A small-gain theorem for set stability of infinite networks: Distributed observation and ISS for time-varying networks

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Abstract

We generalize a small-gain theorem for a network of infinitely many systems, recently developed in [Kawan et. al, IEEE TAC (2021)]. The generalized small-gain theorem addresses exponential input-to-state stability with respect to closed sets, which enables us to study diverse control-theoretic problems in a unified manner, and it also allows for agents to have infinitely many neighbors. The small-gain condition, expressed in terms of the spectral radius of a gain operator collecting all the information about the internal Lyapunov gains, has several useful characterizations which simplify the verification of the condition. To demonstrate applicability of our small-gain theorem, we apply it to the stability analysis of infinite time-varying networks as well as the design of distributed observers for infinite networks.

Keywords: Interconnected systems, input-to-state stability, small-gain theorem, Lyapunov methods

1. Introduction

Emerging technologies such as the Internet of Things, Cloud computing and 5G communication will let us realize a paradigm shift towards a hyper-connected world composed of a large number of smart networked systems providing us with much more autonomy and flexibility. Examples of such smart networked systems include smart grids, connected vehicles, swarm robotics, and smart cities in which the size of the networks is unknown and possibly time-varying. However, standard tools for stability analysis/stabilization of control systems do not scale well to these large-scale complex systems [1, 2, 3]. A promising approach to address this critical issue is to over-approximate a finite but very large network by an infinite network, and then control this over-approximated system; see, e.g., [4, 3, 5].

A striking progress in the infinite-dimensional input-to-state stability (ISS) theory, see, e.g., [6, 7, 8, 9] (also see [10] for a recent survey on this topic) blended with the powerful nonlinear small-gain criteria for the stability analysis of finite networks of nonlinear systems [11, 12] create a foundation for the development of stability conditions for infinite networks.

For interconnections of finitely many systems of arbitrary nature (possibly infinite-dimensional) the gain operator, collecting the information about the internal gains, acts on a finite-dimensional Euclidean space [13]. The case of infinite networks is much more complex, as the gain operator now acts on an infinite-dimensional space, which calls for a careful choice of the infinite-dimensional state space of the overall network, and motivates the use of the theory of positive operators on ordered Banach spaces for the small-gain analysis (cf. Section 3.1).

The development of small-gain theorems for infinite networks has recently received considerable attention [14, 15, 16, 17]. All of these small-gain theorems for infinite networks address ISS with respect to the origin. A more general notion of input-to-state stability with respect to a closed set covers several further stability notions such as incremental stability, robust consensus/synchronization, ISS of time-varying systems as well as variants of input-to-output stability in a unified and generalized manner; see for instance [18]. On the other hand, for large-but-finite networks, dissipative small-gain conditions are widely used for various control problems such as distributed control design [19], compositional construction of (in)finite-state abstractions [20, 21], cyber-security of networked systems [22], and networked control systems with asynchronous communication [23].

To provide a tool to study the above problem for infinite networks (or networks of possibly unknown size), we develop in this paper a dissipative small-gain theorem addressing exponential ISS of infinite networks with respect to closed sets. We assume that all subsystems of

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an infinite network are exponentially ISS, and the associated exponential ISS Lyapunov functions in a dissipative form are known. Based on this information, we define a gain operator, which describes the interconnection structure of the systems and the influence of individual subsystems on each other. Finally, we show that if the spectral radius of this operator is less than one, then the whole network is exponentially ISS with respect to a certain closed set. In particular, we extend the main result of our recent work [14, Thm. VI.1] to ISS with respect to closed sets and to networks in which any agent can be connected to *infinitely* many other agents. This generalization extends the applicability of the small-gain result to several control-theoretic problems, including stability analysis of infinite time-varying networks and the design of distributed observers for infinite networks.

As the first application of the proposed small-gain theorem, we study ISS for time-varying infinite networks. Existing results on infinite networks are developed for time-invariant systems, although, practically speaking, time-variance is a more realistic assumption. In this paper, we address exponential ISS for both time-invariant and time-varying infinite networks within a unified framework.

As the second application area, we provide a methodology to address scalability in distributed estimation problems. We assume that each subsystem has a local observer whose states exponentially converge to those of the subsystem, given estimates of the state of neighboring subsystems. Formulating the state estimation as a stabilization problem with respect to a certain closed set, we show that if the couplings between the subsystems are sufficiently weak, which is quantitatively expressed by our small-gain condition, then the state estimation problem can be tackled.

This paper is organized as follows: first, relevant notation, a discussion of well-posedness of infinite networks and auxiliary results on distance functions with respect to a closed set in an infinite-dimensional state space are given in Section 2. The notion of exponential ISS with respect to a closed set for infinite-dimensional systems and a related Lyapunov criterion are presented in Section 2.4. In Section 3, the small-gain theorem for ISS with respect to closed sets is presented. Applications to ISS for time-varying systems and distributed observers are respectively given in Sections 4 and 5. Section 6 concludes the paper.

2. Preliminaries

2.1. Notation

We write $\mathbb{N} = \{1, 2, 3, \dots\}$ for the set of positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$; $\mathbb{R}$ denotes the reals and $\mathbb{R}_+ := \{t \in \mathbb{R} : t \geq 0\}$ the nonnegative reals. For vector norms on finite- and infinite-dimensional vector spaces, we write $|\cdot|$. For associated operator norms, we use the notation $\|\cdot\|$. By ℓ^p, $p \in [1, \infty]$, we denote the Banach space of all real sequences $x = (x_i)_{i\in\mathbb{N}}$ with finite ℓ^p-norm $|x|_p < \infty$, where $|x|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$ for $p < \infty$ and $|x|_\infty = \sup_{i\in\mathbb{N}} |x_i|$. Given $x, y \in \mathbb{R}^n$, the inner product of x and y are denoted by $\langle x, y\rangle$.

A more general class of ℓ^p-spaces is defined as follows. Let $p \in [1, \infty)$, let $(n_i)_{i\in\mathbb{N}}$ be a sequence of positive integers and fix a norm $|\cdot|_i$ on $\mathbb{R}^{n_i}$ for every $i \in \mathbb{N}$. Then

$$\ell^p(\mathbb{N}, (n_i)) := \Big\{ x = (x_i)_{i\in\mathbb{N}} : x_i \in \mathbb{R}^{n_i},\ \sum_{i=1}^{\infty} |x_i|_i^p < \infty \Big\}$$

equipped with the norm $|x|_p := (\sum_{i=1}^{\infty} |x_i|_i^p)^{\frac{1}{p}}$ is a separable Banach space (which is proved by standard arguments, see, e.g., [24]). Usually, we drop the index i from the norm. We define $\ell^\infty(\mathbb{N}, (n_i))$ in a similar fashion.

We write $L^\infty(\mathbb{R}_+, \mathbb{R}^n)$ for the Banach space of essentially bounded measurable functions from $\mathbb{R}_+$ to $\mathbb{R}^n$. If X is a Banach space, we write $r(T)$ for the spectral radius of a bounded linear operator $T : X \to X$. The notation $C^0(X, Y)$ stands for the set of all continuous mappings $f : X \to Y$ between metric spaces X and Y. The right upper Dini derivative of a function $\gamma : \mathbb{R} \to \mathbb{R}$ at $t \in \mathbb{R}$ is defined by $\mathrm{D}^+\gamma(t) := \limsup_{h\to 0+} \frac{1}{h}\big(\gamma(t + h) - \gamma(t)\big)$, and is allowed to assume the values $\pm\infty$. Analogously, the right lower Dini derivative of γ at t is defined by $\mathrm{D}_+\gamma(t) := \liminf_{h\to 0+} \frac{1}{h}\big(\gamma(t+h)-\gamma(t)\big)$. We will consider $\mathcal{K}, \mathcal{K}_\infty$, and $\mathcal{KL}$ comparison functions, see [25, Ch. 4.4] for definitions.

2.2. Infinite interconnections

We study interconnections of countably many systems, each given by a finite-dimensional ordinary differential equation (ODE). Using $\mathbb{N}$ as the index set (by default), the ith subsystem is written as

$$\Sigma_i : \quad \dot{x}_i = f_i(x_i, \bar{x}, u_i). \tag{1}$$

The family $(\Sigma_i)_{i\in\mathbb{N}}$ comes together with a number $p \in [1, \infty]$ and sequences $(n_i)_{i\in\mathbb{N}}$, $(m_i)_{i\in\mathbb{N}}$ of positive integers so that the following assumptions hold with $X := \ell^p(\mathbb{N}, (n_i))$ for a specified sequence of norms on the spaces $\mathbb{R}^{n_i}$:

- The state vector x_i of Σ_i is an element of $\mathbb{R}^{n_i}$.
- The internal input vector $\bar{x} = (\bar{x}_i)_{i\in\mathbb{N}}$ is an element of X.
- The external input vector u_i is an element of $\mathbb{R}^{m_i}$.
- The right-hand side $f_i : \mathbb{R}^{n_i} \times X \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ is a continuous function.
- Unique local solutions of the ODE (1) exist for all initial states $x_{i0} \in \mathbb{R}^{n_i}$ and all continuous $\bar{x}(\cdot)$ and locally essentially bounded $u_i(\cdot)$ (which are regarded as time-dependent inputs). We denote the corresponding solutions by $\phi_i(\cdot, x_{i0}, (\bar{x}, u_i))$.

The values of the function f_i can be independent of certain components of the input vector $\bar{x}$. We write I_i for the set of indices $j \in \mathbb{N}$ so that $f_i(x_i, \bar{x}, u_i)$ is non-constant with respect to the component $\bar{x}_j$ of $\bar{x}$ (i.e. only dependent on the components $\bar{x}_j$ with $j \in I_i$), and we assume that $i \notin I_i$.

In the ODE (1), we consider $\bar{x}(\cdot)$ as an *internal input* and $u_i(\cdot)$ as an *external input* (which may be a disturbance or a control input). The interpretation is that the subsystem Σ_i is affected by a certain set of neighbors, indexed by I_i, and its external input. We note that the set I_i does *not* have to be finite, implying that subsystem i can be connected to infinitely many other subsystems. Moreover, we note that I_i is allowed to be empty.

To define the interconnection of the subsystems Σ_i, we consider the state vector $x = (x_i)_{i\in\mathbb{N}} \in X = \ell^p(\mathbb{N}, (n_i))$, the input vector $u = (u_i)_{i\in\mathbb{N}} \in U := \ell^q(\mathbb{N}, (m_i))$ for some $q \in [1, \infty]$ (possibly different from p), and the right-hand side

$$f(x, u) := (f_1(x_1, x, u_1), f_2(x_2, x, u_2), \ldots). \tag{2}$$

The interconnection is then written as

$$\Sigma : \quad \dot{x} = f(x, u). \tag{3}$$

The class of admissible input functions is defined as

$$\mathcal{U} := \big\{u : \mathbb{R}_+ \to U : u \text{ is strongly measurable and essentially bounded}\big\}, \tag{4}$$

and we equip this space with the L^∞-norm

$$|u|_{q,\infty} := \operatorname*{ess\,sup}_{t\geq 0} |u(t)|_q.$$

A continuous mapping $\xi : J \to X$, defined on an interval $J = [0, T_*)$ with $T_* \in (0, \infty]$, is called a *solution* of the infinite-dimensional ODE (3) with initial value $x^0 \in X$ for the external input $u \in \mathcal{U}$ provided that the function $s \mapsto f(\xi(s), u(s))$ is an X-valued locally integrable function and

$$\xi(t) = x^0 + \int_0^t f(\xi(s), u(s))\, \mathrm{d}s$$

holds for all $t \in J$, where the integral is the Bochner integral for Banach space valued functions. For the theory of Bochner integration, readers may consult [26].

If for each $x^0 \in X$ and $u \in \mathcal{U}$ a unique (local) solution exists, we say that the system is *well-posed* and write $\phi(\cdot, x^0, u)$ for the corresponding maximal solution (in the positive direction) and $J_{\max}(x^0, u)$ for its interval of existence. We say that the system is *forward complete* if $J_{\max}(x^0, u) = \mathbb{R}_+$ for all $(x^0, u) \in X \times \mathcal{U}$.

In the rest of the paper, unless otherwise stated, we assume that the following is satisfied.

Assumption 1. *The system Σ is well-posed with state space $X = \ell^p(\mathbb{N}, (n_i))$ and external input space $U = \ell^q(\mathbb{N}, (m_i))$, with finite $p, q \geq 1$. Furthermore, all of its uniformly bounded maximal solutions $\phi(\cdot, x, u)$ are global, i.e., exist on $\mathbb{R}_+$ (this latter property is also called boundedness-implies-continuation (BIC) property).*

We note that [14, Thm. III.2] provides sufficient conditions for well-posedness of Σ. The BIC property is satisfied, e.g., if additionally the vector field f is uniformly bounded on bounded sets and Lipschitz continuous on bounded sets with respect to the first argument.

Note that our constructions in this paper essentially rely on the fact that the parameters p and q are finite. For small-gain theorems in the case of $p = q = \infty$, we refer the interested reader to [15, 17, 27, 28].

2.3. Distances in sequence spaces

Let $X = \ell^p(\mathbb{N}, (n_i))$ for a certain $p \in [1, \infty)$. Consider nonempty closed sets $\mathcal{A}_i \subset \mathbb{R}^{n_i}$, $i \in \mathbb{N}$. For each $x_i \in \mathbb{R}^{n_i}$, we define the distance of x_i to the set $\mathcal{A}_i$ by $|x_i|_{\mathcal{A}_i} := \inf_{y_i\in\mathcal{A}_i} |x_i - y_i|$. Now we define the set

$$\mathcal{A} := \{x \in X : \ x_i \in \mathcal{A}_i, \ i \in \mathbb{N}\} = X \cap (\mathcal{A}_1 \times \mathcal{A}_2 \times \ldots). \tag{5}$$

If $\mathcal{A} \neq \emptyset$, we define the distance from any $x \in X$ to $\mathcal{A}$ as

$$|x|_{\mathcal{A}} := \inf_{y\in\mathcal{A}} |x - y|_p = \inf_{y\in\mathcal{A}} \Big(\sum_{i=1}^{\infty} |x_i - y_i|^p\Big)^{\frac{1}{p}}. \tag{6}$$

We note that $|x|_{\{0\}} = |x|_p$. The following lemma shows that the infimum and the sum in (6) can be exchanged.

Lemma 2. *Let $X = \ell^p(\mathbb{N}, (n_i))$ for a certain $p \in [1, \infty)$. Assume that $\mathcal{A}$ defined by (5) is nonempty. Then for any $x \in X$ we have*

$$|x|_{\mathcal{A}} = \Big(\sum_{i=1}^{\infty} |x_i|^p_{\mathcal{A}_i}\Big)^{\frac{1}{p}} < \infty. \tag{7}$$

Moreover, the set $\mathcal{A}$ is closed in X.

Proof. First of all, for any $x_i \in \mathbb{R}^{n_i}$ and $z_i \in \mathcal{A}_i$ it holds that

$$|x_i|_{\mathcal{A}_i} = \inf_{y_i\in\mathcal{A}_i} |x_i - y_i| \leq |x_i - z_i| \leq |x_i| + |z_i|.$$

As $\mathcal{A} \neq \emptyset$, we can choose $z_i \in \mathcal{A}_i$ so that $z = (z_1, z_2, \ldots) \in \mathcal{A} \subset X$. Now, for each $N > 0$ the inequality $\gamma(a + b) \leq \gamma(2a) + \gamma(2b)$, which holds for any $\gamma \in \mathcal{K}$ and all $a, b \geq 0$, can be used to show that

$$\sum_{i=1}^{N} |x_i|^p_{\mathcal{A}_i} \leq \sum_{i=1}^{N} (|x_i| + |z_i|)^p \leq \sum_{i=1}^{N} (2^p|x_i|^p + 2^p|z_i|^p). \tag{8}$$

As both $x, z \in X$, the limit for $N \to \infty$ of the right-hand side exists, and thus

$$\sum_{i=1}^{\infty} |x_i|^p_{\mathcal{A}_i} \leq 2^p(|x|^p_p + |z|^p_p) < \infty. \tag{9}$$

Now, let us show the equality in (7). Pick any $x \in X$ and any $\tilde{y} \in \mathcal{A}$. Then for every $\varepsilon > 0$ there is $N = N(\varepsilon)$ so that

$$\sum_{i=N+1}^{\infty} |x_i|^p < \frac{\varepsilon}{2^{p+1}}, \qquad \sum_{i=N+1}^{\infty} |\tilde{y}_i|^p < \frac{\varepsilon}{2^{p+1}}. \tag{10}$$

The following holds:

$$\begin{aligned} |x|_{\mathcal{A}} &= \inf_{y\in\mathcal{A}} \Big(\sum_{i=1}^{N} |x_i - y_i|^p + \sum_{i=N+1}^{\infty} |x_i - y_i|^p \Big)^{\frac{1}{p}} \\ &\leq \inf_{y_i\in\mathcal{A}_i,\ 1\leq i\leq N} \Big(\sum_{i=1}^{N} |x_i - y_i|^p + \sum_{i=N+1}^{\infty} |x_i - \tilde{y}_i|^p \Big)^{\frac{1}{p}}, \end{aligned} \tag{11}$$

where in the last transition we reduce the set of y over which we take the infimum from $\mathcal{A}$ to $\{(y_1, \ldots, y_N, \tilde{y}_{N+1}, \tilde{y}_{N+2}, \ldots) : y_i \in \mathcal{A}_i, 1 \leq i \leq N\} \subset \mathcal{A}$.

Estimating the last term in (11) similarly to (8), and using (10), we obtain

$$\begin{aligned} |x|_{\mathcal{A}} &\leq \inf_{y_i\in\mathcal{A}_i,\ 1\leq i\leq N} \Big(\sum_{i=1}^{N} |x_i - y_i|^p + 2^p \sum_{i=N+1}^{\infty} (|x_i|^p + |\tilde{y}_i|^p) \Big)^{\frac{1}{p}} \\ &\leq \Big(\sum_{i=1}^{N} \inf_{y_i\in\mathcal{A}_i} |x_i - y_i|^p + \varepsilon \Big)^{\frac{1}{p}} = \Big(\sum_{i=1}^{N} |x_i|^p_{\mathcal{A}_i} + \varepsilon \Big)^{\frac{1}{p}}. \end{aligned}$$

By using (9), we can estimate the last term by

$$|x|_{\mathcal{A}} \leq \Big(\sum_{i=1}^{\infty} |x_i|^p_{\mathcal{A}_i} + \varepsilon \Big)^{\frac{1}{p}} < \infty.$$

Now, as $\varepsilon > 0$ has been chosen arbitrarily, we can take the limit $\varepsilon \to 0$ to obtain

$$|x|_{\mathcal{A}} \leq \Big(\sum_{i=1}^{\infty} |x_i|^p_{\mathcal{A}_i} \Big)^{\frac{1}{p}}. \tag{12}$$

On the other hand, as taking the infimum over all $x \in \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots$ gives a value not larger than taking the infimum over $\mathcal{A}$, it holds that $|x|_{\mathcal{A}} \geq \inf_{y_i\in\mathcal{A}_i,\ i\in\mathbb{N}} (\sum_{i=1}^{\infty} |x_i - y_i|^p)^{\frac{1}{p}} = (\sum_{i=1}^{\infty} \inf_{y_i\in\mathcal{A}_i} |x_i - y_i|^p)^{\frac{1}{p}} = (\sum_{i=1}^{\infty} |x_i|^p_{\mathcal{A}_i})^{\frac{1}{p}}$, which together with (12) completes the proof of (7). Writing $\mathcal{A}$ as the intersection of the preimages $\pi_i^{-1}(\mathcal{A}_i)$, $i \in \mathbb{N}$, under the canonical projection maps $\pi_i : X \to \mathbb{R}^{n_i}$, we see that $\mathcal{A}$ is closed. □

2.4. Exponential input-to-state stability

We continue to suppose that Assumption 1 (well-posedness + BIC property) for Σ holds and that the parameters p and q are finite. Our aim is to study the stability of the interconnected system with respect to a closed set $\mathcal{A} \subset X$. For this purpose, we introduce the notion of exponential input-to-state stability (exponential ISS) with respect to a set $\mathcal{A}$.

Definition 3. *Given a nonempty closed set $\mathcal{A} \subset X$, the system Σ is called exponentially input-to-state stable (eISS) w.r.t. $\mathcal{A}$ if it is forward complete and there are constants $a, M > 0$ and $\gamma \in \mathcal{K}$ such that for any initial state $x^0 \in X$ and any $u \in \mathcal{U}$ the corresponding solution satisfies*

$$|\phi(t, x^0, u)|_{\mathcal{A}} \leq M e^{-at} |x^0|_{\mathcal{A}} + \gamma(|u|_{q,\infty}) \quad \forall t \geq 0. \tag{13}$$

For any function $V : X \to \mathbb{R}$, which is continuous on $X \backslash \mathcal{A}$, we define the *orbital (right upper Dini) derivative* at $x \in X \backslash \mathcal{A}$ for the external input $u \in \mathcal{U}$ by $\mathrm{D}^+ V_u(x) := \mathrm{D}^+ V(\phi(t, x, u))_{|t=0}$, where the right-hand side is the right upper Dini derivative of the function $t \mapsto V(\phi(t, x, u))$, evaluated at $t = 0$.

Exponential input-to-state stability is implied by the existence of an exponential ISS Lyapunov function, which we define in a dissipative form as follows.

Definition 4. *Let a nonempty closed set $\mathcal{A} \subset X$ be given. A function $V : X \to \mathbb{R}_+$, which is continuous on $X \backslash \mathcal{A}$, is called an* eISS Lyapunov function *for Σ* w.r.t. *$\mathcal{A}$ if there exist constants $\underline{\omega}, \overline{\omega}, b, \kappa > 0$ and $\gamma \in \mathcal{K}_\infty$ such that*

$$\underline{\omega} |x|^b_{\mathcal{A}} \leq V(x) \leq \overline{\omega} |x|^b_{\mathcal{A}}, \qquad \forall x \in X, \tag{14a}$$

$$\mathrm{D}^+ V_u(x) \leq -\kappa V(x) + \gamma(|u|_{q,\infty}), \quad \forall x \in X \backslash \mathcal{A}, u \in \mathcal{U}. \tag{14b}$$

Proposition 5. *Let Assumption 1 hold. Also assume that $\mathcal{A}$ is bounded or Σ is forward complete. If there exists an eISS Lyapunov function for Σ w.r.t. $\mathcal{A}$, then Σ is eISS w.r.t. $\mathcal{A}$.*

The proof follows similar steps as those in the proof of [14, Prop. IV.4]. In particular, we obtain the validity of the eISS estimate (13) on the whole domain of the solutions of the system. Furthermore, the BIC property guarantees that the solutions satisfying the eISS estimate can be extended to the whole positive semi-axis.

Example 6. *If $\mathcal{A}$ is unbounded and Σ is not forward complete, then Proposition 5 does not hold in general. In particular, $\mathcal{A}$ may contain trajectories with a finite escape time. Even if we assume that trajectories starting in $\mathcal{A}$ are all forward complete (as the system is linear in the neighborhood of $\mathcal{A}$), Proposition 5 may still not hold.*

Consider the following planar system:

$$\dot{x} = f(x, y), \tag{15a}$$

$$\dot{y} = f(x, y) + (x - y), \tag{15b}$$

where $f(x, y) = x$ whenever $x - y \in [-1, 1]$, and $f(x, y) = x + (k - 1)x^2$ whenever $|x - y| = k$, $k \geq 1$.

Let us study the stability of this system with respect to $\mathcal{A} := \{(x, x) : x \in \mathbb{R}\}$. Take $V(x, y) := (x - y)^2$. As $|(x, y)|_{\mathcal{A}} = \frac{1}{\sqrt{2}}|x - y|$, it holds that $V(x, y) = 2|(x, y)|^2_{\mathcal{A}}$. Since

$$\frac{d}{dt}(x - y) = -(x - y),$$

we obtain

$$\dot{V}(x, y) = -2V(x, y).$$

Hence, V is an exponential Lyapunov function for (15). Furthermore, all trajectories starting in $\mathcal{A}$ or in a neighborhood of $\mathcal{A}$ are forward complete. At the same time, there are trajectories of (15) with a finite escape time.

3. Small-gain theorem

In [14], a small-gain theorem for interconnections of countably many eISS subsystems (1) with linear internal gains has been developed, which allows to analyze stability of infinite networks consisting of individually eISS subsystems. Here, we extend this result to the case of eISS with respect to closed sets. Although mathematically its proof follows similar steps as those in that of the original eISS small-gain theorem in [14], this extension enables us to develop a framework for control and observation of infinite networks, cf. Section 5 below.

3.1. The gain operator and its properties

We assume that each Σ_i is eISS, and for Σ_i there exists a continuous eISS Lyapunov function w.r.t. $\mathcal{A}_i$ with linear gains as introduced next.

Assumption 7. *For each $i \in \mathbb{N}$, there is a nonempty closed set $\mathcal{A}_i \subset \mathbb{R}^{n_i}$ and a continuous function $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ satisfying the following properties.*

- *There are constants $\underline{\alpha}_i, \overline{\alpha}_i > 0$ so that for all $x_i \in \mathbb{R}^{n_i}$*
$$\underline{\alpha}_i |x_i|^p_{\mathcal{A}_i} \le V_i(x_i) \le \overline{\alpha}_i |x_i|^p_{\mathcal{A}_i}. \tag{16}$$
- *There are constants $\lambda_i > 0$, $\gamma_{ij} > 0$ ($j \in I_i$) and $\gamma_{iu} > 0$ so that the following holds: for each $x_i \in \mathbb{R}^{n_i} \setminus \mathcal{A}_i$, $u_i \in L^\infty(\mathbb{R}_+, \mathbb{R}^{m_i})$, each internal input $\bar{x} = (\bar{x}_j)_{j\in\mathbb{N}} \in C^0(\mathbb{R}_+, X)$ and for almost all t in the maximal interval of existence of $\phi_i(t) := \phi_i(t, x_i, (\bar{x}, u_i))$, the following inequality holds:*
$$\mathrm{D}^+(V_i \circ \phi_i)(t) \le -\lambda_i V_i(\phi_i(t)) + \sum_{j\in I_i} \gamma_{ij} V_j(\bar{x}_j(t)) + \gamma_{iu}|u_i(t)|^q. \tag{17}$$
- *For all t in the maximal interval of the existence of ϕ_i, it holds that $\mathrm{D}_+(V_i \circ \phi_i)(t) < \infty$.*

The gains γ_{ij} and decay rates λ_i characterize the response of the i-th subsystem on the internal inputs from other subsystems.

We furthermore assume that the following uniformity conditions hold for the constants introduced above.

Assumption 8. *(a) There are constants $\underline{\alpha}, \overline{\alpha} > 0$ so that*
$$\underline{\alpha} \le \underline{\alpha}_i \le \overline{\alpha}_i \le \overline{\alpha}, \quad i \in \mathbb{N}. \tag{18}$$

(b) There is a constant $\underline{\lambda} > 0$ so that for all $i \in \mathbb{N}$
$$\underline{\lambda} \le \lambda_i. \tag{19}$$

(c) There is a constant $\overline{\gamma}_u > 0$ so that for all $i \in \mathbb{N}$
$$\gamma_{iu} \le \overline{\gamma}_u. \tag{20}$$

In order to formulate a small-gain condition, we further introduce the following infinite nonnegative matrices by collecting the coefficients from (17)
$$\Lambda := \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots), \quad \Gamma := (\gamma_{ij})_{i,j\in\mathbb{N}},$$
where we put $\gamma_{ij} := 0$ whenever $j \notin I_i$. We also introduce the infinite matrix
$$\Psi := \Lambda^{-1}\Gamma = (\psi_{ij})_{i,j\in\mathbb{N}}, \quad \psi_{ij} = \frac{\gamma_{ij}}{\lambda_i}. \tag{21}$$

Under an appropriate boundedness assumption, the matrix Ψ acts as a linear operator on ℓ^1 by $(\Psi x)_i = \sum_{j=1}^\infty \psi_{ij} x_j$ for all $i \in \mathbb{N}$.

We call $\Psi : \ell^1 \to \ell^1$ the *gain operator* associated with the decay rates λ_i and coefficients γ_{ij}.

We make the following assumption which is equivalent to Γ being a bounded operator from ℓ^1 to ℓ^1.

Assumption 9. *The matrix $\Gamma = (\gamma_{ij})$ satisfies*
$$\|\Gamma\|_{1,1} = \sup_{j\in\mathbb{N}} \sum_{i=1}^\infty \gamma_{ij} < \infty, \tag{22}$$
where the double index on the left-hand side indicates that we consider the operator norm induced by the ℓ^1-norm both on the domain and codomain of the operator Γ.

Clearly, under Assumptions 9 and 8(b), the gain operator Ψ is bounded (see also [14, Lem. V.7]). Moreover, clearly Ψ is a positive operator with respect to the standard positive cone $\ell^1_+ := \{x = (x_1, x_2, \ldots) \in \ell^1 : x_i \ge 0,\ \forall i \in \mathbb{N}\}$ in ℓ^1, i.e., it maps ℓ^1_+ into itself.

3.2. Small-gain theorem: eISS with respect to closed sets

In this section, we prove that the interconnected system Σ is exponentially ISS under the given assumptions, provided that the spectral radius of the gain operator satisfies $r(\Psi) < 1$. In particular, we construct an overall eISS Lyapunov function as a linear combination of individual ones given by Assumption 7. This is accomplished by the following *small-gain theorem*.

Theorem 10. *Consider the infinite interconnection Σ, composed of the subsystems Σ_i, $i \in \mathbb{N}$, satisfying Assumption 1. That is, Σ is well-posed as a system with state space $X = \ell^p(\mathbb{N}, (n_i))$, space of input values $U = \ell^q(\mathbb{N}, (m_i))$ with finite p, q, and the external input space $\mathcal{U}$, as defined in (4), and has the BIC property. Consider nonempty closed sets $\mathcal{A}_i \subset \mathbb{R}^{n_i}$ and assume that $\mathcal{A} := X \cap (\mathcal{A}_1 \times \mathcal{A}_2 \times \ldots)$ is nonempty. Also assume that $\mathcal{A}$ is bounded or Σ is forward complete.*

Furthermore, suppose that the following conditions hold:

(i) Each Σ_i admits a continuous eISS Lyapunov function V_i w.r.t. $\mathcal{A}_i$ so that Assumptions 7 and 8 are satisfied with $\mathcal{A}_i$ as above, and with p, q as in the definition of X, U above.

(ii) The operator $\Gamma : \ell^1 \to \ell^1$ is bounded, i.e., Assumption 9 holds.

(iii) The spectral radius of Ψ satisfies $r(\Psi) < 1$.

Then there exists a vector $\mu = (\mu_i)_{i\in\mathbb{N}} \in \ell^\infty$ satisfying $\underline{\mu} \le \mu_i \le \overline{\mu}$ with constants $\underline{\mu}, \overline{\mu} > 0$ such that

$$\mu^\top(-\Lambda + \Gamma) \le -\lambda_\infty \mu^\top \tag{23}$$

with a constant $\lambda_\infty > 0$. Moreover, the function $V : X \to \mathbb{R}_+$, given by

$$V(x) = \sum_{i=1}^{\infty} \mu_i V_i(x_i), \tag{24}$$

is an eISS Lyapunov function for the network Σ w.r.t. $\mathcal{A}$ and has the following properties:

(a) V is continuous on $X \setminus \mathcal{A}$.

(b) For all $x^0 \in X \setminus \mathcal{A}$ and $u \in \mathcal{U}$

$$\mathrm{D}^+ V_u(x^0) \le -\lambda_\infty V(x^0) + \overline{\mu}\,\overline{\gamma}_u |u|^q_{q,\infty}.$$

(c) For all $x \in X$ the following inequalities hold:

$$\underline{\mu}\,\underline{\alpha}|x|^p_{\mathcal{A}} \le V(x) \le \overline{\mu}\,\overline{\alpha}|x|^p_{\mathcal{A}}. \tag{25}$$

In particular, Σ is eISS w.r.t. $\mathcal{A}$.

Proof. According to [14, Lem. V.10], the condition $r(\Psi) < 1$ implies that there exists a vector $\mu = (\mu_i)_{i\in\mathbb{N}} \in \ell^\infty$ satisfying $\underline{\mu} \le \mu_i \le \overline{\mu}$ such that (23) holds.

An eISS Lyapunov function for Σ is defined as in (24) with $\mu \in \ell^\infty$ satisfying (23). It is well-defined, because

$$0 \le V(x) \le \sum_{i=1}^{\infty} \mu_i \overline{\alpha}_i |x_i|^p_{\mathcal{A}_i} \le \overline{\alpha}|\mu|_\infty \sum_{i=1}^{\infty} |x_i|^p_{\mathcal{A}_i} = \overline{\alpha}|\mu|_\infty |x|^p_{\mathcal{A}} < \infty,$$

where we used Lemma 2. This also proves the upper bound for (25). The lower bound for (25) is obtained analogously, and thus inequality (14a) holds for V (with $b = p$). The rest of the proof is a straightforward extension of the proof of [14, Thm. VI.1]. We omit the details, but only mention that the inequality (23) is crucial for the proof of the Lyapunov estimate (14b).

Finally, since we assume that $\mathcal{A}$ is bounded or Σ is forward complete, then Σ is eISS in view of Proposition 5. □

Set stability of time-invariant systems covers several important problems such as stability analysis of time-varying systems and observer design, among others. The next section specifically discusses these two applications.

4. Small-gain theorem for infinite time-varying networks

Although Theorem 10 only considers time-invariant systems, it can also be applied to time-varying systems by transforming them into time-invariant systems of the form (3). To see this, consider the time-varying system

$$\dot{x} = f(t, x, u), \tag{26}$$

where $x \in X$, $u \in U$ and $f : \mathbb{R} \times X \times U \to X$ is continuous with $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}$.

We assume that the state space X and the input space U are chosen as $X = \ell^p(\mathbb{N}, (n_i))$ and $U = \ell^q(\mathbb{N}, (m_i))$, respectively, for fixed $p, q \in [1, \infty)$. The same class of admissible input functions as in (4) with $\mathbb{R}$ in place $\mathbb{R}_+$ is considered here.

Similarly as we did in Section 2.2, one can introduce the concepts of solution, well-posedness, and forward completeness for the system (26). If (26) is well-posed, for any initial time $t^0 \in \mathbb{R}$, initial value $x^0 \in X$ and input $u \in \mathcal{U}$, the corresponding maximal solution in the positive direction of the system (26) with $x(t^0) = x^0$ is denoted by $\phi(\cdot, t^0, x^0, u)$.

In this section, we assume:

Assumption 11. *The system (26) is well-posed with state space $X = \ell^p(\mathbb{N}, (n_i))$ and external input space $U = \ell^q(\mathbb{N}, (m_i))$ for some $p, q \in [1, \infty)$. Furthermore, all of its uniformly bounded maximal solutions in the positive direction $\phi(\cdot, t^0, x^0, u)$ are global, i.e., exist on $[t_0, \infty)$.*

Definition 12. *The system (26) is called* uniformly exponentially input-to-state stable (UeISS) *if it is forward complete and there are constants $a, M > 0$, independent of t^0, and $\gamma \in \mathcal{K}$ such that for any initial time $t^0 \in \mathbb{R}$, initial state $x^0 \in X$ and external input $u \in \mathcal{U}$ the corresponding solution of (26) satisfies for all $t \ge t^0$ that*

$$|\phi(t, t^0, x^0, u)|_p \le M \mathrm{e}^{-a(t-t^0)}|x^0|_p + \gamma(|u(t^0 + \cdot)|_{q,\infty}). \tag{27}$$

The uniformity here means that the transient term on the right-hand side of (27) depends on $t - t_0$, and not on t and t_0 individually.

By adding a "clock", one can (see [29]) transform (26) into

$$\begin{aligned} \dot{y} &= 1, \\ \dot{z} &= f(y, z, u), \end{aligned} \tag{28}$$

where $y \in \mathbb{R}$, $z \in X$, $u \in U$. We equip $\mathbb{R}$ with an arbitrary norm $|\cdot|$ and turn $\mathbb{R} \times X$ into an ℓ^p-space by putting

$$|(y, z)|_p := (|y|^p + |z|^p_p)^{1/p}.$$

Denoting the transition map of (28) by $\tilde{\phi} = \tilde{\phi}(t, (y, z), u)$, and its z-component by $\tilde{\phi}_2$, we see that the following holds:

$$\phi(t, t^0, x, u) = \tilde{\phi}_2(t - t^0, (t^0, x), u(t^0 + \cdot)) \quad \forall t \ge t^0. \tag{29}$$

The stability properties of (26) and (28) are related in the following way:

Proposition 13. *The system* (26) *is UeISS if and only if* (28) *is eISS with respect to the closed set* $\mathcal{A} = \mathbb{R} \times \{0\}$.

The proof is straightforward and thus omitted here.

Assume that the system (26) can be decomposed into infinitely many interconnected subsystems

$$\dot{x}_i = f_i(t, x_i, \bar{x}, u_i), \quad i \in \mathbb{N}, \tag{30}$$

with $t \in \mathbb{R}$, $x_i \in \mathbb{R}^{n_i}$, $\bar{x} \in X$ and $u_i \in \mathbb{R}^{m_i}$. Also, let $f_i\colon \mathbb{R} \times \mathbb{R}^{n_i} \times X \times \mathbb{R}^{m_i} \to \mathbb{R}^{n_i}$ be continuous with $f_i(t, 0, 0, 0) = 0$ for all $t \in \mathbb{R}$.

With each of the systems (30), we associate a time-invariant system by

$$\dot{z}_i = \tilde{f}_i(z_i, (y, \bar{z}), u_i) := f_i(y, z_i, \bar{z}, u_i), \tag{31}$$

where the time t now becomes an additional internal input y.

Define $\mathcal{A}_0 := \mathbb{R}$ and $\mathcal{A}_i := \{0\} \subset \mathbb{R}^{n_i}$ for all $i \geq 1$. Aggregating all subsystems (31), $i \in \mathbb{N}$, and adding the clock $\dot{y} = 1$ as the 0th subsystem, we obtain an infinite network of the form (28), modeled on the state space $\ell^p(\mathbb{N}_0, (n_i))$ with $n_0 := 1$.

To enable the stability analysis of the composite system, we make the following assumption.

Assumption 14. *For each* $i \in \mathbb{N}$*, there exists a continuous function* $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$*, satisfying the following properties:*

- *There are constants* $\underline{\alpha}_i, \overline{\alpha}_i > 0$ *so that for all* $z_i \in \mathbb{R}^{n_i}$

$$\underline{\alpha}_i |z_i|^p \leq V_i(z_i) \leq \overline{\alpha}_i |z_i|^p. \tag{32}$$

- *There are constants* $\lambda_i, \gamma_{ij}, \gamma_{iu} > 0$ *so that the following holds: for each* $z_i \in \mathbb{R}^{n_i}$*,* $u_i \in L^\infty(\mathbb{R}_+, \mathbb{R}^{m_i})$*, each internal input* $(y, \bar{z}) \in C^0(\mathbb{R}, \mathbb{R} \times X)$ *and for almost all* t *in the maximal interval in the positive direction of existence of* $\phi_i(t) := \phi_i(t, z_i, (y, \bar{z}), u_i)$*, one has*

$$\mathrm{D}^+(V_i \circ \phi_i)(t) \leq -\lambda_i V_i(\phi_i(t)) + \sum_{j \in I_i} \gamma_{ij} V_j(\bar{z}_j(t)) + \gamma_{iu} |u_i(t)|^q, \tag{33}$$

where we denote the components of $\bar{z}$ *by* $\bar{z}_j(\cdot)$.

- *For all* t *in the maximal interval in the positive direction of the existence of* ϕ_i*, one has* $\mathrm{D}_+(V_i \circ \phi_i)(t) < \infty$.

Note that due to the inequalities (16) and $\mathcal{A}_0 = \mathbb{R}$, we necessarily have $V_0 = 0$ for the eISS Lyapunov function of the 0th subsystem (the clock). Furthermore, we can choose λ_0 as an arbitrary positive number and $\gamma_{0j} := 0$ for all $j \in \mathbb{N}$.

The following corollary of Theorem 10 is our small-gain theorem for time-varying networks.

Corollary 15. *Consider networks* (26) *and* (28) *and assume that they are well-posed in the sense of Assumption 11 and Assumption 1, respectively. Further suppose the following:*

(i) Assumption 14 holds.

(ii) The constants in Assumption 14 are uniformly bounded as in Assumption 8.

(iii) Assumption 9 holds.

(iv) The spectral radius of Ψ *satisfies* $r(\Psi) < 1$.

Then the composite system (26) *is uniformly eISS.*

5. Distributed observers

We consider the problem of constructing *distributed observers for networks* of control systems. For simplicity, we set the external inputs u_i to zero (i.e., $u_i(t) \equiv 0$ for all $i \in \mathbb{N}$) and focus on the network interconnection aspect, rather than discussing the construction of individual local observers. For more details on observer theory for nonlinear systems, an interested reader is referred to [30, 31, 32, 33, 34].

Our basic assumption is that in a network context, we have local observers for all subsystems. We assume that the states of these *local observers* asymptotically converge to those of subsystems, given perfect knowledge of the states of neighboring subsystems. Of course, such information will be unavailable in practice, and instead each local observer will at best have the state estimates produced by other, neighboring observers available for its operation.

5.1. The distributed system to be observed

Let the distributed nominal system consist of infinitely many interconnected subsystems

$$\Sigma_i\colon \begin{cases} \dot{x}_i = f_i(x_i, \bar{x}_i) \\ y_i = h_i(x_i, \bar{x}_i) \end{cases}, \quad i \in \mathbb{N}. \tag{34}$$

While $x_i \in \mathbb{R}^{n_i}$ is the state of the system Σ_i, the quantity $y_i \in \mathbb{R}^{p_i}$ (for some $p_i \in \mathbb{N}$) is the output that can be measured locally and serves as an input for a state observer. We denote by $\bar{x}_i$ the vector composed of the state variables x_j, $j \in I_i$. Although our general setting allows each subsystem to directly interact with infinitely many other subsystems, in distributed sensing normally each subsystem is only connected to a finite number of subsystems. Therefore, the set I_i is assumed to be finite in this application. To make this observation as clear as possible, in (34), as opposed to the main body of the paper, we use the notation $\bar{x}_i$ in place of $\bar{x}$. Further, we assume that $f_i\colon \mathbb{R}^{n_i} \times \mathbb{R}^{N_i} \to \mathbb{R}^{n_i}$ and $h_i\colon \mathbb{R}^{n_i} \times \mathbb{R}^{N_i} \to \mathbb{R}^{p_i}$ are both continuous, where $N_i := \sum_{j \in I_i} n_j$.

5.2. The structure of the distributed observers

It is reasonable to assume that a local observer $\mathcal{O}_i$ for a system Σ_i has access to y_i and produces an estimate $\hat{x}_i$ of x_i for all $t \geq 0$. Moreover, we essentially need to know x_j for all $j \in I_i$ to reproduce the dynamics (34). Access to this kind of information is unrealistic, so instead it is assumed that it has access to the outputs y_j of neighboring subsystems and/or the estimates $\hat{x}_j$ for $j \in I_i$ produced by neighboring observers. For more details, one may consult the literature on distributed observation and filtering; see e.g. [35] for distributed observers in which the outputs and the state estimates are exchanged among local observers and [36] for those in which only state estimates are shared.

Here, we suppose that each *local observer* is represented by

$$\mathcal{O}_i\colon \begin{cases} \dot{\hat{x}}_i = \hat{f}_i(\hat{x}_i, y_i, \overline{y}_i, \bar{\hat{x}}_i) \\ \hat{y}_i = \hat{h}_i(\hat{x}_i, \bar{\hat{x}}_i) \end{cases}, \quad i \in \mathbb{N}, \tag{35}$$

for some appropriate continuous functions $\hat{f}_i$ and $\hat{h}_i$, respectively. Here, $\overline{y}_i$ (resp. $\bar{\hat{x}}_i$) is composed of the outputs y_j (resp. state variables $\hat{x}_j$), $j \in I_i$.

Necessarily, the observers are coupled in the same directional sense as the original distributed subsystems. Based on the small-gain theorem introduced above, this leads us to a framework for the design of distributed observers that guarantees that an interconnection of local observers exponentially tracks the true system state. Thus, we consider the composite system given by

$$\dot{x}_i = f_i(x_i, \bar{x}_i), \quad y_i = h_i(x_i, \bar{x}_i), \tag{36a}$$

$$\dot{\hat{x}}_i = \hat{f}_i(\hat{x}_i, y_i, \overline{y}_i, \bar{\hat{x}}_i), \quad \hat{y}_i = \hat{h}_i(\hat{x}_i, \bar{\hat{x}}_i), \quad i \in \mathbb{N}. \tag{36b}$$

5.3. A consistency framework for the design of distributed observers

Denote by ϕ_i and $\hat{\phi}_i$ the transition maps of the x_i-subsystem and $\hat{x}_i$-subsystem of (36), respectively, and define

$$\mathcal{A}_i := \{(x_i, \hat{x}_i) \in \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} : x_i = \hat{x}_i\}, \quad i \in \mathbb{N}.$$

Denote also by ϕ and $\hat{\phi}$ the transition maps of x-subsystem and $\hat{x}$-subsystem of (36), respectively.

Assumption 16. *We assume that the sequence of local observers $\mathcal{O} = (\mathcal{O}_i)_{i\in\mathbb{N}}$ for $\Sigma = (\Sigma_i)_{i\in\mathbb{N}}$ is given. Further, there is $p \in [1, \infty)$ so that for each $i \in \mathbb{N}$ there exist a continuous function $V_i\colon \mathbb{R}^{n_i} \times \mathbb{R}^{n_i} \to \mathbb{R}_+$, as well as constants $\overline{\alpha}_i, \underline{\alpha}_i > 0$ and $\lambda_i, \gamma_{ij} > 0$, $j \in I_i$ such that for all $x_i, \hat{x}_i \in \mathbb{R}^{n_i}$ the following holds:*

$$\underline{\alpha}_i |(x_i, \hat{x}_i)|^p_{\mathcal{A}_i} \le V_i(x_i, \hat{x}_i) \le \overline{\alpha}_i |(x_i, \hat{x}_i)|^p_{\mathcal{A}_i}. \tag{37}$$

Here, we endow the state space $\mathbb{R}^{n_i}$ of Σ_i, the state space $\mathbb{R}^{n_i}$ of the observer $\mathcal{O}_i$, and the state space $\mathbb{R}^{n_i} \times \mathbb{R}^{n_i}$ of the composite i-th subsystem with the ℓ_p-norm.

Furthermore, we assume that the dissipative estimates

$$\begin{aligned} \mathrm{D}^+(V_i \circ (\phi_i, \hat{\phi}_i))(t) \le & -\lambda_i V_i(\phi_i(t), \hat{\phi}_i(t)) \\ & + \sum_{j\in I_i} \gamma_{ij} V_j(x_j(t), \hat{x}_j(t)) \end{aligned} \tag{38}$$

hold for all $i \in \mathbb{N}$. Furthermore, for all t in the maximal interval in the positive direction of the existence of ϕ_i and $\hat{\phi}_i$ we have $\mathrm{D}_+(V_i \circ (\phi_i, \hat{\phi}_i))(t) < \infty$.

Following our general framework, we choose the state space for the system $\Sigma = (\Sigma_i)_{i\in\mathbb{N}}$ and for the network of local observers $\mathcal{O} = (\mathcal{O}_i)_{i\in\mathbb{N}}$ as $X := \ell^p(\mathbb{N}, (n_i))$ for p as in Assumption 16.

We would like to derive conditions which ensure that the network of local observers $\mathcal{O} = (\mathcal{O}_i)_{i\in\mathbb{N}}$ is a distributed observer for the whole system Σ, i.e., the error dynamics of the composite system (36) is globally exponentially stable.

Following our approach, the state space of the total composite system consists of sequences $(x_i, \hat{x}_i)_{i=1}^\infty$ with a finite ℓ_p-norm:

$$\begin{aligned} |(x_i, \hat{x}_i)_{i=1}^\infty|_p &= \Big(\sum_{i=1}^\infty |(x_i, \hat{x}_i)|^p\Big)^{1/p} \\ &= \Big(\sum_{i=1}^\infty |x_i|^p + |\hat{x}_i|^p\Big)^{1/p} < \infty. \end{aligned}$$

Hence, we can view the state space of the total composite system as the Banach space $X \times X$ with the norm $\|(x, y)\|_{X\times X} := \Big(|x|_p^p + |y|_p^p\Big)^{1/p}$, $(x, y) \in X \times X$. Define

$$\begin{aligned} \mathcal{A} &:= \{(x, \hat{x}) \in X \times X : x = \hat{x}\} \\ &= (X \times X) \cap (\mathcal{A}_1 \times \mathcal{A}_2 \times \ldots). \end{aligned} \tag{39}$$

Here is the main result of this section.

Theorem 17. *Consider the infinite interconnection Σ, given by equations (34), and the corresponding composite system (36), with fixed $p \in [1, \infty)$. Let the following hold:*

(i) (36) is well-posed and forward complete as a system on $X \times X$.

(ii) Each Σ_i admits a continuous eISS Lyapunov function V_i so that Assumptions 8 and 16 are satisfied.

(iii) Assumption 9 holds.

(iv) The spectral radius of Ψ satisfies $r(\Psi) < 1$.

Then the composite system (36) admits a Lyapunov function w.r.t. $\mathcal{A}$ as defined in (39) of the form

$$V(x, \hat{x}) = \sum_{i=1}^\infty \mu_i V_i(x_i, \hat{x}_i), \quad V : X \times X \to \mathbb{R}_+ \tag{40}$$

for some $\mu = (\mu_i)_{i\in\mathbb{N}} \in \ell^\infty$ satisfying $\underline{\mu} \le \mu_i \le \overline{\mu}$ with some constants $\underline{\mu}, \overline{\mu} > 0$. In particular, the function V has the following properties:

(a) V is continuous on $(X \times X) \setminus \mathcal{A}$.

(b) There is a $\lambda_\infty > 0$ so that for all $x^0 \in (X \times X) \setminus \mathcal{A}$

$$\mathrm{D}^+ V(x^0) \le -\lambda_\infty V(x^0).$$

(c) For all $x, \hat{x} \in X$, the following inequalities hold:

$$\underline{\mu}\,\underline{\alpha} |(x, \hat{x})|^p_{\mathcal{A}} \le V(x, \hat{x}) \le \overline{\mu}\,\overline{\alpha} |(x, \hat{x})|^p_{\mathcal{A}}. \tag{41}$$

Consequently, the error dynamics of (36) *is globally exponentially stable, i.e., there are* $M, a > 0$ *so that the following holds for all* $x, \hat{x} \in X$ *and all* $t \geq 0$*:*

$$|\phi(t, x) - \hat{\phi}(t, \hat{x})|_p \leq M e^{-at} |x - \hat{x}|_p, \tag{42}$$

which in turn means that $\mathcal{O} = (\mathcal{O}_i)_{i\in\mathbb{N}}$ *is a distributed observer for* Σ.

Proof. Applying Theorem 10, we obtain that V is an exponential Lyapunov function for the composite system (36) with respect to the set $\mathcal{A}$.

The distance of $(x, y) \in X \times X$ to the set $\mathcal{A}$ can be computed as

$$\begin{aligned}|(x,y)|_{\mathcal{A}} &:= \inf_{z\in X} \|(x,y) - (z,z)\|_{X\times X} \\ &= \inf_{z\in X} \left(|x-z|_p^p + |y-z|_p^p\right)^{1/p} = 2^{-\frac{p-1}{p}} |x-y|_p,\end{aligned} \tag{43}$$

where the infimum is achieved at $z = \frac{1}{2}(x+y)$.

To see this, note that for every $p \geq 1$ the function $f : x \mapsto x^p$ is convex, so in particular

$$f(x/2 + y/2) \leq f(x)/2 + f(y)/2.$$

Using the triangle inequality, we obtain

$$\begin{aligned}|x-y|_p^p &\leq \sum_i (|x_i - z_i| + |z_i - y_i|)^p \\ &= \sum_i f(|x_i - z_i| + |z_i - y_i|) \\ &\leq \frac{1}{2}\sum_i f(2|x_i - z_i|) + \frac{1}{2}\sum_i f(2|z_i - y_i|) \\ &= 2^{p-1}\Big(\sum_i |x_i - z_i|^p + \sum_i |z_i - y_i|^p\Big).\end{aligned}$$

This implies

$$2^{-(p-1)/p}|x-y|_p \leq (|x-z|_p^p + |z-y|_p^p)^{1/p},$$

and allows us to represent the norm of the error

$$e(t, x, \hat{x}) := \phi(t, x) - \hat{\phi}(t, \hat{x})$$

of the observer system (36) as

$$\begin{aligned}|e(t,x,\hat{x})|_p &= |\phi(t,x) - \hat{\phi}(t,\hat{x})|_p \\ &= 2^{\frac{p-1}{p}} \left|\left(\phi(t,x), \hat{\phi}(t,\hat{x})\right)\right|_{\mathcal{A}}.\end{aligned} \tag{44}$$

Hence, global exponential stability of (36) w.r.t. $\mathcal{A}$ implies global exponential stability of the error dynamics (w.r.t. the X-norm). □

5.4. Example

Consider an infinite network with sector nonlinearities, whose subsystems are described by

$$\Sigma_i : \begin{cases} \dot{x}_i = A_i x_i + E_i \varphi_i(G_i x_i) + D_i \bar{x}_i \\ y_i = C_i x_i \end{cases}, \quad i \in \mathbb{N}, \tag{45}$$

where $A_i \in \mathbb{R}^{n_i\times n_i}$, $E_i \in \mathbb{R}^{n_i}$, $G_i^\top \in \mathbb{R}^{n_i}$, $D_i \in \mathbb{R}^{n_i\times N_i}$, $C_i \in \mathbb{R}^{p_i\times n_i}$, $N_i = \sum_{j\in I_i} n_j$, $I_1 = \{2\}$, and $I_i = \{i-1, i+1\}$ for all $i \geq 2$. Let us take the standard Euclidean norm on each $\mathbb{R}^{n_i}$, $\mathbb{R}^{m_i}$ and $\mathbb{R}^{N_i}$. That is, we choose $p = 2$.

We assume that the pair (A_i, C_i) is detectable for all $i \in \mathbb{N}$. This ensures the existence of matrices $L_i \in \mathbb{R}^{n_i\times p_i}$ such that $\tilde{A}_i := A_i + L_i C_i$ are Hurwitz for all $i \in \mathbb{N}$.

Let the observer $\mathcal{O}_i$ be given for all $i \in \mathbb{N}$ by

$$\mathcal{O}_i : \begin{cases} \dot{\hat{x}}_i = A_i \hat{x}_i + E_i \varphi_i(G_i \hat{x}_i) + D_i \bar{\hat{x}}_i - L_i(y_i - \hat{y}_i), \\ \hat{y}_i = C_i \hat{x}_i. \end{cases} \tag{46}$$

Assume that A_i, E_i, G_i, D_i, L_i and C_i are uniformly bounded for all $i \in \mathbb{N}$, i.e., $\|A_i\| \leq a, \|E_i\| \leq e, \|G_i\| \leq g, \|B_i\| \leq b, \|D_i\| \leq d$, $\|L_i\| \leq l$ and $\|C_i\| \leq c$. We assume that the functions $\varphi_i : \mathbb{R} \to \mathbb{R}$ have some regularity properties (e.g. local Lipschitz continuity) such that well-posedness of the overall network (45) and (46) and the BIC property with the state space $\ell^2(\mathbb{N}, (n_i)) \times \ell^2(\mathbb{N}, (n_i))$ are ensured.

Now, for all $i \in \mathbb{N}$ define the function V_i as

$$V_i(x_i, \hat{x}_i) := (x_i - \hat{x}_i)^\top P_i (x_i - \hat{x}_i),$$

where $P_i \in \mathbb{R}^{n_i\times n_i}$, $i = 1, \ldots, n$ are symmetric, positive definite and uniformly bounded matrices, i.e.

$$\lambda_{\min}(P_i)|x_i - \hat{x}_i|^2 \leq V_i(x_i, \hat{x}_i) \leq \lambda_{\max}(P_i)|x_i - \hat{x}_i|^2, \tag{47}$$

where $0 < \underline{p} \leq \lambda_{\min}(P_i) \leq \lambda_{\max}(P_i) \leq \overline{p} < \infty$. $\lambda_{\min}$ and $\lambda_{\max}$ are, respectively, the smallest and largest eigenvalues. In that way, conditions (37) and (18) hold with $\underline{\alpha}_i = \lambda_{\min}(P_i)$, $\overline{\alpha}_i = \lambda_{\max}(P_i)$, $\underline{\alpha} = \underline{p}$, and $\overline{\alpha} = \overline{p}$.

Moreover, assume that for all $i \in \mathbb{N}$, $x_i \in \mathbb{R}^{n_i}$, the inequality

$$\begin{aligned}2(x_i - \hat{x}_i)^\top P_i\big(\tilde{A}_i(x_i - \hat{x}_i) + E_i(\varphi_i(G_i x_i) - \varphi_i(G_i\hat{x}_i))\big) \leq \\ -\kappa_i (x_i - \hat{x}_i)^\top P_i (x_i - \hat{x}_i)\end{aligned} \tag{48}$$

holds with $\kappa_i \geq \underline{\kappa} > 0$ for some $\underline{\kappa}$. We have

$$\begin{aligned}&\Big\langle \frac{\partial V_i(x_i,\hat{x}_i)}{\partial x_i}, f_i(x_i, \bar{x}_i)\Big\rangle + \Big\langle \frac{\partial V_i(x_i,\hat{x}_i)}{\partial \hat{x}_i}, \hat{f}_i(\hat{x}_i, y_i, \bar{y}_i, \bar{\hat{x}}_i)\Big\rangle = \\ &2(x_i - \hat{x}_i)^\top P_i\big(\tilde{A}_i(x_i - \hat{x}_i) + E_i(\varphi_i(G_i x_i) - \varphi_i(G_i\hat{x}_i))\big) \\ &+ 2(x_i - \hat{x}_i)^\top P_i D_i (\bar{x}_i - \bar{\hat{x}}_i).\end{aligned}$$

Since P_i is positive definite, $\sqrt{P_i}$ is well-defined, and we have

$$2(x_i - \hat{x}_i)^\top P_i D_i (\bar{x}_i - \bar{\hat{x}}_i)$$

$$
\begin{aligned}
&= 2(x_i - \hat{x}_i)^\top \sqrt{P_i}\sqrt{P_i} D_i (\bar{x}_i - \bar{\hat{x}}_i) \\
&\le 2|\sqrt{P_i}(x_i - \hat{x}_i)| \cdot |\sqrt{P_i} D_i (\bar{x}_i - \bar{\hat{x}}_i)| \\
&\le \epsilon_i |\sqrt{P_i}(x_i - \hat{x}_i)|^2 + \frac{1}{\epsilon_i}\|\sqrt{P_i} D_i\|^2 |\bar{x}_i - \bar{\hat{x}}_i|^2 \\
&= \epsilon_i (x_i - \hat{x}_i)^\top P_i (x_i - \hat{x}_i) + \frac{1}{\epsilon_i}\|\sqrt{P_i} D_i\|^2 |\bar{x}_i - \bar{\hat{x}}_i|^2.
\end{aligned}
$$

Using (48), and choosing $\epsilon_i \in (0, \kappa_i)$, we obtain

$$
\begin{aligned}
&\Big\langle \frac{\partial V_i(x_i,\hat{x}_i)}{\partial x_i}, f_i(x_i,\bar{x}_i) \Big\rangle + \Big\langle \frac{\partial V_i(x_i,\hat{x}_i)}{\partial \hat{x}_i}, \hat{f}_i(\hat{x}_i, y_i, \bar{y}_i, \bar{\hat{x}}_i) \Big\rangle \\
&\le -(\kappa_i - \epsilon_i)(x_i - \hat{x}_i)^\top P_i (x_i - \hat{x}_i) + \frac{\|\sqrt{P_i} D_i\|^2}{\epsilon_i} |\bar{x}_i - \bar{\hat{x}}_i|^2.
\end{aligned}
$$

From the first inequality of (47) and since $I_i = \{i-1, i+1\}$, one has

$$
\begin{aligned}
&\Big\langle \frac{\partial V_i(x_i,\hat{x}_i)}{\partial x_i}, f_i(x_i,\bar{x}_i) \Big\rangle + \Big\langle \frac{\partial V_i(x_i,\hat{x}_i)}{\partial \hat{x}_i}, \hat{f}_i(\hat{x}_i, y_i, \bar{y}_i, \bar{\hat{x}}_i) \Big\rangle \le \\
&- (\kappa_i - \epsilon_i) V_i(x_i, \hat{x}_i) \\
&+ \frac{\|\sqrt{P_i} D_i\|^2}{\epsilon_i} \Big(\frac{V_{i-1}(x_{i-1}, \hat{x}_{i-1})}{\lambda_{\min}(P_{i-1})} + \frac{V_{i+1}(x_{i+1}, \hat{x}_{i+1})}{\lambda_{\min}(P_{i+1})} \Big).
\end{aligned}
$$

Clearly, condition (38) is fulfilled with $\lambda_i = \kappa_i - \epsilon_i$, $\lambda_{ij} = \frac{\|\sqrt{P_i} D_i\|^2}{\epsilon_i \lambda_{\min}(P_j)}$ for all $j \in I_i$. To verify (19) and (22), we choose ϵ_i's so that there exist $\underline{\epsilon}, \bar{\epsilon} > 0$ with $\underline{\epsilon} \le \epsilon_i \le \bar{\epsilon} < \infty$. In that way, $\underline{\lambda} = \underline{\kappa} - \bar{\epsilon}$ and $\lambda_{ij} \le \frac{\bar{p} d^2}{\underline{p}\underline{\epsilon}}$. Finally, we need to verify the spectral radius condition for which we use the following sufficient condition

$$
r(\Psi) \le \|\Psi\| < 1, \tag{49}
$$

with $\psi_{ij} = \frac{\|\sqrt{P_i} D_i\|^2}{(\kappa_i - \epsilon_i)\epsilon_i \lambda_{\min}(P_j)}$ for $j \in I_i$ and $\psi_{ij} = 0$ otherwise. The second inequality of (49) is clearly satisfied if

$$
\frac{\bar{p} d^2}{(\underline{\kappa} - \bar{\epsilon})\underline{p}\underline{\epsilon}} < 1,
$$

which, in turn, holds for appropriate choices of ϵ_i and sufficiently small d. The latter particularly implies sufficiently small coupling between subsystems.

6. Conclusions

We developed a small-gain theorem ensuring exponential ISS with respect to a closed set for infinite networks. The small-gain condition was given in terms of the spectral radius representing the strength of the couplings between participating subsystems, for which there exist several insightful criteria, see [16, 37]. We illustrated the applicability of our small-gain theorem by applying it to the stability problem for time-varying infinite networks and distributed state estimation.

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Conflict of Interest and Authorship Conformation Form

Please check the following as appropriate:

- X All authors have participated in (a) conception and design, or analysis and interpretation of the data; (b) drafting the article or revising it critically for important intellectual content; and (c) approval of the final version.
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