Some Improved Bounds on the Number of 1-Factors of n-Connected Graphs

by

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1. Introduction

There is an interesting relation between connectivity and the number of distinct 1-factors in a graph. Beineke and Plummer [1] showed that if a graph has a 1-factor and is n-connected, it has n 1-factors. Zaks [3] sharpened the bound on the number of 1-factors to $n(n-2)(n-4)\cdot\ldots\cdot 5\cdot 3$ for n odd and $n(n-2)(n-4)\cdot\ldots\cdot 4\cdot 2$ for n even. This bound is exact for the complete graphs K_{n+1} (n odd) and the party graph P_6 (n=4). We show these are the only graphs that achieve the bound for $n \geq 3$, and improve the bound for the remaining cases.

2. Preliminaries

This section gives some definitions and previous results. For terms not defined here, see [2].

A graph G that has two or more vertices is <u>connected</u> (<u>1-connected</u>) if there is a path between any two vertices. For a positive integer n, G is <u>n-connected</u> if, when any m vertices v_i, $0 \leq m < n$, are deleted, the resulting graph $G - \{v_i | i=1,\ldots,m\}$ is connected. (Note an n-connected graph has at least n+1 vertices.) G has <u>connectivity n</u> if it is n-connected but not (n+1)-connected. If G is connected but $G - \{v\}$ is not, v <u>separates</u> G.

A <u>1-factor</u> (<u>perfect matching</u>) of G is a subgraph with exactly one edge incident to every vertex of G. A vertex v is <u>totally covered</u> if every edge vw incident to v is in some 1-factor of G. A fundamental result is the following:

<u>Lemma 1</u> [3]: A 2-connected graph that has a 1-factor has at least two totally covered vertices.

Define <u>F(G)</u> as the number of distinct 1-factors of G. For any positive integer n, define <u>f(n)</u> as the largest integer such that if G is an n-connected graph with a 1-factor, then $F(G) \geq f(n)$. Define g(n), a variant of the factorial function, by $g(n) = n$ for $n=1,2$, and $g(n) = ng(n-2)$ for $n \geq 3$. Thus $g(n) = n(n-2)\cdot\ldots\cdot 5\cdot 3$ or $n(n-2)\cdot\ldots\cdot 4\cdot 2$, depending on whether n is odd or even. An induction using Lemma 1 shows $f(n) \geq g(n)$ for $n \geq 1$.

Define f*(n) as the second-lowest possible number of 1-factors for an n-connected graph. That is, $f^*(n)$ is the largest integer such that if G is an n-connected graph with $F(G) > f(n)$, then $F(G) \geq f^*(n)$.

The complete graph K_n consists of n vertices and all possible edges between them. If n is odd, K_{n+1} is n-connected and has a 1-factor. In fact, $F(K_{n+1})=g(n)$; so $f(n)=g(n)$ for n odd.

If n is even, the party graph P_n is K_n with the edges of a 1-factor deleted. P_n is (n-2)-connected. It is easy to check $F(P_4)=2$, $F(P_6)=8$, and $F(P_{n+2})=n(F(P_n) + F(P_{n-2}))$ for $n \geq 6$.

3. The New Bounds

We begin by analyzing 3- and 4- connected graphs, confirming conjectures of Zaks on $f^*(3)$ and $f^*(4)$.

Lemma 2: $f(3) = 3$; K_4 is the only 3-connected graph G with $F(G) = 3$; $f^*(3) = 4$.

Proof: Since $F(K_4) = 3$, we have $f(3) \leq 3$. Similarly, $f^*(3) \leq 4$ is shown by the graph consisting of a cycle on six vertices, $v_1v_2v_3v_4v_5v_6$, plus the edges v_1v_4, v_2v_6, v_3v_5. Lemma 1 implies $f(3) \geq 3$, so $f(3) = 3$.

It remains only to show that if $G \neq K_4$ is a 3-connected graph with a 1-factor, then $F(G) \geq 4$. We assume $F(G) = 3$ and derive a contradiction.

G contains a totally covered vertex t, by Lemma 1. The assumptions imply t is adjacent to exactly three vertices v_i, $i=1,2,3$. Further, each graph $G - \{t,v_i\}$ has exactly one 1-factor, i.e., $F(G - \{t,v_i\}) = 1$.

Using Lemma 1, we see $G - \{t,v_i\}$ is not 2-connected; however it is 1-connected. Thus there is a vertex c_i separating $G - \{t,v_i\}$ i.e., $G - \{t,v_i,c_i\}$ is not connected. Note $c_i \neq v_j$ for any i,j, since $G - \{t,v_i,v_j\}$ is connected. (The assumption $G \neq K_4$ is used here.)

Let v_i,v_j,v_k be the vertices adjacent to t, in some order. Graph $G - \{v_i,c_i\}$ is separated by t. Thus v_j and v_k are in different components of $G - \{t,v_i,c_i\}$.

The proof is completed by contradicting this fact, i.e., we find a path between v_j and v_k in $G - \{t,v_i,c_i\}$. We do this by finding paths between v_j and c_k, v_k and c_j, and c_j and c_k.

Graph $G - \{t,c_i\}$ is separated by v_i. So there is a simple path P between v_j and v_k, containing v_i. The part of P between v_j and v_i is in $G - \{t,v_k\}$; it contains c_k, since v_j and v_i are in different components of $G - \{t,v_k,c_k\}$. This gives the desired path between v_j and c_k. Similarly the part of P between v_k and v_i gives the desired path between v_k and c_j. Note further, vertices c_i,c_j, and c_k are distinct.

Now we find the desired path between c_j and c_k. Since $G - \{t,v_i,v_j\}$ is connected, it contains a path Q between c_j and c_k, and also a path between c_j and c_i. Assume without loss of generality that Q does not contain c_i. (If it does, interchange indices i and j.) So Q is in $G - \{t,v_i,c_i\}$, and is the last desired path. QED

Lemma 3: $f(4) = 8$; P_6 is the only 4-connected graph G with $F(G) = 8$; $f^*(4) = 10$.

Proof: We see $f(4) \leq 8$ and $f^*(4) \leq 10$ by counting the number of 1-factors in P_6 and P_6+e, where the latter graph is P_6 plus one extra edge.

Let G be a 4-connected graph with a 1-factor. Let t be a totally covered vertex and let tv be an edge. Then $G - \{t,v\}$ is 2-connected, so $F(G - \{t,v\}) \geq 2$. Since t has degree at least four, we see $F(G) \geq 8$. Thus $f(4) = 8$.

It remains only to show that $F(G) \geq 10$ if $G \neq P_6$. The argument divides into two cases.

Case 1: There is a totally covered vertex t and an edge tv, such that $G - \{t,v\}$ is 3-connected.

Apply the results of Lemma 2 to $G - \{t,v\}$. If $G - \{t,v\} \neq K_4$, then $F(G - \{t,v\}) \geq 4$, so $F(G) \geq 4+3 \cdot 2 = 10$, as desired. Otherwise if $G - \{t,v\} = K_4$, graph G has six vertices and contains at least the edges of P_6+e. Thus $F(G) \geq F(P_6+e) = 10$.

Case 2: For every totally covered vertex t and every edge tv, $G - \{t,v\}$ has connectivity 2.

If a totally covered vertex has degree five or more, then $F(G) \geq 5 \cdot 2 = 10$, as desired. Thus assume all totally covered vertices have degree four. Assume further that $F(G) < 10$. Below we deduce $G = P_6$.

Let t be a totally covered vertex, adjacent to vertices v_i, $i=1,2,3,4$. We first show vertices v_i form a cycle on four vertices, with no other edges joining them.

No three vertices v_i form a cycle. For suppose $v_1v_2v_3$ is a cycle. Then since $G - \{v_4\}$ is 3-connected, $G - \{t,v_4\}$ is 3-connected. But this violates the assumption of Case 2.

Since $F(G) < 10$, assume without loss of generality that $F(G - \{t,v_i\}) = 2$, for $i=1,2,3$. Since $G - \{t,v_i\}$ is 2-connected, it has two totally covered vertices. Both have degree two in $G - \{t,v_i\}$. Any vertex has degree at least four in G. So both totally covered vertices of $G - \{t,v_i\}$ are adjacent to t and v_i. In particular, both are among the vertices v_j.

Thus each vertex v_i, $i=1,2,3$, is adjacent to two other vertices v_j. Since no three vertices v_j form a cycle, it is easy to see the four vertices v_j form a cycle, with no other edges.

Now we find other totally covered vertices besides t. Without loss of generality, assume the cycle found above is $v_1v_2v_3v_4$. Vertex v_2 is totally covered in G $-\{t,v_i\}$, for i=1,3. Thus v_2 is totally covered in G. This in turn implies v_1 is totally covered in G.

The above argument shows the totally covered vertex v_2 is adjacent to four vertices joined in a cycle. This cycle is v_1tv_3s, where s is a vertex in G. Similarly, v_1 is adjacent to four vertices joined in a cycle, v_2tv_4s. Thus the six vertices s,t,v_i, i=1,2,3,4, and their interconnecting edges, form the party graph P_6.

G contains no other vertices, since G $-\{s,v_3,v_4\}$ is connected. Thus $G=P_6$. QED

Now we extend these results to higher connected graphs. Zaks conjectured that K_{n+1} (n odd) and P_6 (n=4) are the only graphs with exactly g(n) 1-factors, for $n \geq 3$. Theorems 4 and 5 confirm this.

Theorem 4: Let $n \geq 3$ be odd. Then f(n) = g(n); K_{n+1} is the only n-connected graph G with F(G) = g(n); $f^*(n) \geq \frac{4}{3}\, g(n)$.

Proof: We show by induction on odd n that if $G \neq K_{n+1}$ is an n-connected graph with a 1-factor, then $F(G) \geq \frac{4}{3}\, g(n)$. Lemma 2 establishes the base case, n=3. So assume the assertion holds for some $n \geq 3$. Let $G \neq K_{n+3}$ be an (n+2)-connected graph with a 1-factor. Let t be a totally covered vertex, and let tv be an edge.

Graph G $-\{t,v\}$ is n-connected and has a 1-factor. Further, it is not K_{n+1}. For otherwise, each vertex w in G $-\{t,v\}$ is adjacent to both t and v, since w has degree at least n+2 in G. This implies $G = K_{n+3}$, contrary to assumption.

The inductive assertion shows $F(G - \{t,v\}) \geq \frac{4}{3}\, g(n)$. Since there are at least n+2 vertices v, $F(G) \geq \frac{4}{3}\, g(n+2)$. QED

Theorem 5: Let $n \geq 6$ be even. Then $f(n) \geq \frac{5}{4} g(n)$.

Proof: We first show for $n=6$, $f(6) \geq \frac{5}{4} g(6) = 60$. Let G be a 6-connected graph with a 1-factor. It is easy to see a totally covered vertex t has at least six edges tv such that $G - \{t,v\} \neq P_6$. So Lemma 3 shows $F(G) \geq 6 \cdot 10 = 60$, whence $f(6) \geq 60$.

The general case now follows by induction, with $n=6$ as the base. QED

Theorem 5 gives an exact bound for $n=6$, as shown by the party graph P_8. We conjecture that for any even $n \geq 2$, $f(n) = F(P_{n+2})$.

References

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