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GENERAL THEORY OF SURFACE-WAVE PROPAGATION ON A CURVED OPTICAL WAVEGUIDE OF ARBITRARY CROSS-SECTION†

by

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ABSTRACT

A general scheme for calculating the change (both the real part and imaginary part) of the propagation constant of a surface-wave node on a general open waveguide structure due to curvature of its axis is presented. By carefully keeping track of lower-order terms in the analysis, it is shown that approximations commonly made in other analyses can result in quite large errors in the calculation of radiation loss: Specific results for the asymmetric slab and the step-index fiber are presented and compared to other solutions in the literature. Marcuse's result for the attenuation of an asymmetric slab is found to lack an important term which depends on the extent of asymmetry. Arnaud's result for the fiber is found to be in serious error, whereas good agreement is found with that of Lewin. Finally, the general result for a finite cross-section guide indicates the necessity of an inverse square root dependence on the radius of curvature for the attenuation which is absent from Maractili's expression for the rectangular dielectric guide.

SECTION I

INTRODUCTION

One of the practical limitations on the use of open structures such as optical fibers, or channel waveguides in integrated optics for the guided transmission of information is that such structures are much more susceptible to radiation losses than are more conventional closed waveguides [1]. In particular radiation from curved surface waveguides has received a good deal of attention [2-14]. The earliest treatment of this problem seems to have been Richtmyer's [15] with additional development by Miller and Talanov [16]. Of these treatments, only [2-3,5,7,10-11,13,15] treat structures of finite cross-section. Of these [2,3] involve assumptions about the field outside the waveguide whose validity is not easily assessed, [5,13,15] rely on field approximations to the known forms in guides of circular cross-section whose applicability is also open to question, and [4,8,11] utilize ray treatments whose application to guides of general cross-section is not obvious. Unique among these is Lewin's work [10] in which the attenuation of a uniform round fiber is obtained by first constructing an integral representation of the fields, then showing that the necessary boundary conditions are satisfied, and finally evaluating the integral asymptotically. Unfortunately it is unclear how this approach could be extended to more complicated structures.

Arnaud [13] has recently presented a more physical scheme based upon the coupling between a surface-wave mode and a somewhat artifically-introduced whispering-gallery mode. His result for the uniform fiber, however, does not agree with Lewin's. In the present report, a more general and less awkward formulation applicable to a waveguide of arbitrary inhomogeneous cross-section is obtained, which retains much of the simple physical interpretation of the slab case. By carefully keeping track of small-order terms, significant factors may be found in both the attenuation and phase shift which are often incorrectly neglected.

Finally, in none of the previous treatments (with the exception of [9] for the case of the slab guide) is specific attention given to the case where the guided mode does not possess any symmetry with respect to the bending axis. As will be detailed in this treatment, the change in the propagation constant for such a situation can cause not only a significant additional phase shift around the bend, but also a quite sizeable enhancement or reduction in the amount of continuous radiation loss, depending on the extent of the asymmetry.

SECTION II

REVIEW OF THE CURVED SLAB PROBLEM [9]

We consider first a curved homogeneous dielectric slab waveguide of thickness D having a radius of curvature R and a refractive index n (with respect to the surroundings) as depicted in Fig. 1. All quantities are assumed independent of z, and we search for solutions of the form $\exp(i\omega t - ik_o \nu R\theta)$, where $k_o^2 = \omega^2 \mu_o \varepsilon_o$, which satisfy the usual boundary conditions at the slab and the radiation condition at infinity. Here the (normalized) propagation constant ν, although as yet undetermined, has to approach the value ν_o corresponding to the straight guide as $R \to \infty$.

We now define a local coordinate system $\hat{x} = R\,\ell n\,(\rho/R)$ and $\hat{y} = R\theta$ so that the governing wave equation becomes

$$\{\frac{d^2}{dx^2} + k_o^2[n_j^2 \exp(2\hat{x}/R) - \nu^2]\}\ E_z(\hat{x}) = 0 \qquad (1)$$

where $n_j = 1$ or n for $j = 1$ or 2, corresponding to the medium outside or inside the slab. The slab boundaries have now become $\hat{x} = 0$ and $\hat{x} = d = R\,\ell n(1 + D/R) \simeq D$. Thus for all practical purposes we can replace the curved slab of Fig.1(a) by a straight one of virtually the same thickness but with an inhomogeneous refractive index profile as in Fig.1(b). For the case of a straight homogeneous slab, the propagation constant for a propagating surface-wave mode satisfies $1 < \nu_o < n$,

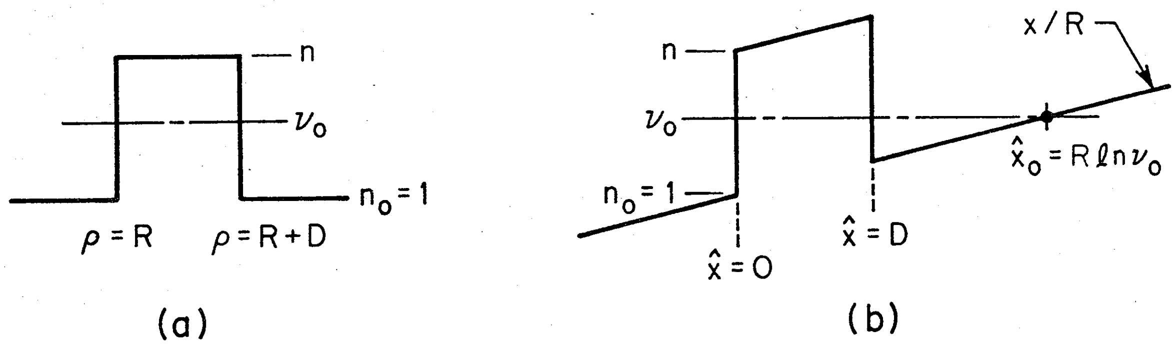

Fig. 1

(a) Straight homogeneous slab

(b) Straightened slab with exponential profile.

Thus inside the slab the solution is a standing wave - a linear combination of $\sin(n^2-\nu_o^2)^{\frac{1}{2}}\hat{x}$ and $\cos(n^2-\nu_o^2)^{\frac{1}{2}}\hat{x}$. Outside the slab, $n_j^2-\nu_o^2 < 0$ so that the solution is an evanescent wave of the form $\exp[-(\nu_o^2-1)^{\frac{1}{2}}|\hat{x}|]$.

Now in the case of the curved slab, if ν_o is essentially unchanged, the character of the solutions is unchanged everywhere except the region where $\hat{x} > \hat{x}_o = R\,\ell n\,\nu_o$ corresponding to $\rho_o = \nu_o R$. Beyond this point (which is known as the turning point) the effective refractive index $n_j \exp(\hat{x}/R) > \nu_o$ and the solution of (1) satisfying the radiation condition is an outgoing unattenuated cylindrical wave. Thus, in spite of the fact that between $\hat{x} = d$ and $\hat{x} = \hat{x}_o$ the fields must be evanescent, the character of the solution at $\hat{x} = \hat{x}_o$ must change so that the field is partially transmitted and partially reflected from the turning point, returning towards the slab as an "incoming" evanescent wave. Near the slab-surface $\hat{x} = d$ then, the electric field must have a finite, although small, exponentially growing component [9]:

$$E_z \simeq E_o(e^{-k_o\lambda\hat{x}} + \sigma_o e^{k_o\lambda\hat{x}}) \tag{2}$$

where the reflection coefficient is found by the WKB method to be

$$\sigma_o = -(\tfrac{i}{2})e^{-2\lambda^3 k_o R/3\nu^2} \quad . \tag{3}$$

Here $\lambda = (\nu^2-1)^{\frac{1}{2}}$, making $(k_o\lambda)^{-1}$ the penetration depth of the surface wave into the outside medium. It should be noted that the reflection coefficient decreases exponentially with R, so that when $R \to \infty$, $\sigma_o \to 0$ and (2) reduces to the field outside a homogeneous straight slab. In [9], the attenuation coefficient for this structure was calculated in a straightforward manner by considering the reflected field in (2) as a perturbation, and calculating the resultant change in impedance at the slab surfaces.

SECTION III

SPECTRAL REPRESENTATION OF FIELDS IN FINITE CROSS-SECTION WAVEGUIDES

We now attack the problem of a curved section of a dissipationless three-dimensional optical waveguide, shown in Fig.2. We allow the guide to be of arbitrary cross-sectional shape, and possibly inhomogeneous, but the outside medium is required to be homogeneous with refractive index n_o for $\rho > \rho_{max}$. We construct four coordinate systems for this geometry as shown: two global ones (Cartesian (x,y,z) and cylindrical (ρ,θ,z) as usual) and two local ones (local Cartesian $(\hat{x},\hat{y},\hat{z})$, where $\hat{x} = R \ln(\rho/R)$, $\hat{y} = R\theta$, $\hat{z} = z$, and local cylindrical $(\hat{r},\hat{\phi},\hat{y})$, where $\hat{r}\cos\hat{\phi} = \hat{x}$ and $\hat{r}\sin\hat{\phi} = -\hat{z}$). The radius of curvature R is chosen as the distance between the origins of the local and global systems.

Now any Cartesian field component Φ in the region $\rho > \rho_{max}$ (the largest value of ρ in the guide cross-section - see Fig.2) must satisfy the scalar wave equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{\partial^2\Phi}{\partial z^2} + k_o^2\left[n_o^2 - \frac{\nu^2R^2}{\rho^2}\right]\Phi = 0 \qquad (4)$$

where an $\exp(i\omega t - ik_o\nu R\theta)$ dependence has been assumed as before. Because the medium in this region is uniform in the z-direction between $\pm\infty$, we may further reduce (4) using the Fourier transform pair

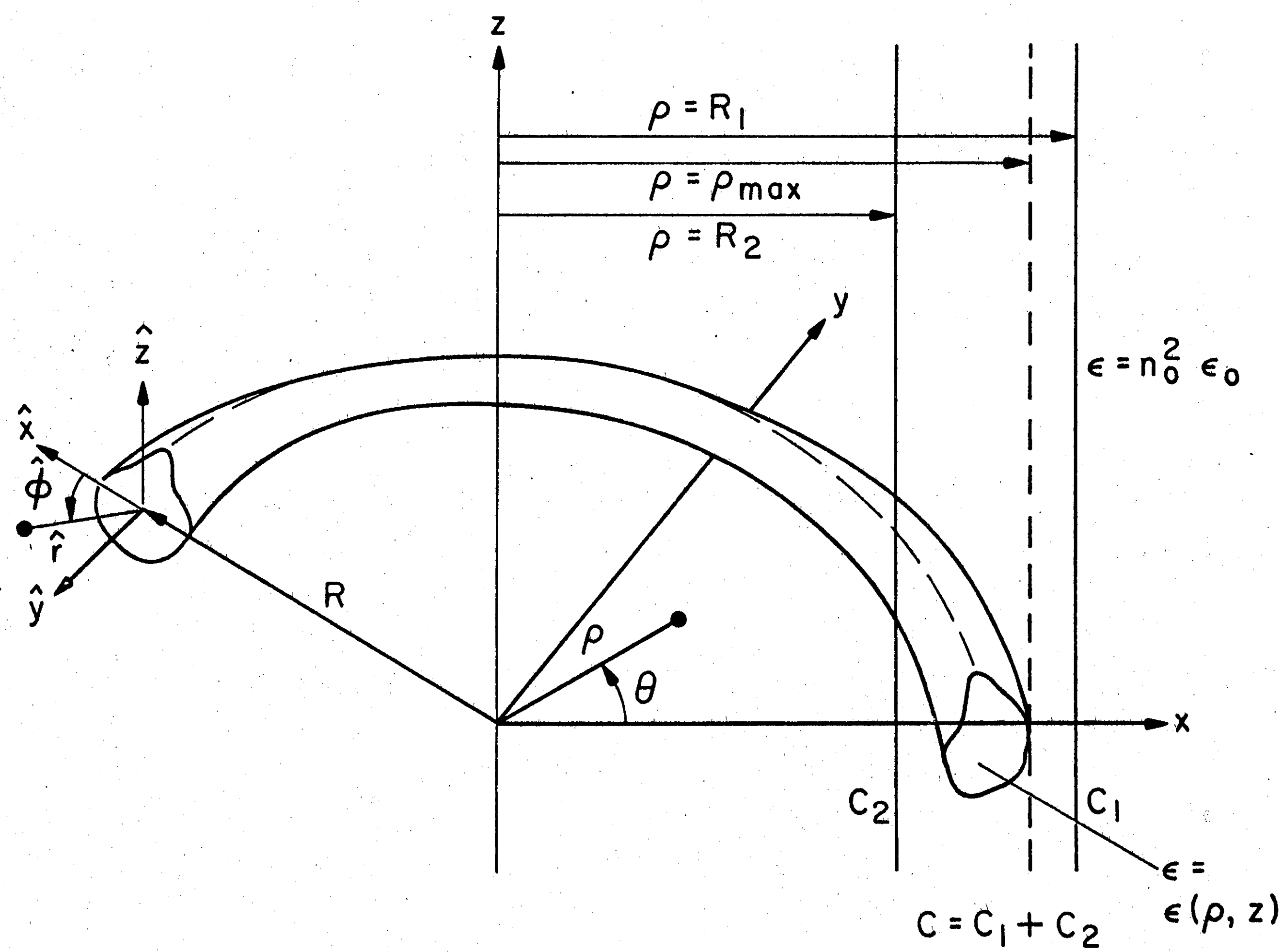

Fig. 2

Section of curved optical waveguide with Cartesian coordinate system (x,y,z), cylindrical system (ρ,θ,z), local Cartesian system $(\hat{x},\hat{y},\hat{z})$, and local cylindrical system $(\hat{r},\hat{\phi},\hat{y})$.

$$\Phi(\rho,z) = \int_{-\infty}^{\infty} \tilde{\Phi}(\rho,s) e^{-iksz}\, ds$$
$$\tilde{\Phi}(\rho,s) = \frac{k}{2\pi} \int_{-\infty}^{\infty} \Phi(\rho,z)\, e^{iksz}\, dz \tag{5}$$

where $k = k_o n_o$ is the wave number in the outer medium. The spectrum function $\tilde{\Phi}$ now satisfies the equation:

$$\frac{1}{\rho} \frac{\partial}{\partial\rho}(\rho \frac{\partial\tilde{\Phi}}{\partial\rho}) + k_o^2[n_o^2(1-s^2) - \frac{\nu^2 R^2}{\rho^2}]\tilde{\Phi} = 0 \tag{6}$$

The exact solution of (6) which satisfies the radiation condition at $\rho = \infty$ is well-known:

$$\begin{aligned} \tilde{\Phi}(\rho,s) &= A_1 H^{(2)}_{\nu k_o R}[k\rho(1-s^2)^{\frac{1}{2}}] && s^2 < 1 \\ &= A_2 (k\rho)^{-\nu k_o R} && s^2 = 1 \\ &= A_3 K_{\nu k_o R}[k\rho(s^2-1)^{\frac{1}{2}}] && s^2 > 1 \end{aligned}$$

where the A_i are independent of ρ, $H^{(2)}$ is the outgoing Hankel function, and K is the modified Bessel function of the second kind.

As in [9], we shall find it more convenient to use approximate solutions found by the WKB method than these exact ones. Changing to the local Cartesian system and calling $v = \hat{x}/R$, we have

$$\{\frac{d^2}{dv^2} + k_o^2 R^2 [n_o^2(1-s^2)e^{2v} - \nu^2]\}\tilde{\Phi} = 0 . \tag{7}$$

For conciseness, let us call

$$g(v) = n_o^2(1-s^2)e^{2v} - \nu^2 = n_o^2(1-s^2)\frac{\rho^2}{R^2} - \nu^2 = g(\rho)$$

The function $g(v)$ or $g(\rho)$ will always be written out to avoid confusion. We note that if ν were real, a turning point [9] would occur at $v = v_t = \ln[\nu/n_o(1-s^2)^{\frac{1}{2}}]$ or $\rho = \rho_t = \nu R/n_o(1-s^2)^{\frac{1}{2}}$, and would be real only for $s^2 < 1$. The necessity of radiation from curved open structures, however, means that ν will actually be complex, with a small negative imaginary part accounting for the consequent attenuation. As a result, no real value of ρ (or v) will cause g to vanish, but some complex value of the turning poing will exist [17, pp.326-333]. The nature of the asymptotic (WKB) solution of (7) for k_oR large is treated in Appendix A for the case of complex ν.

If we define [18] $w = [g(v)]^{\frac{1}{2}}$ and

$$\xi = \int_{v_t}^{v} w(v')dv' = w - i\,\frac{\nu}{2}\{\ln(w+i\nu) - \ln(w-i\nu)\} - \frac{\nu\pi}{2} \tag{8}$$

(for specification of the branch cuts, see Appendix A), we may state the results for two cases:

<u>Case I:</u> $s^2 < 1$. Except for a region surrounding the turning point, we have

$$\Phi \sim D^i(s)w^{-\frac{1}{2}}\{[1-\frac{1}{24k_oR}(\frac{3i}{w} + \frac{5i\nu^2}{w^3})+ \ldots]\exp[+ik_oR(\xi-\xi_o)]$$

$$+ \sigma_s\,[1 + \frac{1}{24k_oR}(\frac{3i}{w} + \frac{5i\nu^2}{w^3})+\ldots]\exp[-ik_oR(\xi-\xi_o)]$$

$$\rho < |\rho_t| \tag{9}$$

and

$$\Phi \sim D^{+}(s)w^{-\frac{1}{2}} \exp(-ik_o R\xi) \qquad \rho > |\rho_t|$$

where

$$\xi_o = \xi(v=0) = i\nu\{\eta_o + \frac{1}{2} \ln(\frac{1-\eta_o}{1+\eta_o})\} \tag{10}$$

$$\eta_o = [1 - \frac{n_o^2(1-s^2)}{\nu^2}]^{\frac{1}{2}}$$

This quantity was inserted into (9) in order to make

$$\sigma_s = - \frac{i}{2} \exp(-2ik_o R\xi_o) \tag{11}$$

the ratio of the incident field and that reflected from the turning point in the local coordinate system.

Case II: $s^2 > 1$. In this case, the turning point is away from the real axis, so that

$$\Phi \sim D^{i}(s)w^{-\frac{1}{2}}[1 - \frac{1}{24k_o R} (\frac{3i}{w} + \frac{5i\nu^2}{w^3}) + \cdots] \exp[+ik_o R(\xi-\xi_o)] \tag{12}$$

for all $\rho > \rho_{max}$ (see Appendix A) .

In summary, then, the fields of a curved guide outside of the guide can be represented by a spectral expansion in the z-direction, and the spectrum function $\tilde{\Phi}(\rho,s)$ can be represented for large $k_o R$ by its WKB approximation. For those components with $s^2 > 1$, formula (12) applies; i.e., locally decaying field spectrum components of this type remain evanescent all the way until $\rho = \infty$.[1] On the other hand, those components with $s^2 < 1$, which appear evanescent (locally to the guide) must in fact possess a small exponentially growing wave below the turning point as in (9), and beyond the turning point become an outwardly propagating rather than evanescent field. It is this, as in the slab case, which accounts for the attenuation in the curved waveguide.

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[1] These are the so-called "stable waves" described by Miller and Talanov [16]. See also Lewin [10].

SECTION IV

PERTURBATION FORMULA FOR THE PROPAGATION CONSTANT

Having now a few general ideas about how the fields in a curved waveguide must behave, we ask how to determine the change in ν, knowing the value ν_o and the fields in the corresponding straight structure, into which the curved guide quantities must go as $R \to \infty$. If an analytic form of the eigenvalue equation is available, as it is for the slab [9], it is a straightforward matter to perform a perturbation calculation of the propagation constant of the curved structure. No such equation seems available in the finite cross-section case, however (since no explicit matching of the boundary conditions at a finite number of coordinate surfaces is possible), and so some alternative perturbation technique must be found. In this section we develop a mixed-field perturbation formula applicable to waveguides of uniformly curved axis, similar to formulas given in [21, pp. 326-331] for straight, uniformly perturbed waveguides.

Referring again to Fig. 2, and assumiming $\exp[i\omega t - ik_o\nu R\theta]$ field dependence as before, we may write Maxwell's equations in the familiar "longitudinal-transverse" form for guided waves [21, p.346], where now longitudinal refers to the θ-direction, and transverse to a ρz-plane in which θ = constant:

$$\bar{a}_\theta \cdot \mathrm{curl}_t\, \bar{E}_t^+ = -i\omega\mu_o H_\theta^+$$

$$\bar{a}_\theta \cdot \mathrm{curl}_t \bar{H}_t^+ = i\omega\varepsilon E_\theta^+ \tag{13}$$

$$-\frac{\bar{a}_\theta}{\rho} \times \mathrm{grad}_t(\rho E_\theta^+) - i\,\frac{k_o \nu R}{\rho}\,\bar{a}_\theta \times \bar{E}_t^+ = -i\omega\mu_o \bar{H}_t^+$$

$$-\frac{\bar{a}_\theta}{\rho} \times \mathrm{grad}_t(\rho H_\theta^+) - i\,\frac{k_o \nu R}{\rho}\,\bar{a}_\theta \times \bar{H}_t^+ = i\omega\varepsilon \bar{E}_t^+ \;.$$

Here $\bar{a}_\theta$ is the unit vector along the θ-direction. Any vector $\bar{A}$ is represented in terms of its longitudinal and transverse components as $\bar{A}_t + \bar{a}_\theta A_\theta$, and the various "true" transverse differential operators (i.e., in the global cylindrical coordinate system) are given by

$$\mathrm{grad}_t\, f = \mathrm{grad}\, f = \bar{a}_\theta[\frac{1}{\rho}\,\frac{df}{d\theta}]$$

$$= \bar{a}_\rho[\frac{\partial f}{\partial \rho}] + \bar{a}_z\,\frac{\partial f}{\partial z}$$

$$\mathrm{div}_t\, \bar{A} = \mathrm{div}\, \bar{A} - \frac{1}{\rho}\,\frac{\partial A_\theta}{\partial \theta}$$

$$= \frac{1}{\rho}\,\frac{\partial}{\partial \rho}\,(\rho\, A_\rho) + \frac{\partial A_z}{\partial z}$$

$$\mathrm{curl}_t \bar{A} = \mathrm{curl}\, \bar{A} - \frac{\bar{a}_\theta}{\rho} \times \frac{\partial \bar{A}}{\partial \theta}$$

$$= \bar{a}_\rho[-\frac{\partial A_\theta}{\partial z}] + \bar{a}_\theta[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}] + \bar{a}_z[\frac{1}{\rho}\,\frac{\partial}{\partial \rho}(\rho A_\theta)]$$

$$= \bar{a}_\theta(\bar{a}_\theta \cdot \mathrm{curl}_t \bar{A}_t) - \frac{\bar{a}_\theta}{\rho} \times \mathrm{grad}_t(\rho A_\rho) \;.$$

It is easily verified that the transpose fields given by

$$\bar{E}_t^- = \bar{E}_t^+ \qquad E_\theta^- = -E_\theta^+ \qquad \bar{H}_t^- = -\bar{H}_t^+ \qquad H_\theta^- = H_\theta^+$$

also satisfy (13) if $+\nu$ is replaced by $-\nu$.

Finally, we will require a generalization of the two-dimensional divergence theorem in order to obtain our perturbation formula. Consider an arbitrary surface S bounded by a closed contour C, both lying in a constant θ-plane as shown in Fig.3. Let this plane be rotated about the z-axis so that S moves to S' in the plane $\theta + \Delta\theta$, tracing out the volume ΔV, and similarly C moves to C' tracing out the surface ΔS as shown. Applying the three-dimensional divergence theorem to an arbitrary suitably differentiable vector function $\bar{F}$ on ΔV, we have

$$\int_{\Delta V} \operatorname{div} \bar{F}\, dV = \int_{S'} \bar{F}\cdot\bar{a}_\theta dS - \int_S \bar{F}\cdot\bar{a}_\theta dS + \int_{\Delta S} \bar{F}\cdot\bar{a}_n dS$$

where $\bar{a}_n$ is the unit normal on ΔS. Dividing through by $\Delta\theta$ and letting $\Delta\theta \to 0$, we have

$$\frac{1}{\Delta\theta}\int_{\Delta V} \operatorname{div} \bar{F}\, dV \to \int_S \rho \operatorname{div} \bar{F}\, dS$$

$$\frac{1}{\Delta\theta}\int_{\Delta S} \bar{F}\cdot a_n dS \to \oint_C \rho\, \bar{F}\cdot a_n d\ell$$

resulting in either of two equivalent forms of the desired theorem:

$$\int_S \rho \operatorname{div} \bar{F}\, dS = \frac{d}{d\theta}\int_S \bar{F}\cdot\bar{a}_\theta dS + \oint_C \rho\, \bar{F}\cdot\bar{a}_n d\ell \tag{14}$$

or

$$\int_S \rho \operatorname{div}_t \bar{F}\, dS = \oint_C \rho\, \bar{F}\cdot a_n d\ell \tag{15}$$

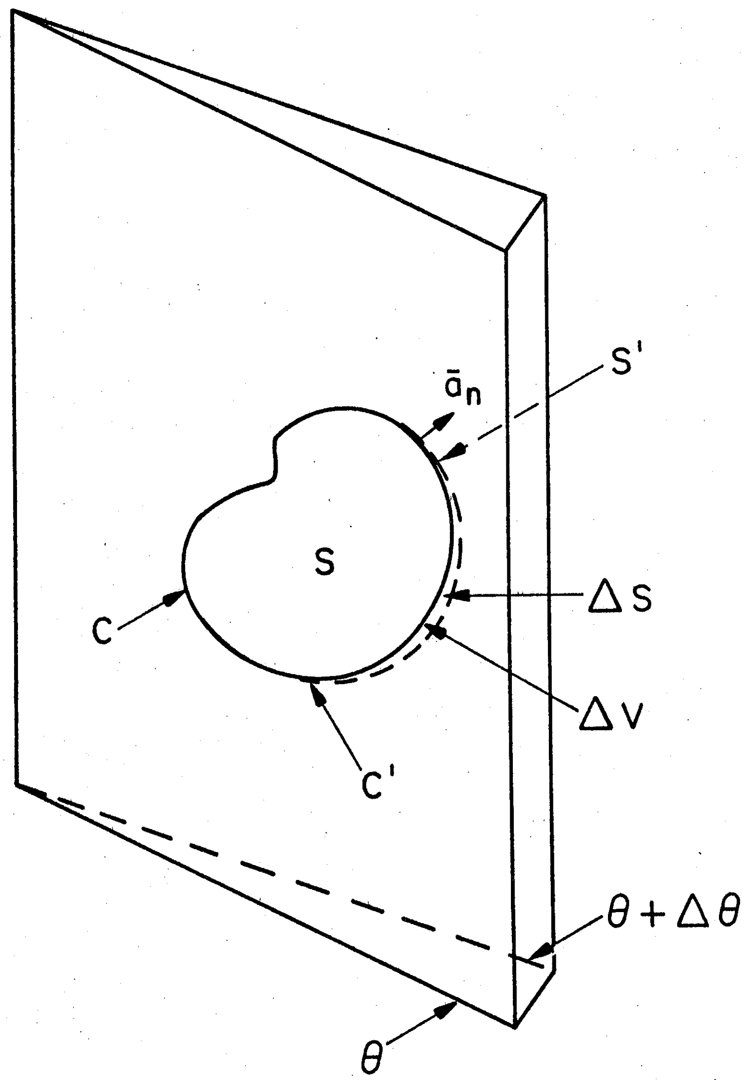

Fig. 3

Diagram for proof of 2-dimensional divergence theorem transverse to θ.

Now if $\bar{E}_0$, $\bar{H}_o$, ν_o are the unperturbed fields and propagation constant of the straight guide, they will satisfy (13) with $R \to \infty$, in which case $(\hat{x},\hat{y},\hat{z})$ become a genuine set of Cartesian coordinates:

$$\begin{aligned} \hat{\nabla}_t \times \bar{E}_o^{\pm} + i\omega\mu_o \bar{H}_o^{\pm} &= \pm i k_o \nu_o \bar{a}_{\hat{y}} \times \bar{E}_o^{\pm} \\ \hat{\nabla}_t \times \bar{H}_o^{\pm} - i\omega\varepsilon \bar{E}_o^{\pm} &= \pm i k_o \nu_o \bar{a}_{\hat{y}} \times \bar{H}_o^{\pm} \end{aligned} \tag{16}$$

where

$$\hat{\nabla}_t \times \bar{A} = a_{\hat{x}}\left[-\frac{\partial A_{\hat{y}}}{\partial \hat{z}}\right] + \bar{a}_{\hat{y}}\left[\frac{\partial A_{\hat{x}}}{\partial \hat{z}} - \frac{\partial A_{\hat{z}}}{\partial \hat{x}}\right] + \bar{a}_{\hat{z}}\left[\frac{\partial A_{\hat{y}}}{\partial \hat{x}}\right]$$

is the "local" transverse curl operator, becoming $\text{curl}_t \bar{A}$ only in the limit as $R \to \infty$. In (16), the fields $\bar{E}_o$, $\bar{H}_o$ may be taken to have meaning for finite R, for the purposes of use in a perturbational formula, so that the field in the curved guide are given by $\bar{E} = \bar{E}_o + \bar{E}_p$; $\bar{H} = \bar{H}_o + \bar{H}_p$, where $\bar{E}_p$ and $\bar{H}_p$ are perturbations resulting from the curvature of the guide. Such a perturbational formula is derived in Appendix B, from (13), (15) and (16) by considering $\bar{F} = \bar{E}_o^- \times \bar{H}^+ - \bar{E}^+ \times \bar{H}_o^-$ which, to terms of lowest order in each of the real and imaginary parts, is:

$$k_o(\nu - \nu_o) \simeq -i\,\frac{c}{P} + i\,\frac{\Delta}{P} \tag{17}$$

where

$$c = \oint_C \frac{\rho}{R}\,[\bar{E}_o^- \times \bar{H}_p^+ - \bar{E}_p^+ \times \bar{H}_o^-]\cdot \bar{a}_n d\ell \tag{18}$$

$$P = 2 \int_S \bar{a}_\theta \cdot [\bar{E}_o^+ \times \bar{H}_o^+] dS \tag{19}$$

$$\begin{aligned}\Delta = \int_S \{ & \tfrac{1}{R}[H_{o\hat{y}}E_{o\hat{z}} - E_{o\hat{y}}H_{o\hat{z}}] - 2ik_o\nu_o(\tfrac{\hat{x}}{R})[E_{o\hat{z}}H_{o\hat{x}} - E_{o\hat{x}}H_{o\hat{z}}] \\ & + (\tfrac{\hat{x}}{R})\,[H_{o\hat{y}}\frac{\partial E_{o\hat{z}}}{\partial \hat{x}} - E_{o\hat{z}}\frac{\partial H_{o\hat{y}}}{\partial \hat{x}} + H_{o\hat{z}}\frac{\partial E_{o\hat{y}}}{\partial \hat{x}} - E_{o\hat{y}}\frac{\partial H_{o\hat{z}}}{\partial \hat{x}}]\} dS\end{aligned} \tag{20}$$

These first order corrections can be logically grouped into two types - those involving Δ which are independent of the perturbation fields $\bar{E}_p$, $\bar{H}_p$ and reflect only a geometric influence, and those resulting from c which are directly a result of the kind of field distortion which occurs when the guide is bent. The next two sections are devoted to an examination of each of these corrections.

SECTION V

GEOMETRIC CORRECTIONS TO THE PROPAGATION CONSTANT

Since we have assumed for simplicity that the waveguide is lossless, we can show that the phase relationship of the various field components is such that $+i\Delta/P$ is purely real [22, Appendix C therein], and so this term is a correction to the phase constant only which contributes no attenuation.[2] It can in fact be shown (Appendix C) that this $(1/R)$ correction is zero for a symmetrical structure, provided that the mode itself possesses certain symmetry properties. This result is well-known for certain closed waveguides [24] and open slab waveguides [9], but not in the general case.

For those asymmetrical waveguide modes where the $(1/R)$ correction is not equal to zero, one can in principle compute Δ from (20) for any guide configuration once the fields of the mode on the straight guide are known. Because of the dependence $\exp[-ik_0\nu R\theta]$, any finite angle θ of bend will cause accumulation of an appreciable excess phase shift in comparison with the straight guide. When the $(1/R)$ correction does

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[2] In the lossy case, any attenuation contributed by this term would be strictly of geometric origin, resulting from a rearrangement of the field pattern relative to the positions of any loss [23]. This effect is quite separate from the radiation loss considered in the next section.

vanish, at most a $(1/R^2)$ correction will occur; however, the total resulting excess phase shift will be of order $(1/R)$ and can thus reasonably be neglected. This phase shift may contribute substantially to both single-mode and multimode pulse distortion if its frequency dependence is strong enough; it is also a major factor in determining the endfire radiation from a curved section of waveguide, inserted between two semi-infinite straight guides [25]. More importantly, however, this phase shift can also affect the amount of radiation loss as detailed in section VII.

SECTION VI

RADIATION CORRECTIONS TO THE PROPAGATION CONSTANT

The remaining correction to the propagation constant is given by $-ic/P$. In contrast to the geometric correction, it depends critically upon the perturbation fields $\overline{E}_p$ and $\overline{H}_p$, and will in addition be essentially an attenuation term. Thus, we will need the more detailed description of $\overline{E}_p$ and $\overline{H}_p$ given in Appendix B, as well as to specify the surface S (hence the boundary C as well) we are to use in performing the integrations.[3] It is interesting to note the similarity of the term $-ic/P$ to a power balance relation [26], as well as to a modal coupling coefficient for surface waves [13,22]. As a matter of fact, Arnaud [13] obtains a related expression for the attenuative part of ν, which he interprets as coupling to a whispering gallery mode propagating along an artificially introduced perfectly conducting cylinder. The cylinder is then allowed to approach infinity, circumventing in the process the mathematical difficulty that such modes in the absence of the cylinder are not normalizable. The authors prefer to interpret (18) as co-directional coupling to a fictitious,

[3] Some restriction on this choice has already been made in Appendix C.

second guide as whose fields are $\bar{E}_{pr}$ and $\bar{H}_{pr}$, the portions of $\bar{E}_p$ and $\bar{H}_p$ reflected from the turning point. These fields can be thought of as produced by a mirror-image guide whose distance from the first guide can actually be identified (see section VII). The analogy cannot, however, be pursued too closely, since the phase relationship of the fields which results in the attenuation correction to ν_0 could not be realized by any physical second guide.

In order to obtain from (18) a useful expression for c, we first note that the only field components which are Cartesian in all of the coordinate systems of Fig. 2 are E_z and H_z. We will find it useful, then, to write (18), as far as possible, solely in terms of these. Furthermore, to use the spectrum function to the greatest possible advantage, we choose the surface S so that its boundary C consists of the two infinite lines at $\rho = R_1$ and $\rho = R_2$, between $z = -\infty$ and $z = +\infty$. R_1 and R_2 must be away from the waveguide, on the outer and inner side of the bend, respectively, but are otherwise as yet arbitrary. The resulting expression consists of two contributions of the following form (one for each part of the boundary C--see Appendix D):

$$c_1 = \frac{-2\pi}{k_0 n_0} \int_{-\infty}^{\infty} \frac{1}{1-s^2} \left| \frac{\tilde{E}_{0\hat{z}}(\hat{x},s)}{i\omega\mu_0} \frac{\partial \tilde{E}_{p\hat{z}}(\hat{x},-s)}{\partial \hat{x}} - i \frac{\lambda(\nu_0,s)}{\zeta_0} \tilde{E}_{0\hat{z}}(\hat{x},s)\tilde{E}_{p\hat{z}}(\hat{x},-s) \right.$$

$$\left. + \frac{\tilde{H}_{0\hat{z}}(\hat{x},s)}{i\omega\varepsilon_0 n_0^2} \frac{\partial \tilde{H}_{p\hat{z}}(\hat{x},-s)}{\partial \hat{x}} - i \frac{\zeta_0 \lambda(\nu_0,s)}{n_0^2} \tilde{H}_{0\hat{z}}(\hat{x},s)\tilde{H}_{p\hat{z}}(\hat{x},-s) - \right.$$

$$i\lambda(\nu_0,s)(e^{\hat{x}/R}-1)[\frac{1}{\zeta_0}\,\tilde{E}_{0\hat{z}}(\hat{x},s)\tilde{E}_{0\hat{z}}(\hat{x},-s) + \frac{\zeta_0}{n_0^2}\,\tilde{H}_{0\hat{z}}(\hat{x},s)\tilde{H}_{0\hat{z}}(\hat{x},-s)]$$

$$+ \frac{s}{n_0}\,(\nu-\nu_0 e^{\hat{x}/R})[\tilde{H}_{0\hat{z}}(\hat{x},-s)\tilde{E}_{0\hat{z}}(\hat{x},s)-\tilde{E}_{0\hat{z}}(\hat{x},-s)\tilde{H}_{0\hat{z}}(\hat{x},s)]\}\;ds \tag{21}$$

Here we have set

$$\lambda(\nu,s) = [\nu^2 + n_0^2(s^2 - 1)]^{\frac{1}{2}}$$

At this point it is appropriate to introduce an approximation which will appear several times in what follows and greatly simplifies the analysis. We assume R_1 and R_2 are taken far enough away from the guide so that essentially all of the "power flow" is included in (19) (strictly speaking, P is not a power since the integration is performed in "real" space and not in the "natural" $(\hat{x}\hat{z})$ coordinates for the unperturbed mode. However, by arguments similar to those in Appendix B, the difference has only a second-order effect on $\nu-\nu_0$), but not so far that R_1 is near or past the WKB turning point. This requirement may be stated in the form $k_0\lambda(\nu_0,s)R>>1$ for all s, or simply $k_0\lambda_0 R>>1$, where we have abbreviated

$$\lambda_0 = \lambda(\nu_0,0)$$

Thus, in equation (21), all terms involving $\exp[-2k_0\lambda(\nu_0,s)\hat{x}]$ may be considered as smaller than any inverse power of R, and the only significant remaining terms are those in which the exponential dependence has been cancelled, i.e., terms involving the product of a locally growing and a locally evanescent wave. To first-order then, we have a contribution only from C_1:

$$C = -\frac{4\pi}{n_0}\int_{-1}^{1}\frac{\lambda(\nu_0,s)\sigma_s}{1-s^2}\left\{\frac{\tilde{E}(s)\tilde{E}(-s)}{i\omega\mu_0}+\frac{\tilde{H}(s)\tilde{H}(-s)}{i\omega\varepsilon_0 n_0^2}\right\} ds \tag{22}$$

where $\tilde{E}$ and $\tilde{H}$ are the spectrum functions of $E_{0\hat{z}}$ and $H_{0\hat{z}}$ after the $\hat{x}$-dependence is removed:

$$\left.\begin{matrix}\tilde{E}(s)\\ \tilde{H}(s)\end{matrix}\right\} = \frac{k_0 n_0}{2\pi}\int_{-\infty}^{\infty}\left.\begin{matrix}E_{0\hat{z}}(\hat{x},\hat{z})\\ H_{0\hat{z}}(\hat{x},\hat{z})\end{matrix}\right\} e^{k_0\lambda(\nu_0,s)\hat{x}+iks\hat{z}}\, d\hat{z} \tag{23}$$

The integral (24) is only from -1 to +1 since we had set $\sigma_s = 0$ outside this range. It should be further recognized that while $\tilde{E}$ and $\tilde{H}$ have no explicit dependence on R, there is an indirect variation which arises when the choice of the origin of the $\hat{x}$-axis is varied, and hence this choice will affect the perturbed value of ν to some extent.

SECTION VII

STEEPEST-DESCENT EVALUATION OF THE ATTENUATION

Consistent with the foregoing approximations involving the magnitude of R, the appropriate method to evaluate (22) is the method of steepest descents [17, pp. 300-302]. We have, by substitution of (10) and (11) into (22), the following expression:

$$C = \frac{2\pi\omega}{k_0^2 n_0^3} \int_{-1}^{1} [\varepsilon_0 n_0^2 \tilde{E}(s)\tilde{E}(-s) + \mu_0 \tilde{H}(s)\tilde{H}(-s)] \exp[k_0 R q(s)] \frac{\lambda(\nu_0, s)\,ds}{1-s^2} \qquad (24)$$

where $$q(s) = 2[\lambda(\nu,s) + \frac{\nu}{2}\,\ell n\,(\frac{\nu-\lambda(\nu,s)}{\nu+\lambda(\nu,s)})].$$

It is easily verified that the steepest-descent path is essentially the real axis between $s = \pm 1$, and that the exponent goes to $-\infty$ as $s^2 \to 1$. Choosing k_0R as the large parameter, we have by the usual techniques that the saddle-point (which satisfies $q'(s)=0$) is located at the point $s=0$.[4] Thus the first-order steepest-descent asymptotic approximation is:

$$C \simeq \frac{2\pi\omega}{k_0^2 n_0^3}\left(\frac{\pi\lambda_0}{k_0 R}\right)^{\frac{1}{2}} [\varepsilon_0 n_0^2 \tilde{E}^2(0) + \mu_0 \tilde{H}^2(0)] \exp(-2\tau) \qquad (25)$$

---

[4]Two spurious complex zeroes of $q'(s)$ also occur; these, however, are branch points of $q(s)$ and will in fact provide the limitation on R which insures the validity of the steepest-descent approximation.

where we abbreviate $\lambda = \lambda(\nu,0) = [\nu^2-n_0^2]^{\frac{1}{2}}$, and

$$\tau = -k_0R[\lambda + \frac{\nu}{2}\,\ell n(\frac{\nu-\lambda}{\nu+\lambda})] \tag{26}$$

In the special case when $|\lambda/\nu| << 1$ (quite typical of both fibers and integrated optics), we have approximately

$$\tau \simeq \frac{1}{3}k_0R\frac{\lambda^3}{\nu^2} \simeq \frac{2}{3}k_0\lambda\;(\rho_{t0}-R) \tag{27}$$

where ρ_{t0} is the WKB turning point for s=0. This way of expressing τ, in addition to the well-known dependence of $\exp(-k_0\lambda W)$ of the parallel-guide coupling coefficient [22], where W is the separation between guides, allows us to roughly identify the distance between the first guide and the fictitious second guide as $W = \frac{4}{3}\;(\rho_{t0}-R)$. Finally, for (25) to be valid, we must require the distance from the saddle-point to the nearest singularities of q(s) (see footnote 4) to be large. The condition is easily shown to be

$$|\tau| >> 1 \tag{28}$$

which, for the case $|\lambda/\nu| << 1$, is obviously more stringent than the condition $k_0\lambda_0R>>1$ assumed in section VI. Thus (28) is the single criterion of applicability for any finite cross-section guide.

It will not ordinarily be sufficient to evaluate τ by replacing ν everywhere by ν_0. In cases such as for a symmetric mode when the difference $(\nu-\nu_0)$ is of order $(1/R^2)$ in its real part and exponentially small as described above in its imaginary part, this is allowed, since only a term of order (1/R) will be changed in the exponent of (25). If the (1/R) correction of section V does not vanish, however, we must take it into account, since a significant factor may be generated into this exponent [9]. Thus, if we call $\lambda_0 = \lambda(\nu=\nu_0)$ and $\tau_0 = \tau(\nu=\nu_0)$, we have by a Taylor series expansion:

$$\tau \simeq \tau_0 - \tfrac{1}{2} k_0 (\nu-\nu_0) R \, \ell n \left(\frac{\nu_0-\lambda_0}{\nu_0+\lambda_0} \right) \tag{29}$$

or, again for $|\lambda_0/\nu_0| << 1$,

$$\tau \simeq \tau_0 + k_0 (\nu-\nu_0) R \frac{\lambda_0}{\nu_0} + \frac{1}{3} k_0 (\nu-\nu_0) R \frac{\lambda_0^3}{\nu_0^3} + \ldots \tag{30}$$

The appropriate procedure, then, for calculating $(\nu-\nu_0)$ is to first calculate the real (1/R) correction as in section V, and then, using this quantity in (29) or (30), proceed to calculate the attenuation from (25). The only fields required in (25) are actually "averaged" in the $\hat{z}$-direction by the integration of (23); thus it is to be expected that less error will be incurred as a result of using inexact fields at this point than would be if the "naked" inexact fields were used to calculate the loss.[5]

For the case of a symmetrical mode, $\nu \simeq \nu_0$ so that the expression for τ reduces to

$$\tau \simeq \tau_0 = \frac{1}{3} k_0 R \frac{\lambda_0^3}{\nu_0^2} \tag{31}$$

[5]Marcatili's [2] field approximations for the rectangular dielectric waveguide are thus utterly incapable of predicting the $R^{-\frac{1}{2}}$ dependence of the attenuation which by (25) should occur in all finite cross-section structures. Not only does there exist experimental evidence that this term should be present [27], but similar studies of coupling between such waveguides show considerable error using these field approximations [28].

SECTION VIII

DISCUSSION AND CONCLUSIONS

To summarize the technique, then, given a straight surface waveguide of arbitrary cross-section whose fields $\bar{E}_0$ and $\bar{H}_0$ and propagation constant ν_0 are known, we may calculate the shift in the real part of the propagation constant as $i\Delta/P$, where Δ and P are given in equations (19) and (20). This done, we may calculate the attenuation due to radiation as ic/P, where c is given by (25), (23), and (29) or (30), using the previously determined value for the real part of the correction to ν_0 in (29) or (30).

As applied to the asymmetric slab waveguide in Appendix E, this procedure yields

$$\mathrm{Re}[k_0(\nu-\nu_0)] = \frac{k_0\nu_0}{2R}\left(\frac{1}{k_0\lambda_0} - \frac{1}{\gamma_0}\right)$$

$$\mathrm{Im}[k_0(\nu-\nu_0)] = -\frac{\lambda_0(n_1^2-\nu_0^2)}{\nu_0 L_e(n_1^2-n_0^2)} \exp\left[-2\tau_0+k_0\lambda_0 d-k_0\lambda_0\left(\frac{1}{k_0\lambda_0} - \frac{1}{\gamma_0}\right)\right]$$

where the various quantities appearing in these expressions are defined in Appendix E. The term $[(k_0\lambda_0)^{-1} - \gamma_0^{-1}]$ is a measure of the asymmetry of the guide, and may in fact be an extremely sensitive function of the guide parameters [9,23]. These results agree with those of [9] obtained by derectly perturbing the eigenvalue equation. The result of Marcuse [7] omits the asymmetry term in the attenuation constant, and the possibility of a factor of 2 or 3 difference between the two results indicates the importance of retaining the first-order correction in (29). In fact, it can be shown that the choice of the location of $\hat{x}=0$ (i.e., the definition of R) relative to the slab will have an effect on the phase constant which amounts to a referral of the phase velocity to the location $\hat{x}=0$. However, the only change in the attenuation constant is a change in R (occurring in τ_0) by the appropriate amount, so that it is clear that the $k_0\lambda_0 d$ term appearing in the exponent is not removable by a redefinition of R.

In Appendix F the circular fiber is treated, which, being a symmetric structure, has only the attenuation

$$\mathrm{Im}[k_0(\nu-\nu_0)] = -k_0\left(\frac{\pi}{k_0R}\right)^{\frac{1}{2}} \frac{(n_1^2-\nu_0^2)\; n_0\; \exp[-2\tau_0]}{4\nu_0(k_0a)^2(n_1^2-n_0^2)\;\lambda_0^{3/2}\; K_{m+1}(k_0\lambda_0 a)K_{m-1}(k_0\lambda_0 a)}$$

independently of mode orientation or polarization, for $HE_{m+1,n}$ or $EH_{m-1,n}$ modes. The quantities appearing in this expression are defined in Appendix F. This result agrees with Lewin's analysis [10] with the exception that the present result is less by a factor ½. The discrepancies with Arnaud's result [13], in which a limiting case is treated, are more serious. The most serious one is the omission of a factor $\exp(2k_0\lambda_0 a)$ in his expression. This very term will be quite large for the case he has considered ($k_0\lambda_0 a \gg 1$). As discussed for the slab, this term cannot be removed by a redefinition of R, and, for Arnaud's geometry, again points up the importance of the correction term in (29), since it appears to have been neglected in [13]. The remaining discrepancy can be explained by Arnaud's use of an approximate spectrum function instead of the exact one obtained in Appendix F.

Finally, it will be noted that (25) predicts a dependence of $R^{-\frac{1}{2}}$ for the attenuation of any finite cross-section guide. As pointed out in footnote 5, field approximations which are not somehow averaged (as, e.g., in equation (23) with s=0) do not generally give satisfactory results, and in particular Marcatili's analysis of the rectangular cross-section guide [2] is in error by at least this term (in fact, to his approximations, the bending loss behaves as if the guide were a slab).

APPENDIX A

In this Appendix, we examine closely the WKB solution to equation (7). The solutions in terms of the integral (8) are well-known [18], and herein we shall discuss what branch cuts need to be chosen in the process. In the first place, the imaginary parts of the logarithms in (8) may be taken to be between $-\pi$ and π, since any other would amount to multiplying the solution by a constant factor. To look at the definition of w, we must consider the two cases separately.

Case I: $s^2 < 1$. The turning points are located at

$$\rho_t = \pm\nu R/n_0(1-s^2)^{\frac{1}{2}}$$

as shown in Fig. A.1(a), which correspond to an infinite set of turning points in the v-plane, $v_t = v_{tr} + iv_{ti}$ with

$$v_{tr} = \ln[|\nu|/n_0(1-s^2)^{\frac{1}{2}}] \quad ; \quad v_{ti} = \arg\nu + n\pi i,\ n=0,\pm 1,\pm 2,\ \ldots$$

Along the positive real ρ-axis, for ρ sufficiently large, recalling that $w = [n_0^2(1-s^2)\frac{\rho^2}{R^2} - \nu^2]^{\frac{1}{2}}$, we can choose w so that $\mathrm{Re}(w) > 0$, and we have then

$$w \simeq \frac{n_0\rho}{R}(1-s^2)^{\frac{1}{2}} - \frac{\nu^2 R}{2n_0\rho(1-s^2)^{\frac{1}{2}}}$$

$$\xi \simeq \frac{n_0\rho}{R}(1-s^2)^{\frac{1}{2}} + \frac{\nu^2 R}{2n_0\rho(1-s^2)^{\frac{1}{2}}} - \frac{\nu\pi}{2}$$

and as $\rho \to +\infty$, $\mathrm{Re}(\xi) \to +\infty$.

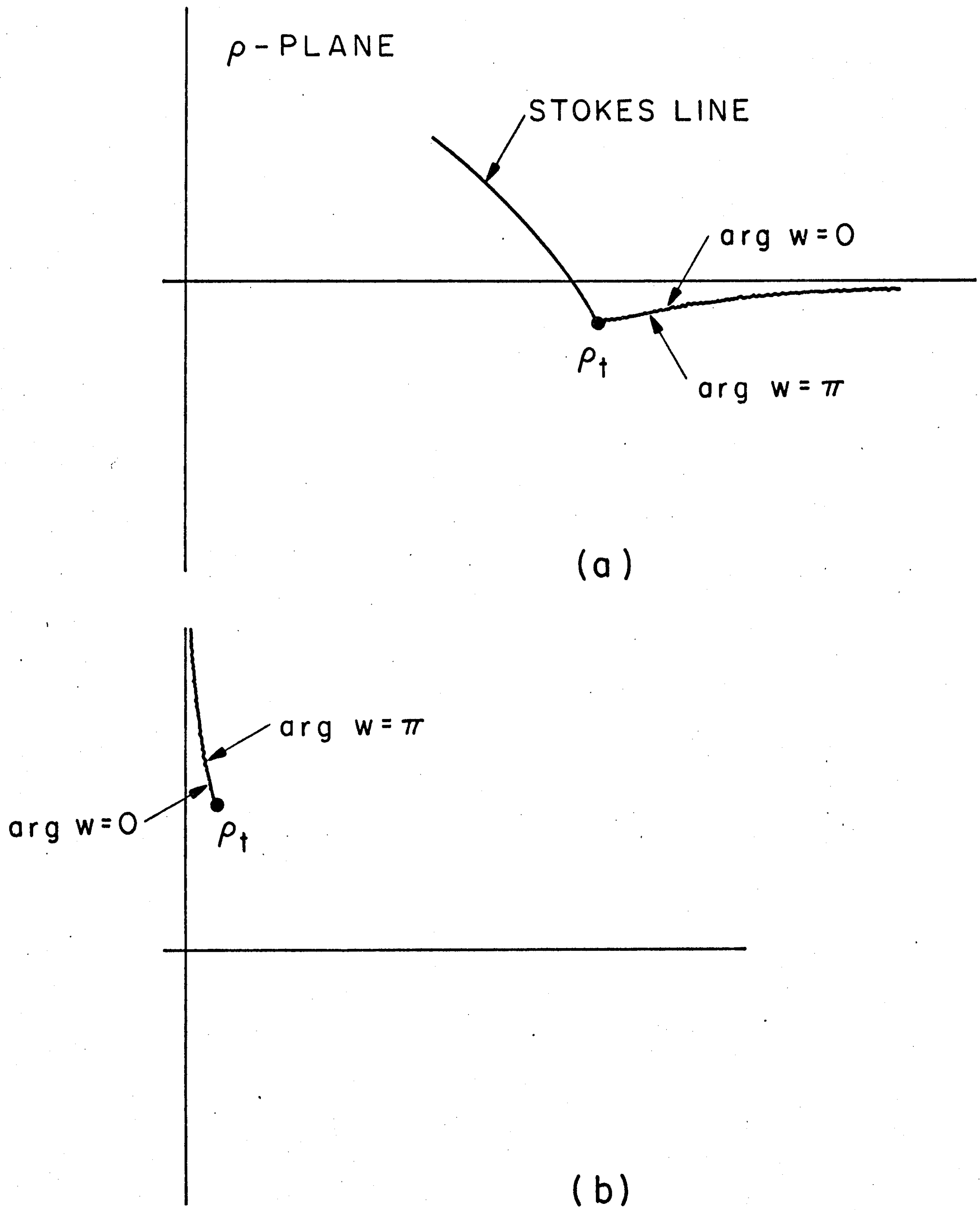

Fig. A.1

Turning points and branch cuts in ρ-plane. Only points in right half-plane are shown. (a) $s^2 < 1$, (b) $s^2 > 1$. In (a), the Stokes line is shown crossing the real axis.

Thus ξ has a small, negative imaginary part, and $\exp(-ik_0R\xi)$ represents an outgoing wave which is bounded as $\rho \to \infty$. Decreasing ρ along the real axis now, we pass by the turning point ρ_t, and arg w increases from near 0 to near $\pi/2$, so that as $\rho \to 0^+$ we can write

$$w \simeq i\nu - i\,\frac{n_0^2(1-s^2)\rho^2}{2\nu R^2}$$

$$\xi \simeq i\nu\left[1+ \tfrac{1}{2}\ln\left(\frac{n_0^2(1-s^2)\rho^2}{4\nu^2R^2}\right)\right] - i\,\frac{3}{8}\,\frac{n_0^2(1-s^2)\rho^2}{\nu R^2}$$

Thus $\mathrm{Re}(\xi) \sim (\mathrm{Im}\,\nu)\,(\infty) \to -\infty$ as $\rho \to 0^+$, and it is apparent that on the interval $[0,+\infty)$, there is a point where $\mathrm{Re}(\xi) = 0$, indicating the crossing of the positive real ρ-axis by a Stokes line [17, p. 293] (defined by $\mathrm{Re}(\xi) = 0$). At this point, there must occur a discontinuous transition between two different asymptotic (WKB) solutions for Φ. Let us examine how this occurrence (known as Stokes' phenomenon) leads to "reflection" of the fields from the complex turning point.

For small $|w|$ (i.e., near the turning point),

$$\xi \simeq \frac{1}{3}\,\frac{w^3}{\nu^2} \tag{A.1}$$

so that as arg w increases from 0 to $\pi/2$, arg ξ increases from about 0 to about $3\pi/2$. As a result, $\exp(+ik_0R\xi)$ represents an outwardly-decaying field as long as ρ is below the Stokes line crossing of the real axis, since ξ is (essentially) negative imaginary and decreases in magnitude as ρ approaches ρ_t from below. Now, although for sufficiently large ρ only the outgoing WKB solution is permitted in order to satisfy the radiation condition at infinity, below the Stokes line we must in general expect a linear combination of the decaying solution (here called "incident") and a locally growing "reflected" field as given by equation (9), directly as a consequence of the Stokes phenomenon [17, p. 325].

The reflection coefficient σ_s may be determined as in [17, pp. 322-326] by matching the behavior of the WKB solution to that of the appropriate Airy function which describes the beahvior of Φ near the turning point. In the present case, the appropriate behavior for large positive v is given in terms of the outgoing Fock-Airy function W_1 [19, p. 112]:

$$\Phi \simeq W_1[-(2k_0^2\nu^2R^2)^{1/3}(v-v_t)]$$

$$\simeq W_1[-(k_0^2R^2/4\nu^4)^{1/3}w^2]$$

As v decreases along the positive real axis, the argument of the function W_1 changes in phase from about $-\pi$ to about 0, and in the latter range the asymptotic form of Φ is [19, pp. 76,113]:

$$\Phi \sim [-(k_0^2R^2/4\nu^4)^{1/3}w^2]^{-1/4} \cdot$$

$$\cdot\{\exp[i\,\frac{k_0R}{3}\,\frac{w^3}{\nu^2}] - \frac{i}{2}\exp[-i\,\frac{k_0R}{3}\,\frac{w^3}{\nu^2}]\}$$

and so the reflection coefficient may be identified using (A.1) by comparison with (9):

$$\sigma_2 = \frac{i}{2}\exp[-2ik_0R\xi_0] \tag{11}$$

It is now clear that we are free to choose the branch cut defining w in any manner so long as the range of arguments from 0 to $\pi/2$ is included. For simplicity, then, we choose $0 \leq$

arg $w < \pi$, so that the branch cut appears as in Fig. A.1(a).

The higher order terms in the asymptotic expansion appearing in (9) can be found as in [18], and in fact coincide with Debye's expansion for the Hankel function, given for real order in [20, p. 366].

Case II: $s^2 > 1$. The turning points are now located at

$$\rho_t = \pm i\nu R/n_0(s^2-1)^{\frac{1}{2}}$$

as in Fig. A.1(b), corresponding to the infinite set of turning points in the v-plane:

$$v_{tr} = \ln[|\nu|/n_0(s^2-1)^{\frac{1}{2}}] \quad ; \quad v_{ti} = \arg\nu + i\pi/2 + n\pi i,\ n=0,\pm 1,\pm 2,\ldots$$

In this case it is evident that w is predominantly imaginary along the entire positive real ρ-axis. If we define w with the same range of argument A.1(b), arg w is approximately $\pi/2$ on all of the positive real ρ-axis. Thus as $\rho \to +\infty$ on this axis

$$w \simeq i\,\frac{n_0\rho}{R}(s^2-1)^{\frac{1}{2}} + i\,\frac{\nu^2 R}{2n_0\rho(s^2-1)^{\frac{1}{2}}}$$

$$\xi \simeq i\,\frac{n_0\rho}{R}(s^2-1)^{\frac{1}{2}} - i\,\frac{\nu^2 R}{2n_0\rho(s^2-1)^{\frac{1}{2}}} - \frac{\nu\pi}{2}$$

so that $\exp(+ik_0R\xi)$ again represents an outwardly-decaying wave. As $\rho \to 0^+$ we have

$$w \simeq i\nu + i\,\frac{n_0^2(s^2-1)\rho^2}{2\nu R^2}$$

$$\xi \simeq i\nu[1+\tfrac{1}{2}\ln(\frac{n_0^2(s^2-1)\rho^2}{4\nu^2R^2})] + i\,\frac{3}{8}\,\frac{n_0^2(s^2-1)\rho^2}{\nu R^2} - \nu\,\frac{\pi}{2}$$

And so $\mathrm{Re}(\xi) \sim (\mathrm{Im}\,\nu)(\infty) \to -\infty$ as $\rho \to 0^+$, and $\mathrm{Re}(\xi) \to -(\mathrm{Re}\,\nu)\,\pi/2$ as $\rho \to +\infty$. Since $\mathrm{Re}(d\xi/dv) = \mathrm{Re}(w) > 0$ over the entire positive real ρ-axis, there is no non-negative value of ρ for which $\mathrm{Re}(\xi) = 0$; therefore no Stokes line crosses the real axis, and only the outwardly decaying WKB solution, eqn. (12), occurs.

The higher order terms in (12) are again found as in [18], and coincide with Debye's expansion for the modified Bessel function, given for real order in [20, p. 378].

APPENDIX B

Let $\overline{E}^+$, $\overline{H}^+$, ν be the guide fields and propagation constant for a curved waveguide, which satisfy (13), and $\overline{E}_0^-$, $\overline{H}_0^-$, ν_0 the corresponding straight waveguide fields (as functions of the local Cartesian coordinates) and propagation constant (transposed) which satisfy (16). Then

$$\frac{\rho}{R}\,\mathrm{div}_t[\overline{E}_0^- \times \overline{H}^+ - \overline{E}^+ \times \overline{H}_0^-] = ik_0(\nu-\nu_0)\overline{a}_\theta\cdot[\overline{E}_0^- \times \overline{H}^+ - \overline{E}^+ \times \overline{H}_0^-]$$
$$+ [\overline{H}^+\cdot D_R \times \overline{E}_0^- + \overline{E}^+\cdot D_R \times \overline{H}_0^-] \qquad \text{(B.1)}$$

where

$$D_R \times \overline{A} = \frac{\rho}{R}\,[\mathrm{curl}_t\,\overline{A} - \hat{\nabla}_t \times \overline{A}] - ik_0\nu_0(e^{\hat{x}/R}-1)\,\overline{a}_{\hat{y}} \times \overline{A}$$
$$= (e^{\hat{x}/R}-1)\,\overline{a}_{\hat{y}}\,\frac{A_{\hat{z}}}{\partial_{\hat{x}}} + \overline{a}_{\hat{z}}\,\{(1-e^{\hat{x}/R})\,\frac{\partial A_{\hat{y}}}{\partial_{\hat{x}}} + \frac{1}{R}\,A_{\hat{y}}\}$$
$$- ik_0\nu_0(e^{\hat{x}/R}-1)\,\{\overline{a}_{\hat{x}}A_{\hat{z}} - \overline{a}_{\hat{z}}A_{\hat{x}}\}$$

Note that D_R x $\overline{A}$ is of order $1/R$ compared with $\overline{A}$ itself. Applying (15) to (B.1), we have

$$\oint_C \frac{\rho}{R}\,[\overline{E}_0^- \times \overline{H}^+ - \overline{E}^+ \times \overline{H}_0^-]\cdot\overline{a}_n d\ell = ik_0(\nu-\nu_0)\int_S \overline{a}_\theta\cdot[\overline{E}_0^+ \times \overline{H}^+ + \overline{E}^+ \times \overline{H}_0^+]dS$$
$$+ \int_S [H^+\cdot D_R \times \overline{E}_0^- + \overline{E}^+\cdot D_R \times \overline{H}_0^-]dS \qquad \text{(B.2)}$$

For the moment, we will not specify S or C. Let us call the perturbation fields in the curved guide $\overline{E}_p$ and $\overline{H}_p$, so that $\overline{E} = \overline{E}_0 + \overline{E}_p$ and $\overline{H} = \overline{H}_0 + \overline{H}_p$. Then (B.2) becomes

$$\oint_C \frac{\rho}{R} [\overline{E}_0^- \times \overline{H}_p^+ - \overline{E}_p^+ \times \overline{H}_0^-]\cdot\overline{a}_n d\ell = 2ik_0(\nu-\nu_0) \int_S \overline{a}_\theta\cdot[\overline{E}_0^+ \times \overline{H}_0^+]dS$$

$$+ik_0(\nu-\nu_0) \int_S \overline{a}_\theta\cdot[\overline{E}_0^- \times \overline{H}_p^+ - \overline{E}_p^+ \times \overline{H}_0^-]dS$$

$$+ \int_S [\overline{H}_0^+\cdot D_R \times \overline{E}_0^- + \overline{E}_0^+\cdot D_R \times \overline{H}_0^-]dS \tag{B.3}$$

$$+ \int_S [\overline{H}_p^+\cdot D_R \times \overline{E}_0^- + \overline{E}_p^+\cdot D_R \times \overline{H}_0^-]dS$$

$$= 2ik_0(\nu-\nu_0) \int_S \overline{a}_\theta\cdot[\overline{E}_0^+ \times \overline{H}_0^+]dS + \int_S [\overline{H}_p^+\cdot D_R' \times \overline{E}_0^- + \overline{E}_p^+\cdot D_R' \times \overline{H}_0^-]dS$$

$$+ \int_S \{\frac{1}{R}[H_{o\hat{y}}E_{o\hat{z}} - E_{o\hat{y}}H_{o\hat{z}}] + (e^{\hat{x}/R}-1)[H_{o\hat{y}} \frac{E_{o\hat{z}}}{\partial\hat{x}} + H_{o\hat{z}} \frac{E_{o\hat{y}}}{\partial\hat{x}} - E_{o\hat{y}} \frac{H_{o\hat{z}}}{\partial\hat{x}}] -$$

$$- 2ik_0\nu_0 \ (e^{\hat{x}/R}-1)[E_{o\hat{z}}H_{o\hat{x}} - E_{o\hat{x}}H_{o\hat{z}}]\} \ dS \tag{B.4}$$

where

$$D_R' \times \overline{A} = D_R \times \overline{A} + ik_0(\nu-\nu_0)\overline{a}_\theta \times \overline{A} \tag{B.5}$$

From the discussion of section III we are able to say a few things about the general characteristics of the perturbation fields. The spectrum function for a Cartesian component of $\overline{E}_0$ or $\overline{H}_0$ is given by $A^i(s)\ \exp[-k_0\lambda(\nu_0,s)\hat{x}]$, where

$$\lambda(\nu,s) = [\nu^2 + n_0^2(s^2-1)]^{\frac{1}{2}} \tag{B.6}$$

As $R \to \infty$, we require that $\overline{E}$ and $\overline{H}$ go over into the unperturbed fields, so that from (9) and (12), the spectrum function amplitude D^i can be given in terms of A^i as

$$D^i(s) = [\lambda(\nu,s)]^{\frac{1}{2}}\ e^{i\pi/4}A^i(s) \tag{B.7}$$

because

$$[\frac{\lambda(\nu,s)}{w}]^{\frac{1}{2}} \exp[\underline{+}ik_0R(\xi-\xi_0) \pm k_0\lambda(\nu,s)x + i\pi/4] =$$

$$= 1 + \frac{e^{i\pi/4}}{2} \frac{n_0^2(1-s^2)}{[\lambda(\nu,s)]^{3/2}} \frac{\hat{x}}{R} + \ldots \tag{B.8}$$

as can be shown by expansion in a Taylor series about $\hat{x} = 0$. In like manner, $\exp[-k_0\lambda(\nu,s)\hat{x}]$ can be expanded about its value at ν_0:

$$\exp[-k_0\lambda(\nu,s)\hat{x}] = \exp[-k_0\lambda(\nu_0,s)\hat{x}]\ \{1-k_0\hat{x}\ \frac{\nu_0(\nu-\nu_0)}{\lambda(\nu_0,s)} + \ldots\} \tag{B.9}$$

From (B.7) - (B.9), (9) and (12) it can be seen that $\bar{E}_p$, $\bar{H}_p$, and $|\nu-\nu_0|$ are all of order (at most) 1/R, and we can in fact write:

$$\tilde{\Phi}\ (\hat{x},s) \sim A^i(s)\ \{[1 + 0(\tfrac{1}{R})]\ \exp[-k_0\lambda(\nu_0,s)\hat{x}] + +\sigma_s[1 + 0(\tfrac{1}{R})]\ \exp[+k_0\lambda(\nu_0,s)\hat{x}]\} \tag{B.10}$$

$$(s^2 < 1)$$

with the same expression holding for $s^2 > 1$ if we define $\sigma_s=0$ for these values of s. The perturbation fields thus consist of evanescent waves which are $0(\frac{1}{R})$ compared with $\bar{E}_0$ or $\bar{H}_0$, and locally exponentially growing fields (for $s^2 < 1$) as described by the second term of (B.10).

It can be seen, then, that if (as shown in sections V and VI) the first-order correction to the real part of ν is given by a (1/R) term, and the first order correction to the imaginary part of is given by a term exponentially dependent on R, then terms such as $\bar{H}_0^+\cdot D_R \times \bar{E}_0^-$, etc., are all of second-order, and so the first-order corrections are given by (17).

APPENDIX C

In this Appendix we show that the geometrical 1/R correction term $i\Delta/P$ in (17) is zero for waveguide modes antisymmetric or symmetric in the x-direction. We assume that $\varepsilon(-\hat{x},\hat{z}) = \varepsilon(\hat{x},\hat{z})$ for all $\hat{x}$ and $\hat{z}$. Now by the symmetry properties of Maxwell's equations (16) (which also give rise to the "transpose" field relationships between $\overline{E}_0^{\pm}$ and $H_0^{\pm}$), we can always choose a given Cartesian field component to be either an even or an odd function of $\hat{x}$. Equations (16) will then force all other field components to be either even or odd functions as follows:

$E_{\hat{y}}$	even	(odd)
$H_{\hat{y}}$	odd	(even)
$E_{\hat{z}}$	even	(odd)
$H_{\hat{z}}$	odd	(even)
$E_{\hat{x}}$	odd	(even)
$H_{\hat{x}}$	even	(odd)

An inspection of the integrand in (20) shows that all integrands are odd functions of $\hat{x}$, and so if the surface S is chosen to be symmetric with respect to $\hat{x}=0$, it is seen that $\Delta=0$.

In certain waveguides of high symmetry (e.g. the circular fiber), it is possible that modes of both types above may exist and be degenerate so that a linear combination of them (with no particular symmetry) may be considered as a single mode.

In this case, it may be seen that the cross terms so introduced into (20) will not in general vanish upon integration, so that when such degeneracy exists, the phase correction term is in general dependent on the polarization of the mode in question.

APPENDIX D

Let us consider first the portion C_1 of the contour C shown in Fig. 2. Here the contribution c_1 to c is

$$\bar{c}_1 = \int_{C_1} \frac{\rho}{R}[\bar{E}_0^- \times \bar{H}_p^+ - \bar{E}_p^+ \times \bar{H}_0^-]\cdot \bar{a}_n d\ell$$

$$= + \frac{R_1}{R}\int_{-\infty}^{\infty}\{E_{o\hat{y}}^- H_{p\hat{z}}^+ - E_{o\hat{z}}^- H_{p\hat{y}}^+ - E_{p\hat{y}}^+ H_{o\hat{z}}^- + E_{p\hat{z}}^+ H_{o\hat{y}}^- \}dz$$

$$= - \frac{R_1}{R}\int_{-\infty}^{\infty}\{E_{o\hat{y}}^+ H_{p\hat{z}}^+ + E_{o\hat{z}}^+ H_{p\hat{y}}^+ - E_{p\hat{y}}^+ H_{o\hat{z}}^+ - E_{p\hat{z}}^+ H_{o\hat{y}}^+ \}dz \tag{D.1}$$

From here on, only "+" fields will be involved, and we drop the superscript as unnecessary. The $\hat{y}$-(or θ-) components of the fields can be given in terms of the z-components by means of the appropriate set of Maxwell's equations, (13) or (16):

$$\left(\frac{\partial^2}{\partial \hat{z}^2} + k^2\right) H_{o\hat{y}} = -i\omega\varepsilon_0 n_0^2 \frac{\partial E_{o\hat{z}}}{\partial \hat{x}} - ik_0\nu_0 \frac{H_{o\hat{z}}}{\partial \hat{z}}$$

$$\left(\frac{\partial^2}{\partial \hat{z}^2} + k^2\right) E_{o\hat{y}} = i\omega\mu_0 \frac{\partial H_{o\hat{z}}}{\partial \hat{x}} - ik_0\nu_0 \frac{\partial E_{o\hat{z}}}{\partial \hat{z}} \tag{D.2}$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) H_\theta = -i\omega\varepsilon_0 n_0^2 \frac{\partial E_z}{\partial \rho} - i\frac{k_0\nu R}{\rho}\frac{\partial H_z}{\partial z}$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_\theta = i\omega\varepsilon_0 \frac{\partial H_z}{\partial \rho} - i\frac{k_0\nu R}{\rho}\frac{\partial E_z}{\partial z} \tag{D.3}$$

valid for $\rho > \rho_{max}$. Subtracting (D.2) from (D.3) gives

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) H_{p\hat{y}} = -i\omega\varepsilon_0 n_0^2 \frac{R}{\rho}\frac{\partial E_{p\hat{z}}}{\partial \hat{x}} + i\omega\varepsilon_0 n_0^2 (1-e^{-\hat{x}/R})\frac{\partial E_{o\hat{z}}}{\partial \hat{x}}$$

$$- i\frac{k_0\nu R}{\rho}\frac{\partial H_{p\hat{z}}}{\partial \hat{z}} - ik_0\left(\frac{\nu R}{\rho} - \nu_0\right)\frac{\partial H_{o\hat{z}}}{\partial \hat{z}}$$

$$\left(\frac{\partial^2}{\partial z^2} + k^2\right) E_{p\hat{y}} = i\omega\mu_0 \frac{R}{\rho} \frac{\partial H_{p\hat{z}}}{\partial \hat{x}} - i\omega\mu_0 (1-3^{-\hat{x}/R}) \frac{\partial H_{o\hat{z}}}{\partial \hat{x}}$$

$$- i \frac{k_0 \nu R}{\rho} \frac{\partial E_{p\hat{z}}}{\partial \hat{z}} - ik_0 \left(\frac{\nu R}{\rho} - \nu_0\right) \frac{\partial E_{o\hat{z}}}{\partial \hat{z}} \qquad (D.4)$$

With the contour chosen as it is, we are able to make use of the spectrum description of the $\hat{z}$-components of the field to find the corresponding description of the $\hat{y}$-components. This done, we may rewrite (D.1) using the convolution theorem for Fourier transforms. Now

$$\tilde{H}_{o\hat{y}} = \frac{1}{1-s^2} \left[\frac{i}{0} \lambda(\nu_0,s)\tilde{E}_{o\hat{z}} - \frac{s\nu_0}{n_0} \tilde{H}_{o\hat{z}}\right]$$

$$\tilde{E}_{o\hat{y}} = \frac{1}{1-s^2} \left[- \frac{i\zeta_0}{n_0^2} \lambda(\nu_0,s) \tilde{H}_{o\hat{z}} - \frac{s\nu_0}{n_0} \tilde{E}_{o\hat{z}}\right]$$

$$\tilde{H}_{p\hat{y}} = \frac{1}{1-s^2} \left[\frac{R}{i\omega\mu_0\rho} \frac{\partial \tilde{E}_{p\hat{z}}}{\partial \hat{x}} - \frac{i}{\zeta_0} \lambda(\nu_0,s)(1-e^{-\hat{x}/R})\tilde{E}_{o\hat{z}}\right. \qquad (D.5)$$

$$\tilde{E}_{p\hat{y}} = \frac{1}{1-s^2} \left[- \frac{R}{i\omega\varepsilon_0 n_0^2 \rho} \frac{\partial H_{p\hat{z}}}{\partial \hat{x}} + \frac{i\zeta_0}{n_0^2} \lambda(\nu_0,s) (1-e^{-\hat{x}/R})\tilde{H}_{o\hat{z}}\right.$$

$$\left. - \frac{s\nu R}{n_0\rho} \tilde{E}_{p\hat{z}} - \frac{s}{n_0} \left(\frac{\nu R}{\rho} - \nu_0\right)\tilde{E}_{o\hat{z}}\right]$$

where $\zeta_0 = \sqrt{\mu_0/\varepsilon_0}$ is the wave impedance of free space, so that application of the convolution theorem to (D.1) gives the expression in (21), in addition to some second order terms. Although we have not explicitly looked into the character of the solutions on the inner side of the bend, it can be seen that a similar expression will hold for the contribution from C_2, although no WKB reflection occurs for any value of s on this side of the bend, and so as discussed in section VI, the contribution from C_2 may be neglected.

APPENDIX E

The treatment of slab waveguides by the present method is simplified somewhat since the surface integrals go over into line integrals and the line integral expression for c is no longer an integral at all, so that no steepest-descent evaluation is necessary. Here we examine the TE modes of an asymmetric slab waveguide. If we locate $\hat{x}=0$ at the center of the slab, the fields are given by [29, p. 9]:

$$\tilde{E}_{o\hat{z}} = -Ae^{-k_0\lambda_0(\hat{x}-d/2)} \qquad \hat{x} \geq d/2 \qquad \text{(I)}$$

$$-A[\cos K_0(\hat{x}-d/2) - \frac{k_0\lambda_0}{K_0} \sin K_0(\hat{x}-d/2)] \qquad -d/2 \leq x \leq d/2 \qquad \text{(II)}$$

$$-A[\cos K_0 d + \frac{k_0\lambda_0}{K_0} \sin K_0 d]\, e^{\gamma_0(\hat{x}+d/2)} \qquad \hat{x} \leq -d/2 \qquad \text{(III)}$$

with

$$H_{o\hat{y}} = \frac{1}{i\omega\mu_0}\frac{\partial E_{o\hat{z}}}{\partial \hat{x}} \quad ; \quad H_{o\hat{x}} = \frac{k_0\nu_0}{\omega\mu_0} E_{o\hat{z}}$$

and all other field components vanishing. Here

$$K_0 = k_0(n_1^2 - \nu_0^2)^{\frac{1}{2}}$$

$$\gamma_0 = k_0(\nu_0^2 - n_2^2)^{\frac{1}{2}} \quad ; \quad \lambda_0 = (\nu_0^2 - n_0^2)^{\frac{1}{2}}$$

where n_0, n_1, and n_2 are the indices of refraction in regions I, II, and III, respectively. The parameters γ^{-1} and $(k_0\lambda_0)^{-1}$ therefore have the physical meaning of penetration depths of a specific mode into the two exterior regions of the waveguide.

Continuity of $H_{o\hat{y}}$ results in the eigenvalue equation

$$k_0\lambda_0(K_0\cos K_0 d + \gamma_0 \sin K_0 d) = K_0(K_0 \sin K_0 d - \gamma_0 \cos K_0 d)$$

The P-integral is calculated in [29, p. 14]:

$$P = \frac{A^2 k_0 \nu_0 L_e (n_1^2 - n_0^2)}{\omega\mu_0 (n_1^2 - \nu_0^2)} \quad ; \quad L_e = d + \frac{1}{\gamma_0} + \frac{1}{k_0\lambda_0}$$

From equation (20), Δ is

$$\Delta = \frac{1}{R}\{\int_{\infty}^{\infty} [H_{0\hat{y}}E_{0\hat{z}} - 2ik_0\nu_0\hat{x}E_{0\hat{z}}H_{0\hat{x}} + x(H_{0\hat{y}}\frac{\partial E_{0\hat{z}}}{\partial \hat{x}} - E_{0\hat{z}}\frac{\partial H_{0\hat{y}}}{\partial \hat{x}}]d\hat{x}\}$$

$$= \frac{1}{R}\{\frac{1}{i\omega\mu_0}\int_{\infty}^{\infty} \hat{x}[k_0^2(\nu_0^2+n^2)E_{0\hat{z}}^2 + (\frac{\partial E_{0\hat{z}}}{\partial \hat{x}})^2]d\hat{x}\}$$

After considerable algebra, this can be shown to be

$$\Delta = \frac{k_0^2\nu_0^2 A^2 (n_1^2 - n_0^2) L_e}{2i\omega\mu_0 R(n_1^2 - \nu_0^2)} [\frac{1}{k_0\lambda_0} - \frac{1}{\gamma_0}]$$

Thus, the real part $i\Delta/P$ of the correction to $k_0\nu_0$ in (17) is simply

$$i\frac{\Delta}{P} = \frac{k_0\nu_0}{2R}(\frac{1}{k_0\lambda_0} - \frac{1}{\gamma_0}) \qquad \text{(E.1)}$$

As expected, this correction is zero in the symmetric case ($k_0\lambda_0 = \gamma_0$). Equation (E.1) is identical to the phase correction found by Chang and Barnes [9] for the asymmetric slab.

The quantity c can be found from (D.1) with the line integral having degenerated into an evaluation at a single point:

$$c = -\frac{2\sigma_0}{i\omega\mu_0} k_0\lambda_0 A^2 e^{k_0\lambda_0 d}$$

$$= \frac{A^2}{\omega\mu_0} k_0\lambda_0 e^{k_0\lambda_0 d} e^{-2\tau}$$

where τ is given by (30) and (27). From (E.1), then, the attenuative correction becomes, for $\lambda_0/\nu_0 << 1$:

$$-i\,\frac{C}{P} = -i\,\frac{\lambda_0(n_1^2 - \nu_0^2)}{\nu_0 L_e(n_1^2 - n_0^2)} \exp\left[-\frac{2}{3}k_0 R\,\frac{\lambda_0^3}{\nu_0^2} - k_0\lambda_0\left(\frac{1}{k_0\lambda_0} - \frac{1}{\gamma_0}\right) + k_0\lambda_0 d\right] \tag{E.2}$$

Once again, this is in agreement with [9], if the reference point $\hat{x}=0$ is changed, and the errors in eqn. (26) of [9] are noted.† As noted in [9], the extra term in the exponent resulting from the asymmetry, which does not appear in [7], can be quite significant.

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†Some typographical mistakes are seen in their expression: specifically, the exponential term in (26) should be $\exp(-k\lambda_0\delta)$ instead of $\exp(-2k\nu_0\lambda_0\delta/3)$.

APPENDIX F

We consider here the "simplified" modes of the step-index optical fiber with $n=n_0$ for $\hat{r} > a$ and $n=n_1$ for $\hat{r} < a$, as discussed by Marcuse [29]. These simplified modes assume that $(n_1-n_0)/n_0 << 1$, and what is more, assume convenient forms quite naturally in the Cartesian coordinate system [29, pp. 62-77]. There exist two orthogonally polarized mode types in this approximation. One is polarized with the electric field in the $\hat{z}$-direction:

$$E_{o\hat{z}} = -AJ_m(\kappa\hat{r})\cos(m\hat{\phi}-\phi_0) \qquad \hat{r} < a$$

$$= -AQ_m(\nu_0)K_m(k_0\lambda_0\hat{r})\cos(m\hat{\phi}-\phi_0) \qquad \hat{r} > a$$

$$H_{o\hat{x}} = -\frac{\nu_0 A}{\zeta_0} J_m(\kappa\hat{r})\cos(m\hat{\phi}-\phi_0) \qquad \hat{r} < a$$

$$= -\frac{\nu_0 A}{0} Q_m(\nu_0)K_m(k_0\lambda_0\hat{r})\cos(m\hat{\phi}-\phi_0) \qquad \hat{r} > a$$

$$E_{o\hat{y}} = \frac{iA\kappa}{2k_0\nu_0}[J_{m+1}(\kappa\hat{r})\sin((m+1)\hat{\phi}-\phi_0)+J_{m-1}(\kappa\hat{r})\sin((m-1)\hat{\phi}-\phi_0)] \qquad \hat{r} < a$$

$$= \frac{iA\lambda_0}{2\nu_0} Q_m(\nu_0)[K_{m+1}(k_0\lambda_0\hat{r})\sin((m+1)\hat{\phi}-\phi_0)-K_{m-1}(k_0\lambda_0\hat{r})\sin((m-1)\hat{\phi}-\phi_0)] \qquad \hat{r} > a$$

$$H_{o\hat{y}} = \frac{iA\kappa}{2\omega\mu_0}[J_{m+1}(\kappa\hat{r})\cos((m+1)\hat{\phi}-\phi_0)-J_{m-1}(\kappa\hat{r})\cos((m-1)\hat{\phi}-\phi_0)] \qquad \hat{r} < a$$

$$= -\frac{iA\lambda_0}{2\zeta_0} Q_m(\nu_0)[K_{m+1}(k_0\lambda_0\hat{r})\cos((m+1)\hat{\phi}-\phi_0)+K_{m-1}(k_0\lambda_0\hat{r})\cos((m-1)\hat{\phi}-\phi_0)] \qquad \hat{r} > a$$

Here

$$Q_m(\nu_0) = J_m(\kappa a)/K_m(k_0\lambda_0 a);$$

$$\zeta_0 = [\mu_0/\varepsilon_0]^{\frac{1}{2}} \quad ; \quad \kappa = k_0(n_1^2-\nu_0^2)^{\frac{1}{2}}$$

and ϕ_0 is an angle defining the orientation of the field maxima. The other mode type is polarized with the electric field in the $\hat{x}$-direction:

$$E_{o\hat{x}} = AJ_m(\kappa\hat{r})\cos(m\hat{\phi}-\phi_1) \qquad \hat{r} < a$$

$$= AQ_m(\nu_0)K_m(k_0\lambda_0 r)\cos(m\hat{\phi}-\phi_1) \qquad \hat{r} > a$$

$$H_{o\hat{z}} = \frac{\nu_0 A}{\zeta_0} J_m(\kappa\hat{r})\cos(m\hat{\phi}-\phi_1) \qquad \hat{r} < a$$

$$= -\frac{\nu_0 A}{\zeta_0} Q_m(\nu_0)K_m(k_0\lambda_0\hat{r}) \cos(m\hat{\phi}-\phi_1) \qquad \hat{r} > a$$

$$E_{o\hat{y}} = \frac{iA\kappa}{2k_0\nu_0} [J_{m+1}(\kappa\hat{r})\cos((m+1)\hat{\phi}-\phi_1)-J_{m-1}(\kappa\hat{r})\cos((m-1)\hat{\phi}-\phi_1)]$$
$$\hat{r} < a$$

$$= \frac{iA\lambda_0}{2\nu_0} Q_m(\nu_0) [K_{m+1}(k_0\lambda_0\hat{r})\cos((m+1)\hat{\phi}-\phi_1)+K_{m-1}(k_0\lambda_0\hat{r})\cos((m-1)\hat{\phi}-\phi_1)]$$
$$\hat{r} > a$$

$$H_{o\hat{y}} = \frac{iA\kappa}{2\omega\mu_0} [J_{m+1}(\kappa\hat{r})\sin((m+1)\hat{\phi}-\phi_1)+J_{m-1}(\kappa\hat{r})\sin((m-1)\hat{\phi}-\phi_1)]$$
$$\hat{r} < a$$

$$= \frac{iA\lambda_0}{\zeta_0} Q_m(\nu_0) [K_{m+1}(k_0\lambda_0\hat{r})\cos((m+1)\hat{\phi}-\phi_1)-K_{m-1}(k_0\lambda_0\hat{r})\cos((m-1)\hat{\phi}-\phi_1)]$$
$$\hat{r} > a$$

where ϕ_1 is another orientation angle. For either mode type the eigenvalue equation can be given in two equivalent forms [29, p. 68]:

$$\kappa J_{m+1}(\kappa a)/J_m(\kappa a) = k_0\lambda_0 K_{m+1}(k_0\lambda_0 a)/K_m(k_0\lambda_0 a)$$

$$\kappa J_{m-1}(\kappa a)/J_m(\kappa a) = -k_0\lambda_0 K_{m-1}(k_0\lambda_0 a)/K_m(k_0\lambda_0 a)$$

and the P-integral (19) is also known [29, p. 70]:

$$P = e_m \frac{A^2}{\zeta_0 \lambda_0^2} \pi a^2 \nu_0 (n_1^2 - n_0^2) \; |J_{m-1}(\kappa a) J_{m+1}(\kappa a)| \qquad \text{(F.1)}$$

where $e_m = 1$ for $m \neq 0$, and $e_0 = 2 \cos^2 \phi_{0,1}$. Note that a lowest order mode of the m=0 class corresponds to the dominant HE_{11} mode. Arbitrarily polarized modes may be constructed by multiplying an $\hat{x}$-mode by $\cos \phi_2$ and adding it to the corresponding $\hat{z}$-mode multiplied by $-\sin \phi_2$ where ϕ_2 is the angle of polarization. Such a mode will have the same P value as the $\hat{x}$ or $\hat{z}$ modes separately, except for the case m=0, where we must take $e_0 = 2[\cos^2\phi_0 \sin^2\phi_2 + \cos^2\phi_1 \cos^2\phi_2]$.

Since m is an integer, it may be readily seen by examining (20) that Δ must vanish by reason of the angular dependences of the various field components. Thus we may immediately go about calculating the attenuation. We shall need the spectrum function for $K_m(k_0\lambda_0\hat{r})\exp(\pm\, im\phi)$ in order to calculate $\tilde{E}$ and $\tilde{H}$. We may start from an integral representation for K_m [30]:

$$K_m(k_0\lambda_0\hat{r}) = \tfrac{1}{2}\int_{-\infty}^{\infty} e^{-k_0\lambda_0\hat{r}\cosh t - mt}\, dt$$

Letting $t = \Psi - i\hat{\phi}$, $\hat{x} = \hat{r}\cos\hat{\phi}$, $\hat{z} = -\hat{r}\sin\hat{\phi}$:

$$K_m(k_0\lambda_0\hat{r}) = \tfrac{1}{2}\int_{-\infty+i\hat{\phi}}^{\infty+i\hat{\phi}} e^{-k_0\,{}_0(\hat{x}\cosh\Psi + i\,\hat{z}\sinh\Psi) - m\Psi + im\hat{\phi}} d\Psi$$

Further changing variables to $n_0 s = \lambda_0 \sinh\Psi$, so that $\lambda_2 = \lambda_0 \cosh\Psi$ and $n_0 ds = \lambda_s\, d\Psi$, gives

$$K_m(k_0\lambda_0\hat{r}) = \tfrac{1}{2}\int_{-\infty e^{-i\hat{\phi}}}^{\infty e^{i\hat{\phi}}} e^{-k_0\lambda_s\hat{x} - iks\hat{z} - m\,\mathrm{arcsinh}(n_0 s/\lambda_0) + im\phi} \left(\frac{n_0 ds}{\lambda_s}\right)$$

We choose the branch cuts in the s-plane from the branch points at $\lambda_s = 0$ to go along the positive and negative imaginary axes as shown in Fig. F.1. The present integration path may be deformed back to the real axis because we are interested only in angles $|\hat{\phi}| < \pi/2$. Thus

$$K_m(k_0\lambda_0\hat{r}) = \int_{-\infty}^{\infty} e^{-k_0\lambda_s\hat{x}-iks\hat{z}+im\hat{\phi}} \left(\frac{n_0}{2\lambda_s}\right)\left(\frac{\lambda_0}{n_0 s+\lambda_s}\right)^m ds \ .$$

Thus the appropriate spectrum function for

$$K_m(k_0\lambda_0\hat{r})e^{\pm im\hat{\phi}}$$

is

$$\frac{n_0}{2\lambda_s}\left(\frac{n_0 s+\lambda_s}{\lambda_0}\right)^{\pm m} e^{-k_0\lambda_s\hat{x}}$$

which allows us to construct $\tilde{E}(s)$ and $\tilde{H}(s)$ for the arbitrarily polarized mode described in the previous paragraph:

$$\tilde{E}(s) = A\sin\phi_2 Q_m(\nu_0)\,\frac{n_0}{4\lambda_s}\left[\left(\frac{n_0 s+\lambda_s}{\lambda_0}\right)^m e^{-i\phi_0} + \left(\frac{\lambda_0}{n_0 s+\lambda_s}\right)^m e^{i\phi_0}\right]$$

$$\tilde{H}(s) = -\frac{\nu_0}{\zeta_0} A\cos\phi_2 Q_m(\nu_0)\,\frac{n_0}{4\lambda_s}\left[\left(\frac{n_0 s+\lambda_s}{\lambda_0}\right)^m e^{-i\phi_1} + \left(\frac{\lambda_0}{n_0 s+\lambda_s}\right)^m e^{i\phi_1}\right]$$

As a result, from (25) and since no correction from (30) need be made,

$$c = \frac{A^2}{\omega\mu_0}\left(\frac{\pi\lambda_0}{k_0 R}\right)^{\frac{1}{2}} Q_m^2(\nu_0)\,\frac{\pi}{2n_0\lambda_0^2}\left[n_0^2\sin^2\phi_2\cos^2\phi_0+\nu_0^2\cos^2\phi_2\cos^2\phi_1\right]\exp(-2\tau_0) \tag{F.2}$$

where as always, $\tau_0 \simeq \frac{1}{3}k_0 R\lambda_0^3/\nu_0^2$.

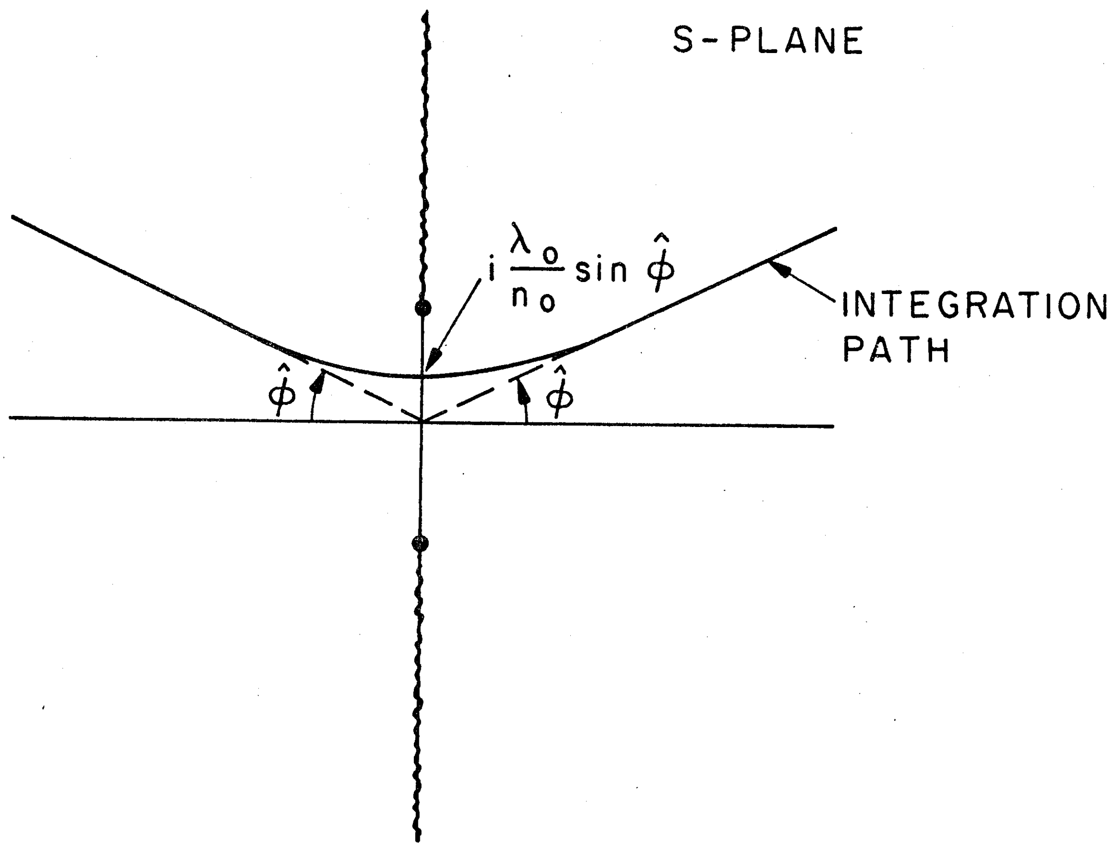

Fig. F.1

Integration path in s-plane for $K_m(k_0\lambda_0\hat{r})$. Branch points occur at $s = \pm i\lambda_0/n_0$.

For the case m = 0, since $n_0 \simeq \nu_0$, we have that the attenuation is essentially independent of polarization and mode orientation, and is given by (making use of the eigenvalue equation)

$$- \frac{ic}{p} = -ik_0 \left(\frac{\pi}{k_0 R}\right) e^{-2\tau_0} \frac{(n_1^2 - \nu_0^2) n_0}{4\nu_0 k_0^2 a^2 (n_1^2 - n_0^2) \lambda_0^{\frac{3}{2}} K_1^2(k_0 \lambda_0 a)} f(\phi_0, \phi_1, \phi_2) \tag{F.3}$$

where

$$f(\phi_0, \phi_1, \phi_2) = 1 + \frac{\lambda_0^2}{n_0^2} \frac{\cos^2\phi_2 \cos^2\phi_1}{\sin^2\phi_2 \cos^2\phi_0 + \cos^2\phi_2 \cos^2\phi_1} \simeq 1$$

since $\lambda_0 / n_0 \ll 1$. This is half of Lewin's result for the HE_{11} mode [10]. For the mode very far from cutoff, $k_0 \lambda_0 a \gg 1$ and $\kappa a \simeq p_{0,1}$, the first zero of J_0. In this case we have

$$- i \frac{c}{\rho} \simeq - \frac{i}{2} k_0 \left(\frac{1}{\pi k_0 R}\right)^{\frac{1}{2}} e^{-2\tau_0 + 2k_0 \lambda_0 a} \frac{\rho_{0,1}^2 n_0}{n_1 k_0^3 a^3 \lambda_0^{\frac{5}{2}}}$$

This result differs from that of Arnaud [13] in two significant ways: (a) the additional $2k_0 \lambda_0 a$ term in the exponent and (b) the presence of R instead of $R + a/\lambda_0^2$ under the square root sign.

From $m \neq 0$, a similar procedure gives

$$- i\frac{c}{p} = - ik_0 \left(\frac{\pi}{k_0 R}\right)^{\frac{1}{2}} e^{-2\tau_0} \frac{(n_1^2 - \nu_0^2) n_0 \; g(\phi_0, \phi_1, \phi_2)}{2\nu_0 k_0^2 a^2 (n_1^2 - n_0^2) \lambda_0^{\frac{3}{2}} |K_{m+1}(k_0 \lambda_0 a) K_{m-1}(k_0 \lambda_0 a)|} \tag{F.4}$$

where

$$g(\phi_0,\phi_1,\phi_2) = \sin^2\phi_2\cos^2\phi_0 + \frac{\nu_0^2}{n_0^2}\cos^2\phi_2\,\cos^2\phi_1 \tag{F.5}$$

Except for the polarization/orientation function g, this result coincides with Lewin's [10] for the HE_{mn} modes. This result seems to indicate that, unlike the $m = 0$ modes, these modes have both polarization and orientation dependent losses. However [31], we may recover the approximate $EH_{m-1,n}$ (respectively $HE_{m+1,n}$) modes by setting $\phi_1 = \phi_0 + \pi/2$ or $\phi_1 = \phi_0 + 3\pi/2$ and $\phi_2 = \pi/4$ or $5\pi/4$ (respectively $\phi_2 = 3\pi/4,\ 7\pi/4$); so that for these cases

$$g(\phi_0,\phi_1,\phi_2) = \frac{1}{2}[1 + \frac{\lambda_0^2}{n_0^2}\sin^2\phi_0] \approx \frac{1}{2}$$

and again this result is half that obtained by Lewin [10]. The TE_n and TM_n modes are included as a special case of the $EH_{m-1,n}$ modes for $m = 1$. Note that all exact modes of the fiber have loss which is essentially independent of the polarization and orientation, but that approximate composite $m \geq 1$ modes can be constructed using (F.5) to minimize the radiation (e.g., $\phi_0 = \phi_1 = \pi/2$).

REFERENCES

[1] F. P. Kapron et al., "Radiation losses in glass optical waveguides," Appl. Phys. Lett. 17, pp. 423-425 (1970).

[2] E. A. J. Marcatili, "Bends in optical dielectric guides," Bell Syst. Tech. J. 48, pp. 2103-2132 (1969).

[3] E. A. J. Marcatili, "Dielectric guide with curved axis and truncated parabolic index," Bell Syst. Tech. J. 49, pp. 1645-1663 (1970).

[4] S. J. Maurer and L. B. Felsen, "Ray methods for trapped and slightly leaky modes in multilayered or multiwave regions," IEEE Trans. MTT 18, pp. 584-595 (1970).

[5] V. V. Shevchenko, "Radiation losses in bent waveguides for surface waves," Izv. VUZ Radiofizika 14, pp. 768-777 (1971) [in Russian; English translation in Radiophysics and Quantum Electronics 14, pp. 607-614 (1971)].

[6] R. H. T. Bates and F. L. Ng, "Contributions to the theory of the azimuthal surface wave," Alta Frequenza 40, pp. 658-666 (1971).

[7] D. Marcuse, "Bending losses of the asymmetric slab waveguide," Bell Syst. Tech. J. 50, pp. 2551-2563 (1971).

[8] D. Gloge, "Bending loss in multimode fibers with graded and ungraded core index," Appl. Opt. 11, pp. 2506-2513 (1972).

[9] D. C. Chang and F. S. Barnes, "Reduction of radiation loss in a curved dielectric slab waveguide," Sci. Rept. No. 2 (AFOSR-72-2417) Dept. of Elec. Eng., Univ. of Colo., Boulder, Colo. (1973).

[10] L. Lewin, "Radiation from curved dielectric slabs and fibers," IEEE Trans. MTT 22, pp. 718-727 (1974).

[11] A. W. Snyder and D. J. Mitchell, "Generalized Fresnel's laws for determining radiation loss from optical waveguides and curved dielectric structures," Optik 40, pp. 438-459 (1974).

[12] D. Marcuse, "Bent optical waveguide with lossy jacket," Bell Syst. Tech. J. 53, pp. 1079-1101 (1974).

[13] J. A. Arnaud, "Transverse coupling in fiber optics--Part III: Bending losses," Bell Syst. Tech. J. 53, pp. 1379-1397 (1974).

[14] M. Heiblum and J. H. Harris, "Analysis of curved optical waveguides by conformal transformation," IEEE J. Quant. Elec. 11, pp. 75-83 (1975).

[15] R. D. Richtmyer, "Dielectric resonators," J. Appl. Phys. 10, pp. 391-398 (1939).

[16] M. A. Miller and V. I. Talanov, "Electromagnetic surface waves guided by small curvature boundaries," Zh. Tekh. Fiz. 26, pp. 2755-2765 (1956) [in Russian; English translation in Sov. Phys. Tech. Phys. 1, pp. 2665-2673 (1956)].

[17] K. G. Budden, Radio Waves in the Ionosphere. Cambridge: University Press, 1961.

[18] F. W. J. Olver, Asymptotics and Special Functions. New York: Academic Press, 1974, chapters 10 and 11.

[19] J. R. Wait, Electromagnetic Waves in Stratified Media. Oxford: Pergamon Press, 1970.

[20] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions. New York: Dover, 1965.

[21] R. F. Harrington, Time-Harmonic Electromagnetic Fields. New York: McGraw-Hill, 1961.

[22] E. F. Kuester and D. C. Chang, "Non-degenerate surface-wave mode coupling of a system of dielectric waveguides," Sci. Rept. No. 9 (AFOSR-72-2417) Dept. of Elec. Eng., Univ. of Colo., Boulder, Colo. (1974).

[23] E. F. Kuester and D. C. Chang, "Propagation, attenuation, and dispersion characteristics of inhomogeneous dielectric slab waveguides," IEEE Trans MTT 23, pp. 98-106 (1975).

[24] L. Lewin, Theory of Waveguides. London: Newnes-Butterworth, 1975, pp. 91-96.

[25] S. W. Maley, "Radiation from a circular bend between two discontinuities in a dielectric slab waveguide," Sci. Rept. No. 10 (AFOSR-72-2417) Dept. of Elec. Eng., Univ. of Colo., Boulder, Colo. (1974).

[26] A. W. Snyder and D. J. Mitchell, "Ray attenuation in lossless dielectric structures," J. Opt. Soc. Am. 64, pp. 956-963 (1974).

[27] E. G. Neumann and H. D. Rudolph, "Radiation from bends in dielectric rod transmission lines," IEEE Trans. MTT 23, pp. 142-149 (1975).

[28] E. Vogls, "Directional couplers with rectangular dielectric waveguides," AEÜ 28, pp. 478-479 (1974).

[29] D. Marcuse, Theory of Dielectric Optical Waveguides. New York: Academic Press, 1974.

[30] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products. New York: Academic Press, 1965, p. 309.

[31] D. Marcuse, Light Transmission Optics. New York: Van Nostrand Reinhold, 1972, pp. 337-339.