Powersum Functions in the Hopf Monoid of Superclass Functions on $\mathrm{UT}_n(q)$

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Abstract

The character theory of the group $\mathrm{UT}_n(q)$ of unipotent upper triangular matrices over a finite field of order q is known to be wild. However, in a generalization of character theory called supercharacter theory, one finds that there is a connection between the representation theory of $\mathrm{UT}_n(q)$, the combinatorics of set partitions, and the algebra of symmetric function in non-commuting variables. The relationship is reminiscent of the relationship between the symmetric group S_n, integer partitions and the algebra of symmetric functions.

In this thesis I begin by giving a brief review of representation theory and Hopf monoids. I then introduce a particular supercharacter theory of $\mathrm{UT}_n(q)$ and its connection to set partitions. A Hopf monoid is then constructed out of supercharacters of the infinite family of groups, $\mathrm{UT}_n(q)$, and the powersum basis of this Hopf monoid is reviewed. The product, coproduct, pointwise product, and antipode are then computed for the powersum basis and a q-deformation of this basis.

1 Introduction

Let $\mathrm{UT}_n(q)$ be the group of unipotent upper triangular $n \times n$ matrices over a finite field $\mathbb{F}_q$ of order q,

$$\mathrm{UT}_n(q) = \{u \in \mathrm{GL}_n(\mathbb{F}_q) | u_{ii} = 1, 1 \leq i \leq n, u_{ij} = 0 \text{ if } i > j\} \tag{1}$$

where, for example, a typical element of $UT_4(5)$ is

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The representation theory of this infinite family of groups is known to be an extremely difficult, so called wild problem, and has prompted the introduction of supercharacter theory. Supercharacter theory generalizes the character theory of a finite group by replacing irreducible characters with certain sums of irreducible characters called supercharacters and conjugacy classes with certain unions of conjugacy classes called superclasses.

For a fixed n and q, the set of supercharacters of $\mathrm{UT}_n(q)$ span the space of superclass functions $f : \mathrm{UT}_n(q) \to \mathbb{C}$ which are constant on superclasses. In certain supercharacter theories, including the one we work with here, there is a relation between the supercharacters and the set partitions of $[n] = \{1, 2 \ldots, ...n\}$ which connects the representation theory of $\mathrm{UT}_n(q)$ to the combinatorics of the set partitions. In the supercharacter theory used here, the supercharacters and superclasses are indexed by set partitions in an explicit manner in which representatives of the superclasses can be chosen so that their nonzero entries correspond to arcs in an arc diagram, a combinatorial object representing a set partition.

In representation theory, one has methods of taking characters of a subgroup to characters of the larger group and vice versa. The subgroups of interest here are direct products of smaller copies of $\mathrm{UT}_n(q)$. By the nature of matrix multiplication, $\mathrm{UT}_n(q)$ contains as subgroups groups that are isomorphic to direct products $\mathrm{UT}_{m_1}(q) \times \mathrm{UT}_{m_2}(q) \times \ldots \times \mathrm{UT}_{m_\ell}(q)$ where $\sum_{i=1}^{\ell} m_i = n$. In fact there is one such group for each set partition of $[n]$. Using representation theory to take superclass functions on these subgroups to superclass functions on the larger group and vice versa allows us to endow these vector spaces of functions with additional structure.

Taken as an infinite dimensional graded vector space, the space of all superclass functions on $\mathrm{UT}_n(q)$ for all n and fixed q is a graded Hopf algebra under the operations of inflation and restriction. This Hopf algebra is known to be isomorphic to the Hopf algebra of symmetric functions in non-commuting variables. A related object, the Hopf monoid of supercharacter functions, has a finer structure because it includes information on the order on the set $[n]$ and working with it instead of the algebra simplifies some computations.

Various bases for these spaces are known, including the defining basis of supercharacters and the superclass indicator functions, but we will be primarily concerned with the powersum basis. The powersum basis is defined in terms of a partial order on the set of set partitions

of $[n]$ (or later of any finite set I), and it was shown in [4] that certain q-deformations of this basis give a triangular decomposition of the supercharacter table of $\mathrm{UT}_n(q)$. In this thesis explicit formulas in the will be proven for the product, coproduct, pointwise product, and antipode of powersum functions defined by the inclusion order on arc diagrams.

Section 2 of this thesis will cover some basic definitions and theory of Hopf monoids and representation theory that are helpful in understanding later material. Further information on Hopf monoids can be found in [1] and further information on representation theory of finite groups in [6].

Section 3 sets up the context of our topic by outlining the particular supercharacter theory of $\mathrm{UT}_n(q)$ that is used in the final section. In this section, a brief outline of the relationship between this supercharacter theory and a known Hopf algebra is given.

Section 4 contains the theorems that are the main focus of this thesis and are the only original contributions to this theory. There we compute the structure constants of the Hopf monoid of supercharacters for various products and coproducts as well as the antipode. In particular, the computation of the restriction, pointwise product, and q-powersum functions are original results.

2 Background Material

When constructing the Hopf monoid of supercharacters, it is helpful to have an understanding of the representation theory of finite groups and of the general theory of Hopf monoids. Here we give only a very brief introduction which includes some of the more relevant theorems.

2.1 Representations and Characters of Finite Groups

Since the definition of supercharacters and the operations in the Hopf monoid rely ultimately on some ideas from representation theory we give a brief recollection of the basic definitions and theorems. A basic reference is [6]. Assume throughout this section that G is a finite group.

Definition 2.1. *A* representation (ρ, V) *of degree* n *of a group* G *is a group homomorphism* $\rho : G \to GL_n(\mathbb{C})$ *into the matrix algebra* $GL_n(\mathbb{C})$ *of invertible* $n \times n$ *matrices over* $\mathbb{C}$.

One can also view a representation as a module. To do so, we first need a ring.

Definition 2.2. *For a group* G *and a field* $\mathbb{C}$*, the* group ring $\mathbb{C}G$ *is defined to be the set of formal finite linear combinations of elements of* G

$$\mathbb{C}G = \left\{ \sum_{g \in G} \lambda_g g | \lambda_g \in \mathbb{F} \right\}$$

together with an addition and multiplication given by

1. $\sum_{g\in G} \lambda_g g + \sum_{g\in G} \mu_g g = \sum_{g\in G} (\lambda_g + \mu_g) g$
2. $\left(\sum_{g\in G} \lambda_g g\right)\left(\sum_{g\in G} \mu_g g\right) = \sum_{\substack{g\in G \\ h\in G}} \lambda_g \mu_h gh.$

Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be a representation of G. Then we can associate with it a module in the following way. Let V be an n-dimensional vector space over $\mathbb{C}$ and choose a basis $\mathcal{B}$ of V. Since $\mathbb{C}^n \cong V$ via this basis, every matrix $A \in \mathrm{GL}_n(\mathbb{C})$ corresponds to a unique linear transformation on V in the usual way. Thus we can view $\mathbb{C}G$ as acting on V via $(\lambda g) \cdot v = \lambda \psi_g(v)$ for $v \in V, g \in G, \lambda \in \mathbb{C}$ where ψ_g is the linear map on V associated to the matrix $\rho(g)$. This action makes V in to an $\mathbb{C}G$ module.

One can also go the other way. Given an $\mathbb{C}G$ module V, choosing a basis gives an action on $\mathbb{C}^n$ and hence a representation. Because of this correspondence we will often refer to an $\mathbb{C}G$-module V as a representation.

Definition 2.3. *A $\mathbb{C}G$-module M is said to be irreducible if it has no $\mathbb{C}G$-submodules other than $\{0\}$ and M itself.*

In our case, this means that if V is an $\mathbb{C}G$ module, then it has no proper, nontrivial subspaces that are invariant under the action of all $g \in G$. That is, a subspace of $W \subseteq V$ is a submodule if and only if $g \cdot w \in W$ for all $g \in G$ and $w \in W$. These irreducible $\mathbb{C}G$ modules are the basic building blocks of representations since it turns out that for finite groups every finite dimensional $\mathbb{C}G$ module is isomorphic to a direct sum of irreducible modules of which there is a finite list. These irreducible modules are also referred to as *irreducible representations.*

It turns out that a great deal of the information about $\mathbb{C}G$ modules is contained in relatively simple maps from G to $\mathbb{C}$ called characters.

Definition 2.4. *A character of a group G is a function $\theta : G \to \mathbb{C}$ such that $\theta(g) = \mathrm{tr}(\varphi(g))$ for some representation φ of G where* tr *denotes the trace of the matrix $\varphi(g)$.*

There is a notion of irreducible character corresponding to that of irreducible representations.

Definition 2.5. *An irreducible character of a finite group G is the trace of some irreducible representation of G.*

Every character is constant on the conjugacy classes of G. This is a consequence of the fact that similar matrices have the same trace.

Definition 2.6. *A class function on a group G is a function that is constant on conjugacy classes. Equivalently, a class function $\psi : G \to \mathbb{C}$ satisfies $\psi(ghg^{-1}) = \psi(h)$ for all $g, h \in G$.*

Some of the important properties of characters are summarized in the following theorem, which is well known in representation theory.

Theorem 2.1. *Let G be a finite group. Then*

1. *There are the same number of conjugacy classes of G as there are irreducible characters.*
2. *The set of irreducible characters of G form a basis of the vector space of class functions.*
3. *The set of irreducible characters is orthonormal with respect to the inner product*

$$\langle \psi, \phi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g)\overline{\phi(g)}$$

for ψ, ϕ class functions where the bar denotes complex conjugation.

A $\mathbb{C}G$ module is determined up to isomorphism by its character. The character information of a finite group G can be summarized in the form of a matrix called the character table.

Definition 2.7. *The character table S of a finite group G is the matrix whose rows are indexed by irreducible characters and whose columns are indexed by conjugacy classes of G and for C a conjugacy class, χ an irreducible character, $S_{\chi,C} = \chi(g)$ where $g \in C$. It is also customary to put the trivial character as the first row and the conjugacy class $\{1\}$ as the first column.*

There are several important operations one can do on characters and/or representations of a group G.

Given a group G, it is a natural enough question to ask what the relationship is between representations of subgroups $H \leq G$ and representations of G. This leads to the notion of restriction. The key observation is to note that $\mathbb{C}H$ is a subring (even subalgebra) of $\mathbb{C}G$, the group ring, so $\mathbb{C}H$ acts on V giving a $\mathbb{C}H$ module.

Theorem 2.2 (Restriction)**.** *Let G be a finite group with subgroup $H \leq G$ and ψ a character of G. Then the restriction of this character to H, $Res^G_H(\psi)$ is a character of H.*

Two other useful operations for us are pointwise product and inflation.

Theorem 2.3 (Pointwise Product)**.** *Let G be a finite group and ψ a character of G. Then the pointwise product of them $\psi \odot \varphi$, defined by*

$$\psi \odot \varphi(g) = \psi(g)\varphi(g).$$

for all $g \in G$, is a character of G.

Theorem 2.4 (Inflation)**.** *Let G be a finite group with subgroup H and ρ a surjective group homomorphism $\phi : G \to H$. Let ψ be a character of H. Then the inflation of ψ from H to G, $Inf^G_H(\psi)$ defined by*

$$\mathrm{Inf}^G_H(\psi)(h) = \psi(\rho(g)).$$

is a character of G.

A supercharacter theory of a finite group is similar to its character theory, but is derived by clumping conjugacy classes and irreducible characters in a compatible manner. The definitions given here follow [5].

Definition 2.8. *Let G be a finite group and $\mathrm{Irr}(G) = \{\psi_1, \psi_2, \ldots, \psi_\ell\}$ be the set of distinct irreducible characters of G. A* supercharacter theory *for G is defined by the following data*

1. *A set of* superclasses *of G which consists of a partition $\mathcal{K}$ of G into nonempty subsets such that for for every set $K \in \mathcal{K}$, K is a union of conjugacy classes, and $\{1\} \in \mathcal{K}$*
2. *A partition $\mathcal{X}$ of $\mathrm{Irr}(G)$ into nonempty subsets such that $|\mathcal{X}| = |\mathcal{K}|$*
3. *A set of* supercharacters *χ_X, one for each $X \in \mathcal{X}$ such that the χ_X are constant on superclasses.*

Definition 2.9. *A* superclass *function of a finite group G with a supercharacter theory $(\mathcal{K}, \mathcal{X}, \{\chi_X\})$ is a function $f : G \to \mathbb{C}$ such that if $K \in \mathcal{K}$ and $g, h \in K$ then $f(g) = f(h)$.*

In particular supercharacters are superclass functions. Some important properties of supercharacters are carried over from the character theory.

Theorem 2.5. *Let G be a finite group with supercharacter theory $(\mathcal{K}, \mathcal{X}, \{\chi_X\})$. Then*

1. *$|\mathcal{K}| = |\{\chi_X\}|$*
2. *The set of supercharacters χ_X of G form a basis of the vector space of superclass functions.*
3. *The set of supercharacters characters is orthogonal with respect to the inner product given for characters.*

2.2 Hopf Monoids

A Hopf Monoid is in some sense a generalization of a Hopf algebra and is defined in the language of category theory.

A monoidal category is a category together with a "bifunctor" $\otimes : C \times C \to C$ which is a functor on the product category. The functor must satisfy various properties that loosely amount to

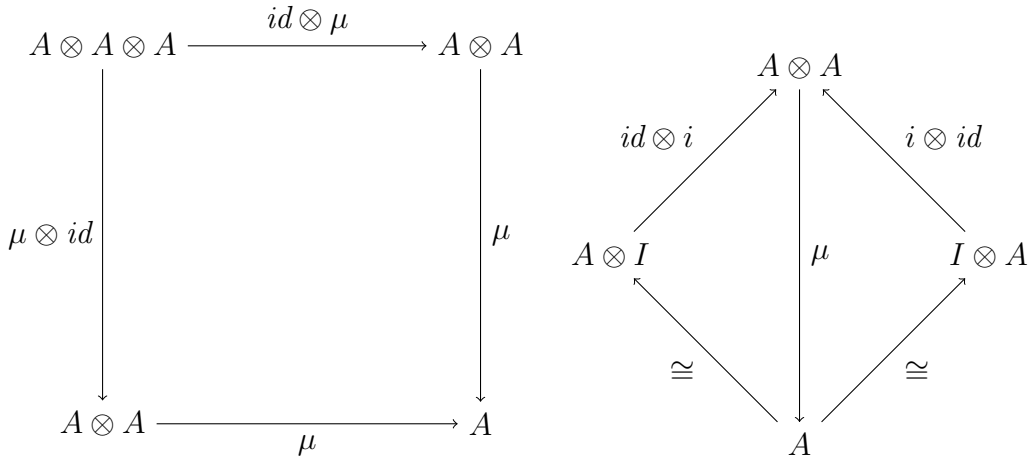
1. Multiplication of objects is associative, $(c \otimes d) \otimes e \cong c \otimes (d \otimes e)$ for c, d, e in C
2. There is a unit object I in C such that $I \otimes c \cong c \otimes I \cong c$

The important examples of monoidal categories for us here are the categories $\mathbb{C}-\mathsf{Vect}$ of vector spaces over $\mathbb{C}$ and linear transformations with the tensor product $\otimes$, and **Sp** the category of vector species (defined later) and natural transformations.

In algebra a monoid is typically presented as a set M with one distinguished element $1 \in M$ and an associative binary operation $* : M \times M \to M$ usually though of as a multiplication. The distinguished element 1 acts as an identity. For any $m \in M$, $1 * m = m * 1 = m$. In this sense, a monoid is simply a group in which inverses are no longer guaranteed.

If C is a monoidal category, we can choose an object A in C and know that there exists an object $A \otimes A$ in C. It may happen that there is a morphism $\mu : A \otimes A \to A$ that satisfies a diagram much like the associative axiom in the definition of an algebra. Such a pair (A, μ) would then be a nonunital monoid in the monoidal category C. A unital monoid is similar to an algebra as it also requires a morphism $i : I \to A$ from the unit object that satisfies an almost identical diagram to the unit axiom in an algebra. Such a triple (A, μ, i) is called a monoid in the monoidal category C.

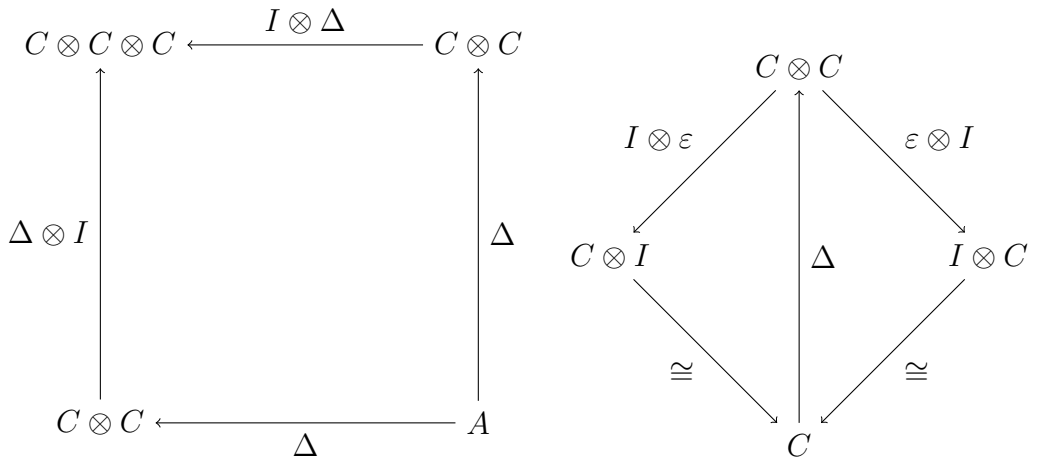
Definition 2.10. *A monoid A in a monoidal category C is a triple (A, μ, i) with A an object of C, and morphisms $\mu : A \otimes A \to A$, and $i : I \to A$ such that the following diagrams commute*



Here the isomorphism $\cong$ represents an isomorphism in the category. A dual construction leads to the notion of a comonoid.

Definition 2.11. *A comonoid A in a monoidal category C is a triple (A, Δ, ε) with A an object in C, and morphisms $\Delta : A \to A \otimes A$ and $\varepsilon : A \to I$ such that the following diagrams*

commute



One should now ask what the morphisms of monoids and comonoids are. A morphism $f : (A, \mu, i) \to (A', \mu', i')$ in the category C is a morphism of monoids if $f \circ \mu = \mu' \circ (f \otimes f)$.. Similarly a morphism $g : (A, \Delta, \varepsilon) \to (A', \Delta', \varepsilon')$ in the category C is a morphism of comonoids if $(g \otimes g) \circ \Delta = \Delta' \circ g$.

If A is a monoid (comonoid), we will need a monoid (comonoid) structure on $A \otimes A$ in order to define a bimonoid. This requires us to work in a slightly more restrictive type of category called a braided monoidal category. A braiding in a monoidal category C is a natural isomorphism β between $A \otimes B$ and $B \otimes A$ for all objects A, B in the category C. A natural isomorphism is simply a natural transformation between these two functors with every morphism being an isomorphism. We say (C, β) is a braided monoidal category if it is a monoidal category with an associated braiding β.

Definition 2.12. *A bimonoid A in a braided monoidal category (C, β) is a tuple $(A, \mu, \Delta, i, \varepsilon)$ such that (A, μ, i) is a monoid, (A, Δ, ε) is a comonoid, μ, i are comonoid morphisms and Δ, ε are monoid morphisms.*

Definition 2.13. *Let C be a braided monoidal category with bifunctor $\otimes : C \times C \to C$ and $(A, \mu, \Delta, i, \varepsilon)$ a bimonoid in C. Then the set $\mathrm{Hom}(A, A)$ of endomorphisms of A together with a map $* : \mathrm{Hom}(A, A) \to \mathrm{Hom}(A, A)$ given by*

$$f * g = \mu \circ (f \otimes g) \circ \Delta$$

for $f, g \in \mathrm{Hom}(A, A)$ is the convolution monoid associated with A.

Definition 2.14. *A* Hopf monoid *in a braided monoidal category C is a bimonoid A in C such the identity morphism $I_A : A \to A$ has an inverse $S : A \to A$ under $*$ in the convolution monoid $\mathrm{Hom}(A, A)$. The map S is called the antipode.*

We will now look at Hopf monoids in the category of vector species which will be the only Hopf monoids we have occasion to use. $\mathsf{Set}^\times$ is the category of all finite sets in which the

morphisms are bijective functions. A *species* is a functor $T : \mathsf{Set}^\times \to C$ into some other category C. We will be exclusively concerned with finite dimensional *vector species* in which $C = \mathbb{C}\text{-}\mathsf{Vect}$, the category of vector spaces and in which $T(S)$ is finite dimensional for any set $S \in \mathsf{Set}^\times$. There is a category $\mathbf{Sp}$ of all vector species in which the objects are vector species and the morphisms are natural transformations between species.

More concretely, suppose we have two finite dimensional vector species $\mathbf{h}, \mathbf{p} : \mathsf{Set}^\times \to \mathbb{C}\text{-}\mathsf{Vect}$. Given a set I in $\mathsf{Set}^\times$ we we write $h[I]$ and $p[I]$ to denote the vector space that is the image of I under the functors $\mathbf{h}$ and $\mathbf{p}$ respectively. Likewise, for a set map $\sigma : I \to J$ between finite sets I and J, we will write $\mathbf{h}[\sigma]$ to denote the linear isomorphism that is the image of σ under the functor $\mathbf{h}$. A morphism of vector species from $\mathbf{h}$ to $\mathbf{p}$ is a collection of linear maps $\{f_I : \mathbf{h}[I] \to \mathbf{p}[I]\}$, one for each finite set I in $\mathsf{Set}^\times$ such that for any set map $\sigma : I \to J$ between finite sets I and J we have

$$f_J \circ \mathbf{h}[\sigma] = \mathbf{p}[\sigma] \circ f_I.$$

The category $\mathbf{Sp}$ can be made into a monoidal category in various ways, but we will use the *Cauchy product*. Define a multiplication of species such that for a finite set, I and bijective set map $\sigma : I \to K$,

$$(\mathbf{h} \cdot \mathbf{p})[I] = \bigoplus_{S \sqcup T = I} \mathbf{h}[I] \otimes \mathbf{p}[I]$$

and

$$(\mathbf{h} \cdot \mathbf{p})[\sigma] = \bigoplus_{S \sqcup T = I} \mathbf{h}[\sigma|_S] \otimes \mathbf{p}[\sigma|_T].$$

A Hopf monoid in vector species then, is a species $\mathbf{h}$ in $\mathbf{Sp}$ together with morphisms of species $\mu : \mathbf{h} \cdot \mathbf{h} \to \mathbf{h}$, $\Delta : \mathbf{h} \to \mathbf{h} \cdot \mathbf{h}$, $i : \mathbf{1} \to \mathbf{h}$, and $\varepsilon : \mathbf{h} \to \mathbf{1}$ where $\mathbf{1}$ is the identity element in $\mathbf{Sp}$ which is the functor

$$\mathbf{1}[I] = \begin{cases} \mathbb{C} & \text{if } I = \emptyset \\ 0 & \text{if } I \neq \emptyset \end{cases}$$

for all I. Here 0 refers to the trivial, zero dimensional vector space.

3 A Supercharacter Theory of $\mathrm{UT}_n(q)$

As mentioned before, the general representation theory and character theory of $\mathrm{UT}_n(q)$ is known to be equivalent to another problem that is intractable. Supercharacter theory simplifies the problem by coarsening the usual character theory. This is done by replacing characters with sums of characters and conjugacy classes by unions of conjugacy classes. In some supercharacter theories of $\mathrm{UT}_n(q)$, the vector space of superclass functions can be given the structure of a Hopf algebra in which the operations are the inflation and restriction of characters. This algebra in turn has a combinatorial description in terms of the arc diagrams. This connection is due to the fact that the supercharacters are indexed by such diagrams in a fairly natural way that relates to a chosen representative of the superclass a specific

partition of a set. It turns out that the Hopf algebra constructed out of the supercharacters is isomorphic to the Hopf algebra of symmetric functions in non-commuting variables, which in turn is related to the Hopf algebra of symmetric functions in commuting variables, as is shown in [2].

The situation here is reminiscent of the relationship between the representation theory and character theory of the symmetric group S_n and the Hopf algebra of symmetric functions in commuting variables.

3.1 Modules

Recall from the introduction that $\mathrm{UT}_n(q)$ is a matrix subgroup of $GL_n(\mathbb{F}_q)$. The group $\mathrm{UT}_n(q)$ is closely related to the algebra $\mathfrak{n} = UT_n(q) - 1$ of strictly upper triangular matrices with zeros along the main diagonal. The associated algebra $\mathfrak{n}$ can also be made into a Lie algebra with the commutator bracket $[t, s] = ts - st$. There is a bijection between $\mathfrak{n}$ and $UT_n(q)$ which amounts to adding or removing the 1's from the diagonal. The constructions here work for algebra groups in general, see [5].

Since $\mathfrak{n}$ is an algebra it is in particular a vector space and we can consider its dual space. This allows us to use the notions of coadjoint orbits and other such constructions. In particular we can define a right action of $G = UT_n(q)$ on $\mathfrak{n}$ via right multiplication, i.e. if $j \in \mathfrak{n}$, $g \in G$ then $g = 1 + t$ for some other $t \in \mathfrak{n}$ and

$$j \cdot g = j(1 + t) = j + jt$$

We can define in a similar manner a left action, a two-sided action and a conjugation action of G on $\mathfrak{n}$

$$\mathfrak{n}^* = \{\lambda : \mathfrak{n} \to \mathbb{C} | \ \ \lambda \text{ is linear } \}.$$

In addition to this we can have G act on $\mathfrak{n}^*$, the dual space of $\mathfrak{n}$ consisting of all $\mathbb{F}_q$ valued linear functionals on $\mathfrak{n}$. If $\lambda \in \mathfrak{n}^*$, $g = 1 + t$ for $g \in G$, $t \in \mathfrak{n}$, then

$$(\lambda \cdot g)(j) = \lambda(jg^{-1}).$$

Again we can define left, two-sided, and conjugation actions in a similar manner.

Considering $\mathbb{F}_q$ as an abelian group, fix θ a nontrivial complex valued character of $\mathbb{F}_q$. Given $\lambda \in \mathfrak{n}^*$ a linear functional, $\theta \circ \lambda : \mathfrak{n} \to \mathbb{C}$. By pre-composing this with the bijection $g \mapsto g - 1$, we can now associate with every $\mathbb{F}_q$ linear functional $\lambda \in \mathfrak{n}^*$ a function $\tilde{\lambda} : G \to \mathbb{C}$,

$$\tilde{\lambda}(g) = \theta(\lambda(g - 1)).$$

Consider now the negative of the right orbit of a fixed $\lambda \in \mathfrak{n}^*$, $Y_\lambda = -\lambda G$, where we take the negative for convenience. We can transfer this orbit to a subset of $\mathcal{F}(G, \mathbb{C})$ the set of complex valued functions on G by the correspondence

$$\lambda \mapsto \tilde{\lambda}.$$

We can turn $\tilde{Y}_\lambda = \mathbb{C}-\text{span}\{\tilde{\lambda}|\lambda \in Y\}$ into a $\mathbb{C}G$ module by the usual action

$$(\tilde{\lambda} \cdot g)(x) = \tilde{\lambda}(x \cdot g^{-1}).$$

A variant of the Theorem 3.1 below is proven in [5].

Theorem 3.1. *Let $\{u_1, u_2, \dots u_\ell\}$ be a set of representatives for the two sided orbits $\mathrm{UT}_n(q)\backslash\mathfrak{n}/\mathrm{UT}_n(q)$. Then*

1. *$\tilde{Y}_{u_i} \cong \tilde{Y}_\lambda$ if λ, u_i belong to the same two sided orbit, i.e. $Gu_iG = G\lambda G$*
2. *$\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}G}(\tilde{Y}_{u_i}, \tilde{Y}_{u_j})) = 0$ if $i \neq j$ and in particular the the two modules do not contain any of the same isomorphism types of irreducible constituents.*
3. *if M is any irreducible $\mathbb{C}G$ module of $UT_n(q)$, then $\dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}G}(M, \tilde{Y}_{u_i}) \neq 0$ for exactly one i.*

This theorem says that these modules are prime candidates for obtaining supercharacters because they partition the irreducible modules. In fact they do give a supercharacter theory which is finer that the one we will be using. The modules that give our supercharacters are simply direct sums of these modules. The sums are are determined by outer automorphisms of $UT_n(q)$ corresponding to diagonal matrices.

3.2 Supercharacters and Superclass functions

As mentioned before, a supercharacter theory of a finite group G is a partition $\mathcal{K}$ of the group and a partition $\mathcal{X}$ of the irreducible characters $\mathrm{Irr}(G)$. Our superclasses, K_λ, will be unions of the two sided orbits $\mathrm{UT}_n(q)\backslash\mathfrak{n}/\mathrm{UT}_n(q) + 1$, and we will have one for each λ a set partition of $[n]$. The supercharacters are then the trace of modules that are the direct sums of $\tilde{Y}_u$ as constructed in the previous section.

We first investigate the two sided orbits to give a more explicit description of them and to describe their connection with set partitions and arc diagrams. Recall that

$$\mathfrak{n} = \mathrm{UT}_n(q) - 1 = \{u - I_n | u \in UT_n(q)\}$$

where I_n is the $n \times n$ identity matrix and that $|\mathfrak{n}| = |UT_n(q)|$ and the map $\varphi : UT_n(q) \to \mathfrak{n}$ given by $\varphi(g) = g - I_n$ for any $g \in UT_n(q)$ is a bijection between them. Fix $t \in \mathfrak{n}$. Then a typical element of $\mathrm{UT}_n(q)\backslash t/\mathrm{UT}_n(q)$, the two sided orbit of t, is utv for some $u, v \in G$. We can view u, v as performing row and column operations and then "reduce" t to a standard form which we will take to be our superclass representative. Since $u, v \in \mathrm{UT}_n(q)$, there are $\tilde{u}, \tilde{v} \in \mathfrak{n}$ such that $1 + \tilde{u} = u$ and $1 + \tilde{v} = v$. Then

$$tv = t + t\tilde{v}$$

A basis for $\mathfrak{n}$ is given by

$$\{E_{ij} | i < j\}$$

where $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ is the matrix with all zeroes and a 1 in the (i,j) spot. With this basis we can write

$$tv = t + t\sum_{i<j} \tilde{v}_{ij}E_{ij}$$

with $\sum_{i<j} \tilde{v}_{ij}E_{ij} = \tilde{v}$. We now describe an algorithm that takes an element $t \in \mathfrak{n}$ and returns an element $\lambda \in \mathfrak{n}$ with the property that there is at most one nonzero element in each column and each row.

Algorithm 1 (Superclass representatives)**.** Fix $t \in \mathfrak{n}$.
Let $i = 1$.

While: $i \leq n$

If ith column nonzero, **Then** set j = index of last nonzero row in column i. Set $k = j - 1$

While: $k \geq 0$

Set the k, i entry to zero by right multiplying t by $I_n - (t_{ji})^{-1}t_{ki}E_{kj}$

$k = k - 1$

End While.

Set $k = i + 1$

While: $k \leq n$

Set the j, k entry to zero by right multiplying t by $I_n - (t_{ji})^{-1}t_{jk}E_{ik}$

End While.

End If.

$i = i + 1$

End While.

For example, if in $UT_4(5)$

$$t = \begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 2 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which now only has at most one nonzero element in each row and column. Every matrix in $\mathfrak{n}$ can be reduced to a unique matrix of with no more than 1 nonzero element in each row and column. In our supercharacter theory we take this one step further and allow rescaling of columns and rows so that every nonzero entry is a 1. This amounts to conjugating by diagonal matrices with elements in $\mathbb{F}_q$. These conjugations correspond to outer automorphism of $UT_n(q)$ and corresponds to unioning two sided orbits. Thus in our example, the superclass representative for t would be

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

It is now possible to see the connection between superclasses and the arc diagrams of set partitions. Let $[n]$ denote the set $\{1, 2 \dots n\}$. A set partition $(I_1|I_2|\dots|I_n)$ of a finite set I is a set of pairwise disjoint, nonempty subsets of I, $\{I_n\}$ such that

$$\bigcup_{i=1}^{n} I_i = I.$$

If I has a an associated total order τ then we call a pair $(i, j) \in I \times I$ such that $i < j$ an arc in I with respect to τ and depict it pictorially as an arc $i \frown j$ or

.

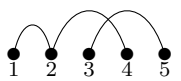
If $\text{arc}(I) \subset I \times I$ is the set $\text{arc}(I) = \{(i, j) \in I \times I | i < j\}$ of all possible arcs in I, then there is a bijection between set partitions $(I_1|I_2|\dots|I_n)$ of I and sets of arcs, $\pi \subseteq \text{arc}(I)$ such that

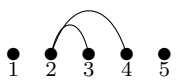
1. if $i \frown j \in \pi$ and $i \frown k \in \pi$, then $j = k$
2. if $m \frown l \in \pi$ and $k \frown l \in \pi$, then $m = k$.

Let

$$S_\tau(I) = \{\text{sets } \mu \text{ of arcs in } I \text{ with respect to } \tau \mid \mu \text{ satisfies 1 and 2}\}$$

be the set of all such sets of arcs. Given a set of arcs μ, the corresponding set partition is found by taking two elements of $i, j \in I$ to be in the same part if there is an arc $i \frown j \in \mu$. For instance, if $I = [5]$ and ε is the usual order, then $\pi = \{1 \frown 2, 2 \frown 4, 3 \frown 5, \} \in S_\varepsilon([5])$, but $\lambda = \{2 \frown 4, 2 \frown 3\} \notin S_\varepsilon([5])$. We would depict these in *arc diagrams* as

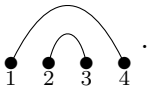
$\pi =$ 

$\lambda =$ 

. (2)

Diagrams such as that of λ where an arc that appears underneath another arc but shares an endpoint are disallowed by conditions 1 and 2. The set partition corresponding to π is $(1,2,4|3,5) = \{\{1,2,4\},\{3,5\}\}$. The parts here correspond exactly to the connected components of the graph. Because of the bijective correspondence between these sets of arcs and set partitions we will often refer to a set of arcs such as π as a set partition.

Let $t \in \mathfrak{n}$ and let $\tilde{t}$ be the reduced matrix that is the superclass representative of the two sided orbit containing t. If we assign to every nonzero entry (i,j) of $\tilde{t}$ the arc $i \frown j$, we end up with a set partition since no row or column can have two nonzero entries. This gives a bijection between the set of strictly upper triangular matrices with at most one entry of 1 in any column or row and the set of all set partitions of $[n]$, or the corresponding arc diagrams. Since these matrices are also in bijection with the two sided orbits and hence our superclasses, this describes a bijection between our superclasses and set partitions of $[n]$. We will index our superclasses by arc diagrams, so for example the superclass with unique representative given by $1+\tilde{t}$ as in the previous example above corresponds to the set partition



We will call this the *partition type* of $1+t$.

We will be using several operations on set partitions, the simplest of which are restriction and concatenation. Given $\mu \in S_\tau(I)$, and a subset $J \subseteq I$, define the *restriction* of μ to J to be

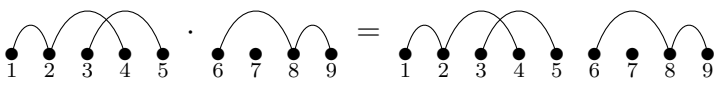
$$\mu|_J = \{i \frown j \in \mu \mid i,j \in J\}.$$

There is also an implied restriction on the total order as well, since $\mu|_J \in S_{\tau|_J}(J)$ where $\tau|_J$ is simply the subposet of I with elements in J. In diagrams, restriction removes all dots that are not in J and removes any arc that does not have both endpoints in J.

Concatenation on the other hand simply joins arc diagrams side by side. Given two disjoint finite sets J_1, J_2 with τ_1, τ_2 total orders on J_1, J_2 respectively, let $J = J_1 \cup J_2$ and τ be the total order determined by $\min(J_2) > \max(J_1)$. Let $\mu_1 \in S_{\tau_1}(J_1)$ and $\mu_2 \in S_{\tau_2}(J_2)$. Then the *concatenation* of μ_1 and μ_2 is given by

$$\mu_1 \cdot \mu_2 = \mu_1 \cup \mu_2$$

as sets of arcs. Let π be as in the above example and let $\gamma = \{6 \frown 8, 8 \frown 9\} \in S_\varepsilon(\{6,7,8,9\})$. Then the concatenation of π and γ would be, in arc diagrams

$\pi \cdot \gamma =$ 

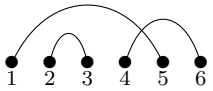
A succinct way to express our superclasses is via an equivalence relation; for $u, v \in \mathrm{UT}_n(q)$ we define u to be equivalent to v if $u = t^{-1}x(v-1)yt + 1$ for some matrices $x, y \in J$ and some diagonal matrix $t \in \mathrm{GL}_n(\mathbb{F}_q)$. Since supercharacters are constant on conjugacy classes, we can define a supercharacter by giving its values on representatives for the superclasses.

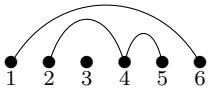
In particular, we can write the values of the supercharacters in terms of the statistics of the set partition types of our group elements. To do so we need some statistics defined for set partitions. Let $\alpha, \mu \in S_\varepsilon([n])$. Then these statistics are defined to be

$$\mathrm{nst}_\alpha^\mu = \#\{i < k < l < j | i \frown j \in \mu, k \frown l \in \alpha\} \tag{3}$$

$$\dim(\mu) = \sum_{i \frown j \in \mu} (j - i) \tag{4}$$

The first counts how many times a set of arcs is nested (lying entirely underneath) within another set of arcs in an arc diagram. The second sums the "sizes" of all the arcs, where size is the number of dots that lie entirely underneath the arc plus one. Let

$\mu =$ 

$\lambda =$ 

.

Then $\mathrm{nst}_\mu^\mu = 1$, $\mathrm{nst}_\mu^\lambda = 1$, $\mathrm{nst}_\lambda^\lambda = 2$, and $\mathrm{nst}_\lambda^\mu = 1$. As well $\dim(\mu) = 4 + 1 + 2 = 7$ and $\dim(\lambda) = 5 + 2 + 1 = 8$.

An important property of nestings is that they are additive in the sense that

$$\mathrm{nst}_\alpha^\beta = \sum_{\substack{i \frown j \in \alpha \\ k \frown l \in \beta}} \mathrm{nst}_{i \frown j}^{l \frown k} \tag{5}$$

In our supercharacter theory the supercharacter indexed by the set partition λ thought of as a set of arcs is given by

$$\chi^\lambda(u) = \begin{cases} \frac{(-1)^{|\lambda \cap \mu|} q^{\dim(\lambda) - |\lambda|} (q-1)^{|\lambda - \mu|}}{q^{\mathrm{nst}_\mu^\lambda}}, & \text{if } i \le j \le k,\ i \frown k \in \lambda \\ & \text{implies } i \frown j, j \frown k \notin \mu, \\ 0, & \text{otherwise.} \end{cases} \tag{6}$$

where to evaluate it on $u \in \mathrm{UT}_n(q)$ we find the partition type μ of u and then calculate as given in terms of arcs. These supercharacters are characters of $\mathrm{UT}_n(q)$, but are not irreducible in general. One that is, is the supercharacter corresponding to the set partition with no arcs (which corresponds to the partition $(1|2|3|\ldots|n-1|n)$). It takes value 1 on all elements and hence is the trivial character of $\mathrm{UT}_n(q)$.

3.3 Hopf Monoid of Superclass functions on $\mathrm{UT}_n(q)$

One can make a Hopf algebra from these superclass functions, but it is more convenient to perform computations in the more general Hopf monoid we now construct. We will later describe how this relates to the associated Hopf algebra. Let I be a finite set and denote by $L[I]$ the set of all linear (total) orders on I. Choose $\phi \in L[I]$. We will write $i \prec_\phi j$ to denote that i is less than j in order ϕ. We also need to define a copy of $\mathrm{UT}_n(q)$ for an arbitrary finite set I and linear order on I, so we let

$$U^\phi = \left\{ u \in UT_{|I|}(q) | \text{ for all } i \in I, u_{ii} = 1, u_{ij} = 0 \text{ if } i \succ_\phi j \right\}$$

which is basically $\mathrm{UT}_n(q)$ with columns and rows indexed by elements of I and written in the order ϕ. These form a group in the same way that $\mathrm{UT}_n(q)$ is a group and so in particular we can consider its supercharacters, $\chi^{(\phi,\lambda)}$, given by formula (6) with $\leq$ replaced by $\preceq_\phi$ and λ a set partition of I.

The Hopf monoid will be a monoid in species, so it is a functor $\mathbf{scf}(U) : \mathsf{Set}^\times \to \mathbb{C}-\mathsf{Vect}$. Let

$$\mathbf{scf}(U^\phi) = \mathbb{C}\text{–span}\{\chi^{(\phi,\lambda)} | \lambda \in S_\phi(I)\}$$

be the vector space spanned by the supercharacter functions. Every function here is a superclass function, and every superclass function is a linear combination of supercharacter functions. The functor $\mathbf{scf}(U)$ is then given by

$$\mathbf{scf}(U)[I] = \bigoplus_{\phi \in L[I]} \mathbf{scf}(U^\phi).$$

This assigns to every finite set a vector space. A typical element of this vector space is a linear combinations of arc diagrams on $|I|$ dots with various linear orders. For example, an element of $\mathbf{scf}(U)[4]$ is

$$\chi \overset{1\,2\,3\,4}{\bullet\bullet\bullet\bullet} + 2i\chi \overset{1\,2\,3\,4}{\bullet\bullet\bullet\bullet} + 4\chi \overset{2\,3\,1\,4}{\bullet\bullet\bullet\bullet} + \chi \overset{4\,3\,2\,1}{\bullet\bullet\bullet\bullet}.$$

Since $\mathbf{scf}(U)$ is a functor if I, J are finite sets with the same cardinality, and $\sigma : I \to J$ is a bijection,

$$\mathbf{scf}(U)[\sigma] : \mathbf{scf}(U)[I] \to \mathbf{scf}(U)[J]$$

is an isomorphism of vector spaces.

Recall, to specify a Hopf monoid we need to specify natural transformations

$$\mu : \mathbf{scf}(U) \cdot \mathbf{scf}(U) \to \mathbf{scf}(U) \tag{7}$$
$$\Delta : \mathbf{scf}(U) \to \mathbf{scf}(U) \cdot \mathbf{scf}(U) \tag{8}$$
$$i : \mathbf{1} \to \mathbf{scf}(U) \tag{9}$$
$$\varepsilon : \mathbf{scf}(U) \to \mathbf{1} \tag{10}$$

The natural transformation μ in turn is specified by providing one map

$$\tilde{\mu}_K : \bigoplus_{I \sqcup J = K} (\mathbf{scf}(U)[I] \otimes \mathbf{scf}(U)[J]) \to \mathbf{scf}(U)[K]$$

for every $K \in \mathsf{Set}^\times$. Since $\tilde{\mu}_K$ is a linear map from a direct sum it suffices to define maps on each of its summands. Because the union of two disjoint sets is unique, this means we can specify one map $\mu_{I,J}$ for each pair of disjoint finite sets to completely specify μ. The notion of inflation given in Theorem 2.4 will provide these maps.

Let $\phi \in L[I]$ for some finite set I. For a partition $(J_1|J_2|\ldots|J_\ell)$ of I, the group U^ϕ has a subgroup isomorphic to

$$U^{\phi|_{J_1}} \times U^{\phi|_{J_2}} \times \ldots \times U^{\phi|_{J_\ell}}$$

with isomorphism given by $\iota : U^{\phi|_{J_1}} \times U^{\phi|_{J_2}} \times \ldots \times U^{\phi|_{J_\ell}} \to U^\phi$

$$[\iota(u_1, u_2, \ldots u_\ell)]_{i,j} = \begin{cases} [u_m]_{i,j} & \text{if } i,j \in J_m \text{ for some } m \\ 0 & \text{otherwise} \end{cases}. \tag{11}$$

For the subgroups that correspond to block diagonal matrices there is a surjective homomorphism,

$$\rho : U^{\phi|_{J_1}\phi|_{J_2}\cdots\phi|_{J_\ell}} \to \iota(U^{\phi|_{J_1}} \times U^{\phi|_{J_2}} \times \ldots \times U^{\phi|_{J_\ell}}) \tag{12}$$

that simply sets a matrix entry i,j to zero if i and j are not both in the same part of J_m of the partition of I. Theorem 2.4 tells us that that inflation will take a character of $U^{\phi|_1} \times U^{\phi_2} \times \ldots \times U^{\phi_\ell}$ and give a character of $U^{\phi_1\phi_2\cdots\phi_\ell}$. It also sends supercharacters to supercharacters.

For two sets S, T let $\mathcal{F}(S,T) = \{f : S \to T\}$ be the set of all functions between them. Given two finite sets, I, J there is a vector space isomorphism $\mathcal{F}(I,\mathbb{C}) \otimes \mathcal{F}(J,\mathbb{C}) \cong \mathcal{F}(I \times J, \mathbb{C})$ given by $(f \otimes g)(i,j) = f(i)g(j)$. Since $\mathbf{scf}(U^\phi) \otimes \mathbf{scf}(U^\tau)$ is a vector subspace of $\mathcal{F}(U^\phi, \mathbb{C}) \otimes \mathcal{F}(U^\tau, \mathbb{C})$ we can use this isomorphism. Given $I \sqcup J = K$, $\phi, \tau \in L[K]$ we define the inflation of two superclass functions to be

$$\mathrm{Inf}_{U^\phi \times U^\tau}^{U^{\phi\tau}} : \mathbf{scf}(U^\phi) \otimes \mathbf{scf}(U^\tau) \to \mathbf{scf}(U^{\phi\tau}) \tag{13}$$

$$\mathrm{Inf}_{U^\phi \times U^\tau}^{U^{\phi\tau}}(\psi \otimes \varphi)(u) = (\psi \otimes \varphi)(\iota^{-1}(\rho(u))) \tag{14}$$

with the identification of $\psi \otimes \varphi$ with a function on $U^\phi \times U^\tau$ via the isomorphism given above.

Thus our maps $\mu_{I,J}$ will be

$$\mu_{I,J}(\psi \otimes \varphi) = \mathrm{Inf}_{U^\phi \times U^\tau}^{U^{\phi\tau}}(\psi \otimes \varphi) \tag{15}$$

Now for the coproduct maps. We need one map for every $K \in \mathsf{Set}^\times$

$$\tilde{\Delta}_K : \mathbf{scf}(U)[K] \to \bigoplus_{I \sqcup J = K} (\mathbf{scf}(U)[I] \otimes \mathbf{scf}(U)[J])$$

and we can define this map by defining a map into each component $\mathbf{scf}(U)[I] \otimes \mathbf{scf}(U)[J]$. Thus we can again completely specify the natural transformation Δ by giving one map $\Delta_{I,J}$ for each pair of disjoint sets $I, J \in \mathsf{Set}^{\times}$. Restriction will give us these maps. Given disjoint finite sets I, J such that $I \sqcup J = K$, we can define $\Delta_{I,J}$ on each of its component subspaces

$$\mathrm{Res}^{\mathbf{scf}(U)[K]}_{\mathbf{scf}(U^{\phi})}(\Delta_{I,J}) : \mathbf{scf}(U^{\phi}) \to \mathbf{scf}(U)[I] \otimes \mathbf{scf}(U)[J] \tag{16}$$

$$\psi \mapsto \mathrm{Res}^{U^{\phi}}_{\iota(U^{\phi|_I} \times U^{\phi|_J})}(\psi), \tag{17}$$

where Res is the restriction of functions.

3.4 Bases for scf(U)

To describe the operations on $\mathbf{scf}(U)$ it is easiest to do so by describing what happens to a basis of each vector space $\mathbf{scf}(U)[I]$. This amounts to specifying a basis for each $\mathbf{scf}(U^{\phi})$ for each $\phi \in L[I]$ and finite set I. We have already given one such basis in defining $\mathbf{scf}(U)$, namely that of the supercharacters $\chi^{(\phi,\lambda)}$. The results in this section are proved [2] and [7].

Proposition 3.1. *Let I, J be finite sets such that $I \cap J = \emptyset$ and $I \sqcup J = K$. Let $\phi \in L[I]$, $\tau \in L[J]$ and $\lambda \in S_{\phi}(I)$, $\nu \in S_{\tau}(J)$. Then*

$$\chi^{(\phi,\lambda)} \cdot \chi^{(\tau,\nu)} = \chi^{(\phi\tau,\lambda\cup\nu)}.$$

The restriction and antipode on the supercharacter basis is considerably more complicated and we will not consider it here.

A second natural basis is given by the superclass characteristic functions.

Definition 3.1. *Given $\phi \in L[I]$ and $\lambda \in S_{\phi}(I)$, the superclass characteristic function corresponding to (ϕ, λ) is given by*

$$\kappa_{(\phi,\lambda)} : U^{\phi} \to \mathbb{C}, \tag{18}$$

$$\textit{where} \quad \kappa_{(\phi,\lambda)}(u) = \begin{cases} 1, & \textit{if } u \textit{ is in the superclass indexed by } \lambda \\ 0, & \textit{otherwise} \end{cases}. \tag{19}$$

Then clearly $\{\kappa_{(\phi,\lambda)} \mid \lambda \in S_{\phi}(I)\}$ is a basis for $\mathbf{scf}(U^{\phi})$ since they span the vector space of superclass functions. The product and coproduct for the κ basis are fairly simple.

Proposition 3.2. *Let K be a finite set such that $I \sqcup J = K$, $\phi \in L[K]$ and $\lambda \in S_{\phi}(K)$. Then*

$$\Delta_{I,J}\left(\kappa_{(\phi,\lambda)}\right) = \begin{cases} \kappa_{(\phi|_I,\lambda|_I)} \otimes \kappa_{(\phi|_J,\lambda|_J)} & \textit{if } \lambda|_I \cup \lambda|_J = \lambda \\ 0 & \textit{otherwise} \end{cases} \tag{20}$$

Proof:
Let $u_1 \in U^{\phi|_I}$ and $u_2 \in U^{\phi|_J}$ and let μ_1, μ_2 be the partition types of u_1 and u_2 respectively, then by definition of κ,

$$\begin{aligned}\Delta_{I,J}(\kappa_{(\phi,\lambda)})(u_1 \otimes u_2) &= \mathrm{Res}^{U^\phi}_{i(U^{\phi|_I} \times U^{\phi|_J})}\left(\kappa_{(\phi,\lambda)}\right)(\iota(u_1,u_2)) \\ &= \kappa_{(\phi,\lambda)}(\iota(u_1,u_2)) = \begin{cases} 1 & \text{if } \mu_1 \cup \mu_2 = \lambda \\ 0 & \text{else} \end{cases}.\end{aligned}$$

If $\lambda|_I \cup \lambda|_J \neq \lambda$, then there is an arc $i \frown j \in \lambda$ such that one endpoint is in I and the other in J. Thus for any μ_1, μ_2, $\mu_1 \cup \mu_2 \neq \lambda$ so in that case $\Delta_{I,J}(\kappa_{(\phi,\lambda)}) = 0$ is the zero map.

If, on the other hand, $\lambda|_I \cup \lambda|_J = \lambda$, then we must have $\mu_1 = \lambda|_I$ and $\mu_2 = \lambda|_J$ in order for the function to be nonzero. This is equivalent to saying

$$\Delta_{I,J}(\kappa_{(\phi,\lambda)}) = \kappa_{(\phi|_I,\lambda|_I)} \otimes \kappa_{(\phi|_J,\lambda|_J)}. \quad \square$$

By virtually the same argument, one can also show that a similar relation holds for arbitrary restrictions.

Proposition 3.3. *If $J_1, J_2, \ldots J_\ell$ are pairwise disjoint subsets of K such that $\bigcup_{s=1}^{\ell} J_s = K$, then restriction on the κ basis is given by*

$$\begin{aligned}&\mathrm{Res}^{UT^\phi_K(q)}_{UT^{\phi|_{J_1}}_{J_1} \times UT^{\phi|_{J_2}}_{J_2} \times \ldots \times UT^{\phi|_{J_\ell}}_{J_\ell}}\left(\kappa_{(\phi,\lambda)}\right) \\ &\quad = \begin{cases} \kappa_{(\phi|_{J_1},\lambda|_{J_1})} \otimes \kappa_{(\phi|_{J_2},\lambda|_{J_2})} \otimes \ldots \otimes \kappa_{(\phi|_{J_l},\lambda|_{J_\ell})} & \text{if } \lambda|_{J_1} \cup \lambda|_{J_2} \ldots \cup \lambda|_{J_\ell} = \lambda \\ 0 & \text{otherwise} \end{cases}.\end{aligned}$$

Proposition 3.4. *Let I, J be disjoint finite sets such that $I \sqcup J = K$, and $\phi \in L[I]$, $\lambda \in S_\phi(I)$, and $\tau \in L[J]$, $\mu \in S_\tau(J)$. Then*

$$\mu_{I,J}\left(\kappa_{(\phi,\lambda)} \otimes \kappa_{(\tau,\mu)}\right) = \sum_{\substack{\nu \in S_{\phi\tau}(K) \\ \nu|_I = \lambda,\ \nu|_J = \mu}} \kappa_{(\phi\tau,\nu)}. \tag{21}$$

Proof:
Let $u \in U^{\phi\tau}$ such that $\pi \in S_{\phi\tau}(K)$ is partition type of u. Then $\iota^{-1}(\rho(u)) = (u_1, u_2)$ for some $u_1 \in U^\phi$ and $u_2 \in U^\tau$ and has partition types $\pi|_I$, $\pi|_J$.

$$\mathrm{Inf}^{U^{\phi\tau}}_{U^\phi \times U^\tau}\left(\kappa_{(\phi,\lambda)} \otimes \kappa_{(\tau,\mu)}\right)(u) = \kappa_{(\phi,\lambda)}(u_1) \cdot \kappa_{(\tau,\mu)}(u_2) = \begin{cases} 1 & \text{if } \pi|_I = \lambda,\ \pi|_J = \mu \\ 0 & \text{otherwise} \end{cases}$$

This is another way of saying

$$\mathrm{Inf}^{U^{\phi\tau}}_{U^\phi \times U^\tau}\left(\kappa_{(\phi,\lambda)} \otimes \kappa_{(\tau,\mu)}\right)(u) = \sum_{\substack{\nu \in S_{\phi\tau}(K) \\ \nu|_I = \lambda,\ \nu|_J = \mu}} \kappa_{(\phi\tau,\nu)}. \ \square$$

3.5 Hopf Algebra of Symmetric Functions in Non-commuting Variables

As the name suggests, Hopf monoids are related to Hopf algebras. Given a Hopf monoid such as $\mathbf{scf}(U)$, one can construct several corresponding Hopf algebras by an operation similar to forming a quotient. In this context it is helpful to think of Hopf monoids in a different light. Essentially all of the information in $\mathbf{scf}(U)$ is contained in the images of $[n]$ for all $n \in \mathbb{N}$. Since $\mathbf{scf}(U)$ is a functor from $\mathsf{Set}^\times$, given any two finite sets I and J of the same cardinality, there is a bijection $\sigma : I \to J$ such that $\mathbf{scf}(U)[\sigma] : \mathbf{scf}(U)[I] \to \mathbf{scf}(U)[J]$ is an isomorphism of vector spaces. A choice of bijection between $[n]$ and each other finite set of size n allows one to identify them all with $[n]$.

In addition, if $I = J = [n]$ these same vector space isomorphisms give an action of the symmetric group S_n on $\mathbf{scf}(U)[n]$ by $\pi \cdot v = \mathbf{scf}(U)[\pi](v)$ where $\pi \in S_n$ is thought of as a bijection $\pi : [n] \to [n]$. Because of this, $\mathbf{scf}(U)$ can be thought of as an collection of $\mathbb{C}S_n$-modules $\{\mathbf{scf}(U)[0], \mathbf{scf}(U)[1], \ldots \mathbf{scf}(U)[n], \ldots\}$ together with module maps corresponding to $\mu_{I,J}$ and $\Delta_{I,J}$. The corresponding Hopf algebras will be the infinite direct sums of these vector spaces or their quotients by the action of S_n together with canonical maps that reindex the sets to keep them in the algebra.

Let $\overline{\mathcal{K}}$ be the functor from Sp to gVect the category of graded vector spaces

$$\overline{\mathcal{K}}(\mathbf{h}) = \bigoplus_{n=0}^{\infty} \mathbf{h}[n]_{S_n}$$

for any species $\mathbf{h}$ where $\mathbf{h}[n]_{S_n}$ denotes the vector space of S_n coinvariants of $\mathbf{h}[n]$,

$$\mathbf{h}[n]_{S_n} = \mathbf{h}[n]/\langle v - \pi v | \pi \in S_n, v \in \mathbf{h}[n] \rangle$$

which identifies an element with its orbit under S_n.

For $\mathbf{scf}(U)$,

$$\overline{\mathcal{K}}(\mathbf{scf}(U)) = \bigoplus_{n=0}^{\infty} \mathbf{scf}(U^{\varepsilon_n})$$

where ε_n is the usual total order on $[n]$, $1 < 2 < 3 < \ldots < n$. For the superclass indicator functions this means $\kappa_{(\phi,\lambda)} \mapsto \kappa_\lambda$, in essence forgetting the total order information.

The coproduct and product in $\overline{\mathcal{K}}(\mathbf{scf}(U))$ are built from the the maps $\mu_{[s],[t]}$ and $\Delta_{S,T}$ where $s + t = n$ and $S \sqcup T = [n]$. There is a reindexing problem that must be sorted out since $\mu_{[s],[t]} : \mathbf{scf}(U^{\varepsilon_s}) \otimes \mathbf{scf}(U^{\varepsilon_t}) \to \mathbf{scf}(U^{\varepsilon_s \varepsilon_t})$ so that the image does not lie in the Hopf algebra. Thus one uses canonical reindexing maps

$$\varphi_{s,t} : \mathbf{scf}(U^{\varepsilon_s}) \otimes \mathbf{scf}(U^{\varepsilon_t}) \to \mathbf{scf}(U^{\varepsilon_s}) \otimes \mathbf{scf}(U^{\varepsilon_{\{s+1,s+2,\ldots,n\}}})$$

$$\psi_{S,T} : \mathbf{scf}(U^{\varepsilon_S}) \otimes \mathbf{scf}(U^{\varepsilon_T}) \to \mathbf{scf}(U^{\varepsilon_{|S|}}) \otimes \mathbf{scf}(U^{\varepsilon_{|T|}})$$

to bring elements back into the Hopf algebra.

The Hopf algebra $\overline{\mathcal{K}}(\mathbf{scf}(U))$ is isomorphic to the Hopf algebra of symmetric functions in non-commuting variables as is shown in [2] and [4]. We give a brief construction of a Hopf algebra of set partitions that is also isomorphic to the Hopf algebra of symmetric functions in non-commuting variables.

For each $n \in \mathbb{Z}_{\geq 1}$ define $\mathcal{S}_n$ to be the free $\mathbb{C}$-vector space over $S_\varepsilon([n])$. That is, $\mathcal{S}_n$ is a complex vector space with a basis given by the set partitions of $[n]$. Any element of $\mathcal{S}_n$ we think of as a formal finite linear combination of arc diagrams. Let

$$\Pi_s = \mathbb{C} \oplus \bigoplus_{i=1}^{\infty} \mathcal{S}_n.$$

In defining the product and coproduct (aka multiplication and comultiplication) on Π_s we will do so on a basis and extend the operation linearly. A basis for this space is the union of all $S_\varepsilon([n])$ together with $1 \in \mathbb{C}$.

The map D will be our reindexing map and reindexes a set partition of $I \subseteq [n]$ to be a set partition of $[|I|]$ and is related to the canonical reindexing maps φ and ψ we mentioned above. For each pair of subsets $I, K \subseteq [n]$ such that $I \cap J = \emptyset$ and $I \cup K = [n]$, hereafter denoted as $I \sqcup J$, define a map

$$\Delta_{I,J} : \mathcal{S}_n \to \mathcal{S}_{|I|} \otimes \mathcal{S}_{|J|}$$

by

$$\Delta_{I,J}(\mu) = \begin{cases} D(\mu|_I) \otimes D(\mu|_J) & \text{if } \ \mu|_I \cup \mu|_J = \mu \\ 0 & \text{otherwise .} \end{cases}$$

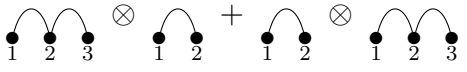
Then define

$$\Delta_n = \sum_{I \sqcup J = [n]} \Delta_{I,J}$$

and finally we can define the coproduct on this basis

$$\Delta(\sum_{i=1}^{n} \lambda_n \mu_n) = \sum_{i=1}^{n} \lambda_i \Delta_{m_i}(\mu_i)$$

Where $\lambda_i \in \mathbb{C}$ and $\mu_i \in S_\varepsilon([m_i])$. Δ is the coproduct on our Hopf Algebra. An example is in order, for the actual operation on a given basis vector is fairly easy despite the multi-layered definition. Consider the π from equation (2). Since π is a set partition of [5], we in principle have to worry about all 52 ways to partition it. However, $\Delta_{I,J}$ is only nonzero when it does not "break any arcs" meaning that every arc in π must have both end points in either I or J. In particular, this means that connected components must stay together, so that

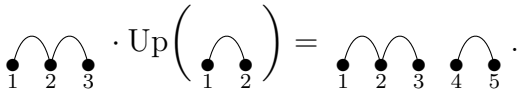
$$\Delta(\pi) = 1 \otimes \pi + \pi \otimes 1 +$$  .

The product $m : \Pi_s \otimes \Pi_s \to \Pi_s$, on the other hand simply concatenates arc diagrams and reindexes the dots to be a partition of $[n]$. Let Up be the map that takes set partitions of

$[n]$ and shifts them up by a given amount m to be a partition of $[n+1, n+2, \ldots n+m]$. If $\mu \in S_\varepsilon([n])$ and $\lambda \in S_\varepsilon([m])$ then

$$m(\mu, \lambda) := \mu \cdot \mathrm{Up}(\lambda)$$

as arc diagrams. For example



The κ basis of $\mathbf{scf}(U)$ gives the isomorphism to this Hopf algebra of set partitions as one might have guess considering their product and coproduct. The antipode on Π_s can be written down in some bases and we will prove one such formula later in the context of the Hopf monoid.

4 The Powersum Basis and its q-Deformations

A third basis is given by the powersum functions and is the main focus of this thesis. One can impose various partial orders on the the set of set partitions of a finite set over some linear order $S_\phi(I)$. In general, powersum functions $P^{\preceq}_{(\phi,\lambda)}$ sum the superclass indicator functions over an upwards closed ideal of $S_\phi(I)$ in some partial order $\preceq$. Perhaps the simplest partial order is that given by inclusions. If $\mu, \nu \in S_\phi(I)$ then we write $\mu \subseteq \nu$ if this is true as sets of arcs (i.e. every arc in μ is also in ν). This defines a partial order on $S_\phi(I)$.

Definition 4.1. *Let $\mu \in S_\phi(I)$ for some finite set I and $\phi \in L[I]$. Then the powersum functions are*

$$P^{\subseteq}_{(\phi,\mu)} = \sum_{\mu \subseteq \alpha} \kappa_\alpha. \tag{22}$$

We will be exclusively concerned with powersum functions in this order and their q-deformations. By q-deformations we mean the coefficients of κ_α in equation (22) are no longer all 1, but is instead some function depending on q, in our case a power of q^{-1}. Recall that q is the order of the finite field $\mathbb{F}_q$ for which $\mathrm{UT}_n(q)$ is a matrix algebra over. The choice of constants given in the definition below are convenient because they provide a triangular decomposition of the supercharacter table of $UT_n(q)$ (see [4]).

Definition 4.2. *Let $\phi \in L[I]$ for some finite set I. Then the q-powersum functions are*

$$P^{\subseteq,q}_{(\phi,\mu)} = \sum_{\alpha \supseteq \mu} \frac{1}{q^{\mathrm{nst}^\mu_{\alpha-\mu}}} \kappa_{(\phi,\alpha)} \tag{23}$$

We will shorthand $P^{\subseteq,q}_{(\phi,\mu)} = P^q_{(\phi,\mu)}$. A useful relation which relates the superclass characteristic functions to the q-powersum functions is given next.

Lemma 4.1 (from [4])**.** *The superclass indicator functions can be written in terms of the q-powersum functions as*

$$\kappa_{(\phi,\mu)} = \sum_{\nu \supseteq \mu} \frac{(-1)^{|\nu-\mu|}}{q^{\mathrm{nst}^{\nu}_{\nu-\mu}}} P^q_{(\phi,\nu)}. \tag{24}$$

4.1 Inflation

The product on the P^q basis is particularly simple and echoes Proposition 3.1.

Proposition 4.1. *Let I, J be finite sets such that $I \cap J = \emptyset$ and $I \sqcup J = K$. Let $\phi \in L[I]$, $\tau \in L[J]$ and $\lambda \in S_\phi(I)$, $\mu \in S_\tau(J)$. Then*

$$\mu_{I,J}\left(P^q_{(\phi,\lambda)} \otimes P^q_{(\tau,\mu)}\right) = P^q_{(\phi\tau,\lambda\cup\mu)}. \tag{25}$$

Proof:
By definition of the product, for $u \in U^{\phi\tau}$

$$\begin{aligned}\mu_{I,J}\left(P^q_{(\phi,\lambda)} \otimes P^q_{(\tau,\mu)}\right)(u) &= \mathrm{Inf}^{U^{\phi\tau}}_{U^\phi \times U^\tau}\left(P^q_{(\phi,\lambda)} \otimes P^q_{(\tau,\mu)}\right)(u) \\ &= (P^q_{(\phi,\lambda)} \otimes P^q_{(\tau,\mu)})(\iota^{-1}(\rho(u)))\end{aligned}$$

where the ι and ρ are from the definitions of inflation and restriction as in equations (11) and (12). By the definiton of the P^q basis then,

$$\begin{aligned}\mu_{I,J}\left(P^q_{(\phi,\lambda)} \otimes P^q_{(\tau,\mu)}\right)(u) &= \left(\sum_{\alpha \supseteq \lambda} \frac{1}{q^{\mathrm{nst}^{\lambda}_{\alpha-\lambda}}} \kappa_{(\phi,\alpha)}\right) \otimes \left(\sum_{\beta \supseteq \lambda} \frac{1}{q^{\mathrm{nst}^{\mu}_{\beta-\mu}}} \kappa_{(\tau,\beta)}\right)(\iota^{-1}(\rho(u))) \\ &= \sum_{\substack{\alpha \supseteq \lambda \\ \beta \supseteq \mu}} \frac{1}{q^{\mathrm{nst}^{\lambda}_{\alpha-\lambda} + \mathrm{nst}^{\mu}_{\beta-\mu}}} \mathrm{Inf}^{U^{\phi\tau}}_{U^\phi \times U^\tau}\left(\kappa_{(\phi,\alpha)} \otimes \kappa_{(\tau,\beta)}\right).\end{aligned}$$

The inflation of the κ basis is given by Proposition 3.4. Therefore

$$\mu_{I,J}\left(P^q_{(\phi,\lambda)} \otimes P^q_{(\tau,\mu)}\right) = \sum_{\substack{\alpha \supseteq \lambda,\ \beta \supseteq \mu \\ \nu \in S_{\phi\tau}(K) \text{ s.t.} \\ \nu|_I = \alpha,\ \nu|_J = \beta}} \frac{1}{q^{\mathrm{nst}^{\lambda}_{\alpha-\lambda} + \mathrm{nst}^{\mu}_{\beta-\mu}}} \kappa_{(\phi\tau,\nu)}.$$

For every $\nu \in S_{\phi\tau}(K)$ with the property that that $\nu \supseteq \lambda \cup \mu$, there is exactly one term in the sum since $\nu|_I$ and $\nu|_J$ are uniquely determined by ν. As well, $\mathrm{nst}^{\lambda}_{\nu|_I - \lambda} + \mathrm{nst}^{\mu}_{\nu|_J - \mu} = \mathrm{nst}^{\lambda\cup\mu}_{\nu-\lambda\cup\mu}$ since, by the additive property given in equation (5)

$$\begin{aligned}\mathrm{nst}^{\lambda\cup\mu}_{\nu-\lambda\cup\mu} = &\sum_{i \frown j \in \nu|_I - \lambda} \mathrm{nst}^{\lambda}_{i\frown j} + \sum_{i \frown j \in \nu|_J - \mu} \mathrm{nst}^{\mu}_{i\frown j} + \sum_{i \frown j \in \nu|_I - \lambda} \mathrm{nst}^{\mu}_{i\frown j} \\ &+ \sum_{i \frown j \in \nu|_J - \lambda} \mathrm{nst}^{\lambda}_{i\frown j} + \sum_{i \frown j \in \nu - \nu|_I - \nu|_J} \mathrm{nst}^{\mu\cup\nu}_{i\frown j}\end{aligned}$$

All the sums here are zero except for the first two since in the total order we are working in, $\phi\tau$, an arc with both endpoints in I cannot be nested in an arc with both endpoints in J since all the dots of J lie strictly to the right of all dots in I in an arc diagram. For a similar reason an arc that has one endpoint in each set cannot be nested in an arc with both endpoints in one of the sets. Therefore we can rewrite the sum as

$$\sum_{\nu \supseteq \lambda \cup \nu} \frac{1}{q^{\mathrm{nst}^{\lambda\cup\mu}_{\nu-\lambda\cup\mu}}} \kappa_{(\phi\tau,\nu)} = P^q_{(\phi\tau,\lambda\cup\mu)}. \quad \square$$

From this we can recover a previously known result.

Corollary 4.1. *Let I, J be finite sets such that $I \cap J = \emptyset$ and $I \sqcup J = K$. Let $\phi \in L[I]$, $\tau \in L[J]$ and $\lambda \in S_\phi(I)$, $\mu \in S_\tau(J)$. Then*

$$\mu_{I,J}\left(P_{(\phi,\lambda)} \otimes P_{(\tau,\mu)}\right) = P_{(\phi\tau,\lambda\cup\mu)}. \tag{26}$$

Proof:
This follows directly from the theorem by setting $q = 1$. $\square$.

4.2 Restriction

The coproduct is not as simple as the product on the P^q basis. It will be convenient for us to denote

$$\Delta_J = \mathrm{Res}^{U^\phi}_{U^{\phi|_{J_1}} \times U^{\phi|_{J_2}} \times \ldots \times U^{\phi|_{J_\ell}}}$$

where $J = (J_1|J_2|\ldots|J_\ell)$ is a set partitions of $\bigsqcup_{i=1}^{\ell} J_i$. Again we first prove a factorization formula for use in calculation of the restriction.

Lemma 4.2. *If $\phi \in L(K)$, $\lambda \in S_\phi(K)$, $J \subseteq K$, and $(\phi|_J, \nu) \supseteq \lambda|_J$ then*

$$\sum_{\nu \supseteq \mu \supseteq \lambda|_J} \frac{(-1)^{|\nu-\mu|}}{q^{\mathrm{nst}^{(\phi|_J,\nu)}_{\nu-\mu} + \mathrm{nst}^{(\phi,\lambda)}_{\mu-\lambda|_J}}} = \prod_{i \frown j \in \nu - \lambda|_J} \left(\frac{1}{q^{\mathrm{nst}^{(\phi|_J,\lambda)}_{i\frown j}}} - \frac{1}{q^{\mathrm{nst}^{(\phi|_J,\nu)}_{i\frown j}}} \right).$$

Proof:
Fix a μ such that $\nu \supseteq \mu \supseteq \lambda|_J$ Then the corresponding term in the sum factors as

$$\frac{(-1)^{|\nu-\mu|}}{q^{\mathrm{nst}^{(\phi|_J,\nu)}_{\nu-\mu} + \mathrm{nst}^{(\phi,\lambda)}_{\mu-\lambda|_J}}} = \prod_{i\frown j \in \nu-\mu} \frac{-1}{q^{\mathrm{nst}^{\nu}_{i\frown j}}} \prod_{i\frown j \in \mu - \lambda|_J} \frac{1}{q^{\mathrm{nst}^{\lambda}_{i\frown j}}}$$

and hence gives

$$\begin{aligned}\prod_{i\frown j \in \nu-\lambda|_J} \left(\frac{1}{q^{\mathrm{nst}^{(\phi|_J,\lambda)}_{i\frown j}}} - \frac{1}{q^{\mathrm{nst}^{(\phi|_J,\nu)}_{i\frown j}}} \right) &= \sum_{\nu \supseteq \mu \supseteq \lambda|_J} \prod_{i\frown j \in \nu-\mu} \frac{-1}{q^{\mathrm{nst}^{\nu}_{i\frown j}}} \prod_{i\frown j \in \mu-\lambda|_J} \frac{1}{q^{\mathrm{nst}^{\lambda}_{i\frown j}}} \\ &= \sum_{\nu \supseteq \mu \supseteq \lambda|_J} \frac{(-1)^{|\nu-\mu|}}{q^{\mathrm{nst}^{(\phi|_J,\nu)}_{\nu-\mu} + \mathrm{nst}^{(\phi,\lambda)}_{\mu-\lambda|_J}}}. \ \square\end{aligned}$$

Theorem 4.1. *For $\phi \in L[K]$, $\lambda \in S_\phi(K)$ and a sequence of pairwise disjoint subsets of K, $J_1, \ldots, J_\ell$ such that $\bigcup_{i=1}^{\ell} J_i = K$,*

$$\Delta_J(P^q_{(\phi,\lambda)}) = \begin{cases} \displaystyle\sum_{\substack{\mu \in S_\phi(K), \mu \supseteq \lambda \\ \mu = \mu|_{J_1} \cup \cdots \cup \mu|_{J_\ell}}} a^\lambda_{J,\mu} P^q_{(\phi|_{J_1}, \mu|_{J_1})} \otimes \cdots \otimes P^q_{(\phi|_{J_\ell}, \mu_{J|_\ell})} & \text{if } \lambda = \lambda_{J_1} \cup \cdots \cup \lambda_{J_\ell}, \\ 0 & \text{otherwise,} \end{cases}$$

where

$$a^\lambda_{J,\mu} = \prod_{k=1}^{\ell} \prod_{i \frown j \in \mu|_{J_k} - \lambda} \left(\frac{1}{q^{\mathrm{nst}^\lambda_{i \frown j}}} - \frac{1}{q^{\mathrm{nst}^{\mu|_{J_k}}_{i \frown j}}} \right).$$

Proof:
By definition,

$$\begin{aligned} \Delta_J(P^q_{(\phi,\lambda)}) &= \sum_{\nu \in S_\phi(K), \nu \supseteq \lambda} \frac{1}{q^{\mathrm{nst}^\lambda_{\nu-\lambda}}} \Delta_J(\kappa_{(\phi,\nu)}) \\ &= \sum_{\substack{\nu \in S_\phi(K), \nu \supseteq \lambda \\ \nu = \nu|_{J_1} \cup \cdots \cup \nu|_{J_\ell}}} \frac{1}{q^{\mathrm{nst}^\lambda_{\nu-\lambda}}} \kappa_{(\phi|_{J_1}, \nu|_{J_1})} \otimes \cdots \otimes \kappa_{(\phi|_{J_\ell}, \nu|_{J_\ell})}. \end{aligned}$$

Thus, if $\lambda \neq \lambda_{J_1} \cup \lambda_{J_2} \cup \cdots \cup \lambda_{J_\ell}$, then $\Delta_J(P^q_{(\phi,\lambda)}) = 0$. Assume that $\lambda = \lambda_{J_1} \cup \lambda_{J_2} \cup \cdots \cup \lambda_{J_\ell}$. Then we can write

$$\begin{aligned} \Delta_J(P^q_{(\phi,\lambda)}) &= \sum_{\substack{\nu_k \in S_{\phi|_{J_k}}(J_k) \\ \nu_k \supseteq \lambda|_{J_k} \\ 1 \le k \le \ell}} \left(\frac{1}{q^{\mathrm{nst}^\lambda_{\nu_1 - \lambda}}} \kappa_{(\phi|_{J_1}, \nu_1)} \right) \otimes \cdots \otimes \left(\frac{1}{q^{\mathrm{nst}^\lambda_{\nu_\ell - \lambda}}} \kappa_{(\phi|_{J_\ell}, \nu_\ell)} \right) \\ &= \bigotimes_{k=1}^{\ell} \left(\sum_{\substack{\nu_k \in S_{\phi|_{J_k}}(J_k) \\ \nu_k \supseteq \lambda|_{J_k}}} \frac{1}{q^{\mathrm{nst}^\lambda_{\nu_k - \lambda}}} \kappa_{(\phi|_{J_k}, \nu_k)} \right). \end{aligned}$$

By Lemma 4.1

$$\kappa_{(\phi,\nu)} = \sum_{\mu \in S_\phi(K), \mu \supseteq \nu} \frac{(-1)^{|\mu - \nu|}}{q^{\mathrm{nst}^\mu_{\mu-\nu}}} P^q_{(\phi,\mu)},$$

so

$$\begin{aligned} \Delta_J(P^q_{(\phi,\lambda)}) &= \bigotimes_{k=1}^{\ell} \left(\sum_{\substack{\mu_k, \nu_k \in S_{\phi|_{J_k}}(J_k) \\ \mu_k \supseteq \nu_k \supseteq \lambda|_{J_k}}} \frac{(-1)^{|\mu_k - \nu_k|}}{q^{\mathrm{nst}^\lambda_{\nu_k - \lambda} + \mathrm{nst}^{\mu_k}_{\mu_k - \nu_k}}} P^q_{(\phi|_{J_k}, \mu_k)} \right) \\ &= \bigotimes_{k=1}^{\ell} \left(\sum_{\substack{\mu_k \in S_{\phi|_{J_k}}(J_k) \\ \mu_k \supseteq \lambda|_{J_k}}} \left(\sum_{\substack{\nu_k \in S_{\phi|_{J_k}}(J_k) \\ \mu_k \supseteq \nu_k \supseteq \lambda|_{J_k}}} \frac{(-1)^{|\mu_k - \nu_k|}}{q^{\mathrm{nst}^\lambda_{\nu_k - \lambda} + \mathrm{nst}^{\mu_k}_{\mu_k - \nu_k}}} \right) P^q_{(\phi|_{J_k}, \mu_k)} \right). \end{aligned}$$

By Lemma 4.2,

$$\Delta_J(P^q_{(\phi,\lambda)}) = \bigotimes_{k=1}^{\ell} \Bigg(\sum_{\substack{\mu_k \in S_{\phi|_{J_k}}(J_k) \\ \mu_k \supseteq \lambda|_{J_k}}} \prod_{i \frown j \in \mu_k - \lambda|_{J_k}} \left(\frac{1}{q^{\mathrm{nst}^{\lambda}_{i\frown j}}} - \frac{1}{q^{\mathrm{nst}^{\mu_k}_{i\frown j}}} \right) P^q_{(\phi|_{J_k},\mu_k)} \Bigg)$$
$$= \sum_{\substack{\mu \in S^{\phi}(K), \mu \supseteq \lambda \\ \mu = \mu|_{J_1} \cup \cdots \cup \mu|_{J_\ell}}} a^{\lambda}_{J,\mu} P^q_{(\phi|_{J_1},\mu_{J_1})} \otimes \cdots \otimes P^q_{(\phi|_{J_\ell},\mu|_{J_\ell})},$$

as desired. □

Corollary 4.2. *Let $I \sqcup J = K$ be finite sets and $\lambda \in S_\phi(K)$. Then the coproduct $\Delta_{I,J}$ of* $\mathbf{scf}(U)$ *on the P^q basis is given by*

$$\Delta_{I,J}(P^q_{(\phi,\lambda)}) = \sum_{\substack{\mu \supseteq \lambda \\ \mu|_I \cup \mu|_J = \mu}} a^{\lambda}_{(I|J),\mu} P^q_{(\phi|_I,\mu|_I)} \otimes P^q_{(\phi|_J,\mu|_J)}. \tag{27}$$

Proof:
This is a special case of Theorem 4.1. □

Another special case results in a previously known result.

Corollary 4.3. *The coproduct on the P basis is given by*

$$\Delta_{I,J}(P_{(\phi,\lambda)}) = \begin{cases} P_{(\phi|_I,\lambda|_I)} \otimes P_{(\phi|_J,\lambda|_J)} & \text{if } \lambda|_I \cup \lambda|_J = \lambda \\ 0 & \text{otherwise .} \end{cases}$$

Proof:
Setting $q = 1$ in the previous corollary and noting that $a^{\lambda}_{(I|J),\mu} = \delta_{\mu,\lambda}$ gives the result. □

The next corollary tells us that in order for a coefficient $a^{\lambda}_{(I|J),\gamma}$, $\gamma \neq \lambda$ in Corollary 4.2 to be non-zero, $\gamma - \lambda$ must have all of its arcs in I nested in an arc in J and vice versa. This is far from a complete characterization of the zero coefficients since the converse is not true.

Corollary 4.4. *Let K be a finite set, $\phi \in L[K]$ and $\lambda \in S_\phi(K)$ with $I \sqcup J = K$, $\gamma \in S_\phi(K)$ with $\gamma \supset \lambda$ and suppose that suppose that there exists an arc $i \frown j \in \gamma - \lambda$ such that either $i, j \in J$, $\mathrm{nst}^{\lambda|_I}_{i\frown j} = 0$ or $i, j \in I$, $\mathrm{nst}^{\lambda|_J}_{i\frown j} = 0$. Then $a^{\lambda}_{(I|J),\gamma} = 0$.*

Proof:
Note we can assume neither γ nor λ have an arc with one endpoint in I and the other in J since then the coefficient is automatically zero. Assume that $i, j \in J$, $\mathrm{nst}^{\lambda|_I}_{i\frown j} = 0$. The argument for the other case is the same replacing I with J and J with I. Since the set of arcs such that such that $i, j \in J$, $\mathrm{nst}^{\lambda|_I}_{i\frown j} = 0$ is nonempty we can choose a maximal such arc

$k \frown l \in \gamma - \lambda$ (i.e. it is not nested in any other arc in $\gamma|_J - \lambda|_J$). Then $\mathrm{nst}^{\lambda}_{k\frown l} = \mathrm{nst}^{\lambda|_J}_{k\frown l} = \mathrm{nst}^{\gamma|_J}_{k\frown l}$ and hence the term in the product in Corollary 4.2 corresponding to $k \frown l$,

$$\left(\frac{1}{q^{\mathrm{nst}^{(\phi,\lambda)}_{k\frown l}}} - \frac{1}{q^{\mathrm{nst}^{(\phi|_J,\lambda|_J)}_{k\frown l}}} \right) = 0. \qquad \square$$

It is worthwhile to note as well that there is no formula for the restriction that is similar to the formula for the supercharacters wherein

$$\Delta_{IJ}(\chi^{\lambda}) = \alpha(Res^{UT_n(q)}_{UT_I(q)}(\chi^{\lambda}) \otimes Res^{UT_n(q)}_{UT_J(q)}(\chi^{\lambda}))$$

for some $\alpha \in \mathbb{C}$ (see [7]).

Take for example the restriction of $\nu = \{1 \frown 2, 3 \frown 4\}$ to $I = \{1, 2\}$ and $J = \{3, 4\}$. We have

$$\Delta_{IJ}(P^q_{(\varepsilon_4,\nu)}) = P^q_{(\varepsilon_2,1\frown 2)} \otimes P^q_{(\varepsilon_{\{3,4\}},3\frown 4)}$$

but note that $\mathrm{Res}^{UT_4(q)}_{UT_I(q)}(P^q_{(\varepsilon_4,\nu)}) = 0$ and $\mathrm{Res}^{UT_4(q)}_{UT_J(q)}(P^q_{(\varepsilon_4,\nu)}) = 0$, so there can be no proportionality constant.

4.3 Pointwise Product

A third operation on superclass functions is the pointwise product. Since it is simply the pointwise product of functions it is clear that the pointwise product of two superclass functions will be constant on superclasses and hence be another superclass function. For any given U^{ϕ},

$$\odot : \mathbf{scf}(U^{\phi}) \otimes \mathbf{scf}(U^{\phi}) \to \mathbf{scf}(U^{\phi})$$
$$\odot(\psi \otimes \varphi)(u) = \psi(u)\varphi(u)$$

and as such the operation is commutative.

Proposition 4.2. *Let I be a finite set, $\phi \in L[I]$ and $\lambda, \mu \in S_{\phi}(I)$. Then the pointwise product of two superclass indicator functions is*

$$\kappa_{(\phi,\lambda)} \odot \kappa_{(\phi,\mu)} = \delta_{\mu,\lambda}\kappa_{(\phi,\lambda)} \tag{28}$$

where $\delta_{\mu,\lambda}$ is the Kroenecker delta.

Proof:
For $u \in U^{\phi}$,

$$(\kappa_{(\phi,\lambda)} \odot \kappa_{(\phi,\mu)})(u) = \kappa_{(\phi,\lambda)}(u)\kappa_{(\phi,\mu)}(u) = \begin{cases} 1 & \text{if } u \text{ in the superclass indexed by both } \mu \text{ and } \lambda \\ 0 & \text{otherwise.} \end{cases}$$

so that by the disjointness of superclasses

$$\kappa_{(\phi,\lambda)} \odot \kappa_{(\phi,\mu)} = \begin{cases} \kappa_{(\phi,\lambda)} & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

$$= \delta_{\lambda,\mu}\kappa_{(\phi,\lambda)}. \quad \square$$

Next we compute an explicit formula for the pointwise product on the P^q basis. A useful factorization which appears in the proof is given as the following lemma.

Lemma 4.3. *Let I be a finite set, $\phi \in L[I]$ and $\mu, \lambda, \nu \in S_\phi(I)$ such that $\lambda \cup \mu \in S_\phi(I)$ and $\nu \supseteq \lambda \cup \mu$. Then*

$$\sum_{\nu \supseteq \beta \supseteq \lambda \cup \mu} \frac{(-1)^{|\nu-\beta|}}{q^{\mathrm{nst}^\lambda_{\beta-\lambda} + \mathrm{nst}^\mu_{\beta-\mu} + \mathrm{nst}^\nu_{\nu-\beta}}} = \left(\frac{1}{q^{\mathrm{nst}^\lambda_{\mu-\lambda} + \mathrm{nst}^\mu_{\lambda-\mu}}}\right) \prod_{i \frown j \in \nu - \lambda \cup \mu} \left(\frac{1}{q^{\mathrm{nst}^\lambda_{i\frown j} + \mathrm{nst}^\mu_{i\frown j}}} - \frac{1}{q^{\mathrm{nst}^\nu_{i\frown j}}}\right) \tag{29}$$

Proof:
Note first that

$$\mathrm{nst}^\lambda_{\beta-\lambda} + \mathrm{nst}^\mu_{\beta-\mu} = \mathrm{nst}^\lambda_{\lambda\cup\mu-\lambda} + \mathrm{nst}^\mu_{\lambda\cup\mu-\mu} + \mathrm{nst}^\lambda_{\beta-\lambda\cup\mu} + \mathrm{nst}^\mu_{\beta-\lambda\cup\mu}$$

by the additive property of nestings in Equation (5) and the fact that $\beta - \lambda = (\beta - \lambda \cup \mu) \cup (\lambda \cup \mu - \lambda)$ (a similar reasoning hold for $\beta - \mu$). Thus

$$\sum_{\nu \supseteq \beta \supseteq \lambda \cup \mu} \frac{(-1)^{|\nu-\beta|}}{q^{\mathrm{nst}^\lambda_{\beta-\lambda} + \mathrm{nst}^\mu_{\beta-\mu} + \mathrm{nst}^\nu_{\nu-\beta}}} = \left(\frac{1}{q^{\mathrm{nst}^\lambda_{\mu-\lambda} + \mathrm{nst}^\mu_{\lambda-\mu}}}\right) \sum_{\nu \supseteq \beta \supseteq \lambda \cup \mu} \frac{(-1)^{|\nu-\beta|}}{q^{\mathrm{nst}^\lambda_{\beta-\lambda\cup\mu} + \mathrm{nst}^\mu_{\beta-\lambda\cup\mu} + \mathrm{nst}^\nu_{\nu-\beta}}}$$

Fix a β such that $\nu \supseteq \beta \supseteq \lambda \cup \mu$, then observe that the the term in the above sum corresponding to β can be factored as

$$\frac{(-1)^{|\nu-\beta|}}{q^{\mathrm{nst}^\lambda_{\beta-\lambda\cup\mu} + \mathrm{nst}^\mu_{\beta-\lambda\cup\mu} + \mathrm{nst}^\nu_{\nu-\beta}}} = \prod_{i\frown j \in \beta - \lambda\cup\mu} \frac{1}{q^{\mathrm{nst}^\lambda_{i\frown j} + \mathrm{nst}^\mu_{i\frown j}}} \prod_{i\frown j \in \nu - \beta} \frac{-1}{q^{\mathrm{nst}^\nu_{i\frown j}}}.$$

Therefore

$$\begin{aligned} \sum_{\nu \supseteq \beta \supseteq \lambda \cup \mu} \frac{(-1)^{|\nu-\beta|}}{q^{\mathrm{nst}^\lambda_{\beta-\lambda\cup\mu} + \mathrm{nst}^\mu_{\beta-\lambda\cup\mu} + \mathrm{nst}^\nu_{\nu-\beta}}} &= \sum_{\nu \supseteq \beta \supseteq \lambda \cup \mu} \prod_{i\frown j \in \beta - \lambda\cup\mu} \frac{1}{q^{\mathrm{nst}^\lambda_{i\frown j} + \mathrm{nst}^\mu_{i\frown j}}} \prod_{i\frown j \in \nu - \beta} \frac{-1}{q^{\mathrm{nst}^\nu_{i\frown j}}} \\ &= \prod_{i\frown j \in \nu - \lambda\cup\mu} \left(\frac{1}{q^{\mathrm{nst}^\lambda_{i\frown j} + \mathrm{nst}^\mu_{i\frown j}}} - \frac{1}{q^{\mathrm{nst}^\nu_{i\frown j}}}\right) \end{aligned}$$

which gives the lemma. $\square$

Theorem 4.2. *Let $\lambda, \mu \in S_\phi(I)$ for some finite set I and total order $\phi \in L[I]$. Then*

$$P^q_{(\phi,\lambda)} \odot P^q_{(\phi,\mu)} = \begin{cases} \displaystyle\sum_{\nu \supseteq \mu \cup \lambda} \left[\left(\frac{1}{q^{\mathrm{nst}^\lambda_{\mu-\lambda} + \mathrm{nst}^\mu_{\lambda-\mu}}} \right) \prod_{i \frown j \in \nu - \lambda \cup \mu} \left(\frac{1}{q^{\mathrm{nst}^\lambda_{i\frown j} + \mathrm{nst}^\mu_{i\frown j}}} - \frac{1}{q^{\mathrm{nst}^\nu_{i\frown j}}} \right) \right] P^q_{(\phi,\nu)} & \text{if } \lambda \cup \mu \in S_\phi(I) \\ 0 & \text{otherwise} \end{cases}$$

Proof:

$$\begin{aligned} P^q_{(\phi,\lambda)} \odot P^q_{(\phi,\mu)} &= \left(\sum_{\alpha \supseteq \mu} \frac{1}{q^{\mathrm{nst}^\lambda_{\alpha-\lambda}}} \kappa_{(\phi,\alpha)} \right) \odot \left(\sum_{\beta \supseteq \mu} \frac{1}{q^{\mathrm{nst}^\mu_{\beta-\mu}}} \kappa_{(\phi,\beta)} \right) \\ &= \sum_{\substack{\alpha \supseteq \lambda \\ \beta \supseteq \mu}} \frac{1}{q^{\mathrm{nst}^\lambda_{\alpha-\lambda} + \mathrm{nst}^\mu_{\beta-\mu}}} \kappa_{(\phi,\alpha)} \odot \kappa_{(\phi,\beta)}. \end{aligned}$$

Now, by Proposition 4.2, $\kappa_\alpha \odot \kappa_\beta$ is the zero function unless $\alpha = \beta$, in which case it is κ_α. This means all terms in the sum are zero except when $\alpha = \beta \supseteq \mu \cup \nu$, which in particular tells us that if $\lambda \cup \mu \notin S_\phi(I)$, the sum is zero. If we now assume that $\lambda \cup \mu$ is a set partition, that is $\lambda \cup \mu \in S_\phi(I)$, then

$$\begin{aligned} P^q_{(\phi,\lambda)} \odot P^q_{(\phi,\mu)} &= \sum_{\substack{\alpha \supseteq \lambda \\ \alpha \supseteq \mu}} \frac{1}{q^{\mathrm{nst}^\lambda_{\alpha-\lambda} + \mathrm{nst}^\mu_{\alpha-\mu}}} \kappa_{(\phi,\alpha)} \\ &= \sum_{\alpha \supseteq \lambda \cup \mu} \frac{1}{q^{\mathrm{nst}^\lambda_{\alpha-\lambda} + \mathrm{nst}^\mu_{\alpha-\mu}}} \kappa_{(\phi,\alpha)}. \end{aligned}$$

By Lemma 4.1,

$$\begin{aligned} P^q_{(\phi,\lambda)} \odot P^q_{(\phi,\mu)} &= \sum_{\alpha \supseteq \lambda \cup \mu} \frac{1}{q^{\mathrm{nst}^\lambda_{\alpha-\lambda} + \mathrm{nst}^\mu_{\alpha-\mu}}} \sum_{\nu \supseteq \alpha} \frac{(-1)^{|\nu-\alpha|}}{q^{\mathrm{nst}^\nu_{\nu-\alpha}}} P^q_{(\phi,\nu)} \\ &= \sum_{\nu \supseteq \lambda \cup \mu} \left(\sum_{\nu \supseteq \alpha \supseteq \lambda \cup \mu} \frac{(-1)^{|\nu-\alpha|}}{q^{\mathrm{nst}^\lambda_{\alpha-\lambda} + \mathrm{nst}^\mu_{\alpha-\mu} + \mathrm{nst}^\alpha_{\nu-\alpha}}} \right) P^q_{(\phi,\nu)}. \end{aligned}$$

Which, using Lemma 4.3 gives

$$P^q_{(\phi,\lambda)} \odot P^q_{(\phi,\mu)} = \sum_{\nu \supseteq \lambda \cup \mu} \left(\frac{1}{q^{\mathrm{nst}^\lambda_{\mu-\lambda} + \mathrm{nst}^\mu_{\lambda-\mu}}} \right) \prod_{i \frown j \in \nu - \lambda \cup \mu} \left(\frac{1}{q^{\mathrm{nst}^\lambda_{i\frown j} + \mathrm{nst}^\mu_{i\frown j}}} - \frac{1}{q^{\mathrm{nst}^\nu_{i\frown j}}} \right) P^q_{(\phi,\nu)}. \quad \square$$

We now give some simple corollaries of Theorem 4.5. Because the coefficients in the decomposition of $P^q_{(\phi,\lambda)} \odot P^q_{(\phi,\mu)}$ are so lengthy it is convenient to use the notation

$$P^q_{(\phi,\lambda)} \odot P^q_{(\phi,\mu)} = \sum_{\nu \in S_\phi(I)} b^\nu_{\lambda\mu} P^q_{(\phi,\nu)} \tag{30}$$

Corollary 4.5. *An alternative expression for $b^{\nu}_{\lambda\mu}$ is given by*

$$b^{\nu}_{\lambda\mu} = \begin{cases} \left(\frac{1}{q^{\mathrm{nst}^{\lambda}_{\mu-\lambda}+\mathrm{nst}^{\mu}_{\lambda-\mu}+\mathrm{nst}^{\lambda\cup\mu}_{\nu-\lambda\cup\mu}}}\right) \prod_{i\frown j\in\nu-\lambda\cup\mu} \left(\frac{1}{q^{\mathrm{nst}^{\lambda\cap\mu}_{i\frown j}}} - \frac{1}{q^{\mathrm{nst}^{\nu-\lambda\cup\mu}_{i\frown j}}}\right) & \text{if } \lambda\cup\mu \in S_{\phi}(I), \nu \supseteq \lambda\cup\mu \\ 0 & \text{otherwise .} \end{cases}$$

Proof:

$$\begin{aligned} &\prod_{i\frown j\in\nu-\lambda\cup\mu} \left(\frac{1}{q^{\mathrm{nst}^{\lambda}_{i\frown j}+\mathrm{nst}^{\mu}_{i\frown j}}} - \frac{1}{q^{\mathrm{nst}^{\nu}_{i\frown j}}}\right) \\ &\quad= \prod_{i\frown j\in\nu-\lambda\cup\mu} \left(\frac{1}{q^{\mathrm{nst}^{\lambda\cup\mu}_{i\frown j}}}\right)\left(\frac{1}{q^{\mathrm{nst}^{\lambda}_{i\frown j}+\mathrm{nst}^{\mu}_{i\frown j}-\mathrm{nst}^{\lambda\cup\mu}_{i\frown j}}} - \frac{1}{q^{\mathrm{nst}^{\nu}_{i\frown j}-\mathrm{nst}^{\lambda\cup\mu}_{i\frown j}}}\right) \\ &\quad= \left(\frac{1}{q^{\mathrm{nst}^{\lambda\cup\mu}_{\nu-\lambda\cup\mu}}}\right) \prod_{i\frown j\in\nu-\lambda\cup\mu} \left(\frac{1}{q^{\mathrm{nst}^{\lambda\cap\mu}_{i\frown j}}} - \frac{1}{q^{\mathrm{nst}^{\nu-\lambda\cup\mu}_{i\frown j}}}\right) \end{aligned}$$

since

$$\mathrm{nst}^{\lambda}_{i\frown j} + \mathrm{nst}^{\mu}_{i\frown j} - \mathrm{nst}^{\lambda\cup\mu}_{i\frown j} = \mathrm{nst}^{\lambda}_{i\frown j} - \mathrm{nst}^{\lambda-\mu}_{i\frown j} = \mathrm{nst}^{\lambda\cap\mu}_{i\frown j}\,. \ \square$$

The advantage of this expression is that it allows one to more readily determine when a coefficient $b^{\nu}_{\lambda\mu}$ is zero.

Corollary 4.6. *Let $\mu,\nu,\beta \in S_{\phi}(I)$ such that $\beta \neq \mu\cup\nu$, and there exists an arc $i\frown j \in \beta - \mu\cup\nu$ such that $\mathrm{nst}^{\mu\cap\nu}_{i\frown j} = 0$. Then $b^{\beta}_{\mu\nu} = 0$.*

Proof:
From Corollary 4.5 we need to show only that the existence of the arc $i\frown j$ implies that there is a arc $k\frown l$ (possibly the same as $i\frown l$) for which $\mathrm{nst}^{\mu\cap\nu}_{i\frown j} = \mathrm{nst}^{\beta-\mu\cup\nu}_{k\frown l} = 0$. Choose an arc $k\frown l \in \{r\frown s \in \beta-\mu\cup\nu |\, \mathrm{nst}^{r\frown s}_{i\frown j} = 1\}\cup\{i\frown j\}$ that is maximal with respect to nestings. Then $\mathrm{nst}^{\mu\cap\nu}_{k\frown l} = 0$ since $i\frown j$ is nested in $k\frown l$. Since $k\frown l$ is maximal, $\mathrm{nst}^{\beta-\mu\cup\nu}_{k\frown l} = 0$. $\square$

4.4 Antipode

A useful formula for computations of the antipode on a basis is Takeuchi's formula for the antipode on a connected Hopf monoid (see [1])

$$S(\psi) = \sum_{(J_1|J_2|\ldots|J_\ell)} (-1)^{\ell} m(\Delta_J(\psi)) \tag{31}$$

where the sum is over all set partitions $(J_1|J_2|\ldots|J_\ell)$ (ℓ can vary) of a finite set K where each J_i is nonempty. The map m is given by the ℓ-fold composition of $\mu_{I,J}$ maps which inflates $\Delta_J(\psi)$ to an element of $\mathbf{scf}(U)[K]$.

The antipode on the P bases are related to a notion of atomic for set partitions. Essentially an atomic set partition cannot be formed by concatenating set partitions.

Definition 4.3. *A set partition $\lambda \in S_\phi(K)$ is* atomic *if there is no set composition $A \sqcup B = K$ such that*

$$\phi = \phi|_A \phi|_B \qquad \text{and} \qquad \lambda = \lambda_A \cup \lambda_B.$$

In addition there is a notion of how atomic a set partition in an order τ is with respect to another order ϕ. If $\phi, \tau \in L[K]$, then we can factor τ into subposets by considering maximal rising subsequences of τ with respect to ϕ. This factorization is found by starting with the minimal element of τ then proceeding up in the linear order of τ until one hits a descent with respect to ϕ. A new sequence starts at this descent. For example if $\tau = 6 < 1 < 2 < 9 < 3 < 5 < 8 < 4 < 7$ and $\phi = 1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9$ then we get the factorization

$$(6|129|358|47).$$

We will call a set partition (τ, λ) ϕ-atomic if after factoring τ with respect to ϕ, the restriction to each subsequence is an atomic set partition and no arcs are lost during this restriction, i.e. there are no arcs passing between different subsequences.

Definition 4.4. *For $\phi, \tau \in L[K]$, a set partition $\lambda \in S_\tau(K)$ is* ϕ-atomic *if the set composition $(J_1, J_2, \ldots, J_\ell)$ of K coming from the factorization $\tau = \tau|_{J_1} \cdots \tau|_{J_\ell}$ into maximal rising subsequences with respect to ϕ satisfies*

(a) $\lambda = \lambda|_{J_1} \cup \lambda|_{J_2} \cup \cdots \cup \lambda|_{J_\ell}$,

(b) $\lambda|_{J_k}$ *is atomic for all* $1 \leq k \leq \ell$.

Using Takeuchi's formula, we now obtain a formula for the antipode. The proof given below is also included in [3].

Theorem 4.3. *For $\phi \in L[K]$, and $\lambda \in S_\phi(K)$,*

$$S(P^q_{(\phi,\lambda)}) = \sum_{\substack{\tau \in L[K], \mu \in S_\tau(K) \\ \mu \supseteq \lambda, \mu \ \phi\text{-atomic}}} (-1)^l \prod_{i \backsim j \in \mu - \lambda} \left(\frac{1}{q^{\mathrm{nst}^{(\phi,\lambda)}_{i \backsim j}}} - \frac{1}{q^{\mathrm{nst}^{(\phi,\mu)}_{i \backsim j}}} \right) P^q_{(\tau,\mu)},$$

where l is the number of subsequences in the factorization of τ into maximal rising subsequences with respect to ϕ.

Proof:

By (31) and Theorem 4.1,

$$\begin{aligned}
S(P^q_{(\phi,\lambda)}) &= \sum_{J=(J_1,J_2,\ldots,J_\ell)} (-1)^\ell m\Bigg(\sum_{\substack{\mu\supseteq\lambda \\ \mu=\mu|_{J_1}\cup\mu|_{J_2}\cup\cdots\cup\mu|_{J_\ell}}} a^\lambda_{J,\mu} P^q_{(\phi|_{J_1},\mu_{J_1})} \otimes \cdots \otimes P^q_{(\phi|_{J_\ell},\mu_{J_\ell})} \Bigg) \\
&= \sum_{\substack{J=(J_1,J_2,\ldots,J_\ell) \\ \mu=\mu|_{J_1}\cup\mu|_{J_2}\cup\cdots\cup\mu|_{J_\ell}\supseteq\lambda}} (-1)^\ell a^\lambda_{J,\mu} P^q_{(\phi|_{J_1}\phi|_{J_2}\cdots\phi|_{J_\ell},\mu)} \\
&= \sum_{\substack{\tau\in L[K],\lambda\in S_\tau(K) \\ \mu\in S_\tau(K),\mu\supseteq\lambda}} \Bigg(\sum_{\substack{J=(J_1,J_2,\ldots,J_\ell) \\ \mu=\mu|_{J_1}\cup\mu|_{J_2}\cup\cdots\cup\mu|_{J_\ell} \\ \tau=\phi|_{J_1}\phi|_{J_2}\cdots\phi|_{J_\ell}}} (-1)^\ell a^\lambda_{J,\mu} \Bigg) P^q_{(\tau,\mu)}.
\end{aligned}$$

Note that $a^\lambda_{J,\mu} = a^\lambda_{J',\mu}$ if $\mu = \mu|_{J_1}\cup\mu|_{J_2}\cup\cdots\cup\mu|_{J_\ell} = \mu|_{J'_1}\cup\mu|_{J'_2}\cup\cdots\cup\mu|_{J'_{\ell'}}$ and $\phi|_{J_1}\phi|_{J_2}\cdots\phi|_{J_\ell} = \phi|_{J'_1}\phi|_{J'_2}\cdots\phi|_{J'_{\ell'}}$, so the coefficients $a^\lambda_{J,\mu}$ do not vary in the inner sum and

$$S(P^q_{(\phi,\lambda)}) = \sum_{\substack{\tau\in L[K],\lambda\in S_\tau(K) \\ \mu\in S_\tau(K),\mu\supseteq\lambda}} \Bigg(\sum_{\substack{J=(J_1,J_2,\ldots,J_\ell) \\ \mu=\mu|_{J_1}\cup\mu|_{J_2}\cup\cdots\cup\mu|_{J_\ell} \\ \tau=\phi|_{J_1}\phi|_{J_2}\cdots\phi|_{J_\ell}}} (-1)^\ell \Bigg) a^\lambda_{J,\mu} P^q_{(\tau,\mu)}.$$

The inner sum can be simplified by noting that it is summing over all factorizations of (μ,τ) into ϕ-decomposable or ϕ-atomic parts. To see this, fix a ϕ-compatible $\mu \in S_\tau(K)$. Note that τ has at least the following two factorizations (which could coincide).

- $\tau = \tau_1\tau_2\ldots\tau_l$ into maximal rising subsequences with respect to ϕ,
- $\tau = \tau'_1\tau'_2\ldots\tau'_L$ where L is maximal such that each τ'_j is a rising subsequence with respect to ϕ and $\mu = \mu|_{\tau'_1}\cup\cdots\cup\mu|_{\tau'_L}$.

If C is the set of positions where new factors start in the first factorization and F is the set of positions where the new factors start in the second, then every factorization of τ into rising sequences that respect the arcs of μ have positions P with $C\subseteq P\subseteq F$. Thus,

$$\begin{aligned}
\sum_{\substack{J=(J_1,J_2,\ldots,J_\ell) \\ \mu=\mu|_{J_1}\cup\mu|_{J_2}\cup\cdots\cup\mu|_{J_\ell} \\ \tau=\phi|_{J_1}\phi|_{J_2}\cdots\phi|_{J_\ell}}} (-1)^\ell &= \sum_{C\subseteq P\subseteq F} (-1)^{|P|} \\
&= (-1)^{|C|} \sum_{C\subseteq P\subseteq F} \mathrm{mb}(P,C).
\end{aligned}$$

where $\mathrm{mb}(P,C)$ is the Möbius function of the subsets of F ordered by inclusion. Thus,

$$\sum_{\substack{J=(J_1,J_2,\ldots,J_\ell) \\ \mu=\mu|_{J_1}\cup\mu|_{J_2}\cup\cdots\cup\mu|_{J_\ell} \\ \tau=\phi|_{J_1}\phi|_{J_2}\cdots\phi|_{J_\ell}}} (-1)^\ell = \begin{cases} (-1)^L & \text{if } l = L, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore only terms where $l = L$ survive. The condition that $l = L$ says that (μ, τ) has been factored as far as possible without breaking atomic parts, that is (τ, μ) is ϕ-atomic. Thus,

$$S(P^q_{(\phi,\lambda)}) = \sum_{\substack{\tau \in L[K], \lambda \in S_\tau(K) \\ \mu \in S_\tau(K), \mu \supseteq \lambda \\ \mu \ \phi\text{-atomic}}} (-1)^l a^\lambda_{J,\mu} P^q_{(\tau,\mu)},$$

as desired. $\square$

Corollary 4.7. *For $\phi \in L[K]$ for some finite set K and $\lambda \in S_\phi(K)$,*

$$S(P_{(\phi,\lambda)} = \sum_{\substack{\tau \in L[K], \lambda \in S_\tau(K) \\ (\tau,\lambda) \ \phi\text{-atomic}}} (-1)^l P_{(\tau,\lambda)}.$$

where l is the number of subsequences in the factorization of τ into maximal rising subsequences with respect to ϕ.

Proof:
This corollary is immediate upon setting $q = 1$. $\square$

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