MOTIVATING A Q-VERSION OF THE EMBEDDING BETWEEN THE CATEGORIES $\mathcal{H}eis$ AND $\mathcal{P}ar(t)$

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Abstract. Within this work, assuming an undergraduate background, we will introduce the material necessary to understand five categories and their relationships. These include the Heisenberg category $\mathcal{Heis}$, the partition category $\mathcal{Par}(t)$, the category of $\Bbbk S_n$-modules S_n**-mod**, the q-deformed Heisenberg category $\mathcal{Heis}(q)$, and $GL_n(\mathbb{F}_q)$**-mod**. The main aim of the work done in this paper is to motivate a generalization of a particular isomorphism of functors constructed by Likeng and Savage [6] between the non-q-deformed categories $\mathcal{Par}(n), \mathcal{Heis}_{\uparrow\downarrow}(n)$, and S_n**-mod** to a similar isomorphism between functors on their q-deformed counterparts. We offer a conjectured embedding into $\mathcal{Heis}_{\uparrow\downarrow}(q)$ and demonstrate which relations of the partition category are respected by the embedding.

Contents

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1. Introduction

In Section 2 we introduce groups, rings, modules, algebras, and representations, which are required to understand functors relating the categories that appear later. In Section 3 we introduce categories, functors, natural transformations, and strict monoidal categories. In Section 4 we introduce the relevant diagrammatic categories such as the partition category, the Heisenberg category, and the q-Heisenberg category. We then introduce the actions these categories have on S_n-modules in Section 5 and relate the action of the partition category to the action of its image through the Heisenberg category via a natural isomorphism, as done in [6]. We then move on to developing the known relationships between the q-categories in Section 6. Finally, in Section 7 we introduce a conjectured embedding which operates under the assumption that a potential q-version of the partition category has a diagrammatic presentation similar to that of the partition category and demonstrate the relations that the embedding respects and breaks.

2. Preliminaries

Here we will introduce concepts that are required to understand the later definitions of the main categories and the functors between them.

2.1. **Groups.** A group is essentially any collection of objects which can be multiplied or composed in an invertible manner. Formally, we have the following;

Definition 2.1 (Group)**.** A *group* is a set G along with a binary operation $\star : G \times G \to G$, which is written on $a, b \in G$ as $\star(a, b) = a \star b$, satisfying:

(1) $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in G$

(2) There is an element $e \in G$ with $a \star e = a = e \star a$

(3) For every $a \in G$ there is an element $a^{-1} \in G$ with $a \star a^{-1} = e = a^{-1} \star a$.

A group is called *abelian* if for all $a, b \in G$ we have $ab = ba$.

Some well known examples of groups are:

- The subsets of the complex numbers: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$ are all groups under addition.
- Any vector space V is a group under addition.
- For a vector space V, The space $GL(V)$ of invertible linear maps $\phi : V \to V$ with the binary operation taken to be composition is a group.
- The set S_n of all bijections $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a group under composition of functions.

To give an example of the elements of S_n, we can take $\sigma, \delta \in S_3$ to be the maps:

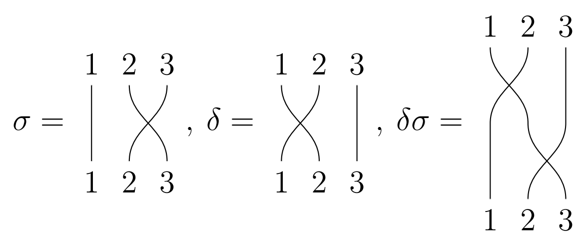
$$\sigma : (1, 2, 3) \mapsto (1, 3, 2)$$

$$\delta : (1, 2, 3) \mapsto (2, 1, 3)$$

so that their composition is given by:

$$\delta\sigma : (1, 2, 3) \mapsto (2, 3, 1).$$

These elements may be diagrammatically pictured as:



where we view composition as stacking diagrams, read from bottom to top.

When introducing algebraic structures, we must also introduce the maps between such structures that preserve relevant algebraic properties. In the case of groups, there is only one algebraic property to preserve, namely $\star$.

Definition 2.2 (Group Homomorphism)**.** A map $\phi : G \to G'$ between groups G with operation $\star$ and G' with operation $\star'$ is a *group homomorphism* if, for all $g_1, g_2 \in G$ we have $\phi(g_1 \star g_2) = \phi(g_1) \star' \phi(g_2)$.

2.2. **Rings.** Algebraic structures often have multiple operations that they come with. For instance, $\mathbb{C}$ has both addition and multiplication. The main algebraic object which contains an operation like addition and an operation like multiplication which behave well with one another is that of a ring.

Definition 2.3 (Ring)**.** A *ring* is a set R along with two binary operations $+ : R \times R \to R$ and $\star : R \times R \to R$ which is a group under $+$ and which satisfies:

(1) $(a \star b) \star c = a \star (b \star c)$ for all $a, b, c \in R$
(2) $a \star (b + c) = a \star b + a \star c$ and $(a + b) \star c = a \star c + b \star c$ for all $a, b, c \in R$.

Some common examples of rings are:

- The subsets of the complex numbers:$\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$ under multiplication and addition form rings.
- The $n \times n$ matrices over $\mathbb{R}$, denoted $M_n(\mathbb{R})$, is a ring under addition and matrix multiplication.
- The collection of all polynomials in an indeterminate x over $\mathbb{R}$ or $\mathbb{C}$ is a ring under multiplication and addition.

An important class of rings are those in which the collection of non-zero elements form an abelian group under the multiplication in the ring. Such rings are called *fields*. The requirement that the non-zero elements form a group is to ensure that we can always invert the multiplication done on objects.

Definition 2.4 (Ring Homomorphism)**.** A map $\phi : R \to R'$ between rings is called a *ring homomorphism* if, suppressing multiplicative notation, for all $r_1, r_2 \in R$ we have $\phi(r_1+r_2) = \phi(r_1)+\phi(r_2)$ and $\phi(r_1r_2) = \phi(r_1)\phi(r_2)$. That is, ϕ is an abelian group homomorphism which respects the multiplicative structure.

2.3. **Modules.** Algebraic objects can act on one another to generate new objects which often gives insight to the behavior of each of the objects separately. Whenever we have a ring acting on a group, we are dealing with a module.

Definition 2.5 (Module)**.** Given a ring R, a (left) R-module M is an abelian group M along with a map $\star : R \times M \to M$ satisfying:

(1) $r \star (m_1 + m_2) = r \star m_1 + r \star m_2$ for all $r \in R, m_1, m_2 \in M$
(2) $(r_1 + r_2) \star m = r_1 \star m + r_2 \star m$ for all $r_1, r_2 \in R, m \in M$
(3) $(r_1r_2) \star m = r_1 \star (r_2 \star m)$ for all $r_1, r_2 \in R, m \in M$
(4) If $1 \in R$ then $1 \star m = m$ for all $m \in M$.

Notice that these are just the axioms for a vector space, but where R is a ring instead of a field. Thus, all vector spaces fit the definition of a module over some field.

Definition 2.6 (R-Module Homomorphism)**.** A map $\phi : M \to M'$ between two R-modules, suppressing the $\star$ notation, is an *R-module homomorphism* if, for all $m_1, m_2 \in M, r \in R$ we have $\phi(m_1 + m_2) = \phi(m_1) + \phi(m_2)$ and $\phi(rm) = r\phi(m)$.

This is just an abelian group homomorphism that respects the R-action. If R is a field this is just a linear map.

The definition of modules above has a ring acting on a group on the *left*, but there is no reason we cannot have a ring acting on a group on the right. The axioms for a *right* module are the same as those for a left module but with the R-action on the right of the module elements. We can have both a left and right module if the left and right actions are compatible, which is stated as the following.

Definition 2.7 (Bimodule)**.** Given two rings S, R, an abelian group M is an (S, R)-*bimodule* if:

(1) M is a left S-module,
(2) M is a right R-module, and
(3) for all $s \in S, r \in R$ and $m \in M$ we have, suppressing notation for the action:

$$(sm)r = s(mr).$$

The relevant structure-preserving map is both a left and right module homomorphism.

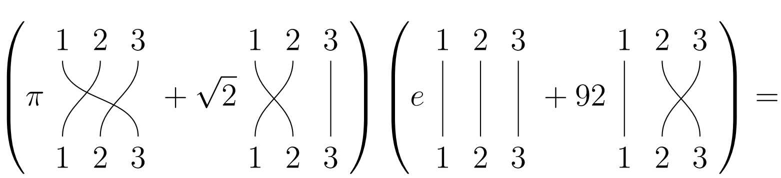
2.4. **Algebras.** Since modules allow rings to act on groups, the natural generalization is allowing rings to act on other rings. The relevant algebraic object here is called an algebra.

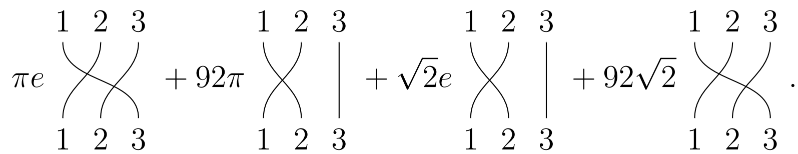
Definition 2.8 (Algebra)**.** Given a commutative, unital ring R, a ring A is called an R-algebra if A is an R-module and the R-module action commutes with multiplication, i.e. for all $r \in R$ and $a, a' \in A$ we have:

$$r(aa') = (ra)a' = a(ra').$$

One example of an R-algebra A is $\text{End}_R(V)$ for any R-module V. Any vector space with a bilinear multiplicative structure is an algebra over the field acting on the vector space.

An important class of R-algebras is the group algebras. Given any group G and a field $\Bbbk$, the $\Bbbk$-algebra $\Bbbk G$ is the $\Bbbk$-vector space with basis G and with multiplication on the basis defined by the operation in G which is extended linearly. For example, if $\Bbbk = \mathbb{R}$ and $G = S_3$ then we could compute:





The structure-preserving map between R-algebras A and B is a map $\phi : A \to B$ that is an R-module homomorphism and ring homomorphism. We will require such morphisms to map multiplicative identity to multiplicative identity. Such maps are called *unital*.

2.5. **Modules and Representations.** Often it is helpful to model algebraic objects in more concrete objects such as matrices. This is done formally by using the relevant structure-preserving map to translate algebraic information of some object into a linear space, which is called a representation.

Definition 2.9 (Algebra Representation)**.** For a field $\Bbbk$, and a $\Bbbk$-algebra A, a representation of A is an algebra homomorphism $\phi : A \to \mathrm{End}_{\Bbbk}(V)$ for some $\Bbbk-$vector space V.

An important result from representation theory tells us that modules and representations coincide, i.e., there is a correspondence:

$$\{A\text{-modules } V \text{ with action } \star : A \times V \to V\} \leftrightarrow \{\text{representations } \phi_\star : A \to \mathrm{End}_{\Bbbk}(V)\}$$

$$(V, \star) \mapsto \phi_\star$$

$$(V, \star_\phi) \leftarrow\!\shortmid\, \phi.$$

If V has the action $\star : A \times V \to V$ given by $(a, v) \mapsto a \star v$, then the map $\phi_\star : A \to \mathrm{End}_{\Bbbk}(V)$ is the one satisfying:

$$(\phi_\star(a))(v) = a \star v$$

for all $a \in A, v \in V$.

The inverse to this is taking any algebra homomorphism $\phi : A \to \mathrm{End}_{\Bbbk}(V)$ and defining an action $\star_\phi : A \times V \to V$ which satisfies, for all $a \in A, v \in V$:

$$\star_\phi(a, v) = (\phi(a))(v).$$

There is an analogous concept for groups.

Definition 2.10 (Group Representation)**.** A representation of G is any group homomorphism $\phi : G \to GL(V)$ for some $\Bbbk$ vector space V.

The natural question is then: How do representations of G relate to representations of $\Bbbk G$? In this case, the representations essentially coincide; There is a correspondence:

$$\{\text{Group representations } \phi : G \to GL(V)\} \leftrightarrow \{\text{Algebra representations } \Phi : \Bbbk G \to \mathrm{End}_{\Bbbk}(V)\}.$$

This is because any group representation $\phi : G \to GL(V)$ induces an algebra representation $\Phi : \Bbbk G \to \mathrm{End}_{\Bbbk}(V)$ since ϕ defines a vector space homomorphism from $\Bbbk G \to \mathrm{End}_{\Bbbk}(V)$ that is also a ring homomorphism. The induced map on $\Bbbk G$ is also a ring homomorphism since the product of any two basis elements $g, g' \in G$ satisfy:

$$\Phi(gg') = \phi(gg') = \phi(g)\phi(g') = \Phi(g)\Phi(g')$$

making Φ an algebra homomorphism into $\mathrm{End}_{\Bbbk}(V)$. Conversely any algebra representation $\Phi : \Bbbk G \to \mathrm{End}_{\Bbbk}(V)$ restricts to a group representation $\phi : G \to GL(V)$, which maps into $GL(V)$ since:

$$\phi(g)\phi(g^{-1}) = \Phi(g)\Phi(g^{-1}) = \Phi(gg^{-1}) = \Phi(e) = 1_V,$$

which shows the image of ϕ is contained in $GL(V)$. Thus, the constructions essentially coincide.

3. Categories and Functors

3.1. **Categories.** Many mathematical objects come with a structure-preserving map. All of the algebraic objects we have introduced so far come with such maps. Collecting related mathematical objects to study them all at once can be useful. This is formalized by category theory.

Definition 3.1 (Category)**.** A *category* $\mathscr{C}$ is a collection of objects $\mathrm{Ob}(\mathscr{C})$ and, for every two objects $c, c' \in \mathrm{Ob}(\mathscr{C})$, a collection of morphisms $\mathrm{Hom}_{\mathscr{C}}(c, c')$ from c to c' such that:

(1) For all $a, b, c \in \mathrm{Ob}(\mathscr{C})$, if $f \in \mathrm{Hom}_{\mathscr{C}}(a, b)$ and $g \in \mathrm{Hom}_{\mathscr{C}}(b, c)$ there is an element $g \circ f \in \mathrm{Hom}_{\mathscr{C}}(a, c)$ and, whenever defined, $h \circ (g \circ f) = (h \circ g) \circ f$.

(2) For all $c \in \mathrm{Ob}(\mathscr{C})$ the set $\mathrm{Hom}_{\mathscr{C}}(c, c)$ contains an element 1_c which satisfies, for all $f \in \mathrm{Hom}_{\mathscr{C}}(a, b)$, $f \circ 1_a = f = 1_b \circ f$.

The collection of all morphisms in a category will be denoted $\mathrm{Ar}(\mathscr{C})$ so that every element f of $\mathrm{Ar}(\mathscr{C})$ carries the data of which set $\mathrm{Hom}_{\mathscr{C}}(a, b)$ it is an element of.

Some examples of categories include:

- The category **Group** has groups as its objects and group homomorphisms as its morphisms.
- The category **Top** has topological spaces as its objects and continuous maps as its morphisms.
- The category **Set** has sets as its objects and functions as its morphisms.
- The category **Cat** has categories as its objects and functors as its morphisms.
- The category R**-mod** has R-modules as its objects and R-module homomorphisms as its morphisms.
- The category (S, R)**-bimod** has (S, R)-modules as its objects and (S, R)-module homomorphisms as its morphisms.

If we have a category $\mathscr{C}$ we can obtain another category by only considering a subcollection of the objects from $\mathscr{C}$ and using the morhpisms between this subcollection that already exist in $\mathscr{C}$. This construction is known as a full subcategory.

Definition 3.2 (Full Subcategory)**.** Given a collection of objects of a category $\mathscr{C}$, the *full subcategory* containing these objects is the category with this collection of objects as its objects and with all morphisms from $\mathscr{C}$ between these objects as its morphisms.

Definition 3.3 (Additive Category)**.** A category $\mathscr{C}$ is *additive* if, for all $c, c' \in \mathrm{Ob}(\mathscr{C})$, we have $\mathrm{Hom}_{\mathscr{C}}(c, c')$ is an abelian group.

A construction that appears later on additive categories is that of the direct sum and direct product.

Definition 3.4 (Direct Sum and Product of Additive Categories)**.** If $\{\mathscr{C}_i\}_{i\in I}$ is a collection of abelian categories, then $\prod_i \mathscr{C}_i$ has objects $(c_i)_{i\in I}$ with $c_i \in \mathscr{C}_i$ and arrows $(f_i) \in \mathrm{Hom}_{\prod_i \mathscr{C}_i}((c_i), (c_i'))$ where $f_i \in \mathrm{Hom}_{\mathscr{C}_i}(c_i, c_i')$ and $\bigoplus_i \mathscr{C}_i$ is the full subcategory of this category in which the objects (c_i) have $c_i = 0$ (whatever the zero object is in the category) for all but finitely many $i \in I$.

3.2. **Functors.** As stated before, most mathematical objects come with structure-preserving maps. The structure required by a category is the ability to compose morphisms and the requirement of the existence of a morphism which does nothing. Thus, the morphism of categories, called a functor, will preserve this structure.

Definition 3.5 (Functor)**.** Given categories $\mathscr{C}, \mathscr{B}$ a *functor* $F : \mathscr{C} \to \mathscr{B}$ is a function $F : \mathrm{Ob}(\mathscr{C}) \to \mathrm{Ob}(\mathscr{B})$ with $c \mapsto Fc$ and a function $F : \mathrm{Ar}(\mathscr{C}) \to \mathrm{Ar}\mathscr{B}$ where a morphism $f : a \to b$ is mapped to $Ff : Fa \to Fb$ which satisfies:

$$F(g \circ f) = Fg \circ Ff, \;\; F\mathrm{id}_c = \mathrm{id}_{Fc}.$$

For all $g, f \in \mathrm{Ar}(\mathscr{C})$ whenever the composition $g \circ f$ is defined and for all $c \in \mathrm{Ob}(\mathscr{C})$.

An example of a functor is the homology functor H_p between the category of pairs of topological spaces and the category of abelian groups. Another example is the forgetful functor $F : \mathbf{Group} \to \mathbf{Set}$ which takes a group to its underlying set and a homomorphism to its underlying function.

A functor $F : \mathscr{C} \to \mathscr{B}$ defines, for every $c, c' \in \mathrm{Ob}(\mathscr{C})$ an association $F_{c,c'} : \mathrm{Hom}_{\mathscr{C}}(c, c') \to \mathrm{Hom}_{\mathscr{B}}(Fc, Fc')$. A functor is said to be *full* whenever every such function is a surjection and is said to be *faithful* whenever every such function is injective.

Two important functors will come into play later, but they require some setup.

3.2.1. *Induction and Restriction.* Recall that, for unital rings S, R and an (S, R)-bimodule M and an R-module N we can form the S-module $M \otimes_R N = \mathbb{Z}(M \times N)/\mathcal{H}$, the quotient of the free abelian group on $M \times N$ by the group $\mathcal{H}$ generated by elements of the form (where $m_1, m_2, m \in M, n_1, n_2, n \in N, r \in R$):

$$\begin{aligned}(m_1 + m_2, n) - (m_1, n) - (m_2, n),\\ (m, n_1 + n_2) - (m, n_1) - (m, n_2), \text{ and}\\ (mr, n) - (m, rn).\end{aligned}$$

The S-action on the cosets $m \otimes n = (m, n) + \mathcal{H}$ is given by $s(m \otimes n) = sm \otimes n$ extended to sums. The verification that this definition extends to the cosets is not immediate but uses the relations between cosets induced by the elements we quotiented out by.

Moreover, given any abelian group L, and any map $\phi : M \times N \to L$ satisfying, for all $m_1, m_2, m \in M, n_1, n_2, n \in N$ and $r \in R$:

$$\phi(m_1 + m_2, n) = \phi(m_1, n) + \phi(m_2, n),$$

$$\phi(m, n_1 + n_2) = \phi(m, n_1) + \phi(m, n_2), \text{ and}$$

$$\phi(m, rn) = \phi(mr, n),$$

we obtain a unique group homomorphism $\Phi : M \otimes_R N \to L$ making the following diagram commute:

$$\begin{array}{ccc} M \times N & \xrightarrow{\iota} & M \otimes_R N \\ & \overset{\phi}{\searrow} & \downarrow \Phi \\ & & L \end{array},$$

where $\iota : M \times N \to M \otimes_R N$ is the projection $(m, n) \mapsto m \otimes n$. Conversely, any group homomorphism $\Phi : M \otimes_R N \to L$ induces the map $\phi = \Phi \circ \iota$ which is a map satisfying the three properties above. Moreover, any abelian group which satisfies this universal property is isomorphic to this particular construction of $M \otimes_R N$. This universal property is the main tool used to define homomorphisms which leave tensor products.

For example, if $\Bbbk$ is a field and G, G' groups, we can form $\Bbbk G \otimes_{\Bbbk} \Bbbk G'$ which has basis $\{g \otimes g' : g \in G, g' \in G'\}$ which is just their tensor product as $\Bbbk$-vector spaces.

More interestingly, if $H \leq G$ then $\Bbbk G$ has a natural $(\Bbbk G, \Bbbk H)$-bimodule structure so that for any $\Bbbk H$-module, V we may form the $\Bbbk G$-module $\Bbbk G \otimes_{\Bbbk H} V$. The map $V \mapsto \Bbbk G \otimes_{\Bbbk_H} V$ is known as *induction*. The tensor product construction is extending the action of $\Bbbk H$ on V in the "best possible" way to an action by $\Bbbk G$ on V. We can do the opposite of this extension of actions by forgetting some structure. This is done by viewing any $\Bbbk G$-module V as a $\Bbbk H$-bimodule and forming $\Bbbk H \otimes_{\Bbbk H} V$ which is essentially just restricting the $\Bbbk G$ action to $\Bbbk H$. This map $V \mapsto \Bbbk H \otimes_{\Bbbk H} V$ is known as *restriction*. There is a precise sense in which induction and restriction are inverses of each other (they are known as biadjoint functors). Thus, have two functors:

Definition 3.6 (Induction)**.** The *induction functor* is given by:

$$\begin{aligned}\mathrm{Ind}_H^G : \Bbbk H\textbf{-mod} &\to \Bbbk G\text{-mod}\\ M &\mapsto \Bbbk G \otimes_{\Bbbk H} M.\end{aligned}$$

Definition 3.7 (Restriction)**.** The *restriction functor* is given by:

$$\begin{aligned}\mathrm{Res}_G^H : \Bbbk G\textbf{-mod} &\to \Bbbk H\text{-mod}\\ M &\mapsto \Bbbk H \otimes_{\Bbbk H} M.\end{aligned}$$

All we have stated so far is these functors' actions on objects. On morphisms we have the following. Given $f : M \to N$ (between $\Bbbk H$-modules or $\Bbbk G$-modules), the induced or restricted map is just the map that acts on the rightmost component of the product as f. For instance, if M and N are $\Bbbk H$-modules, then $f : M \to N$ induces $\mathrm{Ind}_H^G f : \Bbbk G \otimes_{\Bbbk H} M \to \Bbbk G \otimes_{\Bbbk H} N$ via $\mathrm{Ind}_H^G f(g \otimes m) = g \otimes f(m)$. This is somewhat non-trivial.

To show that such an f induces a map on the tensor product, we can use the universal property mentioned above. For instance, in the case of induction, the map $\mathrm{ind} f : \Bbbk G \times M \to \Bbbk G \otimes_{\Bbbk H} N$ given by $\mathrm{ind} f((g, m)) = g \otimes f(m)$ satisfies, for all $g_1, g_2, g \in \Bbbk G$, $m_1, m_2, m \in M$ and $h \in \Bbbk H$:

$$\begin{aligned}\mathrm{ind} f(g_1 + g_2, m) &= \mathrm{ind} f(g_1, m) + \mathrm{ind} f(g_2, m),\\ \mathrm{ind} f(g, m_1 + m_2) &= \mathrm{ind} f(g, m_1) + \mathrm{ind} f(g, m_2),\\ \mathrm{ind} f(g, hm) &= \mathrm{ind} f(gh, m).\end{aligned}$$

This relies on f being a $\Bbbk H$-module homomorphism and properties of $\Bbbk G \otimes_{\Bbbk H} N$. For instance, the third equality holds since:

$$\mathrm{ind} f(g, hm) = g \otimes f(hm)$$

$$= g \otimes hf(m)$$

$$= gh \otimes f(m)$$

$$= \mathrm{ind}f(gh, m).$$

Thus, the universal property above allows us to conclude that defining $\mathrm{Ind}_H^G f$ to be the map given by $\mathrm{Ind}_H^G f(g \otimes m) = g \otimes f(m)$ is a valid group homomorphism $\mathrm{Ind}_H^G f : \Bbbk G \otimes_{\Bbbk H} M \to \Bbbk G \otimes_{\Bbbk H} N$. The fact this group homomorphism is also a $\Bbbk G$-module homomorphism follows from the definition of the $\Bbbk G$ action on the modules since, for all $g, g' \in \Bbbk G$ and $m \in M$ we have:

$$\begin{aligned}
\mathrm{Ind}_H^G f(g'(g \otimes m)) &= \mathrm{Ind}_H^G f((g'g) \otimes m) \\
&= (g'g) \otimes f(m) \\
&= g'(g \otimes f(m)) \\
&= g' \,\mathrm{Ind}_H^G f(g \otimes m).
\end{aligned}$$

3.2.2. *Parabolic Induction and Restriction.* A very similar pair of functors also exists on group algebras. Given a group G and subgroups $L, U \leq G$ with $U \trianglelefteq LU$ and $L \cap U = \{e\}$ we may let $v \in \Bbbk U$ be defined by:

$$v = \frac{1}{|U|} \sum_{u \in U} u,$$

which satisfies

(1) $v^2 = v$,
(2) $uv = vu = v$ for all $u \in U$, and
(3) $lv = vl$ for all $l \in L$.

This element v makes $\Bbbk Gv$ into a $(\Bbbk G, \Bbbk L)$-bimodule and allows us to define the variant of induction.

Definition 3.8 (Parabolic Induction [5])**.** The *parabolic induction functor* is given by:

$$\begin{aligned} \mathrm{IndP}_G^L : \Bbbk L\text{-}\mathbf{mod} &\to \Bbbk G\text{-}\mathbf{mod} \\ M &\mapsto \Bbbk G v \otimes_{\Bbbk L} M. \end{aligned}$$

Similarly, this element v makes $v\Bbbk G$ into a $(\Bbbk L, \Bbbk G)$-bimodule and allows us to define the variant of restriction.

Definition 3.9 (Parabolic Restriction [5])**.** The *parabolic restriction functor* is given by:

$$\begin{aligned} \mathrm{ResP}_G^L : \Bbbk G\text{-}\mathbf{mod} &\to \Bbbk L\text{-}\mathbf{mod} \\ M &\mapsto v\Bbbk G \otimes_{\Bbbk G} M. \end{aligned}$$

3.2.3. *Natural Transformations.* Now that we have defined arrows between categories, we may ask what a morphism of functors is. The morphism of functors is known as a natural transformation.

Definition 3.10 (Natural Transformation)**.** Given parallel functors $S, T : \mathscr{C} \to \mathscr{B}$, a *natural transformation* $\eta : S \to T$ is a function which assigns to every $c \in \mathrm{Ob}(\mathscr{C})$ an arrow $\eta_c : Sc \to Tc$ in $\mathrm{Ar}(\mathscr{B})$ such that for all $f : c \to c'$ we have the following commutative diagram:

$$\begin{array}{ccccc} c & \quad & Sc & \xrightarrow{\eta_c} & Tc \\ {\scriptstyle f}\downarrow & & {\scriptstyle Sf}\downarrow & & \downarrow{\scriptstyle Tf} \\ c' & & Sc' & \xrightarrow[\eta_{c'}]{} & Tc' \end{array} .$$

3.3. **Strict, Monoidal Categories.**

Definition 3.11 (Product of Categories)**.** Given two categories $\mathscr{C}, \mathscr{B}$ we may form the product category $\mathscr{C} \times \mathscr{B}$ whose objects are pairs (c, b) where $c \in \mathrm{Ob}(\mathscr{C}), b \in \mathrm{Ob}(\mathscr{B})$ and whose arrows $(c, b) \to (c', b')$ consist of pairs (f, g) where $f : c \to c'$ and $g : b \to b'$ in $\mathrm{Ar}(\mathscr{C}), \mathrm{Ar}(\mathscr{B})$ respectively.

Some categories carry a unital and associative multiplicative structure. Such categories are called strict monoidal categories.

Definition 3.12 (Strict Monoidal Category [8])**.** We define a *strict monoidal category* to be a category $\mathscr{C}$ along with a functor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ and an identity object $\mathbb{1}$ such that for all $a, b, c \in \mathrm{Ob}(\mathscr{C})$ and $f, g, h \in \mathrm{Ar}(\mathscr{C})$ we have:

$$h \otimes (g \otimes f) = (h \otimes g) \otimes f,\ \mathrm{id}_{\mathbb{1}} \otimes f = f = f \otimes \mathrm{id}_{\mathbb{1}}$$

and

$$a \otimes (b \otimes c) = (a \otimes b) \otimes c,\ \mathbb{1} \otimes a = a = a \otimes \mathbb{1}.$$

The “strict" refers to the strict equalities the functor satisfies above, as it is possible to have a monoidal category in which these equalities hold up to natural isomorphisms.

Some categories can contain a module structure on their morphisms which behaves well with composition.

Definition 3.13 ($\Bbbk$-Linear Category [8])**.** For any commutative ring $\Bbbk$, a $\Bbbk$-*linear category* is a category $\mathscr{C}$ in which every hom set $\mathrm{Hom}_{\mathscr{C}}(a, b)$ is a $\Bbbk$-module along with the requirement that

$$h \circ (\alpha g + \beta f) = \alpha(h \circ g) + \beta(h \circ f)$$

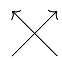
$$(\alpha h + \beta g) \circ f = \alpha(h \circ f) + \beta(g \circ f)$$

for any $\alpha, \beta \in \Bbbk$ and for any $f, g, h \in \mathrm{Hom}_{\mathscr{C}}(a, b)$ for which the compositions are defined.

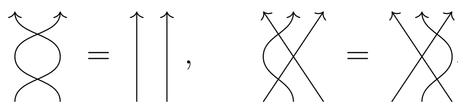
An example of a $\Bbbk$-linear category is the category of $\Bbbk$-modules, since the hom sets form a $\Bbbk$-module under pointwise addition and scalar multiplication and composition of module homomorphisms is bilinear. Notice that this makes $\mathrm{End}_{\mathscr{C}}(a)$ into a $\Bbbk$-algebra.

Definition 3.14 (Strict $\Bbbk$-Linear Monoidal Category)**.** A *strict,* $\Bbbk$*-linear, monoidal category* is a category that is both strict monoidal and $\Bbbk$-linear.

Such categories are often well suited enough to be interpreted via string diagrams. One such example, as in [8], is the strict monoidal category $\mathscr{S}$ generated by the object $\uparrow$ and the morphism:

 $: \uparrow \otimes \uparrow \to \uparrow \otimes \uparrow$

with the relations



This category contains all the algebras $\Bbbk S_n$ via an isomorphism $\mathrm{End}_{\mathscr{S}}(\uparrow^{\otimes n}) \cong \Bbbk S_n$ which is an especially efficient presentation of these algebras.

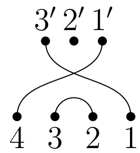
4. Some Diagrammatic Categories

Many of the categories that we have introduced so far have very "large" objects. For instance, the objects in **Group** can be uncountable sets. However, the definition of a category only requires the existence of objects and morphisms which have a nice enough composition. The following categories have objects that are comparatively "small," but this clarifies how some categories can have nice multiplicative and module structures.

4.1. **The Partition Category.**

This section follows the presentation in [6]. Given $n, m \in \mathbb{N}$, an $\binom{n}{m}$-partition is a partition of the set $\{1, \ldots, m, 1', \ldots, n'\}$. There is a bijection between such partitions and diagrams with a lower row labeled from right to left with nodes $1, \ldots, m$ and with an upper row labeled from right to left with nodes $1', \ldots, n'$. For instance, the

$\binom{3}{4}$-partition $\{\{1,3'\},\{2,3\},\{4,1'\},\{2\}\}$ is associated with the diagram:

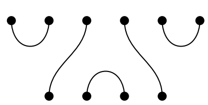
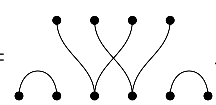


.

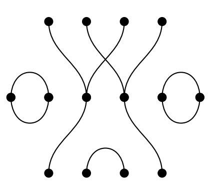
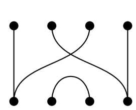
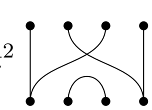
From now on, row labels will be omitted. Any $\binom{n}{m}$-partition D will be denoted $D : m \to n$. Given $D : m \to n$ and $D' : n \to k$ we can form the diagram $D' \Box D$ by stacking their associated diagrams (D' is placed on top of D). Let $\alpha(D'\Box D)$ be the number of connected components in the middle row of $D'\Box D$. Let $D' \star D : m \to k$ be the partition associated with the diagram obtained by stacking D' on top of D and only keeping the connected components in the first and third rows.

Definition 4.1 (The Partition Category [6, §2]). Fix $t \in \Bbbk$. The *partition category* $\mathcal{P}ar(t)$ is the strict, $\Bbbk$-linear, monoidal category with objects $\mathbb{N}$ and with arrows from m to n all finite $\Bbbk$-linear combinations of $\binom{n}{m}$-partitions with composition given by $D' \circ D = t^{\alpha(D'\Box D)} D' \star D$. The functor $\otimes$ on objects is just addition and on arrows is given by horizontal juxtaposition extended linearly.

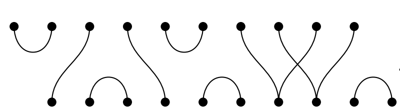
To get a sense of these operations, let D and D' be the following diagrams:

$D =$  and $D' =$  ,

then,

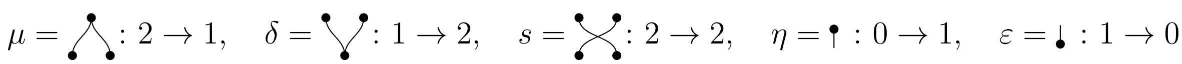
$D'\Box D =$  , $D' \star D =$  , and $D' \circ D = t^2$  .

Their tensor product is given by:

$D \otimes D' =$  .

Often, it is easiest to specify structure-preserving morphisms which leave an object if one has a simple presentation of that object. Whenever this is the case, specifying a morphism boils down to defining its image on the simplest pieces of the object in a way that the relevant structure is preserved between these simple pieces. The partition category has the following simple presentation.

Proposition 4.2 (A Diagrammatic Presentation of the Partition Category [6, §2], [2, §2])**.** *As a strict* $\Bbbk$*-linear monoidal category, the* partition category $\mathcal{P}ar(t)$ *is generated by the object* 1 *and the morphisms*



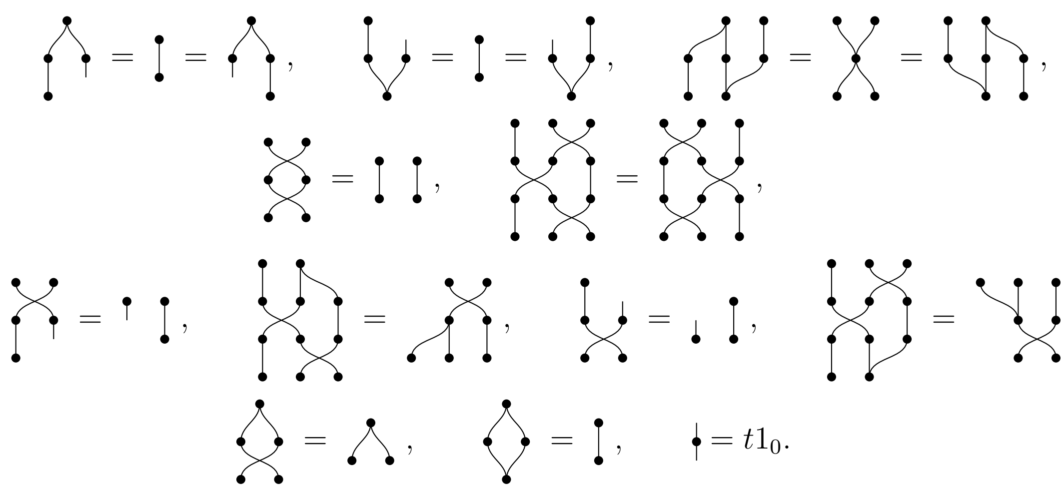
subject to the following relations:

(4.1)

(4.2)

(4.3)

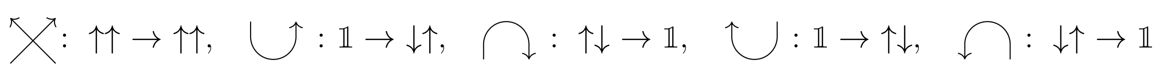
(4.4)



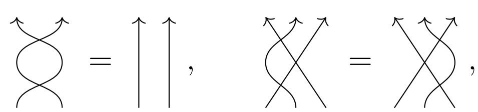
This presentation is given *as a strict* $\Bbbk$*-linear monoidal category.* Thus, we can use all the operations that strict $\Bbbk$-linear monoidal categories are endowed with to generate new objects from these simple few. The tensor product structure allows us to juxtapose these morphisms and the composition structure allows us to stack these morphisms. Often this is done in reverse by first drawing a desired morphism and breaking it up into its simpler parts to apply the local relations given above.

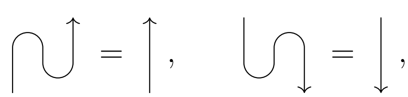
4.2. **The Heisenberg Category.** The Heisenberg category is another diagrammatic category. The representation of its objects can, at first, be confusing since its *objects* are sequences of up arrows and down arrows and its *morphisms* are arrows between these sequences that are the same up to "straightening lines". Thus, throughout this thesis we will try to be careful to use the symbols $\uparrow, \downarrow$ to denote objects in the Heisenberg category, and longer arrows to denote morphisms between the objects. The visual depiction of these morphisms naturally indicates the domain and codomain of the morphisms. A simple presentation of this category is given by the following.

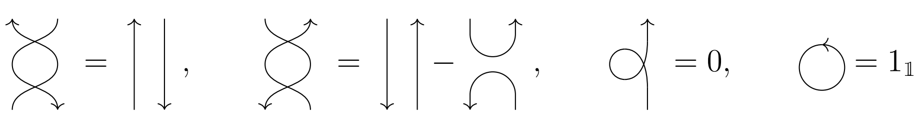
Definition 4.3 (The Heisenberg Category [4][6, §3]). The *Heisenberg category* $\mathcal{H}eis$ is the strict $\Bbbk$-linear monoidal category generated by two objects $\uparrow$, $\downarrow$ (and the unit object $\mathbb{1}$), where we use horizontal juxtaposition to denote the tensor products, and has generating morphisms



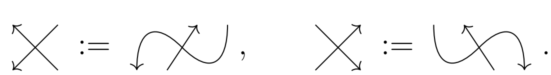
where $\mathbb{1}$ denotes the unit object, subject to the relations

(4.5) 

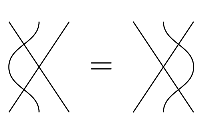
(4.6) 

(4.7) 

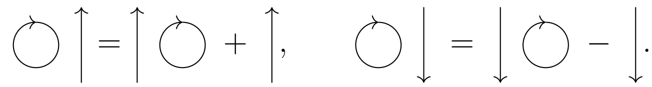
Here the left and right crossings are defined by



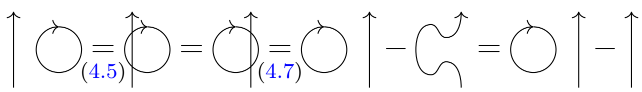
The relations imply that [1, (20)]:

(4.8)  for all possible orientations of the strands

To give a sense of how these relations are applied locally to diagrams, consider commuting a bubble past an arrow:

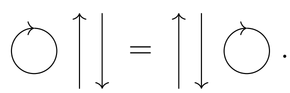
(4.9) 

For instance, to see the first relation, merge the up arrow with the bubble to obtain:

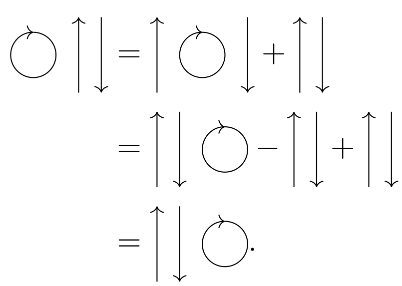


The first equality uses that an upward double crossing can be uncrossed (the first relation in 4.5). The second equality follows since diagrams are equal up to isotopy. The third equality follows by an application of the second relation in 4.7. The last equality follows by equivalence up to isotopy.

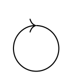
Let $\mathcal{H}eis_{\uparrow\downarrow}$ denote the $\Bbbk$-linear monoidal subcategory of $\mathcal{H}eis$ generated by $\uparrow\downarrow$. The relations imply that



This is not too hard to see by applying the relations in 4.9:

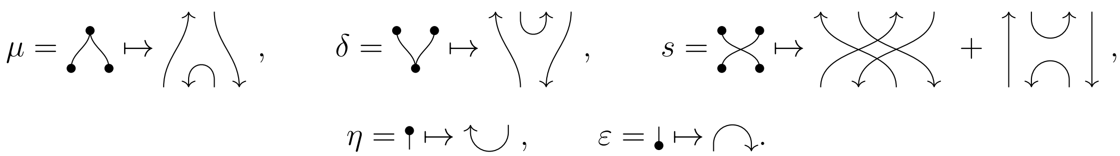


In other words, the clockwise bubble commutes with the morphisms in $\mathcal{H}eis_{\uparrow\downarrow}$. Thus, fixing $t \in \Bbbk$, we can define $\mathcal{H}eis_{\uparrow\downarrow}(t)$ to be the quotient of $\mathcal{H}eis_{\uparrow\downarrow}$ by the additional relation

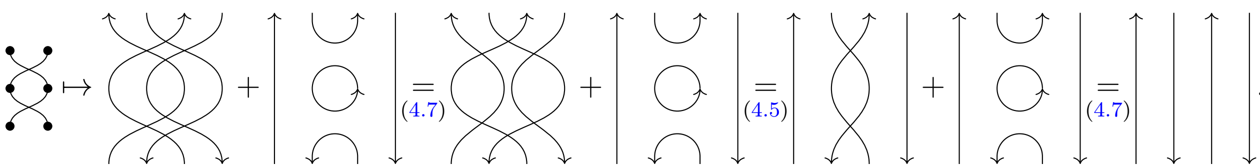
(4.10)  $= t1_{\mathbb{1}}$.

Since we have simple presentations of these categories, we can now describe a functor between them.

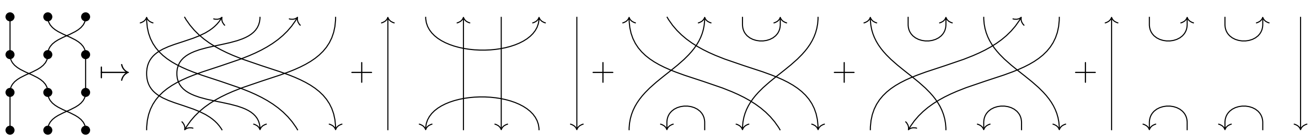
Theorem 4.4 (The Embedding [6, §4])**.** *There is a strict linear monoidal functor* $\Psi_t\colon \mathcal{P}ar(t) \to \mathcal{H}eis_{\uparrow\downarrow}(t)$ *defined on objects by* $k \mapsto (\uparrow\downarrow)^k$ *and on generating morphisms by*



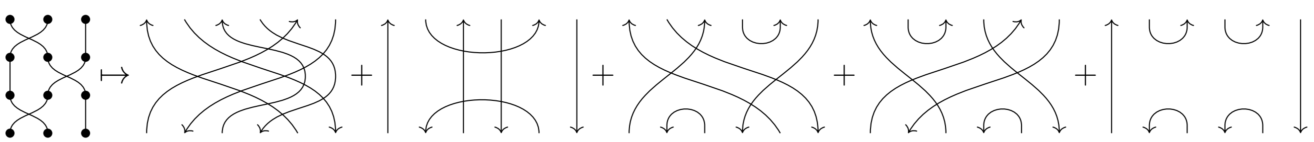
Proof. To prove that such a functor exists, only need to check that all the relations between morphisms in $\mathcal{P}ar(t)$ are respected by this correspondence. We repeat the proof given in [6] to show what is required to check. The first relation in 4.2 holds since:



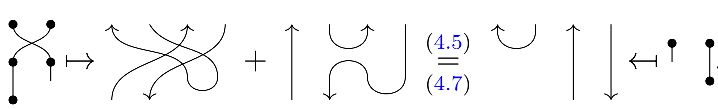
Similarly, one can compute that



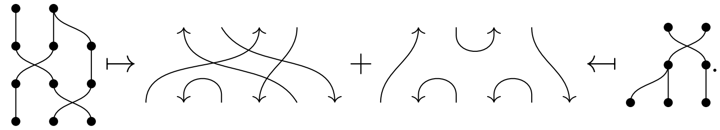
and that



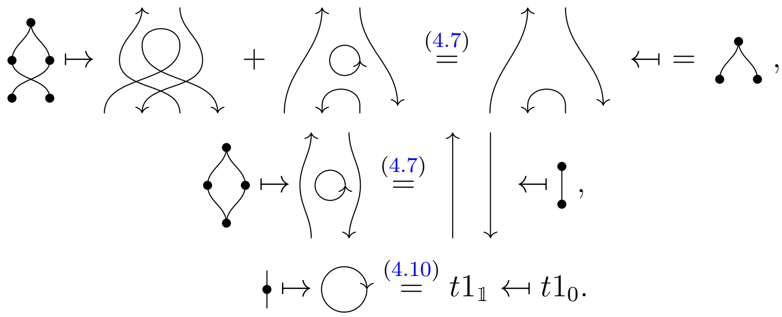
Thus, 4.8 gives that the second relation in 4.2 hols as well. The first relation in 4.3 holds since:



The next relation in 4.3 holds by an application of 4.7:



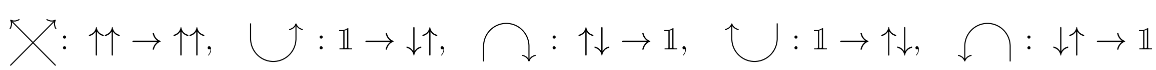
The later relations in 4.3 hold by similar computations. The relations in 4.4 are a bit easier to see:



∎

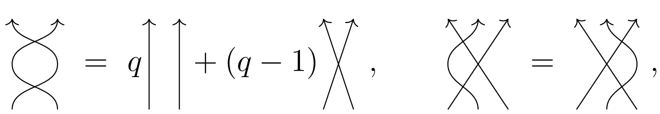
4.3. **The q-Heisenberg Category.** Many mathematical objects have simple relations that might become more interesting by the introduction of a parameter that complicates some relationships. Here we introduce a q-deformation of the Heisenberg category which is essentially the same as the original category, but some of the relations have new terms that are indexed by an invertible parameter $q \in \Bbbk$. If $q = 1$ then the relations return to those found in $\mathcal{H}eis$.

Definition 4.5 (The q-Heisenberg Category)**.** The *q-Heisenberg category $\mathcal{H}eis(q)$* is the strict $\Bbbk$-linear monoidal category generated by two objects $\uparrow, \downarrow$, (we use horizontal juxtaposition to denote the tensor product) and morphisms

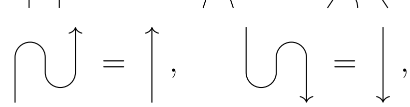


where $\mathbb{1}$ denotes the unit object, subject to the relations

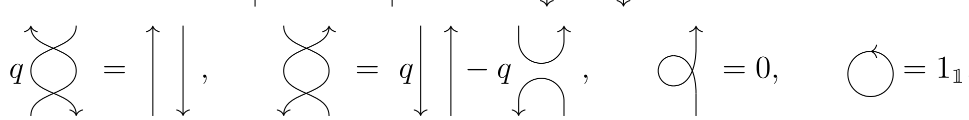
(4.11)



(4.12)

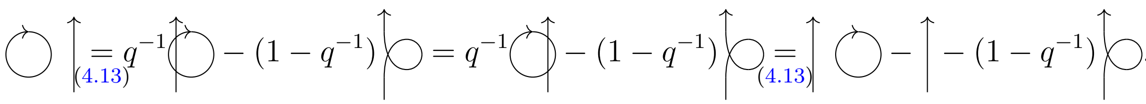


(4.13)



Here the left and right crossings are defined by

To get a sense of how these q-deformations change relations present in $\mathcal{Heis}$, notice that commuting a bubble with an up arrow is no longer as nice:



5. Categorical Actions

The diagrammatic categories presented above contain structure which is well-suited to act on other mathematical objects. $\mathcal{Par}(t)$ and $\mathcal{Heis}$ both can be seen to act on the category S_n**-mod**, and these actions are expressed as functors.

5.1. The Partition Actions.

Proposition 5.1 (The Partition Action [6, §2]). *Suppose* $\Bbbk$ *is* $\mathbb{C}$ *or* $\mathbb{R}$. *Then, there is a strict,* $\Bbbk$*-linear, monoidal, full functor* $\Phi_n : \mathcal{Par}(n) \to S_n$***-mod*** *defined on generators by letting* $\Phi_n(1) = V$ *where* V *is the natural* S_n*-module on which* S_n *permutes the basis, and on morphisms:*

$$\begin{aligned}
&\Phi_n(\mu)\colon V \otimes V \to V, && v_i \otimes v_j \mapsto \delta_{i,j} v_i,\\
&\Phi_n(\eta)\colon \mathbf{1}_n \to V, && 1 \mapsto \textstyle\sum_{i=1}^n v_i,\\
&\Phi_n(\delta)\colon V \to V \otimes V, && v_i \mapsto v_i \otimes v_i,\\
&\Phi_n(\varepsilon)\colon V \to \mathbf{1}_n, && v_i \mapsto 1,\\
&\Phi_n(s)\colon V \otimes V \to V \otimes V, && v_i \otimes v_j \mapsto v_j \otimes v_i.
\end{aligned}$$

Moreover, $(\Phi_n)_{k,l}$ *is an isomorphism if and only if* $k + l \leq n$.

This proof is done by checking that Φ_n respects the relations given in 4.2. For instance, one check is to show that $\mathrm{id}_V = \Phi_n(\mathbb{1}) = \Phi_n(\mu\delta) = \Phi_n(\mu)\Phi_n(\delta)$ but $v_i \mapsto v_i \otimes v_i \mapsto \delta_{i,i} v_i = v_i$ is the identity on V.

Notice that, pictorially, μ is taking two objects to one object, which might make its action $v_i \otimes v_j \mapsto \delta_{ij} v_i$ more natural. Or, for another example, notice that s, pictorially, looks like it is swapping two objects, which makes its action $v_i \otimes v_j \mapsto v_j \otimes v_i$ somewhat intuitive.

5.2. **The Heisenberg Actions.** The Heisenberg category can act on $(S_n, S_m)-$bimodules, but to see this, we will establish some conventions as in [6]. Let $(i, i+1) = s_i \in S_k$. Let $s_i s_{i+1} \cdots s_j = 1$ whenever $i > j$ and, for $i \in \{1, \ldots, n\}$ let $g_i = s_i s_{i+1} \cdots s_{n-1}$ so that $\{g_i\}_{i=1}^n$ forms a set of coset representatives of S_{n-1} in S_n. Let S_0 denote the trivial group. For $k, m \in \{0, \ldots, n\}$ let ${}_k(n)_m$ denote $\Bbbk S_n$ considered to be a $(\Bbbk S_k, \Bbbk S_m)$-bimodule with an empty subscript indicating a subscript of n. Furthermore, denote tensor products of bimodules by juxtaposition, i.e. $(n)_{n-1}(n) = \Bbbk S_n \otimes_{\Bbbk S_{n-1}} \Bbbk S_n$ and, whenever the tensor product of bimodules is undefined, take thier tensor product to be zero. Let 1_n denote the trivial S_n-module and let V denote the standard n-dimensional S_n-module. Now, the action of $\mathcal{H}eis$ on S_n-modules is expressed as a strict, $\Bbbk$-linear, monoidal functor:

Proposition 5.2 (The Heisenberg Action [6, S3])**.** *There exists a strict, $\Bbbk$-linear, monoidal functor*

$$\Theta : \mathcal{H}eis \to \prod_{m \in \mathbb{N}} \left(\bigoplus_{n \in \mathbb{N}} (S_n, S_m)\textbf{-bimod} \right).$$

On objects we have:

$$\Theta(\uparrow) = ((n)_{n-1})_{n \geq 1}, \ \Theta(\downarrow) = ({}_{n-1}(n))_{n \geq 1}.$$

On morphisms, we have:

$$\Theta\left(\nwarrow\!\!\!\!\!\times\!\!\!\!\!\nearrow\right) = \Big((n)_{n-2} \to (n)_{n-2}, \ g \mapsto g s_{n-1} \Big)_{n \geq 2},$$
$$\Theta\left(\smile\!\!\!\uparrow\right) = \Big((n-1) \to {}_{n-1}(n)_{n-1}, \ g \mapsto g \Big)_{n \geq 1},$$

$$\Theta\left(\curvearrowright\right) = \Big((n)_{n-1}(n) \to (n),\ g \otimes h \mapsto gh\Big)_{n\geq 1},$$

$$\Theta\left(\uparrow\!\smile\right) = \Big((n) \to (n)_{n-1}(n),\ g \mapsto \textstyle\sum_{i=1}^{n} g_i \otimes g_i^{-1} g = \sum_{i=1}^{n} g g_i \otimes g_i^{-1}\Big)_{n\geq 1},$$

$$\Theta\left(\curvearrowleft\right) = \left({}_{n-1}(n)_{n-1} \to (n-1),\ g \mapsto \begin{cases} g & \textit{if } g \in S_{n-1}, \\ 0 & \textit{if } g \in S_n \setminus S_{n-1} \end{cases}\right)_{n\geq 1}.$$

The proof that this is actually a functor requires us to check that the relations in the Heisenberg category are respected by the assignment above. For instance, the image of the first relation in 4.5 holds, since:

$$\Theta\left(\nwarrow\!\!\!\!\nearrow\right) \circ \Theta\left(\nwarrow\!\!\!\!\nearrow\right) = \left((n)_{n-2} \to (n)_{n-2}, g \mapsto g s_{n-1} \mapsto g s_{n-1}^2 = g\right)_{n\geq 2} = \Theta\left(\uparrow\ \ \uparrow\right).$$

The restriction of Θ to $\mathcal{H}eis_{\uparrow\downarrow}$ (which will be denoted with the same symbol) induces a functor:

$$\Theta : \mathcal{H}eis_{\uparrow\downarrow} \to \prod_{m\in\mathbb{N}} (S_m, S_m)\textbf{-bimod}.$$

The composition with the functor $- \otimes_{S_n} 1_n$ gives a functor

$$\mathcal{H}eis_{\uparrow\downarrow} \xrightarrow{\Theta} \prod_{m\in\mathbb{N}} (S_m, S_m)\textbf{-bimod} \xrightarrow{-\otimes_{S_n} 1_n} S_n\textbf{-mod},$$

which actually factors through $\mathcal{H}eis_{\uparrow\downarrow}(n)$ to give an action on S_n**-mod**:

$$\Omega_n : \mathcal{H}eis_{\uparrow\downarrow}(n) \to S_n\textbf{-mod}.$$

The fact that this factors through can be seen by checking the composition induced by the clockwise bubble respects the relation we would like to quotient out by. The clockwise bubble induces a map $(n)_{n-1}(n) \to (n) \to (n)_{n-1}(n-1)$ that is given by:

$$g \mapsto \sum_{i=1}^{n} g_i \otimes g_i^{-1} g \mapsto \sum_{i=1}^{n} g_i g_i^{-1} g = ng.$$

5.3. **The Natural Isomorphism.** Notice we have defined two actions on the same object S_n**-mod**. One is given by the partition category, and the other by the Heisenberg category. Also notice that there is functor $\Psi_t : \mathcal{P}ar(t) \to \mathcal{H}eis_{\uparrow\downarrow}$. Thus, it is natural to consider how factoring the partition action through the Heisenberg category is different from not going through the Heisenberg category at all. It turns out that these actions are naturally isomorphic. That is, there is a natural way to translate (in an invertible manner) between the original and factored action. To do this, we must introduce the isomorphism that will do this translation.

Proposition 5.3 ([6, §5])**.** *Let $V = \Bbbk^n$ be the standard S_n-module with a basis $v_1, \dots, v_n$ and let $V_n^{\otimes \Bbbk}$ be the S_n-module on which S_n acts componentwise. With notation as before, we have:*

$$(n)_{n-1} 1_{n-1} = \mathrm{Ind}_{n-1}^{n}(1_{n-1}) \cong V$$

$$g_i \otimes_{n-1} 1 \mapsto g_i v_n = v_i.$$

Moreover, there is an isomorphism of S_n-modules:

$$\underbrace{(n)_{n-1}(n)_{n-1} \cdots_{n-1} (n)_{n-1}}_{k \text{ factors}} 1_{n-1} \cong_{\beta_k^{-1}} V^{\otimes k}$$

$$g_k \otimes \cdots \otimes g_1 \otimes 1 \mapsto g_k v_n \otimes g_k g_{k-1} v_n \otimes \cdots \otimes g_k g_{k-1} \cdots g_1 v_n$$

with inverse:

$$V^{\otimes k} \cong_{\beta_k} \underbrace{(n)_{n-1}(n)_{n-1} \cdots_{n-1} (n)_{n-1}}_{k \text{ factors}} 1_{n-1}$$

$$v_{i_1} \otimes \cdots \otimes v_{i_k} \mapsto g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1.$$

The following is just the statement that these maps actually do translate the actions defined earlier. That is, the diagram below commutes in a natural way.

$$\begin{array}{ccc} \mathcal{P}ar(n) & \xrightarrow{\Psi_n} & \mathcal{H}eis_{\uparrow\downarrow}(n) \\ & \searrow^{\Phi_n} & \downarrow{\scriptstyle \Omega_n} \\ & & S_n\textbf{-mod} \end{array}$$

Theorem 5.4 (The Isomorphisms are Natural [6, §5])**.** *The maps* β_k *provide a natural isomorphism* $\Phi_n \cong_{\beta_n} \Omega_n \circ \Psi_n$*. That is, for all* $k, k' \in \mathbb{N}$*, we have the following diagram:*

$$\begin{array}{ccccc} k & & \Omega_n\Psi_n k & \xrightarrow{\beta_k^{-1}} & \Phi_n k \\ {\scriptstyle \gamma}\downarrow & & \downarrow & & \downarrow \\ k' & & \Omega_n\Psi_n k' & \xrightarrow[\beta_{k'}^{-1}]{} & \Phi_n k' \end{array} .$$

Proof. The proof of this proposition is checking that the diagram commutes on generators of $\mathcal{P}ar(n)$ considered as a $\Bbbk$-*linear category* (since Ω_n is not monoidal). Thus, it would not be sufficient to check that relations are satisfied only on generators as a monoidal category. However, for an easy example if $\gamma = \mu$, then the diagram becomes:

$$\begin{array}{ccccc} 2 & & (n)_{n-1}(n)_{n-1}1_n & \xrightarrow{\beta_2^{-1}} & V \otimes V \\ {\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \Omega_n\Psi_n\mu} & & \downarrow{\scriptstyle \Phi_n\mu} \\ 1 & & (n)_{n-1}1_n & \xrightarrow[\beta_1^{-1}]{} & V, \end{array}$$

which holds since

$$\begin{aligned} \beta_1^{-1}\Omega_n\Psi_n\mu\beta_2(v_{i_1} \otimes v_{i_2}) &= (\Omega_n\Psi_n\mu)(\delta_{i_1,i_2}g_{i_1} \otimes 1) \\ &= \delta_{i_1,i_2}v_{i_1} \\ &= \Phi_n\mu(v_{i_1} \otimes v_{i_2}). \end{aligned}$$

To give a sense of how the general proof goes, I repeat one of the computations done in [6, §5] which verifies that, for $\gamma = 1_k \otimes \mu \otimes 1_j, k, j \in \mathbb{N}$, the diagram above commutes.

Let $j \in \{1, 2, \ldots, n-1\}$. In [6, §5], Likeng and Savage compute that:

$$\beta_{k-1}^{-1} \circ \left(\Omega_n \circ \Psi_n \left(1_{k-j-1} \otimes \mu \otimes 1_{j-1}\right)\right) \circ \beta_k \colon V^{\otimes k} \to V^{\otimes k-1}$$

is the S_n-module map given by

$$\begin{aligned} v_{i_k} \otimes \cdots \otimes v_{i_1} &\mapsto g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\ &\mapsto \ \delta_{i_j, i_{j+1}} g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_{j+3}}^{-1} g_{i_{j+2}} \otimes g_{i_{j+2}}^{-1} g_{i_j} \otimes g_{i_j}^{-1} g_{i_{j-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\ &\mapsto \delta_{i_j, i_{j+1}} v_{i_k} \otimes \cdots \otimes v_{i_{j+2}} \otimes v_{i_j} \otimes \cdots \otimes v_{i_1}. \end{aligned}$$

This is the map $\Phi_n(1_{k-j-1} \otimes \mu \otimes 1_{j-1})$.

The verification for the maps δ, ϵ, and η are all very similar. For instance, they also compute that

$$\beta_{k+1}^{-1} \circ \left(\Omega_n \circ \Psi_n \left(1_{k-j} \otimes \delta \otimes 1_{j-1}\right)\right) \circ \beta_k \colon V^{\otimes k} \to V^{\otimes k+1}$$

is the S_n-module map given by

$$\begin{aligned} v_{i_k} \otimes \cdots \otimes v_{i_1} &\mapsto g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\ &\mapsto g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_{j+1}}^{-1} g_{i_j} \otimes 1 \otimes g_{i_j}^{-1} g_{i_{j-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\ &= g_{i_k} \otimes g_{i_k}^{-1} g_{i_{k-1}} \otimes \cdots \otimes g_{i_{j+1}}^{-1} g_{i_j} \otimes g_{i_j}^{-1} g_{i_j} \otimes g_{i_j}^{-1} g_{i_{j-1}} \otimes \cdots \otimes g_{i_2}^{-1} g_{i_1} \otimes 1 \\ &\mapsto v_{i_k} \otimes \cdots \otimes v_{i_{j+1}} \otimes v_{i_j} \otimes v_{i_j} \otimes v_{i_{j-1}} \otimes \cdots \otimes v_{i_1}. \end{aligned}$$

This is the map $\Phi_n(1_{k-j} \otimes \delta \otimes 1_{j-1})$.

However, the case for the map s requires decomposing the map $\Psi_n(s)$ into generating morphisms and computing Θ's effect on these simpler parts and then writing out the map. Overall, they compute that

$$\Theta_n(\Psi_n(s))(hg_i \otimes g_i^{-1} g_j \otimes g_j^{-1} h') = hg_j \otimes g_j^{-1} g_i \otimes g_i^{-1} h'. \tag{5.1}$$

This gives that

$$\beta_k^{-1} \circ \left(\Omega_n \circ \Psi_n \left(1_{k-j-1} \otimes s \otimes 1_{j-1}\right)\right) \circ \beta_k = \beta_k^{-1} \circ \Omega_n \left(1_{\uparrow\downarrow}^{\otimes(k-j-1)} \otimes (\Psi_n(s)) \otimes 1_{\uparrow\downarrow}^{\otimes(j-1)}\right) \circ \beta_k$$

is the map given by

$$v_{i_k} \otimes \cdots \otimes v_{i_1} \mapsto v_{i_k} \otimes \cdots \otimes v_{i_{j+2}} \otimes v_{i_j} \otimes v_{i_{j+1}} \otimes v_{i_{j-1}} \otimes \cdots \otimes v_{i_1},$$

which is the map $\Phi_n \left(1_{k-j-1} \otimes s \otimes 1_{j-1}\right)$. □

6. The q Picture

6.1. **Overall Goal.** In the classical picture, we have the following diagram of functors which are related by a natural isomorphism:

$$\begin{array}{ccc} \mathcal{Par}(n) & \xrightarrow{\Psi_n} & \mathcal{Heis}_{\uparrow\downarrow}(n) \\ & \searrow^{\Phi_n} & \downarrow \Omega_n \\ & & S_n\textbf{-mod} \end{array} .$$

In [7], a very similar diagram holds:

$$\begin{array}{ccc} \mathcal{Par}(G) & \xrightarrow{\Psi_n} & \mathcal{Heis}_{\uparrow\downarrow}(G) \\ & \searrow^{\Phi_n} & \downarrow \Omega_n \\ & & A_n\textbf{-mod} \end{array} .$$

Here G is a group, $\mathcal{Par}(G)$ is a group analogue of the partition category $\mathcal{Par}(t)$, A_n is the algebra $\Bbbk G^n \rtimes S_n$, and $\mathcal{Heis}(G)$ is a group analogue of $\mathcal{Heis}(t)$. Moreover, these functors are related by a natural isomorphism, making this a G-analogue of the classical picture. Motivated by this extension of the diagram to a G analogue of each category, we want to extend the diagram to a q-analogue of each category.

The proper analogue of $\mathcal{Heis}$ has already been introduced: $\mathcal{Heis}(q)$. Since $\mathcal{Heis}_{\uparrow\downarrow}(n)$ acts on S_n**-mod**, the proper analogue of S_n**-mod** should be a category on which $\mathcal{Heis}_{\uparrow\downarrow}(q)$ can act.

Thankfully, this aspect of the diagram is already known, since $\mathcal{Heis}_{\uparrow\downarrow}(q)$ can actually act on the category $GL_n(\mathbb{F}_q)$-**mod**, which indicates that this should replace S_n-**mod**. Thus, we are searching for a diagram of the form:

$$\begin{array}{ccc} \mathcal{Par}(q) & \xrightarrow{q\Psi_n} & \mathcal{Heis}_{\uparrow\downarrow}(q) \\ & \searrow^{q\Phi_n} & \downarrow q\Omega_n \\ & & GL_n(\mathbb{F}_q)\textbf{-mod} \end{array} .$$

Unfortunately, there is no known diagrammatic presentation of a q-analogue to $\mathcal{Par}(t)$. However, there is a known q-analogue to the related partition *algebra* which is actually isomorphic the q-partition algebra. Thus, it is not unreasonable to assume that at least some of the diagrams for a potential category $\mathcal{Par}(q)$ would take the same form as those in $\mathcal{Par}(t)$. Here, we will introduce the relevant known q-action and present our attempt to guess what the analogue of the $q\Psi$ functor might look like.

6.2. **The Known q-Action.** The proper q-analogue of S_n-**mod** actually turns out to be representations of $GL_n(\mathbb{F}_q)$, i.e. $GL_n(\mathbb{F}_q)$-**mod**. There are multiple motivations for this, but the main motivation is that there is a natural action that $\mathcal{Heis}(q)$ has on this category which we will introduce here.

Embed $GL_{n-1}(\mathbb{F}_q) \to GL_n(\mathbb{F}_q)$ by placing an element of $GL_{n-1}(\mathbb{F}_q)$ into the top left corner of the $n \times n$ matrix with zeros everywhere except entry n, n which has a 1. Let $U_n \subseteq GL_n(\mathbb{F}_q)$ be the upper triangular matrices whose upper left $n-1 \times n-1$ block is the identity matrix. Let $P_n = GL_{n-1}(\mathbb{F}_q)U_n$, which satisfies $U_n \trianglelefteq P_n$. Let $v_n \in \Bbbk GL_n(\mathbb{F}_q)$ be given by:

$$v_n = \frac{1}{|U_n|} \sum_{u \in U_n} u.$$

. This element satisfies:

$$v_n^2 = v_n, \tag{6.1}$$

$$xv_n = v_n x = v_n \text{ for all } x \in \Bbbk U_n, \text{and} \tag{6.2}$$

$$yv_n = v_n y \text{ for all } y \in \Bbbk GL_{n-1}(\mathbb{F}_q). \tag{6.3}$$

This is a particular case leading up to parabolic induction and restriction. Let us denote:

- $\Bbbk GL_n(\mathbb{F}_q)v_n$ viewed as a $(\Bbbk GL_n(\mathbb{F}_q), \Bbbk GL_{n-1}(\mathbb{F}_q))$-bimodule by $(n)_{n-1}$,
- $v_n \Bbbk GL_n(\mathbb{F}_q)$ viewed as a $(\Bbbk GL_{n-1}(\mathbb{F}_q), \Bbbk GL_n(\mathbb{F}_q))$-bimodule by ${}_{n-1}(n)$,
- $v_n \Bbbk GL_n(\mathbb{F}_q)v_n$ viewed as a $(\Bbbk GL_{n-1}(\mathbb{F}_q), \Bbbk GL_{n-1}(\mathbb{F}_q))$-bimodule by ${}_{n-1}(n)_{n-1}$, and
- $\Bbbk GL_n(\mathbb{F}_q)$ viewed as a $(\Bbbk GL_n(\mathbb{F}_q), \Bbbk GL_n(\mathbb{F}_q))$-bimodule by (n).

In this way we can apply the construction of parabolic induction and restriction to our particular case:

$$\begin{aligned} \mathrm{IndP}^n_{n-1} : \Bbbk GL_{n-1}(\mathbb{F}_q)\textbf{-mod} &\to \Bbbk GL_n(\mathbb{F}_q)\textbf{-mod} \\ M &\mapsto \Bbbk GL_n(\mathbb{F}_q)v_n \otimes_{\Bbbk GL_{n-1}(\mathbb{F}_q)} M \end{aligned}$$

and

$$\begin{aligned} \mathrm{ResP}^n_{n-1} : \Bbbk GL_n(\mathbb{F}_q)\textbf{-mod} &\to \Bbbk GL_{n-1}(\mathbb{F}_q)\textbf{-mod} \\ M &\mapsto v_n \Bbbk GL_n(\mathbb{F}_q) \otimes_{\Bbbk GL_n(\mathbb{F}_q)} M. \end{aligned}$$

Let $s_n \in GL_n(\mathbb{F}_q)$ be the identity matrix with the last two columns interchanged. Let $t_n = qv_{n+1}s_n v_{n+1} \in \Bbbk S_{n+1}(q)$ which satisfies $t_n^2 = (q-1)t_n + qv_{n+1}v_n$. Let $GL_n(\mathbb{F}_q) = \coprod_{i=1}^s P_n g_i$ be a decomposition of $GL_n(\mathbb{F}_q)$ into right cosets. The element:

$$\sum_{i=1}^{s} g_i^{-1} v_n \otimes v_n g_i \in (n)_{n-1}(n)$$

is independent of the decomposition chosen and is central in $\Bbbk GL_n(\mathbb{F}_q)$. For instance, to show the eleemnt does not depend on the choice of coset representatives chosen, suppose $GL_n(\mathbb{F}_q) = \coprod_{i=1}^s P_n h_i$ with $g_i h_i^{-1} = p_i \in P_n$ for all $i \in \{1, \ldots, s\}$. By definition of P_n there exist elements $G_i \in GL_{n-1}(\mathbb{F}_q)$ and $u_i \in U_n$ with $p_i = G_i u_i$ for all $i \in \{1, \ldots, s\}$. Thus, the elements induced by these decompositions satisfy

$$\begin{aligned}\sum_{i=1}^{s} g_i^{-1} v_n \otimes_{n-1} v_n g_i &= \sum_{i=1}^{s} h_i^{-1} p_i^{-1} v_n \otimes_{n-1} v_n p_i h_i \\ &= \sum_{i=1}^{s} h_i^{-1} u_i^{-1} G_i^{-1} v_n \otimes_{n-1} v_n G_i u_i h_i \\ &\underset{(6.3)}{=} \sum_{i=1}^{s} h_i^{-1} u_i^{-1} v_n \otimes_{n-1} v_n u_i h_i \\ &\underset{(6.2)}{=} \sum_{i=1}^{s} h_i^{-1} v_n \otimes_{n-1} v_n h_i,\end{aligned}$$

which shows that this element is independent of the choice of coset representatives.

Finally, we are ready to state an action of $\mathcal{H}eis(q)$ on $GL_n(\mathbb{F}_q)$**-mod**. This is done in the same way as with $\mathcal{H}eis$ acting on S_n**-mod**, via a strict $\Bbbk$-linear monoidal functor.

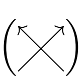
Proposition 6.1 (The q-Action [5, §6])**.** *There is a strict, $\Bbbk$-linear, monoidal functor:*

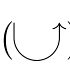
$$q\Theta : \mathcal{H}eis(q) \to \prod_{m\in\mathbb{N}} \left(\bigoplus_{n\in\mathbb{N}} (GL_n(\mathbb{F}_q), GL_m(\mathbb{F}_q))\textbf{-bimod} \right)$$

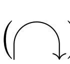
On objects we have:

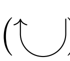
$$q\Theta(\uparrow) = ((n)_{n-1})_{n\geq 1},\ q\Theta(\downarrow) = ({}_{n-1}(n))_{n\geq 1}.$$

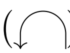
On morphisms, we have:

$q\Theta\Big($  $\Big) = \Big((n)_{n-2} \to (n)_{n-2},\ z v_n v_{n-1} \mapsto z t_{n-1} v_n v_{n-1}\Big)_{n\geq 2},$

$q\Theta\Big($  $\Big) = \Big((n-1) \to {}_{n-1}(n)_{n-1},\ z \mapsto v_n z v_n\Big)_{n\geq 1},$

$q\Theta\Big($  $\Big) = \Big((n)_{n-1}(n) \to (n),\ x v_n \otimes v_n y \mapsto x v_n y\Big)_{n\geq 1},$

$q\Theta\Big($  $\Big) = \Big((n) \to (n)_{n-1}(n),\ z \mapsto z\left(\sum_{i=1}^{s} g_i v_n \otimes v_n g_i^{-1}\right)\Big)_{n\geq 1},$

$q\Theta$ () $= \left({}_{n-1}(n)_{n-1} \to (n-1),\ z \mapsto \begin{cases} \tilde{z} & \textit{if } z \in GL_{n-1}(\mathbb{F}_q), \\ 0 & \textit{if } z \in GL_n(\mathbb{F}_q) \setminus GL_{n-1}(\mathbb{F}_q) \end{cases} \right)_{n \geq 1}.$

Here, $\tilde{z}$ is z with the last row and column removed.

As before, we can restrict to the subcategory generated by $\uparrow$ and $\downarrow$ and compose with the functor which is tensoring by the trivial representation to obtain the functor:

$$\mathcal{H}eis_{\uparrow\downarrow}(q) \xrightarrow{q\Theta} \prod_{m \in \mathbb{N}} (GL_m(\mathbb{F}_q), GL_m(\mathbb{F}_q))\textbf{-bimod} \xrightarrow{-\otimes_{GL_n(\mathbb{F}_q)} 1_n} GL_n(\mathbb{F}_q)\textbf{-mod}.$$

However, notice that in the $\mathcal{H}eis$ case we went a step further and quotiented out by the relation making the bubble act as multiplication by n. In this case, we do not know what the proper collection of diagrams would be to quotient out by, which means that some diagrams may appear to have a complicated action despite their action being quite simple.

6.3. **The q-Partition Category.** This is the aspect of the triangle above 5.3 not yet known. The only real generalization that currently exists is a generalization of the associated partition *algebra* [3]:

$$P_k(n) = \mathrm{End}_{S_n}(V^{\otimes} k) \cong \mathrm{End}_{S_n}(\underbrace{\mathrm{Ind}_{n-1}^n \mathrm{Res}_{n-1}^n \cdots \mathrm{Ind}_{n-1}^n \mathrm{Res}_{n-1}^n}_{2k \text{ functors}}(1_n)).$$

This isomorphism is naturally extended to a q-analogue by replacing S_n with $GL_n(\mathbb{F}_q)$ and replacing the induction and restriction functors with their parabolic counterparts, so that the q-partition algebra can be taken to be [3]:

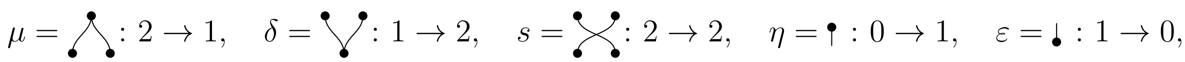
$$P_k(n,q) = \mathrm{End}_{GL_n(\mathbb{F}_q)}(\underbrace{\mathrm{IndP}_{n-1}^n \mathrm{ResP}_{n-1}^n \cdots \mathrm{IndP}_{n-1}^n \mathrm{ResP}_{n-1}^n}_{2k \text{ functors}}(1_n)).$$

7. The Conjectured Embedding

In an attempt to generalize the triangle above 5.3, we must make a guess at what the diagrammatic representation of a q-partition category might look like. Our hope in the following attempt is that some of the generating diagrams for the q-partition category will remain unchanged and will have natural associations with the q-deformations they correspond to in the original Heisenberg category. This is not an unreasonable hope, since we know that $P_k(n, q)$ and $P_k(n)$ are actually isomorphic so that their corresponding categories should contain at least some common generating morphisms.

For ease of access, we repeat the presentation of $\mathcal{Par}(t)$ and $\mathcal{Heis}(q)$ here.

As a strict $\Bbbk$-linear monoidal category, the *partition category* $\mathcal{Par}(t)$ is generated by the object 1 and the morphisms



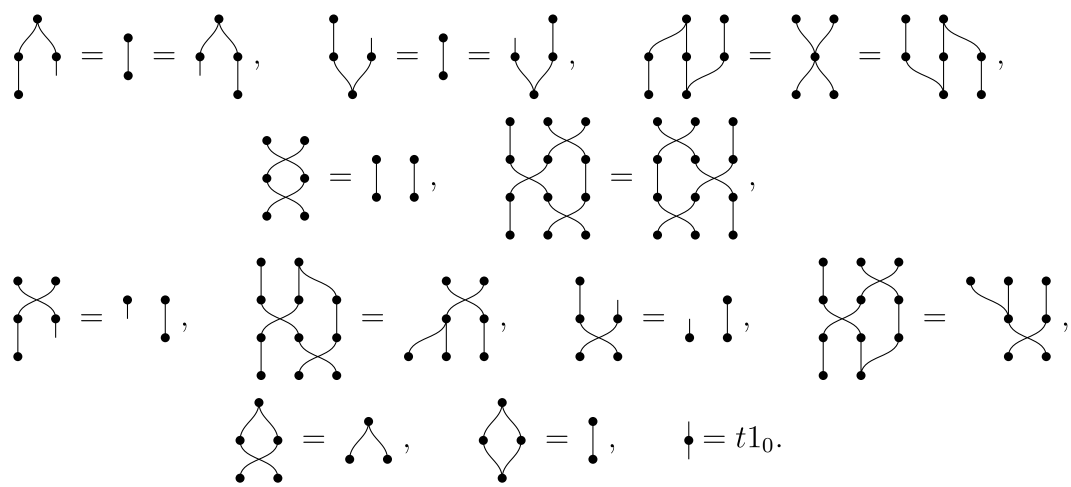
subject to the following relations:

(7.1)

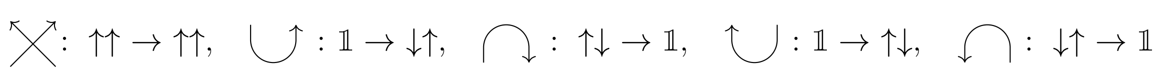
(7.2)

(7.3)

(7.4)

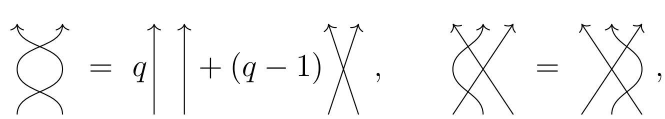


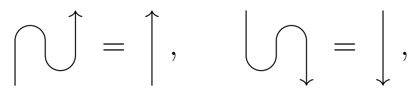
The *q-Heisenberg category* $\mathcal{Heis}(q)$ is the strict $\Bbbk$-linear monoidal category generated by two objects $\uparrow$, $\downarrow$, (we use horizontal juxtaposition to denote the tensor product) and morphisms

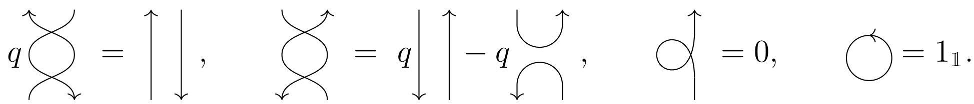


where $\mathbb{1}$ denotes the unit object, subject to the relations

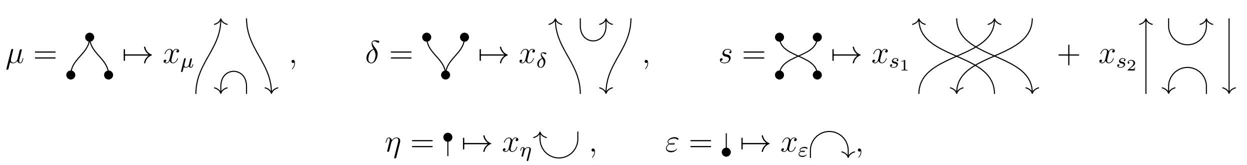
(7.5)



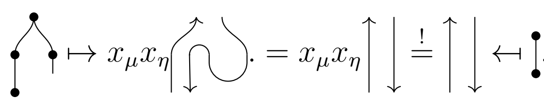
(7.6) 

(7.7) 

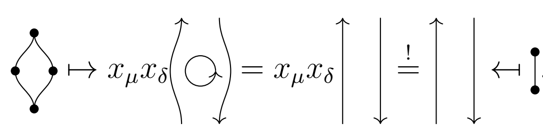
Operating under the assumption that $\mathcal{Par}(q)$ would have a similar presentation, we can attempt an embedding into $\mathcal{Heis}(q)$ under the following assignment:



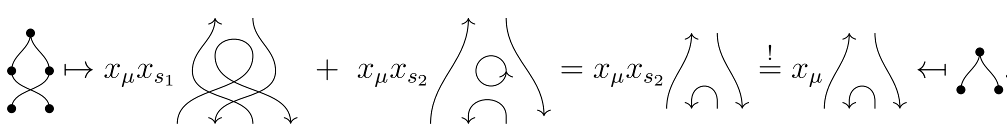
where the diagrams on the right of the assignments are considered as living in $\mathcal{Heis}(q)$ and $x_\mu, x_\delta, x_{s_1}, x_{s_2}, x_\eta$, and x_ϵ are indeterminates. This assignment actually behaves well with several of the morphisms. For instance, the first relation in 7.1 would require that:



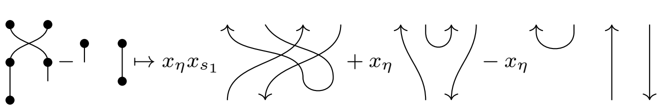
This shows that we would need that $x_\mu = x_\eta^{-1}$ in our assignment. Moreover, the second relation in 7.4 would require that:

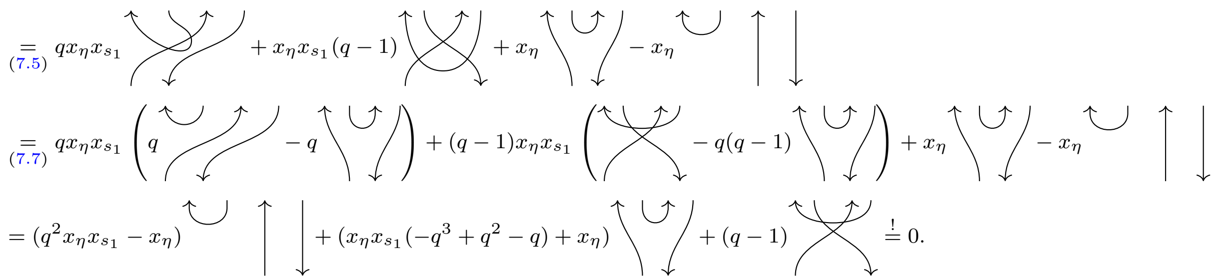


This shows that we would need that $x_\mu = x_\delta$ if we want the assignment to be a functor. Perhaps a more interesting relation is the first appearing in 7.4, which would require that:



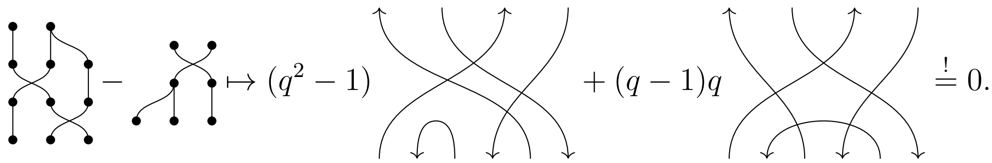
This gives that $x_{s_2} = 1$. Although some relations are respected quite well, we quickly run into problems with many of the relations. For instance, the first relation in 7.3 would need to satisfy:



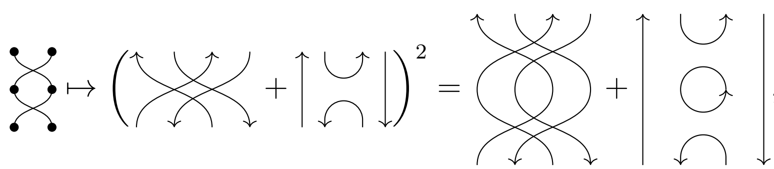


Since we know that $q = 1$ should recover the relations in $\mathcal{Heis}$, we really should have $x_{s_1} = 1$.

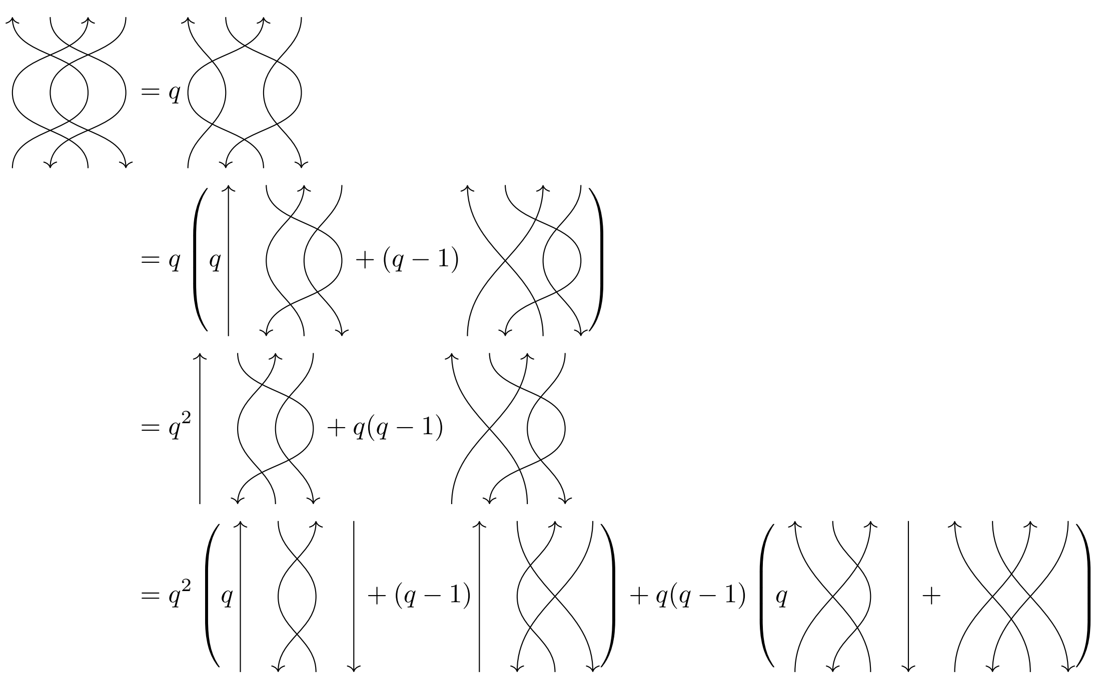
A computation (not worrying about indeterminates) gives another violated relation:

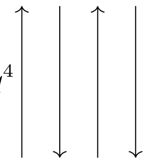
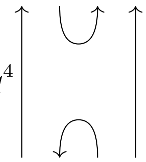
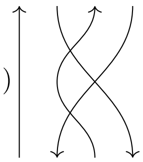
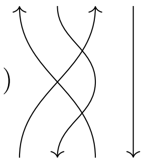
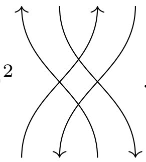


Another diagram which does not behave correctly in the q-picture is the image of s^2:

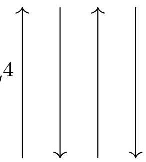
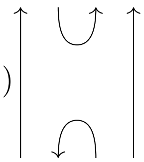
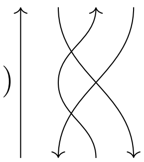
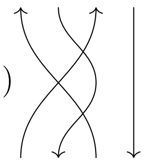
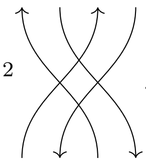


where we have that:



$= q^4$  $- q^4$  $+ q^2(q-1)$  $+ q^2(q-1)$  $+ q(q-1)^2$  .

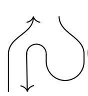
Thus, overall, we have that s^2 maps to:

q^4  $+ (1-q^4)$  $+ q^2(q-1)$  $+ q^2(q-1)$  $+ q(q-1)^2$  .

This computation demonstrates that the proposed embedding does not satisfy the relations that appear in the partition category. This indicates that, likely, relations in a proposed q-partition category would need to modify the s^2 relation.

8. A Few Computations

The following are computations that indicate potential actions a q-partition category should have, assuming that the diagrams embed into $\mathcal{H}eis(q)$ as proposed. Some of the images of potential q-partition diagrams in the q-Heisenberg category have non-zero actions on $GL_n(\mathbb{F}_q)$ modules. These non-trivial actions demonstrate that there is no hidden cancellation happening in the q-Heisenberg category, which shows that the q-partition category would need additional q-relations.



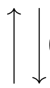
$$(g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} v_n g_3) = \sum_{i=1}^{s} \left[g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} v_n g_3 a_i^{-1} v_n \otimes_{n-1} v_n a_i \right]$$

$$= \sum_{i=1}^{s} \delta_{P_{n-1,n}}(g_3 a_i^{-1}) \left[g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} g_3 a_i^{-1} \otimes_{n-1} v_n a_i \right]$$

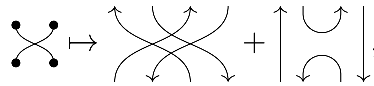
$$= g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} g_3 g_3^{-1} p^{-1} \otimes_{n-1} p g_3$$

$$= g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} 1 \otimes_{n-1} v_n g_3$$

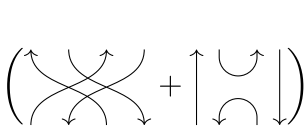
$$= g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} v_n g_3$$

$=$  $(g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} v_n g_3)$

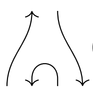
If we let $g = g_1 v_n \otimes_{n-1} v_n g_2 v_n \otimes_{n-1} v_n g_3 \in A_n v_n \otimes_{n-1} v_n A_n v_n \otimes_{n-1} v_n A_n$, then:

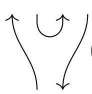


which acts as:

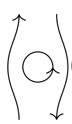
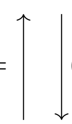
 $(g) = \begin{cases} g_1 v_n \otimes_{n-1} v_n \otimes_{n-1} v_n g_3 & g_2 = 1_n \\ q\sum_i (g_1 a_i t_{n-1} v_n \otimes_{n-1} v_n t_{n-1} v_n \otimes_{n-1} v_n t_{n-1} b_i g_3) & g_2 = \sum_i a_i t_{n-1} b_i. \end{cases}$

We also have the following actions:

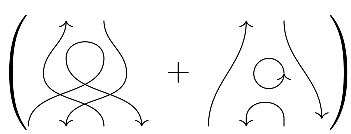
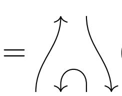
 $(g_1 v_n \otimes_{n-1} v_n g_2 v_n \otimes_{n-1} v_n g_3) = \delta_{P_{n-1,n}}(g_2)(g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} v_n g_3),$

 $(g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} v_n g_3) = g_1 v_n \otimes_{n-1} v_n g_2 v_n \otimes_{n-1} v_n g_3.$

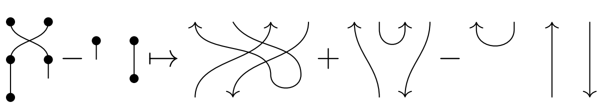
Their composition is given by:

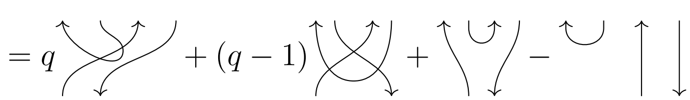
 $(g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} v_n g_3) = g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} v_n g_3 =$  $(g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} v_n g_3).$

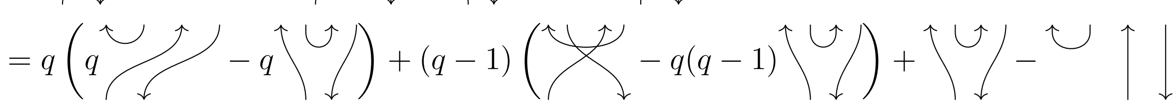
To give an example of a diagram containing the crossing, we have that:

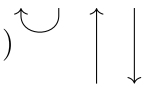
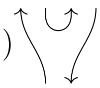
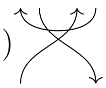
 $(g) = \delta_{P_{n-1,n}}(g_2)(g_1 v_n \otimes_{n-1} g_2 \otimes_{n-1} v_n g_3) =$  $(g).$

The assignment given earlier with indeterminates removed:







$= (q^2 - 1)$  $+ (-q^3 + q^2 - q + 1)$  $+ (q - 1)$ 

gives the following action on an element $1_n \otimes_n g_1 v_n \otimes_{n-1} v_n g_2 \otimes_n 1_n$:

$$(q^2 - 1) \sum_{i=1}^{s} \left(a_i^{-1} v_n \otimes_{n-1} v_n a_i \otimes_n g_1 v_n \otimes_{n-1} v_n g_2 \otimes_n 1_n \right) +$$

$$(-q^3 + q^2 - q + 1)(g_1 v_n \otimes_{n-1} v_n \otimes_{n-1} v_n g_2) +$$

$$(q - 1) \sum_{ij} \left(\delta_{P_{n,n+1}}(t_n a_i g_1 t_n g_2 a_j^{-1} t_n) \left(a_i^{-1} v_n \otimes_{n-1} 1_n \otimes_n v_n t_n a_i g_1 t_n g_2 a_j^{-1} t_n v_n \otimes_n 1_n \otimes_{n-1} v_n a_j \right) \right).$$

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