

Cohomologous 2-cocycles are Homotopic 2-cocycles:
 k -graphs and C^* -algebras

Oliver Orejola

University of Colorado Boulder

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Defense Committee

Dr. Elizabeth Gillaspy, Thesis Advisor, Department of Mathematics

Dr. Nathaniel Thiem, Department of Mathematics

Dr. Oliver DeWolfe, Department of Physics

ABSTRACT

k -graphs equipped with a defined 2-cocycle allow one to construct examples of C^* -algebras. For some k -graph, Λ , there does not necessarily exist a unique choice of 2-cocycle, such that for Λ there may exist multiple C^* -algebras depending on one's choice of 2-cocycle. There exist relations between two defined 2-cocycles on a k -graph, *cohomology* and *homotopy*, that imply features of the C^* -algebras generated by the 2-cocycles and Λ . Cohomology implies that the two C^* -algebras are isomorphic, and homotopy implies the two share the same invariant, K-theory. It is shown here the result that any two 2-cocycles defined on a k -graph which are cohomologous are then homotopic. Also included is the method by which one may construct a matrix equation, $\Psi\vec{x} = \vec{z}$ that encodes the information of a k -graph and 2-cocycle, where the existence of an integer solution to the equation $\Psi\vec{x} = \vec{z}$ implies any two 2-cocycles are homotopic.

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1. INTRODUCTION

The result demonstrated within this paper addresses and provides proof for the following theorem:

“For a k -graph, Λ , if two 2-cocycles are cohomologous then they are homotopic.”

The interest in this conjecture arose from Gillaspy’s result that for homotopic 2-cocycles the C^* -algebras generated share the same K -theory [1], and the known result cohomologous 2-cocycles generate isomorphic C^* -algebras (the latter will be demonstrated in this paper). The notion of two isomorphic C^* -algebras, say $A \cong B$ with A and B both C^* -algebras, states there exists a structure preserving bijective map which implies preservation of invariants between A and B , such as the invariant K -theory. Given this intuition we set out to verify that the properties associated with the 2-cocycles are consistent with our expectation based on the properties on the C^* -algebras generated by their respective 2-cocycles.

In another attempt to study the properties of 2-cocycles, in particular the question of when a 2-cocycle is homotopic to the trivial 2-cocycle, I constructed a formalism for a matrix representation of any (finite) k -graph. Here in this paper I demonstrate the method of generating the matrix equation, $\Psi\vec{x} = \vec{z}$, associated to a 2-cocycle, c , and show that if there exists an integer solution \vec{x} to this equation then the 2-cocycle c is homotopic to the trivial 2-cocycle.

The structure of this paper is as follows. Definitional material and examples span section 1 - 3, results are demonstrated in section 4 & 5, and we end with future directions on the subject. What follows is a brief overview for the general contents of each section 1 - 6.

We continue within the introduction and begin by laying down basic definitions for the subject at hand: k -graphs, 2-cocycles, and C^* -algebras. We then provide simple examples for each, and give a brief description of the invariant which homotopic 2-cocycles preserve, K -theory.

In section 2, we introduce the definitions of the important operators which are needed in the construction of a C^* -algebra from a k -graph, Λ , and 2-cocycle, c . We will then introduce the relations by which one constructs the C^* -algebra $C^*(\Lambda, c)$ associated to Λ and c , and provide a simple example of a k -graph C^* -algebra.

In section 3, we define the relations of interest for a 2-cocycle: cohomology and homotopy. It is then demonstrated that cohomologous 2-cocycles generate isomorphic C^* -algebras, and we then bring attention to Gillaspy's result that homotopic 2-cocycles generate C^* -algebras who share isomorphic K -theory.

In section 4, we demonstrate the proof for our theorem, “*For a k -graph, Λ , if two 2-cocycles are cohomologous then they are homotopic.*” Then in section 5, we demonstrate the construction of the matrix equation, $\Psi\vec{x} = \vec{z}$ based on the pair (Λ, c) . In particular the matrix Ψ depends on Λ and the vectors \vec{x} and \vec{z} depend on the 2-cocycle. We also show the existence of an integer solution \vec{x} to this equation implies one has a 2-cocycle, c , which is homotopic to the trivial 2-cocycle.

We bring the paper to a close in section 6 and discuss implications of our results, and possible directions for future research.

1.1. Introduction to k -graphs and 2-cocycles.

k -graphs, sometimes referred to as *higher-rank graphs*, are category theoretic objects. These k -graphs were first formally defined by Kunjiam and Pask [2] as follows:

Definition 1.1.1. A k -graph is a small category Λ with a functor $d: \Lambda \rightarrow \mathbb{N}^k$ satisfying a *factorization property*: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$, $d(\nu) = n$.

For our purposes a k -graph is most easily understood as a generalization of an edge-colored directed graph, equipped with a set of equivalence classes defined on paths in the graph, where k is a natural number describing the k colors edges of a graph. We can express a k -graph as a skeleton with factorization rules.

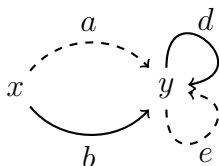
Definition 1.1.2. A *directed graph* is a set of vertices, \mathbb{V} , edges, $\mathbb{E} = \{e = (v_r, v_s) : v_r, v_s \in \mathbb{V}\}$. For an edge $e = (v_r, v_s)$ we define the range of an e to be $r(e) = v_r$ and the source of e as $s(e) = v_s$.

Definition 1.1.3. A *skeleton* is an edge colored directed graph. Let k denote the number of edge colors and c denote the color of the edge for $c \leq k$ then we denote the set of edges $\mathbb{E} = \{(v_r, v_s)^c : v_r, v_s \in \mathbb{V}\}$.

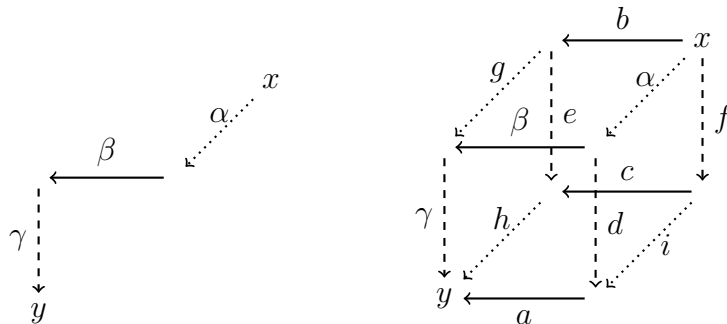
Definition 1.1.4. Let $f, g, h \in \mathbb{E}$. We say the composition of edges $f \circ g$ (denoted fg) is a *path of length 2 from vertex x to y* if $s(f) = r(g)$ and $r(f) = y$ and $s(g) = x$. We specify that the composition of n edges is a *path of length n* . This is generalized for the composition of multiple edges, such that whenever f, g, h are paths or edges in the skeleton we have:

- $r(fg) = r(f)$ and $s(fg) = s(g)$
- $(fg)h = f(gh)$ when $s(f) = r(g)$ and $s(g) = r(f)$

Example 1.1. Below is an example of a skeleton. Let $c = 1$ denote a dotted edge and $c = 2$ denote a solid edge. Then we have $\mathbb{E} = \{a = (y, x)^1, b = (y, x)^2, d = (y, y)^2, e = (y, y)^1\}$, and $\mathbb{V} = \{x, y\}$.



Definition 1.1.5. Let $p_i \in \mathbb{E}$ be an edge with color i on a skeleton with k colors. For any length-3 3 color path $p = p_1p_2p_3$ and any permutation $\phi : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$, a *factorization rule* associates a unique path $q_{\phi(1)}q_{\phi(2)}q_{\phi(3)} = q$, with $r(p) = r(q)$ and $s(p) = s(q)$, such that $q_i \in \mathbb{E}$ with color i . We say two paths are equivalent if they are associated to the same path $p_1p_2p_3 = p$ by this relation. By Theorem 4.4 in Hazelwood, Raburn, Sims, and Webster [3] we get a factorization for paths of all length. We call the set of equivalence classes generated by this relation the *Factorization*, denoted \mathbb{F} . That is, for any 3-path say, $\gamma\beta\alpha$, with $s(\gamma\beta\alpha) = x$ and $r(\gamma\beta\alpha) = y$ we can associate a cube.



We also require that if the same 2-path exists in multiple cubes, the associated squares are the same.

To elaborate upon the notion of factorization we can define factorization on the skeleton shown in example 1.1. Consider the possible length-2 2 color paths: the dashed-solid path ad and the solid-dashed path be from vertex x to vertex y , and the dashed-solid path de and the solid-dashed path ed from vertex y to vertex x .

$$\frac{\text{Length-2 Paths } x \rightarrow y}{\begin{array}{l} \text{dashed-solid path } da \\ \text{solid-dashed path } eb \end{array}} \quad \frac{\text{Length-2 Paths } y \rightarrow x}{\begin{array}{l} \text{dashed-solid path } de \\ \text{solid-dashed path } ed \end{array}}$$

A factorization rule must associate each dashed-solid path to a unique solid-dashed path. In this simple case, we must have the single dashed-solid path equivalent to a solid-dashed path, hence we have the following factorization:

$$\begin{aligned} da &\equiv eb, \mu = [ad], \mu \in \mathbb{F} \\ de &\equiv ed, \eta = [de], \eta \in \mathbb{F} \end{aligned}$$

A useful way to visualize the elements of \mathbb{F} for a 2-graph is with factorization squares. For example 1.1. we have the factorization square for μ ,

$$\mu = \begin{array}{ccc} & & a \\ & y \leftarrow \cdots & x \\ d \downarrow & & \downarrow b \\ & y \leftarrow \cdots & y \\ & & e \end{array}$$

From the definition of a skeleton and factorization we can define a k -graph as the following:

Definition 1.1.6. A k -graph is a skeleton together with a choice of factorization.

With this definition of a k -graph, we have the skeleton shown in Example 1.1 is a valid k -graph as there exists factorization for the skeleton. It is important to note that for some skeleton the choice of factorization is not unique.

To see how the skeleton and factorization give rise to the category-theoretic k -graph, we observe that the factorization rules provide us with the factorization property described in Definition 1.1.1. Suppose we have a skeleton equipped with some factorization, Λ , with the elements of Λ the equivalence classes of paths on the skeleton. Let us construct a map $d : \Lambda \rightarrow \mathbb{N}^k$ such that $d((y, x)^e) = (\delta_{1,e}, \dots, \delta_{k,e})$ with

$\delta_{i,j} = 0$ if $j \neq i$, otherwise $\delta_{i,i} = 1$. For a path $\eta = e_1 e_2 \cdots e_n$ consisting of n edges, we define $d(\eta) = \sum_{i=1}^n d(e_i)$. Suppose we have a arbitrary length- n path $p \in \Lambda$ with $p = [p_1 \dots p_n]$ such that $d(p) = v$ where the sum of the entries of v is equal to n . If $v = u + w$ by the factorization rules we have $p = [p_1 \dots p_n] = [q_1 \dots q_m][j_1 \dots j_l]$ with m equal to the sum of the entries in u and l equal to sum of the entries in w . Let $q = [q_1 \dots q_m]$ and $j = [j_1 \dots j_l]$ such that $d(q) = u$, $d(j) = w$, and $p = qj$, hence our factorization rules satisfy the factorization property in Definition 1.1.1.

To further illustrate how factorization rules provide us with the factorization property, recall the 2-graph in Example 1.1. Consider the path $edda$, this path is composed of two solid and two dashed edges, so

$$d([edda]) = (2, 2) \in \mathbb{N}^2$$

Note that $(2, 2) = (1, 1) + (1, 1)$. By our choice of factorization rules in Example 1.1, we have

$$edda = edbe = deda = debe$$

Immediately we see that we have $[ed][da] = [edda]$ with $d([ed]) = (1, 1)$ and $d([da]) = (1, 1)$ demonstrating that the factorization rules for a skeleton provide us with the category theoretical factorization property.

Definition 1.1.7. We call the *Factorization Squares*, denoted \mathbb{P} , the set of equivalence classes who compose the faces of the factorization cube built from the factorization of a skeleton as shown in Definition 1.1.5.

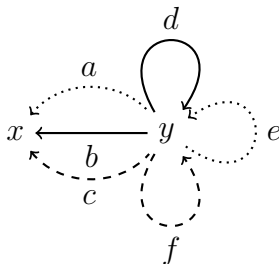
Definition 1.1.8. The *unit circle*, denoted \mathbb{T} , is a subset of complex numbers defined as $\mathbb{T} \equiv \{e^{2\pi i\theta} : \theta \in \mathbb{R}\}$ (note for $k \in \mathbb{Z}$, $e^{i2\pi k} = 1$).

Definition 1.1.9. Let Λ be a k -graph and \mathbb{P} it's factorization squares. A *2-cocycle* is a function $c: \mathbb{P} \rightarrow \mathbb{T}$, $\mu \mapsto e^{2\pi i\tilde{\mu}}$, $\tilde{\mu} \in \mathbb{R}$, such that for every factorization cube generated by the factors \mathbb{F} of Λ .

$$(1.1) \quad c(\mu_{top})c(\mu_{left})c(\mu_{front}) = c(\mu_{bottom})c(\mu_{back})c(\mu_{right})$$

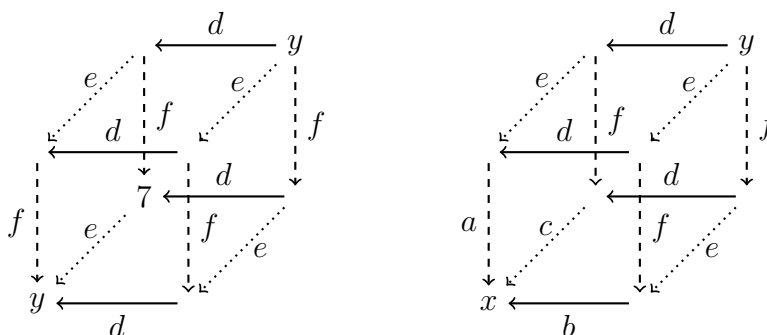
We call the above equation the *2-cocycle condition*, and generally denote $c(\mu) = e^{2\pi i\tilde{\mu}}$. (note $\tilde{\mu}$ is not uniquely defined)

Example 1.2. Consider the following skeleton with 3 edge colors:



We can see we have only 2 paths of 3 colors: a path from $y \rightarrow y$ and $y \rightarrow x$. the cubes pictured below show that we are capable of constructing consistent factorization cubes for this skeleton, demonstrating that this directed graph is a skeleton for a 3-graph.

Example 1.2 Factorization cubes



Hence, this skeleton and factorization gives us a 3-graph. We can visualize the equivalence classes for our 3-graph as the set of all possible paths along the edges of the cubes. From these cubes we we can gather factorization squares, such that we have the set of equivalence classes \mathbb{P}

Example 1.2 Factorization squares

$$\begin{aligned} A = de \equiv ed \quad B = ef \equiv fe \quad C = fd \equiv df \\ D = cf \equiv ae \quad E = ad \equiv bf \quad F = cd \equiv eb \end{aligned}$$

Let us continue this example further by constructing a 2-cocycle $c : \mathbb{P} \rightarrow \mathbb{T}$. Since a 2-cocycle must satisfy the 2-cocycle condition for each factorization cube of a k -graph we have

$$(1.2) \quad c(A)c(B)c(C) = c(A)c(C)c(B)$$

$$(1.3) \quad c(A)c(D)c(E) = c(F)c(C)c(B)$$

With equation (1.2) a trivial relationship we only have to concern ourselves with equation (1.3). Let $c(A) = i$, $c(D) = -i$, $c(E) = -1$, $c(C) = i$, $c(F) = i$, $c(B) = 1$, and observe that equation (1.3) evaluates as

$$(i)(-i)(-1) = -1 = (i)(i)(1)$$

In other words, this choice of c is a 2-cocycle.

A particularly useful, although not very interesting, 2-cocycle is the *trivial 2-cocycle*. Every k -graph admits the *trivial 2-cocycle*, defined by $c_{triv}(\mu) = 1 \forall \mu \in \mathbb{P}$.

Proposition 1.1. Let Λ be a k -graph and let σ and λ be 2-cocycles defined on Λ . The product of two 2-cocycles, $\sigma\gamma : \mathbb{P} \rightarrow \mathbb{T}$, given by $\sigma\gamma(\mu) = \sigma(\mu)\gamma(\mu) \forall \mu \in \mathbb{P}$, is a 2-cocycle.

Proof. With σ and γ 2-cocycles we have $\sigma : \mathbb{P} \rightarrow \mathbb{T}$ and $\gamma : \mathbb{P} \rightarrow \mathbb{T}$, and both σ and γ satisfy the 2-cocycle condition such that for every factorization cube in Λ we have

$$\begin{aligned} \sigma(\mu_{top})\sigma(\mu_{left})\sigma(\mu_{front}) &= \sigma(\mu_{bottom})\sigma(\mu_{back})\sigma(\mu_{right}) \\ \gamma(\mu_{top})\gamma(\mu_{left})\gamma(\mu_{front}) &= \gamma(\mu_{bottom})\gamma(\mu_{back})\gamma(\mu_{right}) \end{aligned}$$

Take the product of these relations; we see that

$$\begin{aligned} \sigma(\mu_{top})\sigma(\mu_{left})\sigma(\mu_{front})\gamma(\mu_{top})\gamma(\mu_{left})\gamma(\mu_{front}) &= \\ \sigma(\mu_{bottom})\sigma(\mu_{back})\sigma(\mu_{right})\gamma(\mu_{bottom})\gamma(\mu_{back})\gamma(\mu_{right}) & \end{aligned}$$

Take $\sigma\gamma(\mu) = \sigma(\mu)\gamma(\mu) = e^{2\pi i(\tilde{\mu}^\sigma + \tilde{\mu}^\gamma)}$ since multiplication in \mathbb{T} is commutative, by rearranging terms we have the 2-cocycle condition satisfied by $\sigma\gamma$

$$\sigma\gamma(\mu_{top})\sigma\gamma(\mu_{left})\sigma\gamma(\mu_{front}) = \sigma\gamma(\mu_{bottom})\sigma\gamma(\mu_{back})\sigma\gamma(\mu_{right})$$

Therefore the product of two 2-cocycles, $\sigma\gamma$, is a 2-cocycle. □

Definition 1.1.10. For a 2-cocycle σ on a k -graph Λ we call the *multiplicative inverse* of σ , σ^{-1} if for every factorization square $\mu \in \mathbb{P}$, $\sigma(\mu)\sigma(\mu)^{-1} = 1$ with $\sigma(\mu) = e^{2\pi i\tilde{\mu}}$ and $\sigma(\mu)^{-1} = e^{-2\pi i\tilde{\mu}}$.

Proposition 1.2. For a 2-cocycle σ on a k -graph Λ we have the multiplicative inverse of σ , $\sigma^{-1} : \mathbb{P} \rightarrow \mathbb{T}$, is a 2-cocycle.

Proof. Suppose we have a 2-cocycle $\sigma : \mathbb{P} \rightarrow \mathbb{T}$, $\mu \mapsto e^{2\pi i\tilde{\mu}}$ on a k -graph Λ . Then with σ a 2-cocycle we have the 2-cocycle condition satisfied for each factorization cube, that is

$$\begin{aligned} \sigma(\mu_{top})\sigma(\mu_{left})\sigma(\mu_{front}) &= \sigma(\mu_{bottom})\sigma(\mu_{back})\sigma(\mu_{right}) \\ &\Downarrow \\ e^{2\pi i(\widetilde{\mu_{top}} + \widetilde{\mu_{left}} + \widetilde{\mu_{front}})} &= e^{2\pi i(\widetilde{\mu_{bottom}} + \widetilde{\mu_{back}} + \widetilde{\mu_{right}})} \end{aligned}$$

Then with the inverse of the 2-cocycle condition for σ we have

$$\begin{aligned} e^{-2\pi i(\widetilde{\mu_{top}} + \widetilde{\mu_{left}} + \widetilde{\mu_{front}})} &= e^{-2\pi i(\widetilde{\mu_{bottom}} + \widetilde{\mu_{back}} + \widetilde{\mu_{right}})} \\ &\Downarrow \\ e^{2\pi i(-\widetilde{\mu_{top}} - \widetilde{\mu_{left}} - \widetilde{\mu_{front}})} &= e^{2\pi i(-\widetilde{\mu_{bottom}} - \widetilde{\mu_{back}} - \widetilde{\mu_{right}})} \\ &\Downarrow \\ \sigma(\mu_{top})^{-1}\sigma(\mu_{left})^{-1}\sigma(\mu_{front})^{-1} &= \sigma(\mu_{bottom})^{-1}\sigma(\mu_{back})^{-1}\sigma(\mu_{right})^{-1} \end{aligned}$$

Hence σ^{-1} satisfies the 2-cocycle condition for the factorization cubes, therefore $\sigma^{-1} : \mathbb{P} \rightarrow \mathbb{T}$ is also a 2-cocycle on Λ . \square

1.2. Introduction to C^* -algebras.

C^* -algebras are intuitively an algebra of operators who act on a finite or infinite dimensional vector space. We begin by introducing the notion of a Hilbert space, a particularly useful vector space on which the elements of a C^* -algebra act.

To define both a C^* -algebra and Hilbert space we must recall the notion of a Cauchy sequence.

Definition 1.2.1. For a metric space \mathcal{X} with a distance function, $\|\cdot\|$, the sequence in \mathcal{X} , call it $\{v_n\}_{n \in \mathbb{N}}$ is *Cauchy* if, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m, n \geq N$ we have $\|v_n - v_m\| < \epsilon$. We say the metric space \mathcal{X} is *complete* if every Cauchy sequence in \mathcal{X} converges in \mathcal{X} .

Definition 1.2.2. A *Hilbert space* is a complete complex vector space \mathcal{H} equipped with a complex valued inner product, namely a map

$$\langle, \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$$

satisfying for all vectors $f, f', g \in \mathcal{H}$ and constants $\lambda \in \mathbb{C}$

- $\langle f, g \rangle = \overline{\langle g, f \rangle}$
- $\langle f + f', g \rangle = \langle f, g \rangle + \langle f', g \rangle$
- $\langle \lambda f, g \rangle = \lambda \langle f, g \rangle$
- $\langle f, f \rangle \geq 0$ with equality if and only if $f = 0$

with a norm defined as

$$\|f\| = \sqrt{\langle f, f \rangle}$$

We require \mathcal{H} to be complete in the norm: that is, for every sequence $\{f_n\}_{n \in \mathbb{N}} \subset \mathcal{H}$ that is Cauchy with respect to the norm, $\|\cdot\|$, we require that the limit of $\{f_n\}$ is in \mathcal{H} .

Definition 1.2.3. If \mathcal{H}, \mathcal{K} are Hilbert space, a function $T : \mathcal{H} \rightarrow \mathcal{K}$ is a *bounded linear operator* if:

- For any $f, g \in \mathcal{H}, \alpha, \beta \in \mathbb{C}, T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$
- $\sup_{v \in \mathcal{H}, v \neq 0} \frac{\|T(v)\|}{\|v\|} < \infty$

The quantity mentioned in the latter requirement is what is known as the operator norm.

Definition 1.2.4. For a two Hilbert spaces \mathcal{H}, \mathcal{K} , and a linear operator $T : \mathcal{H} \rightarrow \mathcal{K}$. We define the *operator norm* as

$$\|T\| = \sup_{\|v\|=1} \|T(v)\| = \inf\{c \in \mathbb{R} : \|T(v)\| \leq c\|v\|\}$$

Then, point 2 in Definition 1.2.3 implies that the operator norm of T is bounded.

We write $\mathcal{B}(\mathcal{H}, \mathcal{K})$ for the bounded linear operators from Hilbert spaces \mathcal{H} to \mathcal{K} . If $\mathcal{H} = \mathcal{K}$, we write $\mathcal{B}(\mathcal{H})$. It is important to note the following properties of $\mathcal{B}(\mathcal{H})$:

- If $T, S \in \mathcal{B}(\mathcal{H})$ then $\alpha T + \beta S \in \mathcal{B}(\mathcal{H})$ for any $\alpha, \beta \in \mathbb{C}$
- $\|\alpha T\| = |\alpha| \|T\|$
- $\|T + S\| \leq \|T\| + \|S\|$
- If $T, S \in \mathcal{B}(\mathcal{H})$ then $T \circ S \in \mathcal{B}(\mathcal{H})$. $\|T \circ S\| \leq \|T\| \|S\|$

Definition 1.2.5. Given an operator $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, we call it's *adjoint* T^* an operator $T^* : \mathcal{K} \rightarrow \mathcal{H}$ such that we have

$$\langle T^*(f), g \rangle = \langle f, T(g) \rangle$$

Proposition 1.3. Let $T \in \mathcal{B}(\mathcal{H})$. Then

$$\|T\|^2 = \|T^*\|^2 = \|T^*T\|$$

We call Proposition 1.3. the C^* -identity.

Definition 1.2.6. A C^* -algebra is norm closed $*$ -subalgebra of $\mathcal{B}(\mathcal{H})$. That is, for $A \subset \mathcal{B}(\mathcal{H})$, we say A is a C^* -algebra if it is closed under taking finite sums, finite products, and adjoints of operators in A ; and for ever Cauchy sequence $\{T_n\}_{n \in \mathbb{N}} \subset A$, $\lim_n T_n \in A$.

Briefly, K -theory associates to each C^* -algebra A a pair of abelian groups $K_0(A)$ and $K_1(A)$ which contain a substantial amount of information about A . The precise definition is irrelevant to this work which is focuses on k -graphs and their C^* -algebras. It is important to restate the relevance of K -theory, namely that it is a particular invariant for a C^* -algebra.

2. CONSTRUCTION OF C^* -ALGEBRA

Definition 2.0.1. The *universal C^* -algebra* associated to a collection of generators $\{t_n\}_{n \in I}$ and relations is the C^* -algebra A with those generators such that, for any C^* -algebra B that is generated by $\{b_n\}_{n \in I}$ that satisfy the same relations, we have an onto $*$ -homomorphism $A \rightarrow C^*(\{b_n\}_{n \in I})$.

Definition 2.0.2. A function $\Phi : A \rightarrow B$ between two C^* -algebras is a *$*$ -homomorphism*, if for all $a, b \in A$,

- $\Phi(\alpha a + \beta b) = \alpha\Phi(a) + \beta\Phi(b)$ for all $\alpha, \beta \in \mathbb{C}$
- $\Phi(ab) = \Phi(a)\Phi(b)$
- $\Phi(a^*) = \Phi(a)^*$

There are two important types of operators in $\mathcal{B}(\mathcal{H})$ that are used in the construction of a C^* -algebra from a k -graph: partial isometries, and projections.

Definition 2.0.3. A *partial isometry* is an element $T \in \mathcal{B}(\mathcal{H})$ where we have

$$TT^*T = T$$

Definition 2.0.4. A *projection* is an operator $P \in \mathcal{B}(\mathcal{H})$ such that

$$P = P^* = P^2$$

Another important class of operator are the unitary operators.

Definition 2.0.5. For a C^* -algebra \mathcal{A} , we call an element $u \in \mathcal{A}$ a *unitary* if

$$u^*u = uu^* = 1$$

2.1. Axioms of Construction.

Suppose we have some k -graph, Λ , and a 2-cocycle c on Λ . For notational purposes we call the set of vertices in Λ , Λ^0 , and the set of edges in Λ , Λ^1 . The C^* -algebra associated to (Λ, c) , written as $C^*(\Lambda, c)$, is the universal C^* -algebra generated by a collection of $\{p_v\}_{v \in \Lambda^0}$ of projections and a collection $\{t_e\}_{e \in \Lambda^1}$ of partial isometries, one for each edge $e \in \Lambda$, satisfying the following properties:

- (1) The projections $\{p_v\}$ are mutually orthogonal:

$$p_v p_w = \delta_{vw} p_v$$

- (2) For any edge e we have $t_e^* t_e = p_{s(e)}$
 (3) If e is an edge of color i and f is an edge of color j , with $i < j$, so that $s(e) = r(f)$, use the factorization rule to write $\mu = f'e' \equiv fe$. Then,

$$t_{f'} t_{e'} = c(\mu) t_f t_e$$

- (4) For each color i of edge, and each vertex v , we have

$$p_v = \sum_{\{e \in \Lambda^1 : r(e) = v\}} t_e t_e^*$$

where the sum ranges over all edges e with range v and color i .

2.2. C^* -algebra generated from a unitary.

What follows is an example of the construction of a C^* -algebra from a k -graph. Consider the following 1-graph



Call this 1-graph, Λ , with one edge and one vertex. Note that this 1-graph does not generate any factorization squares, so there are no 2-cocycles on Λ - so relation (3) in the construction is irrelevant.

By relation (2) we have

$$(2.1) \quad p_x = t_e^* t_e$$

With $s(e) = x$, relation (4) gives us

$$(2.2) \quad p_x = t_e t_e^*$$

Then from these two equations we have

$$p_x = t_e^* t_e = t_e t_e^*$$

In order to have the C^* -algebra generated by $\{p_x\}$ and $\{t_e\}$ be universal, we require that $p_x = 1$, by 1 we mean the identity operator in $\mathcal{B}(\mathcal{H})$, hence we have

$$1 = t_e^* t_e = t_e t_e^*$$

Therefore t_e is a unitary. So $C^*(\Lambda)$ is the universal C^* -algebra generated by one unitary.

3. COHOMOLOGY AND HOMOTOPY

There are two main relations between 2-cocycles, homotopy and cohomology.

Definition 3.0.1. Let Λ be a k -graph. A *homotopy of 2-cocycles* is a family $\{c_t : t \in [0, 1]\}$ of 2-cocycles on Λ , such that for each $\mu \in \mathbb{P}$, the function $f : [0, 1] \rightarrow \mathbb{T}$ defined by $f(t) = c_t(\mu)$ is continuous.

As a result from Alex Kumjian (unpublished) we have that every homotopy of 2-cocycles is of exponential form. In other words, the following is an equivalent definition of a homotopy of 2-cocycles.

Definition 3.0.2. Let Λ be a k -graph. We say c_0 and c_1 are *homotopic 2-cocycles* on Λ if there exists a function $h : \mathbb{P} \rightarrow \mathbb{R}$ such that

$$(3.1) \quad c_0(\mu) = c_1(\mu)e^{2\pi ih(\mu)}$$

and we have $h(\mu)$ satisfying the 2-cocycle condition for \mathbb{R} , namely

$$(3.2) \quad h(\mu_{top}) + h(\mu_{left}) + h(\mu_{front}) = h(\mu_{bottom}) + h(\mu_{back}) + h(\mu_{right})$$

To see that these definitions are equivalent, note that from such a function h , we can define $d_t(\mu) = c_1(\mu)e^{2\pi it h(\mu)}$. Then d_t is a 2-cocycle for all $t \in [0, 1]$, with $d_0 = c_1$ and $d_1 = c_0$. Moreover, $t \mapsto d_t(\mu)$ varies continuously for all $\mu \in \mathbb{P}$. In other words $\{d_t : t \in [0, 1]\}$ is a homotopy in the sense of Definition 3.0.1.

Definition 3.0.3. Let Λ be a k -graph and let c_0 and c_1 be 2-cocycles on Λ . Let \mathbb{E} be the set of edges on the skeleton of Λ . We say c_0 and c_1 are *cohomologous* if there exists a function $\beta : \mathbb{E} \rightarrow \mathbb{T}$ such that, for each factorization square $\mu \in \mathbb{P}$,

$$\mu = \begin{array}{ccc} & \xleftarrow{\mu_{01}} & \\ \mu_{11} \downarrow & & \downarrow \mu_{10} \\ & \xleftarrow{\mu_{00}} & \end{array}$$

we have

$$(3.3) \quad \beta(\mu_{10})\beta(\mu_{00})c_1(\mu) = \beta(\mu_{01})\beta(\mu_{11})c_0(\mu).$$

It is useful for one to identify the condition for when we have a 2-cocycle, c , which is cohomologous to the trivial 2-cocycle, c_{triv} . With c_{triv} defined as $c_{triv}(\mu) = 1, \forall \mu \in \mathbb{P}$ we have

$$(3.4) \quad c(\mu) = \beta(\mu_{01})\beta(\mu_{11})\beta(\mu_{10})^{-1}\beta(\mu_{00})^{-1}$$

for an 2-cocycle which is cohomologous to the trivial 2-cocycle. Also, it is useful to identify the condition for when we have a 2-cocycle, c , which is homotopic to the trivial 2-cocycle. In this case,

$$(3.5) \quad c(\mu) = e^{2\pi i h(\mu)}$$

for a function $h : \mathbb{P} \rightarrow \mathbb{R}$ as in Equation (3.2).

It is important to notice that for a 1-graph these relations do not apply as there are no factorization squares given that 1-graphs consist of a skeleton of only one color. Also note that for a 2-graph, 2-cocycle condition does not make sense considering that there are no factorization cubes. In other words, a 1-graph has no 2-cocycles, and on a 2-graph any function $\sigma : \mathbb{P} \rightarrow \mathbb{T}$ is a 2-cocycle, and all 2-cocycles are homotopic to c_{triv} .

Theorem 3.1. Let Λ be a k -graph and let c_0 and c_1 be 2-cocycles on Λ . If c_0 and c_1 are both homotopic to the trivial 2-cocycle, then c_0 and c_1 are homotopic.

Proof. Suppose we have k -graph Λ and let c_0 and c_1 be 2-cocycles on Λ . Assume that c_0 and c_1 are both homotopic to the trivial 2-cocycle such that we have

$$\begin{aligned} c_0(\mu) &= e^{2\pi i h_0(\mu)} \\ c_1(\mu) &= e^{2\pi i h_1(\mu)} \end{aligned}$$

Let us define $h(\mu) = h_0(\mu) - h_1(\mu)$ such that we can write

$$c_0(\mu) = c_1(\mu)e^{2\pi i(h_0(\mu) - h_1(\mu))} = c_1(\mu)e^{2\pi i h(\mu)}$$

Given that both h_0 and h_1 satisfy the 2-cocycle condition in \mathbb{R} we have $h(\mu)$ satisfies the 2-cocycle condition as well, therefore c_0 and c_1 are homotopic. \square

3.1. 2-cocycle relations and the C^* -algebra relations.

Theorem 3.2. Let σ, γ be cohomologous 2-cocycles on a k -graph, Λ , then $C^*(\Lambda, \gamma) \cong C^*(\Lambda, \sigma)$.

Proof. Let σ, γ be cohomologous 2-cocycles defined on a k -graph, Λ such that we have the function $\beta : \mathbb{E} \rightarrow \mathbb{T}$ such that for a factorization square $\mu \in \mathbb{P}$, $\mu = \mu_{10}\mu_{00} \equiv \mu_{01}\mu_{11}$ we have

$$(3.6) \quad \sigma(\mu)\beta(\mu_{10})\beta(\mu_{00}) = \gamma(\mu)\beta(\mu_{01})\beta(\mu_{11})$$

We denote the generators of $C^*(\Lambda, \gamma)$ $\{t_e^\gamma\}_{e \in \Lambda^1}$ (the collection of partial isometries) $\{p_v\}_{v \in \Lambda^0}$ (the collection of projections), and the generators of $C^*(\Lambda, \sigma)$, $\{t_e^\sigma\}$ and $\{p_v\}$. Note that the generators of $C^*(\Lambda, \gamma)$ and $C^*(\Lambda, \sigma)$ share the same relations (1), (2), and (4) since these relations are independent of the 2-cocycle. By relation (3) we have

$$(3.7) \quad t_{\mu_{10}}^\gamma t_{\mu_{00}}^\gamma = \gamma(\mu) t_{\mu_{01}}^\gamma t_{\mu_{11}}^\gamma$$

$$(3.8) \quad t_{\mu_{10}}^\sigma t_{\mu_{00}}^\sigma = \sigma(\mu) t_{\mu_{01}}^\sigma t_{\mu_{11}}^\sigma$$

Then with equations (3.6), (3.7), and (3.8) we have

$$t_{\mu_{10}}^\gamma \beta(\mu_{01}) t_{\mu_{00}}^\gamma \beta(\mu_{11}) = \sigma(\mu) t_{\mu_{01}}^\gamma \beta(\mu_{10}) t_{\mu_{11}}^\gamma \beta(\mu_{00})$$

$$t_{\mu_{10}}^\sigma \beta(\mu_{10}) t_{\mu_{00}}^\sigma \beta(\mu_{00}) = \gamma(\mu) t_{\mu_{01}}^\sigma \beta(\mu_{01}) t_{\mu_{11}}^\sigma \beta(\mu_{11})$$

Therefore, if we set $s_e^\sigma := t_e^\sigma \beta(e)$ then since s_e^σ is a nonzero scalar multiple of the generator t_e^σ , the set $\{s_e^\sigma\} \cup \{p_v\}$ generates $C^*(\Lambda, \sigma)$ and moreover, these generators satisfy the same relation as the generators of $C^*(\Lambda, \gamma)$. By the universality of $C^*(\Lambda, \sigma)$, we have a homomorphism $\eta : C^*(\Lambda, \gamma) \rightarrow C^*(\Lambda, \sigma)$, where $\eta(t_e^\gamma) = \beta(e) t_e^\sigma$. By symmetry of the argument we can construct a homomorphism $\pi : C^*(\Lambda, \sigma) \rightarrow C^*(\Lambda, \gamma)$, $\pi(t_e^\sigma) = \beta(e)^{-1} t_e^\gamma$. Since $\pi \circ \eta = id_{C^*(\Lambda, \gamma)}$ and $\eta \circ \pi = id_{C^*(\Lambda, \sigma)}$ on the generators, these homomorphisms are isomorphism between $C^*(\Lambda, \gamma)$

and $C^*(\Lambda, \sigma)$. □

Theorem 3.3. For a k -graph Λ and homotopic 2-cocycles c_1 and c_2 we have the K -theory associated to $C^*(\Lambda, c_1)$ isomorphic to the K -theory associated to $C^*(\Lambda, c_2)$

This theorem is proven in Gillaspy's Ph.D thesis Theorem 2.5.5 [1].

4. COHOMOLOGOUS 2-COCYCLES ARE HOMOTOPIC 2-COCYCLES

Here we will prove the theorem

For any two 2-cocycle σ and λ , which are cohomologous, then σ and λ are homotopic.

First we must prove a preliminary result, namely the following theorem.

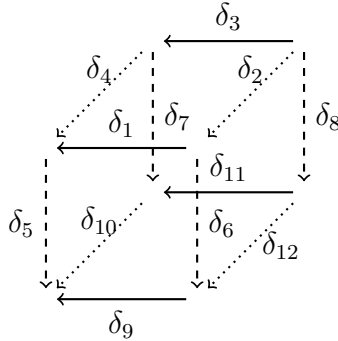
Theorem 4.1. For any 2-cocycle, σ , which is cohomologous to the trivial 2-cocycle, then the 2-cocycle will also be homotopic to the trivial 2-cocycle.

Proof. Consider some k -graph, Λ , ($k \geq 3$) from which we have the set of factorization squares, \mathbb{P} , and the set of edges, \mathbb{E} , on the skeleton of Λ . That is:

$$\forall \mu \in \mathbb{P}, \exists \mu_{ij} \in \mathbb{E} \text{ for } i, j \in \{0, 1\}, \text{ such that } \mu = \mu_{11}\mu_{01} \equiv \mu_{00}\mu_{10}$$

$$\mu = \begin{array}{ccc} & \xleftarrow{\mu_{01}} & \\ \mu_{11} \downarrow & & \downarrow \mu_{10} \\ & \xleftarrow{\mu_{00}} & \end{array}$$

For our k -graph, Λ , we can represent the graph as a set of cubes whose faces are factorization squares from the factorization of Λ , \mathbb{F} . Each cube has the form:



Let us specify the notation of the factorization squares in the following way:

$$\begin{array}{ccc}
 \text{Top} & \text{Left} & \text{Front} \\
 \hline
 a = a_{11}a_{01} \equiv a_{10}a_{00} & b = b_{11}b_{01} \equiv b_{10}b_{00} & c = c_{11}c_{01} \equiv c_{10}c_{00} \\
 \text{Bottom} & \text{Back} & \text{Right} \\
 \hline
 d = d_{11}d_{01} \equiv d_{10}d_{00} & e = e_{11}e_{01} \equiv e_{10}e_{00} & f = f_{11}f_{01} \equiv f_{10}f_{00}
 \end{array}$$

After analyzing the faces of the cube and which edges they share, the following paths on a cube must be equal:

$$\begin{array}{ccc}
 \text{Path-Edge Equalities} \\
 \hline
 \delta_4 \equiv a_{11} = b_{10} & \delta_3 \equiv e_{01} = a_{01} \\
 \delta_5 \equiv b_{00} = c_{11} & \delta_2 \equiv f_{10} = a_{10} \\
 \delta_1 \equiv a_{00} = c_{01} & \delta_{10} \equiv d_{11} = b_{11} \\
 \delta_8 \equiv e_{10} = f_{01} & \delta_9 \equiv d_{00} = c_{00} \\
 \delta_{11} \equiv e_{00} = d_{01} & \delta_6 \equiv f_{00} = c_{10} \\
 \delta_{12} \equiv d_{10} = f_{11} & \delta_7 \equiv e_{11} = b_{01}
 \end{array}$$

Now assume for our k -graph, Λ , we have a 2-cocycle, σ , which is cohomologous to the trivial 2-cocycle. That is, we have a map $\beta: \mathbb{E} \rightarrow \mathbb{T}$ such that β and σ satisfy the following conditions:

$$(4.1) \quad \sigma(\mu_{top})\sigma(\mu_{left})\sigma(\mu_{front}) = \sigma(\mu_{bottom})\sigma(\mu_{back})\sigma(\mu_{right})$$

$$(4.2) \quad \sigma(\mu) = \beta(\mu_{11})\beta(\mu_{01})\beta(\mu_{00})^{-1}\beta(\mu_{10})^{-1}$$

Notice the relation with the unit circle and real numbers given that $\mathbb{T} \cong \mathbb{R}/\mathbb{Z}$:

$$(4.3) \quad e^{2\pi ia} = e^{2\pi ib} \Leftrightarrow \exists k \in \mathbb{Z}, a + k = b$$

For each $\beta(\mu_{ij}), \sigma(\mu) \in \mathbb{T}$ we associate the unique values $\widetilde{\mu}_{ij}$, $\widetilde{\mu} \in [0, 1)$ such that:

$$e^{2\pi i\widetilde{\mu}_{ij}} = \beta(\mu_{ij}), \quad e^{2\pi i\widetilde{\mu}} = \sigma(\mu)$$

From equations (4.2) & (4.3) we have

$$e^{2\pi i\widetilde{\mu}} = e^{2\pi i(\widetilde{\mu}_{11} + \widetilde{\mu}_{01} - \widetilde{\mu}_{00} - \widetilde{\mu}_{10})}$$

$$(4.4) \quad \begin{array}{c} \Downarrow \\ \widetilde{\mu} = \widetilde{\mu}_{11} + \widetilde{\mu}_{01} - \widetilde{\mu}_{00} - \widetilde{\mu}_{10} + k_{\mu}, \quad k_{\mu} \in \mathbb{Z} \end{array}$$

And with equations (4.1) & (4.3)

$$(4.5) \quad \begin{array}{c} \sigma(a)\sigma(b)\sigma(c) = \sigma(d)\sigma(e)\sigma(f) \\ \Downarrow \\ e^{2\pi i(\widetilde{a}+\widetilde{b}+\widetilde{c})} = e^{2\pi i(\widetilde{d}+\widetilde{e}+\widetilde{f})} \\ \Downarrow \\ \widetilde{a} + \widetilde{b} + \widetilde{c} - \widetilde{d} - \widetilde{e} - \widetilde{f} = k, \quad k \in \mathbb{Z} \end{array}$$

Substituting our previous result given by equations (4.4) & (4.5)

$$\begin{array}{c} \widetilde{a}_{11} + \widetilde{a}_{01} - \widetilde{a}_{00} - \widetilde{a}_{10} + \widetilde{b}_{11} + \widetilde{b}_{01} - \widetilde{b}_{00} - \widetilde{b}_{10} + \widetilde{c}_{11} + \widetilde{c}_{01} - \widetilde{c}_{00} - \widetilde{c}_{10} - \\ \widetilde{d}_{11} - \widetilde{d}_{01} + \widetilde{d}_{00} + \widetilde{d}_{10} - \widetilde{e}_{11} - \widetilde{e}_{01} + \widetilde{e}_{00} + \widetilde{e}_{10} - \widetilde{f}_{11} - \widetilde{f}_{01} + \widetilde{f}_{00} + \widetilde{f}_{10} = \widetilde{k} \\ \text{with } \widetilde{k} = k_d + k_e + k_f - k_a - k_b - k_c + k \end{array}$$

With the Path-Edge Equalities the equation above reduces to

$$\begin{array}{c} (\widetilde{\delta}_1 - \widetilde{\delta}_1) + (\widetilde{\delta}_2 - \widetilde{\delta}_2) + (\widetilde{\delta}_3 - \widetilde{\delta}_3) + (\widetilde{\delta}_4 - \widetilde{\delta}_4) + (\widetilde{\delta}_5 - \widetilde{\delta}_5) + (\widetilde{\delta}_6 - \widetilde{\delta}_6) + \\ (\widetilde{\delta}_7 - \widetilde{\delta}_7) + (\widetilde{\delta}_8 - \widetilde{\delta}_8) + (\widetilde{\delta}_9 - \widetilde{\delta}_9) + (\widetilde{\delta}_{10} - \widetilde{\delta}_{10}) + (\widetilde{\delta}_{11} - \widetilde{\delta}_{11}) + (\widetilde{\delta}_{12} - \widetilde{\delta}_{12}) = \widetilde{k} \\ \Downarrow \\ \widetilde{k} = 0 \end{array}$$

Therefore we can construct a function $h : \mathbb{P} \rightarrow \mathbb{R}$ defined as

$$h(\mu) = \widetilde{\mu}_{11} + \widetilde{\mu}_{01} - \widetilde{\mu}_{00} - \widetilde{\mu}_{10}.$$

Note that for any cube we have

$$h(\mu_{top}) + h(\mu_{left}) + h(\mu_{front}) - h(\mu_{right}) - h(\mu_{back}) - h(\mu_{bottom}) = 0$$

is given by the formula in the equation after 4.5. Since $\widetilde{k} = 0$, this tells us that h satisfies the 2-cocycle condition: that is, for any cube in Λ we have

$$h(\mu_{top}) + h(\mu_{left}) + h(\mu_{front}) = h(\mu_{right}) + h(\mu_{back}) + h(\mu_{bottom})$$

Therefore this function h satisfies the requirements for our 2-cocycle to be homotopic to the trivial 2 cocycle. In particular we have

$$e^{2\pi ih(\mu)} = \sigma(\mu)$$

and for any cube in Λ we have

$$h(\mu_{top}) + h(\mu_{left}) + h(\mu_{front}) = h(\mu_{right}) + h(\mu_{back}) + h(\mu_{bottom})$$

given that $\tilde{k} = 0$. □

Theorem 4.2. For any two 2-cocycles σ and λ which are cohomologous, then σ and λ are homotopic.

Proof. Let Λ be a k -graph ($k \geq 3$) and let σ and λ be cohomologous 2-cocycles such that we have

$$\begin{aligned} \beta(\mu_{10})\beta(\mu_{00})\sigma(\mu) &= \beta(\mu_{01})\beta(\mu_{11})\lambda(\mu) \\ &\Downarrow \\ \sigma(\mu)\lambda(\mu)^{-1} &= \beta(\mu_{01})\beta(\mu_{11})\beta(\mu_{10})^{-1}\beta(\mu_{00})^{-1} \end{aligned}$$

By Proposition 1.1 and Proposition 1.2 we have the product of 2-cocycles a 2-cocycle and the multiplicative inverse of a 2-cocycle a 2-cocycle hence here we have $\sigma\lambda^{-1}(\mu) = \sigma(\mu)\lambda(\mu)^{-1}$ a 2-cocycle. In particular $\sigma\lambda^{-1} : \mathbb{P} \rightarrow \mathbb{T}$ is cohomologous to the trivial 2-cocycle, then by Theorem 4.1 there exists $h : \mathbb{P} \rightarrow \mathbb{R}$ such that for every factorization cube in \mathbb{F}

$$h(\mu_{top}) + h(\mu_{left}) + h(\mu_{front}) = h(\mu_{right}) + h(\mu_{back}) + h(\mu_{bottom})$$

and we have

$$\begin{aligned} \sigma\lambda^{-1}(\mu) &= \sigma(\mu)\lambda(\mu)^{-1} = e^{2\pi ih(\mu)} \\ &\Downarrow \\ \sigma(\mu) &= \lambda(\mu)e^{2\pi ih(\mu)} \end{aligned}$$

Therefore the two cocycles σ and λ are homotopic. □

5. CONSTRUCTION OF Ψ MATRIX

In order to check whether a particular 2-cocycle, c , defined on a k -graph, Λ , is homotopic to the trivial 2-cocycle we constructed a matrix Ψ which encodes the information of the k -graph and we can construct a matrix equation from both the k -graph and 2-cocycle

$$(5.1) \quad \Psi \vec{x} = \vec{z}$$

The equation above allows one to quickly verify if a 2-cocycle is homotopic to the trivial 2-cocycle by checking for the existence of an integer solution. What follows outlines the construction for a Ψ matrix equation from a finite k -graph.

Suppose we have k -graph, Λ , ($k \geq 3$), and let c be a 2-cocycle defined on Λ , such that the factorization of Λ , \mathbb{F} , generates n factorization cubes and Λ has m unique factorization squares, such that $\mu_1, \mu_2, \dots, \mu_m \in \mathbb{P}$. With c a 2-cocycle, c satisfies the 2-cocycle condition for each of the n factorization cubes from \mathbb{F} which leaves n equations

$$\begin{aligned} c(\mu_a)c(\mu_b)c(\mu_c) &= c(\mu_d)c(\mu_e)c(\mu_f) \\ &\vdots \\ c(\mu_g)c(\mu_h)c(\mu_i) &= c(\mu_j)c(\mu_k)c(\mu_l) \end{aligned}$$

In the exponential form we have

$$\begin{aligned} e^{2\pi i(\tilde{\mu}_a + \tilde{\mu}_b + \tilde{\mu}_c)} &= e^{2\pi i(\tilde{\mu}_d + \tilde{\mu}_e + \tilde{\mu}_f)} \\ &\vdots \\ e^{2\pi i(\tilde{\mu}_g + \tilde{\mu}_h + \tilde{\mu}_i)} &= e^{2\pi i(\tilde{\mu}_j + \tilde{\mu}_k + \tilde{\mu}_l)} \end{aligned}$$

From this data we can introduce a set of values $z_1, \dots, z_n \in \mathbb{Z}$ by the property of the unit circle mentioned in Equation (4.3) and after some algebra we arrive at a system of equations,

$$\begin{aligned} \widetilde{\mu}_a + \widetilde{\mu}_b + \widetilde{\mu}_c - \widetilde{\mu}_d - \widetilde{\mu}_e - \widetilde{\mu}_f &= z_1 \\ &\vdots \\ \widetilde{\mu}_g + \widetilde{\mu}_h + \widetilde{\mu}_i - \widetilde{\mu}_j - \widetilde{\mu}_k - \widetilde{\mu}_l &= z_n \end{aligned}$$

with each $z_i \in \mathbb{Z}, 1 \leq i \leq n$, relating to one of the n cubes. Associated to this system of n linear equations we have the $n \times m$ matrix with entries $\Psi_{ij} \in [-1, 0, 1]$ which we will call Ψ , such that Ψ has n rows (one for each of the equations) and m columns, one column for each of the factorization squares. We call the vector \vec{j} of dimension m the vector whose elements correspond to each $\widetilde{\mu}_i$ defined by the 2-cocycle for each $\mu_i \in \mathbb{P}$, and $\vec{z} \in \mathbb{Z}^n$ is the vector of dimension n whose elements correspond to each $z_i \in \mathbb{Z}$ for each factorization cube. This leaves us with the matrix equation:

$$(5.2) \quad \Psi \vec{j} = \vec{z}$$

In short, it is important to note that the matrix Ψ can be constructed for any k -graph, and the vectors \vec{j} and \vec{z} depend on the choice of 2-cocycle.

Theorem 5.1. Let Λ be a k -graph and let there be a 2-cocycle, c , defined on Λ such that one can construct the matrix equation in 5.3. If there exists a integer solution $\vec{x} \in \mathbb{Z}^m$ such that $\Psi \vec{x} = \vec{z}$, then the 2-cocycle, c , is homotopic to the trivial 2-cocycle.

Proof. Suppose from some k -graph and 2-cocycle, c , we have constructed the equation(5.3)

$$\Psi \vec{j} = \vec{z}$$

and suppose the equation $\Psi \vec{x} = \vec{z}$ has an integer solution $\vec{x} \in \mathbb{Z}^m$ such that

$$(5.3) \quad \Psi \vec{x} = \vec{z}$$

Consider the difference between equation (5.2) and equation (5.4):

$$\begin{aligned} \Psi \vec{j} - \Psi \vec{x} &= \vec{z} - \vec{z} \\ &\Downarrow \end{aligned}$$

$$(5.4) \quad \Psi(\vec{j} - \vec{x}) = 0$$

We can expand equation (5.4) which leaves the following

$$\begin{aligned} \widetilde{\mu}_a - x_a + \widetilde{\mu}_b - x_b + \widetilde{\mu}_c - x_c - \widetilde{\mu}_d + x_d - \widetilde{\mu}_e + x_e - \widetilde{\mu}_f + x_f &= 0 \\ &\vdots \\ \widetilde{\mu}_g - x_g + \widetilde{\mu}_h - x_h + \widetilde{\mu}_i - x_i - \widetilde{\mu}_j + x_j - \widetilde{\mu}_k + x_k - \widetilde{\mu}_l + x_l &= 0 \end{aligned}$$

Let $\vec{h} = \vec{j} - \vec{x}$ such that for each $\mu_i \in \mathbb{P}$ we can make the association to the function $h : \mathbb{P} \rightarrow \mathbb{R}$, namely

$$(5.5) \quad h(\mu_i) = \widetilde{\mu}_i - x_i$$

From equation (5.5) we can make the association to the 2-cocycle, c , by construction of the vector \vec{j} and \vec{z}

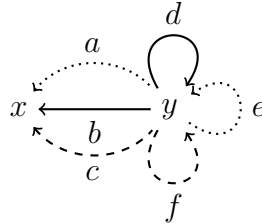
$$(5.6) \quad c(\mu_i) = e^{2\pi i(\widetilde{\mu}_i - x_i)} = e^{2\pi i h(\mu_i)}$$

Moreover we can write the equation $\Psi\vec{h} = 0$ as a system of equations such that for every factorization cube we have

$$\begin{aligned} h(\mu_a) + h(\mu_b) + h(\mu_c) - h(\mu_d) - h(\mu_e) - h(\mu_f) &= 0 \\ &\vdots \\ h(\mu_g) + h(\mu_h) + h(\mu_i) - h(\mu_j) - h(\mu_k) - h(\mu_l) &= 0 \end{aligned}$$

Hence with the existence of an integer vector $\vec{x} \in \mathbb{Z}^m$ we have the 2-cocycle, c , used in the construction of equation (5.2) homotopic to the trivial 2-cocycle. \square

Example 5.1. Consider the 3-graph and 2-cocycle, c , from example 1.2,



Where we had the 2-cocycle conditions for the two cubes,

$$(5.7) \quad c(A)c(B)c(C) = c(A)c(C)c(B)$$

$$(5.8) \quad c(A)c(D)c(E) = c(F)c(C)c(B)$$

and we defined the 2-cocycle, c , as:

$$\begin{aligned} c(A) &= i & i &= e^{2\frac{\pi}{4}i} & \tilde{A} &= \frac{1}{4} \\ c(B) &= 1 & i &= e^{2\pi i 0} & \tilde{B} &= 0 \\ c(C) &= i & i &= e^{2\frac{\pi}{4}i} & \tilde{C} &= \frac{1}{4} \\ c(D) &= -i & i &= e^{2\frac{\pi}{4}i} & \tilde{D} &= \frac{3}{4} \\ c(E) &= -1 & i &= e^{2\frac{\pi}{2}i} & \tilde{E} &= \frac{1}{2} \\ c(F) &= i & i &= e^{2\frac{\pi}{4}i} & \tilde{F} &= \frac{1}{4} \end{aligned}$$

With equation (5.7) a trivial equation we can ignore it. By examining the exponential form of equation (5.8) we have

$$\tilde{A} + \tilde{D} + \tilde{E} - \tilde{F} - \tilde{C} - \tilde{B} = z$$

for some integer z , which with our defined 2-cocycle this evaluates as

$$(5.9) \quad \frac{1}{4} + \frac{3}{4} + \frac{1}{2} - \frac{1}{4} - \frac{1}{4} - 0 = 1$$

We define $\vec{j} = (\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}, \tilde{F})$ such that we get the matrix equation $\Psi \vec{j} = \vec{z}$

$$\begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{A} \\ \tilde{B} \\ \tilde{C} \\ \tilde{D} \\ \tilde{E} \\ \tilde{F} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Immediately we see that there are plenty of integer vectors that are solutions to this equation. Let

$$\vec{x} = [1 \ 0 \ 0 \ 0 \ 0 \ 0]$$

Such that $\Psi(\vec{x}) = \vec{z}$, therefore by Theorem 5.1 this 2-cocycle c is homotopic to the trivial 2-cocycle.

5.1. Use of the Ψ matrix.

From the calculation of multiple examples we became suspicious of the fact that perhaps all 2-cocycles defined on a k -graph are homotopic to the trivial 2-cocycle.

Conjecture 5.1. For any k -graph, Λ , every 2-cocycle defined on Λ is homotopic to the trivial 2-cocycle.

In an attempt to tackle this conjecture, we found that the matrix class of Totally Unimodular matrices may give us a possible solution with the help of a theorem from the realm of linear programming.

Definition 5.1.1. A matrix A is *Totally Unimodular* if the entries in A are $+1$, -1 , or 0 , and each sub-determinant of A is $+1$, -1 , or 0 .

Theorem 5.2. Let A be a $m \times n$ Totally Unimodular matrix and let $\vec{b} \in \mathbb{Z}^m$ such that the equation $A\vec{x} = \vec{b}$ has a solution, then there exists an integer solution to the equation $A\vec{x} = \vec{b}$.

This is proven in Alexander Schrijver's book on the theory of Linear Programming Theorem 19.1 [5].

Proposition 5.1. For a k -graph, Λ , if the constructed Ψ matrix is Totally Unimodular, then all 2-cocycles defined on Λ are homotopic.

Proof. Suppose we have a k -graph, Λ , such that we construct the $m \times n$ matrix Ψ and Ψ is Totally Unimodular. Then for every 2-cocycle which we encode in say, $\vec{j} \in \mathbb{R}^n$, we have $\Psi\vec{j} = \vec{z}$ with $\vec{z} \in \mathbb{Z}^m$. By Theorem 5.2 we have an integer solution to the equation $\Psi\vec{x} = \vec{z}$, and therefore by Theorem 5.1 we have a homotopy between c and the trivial 2-cocycle. Then by Theorem 3.1 all 2-cocycles defined on Λ are homotopic. \square

6. FUTURE DIRECTIONS

In continuation of this research there are multiple outlets we would like to explore. One involves the use of the Ψ matrix formalism to explore the lingering conjecture that “All 2-cocycles on a k -graph are homotopic to the trivial 2-cocycle.” We have also established preliminary results suggesting that not all k -graphs have a Totally Unimodular Ψ matrix, and we would like to characterize or describe k -graphs which have a Totally Unimodular Ψ matrix. We are also interested in the question that for $C^*(\Lambda, c)$, does there exist some k -graph $\tilde{\Lambda}$ such $C^*(\tilde{\Lambda}) \cong C^*(\Lambda, c)$ and $\tilde{\Lambda} \neq \Lambda$? In addition to the fact that it would offer us multiple perspectives on the k -graph C^* -algebra, this questions relates to the idea of using K -theory to classify k -graph C^* -algebras.

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