\left(\frac{p^2}{2m}-\mu\right)^2+\bar{\Delta}_s\Delta_s}=\frac{1}{\lambda}\ , \tag{D.17}$$

where

$$\omega_n=2\pi k_BT\left(1+2n\right)\ , \tag{D.18}$$

instead of Eqs. (D.9) and (D.14). Carrying out the summation over the Matsubara frequencies results in

$$\frac{1}{2V}\sum_{\mathbf{p}}\frac{\tanh\left[\frac{E_s(p)}{2k_BT}\right]}{E_s(p)}=\frac{1}{\lambda}\ , \tag{D.19}$$

which is the standard thermal gap equation [8, 66].