

On the K -Theory of Generalized Bunce-Deddens Algebras

by

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On the K -Theory of Generalized Bunce-Deddens Algebras

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We consider a \mathbb{Z} -action σ on a directed graph – in particular a rooted tree T – inherited from the odometer action. This induces a \mathbb{Z} -action by automorphisms on $C^*(T)$. We show that the resulting crossed product $C^*(T) \rtimes_{\sigma} \mathbb{Z}$ is strongly Morita equivalent to the Bunce-Deddens algebra. The Pimsner-Voiculescu sequence allows us to reconstruct the K -theory for the Bunce-Deddens algebra in a new way using graph methods. We then extend to a \mathbb{Z}^k -action $\tilde{\sigma}$ on a k -graph when $k = 2$, show that $C^*(T_1 \times T_2) \rtimes_{\tilde{\sigma}} \mathbb{Z}^2$ is strongly Morita equivalent to a generalized Bunce-Deddens algebra of type Orfanos, and invoke the Künneth theorem to determine this new crossed product's K -theory. We end by generalizing the results for all k .

Dedication

To those who will never encounter a C^* -algebra.

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Contents

Chapter	
1	Introduction 1
2	Preliminaries 4
2.1	C^* -Algebras 4
2.2	Direct Limits and Tensor Products 8
2.3	Directed Graphs and Higher Rank Graphs 12
2.4	Crossed Products 18
2.5	K -Theory 24
2.6	Strong Morita Equivalence 26
3	Bunce-Deddens Algebras 30
3.1	Classical Approach 30
3.2	Odometer Approach 31
3.3	Graph Approach 32
3.4	Toward a Generalization 38
4	Visualizing K-Groups 45
4.1	Directed Graph Case 45
4.2	2-Graph Case 51

5	Some Generalizations	56
5.1	Motivating Example	56
5.2	k -Graph Case	61
6	Conclusion	67
	Bibliography	69

Chapter 1

Introduction

While we develop much of the C^* -algebra theory needed for this thesis, we assume a working understanding of functional analysis. Relevant sources include [7] and [32]. Chapter 2 begins with basic definitions and properties of C^* -algebras, including the statement of the Gelfand-Naimark theorem. We introduce direct limits and their universal property so we may later define the Bunce-Deddens algebra. The minimal and maximal tensor products are discussed, leading into the definition for nuclearity. We then turn our attention to directed and higher rank graphs, and the universal C^* -algebra associated to each. Crossed products are defined most generally, in terms of any locally compact group G ; for the purposes of this thesis however, G will always be discrete. The K -theory for graph algebras is developed, and since we also compute the K -theory for crossed products later, we require the Pimsner-Voiculescu sequence. Chapter 2 ends with a discussion of strong Morita equivalence – an equivalence relation on C^* -algebras – especially its quality of preserving nuclearity and K -theory.

In Chapter 3, we begin with the original construction of the Bunce-Deddens algebra \mathcal{BD}_n as the direct limit

$$\varinjlim M_{a_k}(C(\mathbb{T})),$$

where n is the supernatural number corresponding to the sequence $\{a_k\}$. We define the odometer action σ of \mathbb{Z} on the Cantor set X such that

$$C(X) \rtimes_{\sigma} \mathbb{Z} \cong \mathcal{BD}_n,$$

where n determines the structure of the pair (X, \mathbb{Z}) , or vice versa. We then construct a tree T , which is a type of directed graph, where \mathbb{Z} acts on the vertices of T via the odometer action; this gives an action – also called σ – on all of T and hence $C^*(T)$ as well. It is proven in [16] that for a row-finite tree T , $C^*(T)$ is Morita equivalent to $C_0(\partial T)$, where ∂T is the boundary of the graph or in our case the Cantor set. Applying a result by Curto, Muhly, and Williams, we establish the Morita equivalence

$$C^*(T) \rtimes_{\sigma} \mathbb{Z} \sim_M \mathcal{BD}_n.$$

At this point, the author would like to acknowledge the contributions of Valentin Deaconu. In a talk at WCOAS 2016, Deaconu presented as an example the above result slightly modified (T being a universal covering of some graph E) and in the case where $n = 2^{\infty}$. This inspired the author to consider all supernatural numbers n . The reasoning behind the example that Deaconu presented stems from results in [16], in which Kumjian and Pask construct a universal covering tree from a directed graph. In this paper, our initial graph already is a tree and so we have no need for universal coverings. We do mimic however some of the techniques of Kumjian and Pask in order to derive our result. Finally, a conversation with Deaconu also inspired our definition of T and the \mathbb{Z} -action on it.

This result serves as the foundation moving forward toward a generalization. When m and n are supernatural numbers, we let $\tilde{\sigma}$ be the action of \mathbb{Z}^2 on the 2-graph $T_m \times T_n$ by declaring each copy of \mathbb{Z} to act on either T_m or T_n while fixing the other. The resulting crossed product

$$C^*(T_m \times T_n) \rtimes_{\tilde{\sigma}} \mathbb{Z}^2$$

is now Morita equivalent to a generalized Bunce-Deddens algebra of type Orfanos.

Chapter 4 concerns itself with the computation of the K -theory for the above crossed products. Letting $\mathbb{Z}_{(n)} \subset \mathbb{Q}$ denote the group of rationals whose denominators “divide” the supernatural number n , the Pimsner-Voiculescu sequence allows us to show that

$$K_0\left(C^*(T) \rtimes_{\sigma} \mathbb{Z}\right) \cong \mathbb{Z}_{(n)} \text{ and } K_1\left(C^*(T) \rtimes_{\sigma} \mathbb{Z}\right) \cong \mathbb{Z}.$$

Because Morita equivalence preserves K -theory, we will have then established the K -groups for \mathcal{BD}_n using graph techniques. We then prove that as abelian groups,

$$\mathbb{Z}_{(m)} \otimes \mathbb{Z}_{(n)} \cong \mathbb{Z}_{(mn)}.$$

Thus, applying the Künneth formula to

$$C^*(T_m \times T_n) \rtimes_{\bar{\sigma}} \mathbb{Z}^2 \sim_M \mathcal{BD}_m \otimes \mathcal{BD}_n$$

yields the K -groups

$$\mathbb{Z}_{(mn)} \oplus \mathbb{Z} \text{ and } \mathbb{Z}_{(m)} \oplus \mathbb{Z}_{(n)}.$$

In Chapter 5, we generalize our results to the k -graph case for all $k \in \mathbb{N}$. Analogous to above, \mathbb{Z}^k acts on $T_{n_1} \times T_{n_2} \times \cdots \times T_{n_k}$ for supernatural numbers n_1, n_2, \dots, n_k . We show that

$$C^*(T_{n_1} \times T_{n_2} \times \cdots \times T_{n_k}) \rtimes_{\bar{\sigma}} \mathbb{Z}^k \cong \left[C^*(T_{n_1} \times \cdots \times T_{n_{k-1}}) \rtimes_{\bar{\sigma}} \mathbb{Z}^{k-1} \right] \otimes \left[C^*(T_{n_k}) \rtimes_{\sigma} \mathbb{Z} \right]$$

is Morita equivalent to a generalized Bunce-Deddens algebra of type Orfanos. And by induction, the Künneth formula allows us to compute the K -theory for

$$C^*(T_{n_1} \times T_{n_2} \times \cdots \times T_{n_k}) \rtimes_{\bar{\sigma}} \mathbb{Z}^k.$$

The proof uses combinatorial arguments.

In Chapter 6, we present some ideas for expanding on our results; however we do not prove any of them. This concludes the thesis.

Chapter 2

Preliminaries

2.1 C^* -Algebras

In 1947, Segal introduced in [34] the term “ C^* -algebra” for what was previously known as a B^* -algebra. As mentioned in [4], the “ C ” refers to “closed” not the commonly assumed word “continuous”. C^* -algebra theory can be thought of as analysis on infinite dimensions. For a comprehensive treatment on the theory, the reader should consult [20], [10], [4], or [11]. This overview of the subject is guided primarily by Murphy’s book ([20]). And as a final note to the reader, the following is intended to be a brief introduction to the subject highlighting the pieces needed for later topics.

Let A be an algebra over \mathbb{C} . An **involution** on A is a conjugate-linear map $A \rightarrow A : a \mapsto a^*$ such that $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. The pair $(A, *)$ is called a **$*$ -algebra**. We say a subset S of A is **self-adjoint** if $S = S^*$ where $S^* = \{a^* : a \in S\}$. And we define a **$*$ -subalgebra** B of A to be a self-adjoint subalgebra; by restricting the involution on A to B , B can be viewed as a $*$ -algebra in its own right. One can check that the intersection of a family of $*$ -subalgebras is itself a $*$ -subalgebra, hence for every subset S of A there is a smallest $*$ -algebra B of A containing S , which we call the $*$ -algebra **generated** by S . When I is a self-adjoint ideal of A , the quotient algebra A/I is a $*$ -algebra where the involution is given by $(a + I)^* = a^* + I$ for $a \in A$. Lastly, if A is unital then $1^* = 1$, and whenever a is an invertible element in A then $(a^*)^{-1} = (a^{-1})^*$.

Definition 2.1.1. Let A be a $*$ -algebra.

- (1) An element $a \in A$ is **self-adjoint** or **hermitian** if $a = a^*$.
- (2) a is **normal** if $a^*a = aa^*$.
- (3) An element $p \in A$ is a **projection** if $p = p^* = p^2$.
- (4) If A is unital, an element $u \in A$ is an **isometry** if $u^*u = 1$.
- (5) u is a **co-isometry** if $uu^* = 1$.
- (6) u is a **unitary** if $u^*u = uu^* = 1$.

Let $\varphi : A \rightarrow B$ be a homomorphism of $*$ -algebras A and B such that $\varphi(a^*) = \varphi(a)^*$ for all $a \in A$. Then φ is called a **$*$ -homomorphism**, $\ker(\varphi)$ is a self-adjoint ideal of A , and $\varphi(A)$ is a $*$ -subalgebra of B . A **$*$ -isomorphism** is a bijective $*$ -homomorphism, and a **$*$ -automorphism** of a $*$ -algebra A is a $*$ -isomorphism $\alpha : A \rightarrow A$. We denote by $\text{Aut}(A)$ the set of $*$ -automorphisms of A . When the context is clear, we drop the $*$ prefix and simply refer to these $*$ -maps as “homomorphisms”, “isomorphisms”, or “automorphisms” respectively.

A **Banach $*$ -algebra** is a $*$ -algebra A equipped with a norm $\|\cdot\|$, under which A is complete, such that $\|ab\| \leq \|a\|\|b\|$ and $\|a^*\| = \|a\|$ whenever $a, b \in A$. If A also has a unit such that $\|1\| = 1$, we say A is a **unital Banach $*$ -algebra**. A **C^* -algebra** is a Banach $*$ -algebra such that for all $a \in A$,

$$\|a^*a\| = \|a\|^2.$$

We define a **C^* -subalgebra** to be a closed $*$ -subalgebra of a C^* -algebra; a C^* -subalgebra is itself a C^* -algebra. It is known that when I is a closed two-sided ideal inside a C^* -algebra A , then I is necessarily self-adjoint and therefore a C^* -subalgebra of A . We call A **simple** if 0 and A are its only closed ideals. Note that C^* -subalgebras of simple C^* -algebras need not be simple themselves. The following example should begin to demonstrate the power of the C^* -algebra identity.

Example 2.1.2. Let A be a unital C^* -algebra. Since

$$\|1\| = \|1^*1\| = \|1\|^2,$$

necessarily $\|1\| = 1$. If p is any nonzero projection,

$$\|p\| = \|p^2\| = \|p^*p\| = \|p\|^2$$

implies that $\|p\| = 1$. And if u is any unitary,

$$1 = \|1\| = \|u^*u\| = \|u\|^2$$

implies that $\|u\| = 1$.

Another consequence of the C^* -algebra identity is the following theorem.

Theorem 2.1.3. [20, Theorem 2.1.7] *A $*$ -homomorphism $\varphi : A \rightarrow B$ from a Banach $*$ -algebra A to a C^* -algebra B is necessarily norm-decreasing.*

Example 2.1.4. The following are all C^* -algebras.

- (1) The complex numbers \mathbb{C} where the involution is given by complex conjugation.
- (2) $C_0(\Omega)$ where Ω is a locally compact Hausdorff space and the involution is given by $f \mapsto \bar{f}$.
- (3) $C_b(X)$ where X is a topological space, with involution $f \mapsto \bar{f}$.
- (4) $M_n(\mathbb{C})$ with involution conjugate transpose.
- (5) $B(H)$, the set of bounded linear operators from a Hilbert space H to itself with involution determined by the adjoint operation.
- (6) $K(H)$, the closed ideal in $B(H)$ of compact operators.
- (7) $\bigoplus_{\lambda \in \Lambda} A_\lambda$, where $\{A_\lambda\}_{\lambda \in \Lambda}$ is any family of C^* -algebras and with the involution defined pointwise.

For a C^* -algebra A denote by $\Omega(A)$ the set of **characters** on A , which are nonzero homomorphisms $\tau : A \rightarrow \mathbb{C}$. When A is abelian, $\Omega(A)$ is a locally compact Hausdorff space. And when A is unital, $\Omega(A)$ is compact. For each $a \in A$, define the function $\hat{a} : \Omega(A) \rightarrow \mathbb{C}$ by

$$\hat{a} : \tau \mapsto \tau(a).$$

We call \hat{a} the **Gelfand transform** of a . By a famous result due to Gelfand, we can completely determine the abelian C^* -algebras.

Theorem 2.1.5. [20, Theorem 2.1.10] (*Gelfand*) *If A is a nonzero abelian C^* -algebra, then*

$$\varphi : A \longrightarrow C_0(\Omega(A)) : a \longmapsto \hat{a}$$

is an isometric $$ -isomorphism.*

The final result of this section is known as the Gelfand-Naimark theorem, which allows us to view every C^* -algebra as a C^* -subalgebra of $B(H)$ for some Hilbert space H . We first start with the necessary background material in order to state it. Note that many of the following definitions and constructions can be done more generally on $*$ -algebras or Banach algebras, but we choose to restrict to the C^* -algebra case.

Start with a C^* -algebra A and set $\tilde{A} = A \oplus \mathbb{C}$ as a vector space. We define a multiplication on \tilde{A} by declaring

$$(a, \lambda)(b, \mu) = (ab + \lambda b + \mu a, \lambda\mu).$$

This turns \tilde{A} into a unital algebra where the unit is $(0, 1)$. We call \tilde{A} the **unitization** of A and the injective homomorphism

$$A \longrightarrow \tilde{A} : a \longmapsto (a, 0)$$

allows us to identify A as an ideal in \tilde{A} . We define an involution by setting $(a, \lambda)^* = (a^*, \bar{\lambda})$, making \tilde{A} a $*$ -algebra and A a self-adjoint ideal in \tilde{A} . We do not describe the C^* -norm in detail but simply state that it exists.

Theorem 2.1.6. [20, Theorem 2.1.6] *If A is a C^* -algebra, then there is a unique norm on its unitization \tilde{A} making it into a C^* -algebra, and extending the norm of A .*

Denote by $\text{Inv}(A)$ the invertible elements of the unital C^* -algebra A . The **spectrum** of an element $a \in A$ is the set

$$\sigma(a) = \sigma_A(a) = \{\lambda \in \mathbb{C} : \lambda 1 - a \notin \text{Inv}(A)\}.$$

When A is not unital, we declare $\sigma_A(a) = \sigma_{\tilde{A}}(a)$.

Example 2.1.7. If $A = M_n(\mathbb{C})$ and $a \in A$, then $\sigma(a)$ is the set of eigenvalues of a .

We say an element a in a C^* -algebra A is **positive** if a is hermitian and $\sigma(a) \subseteq \mathbb{R}^+$. And we denote by A^+ the set of positive elements of A . A linear map $\varphi : A \rightarrow B$ between C^* -algebras is called **positive** if $\varphi(A^+) \subseteq B^+$. A **state** on the C^* -algebra A is a positive linear functional on A of norm one. The set of states on A is denoted $S(A)$. A positive linear functional τ on a C^* -algebra A is **tracial** if $\tau(a^*a) = \tau(aa^*)$ for all $a \in A$; we call a tracial state a **trace**.

A **representation** of a C^* -algebra A is a pair (H, φ) where H is a Hilbert space and $\varphi : A \rightarrow B(H)$ is a homomorphism. We call the representation **faithful** if φ is injective, and **irreducible** if 0 and H are the only closed vector subspaces of H that are invariant for all operators in $\varphi(A)$. Associated to each positive linear functional τ on A is the **Gelfand-Naimark-Segal** (GNS) representation (H_τ, φ_τ) of A , the details of which we omit (see [20, Section 3.4]). When A is nonzero we define the **universal** representation of A to be the direct sum of all the (H_τ, φ_τ) as τ ranges over $S(A)$. We now state the famous result due to Gelfand and Naimark.

Theorem 2.1.8. [20, Theorem 3.4.1] (*Gelfand-Naimark*) *If A is a C^* -algebra, then it has a faithful representation. Specifically, its universal representation is faithful.*

2.2 Direct Limits and Tensor Products

We turn our attention now to two constructions which are used to create new C^* -algebras from old ones. Given a sequence of C^* -algebras we can construct the direct limit, in which each of the old spaces embeds. Or given two C^* -algebras, we can construct the minimal tensor product. Both of these ideas are the C^* -analogues of their algebraic counterparts. Again this section follows the conventions of [20].

A **C^* -seminorm** on the $*$ -algebra A is a seminorm p on A such that for all $a, b \in A$, $p(ab) \leq p(a)p(b)$, $p(a^*) = p(a)$, and $p(a^*a) = p(a)^2$. If p is also a norm, p is called a **C^* -norm**.

Now let

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \dots$$

be a sequence of C^* -algebras A_n connected by the homomorphisms φ_n , for $n \geq 1$. Then $\prod_{n=1}^{\infty} A_n$ is a $*$ -algebra with operations defined pointwise, containing the $*$ -subalgebra

$$A' = \left\{ (a_n) \in \prod_{n=1}^{\infty} A_n : \exists N \in \mathbb{N} \text{ such that } \varphi_n(a_n) = a_{n+1} \forall n \geq N \right\}.$$

By Theorem 2.1.3, each φ_n is norm-decreasing and so $\lim_{n \rightarrow \infty} \|a_n\|$ makes sense. By setting $p((a_n)) = \lim_{n \rightarrow \infty} \|a_n\|$, we get the C^* -seminorm $p : A' \rightarrow \mathbb{R}^+ : a \mapsto p(a)$ on A' . Since $N = p^{-1}\{0\}$ is a self-adjoint ideal in A' , we can consider the $*$ -algebra A'/N with C^* -norm given by letting $\|a + N\| = p(a)$. Taking the completion with respect to this norm yields the C^* -algebra $\overline{A'/N}$, which is called the **direct limit** of the sequence $\{(A_n, \varphi_n)\}_{n=1}^{\infty}$ and denoted $\varinjlim A_n$.

For $a \in A_n$ where $n \in \mathbb{N}$, define $\hat{\varphi}^n(a)$ in A' to be the sequence

$$(0, 0, \dots, 0, a, \varphi_n(a), \varphi_{n+1} \circ \varphi_n(a), \dots),$$

where the first possibly nonzero entry, a , occurs in the n^{th} coordinate. Let $i : A' \rightarrow \varinjlim A_n$ be the canonical embedding. Then we obtain the homomorphism $\varphi^n : a \mapsto i(\hat{\varphi}^n(a))$ from A_n to $\varinjlim A_n$, called the **natural** map. It is straightforward to check that for each $n \in \mathbb{N}$ the diagram

$$\begin{array}{ccc} A_n & \xrightarrow{\varphi_n} & A_{n+1} \\ & \searrow \varphi^n & \downarrow \varphi^{n+1} \\ & & \varinjlim A_n \end{array}$$

commutes. Direct limits also satisfy a universal property.

Theorem 2.2.1. [20, Theorem 6.1.2] *Let $\varinjlim A_n$ be the direct limit of $\{(A_n, \varphi_n)\}_{n=1}^{\infty}$ with natural maps $\varphi^n : A_n \rightarrow \varinjlim A_n$ for each n . If B is a C^* -algebra and there exists a homomorphism $\psi^n : A_n \rightarrow B$ for each n such that the diagram*

$$\begin{array}{ccc}
 A_n & \xrightarrow{\varphi_n} & A_{n+1} \\
 & \searrow \psi^n & \downarrow \psi^{n+1} \\
 & & B
 \end{array}$$

commutes, then there exists a unique homomorphism $\psi : \varinjlim A_n \rightarrow B$ such that for each n the diagram

$$\begin{array}{ccc}
 A_n & \xrightarrow{\varphi^n} & \varinjlim A_n \\
 & \searrow \psi^n & \downarrow \psi \\
 & & B
 \end{array}$$

also commutes.

We now turn our attention to tensor products; for a review of the algebraic construction, the reader should consult [1]. For vector spaces V and W denote their algebraic tensor product by $V \otimes W$, which is generated by the elementary tensors $v \otimes w$ where $v \in V$, $w \in W$. We will omit the details of the following theorem as its statement is not our primary concern, rather a necessity moving forward.

Theorem 2.2.2. [20, Theorem 6.3.1] *Let H and K be Hilbert spaces. Then there is a unique inner product $\langle \cdot, \cdot \rangle$ on $H \otimes K$ such that*

$$\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle,$$

for $x, x' \in H$ and $y, y' \in K$. Taking the completion of the pre-Hilbert space $H \otimes K$ with respect to this inner product yields the **Hilbert space tensor product** $H \hat{\otimes} K$.

We define the ***-algebra tensor product** $A \otimes B$ of two *-algebras A and B to be the tensor product of A and B as algebras where multiplication is given by

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb'$$

and the involution by

$$(a \otimes b)^* = a^* \otimes b^*,$$

for $a, a' \in A$ and $b, b' \in B$. In the following theorem, we will be taking the $*$ -algebra tensor product of two C^* -algebras.

Theorem 2.2.3. [20, Theorem 6.3.3] *Let (H, φ) and (K, ψ) be representations of the C^* -algebras A and B respectively. Then there exists a unique $*$ -homomorphism $\pi : A \otimes B \rightarrow B(H \hat{\otimes} K)$ such that*

$$\pi(a \otimes b) = \varphi(a) \hat{\otimes} \psi(b),$$

where $a \in A$ and $b \in B$. Moreover, if φ and ψ are injective then so is π .

Now let A and B be C^* -algebras with universal representations (H, φ) and (K, ψ) respectively. Denote by π the unique $*$ -homomorphism guaranteed to exist by the preceding theorem. Then

$$\|\cdot\|_{\min} : A \otimes B \longrightarrow \mathbb{R}^+ : c \longmapsto \|\pi(c)\|$$

defines a C^* -norm on $A \otimes B$ called the **minimal** C^* -norm; one can check that $\|a \otimes b\|_{\min} = \|a\| \|b\|$. Taking the completion of $A \otimes B$ with respect to $\|\cdot\|_{\min}$ results in the **minimal tensor product** $A \otimes_{\min} B$ of A and B . While many C^* -norms may exist on $A \otimes B$, the minimal one is always possible and so one may always consider the minimal tensor product of two C^* -algebras.

There is another C^* -norm that always exists on $A \otimes B$. Denote by Γ the set of all C^* -norms γ on $A \otimes B$. For each $c \in A \otimes B$, define

$$\|c\|_{\max} = \sup_{\gamma \in \Gamma} \gamma(c).$$

We omit the details for brevity, but it can be shown that $\|c\|_{\max} < \infty$. Hence

$$\|\cdot\|_{\max} : A \otimes B \longrightarrow \mathbb{R}^+ : c \longmapsto \|c\|_{\max}$$

is a C^* -norm called the **maximal** C^* -norm. Taking the completion of $A \otimes B$ with respect to this norm gives the **maximal tensor product** $A \otimes_{\max} B$ of A and B . As the names would suggest, the minimal and maximal C^* -norms satisfy the following property.

Theorem 2.2.4. [4, II.9.5.2] *Let A and B be C^* -algebras, where $\|\cdot\|$ is any C^* -norm on $A \otimes B$. Then*

$$\|\cdot\|_{\min} \leq \|\cdot\| \leq \|\cdot\|_{\max}.$$

We say a C^* -algebra A is **nuclear** if for each C^* -algebra B , there is only one C^* -norm on $A \otimes B$. In this scenario $\|\cdot\|_{\min} = \|\cdot\|_{\max}$ and we will often simply write $A \otimes B$ in place of $A \otimes_{\min} B = A \otimes_{\max} B$. We end this section by remarking that nuclearity is a fairly common property for elementary examples of C^* -algebras.

Theorem 2.2.5. [20, Theorem 6.3.9] *All finite-dimensional C^* -algebras are nuclear.*

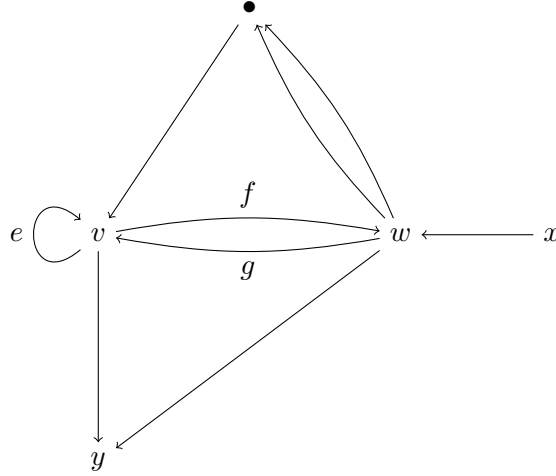
Theorem 2.2.6. [20, Theorem 6.4.15] (*Takesaki*) *Every abelian C^* -algebra is nuclear.*

2.3 Directed Graphs and Higher Rank Graphs

A graph is a combinatorial object made up of vertices and the edges connecting them. By associating the vertices with projections and the edges with partial isometries, we construct a C^* -algebra whose algebraic properties can be realized as patterns in the underlying graph. Known as graph algebras, these C^* -algebras can be represented inside $B(H)$ and generalize the Cuntz-Krieger algebras of [8]. In 2000, Kumjian and Pask introduced in [17] the further generalized higher rank graph C^* -algebras. Details concerning the remarks here on graph algebras can be found in [27].

A **directed graph** $E = (E^0, E^1, r, s)$ is a quadruple consisting of countable sets E^0 and E^1 , and functions $r, s : E^1 \rightarrow E^0$. The elements of E^0 are the **vertices** of the directed graph and the elements of E^1 are its **edges**. For an edge e , $s(e)$ is the **source** of e and $r(e)$ is the **range** of e . When $s(e) = v$ and $r(e) = w$, we say e is an edge from vertex v to vertex w ; in this scenario, v **emits** e and w **receives** e . The presence of these range and source functions are what make a graph directed. While some people consider undirected graphs, that is not the case in this thesis and so we will often shorten “directed graph” to “graph”.

Example 2.3.1. The following is a graph. Observe that we need not label all vertices and edges, especially when they do not matter or are not referenced.

Graph E

In this example, f is an edge from v to w and g is an edge from w to v . An edge, such as e , which begins and ends at the same vertex is called a **loop**. A vertex, such as x , that does not receive any edges is called a **source** (it is standard in the literature to let “source” denote two things). A vertex, such as y , that does not emit any edges is called a **sink**. This graph is an example of a **finite graph** in which case E^0 and E^1 are both finite. The graphs considered later in this thesis are comprised of infinitely many vertices and edges. In that scenario our results require the **row-finiteness** property, which states that each vertex receives at most finitely many edges. Another way to define “row-finite” is via the **adjacency matrix** A_E of E , which is the $E^0 \times E^0$ matrix given by

$$A_E(v, w) := \#\{e \in E^1 : s(e) = w \text{ and } r(e) = v\}.$$

Then the graph E is row-finite if and only if each row of A_E has finite sum. Finally, since there can be many ways to draw the same graph, we say two graphs E and F are **isomorphic** if there are bijections $\phi^0 : E^0 \rightarrow F^0$ and $\phi^1 : E^1 \rightarrow F^1$ such that $s_F \circ \phi^1 = \phi^0 \circ s_E$ and $r_F \circ \phi^1 = \phi^0 \circ r_E$.

Since our goal is to represent a graph by means of operators on a Hilbert space, we first cover some definitions. Recall that a projection $P \in B(H)$, for some Hilbert space H , is characterized by the property that $P = P^* = P^2$. In this case, $h - P(h)$ is orthogonal to the closed subspace $P(H)$ for every $h \in H$. More generally, P is an **orthogonal projection** on the closed subspace M if $P(h) \in M$ and $h - P(h)$ is orthogonal to M for every $h \in H$. We say orthogonal projections

$\{P_i\}_i$ are **mutually orthogonal** if their ranges $\{P_i(H)\}_i$ are mutually orthogonal subspaces of H . An operator $S \in B(H)$ is a **partial isometry** if S restricted to $(\ker S)^\perp$ is an isometry. This is equivalent to the relation $SS^*S = S$, if and only if S^*S is the projection (**initial projection**) on $(\ker S)^\perp$, if and only if SS^* is the projection (**final projection**) on $S(H)$.

Let E be a row-finite graph and H a Hilbert space. A **Cuntz-Krieger E -family** $\{S, P\}$ on H is made up of mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ on H such that

$$S_e^*S_e = P_{s(e)} \text{ for all } e \in E^1$$

and

$$P_v = \sum_{r(e)=v} S_e S_e^* \text{ whenever } v \text{ is not a source.}$$

These relations yield the following consequences.

Proposition 2.3.2. [27, Proposition 1.12] *Let E be a row-finite graph and $\{S, P\}$ a Cuntz-Krieger E -family in some C^* -algebra B . Then*

- (1) *the projections $\{S_e S_e^* : e \in E^1\}$ are mutually orthogonal;*
- (2) *$S_e^* S_f \neq 0$ implies $e = f$;*
- (3) *$S_e S_f \neq 0$ implies $s(e) = r(f)$;*
- (4) *$S_e S_f^* \neq 0$ implies $s(e) = s(f)$.*

A **path of length n** in a graph E is a sequence $\mu = \mu_1 \mu_2 \cdots \mu_n$ of edges in E such that $s(\mu_i) = r(\mu_{i+1})$ for $i = 1, 2, \dots, n-1$. The length of μ is denoted $|\mu| := n$; we take a vertex to be a path of length 0. Denote by E^n the set of paths of length n and write $E^* := \bigcup_{n \geq 0} E^n$. The range and source functions for paths are given by $s(\mu) = s(\mu_{|\mu|})$ and $r(\mu) = r(\mu_1)$ when $|\mu| > 1$; for a vertex v , $s(v) = v = r(v)$. If μ and ν are paths with $s(\mu) = r(\nu)$, then we write $\mu\nu$ for the concatenated path $\mu_1 \cdots \mu_{|\mu|} \nu_1 \cdots \nu_{|\nu|}$. Lastly, for a path $\mu \in E^n$ we set $S_\mu = S_{\mu_1} S_{\mu_2} \cdots S_{\mu_n}$; when $v \in E^0$, we set $S_v = P_v$.

In the following result, $C^*(S, P)$ denotes the C^* -algebra generated by the Cuntz-Krieger family $\{S, P\}$, as viewed inside $B(H)$.

Proposition 2.3.3. [27, Corollary 1.16] *If $\{S, P\}$ is a Cuntz-Krieger E -family for a row-finite graph E , then*

$$C^*(S, P) = \overline{\text{span}}\{S_\mu S_\nu^* : \mu, \nu \in E^*, s(\mu) = s(\nu)\}.$$

By employing the methods of general universal C^* -algebra constructions, we can create a universal C^* -algebra associated to a graph E that mimics the behavior of the spanning set in the above proposition. A C^* -seminorm can be defined on a space of formal linear combinations by taking a supremum over all Cuntz-Krieger E -families. Standard procedure then determines a C^* -norm on the appropriate quotient space. We omit the details, which make up the proof of the following.

Proposition 2.3.4. [27, Proposition 1.21] *For any row-finite directed graph E , there is a C^* -algebra $C^*(E)$ generated by a Cuntz-Krieger E -family $\{s, p\}$ such that for every Cuntz-Krieger E -family $\{T, Q\}$ in a C^* -algebra B , there is a homomorphism $\pi_{T, Q} : C^*(E) \rightarrow B$ such that $\pi_{T, Q}(s_e) = T_e$ for every $e \in E^1$ and $\pi_{T, Q}(p_v) = Q_v$ for every $v \in E^0$.*

The C^* -algebra $C^*(E)$ is called the **graph C^* -algebra** or **Cuntz-Krieger algebra** of E ; we often refer to $C^*(E)$ as a **graph algebra**. In this scenario, $\{s, p\}$ refers to the universal family that generates $C^*(E)$. Since $C^*(E)$ is unique up to isomorphism, it makes sense to talk about “the” graph algebra corresponding to the graph E . We end our discussion on directed graphs with a precise statement of uniqueness, followed by examples.

Corollary 2.3.5. [27, Corollary 1.22] *Let E be a row-finite directed graph and C a C^* -algebra generated by a Cuntz-Krieger E -family $\{w, r\}$ such that for every Cuntz-Krieger E -family $\{T, Q\}$ in a C^* -algebra B , there is a homomorphism $\rho_{T, Q} : C \rightarrow B$ satisfying $\rho_{T, Q}(w_e) = T_e$ for every $e \in E^1$ and $\rho_{T, Q}(r_v) = Q_v$ for every $v \in E^0$. Then there is an isomorphism $\phi : C^*(E) \rightarrow C$ such that $\phi(s_e) = w_e$ for every $e \in E^1$ and $\phi(p_v) = r_v$ for every $v \in E^0$.*

Example 2.3.6. The following are basic graph algebras.

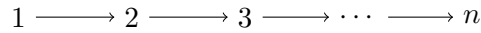
(1) $C^*(E) = \mathbb{C}$ when E is the graph:



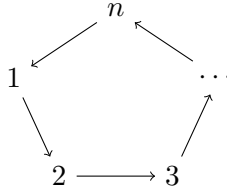
(2) $C^*(E) = C(\mathbb{T})$ when E is the graph:



(3) $C^*(E) = M_n(\mathbb{C})$ when E is the graph:



(4) $C^*(E) = M_n(C(\mathbb{T}))$ when E is the graph:



For the remainder of this section we talk about higher rank graphs, which are defined using the language of category theory. For a reference of categorical terms, the reader is advised to see [19]. Let \mathbb{W} denote the abelian monoid $\mathbb{N} \cup \{0\}$ under addition. Likewise for $k > 1$, \mathbb{W}^k is an abelian monoid with canonical generators e_i for $1 \leq i \leq k$. We also have a partial order $m \leq n$ if and only if $m_i \leq n_i$ for all i , where $m = (m_1, \dots, m_k)$ and $n = (n_1, \dots, n_k)$. This synopsis is based on the original article [17] by Kumjian and Pask, where the notion of higher rank graph was first defined.

Definition 2.3.7. A k -graph (also **rank k graph** or **higher rank graph**) is a pair (Λ, d) consisting of a countable small category Λ – with range map r and source map s – and a functor $d : \Lambda \rightarrow \mathbb{W}^k$ that satisfy the **factorization property**: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{W}^k$ with

$d(\lambda) = m + n$, there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$, $d(\nu) = n$. For $n \in \mathbb{W}^k$ we write $\Lambda^n := d^{-1}(n)$. A morphism between k -graphs (Λ_1, d_1) and (Λ_2, d_2) is a functor $f : \Lambda_1 \rightarrow \Lambda_2$ compatible with the degree maps.

Often we refer to a k -graph simply by Λ . If $\mu\alpha = \nu\alpha$ in Λ , the factorization property says that $\mu = \nu$; the same is true for left cancellation. Hence we identify the objects of Λ , $\text{Obj}(\Lambda)$, with Λ^0 . We use roman letters for objects of Λ and greek letters for morphisms.

Example 2.3.8. Given a directed graph $E = (E^0, E^1, r_E, s_E)$, the set of finite paths E^* can be viewed as a small category with $s = s_E$ and $r = r_E$. If we also let $d = |\cdot|$ be the length function, then (E^*, d) is a 1-graph. Conversely, if we start with a 1-graph Λ , set $E^0 = \Lambda^0$ and $E^1 = \Lambda^1$. Letting $s_E = s$ and $r_E = r$, we get the directed graph $E = (E^0, E^1, r_E, s_E)$.

We say a k -graph Λ is **row-finite** if for each $m \in \mathbb{W}^k$ and $v \in \Lambda^0$ the set

$$\Lambda^m(v) := \{\lambda \in \Lambda^m : r(\lambda) = v\}$$

is finite. And Λ has **no sources** if $\Lambda^m(v) \neq 0$ for all $v \in \Lambda^0$ and $m \in \mathbb{W}^k$. When E is a directed graph, E is row-finite if and only if the 1-graph E^* is row-finite. Similarly, E has no sources if and only if E^* has no sources. For the remainder of this section, we assume all k -graphs are row-finite and have no sources. We now define the universal C^* -algebra associated to a k -graph.

Definition 2.3.9. [17, Definitions 1.5] Let Λ be a k -graph. Then $C^*(\Lambda)$ is defined to be the universal C^* -algebra generated by a family $\{s_\lambda : \lambda \in \Lambda\}$ of partial isometries satisfying:

- (1) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;
- (2) $s_{\lambda\mu} = s_\lambda s_\mu$ for all $\lambda, \mu \in \Lambda$ such that $s(\lambda) = r(\mu)$;
- (3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$;
- (4) $s_v = \sum_{\lambda \in \Lambda^n(v)} s_\lambda s_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{W}^k$.

For all $\lambda \in \Lambda$ define $p_\lambda = s_\lambda s_\lambda^*$; when $v \in \Lambda^0$, $p_v = s_v$. Any family of partial isometries satisfying (1)-(4) in the above definition is called a ***-representation** of Λ . If $\{t_\lambda : \lambda \in \Lambda\}$ is another *-representation of Λ then there is a homomorphism

$$C^*(\Lambda) \longrightarrow C^*(\{t_\lambda : \lambda \in \Lambda\})$$

given by $s_\lambda \mapsto t_\lambda$, establishing universality.

Example 2.3.10. [17, Examples 1.7]

- (1) Let E^* be the 1-graph corresponding to the directed graph E . By restricting a *-representation of E^* to E^0 and E^1 , we get a Cuntz-Krieger E -family. Conversely, every Cuntz-Krieger E -family can be extended uniquely to a *-representation of E^* . Hence

$$C^*(E^*) \cong C^*(E).$$

- (2) For $k \geq 1$, let Ω_k be the small category where $\text{Obj}(\Omega_k) = \mathbb{W}^k$, morphisms given by $\{(m, n) \in \mathbb{W}^k \times \mathbb{W}^k : m \leq n\}$, $r(m, n) = m$, $s(m, n) = n$, and $d(m, n) = n - m$. Then Ω_k is a k -graph and

$$C^*(\Omega_k) \cong K(l^2(\mathbb{W}^k)).$$

2.4 Crossed Products

Let G be a locally compact group acting on a C^* -algebra A and denote this action by α . In this section we define a new C^* -algebra $A \rtimes_\alpha G$, called the crossed product of A by G . It turns out that certain representations of the crossed product are in bijection with those of A and G . In order to define the crossed product, we must define a norm on the *-algebra $C_c(G, A)$. Doing so will require knowledge of vector-valued integration, specifically integration of functions which take values in $B(H)$. The following can be found in more detail in [36].

Recall that a (left) **Haar measure** is a left-invariant Radon measure on a locally compact group G ; we take all Haar measures to be left. When G is discrete, Haar measure is counting

measure. We bring this to the attention of the reader since this thesis is only concerned with crossed products where the group is discrete; therefore, one may view all the integrals that follow as sums. We begin with a basic fact.

Theorem 2.4.1. [36, Theorem 1.57] *Every locally compact group G has a Haar measure μ which is unique up to a strictly positive scalar.*

We must also introduce a property of Haar integration that will make an appearance later.

Lemma 2.4.2. [36, Lemma 1.61] *Let μ be a Haar measure on a locally compact group G . Then there is a continuous homomorphism $\Delta : G \rightarrow \mathbb{R}^+$ such that*

$$\Delta(r) \int_G f(sr) d\mu(s) = \int_G f(s) d\mu(s)$$

for all $f \in C_c(G)$. The function Δ is independent of choice of Haar measure and is called the **modular function** on G .

Denote by $U(H)$ the group of unitaries in $B(H)$. A **unitary representation** of a locally compact group G is a continuous homomorphism $U : G \rightarrow U(H)$ where $U(H)$ is equipped with the strong operator topology. A subspace $V \subseteq H$ is **invariant** for U if $U_s V \subseteq V$ for all $s \in G$. And U is **irreducible** if the only closed invariant subspaces for U are the trivial ones, namely 0 and H . Finally, if $\pi : A \rightarrow B(H)$ is a representation of a C^* -algebra A on a Hilbert space H , then we say π is **nondegenerate** if

$$\{\pi(a)h : a \in A, h \in H\}$$

is dense in H .

We would now like to integrate functions which take values in an arbitrary Banach space, D . Observe that if $f \in C_c(G, D)$, then the map given by $s \mapsto \|f(s)\|$ is in $C_c(G)$ and so by properties of measure theory,

$$\int_G \|f(s)\| d\mu(s) \leq \|f\|_\infty \cdot \mu(\text{supp} f) < \infty.$$

For $f \in C_c(G, D)$, we set

$$\|f\|_1 := \int_G \|f(s)\| d\mu(s).$$

Now if $\varphi \in D^*$, the dual space, then we can define a map $L_f : D^* \rightarrow D^{**}$ into the double dual by

$$L_f(\varphi) := \int_G \varphi(f(s))d\mu(s).$$

Observe that $|L_f(\varphi)| \leq \|\varphi\| \|f\|_1$, hence L_f is a bounded linear functional with $\|L_f\| \leq \|f\|_1$. Now let

$$\iota(a)(\varphi) := \varphi(a)$$

define the natural isometric inclusion of D into D^{**} .

Lemma 2.4.3. [36, Lemma 1.90] *If $f \in C_c(G, D)$, then $L_f \in \iota(D)$.*

This lemma ensures that $\iota^{-1}(L_f)$ is well-defined. Thus for $f \in C_c(G, D)$, we define

$$\int_G f(s)d\mu(s) := \iota^{-1}(L_f).$$

This definition is formalized in the following lemma.

Lemma 2.4.4. [36, Lemma 1.91] *Suppose that D is a Banach space and G is a locally compact group with Haar measure μ . Then there is a linear map*

$$f \mapsto \int_G f(s)d\mu(s)$$

from $C_c(G, D)$ to D which is characterized by

$$\varphi\left(\int_G f(s)d\mu(s)\right) = \int_G \varphi(f(s))d\mu(s)$$

for all $\varphi \in D^$. Furthermore if $L : D \rightarrow Y$ is a bounded linear operator, then*

$$L\left(\int_G f(s)d\mu(s)\right) = \int_G L(f(s))d\mu(s).$$

Even more can be said in the case where D is actually a C^* -algebra.

Lemma 2.4.5. [36, Lemma 1.92] *Suppose that A is a C^* -algebra and that G is a locally compact group with Haar measure μ . Then the integral defined in the previous lemma has the following*

additional properties. Suppose that $f \in C_c(G, A)$. If $\pi : A \rightarrow B(H)$ is a representation and $h, k \in H$, then

$$\left\langle \pi \left(\int_G f(s) d\mu(s) \right) h, k \right\rangle = \int_G \langle \pi(f(s)) h, k \rangle d\mu(s).$$

Furthermore,

$$\left(\int_G f(s) d\mu(s) \right)^* = \int_G f(s)^* d\mu(s).$$

It turns out that we would like to define integrals of the form

$$\int_G f(s) U_s d\mu(s),$$

where $f \in C_c(G, B(H))$ and $U : G \rightarrow U(H)$ is a unitary representation of G . The integrand of this integral is not necessarily a continuous function when $B(H)$ is given the norm topology. For this reason, the previous lemma cannot be applied. The integrand is however continuous in the strong operator topology, and with a little more work we can show that this will suffice. Note that for discrete G the above integrals reduce to sums, which we want to converge in the strong operator topology.

The **multiplier algebra** $M(A)$ of a C^* -algebra A is the largest unital C^* -algebra containing A as an **essential ideal**, that is $A \cap I$ is nontrivial for all ideals $I \subseteq M(A)$. We take for granted the existence of the multiplier algebra; for an explicit construction, see [20]. For $a \in A$, let $\|\cdot\|_a$ be the seminorm on $M(A)$ defined by $\|b\|_a := \|ba\| + \|ab\|$. The **strict topology** on $M(A)$ is the topology generated by the family of seminorms $\{\|\cdot\|_a : a \in A\}$. We denote $M(A)$ with the strict topology by $M_s(A)$. Observe that if $f \in C_c(G, M_s(A))$, then the map $s \mapsto f(s)a$ is in $C_c(G, A)$ for each $a \in A$. We can then define $L_f : A \rightarrow A$ by

$$L_f(a) := \int_G f(s) a d\mu(s)$$

and thus

$$\int_G f(s) d\mu(s) := L_f.$$

That L_f is in $M(A)$ requires a specific construction of $M(A)$ as the adjointable operators on A

viewed as a right Hilbert module over itself; we omit the details. Finally, if ϕ is a map defined on A , then the notation $\bar{\phi}$ represents the extension of ϕ to $M(A)$.

Proposition 2.4.6. [36, Lemma 1.101] *Let A be a C^* -algebra. There is a unique linear map*

$$f \mapsto \int_G f(s) d\mu(s)$$

from $C_c(G, M_s(A))$ to $M(A)$ such that for any nondegenerate representation $\pi : A \rightarrow B(H)$ and all $h, k \in H$,

$$\langle \bar{\pi} \left(\int_G f(s) d\mu(s) \right) h, k \rangle = \int_G \langle \bar{\pi}(f(s)) h, k \rangle d\mu(s).$$

Furthermore if $L : A \rightarrow B$ is a nondegenerate homomorphism into a C^* -algebra B , then

$$\bar{L} \left(\int_G f(s) d\mu(s) \right) = \int_G \bar{L}(f(s)) d\mu(s).$$

One property of the multiplier algebra says that when A is unital, $M(A) = A$. Hence $M(B(H)) = B(H)$ and $M_s(B(H)) = B_s(H)$. Thus for $f \in C_c(G, B(H))$ and $U : G \rightarrow U(H)$ a unitary representation of G , the previous proposition guarantees that

$$\int_G f(s) U_s d\mu(s)$$

is a well-defined element of $B(H)$. This is the integral that we will shortly be concerned with.

A **C^* -dynamical system** is a triple (A, G, α) consisting of a C^* -algebra A , a locally compact group G , and a continuous homomorphism $\alpha : G \rightarrow \text{Aut}(A)$. A **covariant representation** of (A, G, α) is a pair (π, U) consisting of a representation $\pi : A \rightarrow B(H)$ and a unitary representation $U : G \rightarrow U(H)$ on the same Hilbert space H such that

$$\pi(\alpha_s(a)) = U_s \pi(a) U_s^*.$$

We say that (π, U) is a **possibly degenerate** covariant representation if π is a possibly degenerate representation. It is shown in [36, Example 2.14] that covariant representations always exist, a fact needed for our definition of a C^* -norm on $C_c(G, A)$. One can first check that $C_c(G, A)$ is a C^* -algebra with convolution given by

$$(f * g)(s) := \int_G f(r) \alpha_r(g(r^{-1}s)) d\mu(r),$$

involution by

$$f^*(s) := \Delta(s^{-1})\alpha_s(f(s^{-1})^*),$$

and norm by

$$\|f\|_1 := \int_G \|f(s)\| d\mu(s),$$

where $\|f^*\|_1 = \|f\|_1$ and $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Proposition 2.4.7. [36, Proposition 2.23] *Let (π, U) be a possibly degenerate covariant representation of (A, G, α) on H . Then*

$$\pi \rtimes U(f) := \int_G \pi(f(s))U_s d\mu(s)$$

defines a norm-decreasing $$ -representation of $C_c(G, A)$ on H called the **integrated form** of (π, U) .*

Furthermore, $\pi \rtimes U$ is nondegenerate if π is nondegenerate.

The integrand in the above proposition is in $C_c(G, B_s(H)) = C_c(G, M_s(B(H)))$. Hence the integrated form is well-defined by Proposition 2.4.6. Now for each $f \in C_c(G, A)$, define

$$\|f\| := \sup \left\{ \|\pi \rtimes U(f)\| : (\pi, U) \text{ is a possibly degenerate covariant representation of } (A, G, \alpha) \right\}.$$

Then $\|\cdot\|$ is a norm on $C_c(G, A)$ called the **universal norm**. The universal norm is dominated by the $\|\cdot\|_1$ -norm, and the completion of $C_c(G, A)$ with respect to $\|\cdot\|$ is a C^* -algebra called the **crossed product** of A by G and is denoted $A \rtimes_\alpha G$.

Example 2.4.8. (1) [36, Lemma 2.50] Let G be a finite group with $|G| = n$. If we let G act on itself by left translation, then $C(G) \rtimes_{\text{lt}} G \cong M_n(\mathbb{C})$.

(2) [36, Example 2.51] When G is a locally compact group (not necessarily finite) acting on itself by left translation, then $C_0(G) \rtimes_{\text{lt}} G \cong K(L^2(G))$.

We close this section with a final result, in light of which we think of the crossed product as being generated by a universal covariant representation.

Proposition 2.4.9. [36, Proposition 2.40] *If $\alpha : G \rightarrow \text{Aut}(A)$ is a dynamical system, then the map sending a covariant pair (π, U) to its integrated form $\pi \rtimes U$ is a one-to-one correspondence between nondegenerate covariant representations of (A, G, α) and nondegenerate representations of $A \rtimes_\alpha G$.*

2.5 K -Theory

K -theory concerns itself with associating to any C^* -algebra A a pair of abelian groups $K_0(A)$ and $K_1(A)$, which carry information about A . More recently, Elliott's program seeks to use K -theory to construct an invariant for entire classes of C^* -algebras. The scope of this paper is much more narrow; we introduce the basic definitions and results needed for later. Resources on the subject include [3], [35], and [30]. Since we are primarily concerned with the K -theory of graph C^* -algebras, we prefer Raeburn's condensed treatment given in [27, Chapter 7].

Let $\text{Proj}(M_n(A))$ denote the set of projections in the C^* -algebra $M_n(A)$. For any partial isometry $u \in M_n(A)$, we declare its initial projection u^*u to be equivalent to its final projection uu^* ; this determines an equivalence relation on $\text{Proj}(M_n(A))$. A projection $p \in \text{Proj}(M_n(A))$ can be embedded in $\text{Proj}(M_{n+1}(A))$ by adding a row and column of zeros to the bottom and right of p ; this process preserves the equivalence relation. Hence $\text{Proj}(M_n(A)) \subset \text{Proj}(M_{n+1}(A))$, and we define $\text{Proj}_\infty(A) := \bigcup_{n=0}^\infty \text{Proj}(M_n(A))$. The set of equivalence classes $D(A) := \{[p] : p \in \text{Proj}_\infty(A)\}$ is an abelian semigroup, where $[p] + [q]$ is defined to be $[p_1 + q_1]$ satisfying $p_1 \in [p]$, $q_1 \in [q]$, and $p_1 q_1 = 0$. These p_1 and q_1 can always be found by taking larger n if necessary. Thus we define

$$K_0(A) := \left\{ [p] - [q] : p, q \in \text{Proj}_\infty(A) \right\},$$

where

$$([p] - [q]) + ([r] - [s]) = ([p] + [r]) - ([q] + [s]).$$

Example 2.5.1. Let E be a row-finite directed graph. The projections $\{p_v : v \in E^0\}$ give rise to equivalence classes $[p_v] \in D(C^*(E))$. By combining the Cuntz-Krieger relation $p_v = \sum_{r(e)=v} s_e s_e^*$

with the above equivalence relation, we find that

$$[p_v] = \sum_{r(e)=v} [s_e s_e^*] = \sum_{r(e)=v} [s_e^* s_e] = \sum_{r(e)=v} [p_{s(e)}]$$

in $K_0(C^*(E))$. It turns out that $K_0(C^*(E))$ is generated by $\{[p_v] : v \in E^0\}$ (see [27, Example 7.2]).

We now define $K_1(A)$ by dealing first with the case where A is unital. Let $U(M_n(A))$ denote the unitaries in $M_n(A)$. For $u, v \in U(M_n(A))$, we say u and v are equivalent if there is a continuous path $[0, 1] \rightarrow U(M_n(A)) : t \mapsto u_t$ where $u_0 = u$ and $u_1 = v$. We embed $U(M_n(A))$ in $U(M_{n+1}(A))$ by adding a 1 to the bottom-right corner of u and zeros everywhere else in the new right column and bottom row; the equivalence relation is preserved. We then define $U_\infty(A) := \bigcup_{n=0}^\infty U(M_n(A))$, and thus

$$K_1(A) := \left\{ [u] : u \in U_\infty(A) \right\}$$

with

$$[u][v] = [uv].$$

When A is not necessarily unital, we define $K_1(A) := K_1(\tilde{A})$ where \tilde{A} is the unitization of A . One could also do the same for $K_0(A)$, in which case the identity equivalence class would need to be omitted: if $\phi : \tilde{A} \rightarrow \mathbb{C}$ is given by $\phi(a, \lambda) = \lambda$, then take

$$K_0(A) := \ker \left(\phi_* : K_0(\tilde{A}) \longrightarrow K_0(\mathbb{C}) \cong \mathbb{Z} \right).$$

We present a few examples.

Example 2.5.2. [30, Page 234]

- (1) $K_0(\mathbb{C}) \cong K_0(M_n(\mathbb{C})) \cong \mathbb{Z}$ and $K_1(\mathbb{C}) \cong K_1(M_n(\mathbb{C})) \cong 0$.
- (2) $K_0(B(H)) \cong K_1(B(H)) \cong 0$.
- (3) $K_0(C(\mathbb{T}^n)) \cong K_1(C(\mathbb{T}^n)) \cong \mathbb{Z}^{2^{n-1}}$.
- (4) $K_i(A \oplus B) \cong K_i(A) \oplus K_i(B)$ for $i = 0, 1$.

The K -theory for crossed products is quite substantial and we will be frequently using the following result due to Pimsner and Voiculescu, which first appeared in [25].

Theorem 2.5.3. [3, Theorem 10.2.1] (*Pimsner-Voiculescu Exact Sequence*) *Let A be a C^* -algebra and $\alpha \in \text{Aut}(A)$. Then*

$$\begin{array}{ccccc}
 K_0(A) & \xrightarrow{1 - \alpha_*} & K_0(A) & \longrightarrow & K_0(A \rtimes_{\alpha} \mathbb{Z}) \\
 \uparrow & & & & \downarrow \\
 K_1(A \rtimes_{\alpha} \mathbb{Z}) & \longleftarrow & K_1(A) & \xleftarrow{1 - \alpha_*} & K_1(A)
 \end{array}$$

is a cyclic six-term exact sequence.

2.6 Strong Morita Equivalence

While the notion of Morita equivalence already existed for rings, Rieffel introduced the C^* -algebra version in [29]. Although weaker than an isomorphism, Morita equivalence can be thought of as a still useful way to compare two C^* -algebras: Morita equivalent C^* -algebras have the same ideal structure and representation theory. Furthermore, Morita equivalence preserves nuclearity and K -theory, amongst other properties. For these reasons, many general C^* -algebra results can often be phrased in terms of Morita equivalence and not isomorphism. Unless otherwise stated, detailed proofs of the following remarks can be found in [28].

Definition 2.6.1. Let A be a C^* -algebra. A (right) **inner product A -module** is a (right) A -module X with a pairing $\langle \cdot, \cdot \rangle_A : X \times X \rightarrow A$ such that

- (1) $\langle x, \lambda y + \mu z \rangle_A = \lambda \langle x, y \rangle_A + \mu \langle x, z \rangle_A$;
- (2) $\langle x, y \cdot a \rangle_A = \langle x, y \rangle_A a$;
- (3) $\langle x, y \rangle_A^* = \langle y, x \rangle_A$;
- (4) $\langle x, x \rangle_A$ is a positive element of A ;

(5) $\langle x, x \rangle_A = 0$ implies that $x = 0$.

When X is an inner product A -module, then

$$\|x\|_A := \|\langle x, x \rangle_A\|^{\frac{1}{2}}$$

is a norm on X . So we define a **Hilbert A -module** to be an inner product A -module X which is complete with respect to $\|\cdot\|_A$. We say X is **full** if the ideal

$$I = \text{span}\{\langle x, y \rangle_A : x, y \in X\}$$

is dense in A .

Definition 2.6.2. Let A and B be C^* -algebras. Then an $A - B$ **equivalence bimodule** is an $A - B$ -bimodule such that

- (1) X is a full left Hilbert A -module and a full right Hilbert B -module;
- (2) for all $x, y \in X$, $a \in A$, and $b \in B$,

$$\langle a \cdot x, y \rangle_B = \langle x, a^* \cdot y \rangle_B \text{ and } \langle x \cdot b, y \rangle_A = \langle x, y \cdot b^* \rangle_A;$$

- (3) for all $x, y, z \in X$,

$$\langle x, y \rangle_A \cdot z = x \cdot \langle y, z \rangle_B.$$

Then two C^* -algebras A and B are said to be **strongly Morita equivalent** or simply **Morita equivalent**, if there is an $A - B$ equivalence bimodule X . When this is the case, we say that X **implements** the Morita equivalence of A and B . And indeed, Morita equivalence is an equivalence relation on C^* -algebras (though this is a nontrivial fact). Now consider the set $\mathcal{I}(A)$ of closed two-sided ideals in the C^* -algebra A . Let $\mathcal{I}(A)$ be partially ordered by inclusion, forming a **lattice** in the following sense: each pair $I, J \in \mathcal{I}(A)$ has a greatest lower bound $I \cap J$ and a least upper bound, which is the ideal generated by $I \cup J$. A **lattice isomorphism** is a bijection between two lattices that preserves order and the lattice structure. The following is due to Rieffel.

Theorem 2.6.3. [28, Proposition 3.24] (*Rieffel Correspondence*) *If A and B are Morita equivalent C^* -algebras, then the lattices $\mathcal{I}(A)$ and $\mathcal{I}(B)$ are isomorphic.*

Next define a **primitive ideal** of the C^* -algebra A to be an ideal which is the kernel of a nonzero irreducible representation of A . Denote by $\text{Prim}(A)$ the set of primitive ideals in A . If F is a subset of $\text{Prim}(A)$, define the **closure** \overline{F} of F to be

$$\overline{F} := \left\{ P \in \text{Prim}(A) : \bigcap_{I \in F} I \subseteq P \right\}.$$

The subsets F of $\text{Prim}(A)$, for which $F = \overline{F}$, make up the closed sets of the **Jacobson topology** on $\text{Prim}(A)$. The next result is the reasoning behind Morita equivalent C^* -algebras having the same representation theory.

Theorem 2.6.4. [28, Corollary 3.33] *The Rieffel Correspondence restricts to a homeomorphism between $\text{Prim}(A)$ and $\text{Prim}(B)$.*

Finally, we conclude this section with two facts about Morita equivalence that we will use later, the first of which is due to Beer.

Proposition 2.6.5. [2, Proposition 3.2] *The property of nuclearity is invariant under Morita equivalence of C^* -algebras.*

The second fact requires a few definitions. We say two C^* -algebras A and B are **stably isomorphic** if

$$A \otimes K \cong B \otimes K,$$

where K is the compact operators on a separable infinite-dimensional Hilbert space H (the tensor product is unique since K is nuclear by [20, Page 196]). The following proposition combines [30, Proposition 6.4.1] and [30, Proposition 8.2.8].

Proposition 2.6.6. [30] *Let A be a C^* -algebra. Then*

$$K_0(A) \cong K_0(A \otimes K) \text{ and } K_1(A) \cong K_1(A \otimes K).$$

An **approximate identity** for a C^* -algebra A is an increasing net $(u_\lambda)_{\lambda \in \Lambda}$ of positive elements in the closed unit ball of A such that $au_\lambda \rightarrow a$ for all $a \in A$. We say A is **σ -unital** if A has a countable approximate identity. While it can be shown that stable isomorphism implies Morita equivalence for any two C^* -algebras, the converse is not necessarily true unless we add the σ -unital condition. The following amazing result first appeared in [5].

Theorem 2.6.7. [28, Theorem 5.55] (*Brown-Green-Rieffel*) *Two σ -unital C^* -algebras are stably isomorphic if and only if they are Morita equivalent.*

The main C^* -algebras we are concerned with for the rest of this thesis are both separable and unital, either of which implies σ -unital. Therefore, by combining the previous theorem and proposition, we can say K_0 and K_1 are invariant under Morita equivalence.

Chapter 3

Bunce-Deddens Algebras

3.1 Classical Approach

In their 1975 paper [6], Bunce and Deddens introduced a family of simple C^* -algebras, which have since been cited and generalized many times over. We summarize their construction, using the more condensed notation of [10].

Definition 3.1.1. A **weighted shift** is an operator T for which there is an orthonormal basis $\{e_k : k \geq 1\}$ and weights w_n such that $Te_n = w_n e_{n+1}$ for all $n \geq 1$. T is a **periodic** weighted shift if there is an integer n such that $w_{k+n} = w_k$ for all $k \geq 1$.

Suppose that $\{a_k\}$ is a strictly increasing sequence of positive integers such that $a_k | a_{k+1}$ for all $k \geq 1$. The **Bunce-Deddens algebra** $\mathcal{BD}(\{a_k\})$ is the quotient by the compact operators K of the C^* -algebra generated by all weighted shifts (with respect to a fixed basis) of period a_k for $k \geq 1$. We now make this more precise.

Proposition 3.1.2. [10, Corollary V.3.2] *Let $\mathcal{W}(n)$ denote the C^* -algebra of all weighted shifts of period n with respect to a fixed basis. Then the sequence*

$$0 \longrightarrow K \longrightarrow \mathcal{W}(n) \longrightarrow M_n(C(\mathbb{T})) \longrightarrow 0$$

is exact.

Let

$$\mathcal{BD}(a_k) = \mathcal{W}(a_k)/K \cong M_{a_k}(C(\mathbb{T})).$$

Then since a_k -periodic weighted shifts are also a_{k+1} -periodic, there is a natural injection $\iota_k : \mathcal{BD}(a_k) \rightarrow \mathcal{BD}(a_{k+1})$. Hence

$$\mathcal{BD}(\{a_k\}) \cong \varinjlim (M_{a_k}(C(\mathbb{T})), \iota_k).$$

Taking $\{a_k\}$ as above, then for each prime p , there is a unique ϵ_p in $\mathbb{N} \cup \{0, \infty\}$ which is the supremum of the exponents of powers of p which divide a_k as $k \rightarrow \infty$. We define the **supernatural number** associated to $\{a_k\}$ to be the formal product $\delta(\{a_k\}) = \prod_{p \text{ prime}} p^{\epsilon_p}$. For this paper we will assume that $\epsilon_p = 0$ for all but finitely many p .

Theorem 3.1.3. [10, Theorem V.3.5] *Two Bunce-Deddens algebras $\mathcal{BD}(\{a_k\})$ and $\mathcal{BD}(\{b_k\})$ are *-isomorphic if and only if their super natural numbers $\delta(\{a_k\})$ and $\delta(\{b_k\})$ are equal.*

Example 3.1.4. $\mathcal{BD}(\{2 \cdot 3, 2 \cdot 3^2, 2 \cdot 3^3, \dots\}) \cong \mathcal{BD}(\{2 \cdot 3^2, 2 \cdot 3^4, 2 \cdot 3^8, \dots\})$, corresponding to supernatural number $2 \cdot 3^\infty$.

Because of this correspondence, we will from now on refer to a Bunce-Deddens algebra by its equivalence class $\mathcal{BD}_{\delta(\{a_k\})}$.

3.2 Odometer Approach

Around 1990 Putnam ([26]), Exel ([12]), and others were considering another approach to the Bunce-Deddens algebra using what is known as the odometer action. This is relevant to us since we will be concerned with an action on a graph algebra that mimics the odometer. This section also employs the notation of [10].

Let $\{n_i\}$ be a sequence of integers such that each $n_i \geq 2$. Let $X_i = \{0, 1, 2, \dots, n_i - 1\}$ and form the Cantor set $X = \prod_{i \geq 1} X_i$ with the product topology. We write each element $a = (a_i)$ in this set as a formal sum $\sum_{i \geq 1} a_i N_i$ where $N_1 = 1$ and $N_{i+1} = n_i N_i$. In order to define formal addition with carry to the right, we say $(a_i) + (b_i) = (c_i)$ where c_i are the unique integers in X_i such that

$$\sum_{i=1}^n (a_i + b_i) N_i \equiv \sum_{i=1}^n c_i N_i \pmod{N_{n+1}} \text{ for all } n \geq 1.$$

And to acquire $b = -a$, let i_0 be the least integer such that $a_{i_0} \neq 0$ (or ∞ in the case where all $a_i = 0$). Then set $b_i = 0$ for all $i < i_0$, $b_{i_0} = n_{i_0} - a_{i_0}$, and $b_i = n_i - 1 - a_i$ for all $i > i_0$. Noting that \mathbb{Z} -actions are determined by the identity, the **odometer action** σ of \mathbb{Z} on X is obtained by addition of $(1, 0, 0, \dots)$ with carry to the right. Carry occurs whenever the modulo operation is invoked, and it is always a 1 that is carried to the right.

Example 3.2.1. Form the Cantor set X from the sequence $\{2, 3, 4, 2, 2, \dots\}$. Then $1 \in \mathbb{Z}$ acts on $(1, 1, 3, 1, 1, \dots) \in X$ via the computation:

$$\begin{array}{r} (1, 1, 3, 1, 1, \dots) \\ + (1, 0, 0, 0, 0, \dots) \\ \hline (0, 2, 3, 1, 1, \dots) \end{array}$$

Since addition is continuous, this is a homeomorphism. The action of $-1 \in \mathbb{Z}$ is defined to be addition by $-(1, 0, 0, \dots)$ with carry to the right. Above we are told how to compute negatives, from which we deduce that $-(1, 0, 0, \dots) = (1, 2, 3, 1, 1, \dots)$ in this case. Then

$$(0, 2, 3, 1, 1, \dots) - (1, 0, 0, \dots) = (0, 2, 3, 1, 1, \dots) + (1, 2, 3, 1, 1, \dots) = (1, 1, 3, 1, 1, \dots),$$

and we see that the action of -1 undoes the action of 1 .

Lastly, cylinder sets for the product topology on X are given by

$$J(x_1, \dots, x_k) = \{a \in X : a_i = x_i \text{ for } 1 \leq i \leq k\}.$$

We now view the Bunce-Deddens algebra as a crossed product.

Theorem 3.2.2. [10, Theorem VIII.4.1] *Let $X = \prod_{i \geq 1} X_i$ where each X_i has $n_i \geq 2$ points, and let σ be the odometer action. Then the crossed product $C(X) \rtimes_{\sigma} \mathbb{Z}$ is isomorphic to the Bunce-Deddens algebra $\mathcal{BD}_{\prod_{i \geq 1} n_i}$.*

3.3 Graph Approach

We now establish our own approach to the Bunce-Deddens algebra by examining graph algebras. The crossed product we construct is shown to be strongly Morita equivalent to the

classical Bunce-Deddens algebra. This section relies heavily on results from Kumjian and Pask's paper [16].

Choose a sequence $\{n_i\}$ of integers such that each $n_i \geq 2$. Let $X_i = \{0, 1, \dots, n_i - 1\}$ and

$$X^* = \bigcup_{m=0}^{\infty} \{x_1 x_2 \cdots x_m : x_i \in X_i \text{ for all } 1 \leq i \leq m\}.$$

We think of X^* as being the set of finite words where the i^{th} letter is chosen from the alphabet X_i ; in the case $m = 0$, we denote the empty word by \emptyset . We now define an action σ of \mathbb{Z} on X^* . Let \mathbb{Z} fix \emptyset and for any $w = x_1 x_2 \cdots x_m \in X^* \setminus \emptyset$,

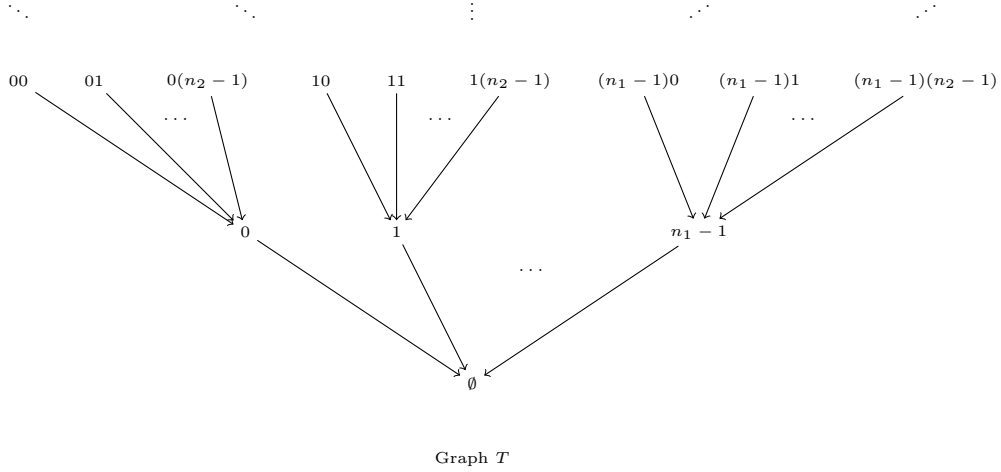
$$1 \cdot (x_1 x_2 \cdots x_m) := \begin{cases} 00 \cdots 0 & \text{if all } x_i = n_i - 1 \\ 00 \cdots (x_{I_w} + 1)x_{I_w+1} \cdots x_m & \text{else} \end{cases},$$

where I_w is the least index such that $x_{I_w} < n_{I_w} - 1$.

Analogous to the odometer case, we think of this action as being addition by 1 in the first letter with carry to the right. While we could define the action of -1 in a similar fashion to what we did in the previous section, we instead define the action of -1 on $x_1 x_2 \cdots x_m$ to be equivalent to the action $\sigma_{n_1 n_2 \cdots n_m - 1}$, or the 1-action repeated $(n_1 n_2 \cdots n_m - 1)$ -many times. The reasoning behind this will be made more clear once we present explicit graphs later, but it stems from the fact that applying the 1-action $(n_1 n_2 \cdots n_m)$ -many times on $x_1 x_2 \cdots x_m$ acts the same as the identity, or 0-action. We will now generate a directed graph T , in particular a row-finite tree.

Definition 3.3.1. [16] Let E be a directed graph and for $e \in E^1$ we formally denote the reverse edge by \bar{e} where $s(\bar{e}) = r(e)$ and $r(\bar{e}) = s(e)$. The set of reverse edges is denoted \bar{E}^1 and we define $\bar{\bar{e}} = e$ for $\bar{e} \in \bar{E}^1$. We call $a = a_1 \cdots a_n$ a **walk** in E if $a_i \in E^1 \cup \bar{E}^1$ are such that $s(a_i) = r(a_{i+1})$ for $1 \leq i \leq n - 1$; we write $s(a) = s(a_n)$ and $r(a) = r(a_1)$. A vertex will be regarded as a trivial walk. A walk $a = a_1 \cdots a_n$ is said to be **reduced** if it does not contain the subword $a_i a_{i+1} = \bar{e} e$ for any $e \in E^1 \cup \bar{E}^1$. Finally, a directed graph T is a **tree** if and only if there is precisely one reduced walk between any two vertices.

Let X^* be the vertex set for the graph T . We declare that two vertices are connected by an edge if they are of the form $w, wx \in X^*$ where $x \in X_{|w|+1}$, in which case the edge goes from wx to w .



Observe that T is row-finite by construction and satisfies the conditions of a tree: between any two vertices is a single reduced walk, that is, a path made up of edges and reverse-direction edges such that at no point does one traverse an edge and then immediately the same edge in the reverse direction. We will often refer to \emptyset as the **root** of this tree.

Definition 3.3.2. [16] Let E be a row-finite directed graph, where E^∞ denotes the space of paths of infinite length. The **shift-tail equivalence** relation for $x, y \in E^\infty$ is given by $x \sim_k y$ if and only if there is an $N \geq 1$ and $k \in \mathbb{Z}$ such that $x_i = y_{i-k}$ for $i \geq N$. Then we define \mathcal{G}_E to be the set of triples (x, k, y) such that $x \sim_k y$ in E^∞ .

Definition 3.3.3. [16] Let T be a tree and \sim denote shift-tail equivalence. For $\mu \in T^*$, let $Z(\mu)$ denote the set of infinite paths of the form μz for some $z \in T^\infty$; the $Z(\mu)$ form a basis of open sets for the topology on T^∞ . Then $\partial T = T^\infty / \sim$ endowed with the quotient topology is called the **boundary** of T .

For each $v \in T^0$, where T is a row-finite tree, define

$$Y(v) = \{[x] \in \partial T : r(x) = v\}.$$

Then it is a result in [16, Lemma 4.2] that $\{Y(v) : v \in T^0\}$ forms a basis of compact open sets for the quotient topology on ∂T ; moreover the quotient map $T^\infty \rightarrow \partial T$ is a local homeomorphism and ∂T is Hausdorff.

Note now that since every vertex emits a single edge, there is a unique finite path connecting any non-root vertex to the root. We say two vertices are at the same level if they are the same path-length distance from the root. Then the map σ induces an action of \mathbb{Z} on the graph T – also called σ – where vertices at the same level are permuted in a cyclic fashion. Specifically, if j -many vertices exist at a given level of the graph T , then j is the least number such that $\sigma_j(v) = v$ for any and all v in that level. Since there is a one-to-one correspondence between edges and the single vertices that emit each one, the action on the edges is determined by the action on the vertices that emit them. Furthermore, the action sends the edges (vertices) that make up a path – finite or infinite – to a new set of edges (vertices) that determine a path of the same length. More precisely, the action sends the path $e_1 e_2 e_3 \cdots$ to $\sigma_g(e_1) \sigma_g(e_2) \sigma_g(e_3) \cdots$ where $g \in \mathbb{Z}$. Hence σ also induces actions on the infinite path space T^∞ , on the boundary ∂T , and on the set \mathcal{G}_T . Observe now that if $x \sim_k y$ in T^∞ then there is only one such $k \in \mathbb{Z}$ with this property since T is a tree and so we may write the elements of \mathcal{G}_T as (x, y) . Furthermore ∂T is simply the set of infinite paths with range the root; this is because there is a unique possibly length 0 finite path μ connecting the range of any infinite path z to the root, and so z can be thought of as a tail of the infinite path μz . Hence there is an obvious map from ∂T into the space $X = \prod_{i \geq 1} X_i$ from the previous section, and thus the boundary is a Cantor set.

Lemma 3.3.4. [16, Proposition 4.3] *Let T be a row-finite tree. Then $C^*(T)$ is strongly Morita equivalent to $C_0(\partial T)$.*

Proof. We summarize the proof because its details will be needed for the main result of this section.

View $C_c(T^\infty)$ as a left $C_c(\mathcal{G}_T)$ -module by defining, for $x \in T^\infty$,

$$(fg)(x) = \sum_{y \in [x]} f(x, y)g(y),$$

where $f \in C_c(\mathcal{G}_T)$ and $g \in C_c(T^\infty)$. Define a $C_c(\mathcal{G}_T)$ -valued inner product on $C_c(T^\infty)$ for $(x, y) \in \mathcal{G}_T$ by

$$\langle f, g \rangle_{C_c(\mathcal{G}_T)}(x, y) = f(x)\overline{g(y)}$$

for $f, g \in C_c(T^\infty)$.

View $C_c(T^\infty)$ as a right $C_c(\partial T)$ -module by defining, for $x \in T^\infty$,

$$(gh)(x) = g(x)h([x]),$$

where $g \in C_c(T^\infty)$ and $h \in C_c(\partial T)$. Define a $C_c(\partial T)$ -valued inner product on $C_c(T^\infty)$ for $[x] \in \partial T$ by

$$\langle f, g \rangle_{C_0(\partial T)}([x]) = \sum_{y \in [x]} \overline{f(y)}g(y)$$

for $f, g \in C_c(T^\infty)$.

Let X be the completion of $C_c(T^\infty)$ in the norm arising from $\langle \cdot, \cdot \rangle_{C_0(\partial T)}$. Then X is a $C^*(T) - C_0(\partial T)$ equivalence bimodule. \square

The induced actions of \mathbb{Z} on T and ∂T further induce actions of \mathbb{Z} by automorphisms on $C^*(T)$ and $C_0(\partial T)$, the former of which restricts to $C_c(\mathcal{G}_T)$ as in [16, Note 4.12]. The action on $C^*(T)$ is given by $\sigma_g(p_v) = p_{\sigma_g(v)}$ and $\sigma_g(s_e) = s_{\sigma_g(e)}$ ($g \in \mathbb{Z}$), where p_v is a projection and s_e is a partial isometry in the Cuntz-Krieger T -family generating $C^*(T)$. The following is due to Curto, Muhly, and Williams.

Lemma 3.3.5. [9, Theorem 1] *Let G be a locally compact group and α, β actions of G on C^* -algebras A, B . Suppose too that X is a complete $A - B$ equivalence bimodule. If there is a strongly continuous action of G on X , $\{\tau_t\}_{t \in G}$, such that for $a \in A$, $b \in B$, and $x, y \in X$,*

$$\langle \tau_t x, \tau_t y \rangle_A = \alpha_t(\langle x, y \rangle_A) \text{ and } \langle \tau_t x, \tau_t y \rangle_B = \beta_t(\langle x, y \rangle_B),$$

then $A \rtimes G$ and $B \rtimes G$ are strongly Morita equivalent.

This lemma allows us to prove the next result, which is the foundation for the rest of this paper.

Theorem 3.3.6. $C^*(T) \rtimes_{\sigma} \mathbb{Z}$ is strongly Morita equivalent to $C(\partial T) \rtimes_{\sigma} \mathbb{Z}$ and thus also to the Bunce-Deddens algebra $\mathcal{BD}_{\prod_{i \geq 1} n_i}$.

Proof. Since ∂T is a Cantor set and thus compact, $C_0(\partial T) = C(\partial T)$. That we get the Bunce-Deddens algebra follows from Theorem 3.2.2. The rest of the proof mimics that of [16, Theorem 4.13].

By the previous lemmas, we require a strongly continuous action τ of \mathbb{Z} on $C_c(T^{\infty})$ that is compatible with the odometer actions on $C_c(\mathcal{G}_T)$ and $C_c(\partial T)$. To that end, for $z \in \mathbb{Z}$, $f \in C_c(T^{\infty})$, and $x \in T^{\infty}$ we set $\tau_z f(x) := f(\sigma_z x)$ where σ is the odometer action on the infinite path space.

Then for $f, g \in C_c(T^{\infty})$, $z \in \mathbb{Z}$, and $(x, y) \in \mathcal{G}_T$ we check

$$\begin{aligned} \langle \tau_z f, \tau_z g \rangle_{C_c(\mathcal{G}_T)}(x, y) &= \tau_z f(x) \overline{\tau_z g(y)} \\ &= f(\sigma_z x) \overline{g(\sigma_z y)} \\ &= \langle f, g \rangle_{C_c(\mathcal{G}_T)}(\sigma_z x, \sigma_z y) \\ &= \sigma_z \langle f, g \rangle_{C_c(\mathcal{G}_T)}(x, y), \end{aligned}$$

and for $[x] \in \partial T$

$$\begin{aligned} \langle \tau_z f, \tau_z g \rangle_{C_0(\partial T)}([x]) &= \sum_{y \in [x]} \tau_z f(y) \overline{\tau_z g(y)} \\ &= \sum_{y \in [x]} f(\sigma_z y) \overline{g(\sigma_z y)} \\ &= \sum_{w \in [\sigma_z x]} f(w) \overline{g(w)} \\ &= \langle f, g \rangle_{C_0(\partial T)}([\sigma_z x]) \\ &= \sigma_z \langle f, g \rangle_{C_0(\partial T)}([x]). \end{aligned}$$

□

3.4 Toward a Generalization

We now want to generalize the construction from the last section using higher rank graphs. The resulting crossed product is strongly Morita equivalent to a generalized Bunce-Deddens algebra, as defined by Orfanos in [21]. We note that others such as Kribs and Solel ([15]) and Kumjian, Pask, and Sims ([18]), have also come up with generalizations for these classical objects. The former construct their spaces using graphs C_j which are single cycles (looped paths) through j vertices. The latter base their approach on coverings of k -graphs, in the sense of Pask, Quigg, and Raeburn ([23]). In turn, Rout has considered the K -theory for those C^* -algebras studied by Kribs and Solel in [31]. This paper, however, concerns itself with those generalized Bunce-Deddens algebras coming from the approach of Orfanos.

Definition 3.4.1. [21] A group G is **residually finite** if it has a separating family of finite index normal subgroups, that is, for every finite set F in G , there is a normal subgroup L of finite index in G such that the quotient map $G \rightarrow G/L$ is injective when restricted to F .

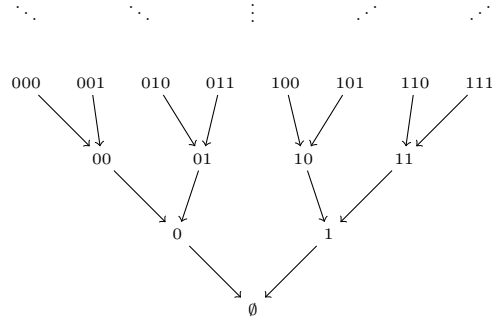
Definition 3.4.2. [21] Let G be a residually finite group and fix a decreasing sequence of finite index normal subgroups L_n that is a separating family. With respect to the homomorphisms $\phi_{nm} : G/L_m \rightarrow G/L_n$ given by $\phi_{nm}(xL_m) = xL_n$ for $n \leq m$, we define the **profinite completion** \tilde{G} of G to be the inverse limit $\varprojlim G/L_n$, that is, the subgroup of $\prod_{n \geq 1} G/L_n$ consisting of sequences $(x_n L_n)_n$ such that $x_m L_n = \phi(x_m L_m) = x_n L_n$ whenever $n \leq m$.

Let π_n denote the canonical projection onto G/L_n , for all $n \geq 1$. By [21, Proposition 5], \tilde{G} is a non-empty totally disconnected compact Hausdorff group and the sets $\pi_n^{-1}(\{xL_n\})$, for $xL_n \in G/L_n$ and $n \geq 1$, form a basis of compact open sets for the topology on \tilde{G} . Now take G as in the above definition and add the condition that G be amenable. Let α denote the action of G by left multiplication on \tilde{G} . The resulting crossed products $C(\tilde{G}) \rtimes_{\alpha} G$ are the **generalized Bunce-Deddens algebras** (of type Orfanos). The following result combines [21, Corollary 7], [21, Theorem 9], and [21, Theorem 12].

Theorem 3.4.3. [21] *The generalized Bunce-Deddens algebras are unital, simple, separable, nuclear, and quasidiagonal; they have real rank zero, stable rank one, comparability of projections, and a unique trace.*

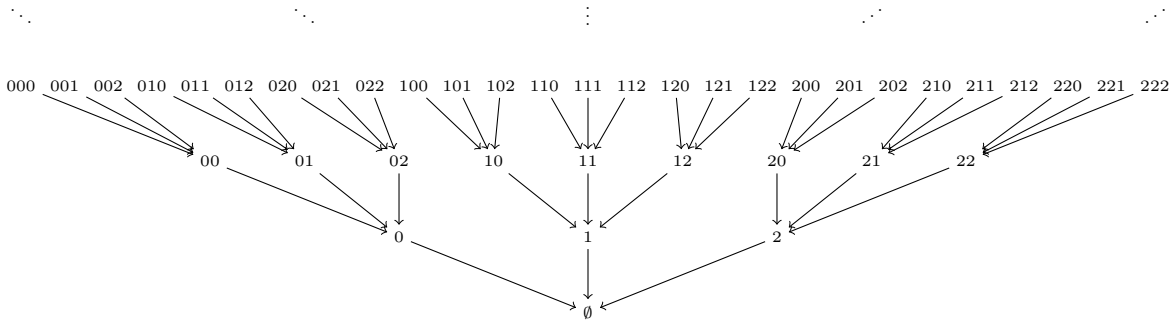
Our goal for now is to generalize the graph approach of the previous section by way of a \mathbb{Z}^2 -action on a 2-graph, and establish strong Morita equivalence with these generalized Bunce-Deddens algebras. We start by considering T_n , the 1-graph T that corresponds to the supernatural number $n = \prod_{i \geq 1} n_i$. It should be mentioned that given a fixed supernatural number, there can be many ways to draw T_n . Fortunately, this won't affect our results since the Bunce-Deddens algebras are determined by their supernatural number and not the ordering of $\{n_i\}$.

Example 3.4.4. The following is the graph T_{2^∞} arising from the sequence $\{2, 2, 2, \dots\}$. A sequence such as $\{4, 4, 4, \dots\}$ would yield a different graph but still correspond to 2^∞ .



Graph T_{2^∞}

Next is the graph T_{3^∞} arising from the sequence $\{3, 3, 3, \dots\}$.



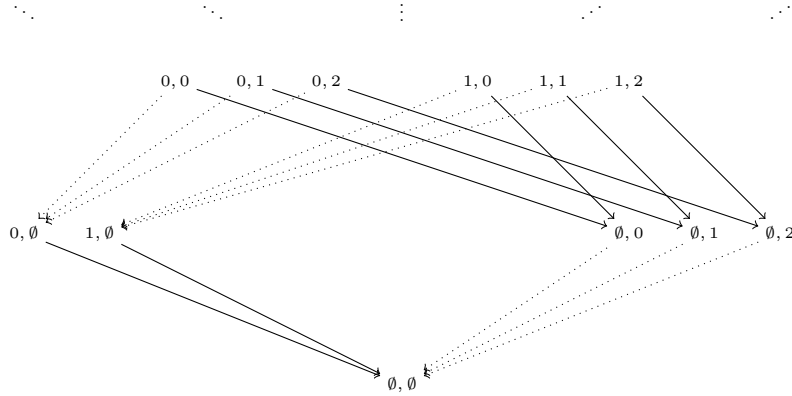
Graph T_{3^∞}

The following result is due to Kumjian and Pask when they first introduced higher rank graphs.

Proposition 3.4.5. [17, Proposition 1.8] *Let (Λ_1, d_1) and (Λ_2, d_2) be rank k_1, k_2 graphs respectively, then $(\Lambda_1 \times \Lambda_2, d_1 \times d_2)$ is a rank $k_1 + k_2$ graph where $\Lambda_1 \times \Lambda_2$ is the product category and $d_1 \times d_2 : \Lambda_1 \times \Lambda_2 \rightarrow \mathbb{W}^{k_1+k_2}$ is given by $d_1 \times d_2(\lambda_1, \lambda_2) = (d_1(\lambda_1), d_2(\lambda_2)) \in \mathbb{W}^{k_1} \times \mathbb{W}^{k_2}$ for $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$.*

This construction will be referred to as the **categorical product** of k -graphs.

Example 3.4.6. $T_{2\infty} \times T_{3\infty}$, a 2-graph, is the categorical product of the two 1-graphs above. In order to visualize a k -graph Λ we consider its **1-skeleton**, which is the colored directed graph $(\Lambda^0, \bigcup_{i=1}^k \Lambda^{e_i}, r, s)$ with the edges in each Λ^{e_i} drawn in a different color. In the picture below, solid edges have degree $e_1 = (1, 0)$ and dotted edges have degree $e_2 = (0, 1)$.



Part of the 1-Skeleton of the 2-Graph $T_{2\infty} \times T_{3\infty}$

In order to define a \mathbb{Z}^2 -action on $T_m \times T_n$, where m and n are supernatural numbers, we simply declare that the first (resp. second) copy of \mathbb{Z} act on T_m (resp. T_n) via the odometer action while fixing T_n (resp. T_m). We call this action the **generalized odometer action** and denote it by $\tilde{\sigma}$. As detailed in Example 2.3.8, we are now viewing T_m and T_n as 1-graphs and so the \mathbb{Z} -action on each is, more precisely, on the finite path space of each. Since these finite paths are determined by the vertices traversed, the action makes sense. We now present a few lemmas which are necessary to prove the main result of this section.

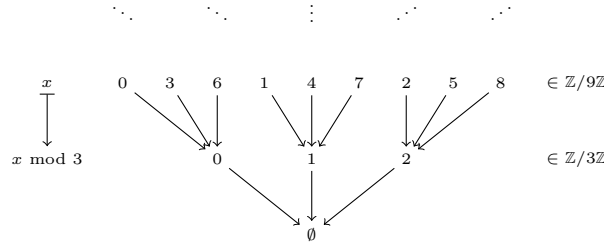
Lemma 3.4.7. Fix any supernatural number m . Let $\{a_i\}$ and $\{b_i\}$ be increasing sequences of positive integers such that $a_i|a_{i+1}$ and $b_i|b_{i+1}$ for all $i \geq 1$, $\delta(\{a_i\}) = m$, and $a_1 = p$ where p is any prime occurring in m . Then

(1) $C^*(T_m) \rtimes_{\sigma} \mathbb{Z} \sim_M C(\varprojlim \mathbb{Z}/a_i\mathbb{Z}) \rtimes_{\alpha} \mathbb{Z}$ (of type Orfanos) and

(2) $\varprojlim \mathbb{Z}/a_i\mathbb{Z} \times \varprojlim \mathbb{Z}/b_i\mathbb{Z} \cong \varprojlim (\mathbb{Z}/a_i\mathbb{Z} \times \mathbb{Z}/b_i\mathbb{Z})$.

Proof. Since inverse limits and direct products are both limits in the language of Category Theory (see [19, Page 69]), and limits commute with limits, the second claim is established.

We now prove the first claim. Utilizing Theorem 3.3.6, if we can show that $\partial T_m \cong \varprojlim \mathbb{Z}/a_i\mathbb{Z}$ with equivalent \mathbb{Z} -actions, then we can identify $C(\varprojlim \mathbb{Z}/a_i\mathbb{Z}) \rtimes \mathbb{Z}$ with $C(\partial T_m) \rtimes \mathbb{Z}$ and the claim holds. To that end, consider for example the following relabeled graph of T_{3^∞}



where $\{a_i\} = \{3^i\}$. By definition, an element of the inverse limit can be regarded as an infinite path terminating at the root \emptyset and vice versa. \mathbb{Z} acts on the above graph via modular addition, permuting vertices at the same level. This is the same as the odometer action and furthermore the vertices are permuted in the same order.

We omit the generalization of this example as it is notationally cumbersome. An important thing to note is distinguishing the sequence $\{m_i\}$, where $m = \prod_{i \geq 1} m_i$, from the sequence $\{a_i\}$, where $m = \delta(\{a_i\})$. For example $3^\infty 5^\infty$ arises from possible sequences $\{3, 5, 3, 5, 3, \dots\}$ or $\{5, 3, 5, 3, 5, \dots\}$, in which case we let $\{a_i\}$ be $\{3, 15, 45, 225, 675, \dots\}$ or $\{5, 15, 75, 225, 1125, \dots\}$ respectively. □

For a more detailed discussion on $\{m_i\}$ versus $\{a_i\}$, see Example 4.2.7.

Lemma 3.4.8. *Suppose $A \rtimes \mathbb{Z}$ and $B \rtimes \mathbb{Z}$ are C^* -algebras both of either form appearing in (1) of the previous lemma. Then \mathbb{Z}^2 acts on $A \otimes B$ in a natural way such that*

$$[A \rtimes \mathbb{Z}] \otimes [B \rtimes \mathbb{Z}] \cong [A \otimes B] \rtimes \mathbb{Z}^2.$$

Proof. We first note that all graph algebras are nuclear by [27, Remark 4.3]. By Lemma 3.3.4, the proof of the previous lemma, and the fact that strong Morita equivalence preserves nuclearity, A and B are necessarily nuclear. Since \mathbb{Z} is amenable, $A \rtimes \mathbb{Z}$ and $B \rtimes \mathbb{Z}$ are also nuclear by [36, Corollary 7.18]. Alternatively, the crossed products are nuclear by Theorem 3.4.3. In any case, all tensor products are understood.

As in the definition of $\tilde{\sigma}$, we declare $(1, 0)$ to act on $A \otimes B$ according to the \mathbb{Z} -action on A and $(0, 1)$ to act on $A \otimes B$ according to the \mathbb{Z} -action on B . Now consider the map

$$\varphi : \overline{C_c(\mathbb{Z}, A)} \otimes \overline{C_c(\mathbb{Z}, B)} \longrightarrow \overline{C_c(\mathbb{Z}^2, A \otimes B)}$$

given by

$$[f] \otimes [g] \longmapsto [f \otimes g],$$

where $[f]$ is the equivalence class of the Cauchy sequence $\{f\}$ and $(f \otimes g)(x, y) := f(x) \otimes g(y)$ for $(x, y) \in \mathbb{Z}^2$. Fixing the second coordinate $[g]$, we check that

$$\begin{aligned} \varphi\left(\alpha[f_1] + [f_2] \otimes [g]\right) &= \varphi\left([\alpha f_1 + f_2] \otimes [g]\right) \\ &= [\alpha f_1 + f_2 \otimes g] \\ &= \alpha[f_1 \otimes g] + [f_2 \otimes g] \\ &= \alpha\varphi\left([f_1] \otimes [g]\right) + \varphi\left([f_2] \otimes [g]\right), \end{aligned}$$

and so φ is linear in the first coordinate; similarly, it is linear in the second. It is a routine exercise to show that φ is a $*$ -homomorphism.

We now introduce some notation. Let D be a C^* -algebra and G a discrete (hence locally compact) group. For $d \in D$ and $x \in G$, define the map $d\delta_x : G \rightarrow D$ by $d\delta_x(g) = d$ if $g = x$ and $d\delta_x(g) = 0$ otherwise. Finite linear combinations of elements of the form $d\delta_x$ (where $x \in \mathbb{Z}$) are

dense in $\overline{C_c(\mathbb{Z}, D)}$ and the linear span of elements of the form $(a \otimes b)\delta_{(x,y)}$ (where $(x, y) \in \mathbb{Z}^2$ and D_1, D_2 are C^* -algebras with $a \in D_1$ and $b \in D_2$) is dense in $\overline{C_c(\mathbb{Z}^2, D_1 \otimes D_2)}$. Hence

$$\varphi : \sum_{i=1}^n \alpha_i a_i \delta_{x_i} \otimes b_i \delta_{y_i} \longmapsto \sum_{i=1}^n \alpha_i (a_i \otimes b_i) \delta_{(x_i, y_i)},$$

where $n \in \mathbb{N}$ and $\alpha_i \in \mathbb{C}$, $a_i \in A$, $b_i \in B$, $x_i, y_i \in \mathbb{Z}$ for $i = 1, \dots, n$. Observe that further finite linear combinations of the combinations already presented in this map are of the same form. And it is clear that we have a one-to-one correspondence.

Now if

$$\pi_1 : [A \rtimes \mathbb{Z}] \otimes [B \rtimes \mathbb{Z}] \longrightarrow B(H_1)$$

is a representation then by the universal properties of tensor products and crossed products by \mathbb{Z}^2 , we can construct a representation

$$\pi'_1 : [A \otimes B] \rtimes \mathbb{Z}^2 \longrightarrow B(H_1)$$

with

$$\pi_1(a_i \delta_{x_i} \otimes b_i \delta_{y_i}) = \pi'_1((a_i \otimes b_i) \delta_{(x_i, y_i)}),$$

in terms of covariant pairs in the sense of [22, Definition 2.4(b)]. Conversely, if

$$\pi_2 : [A \otimes B] \rtimes \mathbb{Z}^2 \longrightarrow B(H_2)$$

is a representation then by the universal properties of tensor products and crossed products by \mathbb{Z} , we can construct a representation

$$\pi'_2 : [A \rtimes \mathbb{Z}] \otimes [B \rtimes \mathbb{Z}] \longrightarrow B(H_2)$$

using covariant pairs by setting

$$\pi'_2(a_i \delta_{x_i} \otimes b_i \delta_{y_i}) = \pi_2((a_i \otimes b_i) \delta_{(x_i, y_i)}).$$

Because of this correspondence of representations we find that φ is an isometry on the above dense subsets, where the C^* -norm on each is obtained as a supremum of norms over all representations.

Thus φ as defined on these dense subsets extends to a $*$ -isomorphism of C^* -algebras. \square

Lemma 3.4.9. [17, Corollary 3.5] *Let (Λ_i, d_i) be k_i -graphs for $i = 1, 2$, then $C^*(\Lambda_1 \times \Lambda_2) \cong C^*(\Lambda_1) \otimes C^*(\Lambda_2)$ via the map $s_{(\lambda_1, \lambda_2)} \mapsto s_{\lambda_1} \otimes s_{\lambda_2}$ for $(\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2$.*

Note that $\tilde{\sigma}$ induces an action of \mathbb{Z}^2 by automorphisms on $C^*(T_m \times T_n)$ and therefore also on $C^*(T_m) \otimes C^*(T_n)$.

Theorem 3.4.10. *Let m and n be supernatural numbers. Then $C^*(T_m \times T_n) \rtimes_{\tilde{\sigma}} \mathbb{Z}^2$ is strongly Morita equivalent to a generalized Bunce-Deddens algebra of type Orfanos.*

Proof. Choose $\{a_i\}$ to be an increasing sequence of positive integers such that $a_i | a_{i+1}$ for all $i \geq 1$, $\delta(\{a_i\}) = m$, and $a_1 = p$ where p is any prime occurring in m . Similarly, choose a $\{b_i\}$ for n . Then

$$\begin{aligned} C^*(T_m \times T_n) \rtimes \mathbb{Z}^2 &\cong \left[C^*(T_m) \otimes C^*(T_n) \right] \rtimes \mathbb{Z}^2 \\ &\cong \left[C^*(T_m) \rtimes \mathbb{Z} \right] \otimes \left[C^*(T_n) \rtimes \mathbb{Z} \right] \\ &\sim_M \left[C(\varprojlim \mathbb{Z}/a_i \mathbb{Z}) \rtimes \mathbb{Z} \right] \otimes \left[C(\varprojlim \mathbb{Z}/b_i \mathbb{Z}) \rtimes \mathbb{Z} \right] \\ &\cong \left[C(\varprojlim \mathbb{Z}/a_i \mathbb{Z}) \otimes C(\varprojlim \mathbb{Z}/b_i \mathbb{Z}) \right] \rtimes \mathbb{Z}^2 \\ &\cong C\left(\varprojlim \mathbb{Z}/a_i \mathbb{Z} \times \varprojlim \mathbb{Z}/b_i \mathbb{Z} \right) \rtimes \mathbb{Z}^2 \\ &\cong C\left(\varprojlim (\mathbb{Z}/a_i \mathbb{Z} \times \mathbb{Z}/b_i \mathbb{Z}) \right) \rtimes \mathbb{Z}^2. \end{aligned}$$

This final space is a generalized Bunce-Deddens algebra of type Orfanos: \mathbb{Z}^2 is an amenable residually finite group and the $a_i \mathbb{Z} \times b_i \mathbb{Z}$ form the separating family, where

$$\mathbb{Z} \times \mathbb{Z} / a_i \mathbb{Z} \times b_i \mathbb{Z} \cong \mathbb{Z}/a_i \mathbb{Z} \times \mathbb{Z}/b_i \mathbb{Z}.$$

□

Corollary 3.4.11. *Let m and n be supernatural numbers. Then $C^*(T_m \times T_n) \rtimes_{\tilde{\sigma}} \mathbb{Z}^2$ is strongly Morita equivalent to $\mathcal{BD}_m \otimes \mathcal{BD}_n$.*

Proof. Apply Theorem 3.3.6 to the fact that

$$C^*(T_m \times T_n) \rtimes \mathbb{Z}^2 \cong \left[C^*(T_m) \rtimes \mathbb{Z} \right] \otimes \left[C^*(T_n) \rtimes \mathbb{Z} \right].$$

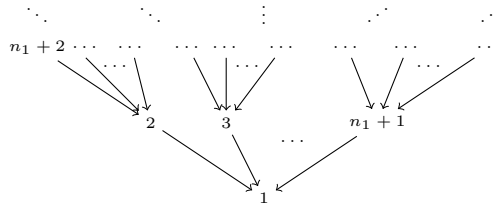
□

Chapter 4

Visualizing K -Groups

4.1 Directed Graph Case

We relabel the vertices of the graph T by counting them



in order to construct the adjacency matrix A_T , given by

$$A_T(v, w) = \#\{e \in T^1 : s(e) = w \text{ and } r(e) = v\},$$

which has the form

$$\begin{pmatrix} 0 & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \mathbf{1} & \cdots & \mathbf{1} & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Every row of this matrix contains some finite number of adjacent 1s corresponding to the fact that every vertex receives a finite number of edges. Furthermore every column except the first contains precisely one 1, since every vertex except the root emits a single edge; the root emits none.

Lemma 4.1.1. [27, Theorem 7.16] *Let E be a row-finite graph with no sources, and let A_E be the adjacency matrix of E . Then $K_1(C^*(E))$ is isomorphic to the kernel of $1 - A_E^t : \mathbb{Z}^{E^0} \rightarrow \mathbb{Z}^{E^0}$, and $K_0(C^*(E))$ is isomorphic to the cokernel.*

The matrix $1 - A_T^t$ has the form

$$\begin{pmatrix} \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\mathbf{1} & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -\mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\mathbf{1} & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\mathbf{1} & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & 0 & \cdots \\ 0 & \vdots & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & 0 & \cdots \\ 0 & -\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 & \cdots \\ 0 & 0 & -\mathbf{1} & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where now we have 1s on the diagonal and a single -1 in each row except for the first. It is then a straightforward computation to show that $\ker(1 - A_T^t) = 0$.

We now consider the following Pimsner-Voiculescu sequence for $C^*(T) \rtimes_{\sigma} \mathbb{Z}$.

$$\begin{array}{ccccc} K_0(C^*(T)) & \xrightarrow{1 - \sigma_*} & K_0(C^*(T)) & \longrightarrow & K_0(C^*(T) \rtimes \mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_1(C^*(T) \rtimes \mathbb{Z}) & \longleftarrow & 0 & \longleftarrow & 0 \end{array}$$

Here σ is the odometer action of \mathbb{Z} on $C^*(T)$ and σ_* is the induced action on $K_0(C^*(T))$, that is, $\sigma_*([p_v]) = [\sigma(p_v)] = [p_{\sigma(v)}]$ for the K_0 -generators $\{[p_v] : v \in T^0\}$. And from this diagram, we observe that

$$K_0(C^*(T) \rtimes \mathbb{Z}) \cong K_0(C^*(T)) / \text{im}(1 - \sigma_*)$$

and

$$K_1(C^*(T) \rtimes \mathbb{Z}) \cong \ker(1 - \sigma_*).$$

Let $\mathbb{Z}_{(n)}$ denote the subgroup of \mathbb{Q} of all rational numbers whose denominators “divide” the

supernatural number $n = \prod_{p \text{ prime}} p^{\epsilon_p}$, that is,

$$\mathbb{Z}_{(n)} = \left\{ \frac{z}{p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}} \in \mathbb{Q} : z \in \mathbb{Z}, m \in \mathbb{N} \right.$$

$$p_1, \dots, p_m \text{ are prime factors of } n,$$

$$\left. k_1, \dots, k_m \in \mathbb{N} \cup \{0\} \text{ such that } k_i \leq \epsilon_{p_i} \text{ for all } 1 \leq i \leq m \right\}.$$

The following theorem gives the K -theory, which is preserved under strong Morita equivalence, for the Bunce-Deddens algebra \mathcal{BD}_n . While the K -theory for the Bunce-Deddens algebra is already known ([3, 10.11.4]), our new method employs graph techniques.

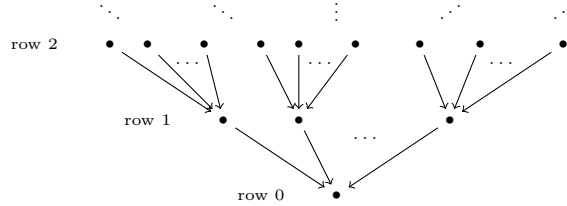
Theorem 4.1.2. *Let n be any supernatural number. Then*

$$K_0(C^*(T_n) \rtimes_{\sigma} \mathbb{Z}) \cong \mathbb{Z}_{(n)}$$

and

$$K_1(C^*(T_n) \rtimes_{\sigma} \mathbb{Z}) \cong \mathbb{Z}.$$

Proof. For this proof we will need to be able to reference certain rows of the graph T_n .



Since T_n is row-finite, $K_0(C^*(T_n))$ is generated by $\{[p_v] : v \in T_n^0\}$ subject only to the relations $[p_v] = \sum_{r(e)=v} [p_{s(e)}]$. Now $(1 - \sigma_*)([p_v]) = [p_v] - [p_{\sigma(v)}] = 0$ when $[p_v] = [p_{\sigma(v)}]$, which is only true for the projection p_\emptyset . In order to establish K_1 , we need to show that $\ker(1 - \sigma_*) = \mathbb{Z}[p_\emptyset] \cong \mathbb{Z}$. We do this by showing that if

$$(1 - \sigma_*) \left(\sum_{i=1}^N \alpha_i [p_{v_i}] \right) = \left(\sum_{i=1}^N \alpha_i [p_{v_i}] \right) - \left(\sum_{i=1}^N \alpha_i [p_{\sigma(v_i)}] \right) = 0,$$

where $N \in \mathbb{N}$ and $\alpha_i \in \mathbb{Z}$ for $1 \leq i \leq N$, then

$$\sum_{i=1}^N \alpha_i [p_{v_i}] = z [p_\emptyset]$$

for some $z \in \mathbb{Z}$. To that end, let row j be the maximum row containing one of the v_i for $1 \leq i \leq N$.

Observe that by the relations $[p_v] = \sum_{r(e)=v} [p_{s(e)}]$, we can write

$$[p_\emptyset] = \sum_{v \text{ in row } j} [p_v].$$

Similarly, we can write

$$\sum_{i=1}^N \alpha_i [p_{v_i}] = \sum_{v \text{ in row } j} \beta_v [p_v],$$

where each $\beta_v \in \mathbb{Z}$. But then, because σ simply permutes the vertices in the same row,

$$\sum_{v \text{ in row } j} \beta_v [p_v] = \sum_{v \text{ in row } j} \beta_v [p_{\sigma(v)}]$$

only when all the β_v are equal to the same integer β , whence

$$\sum_{v \text{ in row } j} \beta [p_v] = \beta [p_\emptyset]$$

as desired. This concludes the proof for K_1 .

Turning now to K_0 , define a map

$$\varphi : K_0(C^*(T_n))/\text{im}(1 - \sigma_*) \longrightarrow \mathbb{Z}_{(n)}$$

determined by $\varphi([[p_\emptyset]]) = 1$, else $\varphi([[p_v]]) = \frac{1}{n_1 n_2 \cdots n_m}$, where m represents the row of the tree in which the vertex v is placed. It is clear that $\varphi([[p_v]]) = \sum_{r(e)=v} \varphi([[p_{s(e)}]])$ (see the example below), so φ is a homomorphism. Since σ permutes the vertices at the same level, $[[p_v]] = [[p_w]]$ whenever v and w are in the same row, establishing well-definedness.

We now check injectivity. Suppose

$$\varphi \left(\sum_{i=1}^N \alpha_i [[p_{v_i}]] \right) = 0,$$

where $N \in \mathbb{N}$ and $\alpha_i \in \mathbb{Z}$ for $1 \leq i \leq N$, then we must show that $\sum_{i=1}^N \alpha_i [[p_{v_i}]] = 0$. Again, we let row j be the maximum row containing one of the v_i for $1 \leq i \leq N$. Since addition in the quotient group is given by $[[p_a]] + [[p_b]] = [[p_a] + [p_b]]$ for vertices a and b , then like above, we can write

$$\sum_{i=1}^N \alpha_i [[p_{v_i}]] = \sum_{v \text{ in row } j} \beta_v [[p_v]]$$

such that each $\beta_v \in \mathbb{Z}$. But now that we are constrained to vertices v in a single row j , all the $[[p_v]]$ are equal and we can further write

$$\sum_{v \text{ in row } j} \beta_v [[p_v]] = \beta [[p_V]],$$

where $\beta \in \mathbb{Z}$ and V is any vertex in row j (say the leftmost). So $\varphi(\beta [[p_V]]) = 0$, hence $\beta = 0$ and thus $\ker \varphi = 0$.

Now consider any generator $\frac{1}{p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}} \in \mathbb{Z}_{(n)}$. Recall that $n = \prod_{i \geq 1} n_i$, where n_i is the number of vertices introduced in row i for each vertex in row $i - 1$. Then let I be the minimum row in the graph T_n such that $p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}$ divides $\prod_{i=1}^I n_i$, and define

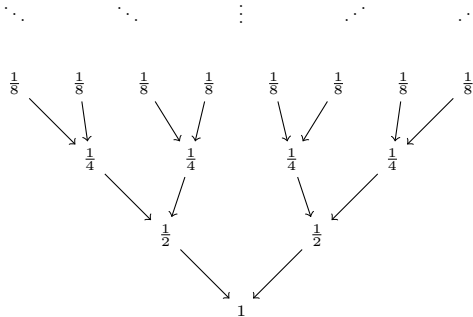
$$d = \frac{\prod_{i=1}^I n_i}{p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}}.$$

Let $[[p_{I_1}]], [[p_{I_2}]], \dots, [[p_{I_d}]]$ refer to the projections corresponding to the d leftmost vertices in row I . Then

$$\begin{aligned} \varphi([[p_{I_1}]] + [[p_{I_2}]] + \cdots + [[p_{I_d}]]) &= \varphi(d[[p_{I_1}]]) \\ &= \frac{d}{\prod_{i=1}^I n_i} \\ &= \frac{1}{p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}}, \end{aligned}$$

so φ is surjective as well. □

Example 4.1.3. We relabel the vertices of T_{2^∞} with their corresponding φ -images in $\mathbb{Z}_{(2^\infty)}$. This gives us a way to visualize the structure of the generators of K_0 for the Bunce-Deddens algebra \mathcal{BD}_{2^∞} by way of $C^*(T_{2^\infty}) \rtimes \mathbb{Z}$.



4.2 2-Graph Case

Definition 4.2.1. [33] Let \mathcal{B} be the smallest subcategory of the category of separable nuclear C^* -algebras which contains the separable Type I algebras and is closed under the operations of taking ideals, quotients, extensions, inductive limits, stable isomorphism, and crossed products by \mathbb{Z} and by \mathbb{R} .

Let $K_*(A)$ denote $K_0(A) \oplus K_1(A)$. The following is known as the Künneth theorem and is due to Schochet.

Theorem 4.2.2. [33, Section 1] (*Künneth Theorem by Schochet*) Let A and B be C^* -algebras with $A \in \mathcal{B}$. Then there is a natural short exact sequence

$$0 \longrightarrow K_*(A) \otimes K_*(B) \xrightarrow{\alpha} K_*(A \otimes B) \xrightarrow{\beta} \text{Tor}(K_*(A), K_*(B)) \longrightarrow 0.$$

The sequence is $\mathbb{Z}/2\mathbb{Z}$ -graded with $\deg \alpha = 0$, $\deg \beta = 1$, where $K_m \otimes K_n$ and $\text{Tor}(K_m, K_n)$ are given degree $m + n$ ($m, n \in \mathbb{Z}/2\mathbb{Z}$).

It is a result ([14, Page 265]) that $\text{Tor}(G, H) = 0$ if either G or H is torsion-free. In all cases we consider G and H are both torsion-free, hence α will always be an isomorphism. The next formula is again due to Schochet.

Formula 4.2.3. [33, Remark 4.11] (*Künneth Formula by Schochet*) Suppose that $N \in \mathcal{B}$ is some fixed C^* -algebra with $K_0(N) = G$, $K_1(N) = 0$. Define

$$K_n(B; G) \cong K_n(B \otimes N).$$

Suppose further that $N \otimes N \otimes K \cong N \otimes K$ (so that $G \otimes G \cong G$). Then we get the exact sequences

$$\begin{aligned} 0 \longrightarrow \sum_m K_m(A; G) \otimes K_{n-m}(B; G) &\longrightarrow K_n(A \otimes B; G) \\ &\longrightarrow \sum_m \text{Tor}(K_m(A; G), K_{n-m-1}(B; G)) \longrightarrow 0 \end{aligned}$$

for $m, n \in \mathbb{Z}/2\mathbb{Z}$.

Let $N = \mathbb{C}$ (so that $G = \mathbb{Z}$) in the above formula. In this thesis our K -groups take the form \mathbb{Z} or $\mathbb{Z}_{(s)}$ (where s is some supernatural number), both of which are clearly torsion-free. Hence in the above formula, Tor is always 0 and so it is a straightforward computation to show that

$$K_0(A \otimes B) \cong [K_0(A) \otimes K_0(B)] \oplus [K_1(A) \otimes K_1(B)]$$

and

$$K_1(A \otimes B) \cong [K_0(A) \otimes K_1(B)] \oplus [K_1(A) \otimes K_0(B)].$$

Lemma 4.2.4. *Let $m = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$ and $n = q_1^{l_1} q_2^{l_2} \cdots q_s^{l_s}$ be supernatural numbers; that is, $r, s \in \mathbb{N}$, $k_1, \dots, k_r, l_1, \dots, l_s \in \mathbb{N} \cup \{\infty\}$, and $p_1, \dots, p_r, q_1, \dots, q_s$ are positive primes. Then as abelian groups,*

$$\mathbb{Z}_{(m)} \otimes \mathbb{Z}_{(n)} \cong \mathbb{Z}_{(mn)}.$$

Proof. Consider the map $\varphi : \mathbb{Z}_{(mn)} \rightarrow \mathbb{Z}_{(m)} \otimes \mathbb{Z}_{(n)}$ given on generators by

$$\frac{z}{(p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s)^k} \mapsto \frac{z}{(p_1 p_2 \cdots p_r)^k} \otimes \frac{1}{(q_1 q_2 \cdots q_s)^k}.$$

By definition an element of $\mathbb{Z}_{(mn)}$ can be expressed using different exponents for each prime in the denominator. Furthermore it is possible that one of the primes p_1, \dots, p_r is equal to one of the primes q_1, \dots, q_s . However, by properties of \mathbb{Q} , such an element is equivalent to one of the form in the above map. Without loss of generality, suppose $k \geq l$ in the remaining computations.

$$\begin{aligned} & \varphi \left(\frac{x}{(p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s)^k} + \frac{y}{(p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s)^l} \right) \\ &= \varphi \left(\frac{x + (p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s)^{k-l} y}{(p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s)^k} \right) \\ &= \frac{x + (p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s)^{k-l} y}{(p_1 p_2 \cdots p_r)^k} \otimes \frac{1}{(q_1 q_2 \cdots q_s)^k} \\ &= \left(\frac{x}{(p_1 p_2 \cdots p_r)^k} + \frac{(q_1 q_2 \cdots q_s)^{k-l} y}{(p_1 p_2 \cdots p_r)^l} \right) \otimes \frac{1}{(q_1 q_2 \cdots q_s)^k} \\ &= \frac{x}{(p_1 p_2 \cdots p_r)^k} \otimes \frac{1}{(q_1 q_2 \cdots q_s)^k} + \frac{(q_1 q_2 \cdots q_s)^{k-l} y}{(p_1 p_2 \cdots p_r)^l} \otimes \frac{1}{(q_1 q_2 \cdots q_s)^k} \\ &= \frac{x}{(p_1 p_2 \cdots p_r)^k} \otimes \frac{1}{(q_1 q_2 \cdots q_s)^k} + \frac{y}{(p_1 p_2 \cdots p_r)^l} \otimes \frac{1}{(q_1 q_2 \cdots q_s)^l} \\ &= \varphi \left(\frac{x}{(p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s)^k} \right) + \varphi \left(\frac{y}{(p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s)^l} \right), \end{aligned}$$

hence φ is a homomorphism. The inverse map φ^{-1} is given by

$$\frac{x}{(p_1 p_2 \cdots p_r)^k} \otimes \frac{y}{(q_1 q_2 \cdots q_s)^l} \mapsto \frac{xy}{(p_1 p_2 \cdots p_r)^k (q_1 q_2 \cdots q_s)^l},$$

and the computations

$$\begin{aligned} \frac{x}{(p_1 p_2 \cdots p_r)^k} \otimes \frac{y}{(q_1 q_2 \cdots q_s)^l} &\stackrel{\varphi^{-1}}{\mapsto} \frac{xy}{(p_1 p_2 \cdots p_r)^k (q_1 q_2 \cdots q_s)^l} \\ &= \frac{(q_1 q_2 \cdots q_s)^{k-l} xy}{(p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s)^k} \\ &\stackrel{\varphi}{\mapsto} \frac{(q_1 q_2 \cdots q_s)^{k-l} xy}{(p_1 p_2 \cdots p_r)^k} \otimes \frac{1}{(q_1 q_2 \cdots q_s)^k} \\ &= \frac{x}{(p_1 p_2 \cdots p_r)^k} \otimes \frac{y}{(q_1 q_2 \cdots q_s)^l} \end{aligned}$$

and

$$\begin{aligned} \frac{z}{(p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s)^k} &\stackrel{\varphi}{\mapsto} \frac{z}{(p_1 p_2 \cdots p_r)^k} \otimes \frac{1}{(q_1 q_2 \cdots q_s)^k} \\ &\stackrel{\varphi^{-1}}{\mapsto} \frac{z}{(p_1 p_2 \cdots p_r q_1 q_2 \cdots q_s)^k} \end{aligned}$$

show that φ is a bijection. □

Theorem 4.2.5. *Let m and n be supernatural numbers. Then*

$$K_0\left(C^*(T_m \times T_n) \rtimes_{\bar{\sigma}} \mathbb{Z}^2\right) \cong \mathbb{Z}_{(mn)} \oplus \mathbb{Z}$$

and

$$K_1\left(C^*(T_m \times T_n) \rtimes_{\bar{\sigma}} \mathbb{Z}^2\right) \cong \mathbb{Z}_{(m)} \oplus \mathbb{Z}_{(n)}.$$

Proof. By Corollary 3.4.11, we know that $C^*(T_m \times T_n) \rtimes \mathbb{Z}^2$ is strongly Morita equivalent to $\mathcal{BD}_m \otimes \mathcal{BD}_n$. Since the K -groups for the Bunce-Deddens algebras are torsion-free, we can invoke the Künneth formula. Hence K_0 is given by the formula

$$[\mathbb{Z}_{(m)} \otimes \mathbb{Z}_{(n)}] \oplus [\mathbb{Z} \otimes \mathbb{Z}] \cong [\mathbb{Z}_{(mn)}] \oplus [\mathbb{Z}]$$

and K_1 by

$$[\mathbb{Z}_{(m)} \otimes \mathbb{Z}] \oplus [\mathbb{Z} \otimes \mathbb{Z}_{(n)}] \cong [\mathbb{Z}_{(m)}] \oplus [\mathbb{Z}_{(n)}],$$

thus giving the result. □

Remark 4.2.6. Let A be a unital C^* -algebra with a unique tracial state τ . Define

$$R(A) := \{\tau(p) : p = p^* = p^2 \in A\}.$$

By [24, Proposition 2.3], due to Pasnicu, $\mathcal{BD}_m \otimes \mathcal{BD}_n$ has a unique tracial state and

$$R(\mathcal{BD}_m \otimes \mathcal{BD}_n) = \mathbb{Z}_{(mn)} \cap [0, 1].$$

The range of the trace on projections is an isomorphism invariant for the tensor product of two Bunce-Deddens algebras.

As a consequence of the above theorem, this gives the K -theory for certain second-order generalized Bunce-Deddens algebras of type Orfanos.

Example 4.2.7. We employ the notation of Theorems 3.4.10 and 4.2.5.

- (1) Start with any supernatural numbers m and n . For example, take

$$m = 2^1 3^\infty \text{ and } n = 3^\infty 5^\infty.$$

- (2) Choose sequences $\{m_i\}$ and $\{n_i\}$ such that each $m_i, n_i \geq 2$, $m = \prod_{i \geq 1} m_i$, and $n = \prod_{i \geq 1} n_i$.

In this example, we can take

$$\{m_i\} = \{3, 6, 3, 3, 3, \dots\} \text{ and } \{n_i\} = \{5, 3, 5, 3, 5, \dots\}.$$

These sequences dictate the structure of the trees T_m and T_n .

- (3) Construct sequences $\{a_i\}$ and $\{b_i\}$ by letting $a_i = m_1 m_2 \cdots m_i$ and $b_i = n_1 n_2 \cdots n_i$ for all i . In this example,

$$\{a_i\} = \{3, 18, 54, 162, 486, \dots\} \text{ and } \{b_i\} = \{5, 15, 75, 225, 1125, \dots\}.$$

By construction, $m = \delta(\{a_i\})$ and $n = \delta(\{b_i\})$.

- (4) The decreasing sequence $\{a_i \mathbb{Z} \times b_i \mathbb{Z}\}$ is a separating family of finite index normal subgroups for \mathbb{Z}^2 . Thus

$$C\left(\varprojlim (\mathbb{Z} \times \mathbb{Z} / a_i \mathbb{Z} \times b_i \mathbb{Z})\right) \rtimes \mathbb{Z}^2$$

is a generalized Bunce-Deddens algebra of type Orfanos.

(5) This is strongly Morita equivalent to $C^*(T_m \times T_n) \rtimes \mathbb{Z}^2$. Thus in this example, K_0 is given by

$$\mathbb{Z}_{(2^1 3^\infty 5^\infty)} \oplus \mathbb{Z}$$

and K_1 by

$$\mathbb{Z}_{(2^1 3^\infty)} \oplus \mathbb{Z}_{(3^\infty 5^\infty)}.$$

Chapter 5

Some Generalizations

5.1 Motivating Example

We begin by extending the definition of the **generalized odometer action** $\tilde{\sigma}$ to general $k \in \mathbb{N}$. Let n_1, n_2, \dots, n_k be supernatural numbers and define a \mathbb{Z}^k -action on the k -graph $T_{n_1} \times T_{n_2} \times \dots \times T_{n_k}$ as follows: for all $1 \leq j \leq k$, let the j^{th} copy of \mathbb{Z} act on T_{n_j} via the odometer action while fixing all other graphs. Like before, this induces a \mathbb{Z}^k -action by automorphisms on $C^*(T_{n_1} \times T_{n_2} \times \dots \times T_{n_k})$. The following lemma and theorem generalize the results of Section 3.4.

Lemma 5.1.1. *Fix any $k \in \mathbb{N}$ and let n_1, n_2, \dots, n_k be supernatural numbers. For each $1 \leq j \leq k$ let $\{a_{j_i}\}$ be an increasing sequence of positive integers such that $a_{j_i} | a_{j_{i+1}}$ for all $i \geq 1$, $\delta(\{a_{j_i}\}) = n_j$, and $a_{j_1} = p$ where p is any prime occurring in n_j . Let A_1, A_2, \dots, A_k be C^* -algebras either all of the form $C^*(T_{n_j})$ or all of the form $C(\varprojlim \mathbb{Z}/a_{j_i}\mathbb{Z})$, for $1 \leq j \leq k$. Then*

$$\left[(A_1 \otimes \dots \otimes A_{k-1}) \rtimes \mathbb{Z}^{k-1} \right] \otimes \left[A_k \rtimes \mathbb{Z} \right] \cong \left[A_1 \otimes A_2 \otimes \dots \otimes A_k \right] \rtimes \mathbb{Z}^k,$$

where the canonical generator $e_l \in \mathbb{Z}^j$, $1 \leq l \leq j$, acts on $A_1 \otimes \dots \otimes A_j$ according to the \mathbb{Z} -action induced by the odometer on A_l .

Proof. As in the proof of Lemma 3.4.8, all spaces are nuclear and so the tensor products are understood. Consider the map

$$\varphi : \overline{C_c(\mathbb{Z}^{k-1}, A_1 \otimes \dots \otimes A_{k-1})} \otimes \overline{C_c(\mathbb{Z}, A_k)} \longrightarrow \overline{C_c(\mathbb{Z}^k, A_1 \otimes \dots \otimes A_k)}$$

given by

$$[f_1 \otimes \cdots \otimes f_{k-1}] \otimes [g] \longmapsto [f_1 \otimes \cdots \otimes f_{k-1} \otimes g],$$

where $(f_1 \otimes \cdots \otimes f_j)(x_1, \dots, x_j) := f_1(x_1) \otimes \cdots \otimes f_j(x_j)$. Fixing $[g]$, we check linearity in the first coordinate:

$$\begin{aligned} & \varphi\left(\alpha[f_1 \otimes \cdots \otimes f_{k-1}] + [g_1 \otimes \cdots \otimes g_{k-1}] \otimes [g]\right) \\ &= \varphi\left([\alpha f_1 \otimes \cdots \otimes f_{k-1}] + (g_1 \otimes \cdots \otimes g_{k-1})\right] \otimes [g] \\ &= \left[(\alpha f_1 \otimes \cdots \otimes f_{k-1}) + (g_1 \otimes \cdots \otimes g_{k-1})\right] \otimes [g] \\ &= \alpha[f_1 \otimes \cdots \otimes f_{k-1} \otimes g] + [g_1 \otimes \cdots \otimes g_{k-1} \otimes g] \\ &= \alpha\varphi\left([f_1 \otimes \cdots \otimes f_{k-1}] \otimes [g]\right) + \varphi\left([g_1 \otimes \cdots \otimes g_{k-1}] \otimes [g]\right). \end{aligned}$$

Linearity in the second coordinate follows similarly. Once again, it is a routine exercise to show that φ is a *-homomorphism. Utilizing the same notation as in the proof of Lemma 3.4.8, we have

$$\varphi : \sum_{i=1}^n \alpha_i (a_{1,i} \otimes \cdots \otimes a_{k-1,i}) \delta_{(x_{1,i}, \dots, x_{k-1,i})} \otimes b_i \delta_{y_i} \longmapsto \sum_{i=1}^n \alpha_i (a_{1,i} \otimes \cdots \otimes a_{k-1,i} \otimes b_i) \delta_{(x_{1,i}, \dots, x_{k-1,i}, y_i)},$$

where $n \in \mathbb{N}$ and $\alpha_i \in \mathbb{C}$; $a_{1,i} \in A_1, \dots, a_{k-1,i} \in A_{k-1}, b_i \in A_k$; $x_{1,i}, \dots, x_{k-1,i}, y_i \in \mathbb{Z}$ for $i = 1, \dots, n$. This is again a one-to-one correspondence on dense subsets. And as before, by the universal properties of tensor products and crossed products by powers of \mathbb{Z} , the C^* -norms are preserved. Thus φ as defined on these dense subsets extends to a *-isomorphism of C^* -algebras. \square

Theorem 5.1.2. *Fix any $k \in \mathbb{N}$ and let n_1, n_2, \dots, n_k be supernatural numbers. Then*

$$C^*(T_{n_1} \times T_{n_2} \times \cdots \times T_{n_k}) \rtimes_{\tilde{\sigma}} \mathbb{Z}^k$$

is strongly Morita equivalent to a generalized Bunce-Deddens algebra of type Orfanos.

Proof. For each $1 \leq j \leq k$ choose $\{a_{j_i}\}$ to be an increasing sequence of positive integers such that

$a_{j_i}|a_{j_{i+1}}$ for all $i \geq 1$, $\delta(\{a_{j_i}\}) = n_j$, and $a_{j_1} = p$ where p is any prime occurring in n_j . Then

$$\begin{aligned}
& C^*(T_{n_1} \times T_{n_2} \times \cdots \times T_{n_k}) \rtimes \mathbb{Z}^k \\
& \cong \left[C^*(T_{n_1}) \otimes C^*(T_{n_2}) \otimes \cdots \otimes C^*(T_{n_k}) \right] \rtimes \mathbb{Z}^k \\
& \cong \left[C^*(T_{n_1}) \rtimes \mathbb{Z} \right] \otimes \left[C^*(T_{n_2}) \rtimes \mathbb{Z} \right] \otimes \cdots \otimes \left[C^*(T_{n_k}) \rtimes \mathbb{Z} \right] \\
& \sim_M \left[C(\varprojlim \mathbb{Z}/a_{1_i}\mathbb{Z}) \rtimes \mathbb{Z} \right] \otimes \left[C(\varprojlim \mathbb{Z}/a_{2_i}\mathbb{Z}) \rtimes \mathbb{Z} \right] \otimes \cdots \otimes \left[C(\varprojlim \mathbb{Z}/a_{k_i}\mathbb{Z}) \rtimes \mathbb{Z} \right] \\
& \cong \left[C(\varprojlim \mathbb{Z}/a_{1_i}\mathbb{Z}) \otimes C(\varprojlim \mathbb{Z}/a_{2_i}\mathbb{Z}) \otimes \cdots \otimes C(\varprojlim \mathbb{Z}/a_{k_i}\mathbb{Z}) \right] \rtimes \mathbb{Z}^k \\
& \cong C\left(\varprojlim \mathbb{Z}/a_{1_i}\mathbb{Z} \times \varprojlim \mathbb{Z}/a_{2_i}\mathbb{Z} \times \cdots \times \varprojlim \mathbb{Z}/a_{k_i}\mathbb{Z} \right) \rtimes \mathbb{Z}^k \\
& \cong C\left(\varprojlim (\mathbb{Z}/a_{1_i}\mathbb{Z} \times \mathbb{Z}/a_{2_i}\mathbb{Z} \times \cdots \times \mathbb{Z}/a_{k_i}\mathbb{Z}) \right) \rtimes \mathbb{Z}^k \\
& \cong C\left(\varprojlim \mathbb{Z}^k / (a_{1_i}\mathbb{Z} \times a_{2_i}\mathbb{Z} \times \cdots \times a_{k_i}\mathbb{Z}) \right) \rtimes \mathbb{Z}^k.
\end{aligned}$$

Note that, when $k = 1$ no products appear in the above computation and many of the steps are redundant. Alternatively, when $k = 1$ the resulting crossed product is equivalent to the classical Bunce-Deddens algebra, which is trivially of type Orfanos. \square

Corollary 5.1.3. *Fix any $k \geq 2$. Let n_1, n_2, \dots, n_k be supernatural numbers. Then*

$$C^*(T_{n_1} \times T_{n_2} \times \cdots \times T_{n_k}) \rtimes_{\tilde{\sigma}} \mathbb{Z}^k \cong \left[C^*(T_{n_1} \times \cdots \times T_{n_{k-1}}) \rtimes_{\tilde{\sigma}} \mathbb{Z}^{k-1} \right] \otimes \left[C^*(T_{n_k}) \rtimes_{\sigma} \mathbb{Z} \right]$$

Proof.

$$\begin{aligned}
C^*(T_{n_1} \times T_{n_2} \times \cdots \times T_{n_k}) \rtimes \mathbb{Z}^k & \cong \left[C^*(T_{n_1}) \rtimes \mathbb{Z} \right] \otimes \left[C^*(T_{n_2}) \rtimes \mathbb{Z} \right] \otimes \cdots \otimes \left[C^*(T_{n_k}) \rtimes \mathbb{Z} \right] \\
& \cong \left[C^*(T_{n_1} \times \cdots \times T_{n_{k-1}}) \rtimes \mathbb{Z}^{k-1} \right] \otimes \left[C^*(T_{n_k}) \rtimes \mathbb{Z} \right].
\end{aligned}$$

\square

Our goal is to again compute the K -theory for generalized Bunce-Deddens algebras of type Orfanos, this time for all $k \in \mathbb{N}$. The following example is not a proof, rather it is meant to motivate the definitions appearing in the next section.

Example 5.1.4. Recall the Künneth formula that holds in our torsion-free scenario:

$$K_0(A \otimes B) \cong [K_0(A) \otimes K_0(B)] \oplus [K_1(A) \otimes K_1(B)],$$

$$K_1(A \otimes B) \cong [K_0(A) \otimes K_1(B)] \oplus [K_1(A) \otimes K_0(B)].$$

Utilizing the previous corollary, we recursively compute

$$K_0\left(C^*(T_{n_1} \times T_{n_2} \times \cdots \times T_{n_k}) \rtimes \mathbb{Z}^k\right) \text{ and } K_1\left(C^*(T_{n_1} \times T_{n_2} \times \cdots \times T_{n_k}) \rtimes \mathbb{Z}^k\right)$$

up to $k = 6$. Keep in mind that for groups, tensor products distribute over direct sums and direct sums satisfy associativity (so final brackets are dropped).

- $C^*(T_m) \rtimes \mathbb{Z}$

$$K_0 : \mathbb{Z}_{(m)}$$

$$K_1 : \mathbb{Z}$$

- $C^*(T_m \times T_n) \rtimes \mathbb{Z}^2$

$$K_0 : \mathbb{Z}_{(mn)} \oplus \mathbb{Z}$$

$$K_1 : \mathbb{Z}_{(m)} \oplus \mathbb{Z}_{(n)}$$

- $C^*(T_m \times T_n \times T_o) \rtimes \mathbb{Z}^3$

We work out the details of the Künneth formula in this case. Here $A = C^*(T_m \times T_n) \rtimes \mathbb{Z}^2$ and $B = C^*(T_o) \rtimes \mathbb{Z}$.

$$\begin{aligned} K_0 &: \left[(\mathbb{Z}_{(mn)} \oplus \mathbb{Z}) \otimes \mathbb{Z}_{(o)} \right] \oplus \left[(\mathbb{Z}_{(m)} \oplus \mathbb{Z}_{(n)}) \otimes \mathbb{Z} \right] \\ &= \left[(\mathbb{Z}_{(mn)} \otimes \mathbb{Z}_{(o)}) \oplus (\mathbb{Z} \otimes \mathbb{Z}_{(o)}) \right] \oplus \left[(\mathbb{Z}_{(m)} \otimes \mathbb{Z}) \oplus (\mathbb{Z}_{(n)} \otimes \mathbb{Z}) \right] \\ &= \mathbb{Z}_{(mno)} \oplus \mathbb{Z}_{(o)} \oplus \mathbb{Z}_{(m)} \oplus \mathbb{Z}_{(n)} \end{aligned}$$

$$\begin{aligned} K_1 &: \left[(\mathbb{Z}_{(mn)} \oplus \mathbb{Z}) \otimes \mathbb{Z} \right] \oplus \left[(\mathbb{Z}_{(m)} \oplus \mathbb{Z}_{(n)}) \otimes \mathbb{Z}_{(o)} \right] \\ &= \left[(\mathbb{Z}_{(mn)} \otimes \mathbb{Z}) \oplus (\mathbb{Z} \otimes \mathbb{Z}) \right] \oplus \left[(\mathbb{Z}_{(m)} \otimes \mathbb{Z}_{(o)}) \oplus (\mathbb{Z}_{(n)} \otimes \mathbb{Z}_{(o)}) \right] \\ &= \mathbb{Z}_{(mn)} \oplus \mathbb{Z} \oplus \mathbb{Z}_{(mo)} \oplus \mathbb{Z}_{(no)} \end{aligned}$$

- $C^*(T_m \times T_n \times T_o \times T_p) \rtimes \mathbb{Z}^4$

$$K_0 : \mathbb{Z}_{(mnop)} \oplus \mathbb{Z}_{(op)} \oplus \mathbb{Z}_{(mp)} \oplus \mathbb{Z}_{(np)} \oplus \mathbb{Z}_{(mn)} \oplus \mathbb{Z} \oplus \mathbb{Z}_{(mo)} \oplus \mathbb{Z}_{(no)}$$

$$K_1 : \mathbb{Z}_{(mno)} \oplus \mathbb{Z}_{(o)} \oplus \mathbb{Z}_{(m)} \oplus \mathbb{Z}_{(n)} \oplus \mathbb{Z}_{(mnp)} \oplus \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(mop)} \oplus \mathbb{Z}_{(nop)}$$

- $\underline{C^*(T_m \times T_n \times T_o \times T_p \times T_q)} \rtimes \mathbb{Z}^5$

$$K_0 : \mathbb{Z}_{(mnopq)} \oplus \mathbb{Z}_{(opq)} \oplus \mathbb{Z}_{(mpq)} \oplus \mathbb{Z}_{(npq)} \oplus \mathbb{Z}_{(mnq)} \oplus \mathbb{Z}_{(q)} \oplus \mathbb{Z}_{(moq)} \oplus \mathbb{Z}_{(noq)}$$

$$\oplus \mathbb{Z}_{(mno)} \oplus \mathbb{Z}_{(o)} \oplus \mathbb{Z}_{(m)} \oplus \mathbb{Z}_{(n)} \oplus \mathbb{Z}_{(mnp)} \oplus \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(mop)} \oplus \mathbb{Z}_{(nop)}$$

$$K_1 : \mathbb{Z}_{(mnop)} \oplus \mathbb{Z}_{(op)} \oplus \mathbb{Z}_{(mp)} \oplus \mathbb{Z}_{(np)} \oplus \mathbb{Z}_{(mn)} \oplus \mathbb{Z} \oplus \mathbb{Z}_{(mo)} \oplus \mathbb{Z}_{(no)}$$

$$\oplus \mathbb{Z}_{(mnoq)} \oplus \mathbb{Z}_{(oq)} \oplus \mathbb{Z}_{(mq)} \oplus \mathbb{Z}_{(nq)} \oplus \mathbb{Z}_{(mnpq)} \oplus \mathbb{Z}_{(pq)} \oplus \mathbb{Z}_{(mopq)} \oplus \mathbb{Z}_{(nopq)}$$

- $\underline{C^*(T_m \times T_n \times T_o \times T_p \times T_q \times T_r)} \rtimes \mathbb{Z}^6$

$$K_0 : \mathbb{Z}_{(mnopqr)} \oplus \mathbb{Z}_{(opqr)} \oplus \mathbb{Z}_{(mpqr)} \oplus \mathbb{Z}_{(npqr)} \oplus \mathbb{Z}_{(mnqr)} \oplus \mathbb{Z}_{(qr)} \oplus \mathbb{Z}_{(moqr)} \oplus \mathbb{Z}_{(noqr)}$$

$$\oplus \mathbb{Z}_{(mnor)} \oplus \mathbb{Z}_{(or)} \oplus \mathbb{Z}_{(mr)} \oplus \mathbb{Z}_{(nr)} \oplus \mathbb{Z}_{(mnp r)} \oplus \mathbb{Z}_{(pr)} \oplus \mathbb{Z}_{(mopr)} \oplus \mathbb{Z}_{(nopr)}$$

$$\oplus \mathbb{Z}_{(mnop)} \oplus \mathbb{Z}_{(op)} \oplus \mathbb{Z}_{(mp)} \oplus \mathbb{Z}_{(np)} \oplus \mathbb{Z}_{(mn)} \oplus \mathbb{Z} \oplus \mathbb{Z}_{(mo)} \oplus \mathbb{Z}_{(no)}$$

$$\oplus \mathbb{Z}_{(mnoq)} \oplus \mathbb{Z}_{(oq)} \oplus \mathbb{Z}_{(mq)} \oplus \mathbb{Z}_{(nq)} \oplus \mathbb{Z}_{(mnpq)} \oplus \mathbb{Z}_{(pq)} \oplus \mathbb{Z}_{(mopq)} \oplus \mathbb{Z}_{(nopq)}$$

$$K_1 : \mathbb{Z}_{(mnopq)} \oplus \mathbb{Z}_{(opq)} \oplus \mathbb{Z}_{(mpq)} \oplus \mathbb{Z}_{(npq)} \oplus \mathbb{Z}_{(mnq)} \oplus \mathbb{Z}_{(q)} \oplus \mathbb{Z}_{(moq)} \oplus \mathbb{Z}_{(noq)}$$

$$\oplus \mathbb{Z}_{(mno)} \oplus \mathbb{Z}_{(o)} \oplus \mathbb{Z}_{(m)} \oplus \mathbb{Z}_{(n)} \oplus \mathbb{Z}_{(mnp)} \oplus \mathbb{Z}_{(p)} \oplus \mathbb{Z}_{(mop)} \oplus \mathbb{Z}_{(nop)}$$

$$\oplus \mathbb{Z}_{(mnop r)} \oplus \mathbb{Z}_{(opr)} \oplus \mathbb{Z}_{(mpr)} \oplus \mathbb{Z}_{(npr)} \oplus \mathbb{Z}_{(mnr)} \oplus \mathbb{Z}_{(r)} \oplus \mathbb{Z}_{(mor)} \oplus \mathbb{Z}_{(nor)}$$

$$\oplus \mathbb{Z}_{(mnoqr)} \oplus \mathbb{Z}_{(oqr)} \oplus \mathbb{Z}_{(mqr)} \oplus \mathbb{Z}_{(nqr)} \oplus \mathbb{Z}_{(mnpqr)} \oplus \mathbb{Z}_{(pqr)} \oplus \mathbb{Z}_{(mopqr)} \oplus \mathbb{Z}_{(nopqr)}$$

We examine the 6-graph scenario (final bullet point) further. From the set of letters $\{m, n, o, p, q, r\}$, where order does not matter, we have in the K_0 case “words” of length 6, 4, 2, and 0; and in the K_1 case words of length 5, 3, and 1. The pattern for the K -groups arises from the following two computations:

$$\binom{6}{6} + \binom{6}{4} + \binom{6}{2} + \binom{6}{0} = 1 + 15 + 15 + 1 = 32$$

and

$$\binom{6}{5} + \binom{6}{3} + \binom{6}{1} = 6 + 20 + 6 = 32.$$

We make this more explicit in the next section.

5.2 k -Graph Case

We start this section by introducing a new notation. Fix any $k \in \mathbb{N}$ and let n_1, n_2, \dots, n_k be not necessarily distinct supernatural numbers. For $0 \leq j \leq k$, let $W_{\binom{k}{j}}$ be the $\binom{k}{j}$ -many element set of j letter words written using the alphabet $\{n_1, n_2, \dots, n_k\}$, where order of the letters does not matter (since the formal product of supernatural numbers is commutative) and without replacement (a given letter can be used at most once in a word). When $j = 0$, $W_{\binom{k}{0}}$ is the singleton containing the empty word.

Remark 5.2.1. The reader may be bothered by our use of the term “word”, since a supernatural number can be thought of as an already infinite string of repeating primes. Nonetheless, we want to view each supernatural number, the formal product of possibly infinite powers of finitely many primes, as a single letter or symbol. Because we will be using counting techniques, we need to keep track of the number of letters in a word. Therefore we do not “simplify” words, that is, if two letters in the same word contain the same prime then we do not “add” exponents. Words are the concatenation of “unique” symbols. Moreover, we have to be extra careful when two letters in the alphabet are the same supernatural number. For example, consider the case when $n_1 = 2^\infty$, $n_2 = 2^\infty$, and $n_3 = 2^\infty$. Since order does not matter, the two letter word $n_1 n_2 = 2^\infty 2^\infty$ is the same as the two letter word $n_2 n_1 = 2^\infty 2^\infty$. However, for the purposes of computing K -theory correctly, $n_1 n_3 = 2^\infty 2^\infty$ must be regarded as a different two letter word.

Now let

$$\bigoplus_{w \in W_{\binom{k}{j}}} \mathbb{Z}_{(w)}$$

denote the direct sum of the $\binom{k}{j}$ -many groups $\mathbb{Z}_{(w)}$; and in the $j = 0$ case, we declare that

$$\bigoplus_{w \in W_{\binom{k}{0}}} \mathbb{Z}_{(w)} = \mathbb{Z}.$$

Theorem 5.2.2. *Fix any $k \in \mathbb{N}$ and let n_1, n_2, \dots, n_k be supernatural numbers. Then*

$$\begin{aligned} & K_0 \left(C^*(T_{n_1} \times T_{n_2} \times \cdots \times T_{n_k}) \rtimes_{\bar{\sigma}} \mathbb{Z}^k \right) \\ & \cong \bigoplus_{w \in W_{\binom{k}{k}}} \mathbb{Z}_{(w)} \oplus \bigoplus_{w \in W_{\binom{k}{k-2}}} \mathbb{Z}_{(w)} \oplus \bigoplus_{w \in W_{\binom{k}{k-4}}} \mathbb{Z}_{(w)} \oplus \cdots \oplus \bigoplus_{w \in W_{\binom{k}{1 \text{ or } 0}}} \mathbb{Z}_{(w)} \end{aligned}$$

and

$$\begin{aligned} & K_1 \left(C^*(T_{n_1} \times T_{n_2} \times \cdots \times T_{n_k}) \rtimes_{\bar{\sigma}} \mathbb{Z}^k \right) \\ & \cong \bigoplus_{w \in W_{\binom{k}{k-1}}} \mathbb{Z}_{(w)} \oplus \bigoplus_{w \in W_{\binom{k}{k-3}}} \mathbb{Z}_{(w)} \oplus \bigoplus_{w \in W_{\binom{k}{k-5}}} \mathbb{Z}_{(w)} \oplus \cdots \oplus \bigoplus_{w \in W_{\binom{k}{0 \text{ or } 1}}} \mathbb{Z}_{(w)}, \end{aligned}$$

depending on whether k is odd or even, respectively.

In computing the K -theory of specific examples, one may simplify words (see above remark) after the fact.

Proof. An important thing to note is that after expanding the \bigoplus -terms, K_0 and K_1 both contain 2^{k-1} total summands of the form $\mathbb{Z}_{(w)}$. When k is odd, K_0 and K_1 also have the same number of \bigoplus -terms. However, when k is even K_1 has one fewer of these terms than K_0 , due to the way the 2^{k-1} summands are grouped into the \bigoplus -terms. We now proceed by induction, noting that the result was established up to $k = 6$ in the previous section. Assume the result holds for k odd. By Corollary 5.1.3,

$$C^*(T_{n_1} \times T_{n_2} \times \cdots \times T_{n_{k+1}}) \rtimes \mathbb{Z}^{k+1} \cong \left[C^*(T_{n_1} \times \cdots \times T_{n_k}) \rtimes \mathbb{Z}^k \right] \otimes \left[C^*(T_{n_{k+1}}) \rtimes \mathbb{Z} \right],$$

and so we apply the Künneth formula to show that K_0 in the $k + 1$ case is given by

$$\begin{aligned}
& \left[\left(\bigoplus_{w \in W_{(k)}} \mathbb{Z}_{(w)} \oplus \bigoplus_{w \in W_{(k-2)}} \mathbb{Z}_{(w)} \oplus \bigoplus_{w \in W_{(k-4)}} \mathbb{Z}_{(w)} \oplus \cdots \oplus \bigoplus_{w \in W_{(1)}} \mathbb{Z}_{(w)} \right) \otimes \mathbb{Z}_{(n_{k+1})} \right] \\
& \oplus \left[\left(\bigoplus_{w \in W_{(k-1)}} \mathbb{Z}_{(w)} \oplus \bigoplus_{w \in W_{(k-3)}} \mathbb{Z}_{(w)} \oplus \bigoplus_{w \in W_{(k-5)}} \mathbb{Z}_{(w)} \oplus \cdots \oplus \bigoplus_{w \in W_{(0)}} \mathbb{Z}_{(w)} \right) \otimes \mathbb{Z} \right] \\
& \cong \left[\bigoplus_{w \in W_{(k)}} \mathbb{Z}_{(wn_{k+1})} \oplus \bigoplus_{w \in W_{(k-2)}} \mathbb{Z}_{(wn_{k+1})} \oplus \bigoplus_{w \in W_{(k-4)}} \mathbb{Z}_{(wn_{k+1})} \oplus \cdots \oplus \bigoplus_{w \in W_{(1)}} \mathbb{Z}_{(wn_{k+1})} \right] \\
& \oplus \left[\bigoplus_{w \in W_{(k-1)}} \mathbb{Z}_{(w)} \oplus \bigoplus_{w \in W_{(k-3)}} \mathbb{Z}_{(w)} \oplus \bigoplus_{w \in W_{(k-5)}} \mathbb{Z}_{(w)} \oplus \cdots \oplus \bigoplus_{w \in W_{(0)}} \mathbb{Z}_{(w)} \right] \\
& \cong \bigoplus_{w \in W_{(k)}} \mathbb{Z}_{(wn_{k+1})} \oplus \left(\bigoplus_{w \in W_{(k-2)}} \mathbb{Z}_{(wn_{k+1})} \oplus \bigoplus_{w \in W_{(k-1)}} \mathbb{Z}_{(w)} \right) \\
& \oplus \left(\bigoplus_{w \in W_{(k-4)}} \mathbb{Z}_{(wn_{k+1})} \oplus \bigoplus_{w \in W_{(k-3)}} \mathbb{Z}_{(w)} \right) \oplus \cdots \oplus \bigoplus_{w \in W_{(0)}} \mathbb{Z}_{(w)} \\
& \cong \bigoplus_{w \in W_{(k+1)}} \mathbb{Z}_{(w)} \oplus \bigoplus_{w \in W_{(k+1)}} \mathbb{Z}_{(w)} \oplus \bigoplus_{w \in W_{(k+1)}} \mathbb{Z}_{(w)} \oplus \cdots \oplus \bigoplus_{w \in W_{(0)}} \mathbb{Z}_{(w)}.
\end{aligned}$$

Likewise, K_1 in the $k + 1$ case is given by

$$\begin{aligned}
& \left[\left(\bigoplus_{w \in W_{(k)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-2)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-4)}^{(k)}} \mathbb{Z}(w) \oplus \cdots \oplus \bigoplus_{w \in W_{(1)}^{(k)}} \mathbb{Z}(w) \right) \otimes \mathbb{Z} \right] \\
& \oplus \left[\left(\bigoplus_{w \in W_{(k-1)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-3)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-5)}^{(k)}} \mathbb{Z}(w) \oplus \cdots \oplus \bigoplus_{w \in W_{(0)}^{(k)}} \mathbb{Z}(w) \right) \otimes \mathbb{Z}(n_{k+1}) \right] \\
& \cong \left[\bigoplus_{w \in W_{(k)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-2)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-4)}^{(k)}} \mathbb{Z}(w) \oplus \cdots \oplus \bigoplus_{w \in W_{(1)}^{(k)}} \mathbb{Z}(w) \right] \\
& \oplus \left[\bigoplus_{w \in W_{(k-1)}^{(k)}} \mathbb{Z}(wn_{k+1}) \oplus \bigoplus_{w \in W_{(k-3)}^{(k)}} \mathbb{Z}(wn_{k+1}) \oplus \bigoplus_{w \in W_{(k-5)}^{(k)}} \mathbb{Z}(wn_{k+1}) \oplus \cdots \oplus \bigoplus_{w \in W_{(0)}^{(k)}} \mathbb{Z}(wn_{k+1}) \right] \\
& \cong \left(\bigoplus_{w \in W_{(k)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-1)}^{(k)}} \mathbb{Z}(wn_{k+1}) \right) \oplus \left(\bigoplus_{w \in W_{(k-2)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-3)}^{(k)}} \mathbb{Z}(wn_{k+1}) \right) \\
& \oplus \left(\bigoplus_{w \in W_{(k-4)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-5)}^{(k)}} \mathbb{Z}(wn_{k+1}) \right) \oplus \cdots \oplus \left(\bigoplus_{w \in W_{(1)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(0)}^{(k)}} \mathbb{Z}(wn_{k+1}) \right) \\
& \cong \bigoplus_{w \in W_{(k+1)}^{(k+1)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k+1)}^{(k+1)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k+1)}^{(k+1)}} \mathbb{Z}(w) \oplus \cdots \oplus \bigoplus_{w \in W_{(k+1)}^{(k+1)}} \mathbb{Z}(w).
\end{aligned}$$

Now assume the result holds for k even. Then K_0 in the $k+1$ case is given by

$$\begin{aligned}
& \left[\left(\bigoplus_{w \in W_{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-2)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-4)}} \mathbb{Z}(w) \oplus \cdots \oplus \bigoplus_{w \in W_{(0)}} \mathbb{Z}(w) \right) \otimes \mathbb{Z}(n_{k+1}) \right] \\
& \oplus \left[\left(\bigoplus_{w \in W_{(k-1)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-3)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-5)}} \mathbb{Z}(w) \oplus \cdots \oplus \bigoplus_{w \in W_{(1)}} \mathbb{Z}(w) \right) \otimes \mathbb{Z} \right] \\
& \cong \left[\bigoplus_{w \in W_{(k)}} \mathbb{Z}(wn_{k+1}) \oplus \bigoplus_{w \in W_{(k-2)}} \mathbb{Z}(wn_{k+1}) \oplus \bigoplus_{w \in W_{(k-4)}} \mathbb{Z}(wn_{k+1}) \oplus \cdots \oplus \bigoplus_{w \in W_{(0)}} \mathbb{Z}(wn_{k+1}) \right] \\
& \oplus \left[\bigoplus_{w \in W_{(k-1)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-3)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-5)}} \mathbb{Z}(w) \oplus \cdots \oplus \bigoplus_{w \in W_{(1)}} \mathbb{Z}(w) \right] \\
& \cong \bigoplus_{w \in W_{(k)}} \mathbb{Z}(wn_{k+1}) \oplus \left(\bigoplus_{w \in W_{(k-2)}} \mathbb{Z}(wn_{k+1}) \oplus \bigoplus_{w \in W_{(k-1)}} \mathbb{Z}(w) \right) \\
& \oplus \left(\bigoplus_{w \in W_{(k-4)}} \mathbb{Z}(wn_{k+1}) \oplus \bigoplus_{w \in W_{(k-3)}} \mathbb{Z}(w) \right) \oplus \cdots \oplus \left(\bigoplus_{w \in W_{(0)}} \mathbb{Z}(wn_{k+1}) \oplus \bigoplus_{w \in W_{(1)}} \mathbb{Z}(w) \right) \\
& \cong \bigoplus_{w \in W_{(k+1)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k+1)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k+1)}} \mathbb{Z}(w) \oplus \cdots \oplus \bigoplus_{w \in W_{(1)}} \mathbb{Z}(w),
\end{aligned}$$

and K_1 in the $k + 1$ case by

$$\begin{aligned}
& \left[\left(\bigoplus_{w \in W_{(k)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-2)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-4)}^{(k)}} \mathbb{Z}(w) \oplus \cdots \oplus \bigoplus_{w \in W_{(0)}^{(k)}} \mathbb{Z}(w) \right) \otimes \mathbb{Z} \right] \\
& \oplus \left[\left(\bigoplus_{w \in W_{(k-1)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-3)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-5)}^{(k)}} \mathbb{Z}(w) \oplus \cdots \oplus \bigoplus_{w \in W_{(1)}^{(k)}} \mathbb{Z}(w) \right) \otimes \mathbb{Z}(n_{k+1}) \right] \\
& \cong \left[\bigoplus_{w \in W_{(k)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-2)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-4)}^{(k)}} \mathbb{Z}(w) \oplus \cdots \oplus \bigoplus_{w \in W_{(0)}^{(k)}} \mathbb{Z}(w) \right] \\
& \oplus \left[\bigoplus_{w \in W_{(k-1)}^{(k)}} \mathbb{Z}(wn_{k+1}) \oplus \bigoplus_{w \in W_{(k-3)}^{(k)}} \mathbb{Z}(wn_{k+1}) \oplus \bigoplus_{w \in W_{(k-5)}^{(k)}} \mathbb{Z}(wn_{k+1}) \oplus \cdots \oplus \bigoplus_{w \in W_{(1)}^{(k)}} \mathbb{Z}(wn_{k+1}) \right] \\
& \cong \left(\bigoplus_{w \in W_{(k)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-1)}^{(k)}} \mathbb{Z}(wn_{k+1}) \right) \oplus \left(\bigoplus_{w \in W_{(k-2)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-3)}^{(k)}} \mathbb{Z}(wn_{k+1}) \right) \\
& \oplus \left(\bigoplus_{w \in W_{(k-4)}^{(k)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k-5)}^{(k)}} \mathbb{Z}(wn_{k+1}) \right) \oplus \cdots \oplus \bigoplus_{w \in W_{(0)}^{(k)}} \mathbb{Z}(w) \\
& \cong \bigoplus_{w \in W_{(k+1)}^{(k+1)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k+1)}^{(k+1)}} \mathbb{Z}(w) \oplus \bigoplus_{w \in W_{(k+1)}^{(k+1)}} \mathbb{Z}(w) \oplus \cdots \oplus \bigoplus_{w \in W_{(0)}^{(k+1)}} \mathbb{Z}(w).
\end{aligned}$$

□

Chapter 6

Conclusion

We conclude this thesis with a few ideas on how to expand on our results, noting that nothing appearing in this section has been validated. Firstly, we emphasize that this paper has not addressed all of the generalized Bunce-Deddens algebras of type Orfanos. This is because the original Bunce-Deddens algebras, and the graph versions we showed were Morita equivalent, are in one way or another dependent on sequences that have restrictions on them (for example, the divisibility condition). An obvious next step would be to see if our results or methods could be extended to cover more cases of type Orfanos.

In the definition of a supernatural number, we insisted that only finitely many primes make up the factorization. It would be interesting to loosen this restriction. Let

$$\{a_k\} = \left\{ (2), (2 \cdot 3), (2 \cdot 3 \cdot 5), \dots \right\},$$

where $m_1 := \delta(\{a_k\}) = \prod_p \text{prime } p$; and let

$$\{b_k\} = \left\{ (2)^1, (2 \cdot 3)^2, (2 \cdot 3 \cdot 5)^3, \dots \right\},$$

where $m_2 := \delta(\{b_k\}) = \prod_p \text{prime } p^\infty$. The K_0 -group $\mathbb{Z}_{(n)}$, by definition, is made up of rationals whose numerator is an integer and whose denominator is the product of powers of finitely many primes. Thus, even though m_1 and m_2 contain infinitely many primes, we need not consider them all simultaneously and so Theorem 4.1.2 should still be applicable. If this is the case, we expect $K_0(\mathcal{BD}_{m_2})$ to be given by

$$\mathbb{Z}_{(m_2)} = \mathbb{Q}.$$

If we then wish to extend our generalized results to the infinite-prime case, modifications to Lemma 4.2.4 are necessary. The map used in the proof of this lemma is dependent on the existence of only finitely many primes. However, if n is any supernatural number, we would at the very least hope that

$$\mathbb{Z}_{(m_1)} \otimes \mathbb{Z}_{(n)} \cong \mathbb{Z}_{(m_1 n)}$$

and

$$\mathbb{Z}_{(m_2)} \otimes \mathbb{Z}_{(n)} \cong \mathbb{Z}_{(m_2)}.$$

Another interesting idea concerns a result in [13] due to Farthing, Pask, and Sims. Given a \mathbb{Z}^l -action by automorphisms on a k -graph C^* -algebra, the authors construct a $(k+l)$ -graph whose C^* -algebra is isomorphic to the crossed product of the original k -graph C^* -algebra by \mathbb{Z}^l . By applying this result to our scenario ($l = k$), one could attempt to reconstruct the K -theory of our generalized Bunce-Deddens algebras of type Orfanos using higher rank graph techniques, or at the very least, find interesting connections between our k -graph approach and their $(k+l)$ -graphs.

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