# Wavepacket-Mean Flow Interactions in the Kadomtsev–Petviashvili Equation

by

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B.S., University of Colorado, 2024

A thesis submitted to the
Faculty of the Graduate School of the
University of Colorado in partial fulfillment
of the requirements for the degree of
Master of Science
Department of Applied Mathematics
2025

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Thesis directed by Prof. Mark Hoefer

The interaction between a two-dimensional, vanishing amplitude (linear) wavepacket and a nonlinear, one-dimensional expansion wave is analyzed. Utilizing wave modulation theory in which the wavepacket's oscillation period is much shorter than the total evolution time, approximate solutions are found to the governing partial differential equation, the Kadomtsev-Petviashvili (KP) equation, which models a variety of nonlinear wave phenomena. The conditions on the incident wavepacket's wave vector that lead to either trapping or transmission of the wavepacket by the background mean flow are analyzed, and the evolution of the wavepacket amplitude is also determined. These solutions of the two-dimensional KP-Whitham modulation system in the harmonic limit are compared to analogous solutions of the one-dimensional Korteweg-deVries-Whitham modulation system.

# Acknowledgements

I would first like to thank my advisor, Prof. Mark Hoefer. Your continued support and guidance over the past two years have been instrumental in my development as an applied mathematician. I could not have asked for a better mentor to introduce me to the world of research, and prepare me for any endeavors to come.

I would also like to thank the rest of my committee members, Prof. Ian Grooms and Prof. Stephen Becker, for their participation in my thesis.

I would like to thank my friends and colleagues in the Dispersive Hydrodynamics Laboratory. Our weekly discussions were always a source of encouragement and insight, and your feedback has been invaluable in this process.

Finally, I would like to thank my friends, my family, and Olivia for their never-ending belief and confidence in me. Without you, I would not have made it half as far as I did.

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# Chapter 1

# Introduction

The wave-mean flow interaction problem is a widely studied problem, with many interesting and fundamental applications and contexts. In this thesis, wave-mean flow interaction for the Kadomtsev-Petviashvili (KP) equation,

$$(u_t + uu_x + u_{xxx})_x + \sigma u_{yy} = 0, \tag{1.1}$$

is studied. This partial differential equation (PDE) governs small amplitude, weakly nonlinear, long (shallow) water waves with weak transverse variation [10]. When $\sigma = 1$, gravity is the dominating force and the equation is referred to as KPII. When $\sigma = -1$, surface tension dominates gravity, and the equation is referred to as KPI – this can describe capillary waves. The KP equation is a $(2+1)$-dimensional generalization of the $(1+1)$-dimensional Korteweg-deVries (KdV) equation [1],

$$u_t + uu_x + u_{xxx} = 0. \tag{1.2}$$

We seek special solutions to the KP-Whitham modulation equations in the harmonic limit, to analyze linear wavepacket-mean flow interactions. The KP-Whitham modulation equations were derived in [2], which describe the slow evolution of parameters like amplitude, wavenumber, and frequency of nonlinear periodic waves. In the harmonic limit, the amplitude is vanishing – what remains is a decoupled evolution equation for the mean flow, and equations for the slowly varying wave vector. [4]. The solutions to these are approximate solutions of the original (1.1). Some of the seminal works in developing the modulation theory for the interaction of slowly varying (in space and/or time) mean flows and waves are seen in [16, 5]. Here, we study the rarefaction wave (RW) solution to the Riemann problem of (1.1) with step initial condition [2]

$$u(x,y,0) = \begin{cases} u_- & x<0 \\ u_+ & x>0 \end{cases}, \quad y \in \mathbb{R} \tag{1.3}$$

given by [7]

$$u(x,y,t) = \begin{cases} u_- & x<u_-t \\ \frac{x}{t} & u_-t<x<u_+t \\ u_+ & x>u_+t \end{cases}, \tag{1.4}$$

which represents the mean flow in the interaction problem. Note that the partial derivatives $(\partial_x u, \partial_t u)$ of (1.4) are not continuous, and thus it is considered an approximate solution to (1.1). When solving the modulation equations, we will construct piecewise solutions with the same boundaries as (1.4), and we require continuity at the solution edges, though not continuity of partial derivatives at the edges. A picture of the initial setup of the problem, pulled from a numerical simulation, can be seen in figure 1. Note that the figure shows the solution only along the $x$-axis, but the wavepacket is two-dimensional in space.

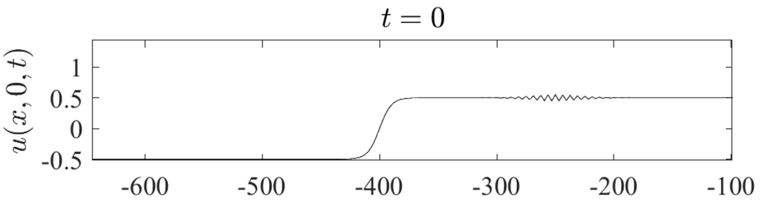


Figure 1.1: Setup of the wavepacket-mean flow interaction problem along the $x$-axis at time $t = 0$.

The wave-mean flow interaction problem has been studied in depth in various contexts, particularly in the flow of geophysical fluids. In [8], statistical dynamical theories are extended to interactions between turbulence and waves, notably discussing the context of oceanic flows, Rossby waves, and topography. There are many applications of wave-mean flow interactions to furthering the understanding of atmospheric and oceanographic modeling [11]. These problems often involve nonuniform media and small amplitude eddies, though the interacting waves are often nonlinear and the mean flow largely different than the RW type discussed here. Internal gravity waves, shear flows, and again Rossby waves are featured interaction waves relevant to large scale modeling in midlatitudes [6, 15]. In this thesis, we use scale separation to solve a wave-mean flow interaction problem where both the waves and the mean flow satisfy (approximately) the same equation, (1.1).

The application of the KP equation is not confined to water waves. Specifically, KPI admits lump solutions that are relevant to Bose-Einsten Condensates (BEC), and can be derived through a multiple scales analysis from the Gross-Pitaevskii equation. It's been found that for certain atomic interactions, the weakly nonlinear, two-dimensional, lump-like excitations are controlled by the KPI equation [9].

In [7], the wave-mean flow problem was studied under the KdV equation (1.2), where two interaction scenarios were found to occur: transmission or trapping of the wavepacket. We define transmission as the event occurring if the wavepacket interacts with the RW and reaches the third solution region; either $u_-$ if the wavepacket approaches the RW from the right, or $u_+$ if it approaches from the left. In other words, the wavepacket must cross the edge of the RW that it did not initially interact with. An interaction occurs if the wavepacket crosses the respective edge of the RW and enters the second solution region, where the RW is described by $\frac{x}{t}$. We define trapping as the event occurring if the wavepacket remains in the second region indefinitely. Ejection of the wavepacket is not observed in KdV, which is defined as the event occurring if the wavepacket crosses the same edge of the RW wave twice; it never reaches the third region, yet doesn't remain in the second region indefinitely.

In [7], it was found that in KdV, transmission occurs if the initial wave number $k_+$ satisfies

$$k_+ > \sqrt{\frac{2(\overline{u}_+ - \overline{u}_-)}{3}} \tag{1.5}$$

when the wavepacket approaches the RW from the right, and the wavepacket is trapped otherwise.

Here, $\overline{u}$ represents a slowly-varying mean flow. This was found using the Riemann invariant $\overline{u} - \frac{3}{2}k^2$. The time of interaction in terms of the initial position of the wavepacket, $X_+$, and initial wave number, $k_+$, is given by

$$t_1 = \frac{X_+}{3k_+^2}. \tag{1.6}$$

The time of transmission in terms of the initial position, initial wave number, and the wave number of the wavepacket after transmission, $k_-$, is described by

$$t_2 = \frac{X_+}{3k_+k_-}. \tag{1.7}$$

The trajectory of the wavepacket is given by

$$X(t) = \begin{cases} v_g(k_+, \overline{u}_+)t + X_+, & 0 \le t \le t_1 \\ (\overline{u}_+ - \frac{3}{2}k_+^2)t + \frac{X_+ t_1}{2t}, & t_1 \le t \le t_2 \\ v_g(k_-, \overline{u}_-)t + 3k_-^2 t_2, & t_2 \le t \end{cases}, \tag{1.8}$$

where the wavepacket travels with the group velocity $v_g(k, \overline{u}) = \overline{u} - 3k^2$. The evolution of the wave number of the wavepacket during interaction is given by

$$k(X(t), t) = \frac{k_+ t_1}{t}. \tag{1.9}$$

The amplitude of a plane wave interacting with the RW is conserved in KdV – the relation $a_+ = a_-$ is satisfied, where $a_+$ and $a_-$ are the amplitudes of the wave before and after interaction, respectively.

In this thesis, we extend this problem to include two spatial dimensions. We find that the introduction of a nonzero $y$ spatial component, i.e. where the wave number of the $y$ component satisfies $l \neq 0$, leads to interaction results that differ from those found in KdV [7]. We find that trapping does not occur in any case. Transmission always occurs in KPII when the wavepacket approaches from the right, and no interaction occurs when approaching from the left. In KPI, when approaching from the right, the wavepacket can be either transmitted, ejected, or avoid interaction altogether depending on the initial value of $k_+$. When approaching from the left, the wavepacket will always transmit, if the requirements for interaction are satisfied. These interaction cases, along with the trapping and transmission cases in KdV, are summarized in table 4.1. We also do not find that the amplitude of a plane wave is conserved during an interaction, and the evolution equation for the amplitude is calculated.

The structure of the remainder of this thesis is organized as follows. In the rest of this chapter, a derivation of the KP equation is carried out using asymptotic expansions and the method of multiple scales [1]. In Chapter 2, we use similar techniques as well as averaged Lagrangian methods to reproduce the derivations of the KP-Whitham system in the harmonic limit and discuss their mathematical structure. The benefits and limitations of each approach are discussed, followed by the calculation of KP-Whitham Riemann invariants. Chapter 3 presents explicit solutions to the wavepacket-mean flow interaction problem using the method of characteristics [4, 12]. We discuss the resulting possibilities of transmission and ejection through an analysis of the characteristic curves. Conditions for these different interaction behaviors are summarized. Chapter 4 provides a discussion of the theoretical results, incorporating comparisons with numerical simulations and results obtained for the analogous KdV problem.

## 1.1 Derivation of KP

The KP equation was first derived in 1970 by Kadomtsev and Petviashvili [10]. We follow the derivation of KdV in [1], while allowing for weak transverse variation to incorporate a $y$ spatial component. We first assume that gravity is the dominating force over surface tension to derive KPII, then study the dimensional version of KP to see the effects of including surface tension. We assume that there is constant density throughout an inviscid fluid with irrotational flow. We let $z$ be the vertical axis with a flat, constant bottom boundary through which no fluid can flow at $z = -h$, a distance $h$ from the mean level $z = 0$. We let $\eta(x, y, t)$ represent the free surface of the waves around the mean level. A schematic pulled from [1] describing the setup can be seen in figure 1.1. With $\nu(x, y, t)$ the fluid velocity vector, we introduce a velocity potential, $\nu = \nabla\phi$, a convenience when assuming irrotational flow [3]. From these assumptions, we have the governing equations

$$\Delta\phi = 0, \quad -h < z < \eta \tag{1.10a}$$

$$\frac{\partial\phi}{\partial z} = 0, \quad z = -h \tag{1.10b}$$

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}||\nabla\phi||^2 + g\eta = 0, \quad z = \eta \tag{1.10c}$$

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} + \nu \cdot \nabla\eta, \quad z = \eta \tag{1.10d}$$

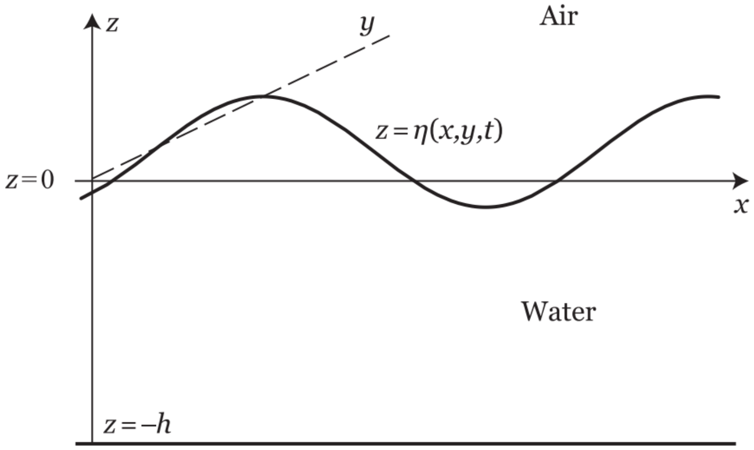


Figure 1.2: Setup of the water wave problem. $z = \eta$ is the free surface, there is a flat bottom at $z = -h$, and the undisturbed water level is at $z = 0$. Reprinted from *Nonlinear Dispersive Waves: Asymptotic Analysis and Solitons* (103), by M. J. Ablowitz, 2011, Cambridge University Press.

where $g$ is the constant of gravitational acceleration. Note that (1.10a) follows from the simplification of the Euler equations inside the fluid region from the assumptions above, (1.10b) follows directly from the assumption that the flat bottom is impenetrable, (1.10c) is Bernoulli's equation or the pressure equation, and (1.10d) is the kinematic condition, following from the assumption that a fluid parcel that starts out on the free surface will remain there. From these equations, the KP equation can be derived. We begin by following the definitions given in [1], non-dimensionalizing in a way that is convenient for shallow water wave problems. Working with non-dimensional variables allows for an easier characterization of the scale of terms. We let $\lambda_x$ and $\lambda_y$ be the characteristic wavelengths of the initial data in the respective $x$ and $y$ directions. Again, $h$ is the distance from the flat bottom to $z = 0$. Let $a$ be the typical or maximum amplitude of the initial data and $c_0 = \sqrt{gh}$ be the shallow water wave speed. We then have

$$x = \lambda_x x', \quad y = \lambda_y y', \quad z = hz' \quad t = \frac{\lambda_x}{c_0} t', \quad \eta = a\eta', \quad \phi = \frac{\lambda_x g a}{c_0}\phi', \tag{1.11}$$

where the primed variables are dimensionless. The respective derivatives are then

$$\partial_x = \frac{1}{\lambda_x}\partial_{x'}, \quad \partial_y = \frac{1}{\lambda_y}\partial_{y'}, \quad \partial_z = \frac{1}{h}\partial_{z'}, \quad \partial_t = \frac{c_0}{\lambda_x}\partial_{t'}. \tag{1.12}$$

We now plug (1.11) into (1.10). We simplify the resulting equations with appropriate scaling from shallow water theory. We seek a small depth relative to the characteristic wavelength $\lambda_x$. We define this ratio to be $\mu \equiv \frac{h}{\lambda_x} \ll 1$. This is the dispersion parameter. We also want small amplitude, or weak nonlinearity, so we define $\epsilon \equiv \frac{a}{h} \ll 1$. Finally, we define $\delta \equiv \frac{\lambda_x}{\lambda_y}$, which measures the size of the transverse variations. As we consider weak transverse variations, we have $\delta \ll 1$. Note that for ease in the subsequent steps, we immediately drop the prime notation. The equations (1.10) in dimensionless form become

$$\phi_{zz} + \mu^2\phi_{xx} + \mu^2\delta^2\phi_{yy} = 0, \quad -1 < z < \epsilon\eta \tag{1.13a}$$

$$\phi_z = 0, \quad z = -1 \tag{1.13b}$$

$$\phi_t + \frac{\epsilon}{2}\left(\phi_x^2 + \delta^2\phi_y^2 + \frac{1}{\mu^2}\phi_z^2\right) + \eta = 0, \quad z = \epsilon\eta \tag{1.13c}$$

$$\mu^2\left(\eta_t + \epsilon\left(\phi_x\eta_x + \delta^2\phi_y\eta_y\right)\right) = \phi_z, \quad z = \epsilon\eta. \tag{1.13d}$$

In the derivation of KdV, one has the maximal balance $\mu^2 = \epsilon$, and neglects any terms featuring $y$ derivatives. For KP, our maximal balance of weak nonlinearity, dispersion, and transverse variation is $\mu^2 = \delta^2 = \epsilon$. We now make an asymptotic expansion in $\phi$,

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots, \quad \epsilon \to 0,$$

and plug this into (1.13a) to get

$$\phi_{0zz} + \epsilon\left(\phi_{0xx} + \phi_{01zz}\right) + \epsilon^2\left(\phi_{1xx} + \phi_{2zz} + \phi_{0yy}\right) + \cdots = 0.$$

At leading order, we have $\phi_{0zz} = 0$. Integrating once yields $\phi_{0z} = B(x, y, t)$, but from (1.13b), we know that $\phi_{0z} = 0$ at $z = -1$, which requires $B = 0$. Integrating again yields the leading order solution of $\phi_0 = A(x, y, t)$. At $O(\epsilon)$, we have $\phi_{1zz} = -\phi_{0xx} \implies \phi_{1zz} = -A_{xx}$. Integrating twice, using (1.13b), and absorbing homogeneous solutions into the leading order term yields $\phi_1 = -\frac{1}{2}A_{xx}\left(z+1\right)^2$. Finally, at $O\left(\epsilon^2\right)$ we have

$$\phi_{1xx} + \phi_{2zz} + \phi_{0yy} = 0 \implies \phi_{2zz} = \frac{1}{2}A_{xxxx}(z+1)^2 - A_{yy}.$$

Following the same procedure as before, we obtain a solution of $\phi_2 = \frac{-1}{2}A_{yy}(z+1)^2 + A_{xxxx}\frac{(z+1)^4}{4!}$. All together, we have a solution for $\phi$ with the expansion

$$\phi = A - \frac{\epsilon}{2}A_{xx}\left(z+1\right)^2 - \frac{\epsilon^2}{2}A_{yy}(z+1)^2 + \frac{\epsilon^2}{4!}A_{xxxx}(z+1)^4 + \cdots, \tag{1.14}$$

which we can use in (1.13c) (noting that the equation is valid along $z = \epsilon\eta$) to get

$$\begin{aligned} A_t - \frac{\epsilon}{2}A_{xxt}(1+\epsilon\eta)^2 - \frac{\epsilon^2}{2}A_{yyt}(1+\epsilon\eta)^2 + \frac{\epsilon^2}{4!}A_{xxxxt}(1+\epsilon\eta)^4 + \cdots \\ +\frac{\epsilon}{2}\left(A_x - \frac{\epsilon}{2}A_{xxx}(1+\epsilon\eta)^2 - \frac{\epsilon^2}{2}A_{yyx}(1+\epsilon\eta)^2 + \frac{\epsilon^2}{4!}A_{xxxxx}(1+\epsilon\eta)^4 + \cdots\right)^2 \\ +\frac{\epsilon^2}{2}\left(A_y - \frac{\epsilon}{2}A_{xxy}(1+\epsilon\eta)^2 - \frac{\epsilon^2}{2}A_{yyy}(1+\epsilon\eta)^2 + \frac{\epsilon^2}{4!}A_{xxxxy}(1+\epsilon\eta)^4 + \cdots\right)^2 \\ +\frac{1}{2}\left(-\epsilon A_{xx}(1+\epsilon\eta) - \epsilon^2 A_{yy}(1+\epsilon\eta) + \frac{\epsilon^2}{6}A_{xxxx}(1+\epsilon\eta)^3 + \cdots\right)^2 + \eta = 0. \end{aligned}$$

We can solve this explicitly for $\eta$. Retaining only terms up to order $\epsilon$, we have

$$\eta = -A_t + \frac{\epsilon}{2}\left(A_{xxt} - A_x^2\right) + O(\epsilon^2). \tag{1.15}$$

We now plug (1.14) into the last equation, (1.13d)

$$\begin{aligned}\epsilon\eta_t + \epsilon^2[\eta_x &\left(A_x - \frac{\epsilon}{2}A_{xxx}(1+\epsilon\eta)^2 - \frac{\epsilon^2}{2}A_{yyx}(1+\epsilon\eta)^2 + \frac{\epsilon^2}{4!}(1+\epsilon\eta)^4 + \cdots\right)\\ +\epsilon\eta_y &\left(A_y - \frac{\epsilon}{2}A_{xxy}(1+\epsilon\eta)^2 - \frac{\epsilon^2}{2}A_{yyy}(1+\epsilon\eta)^2 + \frac{\epsilon^2}{2}A_{xxxxy}(1+\epsilon\eta)^4 + \cdots\right)]\\ &= -\epsilon^2 A_{xx}\eta - \epsilon A_{xx} - \epsilon^3 A_{yy}\eta - \epsilon^2 A_{yy} + \frac{\epsilon^2}{3!}A_{xxxx}(1+\epsilon\eta)^3 + \cdots,\end{aligned}$$

which, when retaining only terms up to order $O\left(\epsilon^2\right)$, reduces to

$$\epsilon\eta_t + \epsilon^2\eta_x A_x = -\epsilon A_{xx} - \epsilon^2 A_{xx}\eta - \epsilon^2 A_{yy} + \frac{\epsilon^2}{6}A_{xxxx}\eta. \tag{1.16}$$

To decouple equations (1.15) and (1.16), we can substitute (1.15) and its appropriate derivatives into (1.16) to get an equation solely in terms of $A(x, y, t)$. Doing so, dividing by $\epsilon$, and retaining only terms of the two lowest orders gives

$$-A_{tt} + A_{xx} = \epsilon\left(-\frac{A_{xxtt}}{2} + 2A_x A_{xt} + A_{xx}A_t - A_{yy} + \frac{A_{xxxx}}{6}\right).$$

To simplify, we make the same note as in [1]:

$$-A_{tt} = -A_{xx} + O(\epsilon) \implies -\epsilon\frac{A_{xxtt}}{2} = -\epsilon\frac{A_{xxxx}}{2} + O\left(\epsilon^2\right),$$

where we differentiate twice with respect to $x$ and multiply by a factor of $\epsilon$ so that our equation becomes

$$A_{tt} - A_{xx} = \epsilon\left(\frac{A_{xxxx}}{3} - 2A_x A_{xt} + A_{xx}A_t + A_{yy}\right). \tag{1.17}$$

As with equations (1.13), this is a PDE not easily solved exactly. Again, we make an asymptotic expansion in $A$ of the form

$$A = A_0 + \epsilon A_1 + \cdots. \tag{1.18}$$

And substitute it into (1.17) to get the leading-order equation

$$A_{0tt} - A_{0xx} = 0.$$

This is the wave equation in one spatial and one temporal dimension with a known solution given by d'Alembert's formula [1]. In this case, we select a rightward-moving wave to get $A_0(x, y, t) = F(x-t, y)$. Now, following the procedure in [1], we include multiple scales in anticipation of secular terms at the next order. We define the new variables

$$\xi = x - t, \quad T = \epsilon t, \tag{1.19}$$

with partial derivatives transforming as

$$\partial_x = \partial_\xi, \quad \partial_t = -\partial_\xi + \epsilon \partial_T. \tag{1.20}$$

Note that we do not rescale or transform $y$. We only consider weak transverse variations, the scale of which have already been defined through the balance $\delta^2 = \epsilon$. Our solution for $A_0$ features evolution in the $x$ direction only, hence the transformation of $x$ to a moving reference frame through $\xi$. We substitute these into the original equation and retain only terms of order $\epsilon$. Note that we know that the right-hand side will contain no $A_1$ components. Our equation becomes

$$\left[(-\partial_\xi + \epsilon\partial_T)^2 - \partial_\xi^2\right] A = \epsilon \left( \frac{1}{3}\partial_\xi^4 A - 2\partial_\xi A \partial_\xi \left(-\partial_\xi + \epsilon\partial_T\right) A - \partial_\xi^2 A \left(-\partial_\xi + \epsilon\partial_T\right) A + \partial_y^2 A \right). \tag{1.21}$$

We plug in $A = F(\xi, y) + \epsilon A_1 + \cdots$ and note that the leading order equation is trivially solved. At order $\epsilon$, we have

$$2F_{\xi T} + \frac{F_{\xi\xi\xi\xi}}{3} + 3F_\xi F_{\xi\xi} + F_{yy} = 0. \tag{1.22}$$

We can make the substitution $U = F_\xi$ and note that $\partial_\xi^{-1} U = F$, where $\partial_\xi^{-1} = \int_{-\infty}^{\xi} \, d\xi$, to get the equation

$$\left(2U_T + \frac{1}{3}U_{\xi\xi\xi} + 3UU_\xi\right)_\xi + U_{yy} = 0, \tag{1.23}$$

the KP equation with $\sigma = 1$, or KPII. We can implement a further rescaling of variables to end with the familiar (and the form referenced henceforth) version of KPII,

$$\left(u_t + u_{xxx} + uu_x\right)_x + u_{yy} = 0, \tag{1.24}$$

where we have kept the original denotation $x, y, t$, though these are non-dimensional and rescaled versions of the original variables.

We recall that we originally chose to consider the case where gravity is the dominating force over surface tension, a choice that affects (1.10c). If we include the surface tension terms that have effects to order $\epsilon$, this becomes

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}||\nabla\phi||^2 + g\eta - \frac{T}{\rho}\left(\eta_{xx} + \eta_{yy}\right) = 0, \quad z = \eta, \tag{1.25}$$

where $T$ is the surface tension coefficient and $\rho$ the fluid density. Before citing the end result, we first re-dimensionalize (1.23). We reinstate the primed notation for the dimensionless variables. Note that

$$U_{t'} = -U_\xi + \epsilon U_T \quad \text{and} \quad U_{x'} = U_\xi \implies \partial_T = \frac{(\partial_{t'} + \partial_{x'})}{\epsilon}.$$

This gives

$$\frac{2}{\epsilon}U_{t'x'} + \frac{2}{\epsilon}U_{x'x'} + \frac{1}{3}U_{x'x'x'x'} + 3UU_{x'} + 3U_{x'}U_{x'} + U_{y'y'} = 0.$$

Substituting the original definitions of the prime variables and their derivatives, (1.11, 1.12), as well as $\epsilon = \frac{a}{h} = \frac{\lambda_x^2}{\lambda_y^2} = \frac{h^2}{\lambda_x^2}$ and noting that $U \sim \eta'$, we have

$$\left(\frac{1}{c_0}\eta_t + \eta_x + \frac{h^2}{6}\eta_{xxx} + \frac{3}{2h}\eta\eta_x\right)_x + \frac{1}{2}\eta_{yy} = 0. \tag{1.26}$$

From [1], we note that the change to this dimensional equation resulting from the inclusion of surface tension is equivalent to the KdV case: the coefficient for the $\eta_{xxx}$ term is altered. When using (1.10c), we can find this coefficient to be

$$\frac{h^2}{6}\left(1-\hat{T}\right), \quad \text{with} \quad \hat{T}=\frac{T}{\rho g h^2}. \tag{1.27}$$

The case where gravity is the dominating force over surface tension occurs when $\hat{T} < 1/3$, resulting in KPII, and when surface tension dominates over gravity we have $\hat{T} > 1/3$, resulting in the KPI equation. In the rescaled, non-dimensional form we have 1.1, where $\sigma = 1$ corresponds to KPII, and $\sigma = -1$ corresponds to KPI.

# Chapter 2

## Mathematical Structure and Derivations

In this chapter, different derivation methods for the KP Whitham system are employed to obtain the modulation equations, conservation equations, and the mean flow equation for KP. The Riemann invariants are then calculated using the derived equations. Differences between the derivations are discussed, as well as the benefits of each respective method. Section 2.1 features the method of multiple scales [2], where a separation between slow and fast variables is used, and different equations are seen at different orders. In section 2.2, the Lagrangian approach is used [17]. The two methods are subsequently compared in section 2.3 before calculating the Riemann invariants for the KP-Whitham equations under the assumption of unidirectional modulations in 2.4.

### 2.1 Asymptotic Derivation of the harmonic KP-Whitham System

One method of deriving the Whitham system involves using the method of multiple scales [2]. We seek solutions of the form

$$u(x,y,t) = \overline{u}\left(X,Y,T\right) + \epsilon^2 \Psi\left(\theta, X, Y, T\right) + \epsilon^3 u_1\left(\theta, X, Y, T\right) + \cdots \tag{2.1}$$

where $0 < \epsilon \ll 1$ and the powers of $\epsilon$ result from maximal balancing. The variables $X, Y, T$, and $\theta$ can be defined as

$$X = \epsilon x, \quad Y = \epsilon y, \quad T = \epsilon t \tag{2.2}$$

$$\theta_x = k\left(X,Y,T\right), \quad \theta_y = l\left(X,Y,T\right), \quad \text{and} \quad \theta_t = -\omega\left(X,Y,T\right), \tag{2.3}$$

where $(k,l) = (\theta_x, \theta_y)$ is the wave vector and $\omega$ is the wave frequency. Via the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} + \frac{\partial X}{\partial x}\frac{\partial}{\partial X}, \quad \frac{\partial}{\partial y} = \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} + \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y}, \quad \text{and} \quad \frac{\partial}{\partial t} = \frac{\partial \theta}{\partial t}\frac{\partial}{\partial \theta} + \frac{\partial T}{\partial t}\frac{\partial}{\partial T},$$

so that the partial derivative operators with respect to the original $x, y, t$ variables can be expanded as

$$\partial_t \to -\omega\partial_\theta + \epsilon\partial_T, \quad \partial_x \to k\partial_\theta + \epsilon\partial_X, \quad \text{and} \quad \partial_y \to l\partial_\theta + \epsilon\partial_Y. \tag{2.4}$$

We can see that the variations in fast oscillations and the slow modulations are separated in this expansion. Hence, $X, Y,$ and $T$ are slowly varying, while $\theta$ is the fast variable. We require continuity of the partial derivatives of $\theta$, i.e.,

$$\left(\theta_x\right)_y = \left(\theta_y\right)_x \implies k_y = l_x \implies k_Y = l_X, \tag{2.5}$$

since, for example,

$$k_y = l_x \implies l\partial_\theta k + \epsilon\partial_Y k = k\partial_\theta l + \epsilon\partial_X l \implies 0 + \epsilon k_Y = 0 + \epsilon l_X \implies k_Y = l_X.$$

From this, we also have

$$\left(\theta_x\right)_t = \left(\theta_t\right)_x \implies k_T + \omega_X = 0, \tag{2.6}$$

$$\left(\theta_y\right)_t = \left(\theta_t\right)_y \implies l_T + \omega_Y = 0, \tag{2.7}$$

the conservation of waves equations. The partial derivative operators can then be applied to (1.1):

$$\begin{aligned}\left(k\partial_\theta + \epsilon\partial_X\right)\left[\left(-\omega\partial_\theta + \epsilon\partial_T\right)\left(\overline{u} + \epsilon^2\Psi + \epsilon^3 u_1\right) + \left(\overline{u} + \epsilon^2\Psi + \epsilon^3 u_1\right)\left(k\partial_\theta + \epsilon\partial_X\right)\left(\overline{u} + \epsilon^2\Psi + \epsilon^3 u_1\right)\right.\\ \left. + \left(k\partial_\theta + \epsilon\partial_X\right)^3\left(\overline{u} + \epsilon^2\Psi + \epsilon^3 u_1\right)\right] + \sigma\left(l\partial_\theta + \epsilon\partial_Y\right)^2\left(\overline{u} + \epsilon^2\Psi + \epsilon^3 u_1\right) = 0.\end{aligned}$$

After distributing and simplifying, if we consider only the terms of $O(1)$, we are left trivially with $0 = 0$, as there are no terms without some degree of $\epsilon$. At order $O(\epsilon)$, we still have $0 = 0$. If we consider only the terms of $O(\epsilon^2)$, we have the equation

$$(\overline{u}_T + \overline{u}\overline{u}_X)_X + \sigma\overline{u}_{YY} - \left(k\omega - k^2\overline{u} - \sigma l^2\right)\Psi_{\theta\theta} + k^4\Psi_{\theta\theta\theta\theta} = 0.$$

We can then define $\Psi(\theta, X, Y, T) = a(X, Y, T)\, e^{i\theta}$, where $a$ is the amplitude function dependent only on the slow variables. Plugging this in gives

$$(\overline{u}_T + \overline{u}\overline{u}_X)_X + \sigma\overline{u}_{YY} + ae^{i\theta}\left(k\omega - k^2\overline{u} - \sigma l^2 + k^4\right) = 0.$$

By setting the terms proportional to $a$ equal to zero, we obtain the linear dispersion relation

$$\omega = \overline{u}k - k^3 + \sigma\frac{l^2}{k}, \tag{2.8}$$

and

$$\overline{u}_T + \overline{u}\overline{u}_X + \sigma\overline{v}_Y = 0. \tag{2.9}$$

This is the equation governing the slowly varying mean flow, $\overline{u}$. This is equivalent to the dispersionless-KP equation – note that there is no longer a third order derivative term. We note that the linear dispersion relation holds locally, even when $(k, l)$ evolve. Before proceeding to the higher order terms, we revisit the conservation of waves. Using the linear dispersion relation, (2.6) becomes

$$k_T + \left(\overline{u} - 3k^2 - \sigma\frac{l^2}{k^2}\right)k_X + 2\sigma\frac{l}{k}k_Y + \overline{u}_X k = 0, \tag{2.10}$$

which can be rewritten by letting $q = \frac{l}{k}$:

$$k_T + \left(\overline{u} - 3k^2 - \sigma q^2\right)k_X + 2\sigma q k_Y + \overline{u}_X k = 0.$$

Similarly, (2.7) becomes

$$l_T + \left(\overline{u} - 3k^2 - \sigma\frac{l^2}{k^2}\right) l_X + 2\sigma\frac{l}{k}l_Y + \overline{u}_Y k = 0. \tag{2.11}$$

Plugging in $q = \frac{l}{k} \implies l = kq$ and dividing by $k$ yields

$$q_T + \left(\overline{u} - 3k^2 - \sigma q^2\right) q_X + 2\sigma q q_Y + \overline{u}_Y - \overline{u}_X q = 0. \tag{2.12}$$

While equation (2.9) governs the slowly varying mean flow, equations (2.10) and (2.11) govern the dynamical modulations of the linear waves. Note that these equations are decoupled from 2.9, allowing one to first solve the mean flow equation, and then use such solutions in the linear wave equations. Now, if we consider the $O(\epsilon^3)$ terms and simplify, we have

$$\begin{aligned} k^4\left(u_{1\theta\theta} + u_{1\theta\theta\theta\theta}\right) = -ie^{i\theta}\big[&ka_T - (\omega a)_X + (k\overline{u}a)_X + k(\overline{u}a)_X - 3k^3 a_X \\ &-3k^2 k_X a - (k^3 a)_X + \sigma(la)_Y + \sigma l a_Y\big], \end{aligned}$$

which implies that the bracketed term on the R.H.S. must be equal to 0. This is a solvability condition, required to ensure that a valid solution to $u_1$ exists, and to prevent secular growth arising from the $e^{i\theta}$ term present in the lower order component $\Psi$. Using (2.8) and simplifying yields

$$a_T + a_X\left(\overline{u} - 3k^2 - \sigma q^2\right) + a_Y\left(2\sigma q\right) + a\left(\overline{u}_X - 3kk_X - \sigma q^2\frac{k_X}{k} - 2\sigma q q_X + \sigma q\frac{k_Y}{k} + \sigma q_Y\right) = 0, \tag{2.13}$$

the equation governing the amplitude of the linear waves, $a(X, Y, T)$.

Using a multiple-scales expansion, we have derived the KP-Whitham modulation system in the harmonic limit. The equation governing the slowly varying mean flow is given by (2.9). From the conservation of waves and the linear dispersion relation (2.8), we have the equations governing the dynamical modulations of the linear waves given by (2.10) and (2.12). In addition to these equations, which comprise the KP-Whitham modulation system in the harmonic limit, at a higher order we obtain the equation that provides the linear approximation to the amplitude, (2.13).

## 2.2 Averaged Lagrangian Derivation

We now seek to derive the system using the averaged Lagrangian method [17]. The variational principle,

$$\delta J = \delta \iiint_R L\left(\psi_t, \psi_x, \psi_y, \psi\right)\ dtdxdy = 0, \tag{2.14}$$

states that an integral over a finite space-time region R should be stationary to small changes of $\psi$. From this, we have the Euler-Lagrange equation

$$-\partial_t L_{\psi_t} - \partial_x L_{\psi_x} - \partial_y L_{\psi_y} + \partial_x^2 L_{\psi_{xx}} + \cdots = 0, \tag{2.15}$$

which must be satisfied for (2.14) to hold true. The KP equation does not have a Lagrangian as is, but introducing a potential term defined as $u = \psi_x$ gives the Lagrangian

$$L = -\frac{1}{2}\psi_t\psi_x - \frac{1}{6}\psi_x^3 + \frac{1}{2}\psi_{xx}^2 - \frac{1}{2}\sigma\psi_y^2. \tag{2.16}$$

Computing the partial derivatives and plugging them into (2.15) gives

$$\frac{1}{2}\psi_{xt} + \frac{1}{2}\psi_{tx} + \psi_x\psi_{xx} + \psi_{xxxx} + \sigma\psi_{yy} = 0.$$

Note that plugging in $u = \psi_x$ gives

$$u_t + uu_x + u_{xxx} + \sigma\partial_x^{-1}[u_{yy}] = 0,$$

and if we differentiate all terms with respect to $x$ we get

$$\left(u_t + uu_x + u_{xxx}\right)_x + \sigma u_{yy} = 0,$$

or if we let $u_y = v_x$ and note that $u_{yy} = v_{xy}$ and $\partial_x^{-1}[u_{yy}] = v_y$, we have

$$u_t + uu_x + u_{xxx} + \sigma v_y = 0,$$

two forms of the KP equation, as expected. We now make an ansatz for $\psi$ that is a periodic traveling wave: $\psi(x,y,t) = \eta(x,y,t) + \Psi(\theta(x,y,t))$. Here, $\eta$ is called the pseudo-phase, counterpart to the genuine phase $\theta$. The introduction of a pseudo-phase is necessary to incorporate variations in the mean flow that are not captured by $\theta$ alone. We require the period of $\Psi(\theta)$ to be $2\pi$, and the period average to be 0, i.e.,

$$\Psi(\theta + 2\pi) = \Psi(\theta), \qquad \frac{1}{2\pi}\int_0^{2\pi} \Psi(\theta)\, d\theta = \overline{\Psi(\theta)} = 0. \tag{2.17}$$

We further define the ansatz with $\eta = \frac{\tilde{S}(X,Y,T)}{\epsilon}$ and $\theta = \frac{S(X,Y,T)}{\epsilon}$, where $0 < \epsilon \ll 1$. Each variable can be defined through its partial derivatives.

**Phase**:

$$\begin{aligned} \theta_x &= S_X = k(X,Y,T), \quad X = \epsilon x \\ \theta_y &= S_Y = l(X,Y,T), \quad Y = \epsilon y \\ \theta_t &= S_T = -\omega(X,Y,T), \quad T = \epsilon t \end{aligned} \tag{2.18}$$

**Pseudo-Phase**:

$$\begin{aligned} \eta_x &= \tilde{S}_X = \beta(X,Y,T) \\ \eta_y &= \tilde{S}_Y = \zeta(X,Y,T) \\ \eta_t &= \tilde{S}_T = -\gamma(X,Y,T) \end{aligned} \tag{2.19}$$

Again, requiring the compatibility of the partial derivatives yields the following systems of equations (and conservation of waves).

**Phase**:

$$\begin{aligned} S_{XT} = S_{TX} &\iff k_T + \omega_X = 0 \\ S_{YT} = S_{TY} &\iff l_T + \omega_Y = 0 \\ S_{XY} = S_{YX} &\iff k_Y - l_X = 0 \end{aligned} \tag{2.20}$$

**Pseudo-Phase**:

$$\begin{aligned} \tilde{S}_{XT} = \tilde{S}_{TX} \quad &\Longleftrightarrow \quad \beta_T + \gamma_X = 0 \\ \tilde{S}_{YT} = \tilde{S}_{TY} \quad &\Longleftrightarrow \quad \zeta_T + \gamma_Y = 0 \\ \tilde{S}_{XY} = \tilde{S}_{YX} \quad &\Longleftrightarrow \quad \beta_Y - \zeta_X = 0 \end{aligned} \tag{2.21}$$

Since $u = \psi_x = \tilde{S}_X + S_X \Psi'(\theta) = \beta + k\Psi'(\theta)$, we have that

$$\overline{u} = \frac{1}{2\pi}\int_0^{2\pi} u(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \big(\beta + k\Psi'(\theta)\big)\, d\theta = \beta + \frac{1}{2\pi}\left( k\Psi(\theta)\Big|_{\theta=0}^{\theta=2\pi} \right) = \beta \tag{2.22}$$

by the periodicity of $\Psi(\theta)$. So, $\overline{u} = \beta$, the mean flow, thus highlighting the need for a pseudo-phase in order to include a slowly varying mean. We now consider the average variational principle [17]

$$\delta \iiint_R \mathcal{L}(-\theta_t, \theta_x, \theta_y, -\eta_t, \eta_x, \eta_y, a)\, dtdxdy = 0, \tag{2.23}$$

where $a = a(X, Y, T)$ is an amplitude parameter and $\mathcal{L} = \frac{1}{2\pi}\int_0^{2\pi} L\, d\theta$. For the integral to be stationary with respect to variations in the amplitude, phase, and pseudo-phase, we require, respectively,

$$\delta a : \quad \mathcal{L}_a = 0, \tag{2.24}$$

$$\delta\theta : \quad \partial_T \mathcal{L}_{-\theta_t} + \partial_X \mathcal{L}_{\theta_x} + \partial_Y \mathcal{L}_{\theta_y} = 0 \implies \partial_T \mathcal{L}_\omega - \partial_X \mathcal{L}_k - \partial_Y \mathcal{L}_l = 0, \tag{2.25}$$

and

$$\delta\eta : \quad \partial_T \mathcal{L}_{-\eta_t} + \partial_X \mathcal{L}_{\eta_x} + \partial_Y \mathcal{L}_{\eta_y} = 0 \implies \partial_T \mathcal{L}_\gamma - \partial_X \mathcal{L}_\beta - \partial_Y \mathcal{L}_\zeta = 0. \tag{2.26}$$

These, along with the six conservation of waves equations (2.20, 2.21), make up the Whitham modulation equations for $\omega$, $k$, $l$, $\gamma$, $\beta$, $\zeta$, and $a$. We now consider an appropriate $\psi$ for small amplitude, linear waves. We let $\psi = \eta + \frac{a}{k}\sin(\theta)$, so that $u = \psi_x = \eta_x + \theta_x \frac{a}{k}\cos(\theta) = \beta + a\cos(\theta)$ and $\psi_{xx} = -ka\sin(\theta)$. Note that no partial derivatives of $\omega$, $k$, $l$, $\gamma$, $\beta$, $\zeta$, or $a$ appear in $\mathcal{L}$ as these are higher order terms. When considering the average of each term, we are interested in the long-term behavior, not the periodic, short-term variations. Calculating the partial derivatives of $\psi$, plugging these into (2.16), averaging over one period and using (2.17), and simplifying gives

$$\mathcal{L} = \frac{a^2}{4}\left(\frac{\omega}{k} - \beta + k^2 - \sigma\frac{l^2}{k^2}\right) + \left(\frac{1}{2}\gamma\beta - \frac{1}{6}\beta^3 - \frac{1}{2}\sigma\zeta^2\right). \tag{2.27}$$

We see that

$$\mathcal{L}_a = \frac{\omega a}{2k} - \frac{\beta a}{2} + \frac{k^2 a}{2} - \sigma\frac{l^2 a}{2k^2} = 0 \implies \omega = \beta k - k^3 + \sigma\frac{l^2}{k},$$

and thus from (2.24) we obtain the linear dispersion relation. Similarly, for (2.26), we compute the partial derivatives to obtain $\beta_T + 2\beta\beta_X + 2\sigma\zeta_Y - \gamma_X = 0$, which is simplified using the fact that $\gamma_X = -\beta_T$ and $\beta_Y = \zeta_X$ from (2.20) to get

$$\left(\beta_T + \beta\beta_X\right)_X + \sigma\beta_{YY} = 0, \tag{2.28}$$

which is the dispersionless-KP equation (2.9) if we set $\overline{u} = \beta$. This is equivalent to (2.9). For (2.25), we again compute the partial derivatives of $\mathcal{L}$ to obtain

$$\left(\frac{a^2}{k}\right)_T + \left(\left(\frac{\omega}{k^2} - 2k - 2\sigma\frac{l^2}{k^3}\right)a^2\right)_X + \left(2\sigma\frac{l}{k^2}a^2\right)_Y = 0. \tag{2.29}$$

We note that $\omega_k = \beta - 3k^2 - \sigma\frac{l^2}{k^3}$, and $\omega_l = 2\sigma\frac{l}{k}$, in addition to the fact that the expression within the second term can be simplified as

$$\frac{\omega}{k^2} - 2k - 2\sigma\frac{l^2}{k^3} = \frac{\beta}{k} - k + \sigma\frac{l^2}{k^3} - 2k - 2\sigma\frac{l^2}{k^3} = \frac{\beta}{k} - 3k - \sigma\frac{l^2}{k^3} = \frac{\omega_k}{k}.$$

Then, (2.25) can be written as

$$\left(\frac{a^2}{k}\right)_T + \left(\omega_k\frac{a^2}{k}\right)_X + \left(\omega_l\frac{a^2}{k}\right)_Y = 0, \tag{2.30a}$$

or, equivalently, as

$$\left(\frac{a^2}{k}\right)_T + \nabla_{X,Y} \cdot \left(\left(\nabla_{k,l}\omega\right)\frac{a^2}{k}\right) = 0. \tag{2.30b}$$

This is the conservation of the wave action $\frac{a^2}{k}$, a quantity not obvious from the multiple-scales derivation. Using this method, equations for wave momentum and energy are immediately available as well. If we actually compute the partial derivatives for (2.30a), we get

$$\partial_T \mathcal{L}_\omega = \frac{aa_T}{2k} - \frac{a^2 k_T}{4k^2}$$

$$\partial_X \mathcal{L}_k = \frac{aa_X}{2}\left(-\frac{\omega}{k^2} + 2k + 2\sigma\frac{l^2}{k^3}\right) + \frac{a^2}{4}\left(-\frac{\omega_X}{k^2} + \frac{2\omega k_X}{k^3} + 2k_X + 4\sigma\frac{ll_X}{k^3} - 6\sigma\frac{l^2 k_X}{k^4}\right)$$

$$\partial_Y \mathcal{L}_l = \sigma\left(-\frac{aa_Y l}{k^2} - \frac{a^2 l_Y}{2k^2} + \frac{a^2 l k_Y}{k^3}\right).$$

Plugging these in, dividing by $a$, and multiplying by $2k$ gives

$$\begin{aligned} a_T - \frac{ak_T}{2k} + a_X\left(\frac{\omega}{k} - 2k^2 - 2\sigma\frac{l^2}{k^2}\right) \quad &+ \quad a\left(\frac{\omega_X}{2k} - \frac{\omega k_X}{k^2} - kk_X - 2\sigma\frac{ll_X}{k^2} + 3\sigma\frac{l^2 k_X}{k^3}\right) \\ &+ \frac{2a_Y l}{k} + \frac{al_Y}{k} - \frac{alk_Y}{k^2} = 0. \end{aligned}$$

We note that $\omega = \beta k - k^3 + \sigma\frac{l^2}{k}$ and thus $\omega_X = \beta_X k + \beta k_X - 3k^2 k_X + 2\sigma\frac{ll_X}{k} - \sigma\frac{l^2 k_X}{k^2}$, and that $k_T = -\omega_X$ to simplify further

$$a_T + a_X\left(-3k^2 + \beta - \sigma\frac{l^2}{k^2}\right) + a_Y\left(2\sigma\frac{l}{k}\right) + a\left(-3kk_X + \beta_X - 2\sigma\frac{l}{k^2}k_Y + \sigma\frac{l^2}{k^3}k_X + \sigma\frac{l_Y}{k}\right) = 0,$$

and finally use $q = \frac{l}{k}$ and $\beta = \overline{u}$ to get

$$a_T + a_X\left(\overline{u} - 3k^2 - \sigma q^2\right) + a_Y\left(2\sigma q\right) + a\left(\overline{u}_X - 3kk_X - \sigma q^2\frac{k_X}{k} - 2\sigma qq_X + \sigma q\frac{k_Y}{k} + \sigma q_Y\right) = 0,$$

the amplitude equation, equivalent to (2.13).

We have now rederived the KP-Whitham system in the harmonic limit using the averaged Lagrangian. We obtain the equation governing the slowly varying mean flow, (2.28), from the variational equation for the pseudo-phase, (2.26). As before, the conservation of waves arises from both requiring the compatibility of the partial derivatives of the phase, and the linear dispersion relation, obtained here from the variational equation for the amplitude, (2.24). In addition to the KP-Whitham system in the harmonic limit, we also have the equation for the linear approximation of the amplitude from the variational equation for the phase, (2.25).

## 2.3 Discussion of Methods

The two derivations presented offer unique perspectives on the KP-Whitham system. Using an asymptotic expansion of $u$ with a scale separation between the slowly varying variables $X, Y, T$, and the fast variable $\theta$ allows for the derivation of multiple equations whose solutions yield approximate solutions of the original KP equation when substituted back into the ansatz. The use of multiple scales allows for a clear definition of the fast and slow variations, which are kept track of throughout the derivation, allowing for a rigorous formulation of the different equations appearing at different orders. In the averaged Lagrangian approach, variational principles are utilized to obtain the same equations. This approach offers a perhaps more intuitive framework, with key equations resulting from requiring certain stationary behavior of variations in the parameters. From this, we also obtain the conservation equation for wave action, with the wave action density itself. This term and corresponding equation are not readily visible from the multiple scales derivation. Similarly, we can obtain equations for energy and wave momentum that do not immediately follow in the other method.

## 2.4 Hyperbolicity and Riemann Invariants of the harmonic KP-Whitham System

The calculation of Riemann invariants enables further understanding of the structure of the KP-Whitham system [2]. With Riemann invariants, we gain insight into what quantities remain constant when moving along the characteristic curves. These quantities can be especially useful when analyzing multiphase interactions, as with a DSW mean flow solution, where methods used in chapter 3 are not directly applicable. In solving for the Riemann invariants, we also gain information on the mathematical structure of the system.

We start by studying the characteristic velocities to determine if the system is hyperbolic and genuinely nonlinear [4]. We have (2.9), (2.10), and (2.11), but take the higher derivative term to zero, i.e., $\overline{u}_{YY} \to 0$, so we can write

$$\mathbf{u}_T + \alpha A(\mathbf{u})\mathbf{u}_X + \beta B(\mathbf{u})\mathbf{u}_Y = 0, \tag{2.31}$$

where

$$\mathbf{u} = \begin{pmatrix} \overline{u} \\ k \\ l \end{pmatrix}, \quad A(\mathbf{u}) = \begin{pmatrix} \overline{u} & 0 & 0 \\ k & \overline{u} - 3k^2 - \sigma\frac{l^2}{k^2} & 0 \\ 0 & 0 & \overline{u} - 3k^2 - \sigma\frac{l^2}{k^2} \end{pmatrix}, \quad \text{and} \quad B(\mathbf{u}) = \begin{pmatrix} 0 & 0 & 0 \\ k & 2\sigma\frac{l}{k} & 0 \\ 0 & 0 & 2\sigma\frac{l}{k} \end{pmatrix}. \tag{2.32}$$

We do not include equation (2.13) as this is not a part of the KP-Whitham system in the harmonic limit; instead it represents a linear approximation of the amplitude. We also have the direction of propagation, $\hat{\mathbf{n}} = \left(\alpha, \beta\right)^T = \left(\cos(\theta), \sin(\theta)\right)^T$ with $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. The eigenvalues of the matrix $\alpha A + \beta B$ are then the characteristic velocities. These eigenvalues and their corresponding eigenvectors are

$$\lambda_1 = \overline{u}\cos(\theta), \quad \mathbf{v}_1 = \begin{pmatrix} \cot(\theta)\left(3k + \sigma\frac{l^2}{k^3}\right) - 2\sigma\frac{l}{k^2} \\ \cot(\theta) \\ 1 \end{pmatrix} \tag{2.33}$$

$$\lambda_{2,3} = \cos(\theta)\left(\overline{u} - 3k^2 - \sigma\frac{l^2}{k^2}\right) + \sin(\theta)\left(2\sigma\frac{l}{k}\right), \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.34}$$

We can see that each eigenvalue is real, but the second eigenvalue has algebraic multiplicity of 2. This system is therefore hyperbolic, though not strictly hyperbolic. We now consider

$$\nabla_{(\overline{u},k,l)}\lambda_1 \cdot \mathbf{v}_1 = \frac{\cos^2(\theta)}{\sin(\theta)}\left(3k + \sigma\frac{l^2}{k^3}\right) - 2\sigma\frac{l}{k^2}\cos(\theta), \tag{2.35}$$

$$\nabla_{(\overline{u},k,l)}\lambda_{2,3} \cdot \mathbf{v}_2 = \cos(\theta)\left(-6k + 2\sigma\frac{l^2}{k^3}\right) - 2\sigma\frac{l}{k^2}\sin(\theta), \tag{2.36}$$

and

$$\nabla_{(\overline{u},k,l)}\lambda_{2,3} \cdot \mathbf{v}_3 = -\cos(\theta)\left(2\sigma\frac{l}{k^2}\right) + \frac{2\sigma}{k}\sin(\theta). \tag{2.37}$$

Because this system is genuinely nonlinear if $\nabla_{(\overline{u},k,l)}\lambda_j \cdot \mathbf{v}_l \neq 0$, we have genuine nonlinearity if

$$\cot(\theta)\left(3k + \sigma\frac{l^2}{k^3}\right) \neq 2\sigma\frac{l}{k^2}, \quad \cot(\theta)\left(-6k + 2\sigma\frac{l^2}{k^3}\right) \neq 2\sigma\frac{l}{k^2}, \quad \text{and} \quad \frac{l}{k} \neq \tan(\theta). \tag{2.38}$$

For the calculation of the Riemann invariants, we can again start with (2.9), (2.10), and (2.11), but we can now drop the terms with $y$-derivatives as we are considering a one-dimensional (RW) mean flow in our analysis. This leaves the system

$$\begin{aligned} \overline{u}_T + \overline{u}\overline{u}_X &= 0, \\ k_T + v_{gx}k_X + k\overline{u}_X &= 0, \\ l_T + v_{gx}l_X &= 0, \end{aligned} \tag{2.39}$$

where $v_{gx} = \overline{u} - 3k^2 - \sigma\frac{l^2}{k^2}$. We can now write these as

$$\mathbf{u}_T + A(\mathbf{u})\mathbf{u}_X = 0. \tag{2.40}$$

$A$ has one eigenvalue of multiplicity two, $\lambda_{1,2} = \overline{u} - 3k^2 - \sigma\frac{l^2}{k^2}$, and one eigenvalue of multiplicity one, $\lambda_3 = \overline{u}$. These eigenvalues have the respective corresponding left eigenvectors,

$$\mathbf{l}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{l}_2 = \begin{pmatrix} \frac{-k}{3k^2+\sigma\frac{l^2}{k^2}} \\ 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{l}_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{2.41}$$

Starting with $\mathbf{l}_1$, we calculate

$\mathbf{l}_1^T \cdot \begin{pmatrix} d\overline{u} \\ dk \\ dq \end{pmatrix} = dl = 0$ along the characteristic $\frac{dx}{dt} = \overline{u} - 3k^2 - \sigma\frac{l^2}{k^2}$. Integrating this gives $l = l_0$ for some constant $l_0$, so the first Riemann invariant is $l$, constant along the given characteristic $v_{gx}$. We set $R_1 = l = l_0$. With the second left eigenvector, we have $\mathbf{l}_2^T \cdot \begin{pmatrix} d\overline{u} \\ dk \\ dq \end{pmatrix} = \left(\frac{-k}{3k^2+\sigma\frac{l^2}{k^2}}\right) d\overline{u} + dk = 0$, along the characteristic $\frac{dx}{dt} = \overline{u} - 3k^2 - \sigma\frac{l^2}{k^2}$. Requiring

$$\frac{dR_2}{dt} = \frac{\partial R_2}{\partial \overline{u}} d\overline{u} + \frac{\partial R_2}{\partial k} dk + \frac{\partial R_2}{\partial l} dl = 0$$

along $\frac{dx}{dt} = v_{gx}$ gives

$$\frac{\partial R_2}{\partial \overline{u}} = \frac{-k}{3k^2 + \sigma\frac{l^2}{k^2}}, \quad \frac{\partial R_2}{\partial k} = 1, \quad \text{and} \quad \frac{\partial R_2}{\partial l} = 0.$$

We have $\frac{-k}{3k^2+\sigma\frac{l^2}{k^2}} d\overline{u} + dk = 0$ along the same characteristic as for the first invariant, and thus we recognize that $l = l_0$ here, too. We can then multiply by an integrating factor of $\mu = -\frac{3k^4+\sigma l_0^2}{k^3}$ to get $d\overline{u} - \frac{3k^4+\sigma l_0^2}{k^3} dk = 0$. Integrating yields $\overline{u} - \frac{3}{2}k^2 + \frac{1}{2}\sigma\frac{l_0^2}{k^2} = c_2$ for some constant $c_2$. We set this constant to be the Riemann invariant $R_2$. For the final eigenvector, we have $d\overline{u} = 0$ along the characteristic $\overline{u}$. Integrating this gives $\overline{u} = c_3$ for some constant $c_3$, which we set as the final Riemann invariant. Thus, we have

$$(\lambda_1, \mathbf{l}_1, R_1) = \left( \overline{u} - 3k^2 - \sigma\frac{l_0^2}{k^2}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, l_0 \right), \tag{2.42a}$$

$$(\lambda_2, \mathbf{l}_2, R_2) = \left( \overline{u} - 3k^2 - \sigma\frac{l_0^2}{k^2}, \begin{bmatrix} \frac{-k}{3k^2+\sigma\frac{l_0^2}{k^2}} \\ 1 \\ 0 \end{bmatrix}, \overline{u} - \frac{3}{2}k^2 + \frac{1}{2}\sigma\frac{l_0^2}{k^2} \right), \tag{2.42b}$$

$$(\lambda_3, \mathbf{l}_3, R_3) = \left( \overline{u}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \overline{u} \right). \tag{2.42c}$$

# Chapter 3

## Solutions to the Wavepacket-Mean Flow Interaction Problem

In this chapter, we present solutions to the equations governing the wavepacket-mean flow interaction problem. We consider the rarefaction wave (RW) solution to the mean flow equation and use it in the decoupled wavepacket equations. The method of characteristics is employed to calculate solutions for these wavepacket equations [12, 4], and continuity conditions are enforced to ensure smoothness at the edges of solution regions. Additional constraints on the solutions are implemented to obtain conditions on the wave numbers of the linear waves for which interaction, transmission, or ejection are possible [7]. Finally, the solution for the amplitude equation is presented and the conservation of the integral of wave action is discussed [17].

### 3.1 Mean Flow Solution

In this interaction problem, the rarefaction wave solution [7] to (2.9) is utilized. This expansion wave occurs when the initial conditions for the mean flow equation feature a jump discontinuity between two constant values [13],

$$\overline{u}(X,Y,0) = \begin{cases} \overline{u}_- & X < 0 \\ \overline{u}_+ & X > 0 \end{cases} \tag{3.1}$$

and $\overline{u}_- < \overline{u}_+$. The resulting RW solution is

$$\overline{u}\,(X,Y,T) = \begin{cases} \overline{u}_- & X < \overline{u}_- T \\ \frac{X}{T} & \overline{u}_- T < X < \overline{u}_+ T \\ \overline{u}_+ & X > \overline{u}_+ T \end{cases}, \quad \overline{v}(X,Y,T) = 0. \tag{3.2}$$

Note that, in the second region of the solution, $\overline{u}_T = -\frac{X}{T^2}$, $\overline{u}_X = \frac{1}{T}$, and $\overline{v}_Y = 0$, so (2.9) becomes $-\frac{X}{T^2} + \frac{X}{T}\frac{1}{T} + 0 = 0$, with the other solution regions yielding a trivial sum of zeros.

## 3.2 Solutions to the Wave Vector Equations

We now seek the solutions to (2.10), and (2.12). The decoupled nature of these equations from the mean flow equation allows for (3.2) to be obtained first, and then plugged into these equations governing $k$ and $q$. It should be noted that variable coefficients of the spatial partial derivatives are equivalent in (2.10) and (2.12), allowing for the use of the method of characteristics in solving these equations [4]. These variable coefficients are the characteristic velocities of the respective spatial component. Via the method of characteristics [12], we can transform these PDE's into a system of first order, linear ODE's called the characteristic equations, which are solvable analytically. The solutions to these characteristic equations are the characteristic curves. In particular, the solutions $X(T)$ and $Y(T)$ represent the trajectories of the wavepacket in each respective spatial component. We follow the procedure used in [4], and consider each region separately. The characteristic equations are

$$\frac{dk}{dT} = -k\overline{u}_X, \quad \frac{dl}{dT} = 0 \quad \frac{dX}{dT} = \overline{u} - 3k^2 - \sigma q^2, \quad \frac{dY}{dT} = 2\sigma q, \tag{3.3}$$

with initial conditions

$$k(X_+,Y_+,0) = k_+, \quad l(X_+,Y_+,0) = l_+, \quad X(0) = X_+ > 0, \quad Y(0) = Y_+. \tag{3.4}$$

if the wavepacket is interacting with the RW from the right, or

$$k(X_-, Y_-, 0) = k_-, \quad l(X_-, Y_-, 0) = l_-, \quad X(0) = X_- < 0, \quad Y(0) = Y_- \tag{3.5}$$

if the wavepacket is interacting with the RW from the left. We go through the calculation using (3.4), then present the results for (3.5) at the end. Numbered subscripts denote the solution region, e.g., $X_1(t)$ represents the solution to the equation for $\frac{dX}{dT}$ in the first region, where $\overline{u} = \overline{u}_+$. We also note that $l(X, Y, T) = l_+$ in each section as it is invariant along the characteristic curve. We begin with the case where $(\overline{u}, \overline{v}) = (\overline{u}_+, 0)$. We can directly integrate the resulting versions of the characteristic equations (3.3) to obtain the characteristic curves

$$k_1\left(X_1(T), Y_1(T), T\right) = f_1\left(X_+, Y_+\right) = k_+,$$

$$X_1\left(T\right) = \left(\overline{u}_+ - 3k_+^2 - \sigma q_+^2\right) T + X_+, \qquad Y_1\left(T\right) = 2\sigma q_+ T.$$

Without loss of generality, initial condition $Y_+$ has been set to 0 – a nonzero initial condition results in a shift with an additional $Y_+$ term added to each $Y$ solution. We consider the $X_1$ and $Y_1$ solutions, specifically their respective coefficients $\overline{u}_+ - 3k_+^2 - \sigma q_+^2$ and $2\sigma q_+$. These are equal to $\nabla_{(k,l)}\omega$, the group velocity of the wavepacket. These solutions represent free propagation of the wavepacket on $\overline{u}_+$ with this group velocity, $\nabla_{(k,l)}\omega$.

For $(\overline{u}, \overline{v}) = \left(\frac{X}{T}, 0\right)$, we can obtain the solutions to the characteristic equations via direct integration (using an integrating factor for the $X_2(T)$ solution), yielding

$$k_2\left(X_2(T), Y_2(T), T\right) = \frac{f_2\left(X_{02}, Y_{02}\right) t_1}{T} = \frac{k_+ t_1}{T},$$

$$X_2\left(T\right) = \left(\overline{u}_+ - \frac{3}{2}k_+^2 + \frac{1}{2}\sigma q_+^2\right) T + \left(\frac{3}{2}k_+^2 t_1^2\right)\frac{1}{T} - \frac{1}{2t_1^2}\sigma q_+^2 T^3, \qquad Y_2\left(T\right) = \frac{\sigma q_+}{t_1}T^2 + \sigma q_+ t_1.$$

Here, we set the initial time of the section of this solution to $t_1$ and recognize that due to the bounds of the piecewise solution of $(\overline{u}, \overline{v})$, we require

$$X_1(t_1) = \overline{u}_+ t_1 = X_2(t_1), \quad Y_1(t_1) = 2\sigma q_+ t_1 = Y_2(t_1), \quad k_1(t_1) = k_+ = k_2(t_+). \tag{3.6}$$

The solutions from these conditions have already been implemented in the curves given above. This also gives us a relation between the initial value $X(0) = X_+$ of the wavepacket and the initial time of interaction $t_1$:

$$X_+ = \left(3k_+^2 + \sigma q_+^2\right) t_1 \tag{3.7}$$

$$t_1 = \frac{X_+}{3k_+^2 + \sigma q_+^2}. \tag{3.8}$$

In the limit that $q_+ \to 0$, we expect to obtain the solution for $t_1$ found in the KdV regime (1.6), which we can see is true. Finally, we have the case where $X < 0$ and $(\overline{u}, \overline{v}) = (\overline{u}_-, 0)$. Here, we have the solutions

$$k_3\left(X_3(T), Y_3(T), T\right) = f_3\left(X_{03}, Y_{03}\right) = k_-,$$

$$X_3\left(T\right) = \left(\overline{u}_- - 3k_-^2 - \sigma q_-^2\right) T + \left(3k_-^2 + \sigma q_-^2\right) t_2, \qquad Y_3\left(T\right) = 2\sigma q_- T - 2\sigma q_- t_- + \frac{\sigma q_+}{t_1} t_-^2 + \sigma q_+ t_1,$$

requiring

$$X_2(t_2) = \overline{u}_- t_2 = X_3(t_2), \quad Y_2(t_2) = \frac{\sigma q_+^2}{t_1} t_2^2 + \sigma q_+ t_1 = Y_2(t_2), \quad k_2(t_2) = k_+ \frac{t_1}{t_2} = k_-, \tag{3.9}$$

where again, the solutions to these conditions have already been implemented. have. We see that $k$ decreases over the interaction with the rarefaction wave, and we know that $l = l_+$ is constant throughout the interaction. $q$ must then increase during the interaction, and thus we have

$$q_- = q_+ \frac{t_2}{t_1}. \tag{3.10}$$

We let $v_g$ denote the respective components of the group velocity of the wavepacket with respect to the arguments i.e., $v_g(K, Q, \overline{u}, \overline{v}) = \left(\overline{u} - 3K^2 - \sigma Q^2, 2\sigma Q\right)^T = (v_{g1}, v_{g2})$. The full solutions are then

$$X(T) = \begin{cases} v_{g_1}(k_+, q_+, \overline{u}_+)T + X_+ & 0 \leq T \leq t_1, \\ \left(\overline{u}_+ - \frac{3}{2}k_+^2 + \frac{1}{2}\sigma q_+^2\right) T + \frac{3k_+^2 t_1^2}{2T} - \frac{\sigma q_+^2 T^3}{2t_1^2} & t_1 \leq T \leq t_2 \\ v_{g_1}(k_-, q_-, \overline{u}_-)T + (3k_-^2 + \sigma q_-^2)t_2 & t_2 \leq T \end{cases} \tag{3.11a}$$

$$Y(T) = \begin{cases} v_{g_2}(q_+, \overline{v})T & 0 \leq T \leq t_1, \\ \frac{\sigma q_+}{t_1}T^2 + \sigma q_+ t_1 & t_1 \leq T \leq t_2, \\ v_{g_2}(q_-, \overline{v})T - v_{g_2}(q_-, \overline{v})t_2 + \frac{\sigma q_+}{t_1}t_2^2 + \sigma q_+ t_1 & t_2 \leq T \end{cases} \tag{3.11b}$$

$$k(X(T), Y(T), T) = \begin{cases} k_+ & 0 \leq T \leq t_1, \\ \frac{k_+ t_1}{T} & t_1 \leq T \leq t_2, \\ k_- & t_2 \leq T \end{cases} \tag{3.11c}$$

$$l = l_+ \tag{3.11d}$$

When the wavepacket is initially to the left of the RW and we use initial conditions (3.5), and obtain

$$X(T) = \begin{cases} v_{g_1}(k_-, q_-, \overline{u}_-)T + X_- & 0 \leq T \leq t_1, \\ \left(\overline{u}_- - \frac{3}{2}k_-^2 + \frac{1}{2}\sigma q_-^2\right) T + \frac{3k_-^2 t_1^2}{2T} - \frac{\sigma q_-^2 T^3}{2t_1^2} & t_1 \leq T \leq t_2 \\ v_{g_1}(k_+, q_+, \overline{u}_+)T + (3k_+^2 + \sigma q_+^2)t_2 & t_2 \leq T \end{cases} \tag{3.12a}$$

$$Y(T) = \begin{cases} v_{g_2}(q_-, \overline{v})T & 0 \leq T \leq t_1, \\ \frac{\sigma q_-}{t_1}T^2 + \sigma q_- t_1 & t_1 \leq T \leq t_2, \\ v_{g_2}(q_+, \overline{v})T - v_{g_2}(q_+, \overline{v})t_2 + \frac{\sigma q_-}{t_1}t_2^2 + \sigma q_- t_1 & t_2 \leq T \end{cases} \tag{3.12b}$$

$$k(X(T), Y(T), T) = \begin{cases} k_- & 0 \leq T \leq t_1, \\ \frac{k_- t_1}{T} & t_1 \leq T \leq t_2, \\ k_+ & t_2 \leq T \end{cases} \cdot \tag{3.12c}$$

In the case of ejection, we denote the time that ejection occurs as $t_{\text{ejec}}$, so that the continuity conditions for the second and third solution edge become

$$X_2(t_{\text{ejec}}) = \overline{u}_+ t_{\text{ejec}} = X_3(t_{\text{ejec}}) \quad Y_2(t_{\text{ejec}}) = \frac{\sigma q_+}{t_1}t_{\text{ejec}}^2 + \sigma q_- t_1 = Y_3(t_{\text{ejec}}) \quad k_2(t_{\text{ejec}}) = \frac{t_1}{t_{\text{ejec}}}, \tag{3.13}$$

yielding the solutions

$$X(T) = \begin{cases} v_{g_1}(k_+, q_+, \overline{u}_+)T + X_+ & 0 \leq T \leq t_1, \\ \left(\overline{u}_+ - \frac{3}{2}k_+^2 + \frac{1}{2}\sigma q_+^2\right)T + \frac{3k_+^2 t_1^2}{2T} - \frac{\sigma q_+^2 T^3}{2t_1^2} & t_1 \leq T \leq t_{\text{ejec}} \\ v_{g_1}(k_-, q_-, \overline{u}_+)T + (3k_-^2 + \sigma q_-^2)t_{\text{ejec}} & t_{\text{ejec}} \leq T \end{cases} \tag{3.14a}$$

$$Y(T) = \begin{cases} v_{g_2}(q_+, \overline{v})T & 0 \leq T \leq t_1, \\ \frac{\sigma q_+}{t_1}T^2 + \sigma q_+ t_1 & t_1 \leq T \leq t_{\text{ejec}}, \\ v_{g_2}(q_-, \overline{v})T - v_{g_2}(q_-, \overline{v})t_{\text{ejec}} + \frac{\sigma q_+}{t_1}t_{\text{ejec}}^2 + \sigma q_+ t_1 & t_{\text{ejec}} \leq T \end{cases} \tag{3.14b}$$

$$k(X(T),Y(T),T)=\begin{cases} k_+ & 0\le T\le t_1, \\ \frac{k_+t_1}{T} & t_1\le T\le t_{\text{ejec}}, \\ k_- & t_{\text{ejec}}\le T \end{cases}. \tag{3.14c}$$

## 3.3 Conditions for Various Interaction Behaviors

We now seek the conditions on the wave numbers that lead to different interaction outcomes. Note that an interaction only occurs if $t_1$ is finite and positive, i.e., if there is a valid solution for $t_1$ in the expression $X_1(t_1)=X_2(t_1)$. We also note that for transmission to be possible, $t_2$ must be finite and greater than $t_1$. We use these constraints to determine when these scenarios are possible. A summary of the results can be found in table 4.1.

### 3.3.1 KPII From the Right

First, we check the requirements needed for an interaction to be possible. In KPII, (3.8) becomes

$$\frac{X_+}{3k_+^2+q_+^2}, \tag{3.15}$$

which we require to be positive. When the wavepacket approaches from the right, $X_+>0$. Because $3k_+^2+q_+^2$ is always positive, $t_1$ is always positive and thus, there will always be an interaction in KPII when $X_+>0$. We now want to solve for $t_2$ in the equation $X_2(t_2)=X_3(t_2)$, which becomes

$$\left(\overline{u}_+-\frac{3}{2}k_+^2+\frac{1}{2}q_+^2\right)t_2+\frac{3k_+^2t_1^2}{2t_2}-\frac{q_+^2t_2^3}{2t_1^2}=\overline{u}_-t_2. \tag{3.16}$$

This yields four solutions for $t_2$, though we can immediately discard two of them as they are negative roots. Using Mathematica's Reduce function, we test the remaining two solutions' ability to satisfy $t_2>t_1$ when $X_+>0$, $k_+$ and $q_+$ are real, and $\overline{u}_+>\overline{u}_-$. From this, we see that there is only one valid solution,

$$t_2 = \frac{X_+\sqrt{\sqrt{6k_+^2\left(2\bar{u}_- - 2\bar{u}_+ + q_+^2\right) + \left(-2\bar{u}_- + 2\bar{u}_+ + q_+^2\right)^2 + 9k_+^4} - 2\bar{u}_- + 2\bar{u}_+ - 3k_+^2 + q_+^2}}{\sqrt{2}\left|3k_+^2 q_+ + q_+^3\right|} \tag{3.17}$$

which satisfies these conditions with any initial $k_+, q_+$. Therefore, in KPII with $X_+ > 0$, we have transmission no matter what initial $k_+, q_+$ is chosen.

### 3.3.2 KPII From the Left

Again, we begin by analyzing $t_1$. In this case, $X_+ < 0$, but $3k_-^2 + q_-^2$ is always positive, so $t_1$ is always negative. Thus, an interaction is impossible when the wavepacket is approaching from the left of the RW in KPII.

### 3.3.3 KPI From the Right

The interaction time $t_1$ in KPI becomes

$$t_1 = \frac{X_+}{3k_+^2 - q_+^2}. \tag{3.18}$$

In this case, $X_+ > 0$, so we require that $3k_+^2 - q_+^2 > 0$, which leads to the interaction condition

$$k_+ > \frac{|q_+|}{\sqrt{3}}. \tag{3.19}$$

We first look for the conditions under which transmission is possible. The equation $X_2(t_2) = X_3(t_2)$ becomes

$$\left(\overline{u}_+ - \frac{3}{2}k_+^2 - \frac{1}{2}q_+^2\right)t_2 + \frac{3k_+^2 t_1^2}{2t_2} + \frac{q_+^2 t_2^3}{2t_1^2} = \overline{u}_- t_2, \tag{3.20}$$

which again leads to four solutions, two of which are negative. We again test the remaining positive solutions for their possibility of satisfying $t_2 > t_1$ when $X_+ > 0$, $\overline{u}_+ > \overline{u}_-$ and $k_+ > \frac{|q_+|}{\sqrt{3}}$. This yields only one solution,

$$t_2 = \frac{X_+ \sqrt{-\sqrt{6k_+^2\left(2\bar{u}_- - 2\bar{u}_+ + q_+^2\right) + \left(-2\bar{u}_- + 2\bar{u}_+ + q_+^2\right)^2 + 9k_+^4} - 2\bar{u}_- + 2\bar{u}_+ - 3k_+^2 + q_+^2}}{\sqrt{2}\left|q_+^3 - 3k_+^2 q_+\right|}, \tag{3.21}$$

valid under the constraint

$$k_+ > \frac{|q_+| + \sqrt{2(\overline{u}_+ - \overline{u}_-)}}{\sqrt{3}}. \tag{3.22}$$

This is the requirement for transmission to occur. We now consider the case where $k_+$ satisfies the interaction condition, but not the transmission condition. Under these conditions, we find that the wavepacket turns around during the interaction. Thus, we solve for $t_{\text{ejec}}$ in the equation

$$\left(\overline{u}_+ - \frac{3}{2}k_+^2 - \frac{1}{2}q_+^2\right) t_{\text{ejec}} + \frac{3k_+^2 t_1^2}{2t_{\text{ejec}}} + \frac{q_+^2 t_{\text{ejec}}^3}{2t_1^2} = \overline{u}_+ t_{\text{ejec}}, \tag{3.23}$$

which again yields only one valid solution under the given conditions for $k_+$, and the requirement that $t_1 < t_{\text{ejec}}$. This solution is given by

$$t_{\text{ejec}} = \frac{\sqrt{3}k_+ t_1}{|q_+|} = \frac{\sqrt{3}k_+ x_+}{|q_+|\left(3k_+^2 - q_+^2\right)}. \tag{3.24}$$

### 3.3.4 KPI From the Left

With $X_+ < 0$ in this case, the condition $t_1 > 0$ now leads to the requirement $3k_-^2 - q_-^2 < 0$, yielding the interaction condition

$$k_- < \frac{|q_-|}{\sqrt{3}}. \tag{3.25}$$

Following the same procedure as in the other case, analyzing the solutions for $t_2$ in the equation

$$\left(\overline{u}_- - \frac{3}{2}k_-^2 - \frac{1}{2}q_-^2\right) t_2 + \frac{3k_-^2 t_1^2}{2t_2} + \frac{q_-^2 t_2^3}{2t_1^2} = \overline{u}_+ t_2, \tag{3.26}$$

and imposing the same requirements (though with $k_+ < \frac{|q_+|}{\sqrt{3}}$ here), we obtain the solution

$$\frac{X_+\sqrt{\sqrt{-6k_-^2\left(2\bar{u}_- - 2\bar{u}_+ + q_-^2\right) + \left(-2\bar{u}_- + 2\bar{u}_+ + q_-^2\right)^2 + 9k_-^4} - 2\bar{u}_- + 2\bar{u}_+ + 3k_-^2 + q_-^2}}{\sqrt{2}\left|q_-^3 - 3k_-^2 q_-\right|}, \quad (3.27)$$

which is valid for any initial $k_-, q_-$ that satisfy the interaction condition. Thus, for any interaction in KPI with the wavepacket approaching from the left, there will be transmission.

### 3.3.5 Transmission Conditions From Riemann Invariants

We note that we can also determine the conditions for transmission by considering the Riemann invariants (2.42a, 2.42b). From these, we have the conservation equation

$$\bar{u}_+ - \frac{3}{2}k_+^2 + \frac{1}{2}\sigma\frac{l_0^2}{k_+^2} = \bar{u}_- - \frac{3}{2}k_-^2 + \frac{1}{2}\sigma\frac{l_0^2}{k_-^2}. \quad (3.28)$$

In order for transmission to occur, the transmitted wavenumber $k_-$ (or $k_+$ if interacting from the left) must be real. Solving for $k_-$ ($k_+$) yields the constraint on $k_+$ ($k_-$) that is exactly equal to the constraints found in table 4.1.

## 3.4 Solution to the Amplitude Equation

To solve the amplitude equation, we consider the form (2.13), which allows for easy extraction of the characteristic equations. These are equivalent to those from (3.3) involving $X$ and $Y$. Their respective solutions are thus given by (3.11a) and (3.11b). The final characteristic equation is

$$\frac{da}{dT} = -a\left(\overline{u}_X - 3kk_X - \sigma q^2\frac{k_X}{k} - 2\sigma q q_X + \sigma q\frac{k_Y}{k} + \sigma q_Y\right). \quad (3.29)$$

Before directly solving, we consider the $k_X$, $k_Y$, and $q_X$ terms present. To solve for $k_X$, we first note that $\frac{\partial k}{\partial X} = \frac{\partial k}{\partial X_+} \cdot \frac{\partial X_+}{\partial X}$. We can see that $\frac{\partial k}{\partial X_+}$ is 0, except for within the second region where $\frac{\partial k}{\partial X_+} = \frac{k_+}{\left(3k_+^2 + \sigma q_+^2\right)T}$. Using $X_2(T)$, the $X$ solution in the corresponding region, and the substitution (3.8), we differentiate implicitly to get

$$1 = \frac{3k_+^2 X_+}{T\left(3k_+^2 + \sigma q_+^2\right)^2}\left(\frac{\partial X_+}{\partial X}\right) + \frac{\sigma q_+^2 T^3\left(3k_+^2 + \sigma q_+^2\right)^2}{X_+^3}\left(\frac{\partial X_+}{\partial X}\right).$$

We can solve this for $\frac{\partial X_+}{\partial X}$ to use for the solution for $k_X$,

$$k_X = \frac{k_+(3k_+^2 + \sigma q_+^2)X_+^3}{3k_+^2 X_+^4 + \sigma q_+^2 T^4 (3k_+^2 + \sigma q_+^2)^4}.$$

We can find the expression for $\frac{\partial q}{\partial x}$ in the same manner. We have $\frac{\partial q}{\partial X_+} = -\frac{q_+ T(3k_+^2 + \sigma q_+^2)}{X_+^2}$, which gives

$$q_X = \frac{-q_+ T^2 X_+ (3k_+^2 + \sigma q_+^2)^3}{3k_+^2 X_+^4 + \sigma q_+^2 T^4 (3k_+^2 + \sigma q_+^2)^4}.$$

We know that $l = l_+ = l_-$, and thus $l_X = 0$. Following (2.5), we know that $k_Y = 0$ as well. From this, (3.29) can be simplified as

$$\frac{da}{dT} = -a\left(\overline{u}_X - 3kk_X + \sigma q^2 \frac{k_X}{k}\right). \tag{3.30}$$

In the first and third regions, $k = k_\pm$ and $q = q_\pm$, respectively, which are constants. Thus, the solutions for these regions are denoted

$$a_1 = a_+, \qquad a_3 = a_-$$

where $a_1(X(0), Y(0), 0) = a_+$ and $a_3(X(t_2), Y(t_2), t_2) = a_-$. In the second region, where $(\overline{u}, \overline{v}) = (\frac{x}{T}, 0)$, we have

$$\frac{da}{dT} = -\frac{a}{T}\left(\overline{u}_+ - \frac{3k_+^2 X_+^4 - \sigma q_+^2 T^4\left(3k_+^2 + \sigma q_+^2\right)^4}{3k_+^2 X_+^4 + \sigma q_+^2 T^4\left(3k_+^2 + \sigma q_+^2\right)^4}\right).$$

Directly integrating and using the fact that $a_1(X(t_1), Y(t_1), t_1) = a_2(X(t_1), Y(t_1), t_1) = a_+$ to ensure continuity yields a final solution of

$$a_2(X(t_1), Y(t_1), t_1) = a_+\left(\frac{X_+^4\left(3k_+^2 + \sigma q_+^2\right)}{3k_+^2 X_+^4 + \sigma q_+^2\left(3k_+^2 + \sigma q_+^2\right)^4 T^4}\right)^{\frac{1}{2}}.$$

We can obtain an expression for $a_-$ using the requirement that $a_2(X(t_2), Y(t_2), t_2) = a_-$:

$$a_- = a_+ \left( \frac{x_+^4 \left(3k_+^2 + \sigma q_+^2\right)}{3k_+^2 x_+^4 + \sigma q_+^2 \left(3k_+^2 + \sigma q_+^2\right)^4 t_2^4} \right)^{\frac{1}{2}}.$$

Our final solution for the evolution of the amplitude of a plane wave is

$$a(X(T),Y(T),T) = \begin{cases} a_+ & 0 \le T \le t_1, \\ a_+ \left( \frac{X_+^4 \left(3k_+^2+\sigma q_+^2\right)}{3k_+^2 X_+^4+\sigma q_+^2 \left(3k_+^2+\sigma q_+^2\right)^4 T^4} \right)^{\frac{1}{2}} & t_1 \le T \le t_2, \cdot \\ a_- & t_2 \le T, \end{cases} \tag{3.31}$$

## 3.5 Integral of Wave Action

Though (3.31) is valid for plane waves, we may also analyze the change in the amplitude of a Gaussian wavepacket after interaction with a RW by studying the conservation of the integral of wave action [17, 7]. We start with the conservation equation for wave action $A(X,Y,T) = \frac{a^2}{k}$ (2.30b). We consider a general domain $\Omega \in \mathbb{R}^2$, and integrate over this domain

$$\int_\Omega \left[A_T + \nabla \cdot (\nabla_{k,l}\omega A)\right] dXdY = 0. \tag{3.32}$$

We break the integral up into its two separate parts via linearity and use Green's theorem [14] to get

$$\int_\Omega A_T dXdY + \int_{\partial\Omega} (\nabla_{k,l}\omega A) \cdot \hat{\mathbf{n}}\ dl = 0, \tag{3.33}$$

where the second term is the flux of wave action across the boundary $\partial\Omega$. We now consider the integral of wave action [7],

$$E(T) = \int_{Y_1(T)}^{Y_2(T)} \int_{X_1(T)}^{X_2(T)} A(X,Y,T) dXdY. \tag{3.34}$$

We use the Leibniz Rule (or Reynold's Transport Theorem) [14] in higher dimensions to differentiate and obtain

$$\frac{dE}{dT} = \int_{Y_1(T)}^{Y_2(T)} \int_{X_1(T)}^{X_2(T)} A_T dX dY + \int_{\partial([X_1,X_2]\times[Y_1,Y_2])} (\mathbf{v}_b \cdot \hat{\mathbf{n}})\, A dl, \tag{3.35}$$

where $\mathbf{v}_b$ is the velocity of the area element. Thus, $\mathbf{v}_b = [\dot{X}_\xi \quad \dot{Y}_\xi]^T$, which is equivalent to the gradient of $\omega$, or $\mathbf{v}_b = \nabla_{k,l}\omega$. Therefore, $(\mathbf{v}_b \cdot \hat{\mathbf{n}})\, A = (\nabla_{k,l}\omega \cdot \hat{\mathbf{n}})\, A = (\nabla_{k,l}\omega A) \cdot \hat{\mathbf{n}}$. Using Green's Theorem again with region $\Omega$ defined by $[X_1, X_2] \times [Y_1, Y_2]$, the second term on the right-hand side becomes

$$\int_{\partial([X_1,X_2]\times[Y_1,Y_2])} (\nabla_{k,l}\omega A) \cdot \hat{\mathbf{n}} dl = \int_{Y_1(T)}^{Y_2(T)} \int_{X_1(T)}^{X_2(T)} \nabla \cdot ((\nabla_{k,l}\omega)\, A)\, dX dY. \tag{3.36}$$

From the original conservation equation, we have that $A_T = -\nabla \cdot (\nabla_{k,l}\omega A)$, and thus the substitution

$$\int_{Y_1(T)}^{Y_2(T)} \int_{X_1(T)}^{X_2(T)} A_T dX dY = -\int_{Y_1(T)}^{Y_2(T)} \int_{X_1(T)}^{X_2(T)} \nabla \cdot (\nabla_{k,l}\omega A)\, dX dY \tag{3.37}$$

can be made into the expression (3.35) to get

$$-\int_{Y_1(T)}^{Y_2(T)} \int_{X_1(T)}^{X_2(T)} \nabla \cdot (\nabla_{k,l}\omega A)\, dX dY + \int_{Y_1(T)}^{Y_2(T)} \int_{X_1(T)}^{X_2(T)} \nabla \cdot (\nabla_{k,l}\omega A)\, dX dY = 0, \tag{3.38}$$

so that $\frac{dE}{dT} = 0$ and thus the wave action integral, (3.34), is conserved. It follows that $E(t_1) = E(t_2)$, where $t_1$ and $t_2$ are the times before and after interaction, so that with $A(X, Y, T) = a^2(X, Y, T)/k(X, Y, T)$, we have the relation

$$\frac{1}{k_+} \int_{Y_1(T)}^{Y_2(T)} \int_{X_1(T)}^{X_2(T)} a^2(X, Y, t_1) dX dY = \frac{1}{k_-} \int_{Y_1(T)}^{Y_2(T)} \int_{X_1(T)}^{X_2(T)} a^2(X, Y, t_2) dX dY. \tag{3.39}$$

If the amplitude profile is Gaussian,

$$a(X, Y, 0) = a_+ \exp\left( -\left(\frac{X - X_+}{w_{x+}}\right)^2 - \left(\frac{Y - Y_+}{w_{y+}}\right)^2 \right), \tag{3.40}$$

this yields an analytical solution of

$$\frac{a_+^2 \pi w_{x+} w_{y+}}{2k_+} = \frac{a_-^2 \pi w_{x-} w_{y-}}{2k_-}. \tag{3.41}$$

# Chapter 4

# Discussion of Results

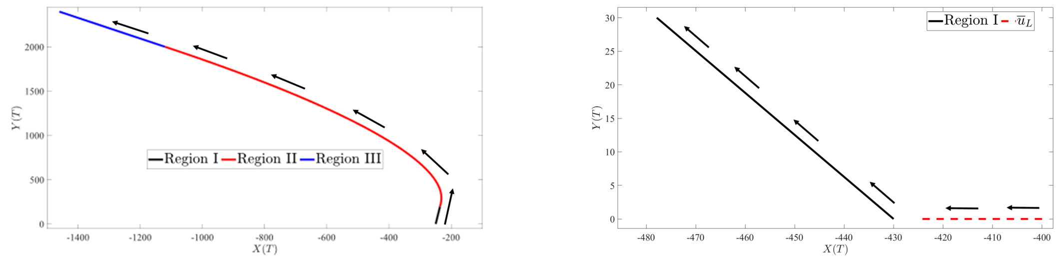
Here, we discuss the results of chapter 3 in further detail. We compare the computed transmission conditions with those found under KdV, discuss differences in trajectory behaviors between the two regimes, and highlight interesting cases of interaction that are unique to KP. The amplitude solution is analyzed further, including the presence of components that deviate from expected behaviors. Some results are compared to their numerically computed counterparts with specific discussion concerning discrepancies between theoretical and numerical solutions.

## 4.1 Transmission Conditions

In [7], the interaction problem studied in this thesis was first studied under the KdV equation. We find changes in the interaction behaviors in KP. In KdV, when the initial position of the wavepacket is rightward of the leading edge of the RW, transmission occurs if (1.5) is satisfied, and the wavepacket is trapped in the RW otherwise. In KPII, a wavepacket initially rightward of the RW is transmitted with any nonzero $k_+, q_+$. The introduction of a nonzero $l$ leads to guaranteed transmission, even if the wavepacket initially moves with a positive (rightward) velocity away from the mean flow. In such a case, the right edge of the RW will always catch up to the wavepacket, eventually causing the wavepacket to turn around and transmit across the left edge. An example of this case can be seen in figure 4.1.

It is clear that in KPI, the $q_+ \to 0$ limiting case of the transmission condition for the interaction from the right is equal to the transmission condition for KdV. However, given that

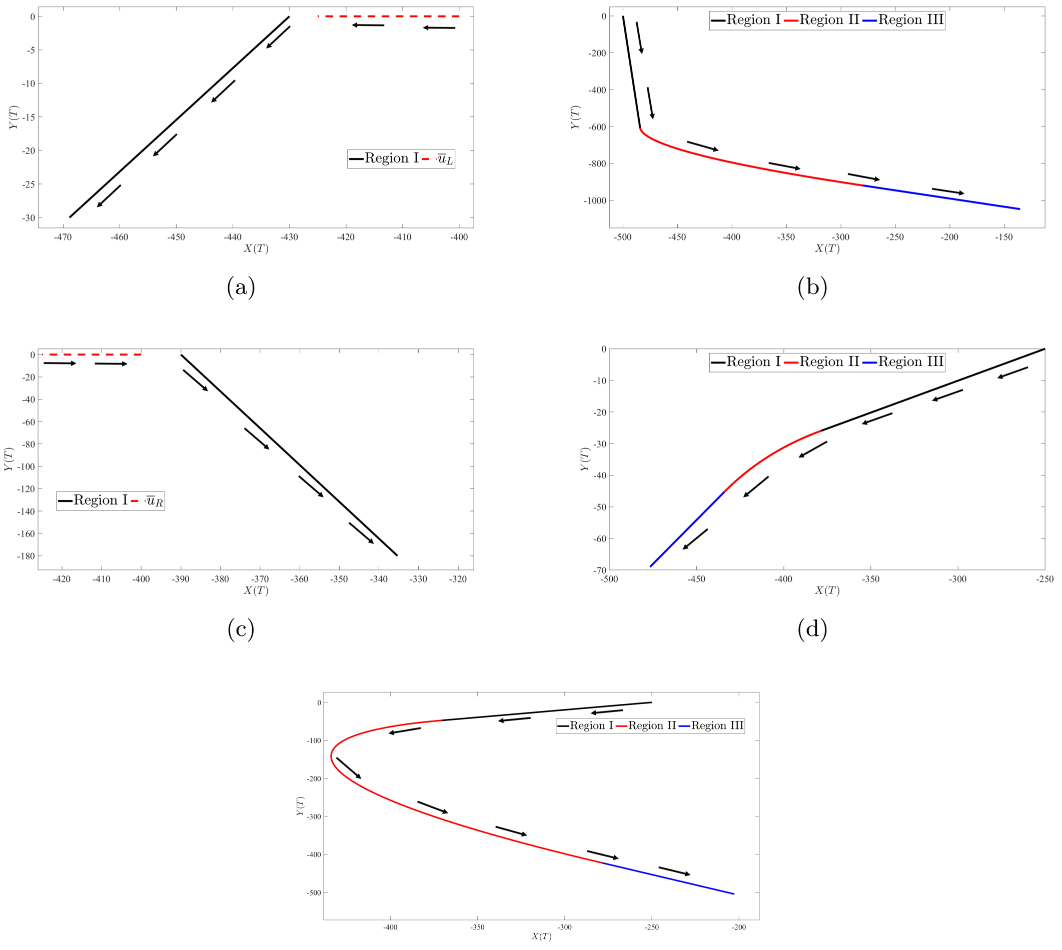
**KPII Theoretical Trajectories**



(a) (b)

Figure 4.1: Wavepacket trajectories. Different colors correspond to different solution regions. $\overline{u}_{\pm} = \pm 1/2$. RW centered at $X = -400$. (a) Transmission. Wavepacket initially to the right of the RW. $k_+ = 0.35$, $q_+ = 0.3$, and $X_+ = -250$. (b) No interaction. Wavepacket initially to the left of the RW. $k_+ = 0.35$, $q_+ = 0.3$, and $X_+ = -430$.

**KPI Theoretical Trajectories**



(a) (b) (c) (d) (e)

Figure 4.2: Wavepacket trajectories. Different colors correspond to different solution regions. $\overline{u}_\pm = \pm 1/2$. RW centered at $X = -400$ (a) **No interaction.** Wavepacket initially to the left of the RW. $k_+ = 0.35$, $q_+ = 0.3$, and $X_+ = -430$. (b) **Transmission.** Wavepacket initially to the left of the RW. $k_+ = \frac{q_+}{\sqrt{3}} - 0.1$, $q_+ = 1.8$, and $X_+ = -500$. (c) **No interaction.** Wavepacket initially to the right of the RW. $k_+ = \frac{q_+}{\sqrt{3}} - 0.1$, $q_+ = 1.8$, and $X_+ = -370$. (d) **Transmission.** Wavepacket initially to the right of the RW. $k_+ = \frac{q_+ + \sqrt{2}}{\sqrt{3}} + 0.1$, $q_+ = 0.3$, and $X_+ = -250$. (e) **Ejection.** Wavepacket initially to the right of the RW. $k_+ = \frac{q_+ + \sqrt{2}}{\sqrt{3}} - 0.1$, $q_+ = 0.4$, and $X_+ = -250$.

there is an interaction, if the transmission condition for KPI is not satisfied, the wavepacket does not remain trapped within the RW. Instead, the wavepacket eventually turns around and once again crosses the right edge of the RW, a result not found under KdV.

We also see differences between the three regimes regarding the requirements for an interaction to occur from the right. While KdV and KPII will always result in a wavepacket-RW interaction, there is a restriction on the values of $k_+$ that yield an interaction in KPI. There is the case where a rightward moving wavepacket will never cross the right edge of the RW. When considering a wavepacket initially to the left of the mean flow, we see more similarity in KPII and KdV. In both cases, there is no initial value for $k_+$ that will result in an interaction. However, for KPI, there is the possibility of an interaction if the $X$ component of the group velocity of the wavepacket is positive. In the case of an interaction, the wavepacket will always transmit. Each result is listed in table 4.1.

To summarize, the introduction of a nonzero $l$ results in different interactions from KdV, with the choice of parameter strongly impacting transmission and the dynamics of the wavepacket-mean flow interaction. We do not observe trapping in either KPI or KPII. We see transmission with any initial wavenumbers in KPII from the right, and the possibility of ejection as well as the possibility of an interaction from the left in KPI. Post-interaction, we see a wavelength upshift with $k_- = k_+t_+/t_-$, or $|\mathbf{k}_-| = |[k_-, l_0]| < |[k_+, l_0]| = |\mathbf{k}_+|$. Again, we observe that $l$ is conserved in all interactions.

## 4.2 Comparison to Numerical Simulations

We now compare the theoretical trajectories of a few cases with their corresponding numerical simulations. The numerical solver for the KP equation employs a fourth-order Runge–Kutta (RK4) method for time integration and a pseudo-spectral Fourier approach for spatial discretization. In this method [13], the spatial derivatives are computed using Fourier transforms, while periodic boundary conditions are imposed to ensure consistency with the problem's periodicity. An integrating factor is implemented, incorporating only the dispersive terms. It employs an initial

Table 4.1: Summary of Outcomes of Wavepacket-Mean Flow Interactions

| | KPII ($\sigma = 1$) | KPI ($\sigma = -1$) | KdV ($l = 0$) |
|---|---|---|---|
| $X_+ > 0$ | Always transmits | Transmits if $k_+ > \frac{\lvert q_+\rvert + \sqrt{2(\overline{u}_+ - \overline{u}_-)}}{\sqrt{3}}$<br>No interaction if $k_+ < \frac{\lvert q_+\rvert}{\sqrt{3}}$<br>Ejection from right edge of RW if $\frac{\lvert q_+\rvert}{\sqrt{3}} < k_+ < \frac{\lvert q_+\rvert + \sqrt{2(\overline{u}_+ - \overline{u}_-)}}{\sqrt{3}}$ | Transmits if $k_+ > \sqrt{\frac{2(\overline{u}_+ - \overline{u}_-)}{3}}$<br>Trapped otherwise |
| $X_+ < 0$ | No interaction | Transmits if $k_- < \frac{\lvert q_-\rvert}{\sqrt{3}}$<br>No interaction otherwise | No interaction |

wavepacket of the form

$$\frac{1}{2}a_+ \exp\left(-\left(\frac{X-X_+}{w}\right)^2-\left(\frac{Y-Y_+}{w}\right)^2\right)\cos\left(k_+\left(X-X_+\right)+l\left(Y-Y_+\right)\right) \tag{4.1}$$

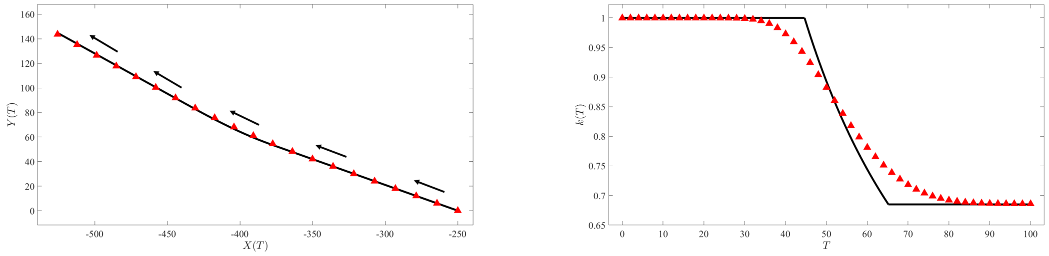
and initial mean flow of the form

$$\frac{1}{2}\left(\tanh\left(-\frac{(X-X_0)}{w}+1\right)\cdot\tanh\left(-\frac{(X+X_0)}{w}+1\right)-1+2\cdot\text{wavepacket}\right), \tag{4.2}$$

where $w$ is the initial width of the wavepacket and $X_0$ is the center of the RW wave at time 0. To extract the position of the wavepacket, we carry out a center of mass calculation with the amplitude, finding the average position [7]

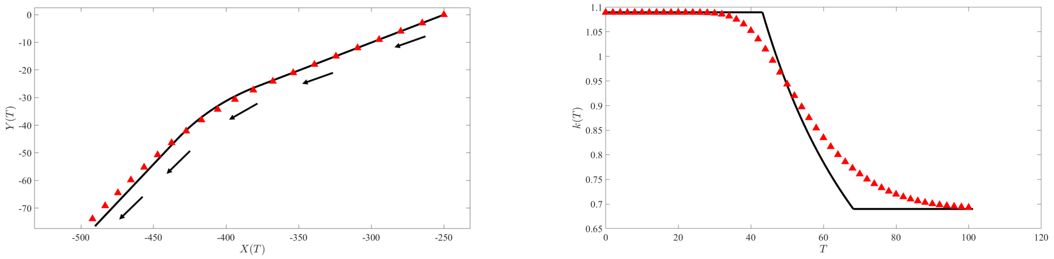
$$X(T)=\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a^2(X,Y,T)X\,dX\,dY}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a^2(X,Y,T)\,dX\,dY},\quad Y(T)=\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a^2(X,Y,T)Y\,dY\,dX}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}a^2(X,Y,T)\,dY\,dX}. \tag{4.3}$$

The integrals are evaluated numerically using a double sum. In cases of transmission, we see agreement of the numerically extracted wavepacket position with the analytically computed positions from the previous chapter. These cases can be seen in figures 4.3, 4.5, and 4.4. A large deviation between the analytical and numerical solutions can be seen in figure 4.6, during the interaction scenario where ejection occurs. We discuss this case further in the next section.



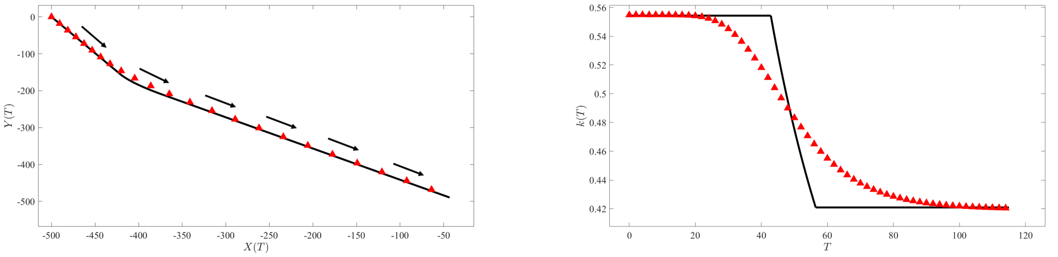
(a) (b)

Figure 4.3: (a) Wavepacket trajectory in KPII with $k_+ = 1$, $q_+ = 0.6$, $X_+ = -250$, $\overline{u}_\pm = \pm 1/2$, and RW centered at $X = -400$. Solid lines correspond to the analytical solution and triangles correspond to the numerical result. Wavepacket transmits. (b) The corresponding temporal variation of the wavepacket wavenumber $k$.



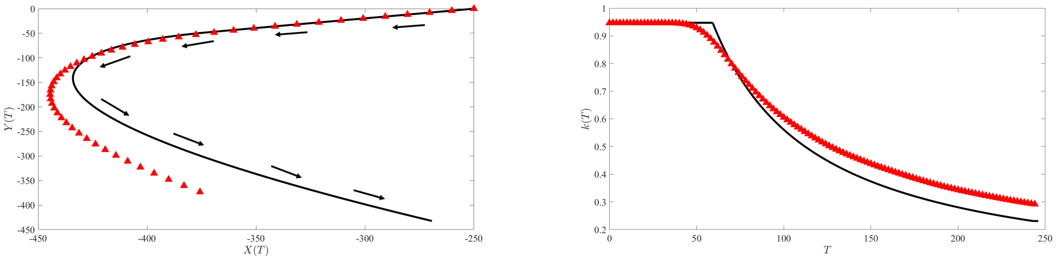
(a) (b)

Figure 4.4: (a) Wavepacket trajectory in KPI with $k_+ = \frac{q_+ + \sqrt{2}}{\sqrt{3}}$, $q_+ = 0.3$, $X_+ = -250$, $\overline{u}_\pm = \pm 1/2$, and RW centered at $X = -400$. Solid lines correspond to the analytical solution and triangles correspond to the numerical result. Wavepacket transmits. (b) The corresponding temporal variation of the wavepacket wavenumber $k$.



(a) (b)

Figure 4.5: (a) Wavepacket trajectory in KPI with $k_+ = \frac{q_+}{\sqrt{3}}$, $q_+ = 0.3$, $X_+ = -250$, $\overline{u}_\pm = \pm 1/2$, and RW centered at $X = -400$. Solid lines correspond to the analytical solution and triangles correspond to the numerical result. Wavepacket transmits. (b) The corresponding temporal variation of the wavepacket wavenumber $k$.



(a) (b)

Figure 4.6: (a) Wavepacket trajectory in KPI with $k_+ = \frac{q_+ + \sqrt{2}}{\sqrt{3}} - 0.1$, $q_+ = 0.4$, $X_+ = -250$, $\overline{u}_\pm = \pm 1/2$, and RW centered at $X = -400$. Solid lines correspond to the analytical solution and triangles correspond to the numerical result. Wavepacket is ejected. (b) The corresponding temporal variation of the wavepacket wavenumber $k$.

## 4.3 Amplitude

Recall the final solution found for the evolution of the amplitude of a plane wave (3.31). We note that if we take the limit of $q_+ \to 0$, the expression becomes

$$a_- = a_+ \left( \frac{3k_+^2 x_+^4}{3k_+^2 x_+^4} \right)^{\frac{1}{2}} = a_+,$$

and we recover the conservation of amplitude found in the KdV case [7]. An unexpected aspect of the solution that is introduced by the nonzero $l$ component is seen in KPI. While the amplitude decreases during the interaction in KPII, (3.31) indicates that the amplitude will increase during an interaction in KPI. When $\sigma = -1$, the term in the denominator decreases in magnitude, thus resulting in amplitude growth during this period. In order for this solution to remain valid, the term inside the square root must remain positive, and the denominator must remain non-zero. It can be seen that, under the conditions that result in an interaction, the denominator will always reach zero before the whole term becomes negative. This time occurs at

$$t_{\text{blowup}} = \left( \frac{3k_+^2 x_+^4}{q_+^2 \left(3k_+^2 - q_+^2\right)^4} \right)^{\frac{1}{4}} = \left( \frac{3k_+^2 t_+^4}{q_+^2} \right)^{\frac{1}{4}}. \tag{4.4}$$

In order to avoid reaching the singularity in the amplitude solution, the interaction time must be less than this value, i.e. $t_-$ or $t_{\text{ejec}} < \left( \frac{3k_+^2 t_+^4}{q_+^2} \right)^{1/4}$. In the case of transmission, this requirement is always satisfied. The wavepacket will always transmit, and thus move to the constant amplitude solution $a_-$, before $T = t_{\text{blowup}}$. However, when the wavepacket is expected to eject, this value is always reached, i.e., $t_{\text{ejec}} < t_{\text{blowup}}$. This unbounded growth in amplitude is not reflected in the numerical simulations run with the initial conditions that result in the singularity in the analytical solution. We do not expect this to point to a physical phenomenon occurring during the interaction. Rather, this singularity may be a result of a breakdown in the linear theory in the non-transmission cases of KPI. Interestingly, we do observe a much larger deviation of the analytical trajectories from the numerically extracted trajectories in the ejection case of KPI (figure 4.6) and the time of significant deviation seems to correspond to the $t_{\text{blowup}}$, though this has not yet been rigorously studied. It should also be noted that the time of ejection seen in the numerical results does not match the analytical solution.

# Chapter 5

# Conclusion

In this thesis, we studied the dynamics of wavepacket-mean flow interactions governed by the KP equation and its approximate solutions. We explored the derivation of the KP equation through the method of multiple scales, and by expanding from the KdV framework with the inclusion of weak transverse variation [1]. We then examined the derivation of the associated Whitham modulation system, both through the method of multiple scales and the averaged Lagrangian approach. Each method provided insight into the structure of the KP Whitham modulation system, and the equations governing the wavepacket-mean flow interaction problem. We then employed these equations with the RW mean flow solution and utilized the method of characteristics to obtain analytical solutions for the wavepacket trajectories, and the plane wave amplitude evolution. We were then able to characterize distinct interaction regimes, characterized by transmission and, uniquely with KP, ejection. The conservation equation computed from the Riemann invariants verified these bifurcations. We found that transmission occurred in all cases of KPII if the wavepacket is approaching from the right, and that no interaction is possible if approaching from the left. In KPI with a wavepacket approaching from the right, we see no interaction if $k_+ < \frac{|q_+|}{\sqrt{3}}$, ejection if $\frac{|q_+|}{\sqrt{3}} < k_+ < \frac{|q_+|+\sqrt{2(\overline{u}_+-\overline{u}_-)}}{\sqrt{3}}$, and transmission if $k_+ > \frac{|q_+|+\sqrt{2(\overline{u}_+-\overline{u}_-)}}{\sqrt{3}}$. If the wavepacket is approaching from the left in KPI, we see transmission if $k_- < \frac{|q_-|}{\sqrt{3}}$, and no interaction otherwise. With the addition of transverse effects, we have introduced the possibility of ejection and eliminated the possibility of trapping. We have also introduced the possibility of interaction when approaching from the left, successfully adding to previous results from purely (1+1)-dimensional

KdV theory. Numerical simulations showed agreement in our analytical trajectory solutions in the cases of transmission, and further validated the conditions under which each type of interaction (or non-interaction) occurs.

There are many possibilities in which further research could extend from this work, such as the study of different mean flow solutions. Namely, the DSW solution would readily follow as this interaction problem exhibits hydrodynamic reciprocity – the transmission conditions can be extracted using time reversibility [7]. Other solutions, such as the steady flow around a corner, could also be studied. Additionally, further, rigorous study into the potential need for a fully nonlinear model in certain KPI regimes could lead to interesting findings not fully explored in this thesis, as well as insight into the limitations of the linear theory employed here.

Further numerical analysis could also be conducted, where the extraction of the dominant wavenumber throughout interaction and computation of the integral of wave action at pre- and post-interaction times could be used for validation of analytical results.

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