Topological Foundations of Tropical Geometry

by

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Topological Foundations of Tropical Geometry

Thesis directed by Professor Jonathan Wise

We construct two subcanonical Grothendieck Topologies on the category of commutative, integral monoids and show that the moduli space of tropical curves is a stack in both topologies. We additionally construct two subcanonical topologies on the category of sharp, saturated, integral, commutative monoids with an eye towards answering outstanding questions of algebraicity of tropical moduli problems.

Dedication

To my parents,

Mitchell and Pamela Willis

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Chapter 1

Introduction

An foundational idea in the study of algebraic geometry is that every commutative ring is a ring of functions for some space; given a commutative ring A the space for which A is a sheaf of functions is the prime ideal spectrum $\mathrm{Spec}(A)$. Every commutative ring has an underlying (additive) abelian group structure, and abelian groups are commutative monoids. Furthermore, any commutative ring has a multiplicative structure which is also a monoid. It turns out that we can often use monoids to encode the relations that define certain types of commutative rings. More concretely, given a monoid P, we can form the monoid algebra $\mathbb{Z}[P]$, which is a commutative ring with generators and relations encoded by P. This defines a functor from the category of monoids to the category of commutative rings. We can then consider the affine monoid scheme $\mathrm{Spec}(\mathbb{Z}[P])$. In the event that the cone associated to P is polyhedral – equivalently, if the monoid is fine, sharp, and saturated – then this monoid scheme will be an affine toric variety. These affine toric varieties are the building blocks of toric varieties, which are the central objects of study in toric geometry. Furthermore, the scheme is naturally equipped with a log structure coming from P, and thus it will be an affine log scheme; these objects are central to the study of log geometry. Thus commutative monoids underlie these two commonly studied areas within algebraic geometry. Another, more recently developed, area where monoids are center stage is tropical geometry. A deep connection between tropical and algebraic geometry can be seen in the Deligne-Mumford compactification of the moduli space of genus g curves. A theorem of F. Kato ([8]) shows that the moduli space of stable log curves of genus g is isomorphic to the Deligne-Mumford compactification of the moduli space of genus g curves. Then, in [1], it is shown that there is a one-to-one, inclusion reversing bijection between the boundary strata of the Deligne Mumford compactification, and the cones that comprise the generalized polyhedral complex structure on the moduli space of genus g tropical curves. This is one illustration of the relationship between the dual cones associated to certain types of monoids (in this case coming from log structures) and the objects of interest coming from tropical geometry.

There have been many recent developments in studying the geometry of the moduli space of tropical curves. In [9], the authors propose a Hodge Bundle on the moduli space of tropical curves. Additionally in [11], the authors construct the tropical Picard group associated to a family of logarithmic curves. These constructions add to the growing list of correspondences between algebraic geometry and tropical geometry. However all of the constructions thus far make use of topologies such as the face topology ([3], Section 2), which are chaotic and consequently not sufficiently robust to generally show that these moduli problems are algebraic. For instance, in showing that the moduli space of tropical curves and the tropical Hodge bundle ([9]) have the structure of a generalized polyhedral complex ([3]), we recognize them as the colimit over a diagram consisting only of face maps – these are also often referred to as "Stacky Fans" (e.g. [4]). The structure of the polyhedral cone complex on the moduli space of curves comes from a natural polyhedral subdvision that decomposes the cone stack (see [3], Section 3); hence the topological aspects of our theory do not really add anything to the description of the objects being parametrized aside from showing that the moduli functor is still a stack in this more refined topology – that the moduli space of curves is algebraic comes from the natural polyhedral cone complex structure, which the authors of [3] use to produce a universal curve over the moduli space. However, there are other situations in studying tropical moduli problems wherein it is of benefit to be "agnostic" about the choice of the polyhedral subdivisions – for instance in the case of the tropical Picard group (see [11]). It is in these situations where the topologies we develop will add something to the story. Our topologies do not require any sort of choice of polyhedral subdivision, which will hopefully prove to be a useful perspective when studying the moduli of these tropical objects. Put another way, finding a cover of a moduli problem by rational polyhedral cones, in such a way that the cones pullback along face maps to cones, is the tropical version of showing that the moduli problem is algebraic – we would like some universal object covering the moduli space that when pulled back along a morphism is still a cover whatever prescribed type, which in this case is a polyhedral cone complex, where the gluing information is via face maps. There are situations, again for example in [11], where the moduli problem is parametrized by a family of algebraic objects, in this case log schemes, and admits a cover by algebraic objects, but the associated tropical moduli problem has not yet been shown to be algebraic – the face maps that are used to define the topology often pullback to subdivided faces and therefore are not comprised of face maps. The issue in these cases is that the topologies that have been used on the category of rational polyhedral cones are too chaotic, and there do not seem to be any "obvious" polyhedral subdivisions as there is in the case of the moduli of curves.

We will be interested in monoids as an avenue to studying tropical geometry. Loosely speaking, a tropical curve is a vertex weighted metric graph, with the metric valued in a commutative monoid. The category of rational polyhedral cones can be equipped with the face topology. However the descent data, in showing that the moduli space of curves is a stack over the category of rational polyhedral cones, end up being trivial as there are no nontrivial covers in the topology. A natural question arises then what other topologies on the opposite category of commutative monoids there are, and whether there exist nontrivial covers as well as questions of descent. We propose two such topologies; much of the intuition for these topologies comes from familiar constructions in algebraic geometry. In particular, we draw from the constructions of the smooth and étale topologies on the category of schemes – once we have a notion of infinitessimal motion, we can inspect morphisms that lift that infinitessimal motion. In section one of chapter 4, we construct two topologies on the category of integral commutative monoids, the exact and étale topologies, both of which are subcanonical.

Theorem 1.0.1. *The exact and étale topologies on the category of commutative integral monoids* *are subcanonical.*

We then go on to show in section 3 of chapter 4 that the moduli space of genus g tropical curves is a stack with respect to both of these topologies.

Theorem 1.0.2. *The moduli space of genus g tropical curves is a stack over the category of commutative, integral monoids with either the smooth or étale topology.*

Some rather special objects that are studied in algebraic geometry – for instance toric varieties – are built from rational polyhedral cones, which in turn can be realized as the dual cones of finitely generated, sharp, saturated monoids. The exact and étale topologies, when restricted to this subcategory of monoids, do not retain the desired descent properties due to issues with torsion in the associated groups. Hence some tweaks are required. In chapter 5 we construct two other topologies, the smooth and étale topologies, on the category of sharp, saturated monoids, with sharp morphisms, and show them to be subcanonical.

Theorem 1.0.3. *The smooth and étale topologies on the category of commutative, integral, sharp, saturated monoids with sharp morphisms, are subcanonical.*

The topologies in this article reflect several of the properties that the ring theoretic topologies by the same name posess. For example, if a morphism of rings $A \to B$ is smooth then the associated morphism on Kahler differentials $\Omega_A \otimes_A B \to \Omega_B$ is a split injection. We will see in Chapter 5 that an analogous result holds in the case of smooth morphisms of monoids. Additionally, smooth morphisms of rings étale locally have sections; we will show, also in Chapter (5), that if a morphism of monoids is formally smooth at a point, then there exists an étale neighborhood of that point such that the morphism admits a section. Finally valuative monoids will provide for us a notion of point in the cone associated to a monoid, along with their infinitessimal extensions, much the same way fields and nilpotent extensions provide analogous notions for the associated scheme to a ring – our topologies are defined by lifting critera that mimic smooth and étale morphisms of rings.

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Chapter 2

Background on Monoids

2.1 Properties of Monoids

In this article, by a monoid P we will mean a set with an associative, commutative binary operation $+ : P \times P \to P$, which we will call addition, containing an identity element, denoted by 0. A morphism between monoids $P \to Q$ is a map of sets that respects the addition operation – i.e. it is a linear map. Let $\mathfrak{Mon}$ denote the category of monoids. Let $\mathfrak{Ab}$ denote the category of abelian groups. There is a functor $(-)^{gp} : \mathfrak{Mon} \to \mathfrak{Ab}$ taking a monoid to its associated group, which we refer to as the groupification of the monoid. The groupification of a monoid is formed by adjoing an inverse for each element of the monoid. We will say that a monoid P is integral provided the morphism $P \to P^{gp}$ is injective. In the event that P is finitely generated and integral, we will say that P is a fine monoid. As we will only be working with commutative integral monoids, we will again use $\mathfrak{Mon}$ to denote the category of commutative integral monoids, henceforth referred to as simply "monoids". Given morphisms $M \to N$ and $M \to P$ in $\mathfrak{Mon}$, the colimit over the diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d] & N \\ P & \end{tikzcd}$$

exists in $\mathfrak{Mon}$. We denote this object by $N \oplus_M P$, the pushout of N with P over M.

We will say that a monoid M is **sharp** if the only invertible element of M is the identity. Denote by M^* the subgroup of units in M. We may form a quotient $M^\sharp = M/M^*$; this is a sharp monoid, which we refer to as the sharpening of M. Sharp monoids form a full subcategory $\mathfrak{Mon}^\sharp$

of $\mathfrak{Mon}$ and the sharpening functor $(-)^{\sharp} : \mathfrak{Mon} \to \mathfrak{Mon}^{\sharp}$ is left adjoint to the inclusion of sharp monoids (see [6], Section 1.9 for details). The pushout of sharp monoids may not be sharp, as the following example illustrates.

Example 2.1.1. Let $e_1, e_2 \in \mathbb{Z}^2$ be defined by $e_1 = (1,0)$ and $e_2 = (0,1)$. Then let $Q = \mathbb{N}e_1 + \mathbb{N}(e_2 - 2e_1)$, and $Q' = \mathbb{N}e_2 + \mathbb{N}(e_1 - 2e_2)$. Let $P = \mathbb{N}e_1 + \mathbb{N}e_2$. Then we may observe that $P^{gp} \cong Q^{gp} \cong Q'^{gp} \cong \mathbb{Z}^2$. Therefore the pushout $Q \oplus_P Q'$ is isomorphic to the sum $Q + Q'$ taken in $\mathbb{Z}^2$. But this is not a sharp monoid since the element $e_1 - e_2$ is invertible.

Therefore to define a pushout in the sharp subcategory $\mathfrak{Mon}^{\sharp}$ of $\mathfrak{Mon}$, we take the pushout in monoids and apply the sharpening functor.

A monoid P is said to be **saturated** if, for any $x \in P^{gp}$ and $n \in \mathbb{N}$, $nx \in P$ implies that $x \in P$ – this is akin to saying that P has all n^{th} roots contained in P^{gp}. Saturated monoids form a full subcategory of $\mathfrak{Mon}$, which we denote by $\mathfrak{Mon}^{sat}$. There is a functor $(-)^{sat} : \mathfrak{Mon} \to \mathfrak{Mon}^{sat}$ taking a monoid to its saturation – given a monoid P, P^{sat} is formed by adjoining all n^{th} roots in P^{gp} to P. This functor is left adjoint to the inclusion of $\mathfrak{Mon}^{sat}$ into $\mathfrak{Mon}$. The pushout of saturated monoids will not necessarily be saturated, and hence we define the pushout in the category of saturated monoids by taking the pushout in $\mathfrak{Mon}$ and applying the saturation functor.

The units inside of the saturation of sharp monoids will play an important role in this article, so we provide the following lemma.

Lemma 2.1.2. *Let P be a sharp monoid. Then $(P^{sat})^*$ is a torsion group.*

Proof. We form P^{sat} by adding all of those elements $x \in P^{gp}$ for which there exists $n \in \mathbb{N}$ such that $nx \in P$. Let $x \in (P^{sat})^*$. Then nx and $-nx$ are elements of P, which implies that $nx \in P^* = \{0\}$ since P is sharp by assumption. □

We can apply the saturation functor followed by the sharpening functor to all monoids in $\mathfrak{Mon}$ to obtain a full subcategory $(\mathfrak{Mon}^{sat})^{\sharp}$ of sharp and saturated monoids. This category has a pushout defined as follows. Given $P \to Q$ and $P \to Q'$ in $(\mathfrak{Mon}^{sat})^{\sharp}$, the pushout of Q and Q' over P is defined by $((Q \oplus_P Q')^{sat})^{\sharp}$ where $Q \oplus_P Q'$ is taken in $\mathfrak{Mon}$.

To any monoid P, there is an associated functor

$$\mathbf{Cone}(P) : \mathfrak{Mon}^{op} \to \mathbf{Sets}$$

defined by $\mathbf{Cone}(P)(Q) = \mathrm{Hom}(P, Q)$ for any $Q \in \mathfrak{Mon}$. Indeed, this is a contravariant functor since given any morphism $P \to P'$, we get a morphism $\mathbf{Cone}(P')(Q) \to \mathbf{Cone}(P)(Q)$, for any Q, by precomposition. We will refer to this functor as the dual cone of P. We may then observe that $\mathrm{Hom}(P, Q) \cong \mathrm{Hom}(\mathbf{Cone}(Q), \mathbf{Cone}(P))$. This is very much in analogous to the case with commutative rings, wherein we consider the functors $\mathrm{Hom}(A, -) = \mathrm{Spec}(A)$ for a commutative ring A. We then have that $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(\mathrm{Spec}\, B, \mathrm{Spec}\, A)$. In the Zariski topology on the category of commutative rings, these representable presheaves turn out to be sheaves. The pushout of two commutative rings A and B is the tensor product $A \otimes_{\mathbb{Z}} B$, and $\mathrm{Spec}(A \otimes_{\mathbb{Z}} B) = \mathrm{Spec}\, A \times_{\mathrm{Spec}\, \mathbb{Z}} \mathrm{Spec}\, A$. Likewise in the case of cones we have that $\mathbf{Cone}(Q \oplus_P Q') = \mathbf{Cone}(Q) \times_{\mathbf{Cone}(P)} \mathbf{Cone}(Q')$. These statements are both proved in the exact same why, by leveraging the convtravariance of the functors in addition to the universal properties of the pushout and the fibered product.

The monoids we work with are the targets of the metrics that show up in the definition of a tropical curve as a vertex weighted metric graph (see [3] for more details]). A certain type of monoid, a **valuative** monoid, plays a very similar role in the study of monoids and cones as that of a local ring in the study of commutative rings and schemes.

Definition 2.1.3. A sharp monoid M is said to be valuative if for any $x \in M^{gp}$, either x or $-x$ is an element of M.

Sharp morphisms from monoids to valuative monoids are of particular interest; sharp morphisms to valuative monoids tell us the points of the associated cones. Given a morphism of monoids $f : P \to Q$, we get an associated morphism of cones $f^* : \mathbf{Cone}(P) \to \mathbf{Cone}(Q)$. Suppose that Q is valuative, say $Q = \mathbb{N}$, then the morphism on cones gives a ray $\mathbf{Cone}(\mathbb{N}) \to \mathbf{Cone}(P)$. Therefore the collection of all morphisms $\mathbf{Cone}(M) \to \mathbf{Cone}(P)$ over all sharp valuative monoids will tell us all of the points in the dual cone of P. To say that f is sharp is to say that this ray does not land in any face of the cone $\mathbf{Cone}(P)$, so sharp morphisms to valuative monoids will recover all points in the relative interior of the cone.

Example 2.1.4. In the event that P is fine, sharp, and saturated monoid then it suffices, in particular because P is fine, to consider $\mathrm{Hom}(P, \mathbb{N})$ to recover the points of the cone – indeed, $\mathbb{N}$ is the only non-trivial finitely generated valuative monoid. This should be reminiscent of considering all morphisms of a ring to a field in order to recover the points of the scheme. But then $\mathbf{Cone}(P)(\mathbb{N})$ consists of all non-negative linear functions on P. Since the monoid is also saturated and sharp, this cone will be polyhedral, and hence is the dual cone of a toric monoid – the cone and the monoid live in the same ambient vector space. Let $x_1, \ldots, x_n$ be generators for P – inside of an n-dimensional $\mathbb{Q}$ vector space. Then there is a pairing that defines the relations that cuts out $\mathbf{Cone}(P)$ inside of $\mathbb{Q}^n$. For each generator x_i we get a face of $\mathbf{Cone}(P)$ defined by the orthogonal subspace $x_i^{\perp}$. Said differently, we can think of x_i as a function and the inequalty $x_i \geq 0$ defines a half space. We intersect all of the half spaces to obtain $\mathbf{Cone}(P)$. If $P = \mathbb{N}^2$, say with generators $x_1 = (1, 0)$ and $x_2 = (-1, 1)$. Then the pairing takes x_1 to the ray $(0, 1)$ and x_2 to the ray $(1, 1)$. (see for example [5] or [12]).

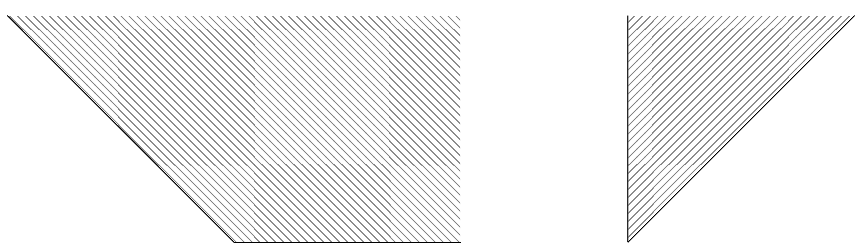


Figure 2.1: The monoid (left) and its dual cone (right)

There is an action of P on P^{gp} by $p \cdot x = p + x$ for any $p \in P$ and $x \in P^{gp}$.

Definition 2.1.5. We will say that a partial order $\leq$ on a set S is weak if when $x \leq y$ and $y \leq x$, it it not necessarily the case that $x = y$.

This action of P on P^{gp} induces a weak partial order on P^{gp}. If P is sharp, this weak partial order is actually partial order on P^{gp} defined as follows. For any $x, y \in P^{gp}$ we say $x \leq y$ provided there exists some $p \in P$ such that $x + p = y$ (see [12], (1.1.7)).

Example 2.1.6. Suppose $P^* \neq \{0\}$ and let $x \in P^*$. Then $x+(-x) = 0$ so that $x \leq 0$ and $-x+x = 0$ so that $-x \leq 0$, which in turn implies that $x \geq 0$. Therefore $x \leq 0$ and $0 \leq x$ but $x \neq 0$. That is, non-trivial units are cmparable to zero but are not zero. Hence the partial order in this case is weak.

We may equivalently define a sharp monoid to be valuative provided the partial order induced on P^{gp} by P is actually a total order.

Lemma 2.1.7. *Let M be a sharp monoid. The following are equivalent:*

(i) M is valuative;

(ii) The partial order induced on M^{gp}, by M, is a total order.

Proof. First assume that M is valuative, and let $x, y \in M^{gp}$. Then $x-y \in M^{gp}$ and hence $x-y \in M$ or $-(x-y) \in M$ by definition of a valuative monoid. Since, under this partial ordering, M is the monoid of elements that are ≥ 0 in M^{gp}, it is either the case that $x - y \geq 0$ or $y - x \geq 0$, that is either $x \geq y$ or $y \geq x$ and hence M^{gp} is totally ordered.

Conversely assume that the partial order induced on M^{gp} by M is a total order and let $x \in M^{gp}$. Then in particular either $x \geq 0$ or $x \leq 0$ which is equivalent to saying that $x \in M$ or $-x \in M$. □

Notice that we did not include the adjective saturated in the definition of a valuative monoid. This is because a valuative monoid is always saturated.

Lemma 2.1.8. *Valuative monoids are saturated.*

Proof. Let M be a valuative monoid. Let $x \in M^{gp}$ such that $nx \in M$ for some $n \in \mathbb{N}$. Then, since we can view M as the monoid of non-negative elements under the total order induced from the action of M on M^{gp}, we have that $nx \geq 0$. It follows that $x \geq 0$ and hence $x \in M$, from which it follows that M is saturated. □

In algebraic geometry, we arrive at the definition of the smooth topology by essentially abstracting the notion of a submersion from differential geometry, at least in the sense that we reclver the fact that there is a surjection of tangent spaces. In order to do so, we study all points of schemes and their infinitessimal extensions. This essentially boils down to studying lifting diagrams associated to nilpotent extensions. In tropical geometry, while the valuative monoids play a similar role to local rings, we have a notion that is akin to an infinitessimal extension: relatively valuative morphisms.

Definition 2.1.9. Let $f : P \to Q$ be a morphism of monoids. We say that f is relatively valuative provided for any $x \in P^{gp}$ we have that $f(x) \in Q$ if and only if x or $-x$ is in P.

The topologies that we construct are thus based on lifting diagrams associated to these relatively valuative morphisms. We can see rather easily from the definitions of realtively valuative morphisms and valuative monoids that any morphism out of a valuative monoid will necessarily be relativey valuative.

Lemma 2.1.10. *Let M be a valuative monoid, P a monoid, and $f : M \to P$ a morphism. Then f is relatively valuative.*

Proof. If $f(x) \in P$ then $x \in M^{gp}$ and hence either $x \in M$ or $-x \in M$ since M is valuative. □

Following our intuition for lifting of infinitessimal motion in the dual cones, we will restrict our attention to the case where f is surjective and P is also valuative – we will think of the target of f as a point of the cone while the source is an infinitessimal extension. We can see this come out of the case of a surjective morphism from an infinitely generated valuative monoid to a finitely generated one. It turns out that there is only one isomorphism class of finitely generated valuative monoids.

Proposition 2.1.11. *Any sharp valuative monoid generated by finitely many irreducible elements is isomorphic to* $\mathbb{N}$.

Proof. Let P be a sharp valuative monoid that is finitely generated, say by $x_1, \ldots, x_n$. Then since P^{gp} is totally ordered, we have that $x_i \leq x_j$ or $x_j \leq x_i$ for each i, j. The x_i are each irreducible elements of P, and hence $x_i \leq x_j$ if and only if $x_j \leq x_i$. Therefore, since P is totally ordered we in fact have that $x_i = x_j$ for all i, j. It follows that $P \cong \mathbb{N}$. $\square$

Let $\mathbb{N}[\epsilon]$ be generated by 1 and $1 - n\epsilon$ for all $n \in \mathbb{N}$. There is a surjective morphism $\mathbb{N}[\epsilon] \to \mathbb{N}$ by $\epsilon \mapsto 0$. Given a commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & \mathbb{N}[\epsilon] \arrow[d] \\ & \mathbb{N} \end{tikzcd}$$

we have the dual diagram on the level of cones

$$\begin{tikzcd} \mathbf{Cone}(P) & \mathbf{Cone}(\mathbb{N}[\epsilon]) \arrow[l] \\ & \mathbf{Cone}(\mathbb{N}) \arrow[u] \arrow[ul] \end{tikzcd}$$

where the morphism $\mathbf{Cone}(\mathbb{N}) \to \mathbf{Cone}(P)$ describes a ray inside of $\mathbf{Cone}(P)$ and the morphism $\mathbf{Cone}(\mathbb{N}[\epsilon])$ describes infinitessimal motion inside of the cone of P anchored at the ray defined by the image of $\mathbf{Cone}(\mathbb{N})$. See (5.1.9) for an example and picture of this exact situation. This is the motivation for the topologies that we develop. We first develop a notion of infinitessimal, and then ask for covering families to lift this infinitessimal motion.

Chapter 3

Background on Grothendieck Topologies

3.1 Grothendieck Topologies

It is rather unclear what exactly it would mean for a category to actually have a topology – classical topology is steeped in the theory of sets. However, following Alexander Grothendieck, we can abstract the properties that we wish a covering family from an topology to have. This is the idea behind Grothendieck topologies. We only need to understand what it means for some family of morphisms in a category to be covering, along with some extra compatibility conditions to ensure that the covers behave in the way that we expect using our intuition from topology – covers should be stable under base change for example. Consider the example of schemes; the Zariski topology is far too coarse to allow for things we wish to have from topologies with a good notion of local neighborhoods, such as homotopy theory for instance. We would like a refinement of this very coarse topology, and we would like for it to have a good notion of local to global. We should be able to get after such a thing by finding a way to construct a good notion of a neighborhood of a point. We can realize the Zariski topology as a Grothendieck topology, at which point it is sensible to ask for a refinement of the notion of covering coming from that topology. One approach is to define the étale topology, where a cover in this topology is a family of jointly surjective étale morphisms. This gives us a much more well behaved notion of a local neighborhood – the étale neighborhoods are much smaller than Zariski open sets. The étale topology is built from morphisms that uniquely lift infinitessimal motion, formalized algebraically by square zero extensions of rings.

We will work only with the definition of a pretopology. This will suffice for our purposes of studying sheaves, and furthermore we will not have to venture off into a discussion covering sieves.

Definition 3.1.1. Let $\mathcal{C}$ be a category that has all fibered products. For any object X of $\mathcal{C}$, we may define the notion of a covering family $\{Y_i \to X\}_{i \in I}$. These covering families form a Grothendieck pretopology precisely when the following conditions hold.

- For any object X of $\mathcal{C}$ and a morphism $Y \to X$ coming from a covering family of X and any other morphism $Z \to X$ for some object Z of $\mathcal{C}$, the fibered product $Z \times_X Y$ exists.
- Given a covering family $\{Y_i \to X\}_{i \in I}$ of some object X of $\mathcal{C}$, and a covering family family $\{Z_{i_j} \to Y_i\}_{j \in J}$ for each i, the composition $\{Z_{i_j} \to X\}_{j \in J}$ forms a covering family of X.
- For any covering family $\{Y_i \to X\}_{i \in I}$ of an object X in $\mathcal{C}$, and any morphism $Z \to X$, the family of morphisms $\{Z \times_X Y_i \to Z\}_{i \in I}$ is covering.
- Any isomorphism $X \xrightarrow{\sim} Y$ is a covering family.

These four properties are enough to ensure that the covering families will generate a Grothendieck topology – this is very much like defining a basis of open sets in classical topology, which then generate a topology when we complete under arbitrary unions, finite intersections, and complements. To see why we are justified in calling this a topology, we refer to a familiar example.

Example 3.1.2. Let X be a topological space and let $\mathbf{Open}(X)$ be the category consisting of objects as open subsets of X with morphisms given by inclusion, that is $U \to V$ is a morphism in $\mathbf{Open}(X)$ provided $U \subseteq V$. Define a family of morphisms $\{U_i \to V\}$ to be covering provided $V = \bigcup_i U_i$. This defines a Grothendieck pretopology on $\mathbf{Open}(X)$ that generates the topology on X.

This example in part might justify why we should call these topologies. However, in this example we already have a sort of notion of "neighborhood" coming from the topology that already exists on the space X. We might wish to pass to different topologies in the event that this notion of local neighborhood is too coarse. Consider again the example of schemes; the obvious topology is the Zariski topology. Every affine scheme, and by extension every scheme, comes equipped naturally with this topology. The Zariski topology can also be realized as a Grothendieck topology on the category of schemes, and once we have passed to this abstraction, we may start inspecting what other types of covering families define topologies and furthermore how the various notions of covering relate to one another – e.g. the fpqc, fppf, étale, flat, and smooth topologies. The Zariski topology has a notion of open neighborhood, since there is a notion of open set and a notion of a point, but the open sets are simply too large to behave as we would expect using intuition from complex geometry, wherein we have an algebraic and analytic structure. One thing, for instance, we might hope for is a notion of covering family where the elements in the cover can be described algebraically, without any reference to analysis, and which give a good enough notion of neighborhood that we can effetictively pass between the analytic and the algebraic topology. It turns out that there are approximation theorems for the étale toology regarding tihs very thing, but there is no hope that the Zariski topology could ever be fine enough to perform such approximations.

Grothendieck topologies give us the necessary tools to talk about descent, which allows us to talk about sheaves on categories and their cohomology. Sheaves, and their cohomology, are ubiquitous in algebraic geometry. A presheaf is a contravariant functor from a category to the category **Sets**. Let $\mathcal{C}$ be a category and let $\mathcal{F} : \mathcal{C}^{op} \to \mathbf{Sets}$ be a presheaf on $\mathcal{C}$. In the event that $\mathcal{C} = \mathbf{Open}(X)$ for some topological space X, we can phrase the sheaf condition as follows. For any object U in $\mathbf{Open}(X)$, and any open cover $\{V_i \to U\}_{i \in I}$ of U the sequence

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(V_i) \to \prod_{i,j} \mathcal{F}(V_i \cap V_j)$$

is exact. Now the intersections $V_i \cap V_j$ can be rephrased as the fibered products $V_i \times_U V_j$, and hence we can rewrite this sequence as

$$0 \to \mathcal{F}(U) \to \prod_i \mathcal{F}(V_i) \to \prod_{i,j} \mathcal{F}(V_i \times_U V_j).$$

This small observation allows us to abstract immediately to the notion of a sheaf on a category with a Grothendieck topology. Let $\mathcal{C}$ be a category with fibered products, and suppose we have a Grothendieck pretopology defined on $\mathcal{C}$. Then a presheaf $\mathcal{F} : \mathcal{C}^{op} \to \mathbf{Sets}$ is a sheaf provided for any object X of $\mathcal{C}$ and any covering family $\{Y_i \to X\}_{i \in I}$ the sequence

$$0 \to \mathcal{F}(X) \to \prod_i \mathcal{F}(Y_i) \to \prod_{i,j} \mathcal{F}(Y_i \times_X Y_j)$$

is exact. A particular example of a presheaf that is often of interest is the contravariant functor represented by an object in $\mathcal{C}$. That is, every functor of the form $\mathrm{Hom}(X, -)$ is a presheaf on $\mathcal{C}$. A property that we often wish a Grothendieck topology to have is that each of these representable presheaves is already a sheaf. This is something that is common, and desirable, enough that we give it a name.

Definition 3.1.3. A Grothendieck topology on a category $\mathcal{C}$ is said to be **subcanonical** provided every representable presheaf is a sheaf.

In the event that we have a subcanonical topology on a category, we can thus identify the category as the category of sheaves with respect to some topology by the Yoneda embedding, which takes an object to the functor that it represents $X \mapsto \mathrm{Hom}(X, -)$. That is, if $\mathcal{C}$ is a category with a Grothendieck topology that is subcanonical, then we get a functor $\mathcal{C} \to \mathbf{Sh}(\mathcal{C})$, where $\mathbf{Sh}(\mathcal{C})$ is the category of sheaves on $\mathcal{C}$. If this functor is an isomorphism then we say that $\mathcal{C}$ is a topos.

Chapter 4

Exact and Étale Topologies for Integral Monoids

The exact topology is generated by covering families $\{M \to N_i\}_i$ where each $M \to N_i$ is injective and the morphism $M \to \prod_i N_i$ is exact. The reader might like to think of this as a tropical analogue to the flat, or smooth, topology on the category of schemes. It is known, by unpublished work of T. Tsuji (see [14], Section 3), that exact morphisms are stable under pushout and composition, and hence it seems probable that these families generate a topology, which is precisely the content of the first main proposition proved in Section 4.1.

Theorem 4.0.1. *Exact and Étale covering families each generate a Grothendieck topology on the category of monoids.*

The étale topology is suboordinate to the exact topology. The families that generate the Grothendieck topology are exact families with the added condition that the associated extensions of groups are Kummer. A Kummer extension of monoids is characterized by adjoining n^{th} roots, which is effectively asking for torsion quotients. This is a kind of analogue for the étale topology on the category of schemes. However, there is a subtle difference in the notion of an étale covering family of monoids from that of schemes. For schemes, one may first define the notion of an étale morphism between two schemes, and then ask for a family of étale morphisms to be covering if that family is jointly surjective; for monoids we define only the notion that the "étale part" of the family is covering. We do not require the notion of an individual étale morphism between monoids in order to construct an étale covering family.

As is the case with schemes, the étale covering families are characterized by a formal lifting criterion. A valuative monoid is a monoid theoretic analogue to a local ring, and is analogously used to produce an infinitessimal lifting criterion for being an étale covering family. Studying all maps of a monoid P into all valuative monoids is effectively studying all possible ways of ordering P. We will propose a lifting criterion in Section 4.2, where we formalize the following proposition.

Proposition 4.0.2. *A family $\{P \to Q_i\}_i$ is étale covering if and only if it has the lifting criterion at M for all valuative monoids M.*

Given a monoid M, we may consider its associated contravariant functor $\mathrm{Hom}(M, -)$ on the opposite category of monoids; these are the representable presheaves. In section 4.1 we prove that every representable presheaf is a sheaf.

Theorem 4.0.3. *The exact and étale topologies on the category of monoids are subcanonical.*

Hence we get an embedding of the opposite category of monoids into the category of sheaves on the exact site by sending a monoid to the functor it represents. This is reminiscent of the case with schemes wherein we can construct affine schemes as the representable sheaves on the Zariski site, and then use descent properties for the topology to glue the representables together into schemes.

In the article [3], the authors introduce a moduli stack of tropical curves by studying the functor

$$\mathcal{M}_{g,n}^{trop} : \mathfrak{Mon} \to \mathbf{Groupoids},$$

which takes a monoid P to the collection of pairs (P, Γ) where Γ is an n-marked, genus g, tropical curve over P. In that article, the face topology is utilized on the category of rational polyhedral cones – so the monoids are all fine, sharp and saturated. We will study the same functor, but with respect to the exact and étale topologies on the category of monoids. Additionally, a main difference in this chapter is that we do not require the monoids to be saturated. In section 4.3, we prove that the moduli space of tropical curves is a stack in both of the topologies.

Theorem 4.0.4. *The functor $\mathcal{M}_{g,n}^{trop}$ is a stack in the exact and étale topologies on $\mathfrak{Mon}$.*

A significant difference between this stack and the one appearing in [3] is that the descent data is nontrivial, and we can see in the proof how the higher descent data is needed in order to descend the underlying graph of a tropical curve.

4.1 Exact and Étale Families of Monoids

In this section we will describe exact and étale families of monoids, which we will later show to be the covering families that generate a Grothendieck topology. We will first define an exact family of monoids, upon which the definition of an étale family will depend.

A morphism of monoids $f : M \to N$ is exact provided $(f^{gp})^{-1}(N) = M$. Diagramatically, this is the same as saying the following diagram is cartesian.

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow & & \downarrow \\ M^{gp} & \xrightarrow{f^{gp}} & N^{gp} \end{array}$$

It is always the case that $(f^{gp})^{-1}(N) \supset M$. In the event that M and N are sharp, exactness is the statement that $x \leq y$ if and only if $f(x) \leq f(y)$.

Definition 4.1.1. A morphism $f : P \to Q$ is universally exact if for any $P \to N$, the morphism $N \to N \oplus_P Q$ is exact.

The category $\mathfrak{Mon}$ has products, and thus given a family of morphisms $\{M \xrightarrow{f_i} N_i\}_i$, there is a unique morphism to the product $M \xrightarrow{(f_i)_i} \prod_i N_i$.

Definition 4.1.2. A collection of monoid morphisms $\{M \to N_i\}_i$ is said to be an exact covering family if there exists a subfamily $\{M \to N_{i_j}\}_j$ such that each $M \to N_{i_j}$ is injective and $(f_{i_j})_j : P \to \prod_j Q_{i_j}$ is universally exact.

An exact family of monoids should be thought of as analogous to a flat or smooth family of commutative rings or algebras. A Kummer extension of rings is one obtained by adjoining n^{th}

roots. For monoids we can make a similar definition, which will be required in order to formulate the notion of an étale covering family.

Definition 4.1.3. A morphism of monoids $M \to N$ is called Kummer provided it is injective and N^{gp}/M^{gp} is torsion.

We now are ready to define an étale family of monoids; this should be thought of as something like an étale family of rings or algebras.

Definition 4.1.4. A family $\{M \to N_i\}_i$ is said to be an étale cover provided there is a subfamily $\{M \to N_{i_j}\}_j$ such that each $M \to N_{i_j}$ is injective, the induced morphism on groups $M^{gp} \to N_{i_j}^{gp}$ is Kummer, and $(f_{i_j})_j : P \to \prod_j Q_{i_j}$ is exact.

Remark 4.1.5. We have only defined the notion of being étale for a family of morphisms. An étale covering of $\mathbf{Cone}(\mathbb{N}^2)$ by two cones, say $\mathbf{Cone}(Q_1)$ and $\mathbf{Cone}(Q_2)$, is shown in Figure 4.1 – the morphism $\mathbf{Cone}(Q_1) \to \mathbf{Cone}(\mathbb{N}^2)$ is not étale at the face of $\mathbf{Cone}(Q_1)$ that lies in the relative interior of $\mathbf{Cone}(Q_2)$, and conversely. The "étale part" of the collection is covering, as opposed to each individual subcone being étale.

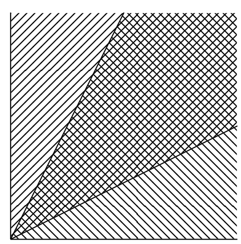


Figure 4.1: An étale covering of $\mathbf{Cone}(\mathbb{N}^2)$.

We will show that there exists a Grothendieck pretopology wherein a family of maps covering precisely when it is an exact family. First, we will need some lemmas.

Lemma 4.1.6. *Exact morphisms are stable under composition.*

Proof. The proof is ommitted in [14, (3.2) (1)], so we present one here. Let $f : M \to N$ and $g : N \to P$ be exact morphisms. Then

$$((g \circ f)^{gp})^{-1}(P) = (g^{gp} \circ f^{gp})^{-1}(P) = (f^{gp})^{-1}(g^{gp})^{-1}(P) = (f^{gp})^{-1}(N) = M,$$

from which it follows that $g \circ f$ is exact. □

Lemma 4.1.7. *Étale covering families are stable under pushout.*

Proof. Let $\{M \to N_i\}_i$ be an étale family and $M \to P$ any morphism of monoids. We have the following diagram

$$\begin{tikzcd} P^{gp} \arrow[r] & (P \oplus_M N_i)^{gp} \arrow[r] & N_i^{gp}/M^{gp} \\ M^{gp} \arrow[u] \arrow[r] & N_i^{gp} \arrow[u] \arrow[r] & N_i^{gp}/M^{gp}, \arrow[u, equal] \end{tikzcd}$$

and hence $P \to P \oplus_M N_i$ is Kummer for each i, since N_i^{gp}/M^{gp} is torsion by assumption. Furthermore exact families are stable under pushout by definition, completing the proof. □

We are now ready to show that both types of families generate a Grothendieck topology on $\mathfrak{Mon}$. We will then show that each is subcanonical.

Proposition 4.1.8. *On the category $\mathfrak{Mon}$ of integral commutative monoids, taking exact covering families to be covers defines a Grothendieck pretopology.*

Proof. Let $\{M \to N_i\}_i$ be an exact family, and let $M \to P$ in $\mathfrak{Mon}$ be any morphism. By replacing the family with the subcollection having the exactness property, we may assume that $M \to \prod_i N_i$ is injective and exact. We need to verify the following conditions.

(I) Each of the pushouts $N_i \oplus_M P$ exist in $\mathfrak{Mon}$.

(II) Exact covers are stable under pushout.

(III) If $\{N_i \to Q_{ij}\}_j$ is a cover for each i, then the family $\{M \to Q_{ij}\}_{i,j}$ induced by composition is a cover.

(IV) Any isomorphism $M \xrightarrow{\sim} M'$ is a covering family.

Let $N = \prod_i N_i$, $Q_i = \prod_j P_{ij}$, and $Q = \prod_{i,j} P_{ij}$. The pushouts $N_i \oplus_M P$ exist for all i as $\mathfrak{Mon}$ has pushouts, proving (I). For (II), that $\{P \to \prod_i N_i \oplus_M P\}_i$ is an exact family follows from the definition of an exact covering family.

To prove (III), the morphisms $M \to Q_{ij}$ are injective for all i, j because the composition of injective morphisms is an injection. Moreover, since pullbacks and products are both limits, they commute, and hence $N \to Q$ is exact. Moreover, $N \to Q$ is an exact morphism by hypothesis. Thus by Lemma 4.1.6, the morphism obtained from the composition $M \to N \to Q$ is exact showing that $\{M \to Q_{ij}\}_{i,j}$ is an exact family.

Finally, to prove (IV), one simply observes that an isomorphism of monoids $\{M \xrightarrow{\sim} M'\}$ is exact. It follows that exact families define a Grothendieck pretopology. □

It the follows immediately that taking étale families to be covers defines a Grothendieck pretopology.

Corollary 4.1.9. *On the category $\mathfrak{Mon}$ of integral commutative monoids, taking covers to be étale covering families defines a Grothendieck pretopology.*

Proof. Lemma 4.1.7 shows that étale covering families are stable under pushout. Given étale covering families $\{M \to N_i\}_i$ and $\{N_i \to P_{ij}\}_j$ for each i, we have the exact sequence of abelian groups:

$$0 \to N_i^{gp}/M^{gp} \to P_{ij}^{gp}/M^{gp} \to P_{ij}^{gp}/N_i^{gp} \to 0.$$

As N_i^{gp}/M^{gp} and P_{ij}^{gp}/N_i^{gp} are both torsion by assumption, it follows that P_{ij}^{gp}/M^{gp} is torsion, and hence that each $M \to P_{ij}$ is Kummer, showing that the family $\{M \to P_{ij}\}_{i,j}$ is étale. Finally an isomorphism of monoids is étale, and thus étale covering families define a Grothendieck pretopology on the category of commutative monoids. □

We will refer to the topology **ex** generated by the exact pretopology as the **exact topology**, and we refer to the site $\mathfrak{Mon}_{\mathbf{ex}}^{op}$ as the **exact site**. Similarly, the topology **ét** generated by the étale pretopology will be referred to as the **étale topology**, and the corresponding site $\mathfrak{Mon}_{\textbf{ét}}^{op}$ the étale site. We now show that each of these topologies is subcanonical.

Theorem 4.1.10. *The exact topology on $\mathfrak{Mon}$ is subcanonical.*

Proof. Let $\{M \to N_i\}_i$ be an exact family. Replace the covering family with the subfamily having the exactness property. Consider the diagram

$$M \to \prod_i N_i \rightrightarrows \prod_{i,j} N_i \oplus_M N_j, \tag{4.1.11}$$

where the morphism $M \to \prod_i N_i$ is the unique morphism coming from the universal property of the product, and the morphisms $\prod_i N_i \to \prod_{i,j} N_i \oplus_M N_j$ from the universal property of the pushout. For any monoid P, there is the associated sequence

$$\mathrm{Hom}(P, M) \to \prod_i \mathrm{Hom}(P, N_i) \rightrightarrows \prod_{i,j} \mathrm{Hom}(P, N_i \oplus_M N_j),$$

the exactness of which will follow from showing that M is the equalizer in (4.1.11).

There is a diagram in the category of abelian groups associated to (4.1.11) obtained by taking the groupification of each entry in the sequence. This induces the following commutative diagram.

$$\begin{tikzcd}
M \arrow[r] \arrow[d] & \prod_i N_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,j} N_i \oplus_M N_j \arrow[d] \\
M^{gp} \arrow[r] & \prod_i N_i^{gp} \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} N_i^{gp} \oplus_{M^{gp}} N_j^{gp}
\end{tikzcd}$$

The bottom row of the diagram is exact because $M^{gp} \to \prod_i N_i^{gp}$ is injective and $N_i^{gp} \oplus_{M^{gp}} N_j^{gp} \cong N_i^{gp} \oplus N_j^{gp}/M^{gp}$. We will show that M is the equalizer of the top row by leveraging the universal property of an equalizer. Let P be a any monoid that equalizes $\prod_i N_i \rightrightarrows \prod_{i,j} N_i \oplus_M N_j$. Then P will also equalize $\prod_i N_i^{gp} \rightrightarrows \prod_{ij} N_i^{gp} \oplus_{M^{gp}} N_j^{gp}$. As M^{gp} is the equalizer of the bottom row, there exists a unique morphism $P \to M^{gp}$, from which there arises a commutative diagram of solid arrows:

$$\begin{tikzcd}
P \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & & \\
& M \arrow[r] \arrow[d] & \prod_i N_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,j} N_i \oplus_M N_j \arrow[d] \\
& M^{gp} \arrow[r] & \prod_i N_i^{gp} \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} N_i^{gp} \oplus_{M^{gp}} N_j^{gp}.
\end{tikzcd}$$

Exactness implies that the left square is cartesian from which we obtain a unique dotted arrow making the diagram commute. It follows that M is the equalizer of the top row in the diagram. $\square$

The following corollary is then an immediate consequence.

Corollary 4.1.12. *The étale topology on* $\mathfrak{Mon}$ *is subcanonical.*

Proof. The étale topology is coarser than the exact topology. □

In the topologies generated by the exact and étale coverings, the functors

$$\mathrm{Hom}(M,-) : (\mathfrak{Mon}^{op})^{op} \to \mathbf{Sets}$$

are sheaves. We refer to the functor $\mathrm{Hom}(M,-)$ as the cone associated to M, denoted by $\mathbf{Cone}(M)$.

4.2 The Lifting Condition

We are able to characterize étale covering families in terms of a lifting condition. This should be reminiscent of the case with schemes wherein a morphism being étale is equivalent to satisfiying a formal lifting criterion, plus a minor technical condition (see [2]). In the case of monoids, there is a very similar situation; the notion of being an étale covering family and that of satisfying some formal lifting criterion, plus an extra minor technical condition, are equivalent. We will make precise "some formal lifting condition" after we do so with "extra minor technical condition"; this extra condition required is that the monoids appearing in the cover be saturated.

Definition 4.2.1. For a family of monoids $\{P \to Q_i\}_i$, if P and every Q_i is saturated, we will refer to the family as a saturated family.

Definition 4.2.2. For a family of monoids $\{P \to Q_i\}_i$, if P and every Q_i is sharp, we will refer to the family as a sharp family.

Definition 4.2.3. When a saturated or sharp family is étale covering, we will refer to it as a saturated étale covering family or sharp étale covering family, respectively.

The lifting condition will depend on valuative monoids, and will serve as a sort of valuative criterion for a family to be étale covering.

Definition 4.2.4. A family $\{P \to Q_i\}_i$ of monoids has the lifting property at M if given any morphism of monoids $P \to M$, there exists some i and a morphism $Q_i \to M$ making the following diagram commute:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & M \\ Q_i \arrow[ur, dashed] & \end{tikzcd} .$$

We will say that the family has the unique lifting property at M if the lift in the diagram is unique.

The sharp and saturated étale covering families have the lifting property at all valuative monoids. Before formalizing this as a proposition, we will need the following two lemmas, which assert that submonoids of a common ambient group containing a valuative monoid are ordered by inclusion, and that the associated group of a valuative monoid is torsion free.

Lemma 4.2.5. *Let M be a valuative monoid and let $\{M_i\}_i$ be a finite family of monoids in M^{gp} such that $\bigcap M_i = M$. Then there exists an i for which $M = M_i$.*

Proof. By induction, it suffices to prove the lemma for the case of two monoids. To this end, suppose K and L are two submonoids of M^{gp} such that $K \cap L = M$. There exist some, possibly infinite, families of elements $\{x_i\}, \{y_j\} \subset M^{gp}$ such that $K = M[x_1, x_2, \ldots]$ and $L = M[y_1, y_2, \ldots]$. As M is valuative, M^{gp} is totally ordered and hence we may adjoin the elements to M in such a way that $x_i \leq x_{i+1}$ and $y_j \leq y_{j+1}$ for all i, j. Then, either there exists some N such that $x_i \leq y_N$ for all i, in which case $L \subset K$, or that for each i there exists a j such that $y_j \leq x_i$, in which case $y_j \in K$ for all j and hence $K \subset L$. Therefore $M = K$ or $M = L$. □

Lemma 4.2.6. *Let M be a valuative monoid. Then M^{gp} is torsion free.*

Proof. Suppose $x \in M^{gp}$ with $nx = 0$ for some $n \in \mathbb{N}$. We show that $x = 0$. As M is valuative, either $x \in M$ or $-x \in M$. Suppose without loss of generality that $x \in M$. Then $-x = (n-1)x \in M$. As valuative monoids are sharp, it follows that $x = 0$. □

The unique lifting property should be reminiscent of the formal lifting condition from algebraic geometry for being formally étale.

Proposition 4.2.7. *A sharp and saturated family $\{P \to Q_i\}_i$ in $\mathfrak{Mon}$ is étale covering if and only if there is some finite subfamily that has the unique lifting property at all valuative monoids M.*

Proof. Suppose that $\{P \to Q_i\}_i$ is sharp and saturated étale covering family. Replace the family with the finite subfamily such that $P \to Q_i$ is injective and Kummer for each i, and $P \to \prod_i Q_i$ is exact. We may identify P with its image in each Q_i. As P is saturated and Q_i^{gp}/P^{gp} is torsion, it follows that $Q_i^{gp} \subset P^{gp} \otimes \mathbb{Q}$. Thus exactness ensures that $P = \bigcap_i Q_i$ in $P^{gp} \otimes \mathbb{Q}$. Let $P \to M$ be an arbitrary morphism of monoids with M valuative; form the pushout diagram for each i:

$$\begin{tikzcd} P \arrow[r] \arrow[d] & M \arrow[d] \\ Q_i \arrow[r] & Q_i \oplus_P M. \end{tikzcd}$$

We then observe that $M = \bigcap_i Q_i \oplus_P M$. By Lemma 4.2.5 it follows that there exists some i for which $M = Q_i \oplus_P M$, and thus the family has the lifting property at M. Now suppose that there are two lifts $h, h' : Q_i \to M$. Denote by $\mathcal{L}_i(M)$ the collection of all lifts $Q_i \to M$. Note that $h' - h : Q_i^{gp} \to M^{gp}$ is identically zero on P^{gp} and hence descends to a morphism $h' - h : Q_i^{gp}/P^{gp} \to M^{gp}$. Thus, given a lift h, we get a map

$$\mathcal{L}_i(M) \to \mathrm{Hom}(Q_i^{gp}/P^{gp}, M^{gp})$$
$$h' \mapsto h' - h.$$

This map has a two-sided inverse: given $\varphi \in \mathrm{Hom}(Q_i^{gp}/P^{gp}, M^{gp})$, one produces $\varphi + h \in \mathcal{L}_i(M)$. In order to make sense of $\varphi + h$ as a morphism $Q_i \to M$, we think of φ as a morphism $Q_i^{gp} \to M^{gp}$ that kills P^{gp}, and then in order to produce a morphism on Q_i, we may simply restrict $\varphi|_{Q_i}$ - this produces a morphism $Q_i \to M$ that kills P. Thus for any $x \in P$, $(\varphi + h)(x) = \varphi(x) + h(x) = h(x)$ which is indeed a lift. It follows that there is a bijection

$$\mathcal{L}_i(M) \cong \mathrm{Hom}(Q_i^{gp}/P^{gp}, M^{gp}), \tag{4.2.8}$$

provided $\mathcal{L}_i(M) \neq \emptyset$. Now M^{gp} is a torsion free group since M is valuative, and hence $\mathrm{Hom}(Q_i^{gp}/P^{gp}, M^{gp}) = 0$ because Q_i^{gp}/P^{gp} is torsion. This shows that $\mathcal{L}_i(M)$ consists of precisely one element, and hence the lift is unique.

Now suppose that there is a finite, sharp, and saturated subfamily $\{P \xrightarrow{f_i} Q_i\}_i$ that satisfies the unique lifting condition for all valuative monoids. As P is sharp and saturated, we may realize P as the intersection of all sharp valuative monoids that contain P inside of P^{gp}. Indeed, for any $x \in P^{gp}$, if $x \in M$ for every valuative M containing P, then since any morphism $P \to M'$ to a valuative monoid factors through $P \to M$, it is the case that the image of x is ≥ 0 in every M'. Therefore $x \geq 0$ and hence $x \in P$, showing that $\bigcap M = P$. For each such valuative monoid M with $P \subset M \subset P^{gp}$, there exists some Q_i that lifts the inclusion $P \hookrightarrow M$. Hence $P \hookrightarrow Q_i \hookrightarrow M$ for all such M. It follows then that $\bigcap_i Q_i = P$; this shows that $P \to \prod_i Q_i$ is exact.

Finally, we show that $P \to Q_i$ is Kummer. Observe that each i for which there is a unique lift at M will have $\mathcal{L}_i(M) = 1$, which by (4.2.8) shows that $\mathrm{Hom}(Q_i^{gp}/P^{gp}, M^{gp}) = 0$. This forces Q_i^{gp}/P^{gp} to be torsion since M^{gp} is torsion free, and thus $P \to Q_i$ is Kummer. It follows that the family is étale covering. $\square$

4.3 The Moduli Space Of Tropical Curves Is A Stack

A tropical curve Γ over a monoid P is a quadruple (G, h, m, d) consisting of a graph G, a vertex weighting $h : V(G) \to \mathbb{Z}_{\geq 0}$, a marking $m : \{1, \ldots, n\} \xrightarrow{\sim} L(G)$ of the legs of G, and a metric $d : E(G) \to P$ (for more details, see [3], Section 3). Given a tropical curve Γ, we will denote its underlying graph by $\mathbb{G}(\Gamma)$. We now may define the main object of study in this section, the functor $\mathcal{M}_{g,n}^{trop} : \mathfrak{Mon} \to \mathbf{Sets}$. Given a monoid P, the set $\mathcal{M}_{g,n}^{trop}(P)$ consists of all n-marked, tropical curves of genus g over P. Given two tropical curves $\Gamma = (G, h, m, d)$ and $\Gamma' = (G', h', m', d')$ over monoids P and P', a morphism $\Gamma \to \Gamma'$ consists of a morphism $f : G \to G'$ of vertex weighted, marked graphs, and a morphism of monoids $g : P \to P'$ such that the following diagram commutes

$$\begin{tikzcd}
G \arrow[r, "f"] \arrow[d, "d"'] & G' \arrow[d, "d'"] \\
P \arrow[r, "g"] & P'
\end{tikzcd}$$

We will consider the category $\mathcal{M}_{g,n}^{trop}(\mathfrak{Mon})$ whose objects are pairs (P, Γ) where Γ is a tropical curve over the commutative monoid P, and whose morphisms are those defined above. This category is fibered in groupoids over $\mathfrak{Mon}$ under the projection $(P, \Gamma) \mapsto P$.

Given a monoid P and an exact cover $\{P \to Q_i\}_i$, consider the exact sequence

$$0 \to P \xrightarrow{(f_i)_i} \prod_i Q_i \rightrightarrows \prod_{i,j} Q_i \oplus_P Q_j \overset{\rightarrow}{\rightrightarrows} \prod_{i,j,k} Q_i \oplus_P Q_j \oplus_P Q_k.$$

Denote by Q_{ij} and Q_{ijk} the respective pushouts $Q_i \oplus_P Q_j$ and $Q_{ijk} = Q_i \oplus Q_j \oplus_P Q_k$.

4.3.1 Proof of Theorem (4.0.4)

Proof. Suppose we are given a curve Γ_i over each Q_i, Γ_{ij} over each Q_{ij}, and Γ_{ijk} over each Q_{ijk} along with isomorphisms $\Gamma_i, \Gamma_j \xrightarrow{\sim} \Gamma_{ij}$, and $\Gamma_{ij}, \Gamma_{ik}, \Gamma_{jk} \xrightarrow{\sim} \Gamma_{ijk}$, for all i, j, k. Then we need to produce a curve Γ over P such that $(f_i)_*\Gamma = \Gamma_i$ for all i. Let G_i be the underlying graph of each curve and d_i the underlying metric. We need to show that we can descend the graph and that we can descend the metric. We will first show that we can produce a G such that $(f_i)_*G = G_i$ for all i. We can break up the diagram into subdiagrams of the form

$$\begin{tikzcd}
 & G_i \arrow[r] \arrow[dr] & G_{ij} \arrow[dr] & \\
P \arrow[ur, dashed, "\alpha_1"] \arrow[r, dashed, "\alpha_2"] \arrow[dr, dashed, "\alpha_3"'] & G_j \arrow[ur] \arrow[dr] & G_{ik} \arrow[r] & G_{ijk} \\
 & G_k \arrow[r] \arrow[ur] & G_{jk} \arrow[ur] &
\end{tikzcd}$$

where the solid arrows form a commutative diagram of isomorphisms; we need to show that there exist dotted arrows – labelled α_1, α_2, and α_3 – making the whole diagram commute. For the sake of notational simplicity, let us assume that $i = 1$, $j = 2$, and $k = 3$, and label the isomorphisms $\beta^i_{ij} : G_i \to G_{ij}$ and $\gamma^{ij}_{ijk} : G_{ij} \to G_{ijk}$. Choose an automorphism α_1 of G, and define the graph

$G = \alpha_1^{-1}(G_1)$. Then we have an isomorpshim $\alpha_1 : G \to G_1$. Now define $\alpha_2 : G \to G_2$ and $\alpha_3 : G \to G_3$ by

$$\alpha_2 = (\beta_{12}^2)^{-1}\beta_{12}^1\alpha_1 \qquad \text{and} \qquad \alpha_3 = (\beta_{23}^3)^{-1}\beta_{23}^2\alpha_2.$$

Using the commuataivity of the solod arrows, we have that

$$\gamma_{123}^{12}\beta_{12}^1\alpha_1 = \gamma_{123}^{23}\beta_{23}^2(\beta_{12}^2)^{-1}\beta_1\alpha_1 = \gamma_{123}^3\beta_{23}^2\alpha_2,$$

and again the commutativity gives

$$\gamma_{123}^3\beta_{23}^2\alpha_2 = \gamma_{123}^{13}\beta_{13}^3(\beta_{23}^3)^{-1}\beta_{23}^2\alpha_2 = \gamma_{123}^{13}\beta_{13}^2\alpha_3.$$

Hence the α_i give isomorphisms $G \xrightarrow{\sim} G_i$ making the whole diagram commute. Therefore the graph decends.

Each metric d_i is a map of sets $d_i : E(G_i) \to Q_i$. Since each Q_i is a monoid, we may then think of d_i as a morphism of monoids $\mathbb{N}E(G_i) \to Q_i$, where $\mathbb{N}E(G_i)$ is the free monoid on the edges of G_i. But we have already seen that $\mathrm{Hom}(\mathbb{N}E(G_i), -)$ is a sheaf by (4.1.10), and hence the metrics descend. Therefore the moduli space of tropical curves is a stack over $\mathfrak{Mon}$ with the exact topology. $\square$

Chapter 5

Smooth and Étale Topologies for Sharp Saturated Monoids

5.1 Formal Smoothness for Sharp and Saturated Monoids

Just as in algebraic geometry, we think of morphisms of local rings as being geometric when they are local; we do not want the morphisms to invert elements that were not already invertible. The analogous notion for monoids is a sharp morphism.

Definition 5.1.1. A morphism $f : P \to Q$ of monoids is said to be sharp provided $f^{-1}(Q^*) = P^*$.

Remark 5.1.2. Other authors refer to this condition as that of being local (e.g. in [7]), while others use local to mean that there is an induced isomorphism $P^* \to Q^*$ (e.g. in [12]). This distinction doesn't matter when working with sharp monoids, since the two notions coincide.

We will work in the category $(\mathfrak{Mon}^{sat})^\sharp$, consisting of sharp, saturated, integral, commutative, monoids as objects together with sharp morphisms. As illustrated in Example (2.1.1), pushouts of sharp morphsims are not necessarily sharp. Thus, following unpublished work by W.D. Gillam wherein he makes the definition of "good pushout", we make the following definition.

Definition 5.1.3. Let $P \to Q$ and $P \to Q'$ be sharp morphisms of monoids. We say that Q and Q' form an overlapping pair over P when $Q \to Q \oplus_P Q'$ and $Q' \to Q \oplus_P Q'$ are sharp.

Remark 5.1.4. We will often omit "over P", and just say that Q and Q' form an overlapping pair when it is clear that we are working over P.

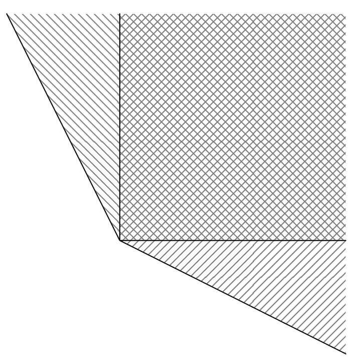
Proposition 5.1.5. *Suppose we are given sharp morphisms $P \to Q$ and $P \to Q'$. Then Q and Q' form an overlapping pair over P if and only if there exists some sharp, saturated monoid and a* *commutative diagram*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ Q' \arrow[r] & R \end{tikzcd}$$

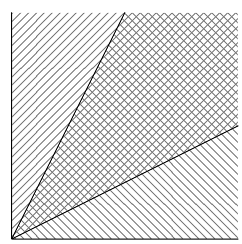
such that both morphisms $Q \to R$ and $Q' \to R$ are sharp.

Proof. Suppose that Q and Q' form an overlapping pair. Then take $R = Q \oplus_P Q'$. Conversely suppose that such an R exists creating a commutative diagram as in the statement of the proposition. Then by the universal property of the pushout, we obtain a morphism $Q \oplus_P Q' \to R$. Since the morphisms $Q \to Q \oplus_P Q'$ and $Q' \to Q \oplus_P Q'$ are sharp, it follows that the morphisms $Q \to R$ and $Q' \to R$ are sharp. $\square$

Geometrically speaking, this is saying that the cones do not intersect in a face. That is to say the monoids Q and Q' form an overlapping pair if and only if the intersection of the relative interiors of **Cone**(Q) and **Cone**(Q') is not empty, hence the name.



An overlapping pair of monoids in $\mathbb{Z}^2$



The dual cones of the overlapping pair.

The very definition of a formally smooth covering family relies on sharp morphisms to valuative monoids. Morphisms to valuative monoids give all points of the assoicated cone, in addition to infinitessimal motion away from the points. This is much like the case in algebraic geometry, where we recover all points, in addition to infinitessimal information and specializations, of an affine scheme by looking at all local morphisms to local rings. Every monoid admits non-trivial morphisms to some non-trivial valuative monoid. In particular, a monoid P admits a morphism to a valuative monoid contained in P^{gp} as the following lemma illustrates.

Lemma 5.1.6. *Any sharp, integral monoid P admits a sharp morphism to a sharp valuative monoid with the same associated group.*

Proof. Extend the partial order on P^{gp} to a total order ([10], pg. 387), call it V. Then take the maximal sharp monoid of non-negative elements inside of V. This will be a sharp valuative monoid admitting a sharp morphism from P. □

By ([12], Lemma 2.2.2), in the event that P is finitely generated, it suffices to take the valuative monoid whose existence is ensured by the lemma to be $\mathbb{N}$. In fact, up to isomorphism, $\mathbb{N}$ is the only finitely generated valuative monoid ([12], Proposition 2.1.16). What is more, every sharp morphism to a sharp valuative monoid $P \to M$ factors through a sharp valuative monoid contained in P^{gp} containing P.

Lemma 5.1.7. *Let $f : P \to M$ be a sharp morphism to a sharp valuative monoid. Then f factors through $P \subseteq M'$ for some sharp valuative monoid M' contained in P^{gp} containing P.*

Proof. The group M^{gp} is totally ordered by definition of a valuative monoid. Hence the morphism $f^{gp} : P^{gp} \to M^{gp}$ induces an order on P^{gp} by $x \leq y$ if and only if $f(x) \leq f(y)$. Let M' be a maximal sharp monoid inside of $(f^{gp})^{-1}(M)$ containing P. Then M' is a sharp valuative monoid contained in P^{gp} containing P such that the diagram

$$\begin{tikzcd} & M' \arrow[dr] & \\ P \arrow[ur] \arrow[rr] & & M \end{tikzcd}$$

commutes, completing the proof. □

Let P, Q be monoids and M a valuative monoid, and let $P \to Q$ be a morphism of monoids. The morphism $P \to Q$ is said to be formally smooth at $Q \to M$ if for any commutative diagram

$$\begin{tikzcd} P \arrow[d] \arrow[dr] & \\ Q \arrow[r] & M, \end{tikzcd}$$

and any surjection of valuative monoids $M' \to M$ producing a commutative diagram of solid arrows

$$\begin{tikzcd} P \arrow[r] \arrow[d] & M' \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & M, \end{tikzcd}$$

there exists a dashed arrow making the diagram commute.

Definition 5.1.8. A family $\{P \to Q_i\}_i$ is said to be formally smooth covering if for any valuative monoid M and sharp morphism $P \to M$, there exists an i such that $P \to Q_i$ is formally smooth at $Q_i \to M$.

Example 5.1.9. There is a way to formalize "infinitessimal motion" in the setting of monoids by using certain valuative monoids. This infinitessimal motion provides some intuition as to what geometric condition the formally smooth covering families should satisfy. To build this intuition, let us work with the dual cones of the monoids. Given a ray lying in the interior of the dual cone of a monoid, there should be some element of the cover that this ray factors through. Moreover, infinitessimal motion away from this ray in any direction should factor through this same element of the cover. For a concrete example, let $\mathbb{N}[\epsilon]$ be the submonoid of $\mathbb{Z} + \mathbb{Z}[\epsilon]$ generated by 1 and $1 - n\epsilon$ for all $n \in \mathbb{N}$. We observe that $\mathbb{N}[\epsilon]^{gp} \cong \mathbb{Z}^2$. Given any $x \in \mathbb{Z}^2$ either x or $-x$ will lie in $\mathbb{N}[\epsilon]$, and thus $\mathbb{N}[\epsilon]$ is a valuative monoid. Let $f : \mathbb{N}^2 \to \mathbb{N}[\epsilon]$ be any monoid morphism. Then, taking dual cones, we will get $f^* : \mathbf{Cone}(\mathbb{N}[\epsilon]) \to \mathbf{Cone}(\mathbb{N}^2)$. We need only say what the generators pull back to under f in order to specify the associated map on cones. Let x_1 and x_2 be a pair of generators for $\mathbb{N}^2$. Define f by $x_1 \mapsto 1 - \epsilon$ and $x_2 \mapsto 1 + \epsilon$. Let e_1 and e_2 be the generators of the dual cone of $\mathbb{N}^2$. In order to describe f^*, we may describe where each of the dual functions ϵ^* and $(1 - n\epsilon)^*$ gets sent in $\mathbf{Cone}(\mathbb{N}^2)$. We see that ϵ pulls back to $e_1 + e_2$, and $1 - n\epsilon$ pulls back to $(n-1)e_1 + (n+1)e_2$ for each n. We may envision this as a ray with a little bit of"infinitessimal motion" in the e_2 direction. The ray will pass through $(1, 1)$ since $f^*(\epsilon^*) = e_1 + e_2$.

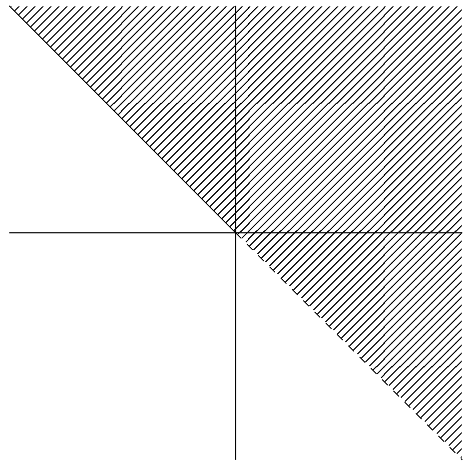


Figure 5.1: The lattice points in the shaded region, along with the solid ray along the line passing through the origin and $(-1,1)$, comprise the monoid $\mathbb{N}[\epsilon]$. We have mapped it into $\mathbb{Z}^2$ here by sending $\epsilon \mapsto (-1,1)$ and $1 \mapsto (1,1)$.

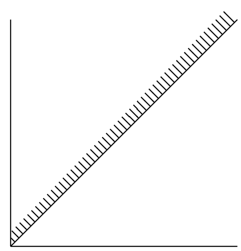


Figure 5.2: The "fuzz" extending from the ray passing through $(1,1)$ is coming from the e_2 direction.

The collection of morphisms from a monoid P to valuative monoids determines all possible ways of ordering P. If two elements $x, y \in P$ are comparable in the same way, e.g. $f(x) \leq f(y)$, for any morphism f to a valuative monoid, then we might expect that $x \leq y$. Indeed, this is the case.

Lemma 5.1.10. *Let P be a monoid. Let $x, y \in P$. If $f(x) \leq f(y)$ for all morphisms $f : P \to M$ to all valuative monoids M, then $x \leq y$.*

Proof. If $x = y$ then there is nothing to show, so suppose that $x \neq y$ and $f(x) \leq f(y)$ for any morphism f to a valuative monoid. By (5.1.7), there exists a valuative monoid M and an injective morphism $f : P \to M$. Therefore $f(x) \neq f(y)$. It follows that $x \leq y$ since otherwise we would have $f(x) \geq f(y)$ and $f(x) \leq f(y)$ which would then imply that $f(x) = f(y)$ since the associated group of a valuative monoid is totally ordered. $\square$

It follows immediately from (5.1.10) that formally smooth covering families are exact.

Proposition 5.1.11. *Let* $\{P \xrightarrow{f_i} Q_i\}_{i\in I}$ *be a formally smooth covering family. Then the morphism* $f = (f_i)_{i\in I} : P \to \prod_{i\in I} Q_i$ *is exact.*

Proof. Let $x, y \in P$ such that $f_i(x) \leq f_i(y)$ for all i. For any valuative monoid M and any morphism $h : P \to M$, there exists some i with a morphism $g : Q_i \to M$ such that $gf_i = h$. Therefore $g(x) \leq g(y)$ for any morphism $g : P \to M$ for any valuative monoid M. By Lemma (5.1.10), it follows that $x \leq y$ and hence f is exact. □

Formally smooth covering families also behave nicely with respect to pushouts along sharp morphisms.

Lemma 5.1.12. *Let* $\{P \to Q_i\}_{i\in I}$ *be a formally smooth covering family. Let* $P \to P'$ *be a sharp morphism. Then* $\{P' \to P' \oplus_P Q_i\}_{i\in I}$ *is also a formally smooth covering family.*

Proof. Let M be a valuative monoid and $P' \to M$ a sharp morphism. Then we get a sharp morphism $P \to M$ by composition. By definition of a formally smooth covering family, there thus exists a commutative diagram

$$\begin{tikzcd} P \arrow[d] \arrow[dr] & \\ Q_i \arrow[r] & M \end{tikzcd}$$

for some i such that for any valuative monoid M' with a surjective morphism $M' \to M$, and any sharp morphism $P \to M'$, there exists a commutative diagram of solid arrows

$$\begin{tikzcd} P \arrow[r] \arrow[d] & M' \arrow[d] \\ Q_i \arrow[r] \arrow[ur, dashed] & M \end{tikzcd}$$

such that the dotted arrow exists making the diagram commute. Now, applying the universal property of the pushout to the commutative diagram of solid arrows

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P' \arrow[d] \arrow[ddr, bend left] & \\ Q_i \arrow[r] \arrow[drr, bend right] & Q_i \oplus_P P' \arrow[dr, dashed] & \\ & & M \end{tikzcd}$$

ensures the existence of the unique dotted arrow making the diagram commute. Again, applying the universal property of the pushout to the commutative diagram of solid arrows

$$\begin{tikzcd} P \arrow[r] \arrow[d] & P' \arrow[r] \arrow[d] & M' \\ Q_i \arrow[r] \arrow[urr] & Q_i \oplus_P P' \arrow[ur, dashed] & \end{tikzcd}$$

ensures the existence of a unique dotted arrow making the diagram commute. Putting these two diagrams together gives a commutative diagram of solid arrows

$$\begin{tikzcd} P' \arrow[r] \arrow[d] & M' \arrow[d] \\ Q_i \oplus_P P' \arrow[r] \arrow[ur, dashed] & M \end{tikzcd}$$

such that the dotted arrow exists making the diagram commute. This completes the proof. □

Let $A \to B$ be a smooth morphism of commutative rings. Then the induced morphism $\Omega_A \otimes_A B \to \Omega_B$ is a split injection. Recall that for a ring A, Ω_A is defined by the following universal property. Let $\delta : A \to \Omega_A$ be the universal derivation. Let M be any A-module and let $D : A \to M$ be a derivation. Then there exists a unique morphism $f : \Omega_A \to M$ such that $f \circ \delta = D$. That is $\mathrm{Hom}(\Omega_A, M) = \mathrm{Der}(A, M)$. Following Quillen (see [13], Section 5.13) we can arrive at this universal property by considering the category of abelian group objects in the slice category over a fixed ring A. Let **CRng** be the category of commutative rings. Let A be a commutative ring and **CRng** $/A$ the category of commutative rings over A. For any A-module we can construct an abelian group object $A \oplus M \xrightarrow{\pi_1} A$, where $(a, x) \cdot (b, y) = (ab, ay + bx)$ – this is the infinitessimal extension of A by M. For a given M, the space of sections of π is in bijection with the space of M-valued derivations $\mathrm{Der}(A, M)$, which in turn is isomorphic to $\mathrm{Hom}(\Omega_A, M)$.

We can perform a similar construction in the case of monoids. Fix a monoid P and let $\mathfrak{Mon}/P$ be the category of monoids with a morphism to P. To define an abelian group object in $\mathfrak{Mon}/P$ is to specify an abelian group V and form the direct sum $P \oplus V$ with the projection to P. Then sections of this map are just $\mathrm{Hom}(P, V) \cong \mathrm{Hom}(P^{gp}, V)$; this isomorphism follows from the fact that a morphism from any integral monoid to an abelian group is determined by its associated group. This makes clear why we can think of P^{gp} as "Ω_P". If we are working in the category of sharp monoids with sharp morphisms, as we have been throughout this chapter, then the infinitessimal extension of P by a sharp monoid is the maximal sharp submonoid M' of $P \oplus M$ such that the morphism to P induced by π is sharp. Indeed, the sections of this morphism will be determined by sections of π and hence we still arrive at "Ω_P" $= P^{gp}$. Thus the following theorem makes the analogous statement for monoids as we have for commutative rings – that is, a formally smooth morphism at a point is a split injection on Kahler differentials at that point.

Theorem 5.1.13. *Let $f : P \to Q$ be a morphism of monoids that is formally smooth at $h : Q \to M$ for some valuative monoid M, let $g : P \to M$ defined by $g = h \circ f$, and assume that g^{gp} is surjective. Then $P^{gp} \to Q^{gp}$ is a split injection at M^{gp}. That is, there exists a commutative diagram of solid arrows*

$$\begin{tikzcd} P^{gp} \arrow[r, equal] \arrow[d, "f^{gp}"] & P^{gp} \arrow[d, "g^{gp}"] \\ Q^{gp} \arrow[r, "h^{gp}"] \arrow[ur, dashed] & M^{gp} \end{tikzcd}$$

and a dotted arrow making the whole diagram commute.

Proof. We will first show that $P^{gp} \to Q^{gp}$ is injective. Let $g : P \to M$ be a sharp morphism to a valuative monoid, with $P^{gp} \to M^{gp}$ surjective, and suppose $f : P \to Q$ is formally smooth at $h : Q \to M$. Let $g' : P \to M \times \epsilon P^{gp}$ by $g'(x) = g(x) + \epsilon x$. Let M be a valuative monoid in $M \times \epsilon P^{gp}$ containing the image of P. Then we obtain an induced surjective morphism $M' \to M$ from $\epsilon \mapsto 0$.

Therefore the following diagram of solid arrows

$$\begin{tikzcd} P \arrow[r] \arrow[d] & M' \arrow[d] \\ Q \arrow[r, "h"] \arrow[ur, dashed, "h'"] & M \end{tikzcd} \tag{5.1.14}$$

commutes, and the dotted arrow exists making the whole diagram commute. Now suppose $x \neq y$ in P^{gp}. Then $g'(x) \neq g'(y)$ since $\epsilon x \neq \epsilon y$. Now, as the diagram commutes we have that $h'(f(x)) = g'(x) \neq g'(y) = h'(f(y))$ from which it follows that $f(x) \neq f(y)$.

To see that $P^{gp} \to Q^{gp}$ is split at M^{gp}, we make use of the diagram (5.1.14) again, but we replace M with a valuative monoid $M[\epsilon]$ in $M \times \epsilon M^{gp}$. Let $\varphi^{gp} : Q^{gp} \to M \times \epsilon M^{gp}$ by $\varphi^{gp}(x) = h(x) + \epsilon h(x)$; let $M[\epsilon]$ be a valuative monoid containing the image of Q under φ^{gp}. This defines a morphism $\varphi : Q \to M[\epsilon]$. Now let $\psi^{gp} : P^{gp} \to M^{gp} \times \epsilon M^{gp}$ be defined by $\psi^{gp}(x) = g^{gp}(x) + \epsilon g^{gp}(x)$. Since $h \circ f = g$ and f is injective, we obtain a morphism $\psi : P \to M[\epsilon]$ such that $\psi = \varphi \circ f$. Finally let $\pi : M^{gp} \times \epsilon P^{gp} \to M^{gp} \times \epsilon M^{gp}$ by $\pi(x + \epsilon y) = x + \epsilon g^{gp}(y)$. Then on the level of monoids we get a surjective morphism $\pi : M' \to M[\epsilon]$ of valuative monoids. Now we may observe that

$$\pi(g'(x)) = \pi(g(x) + \epsilon x) = g(x) + \epsilon g(x) = h(f(x)) + \epsilon h(f(x)) = \varphi(f(x)),$$

from which it follows that we have a commutative diagram of solid arrows

$$\begin{tikzcd} P \arrow[r, "g'"] \arrow[d, "f"] & M' \arrow[d, "\pi"] \\ Q \arrow[r, "\varphi"] \arrow[ur, dashed] & M[\epsilon]. \end{tikzcd}$$

where by assumption, there exists a lift $h' : Q \to M'$ making the whole diagram commute. In particular the diagram commutes on the level of groups, so $h'^{gp} : Q^{gp} \to M^{gp} \times \epsilon P^{gp}$ satisfies

$$h'^{gp}(f^{gp}(x)) = g'^{gp}(x) = g(x) + \epsilon x.$$

Therefore $h'^{gp} = h^{gp} + \epsilon s$ where $s : Q^{gp} \to P^{gp}$ is a section of f^{gp}. Moreover we also have that the lower right triangle commutes, hence also on the level of groups, which is to say that $\pi^{gp} \circ h'^{gp} = \varphi^{gp}$.

Therefore

$$h^{gp}(x) + \epsilon h^{gp}(x) = \varphi^{gp}(x) = \pi^{gp}(h'^{gp}(x)) = \pi^{gp}(h^{gp}(x) + \epsilon s(x)) = h^{gp}(x) + g^{gp}(s(x)).$$

Hence it follows that $g^{gp} \circ s = h^{gp}$, which complete the proof. □

Remark 5.1.15. The proof of Theorem (5.1.13) shows that the space of all sections of $P^{gp} \to Q^{gp}$ at M^{gp} is a torsor under the group $\mathrm{Hom}(Q^{gp}/P^{gp}, K)$ where $K = \ker(P^{gp} \to M^{gp})$, under the assumption that $P^{gp} \to M^{gp}$ is surjective, while the space of all sections, without any reference to a point, is a torsor under $\mathrm{Hom}(Q^{gp}/P^{gp}, P^{gp})$.

As a direct corollary to this lemma, we may observe that formally smooth morphisms have torsion free quotients.

Lemma 5.1.16. *Let $f : P \to Q$ be formally smooth at $Q \to M$. Then Q^{gp}/P^{gp} is torsion free.*

Proof. By Lemma (5.1.13), the morphism $f : P^{gp} \to Q^{gp}$ admits a splitting $s : Q^{gp} \to P^{gp}$. Therefore $Q^{gp} \cong P^{gp} \oplus Q^{gp}/P^{gp}$. Now P and Q are staurated and hence P^{gp} and Q^{gp} are torsion free. It follows that Q^{gp}/P^{gp} is torsion free. □

Example 5.1.17. Let $P \cong \mathbb{Q}^2_{\geq 0}$ generated by x_1 and x_2, and $Q \cong \mathbb{Q}^3_{\geq 0}$ generated by y_1, y_2, and y_3. Define $f : P \to Q$ by $x_1 \mapsto y_1 + y_3/2$ and $x_2 \mapsto y_2 + y_3/2$. Let $M = \mathbb{Q}_{\geq 0}$. Let $g : P \to M$ by $x_i \mapsto 1/2$ and $h : Q \to M$ by $y_i \mapsto 1/3$. A slice of the associated picture on cones is shown in figure 5.3. Then we wish to find a section $s : Q^{gp} \to P^{gp}$ such that $g^{gp} \circ s = h^{gp}$. Now the space of sections is a torsor under $\mathrm{Hom}(Q^{gp}/P^{gp}, P^{gp}) \cong \mathbb{Q}^2$, and hence once we choose a "base point", we get an isomorphism of the space of all sections with $\mathbb{Q}^2$. Define s_0 by $y_1 \mapsto x_1$, $y_2 \mapsto x_2$, and $y_3 \mapsto 0$. Then every other section is of the form $s_0 + \alpha$ for some $\alpha \in \mathrm{Hom}(Q^{gp}/P^{gp}, P^{gp})$. We have that $Q^{gp}/P^{gp} \cong \mathbb{Q}\{y_3 - \frac{1}{2}y_1 - \frac{1}{2}y_2\}$ and thus every section is of the form:

$$y_1 \mapsto (1 - \frac{m}{2})x_1 - \frac{n}{2}x_2$$
$$y_2 \mapsto (1 - \frac{n}{2})x_2 - \frac{m}{2}x_1$$
$$y_3 \mapsto mx_1 + nx_2$$

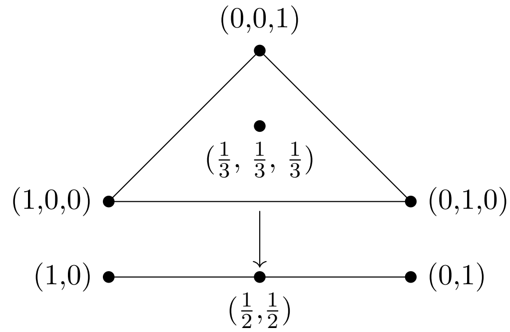


Figure 5.3: Projection from a three dimensional cone onto a two dimensional cone. The point defined by $Q \to M$ is the ray through the point $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ in the $\mathbf{Cone}(Q)$ and projects down to $(\frac{1}{2}, \frac{1}{2})$ in $\mathbf{Cone}(P)$.

where $m, n \in \mathbb{Q}$. We see explicitly the action of $\mathrm{Hom}(Q^{gp}/P^{gp}, P^{gp})$ on the sections here. The further condition that $g^{gp} \circ s = h^{gp}$ tells us that $m + n = \frac{2}{3}$. For instance, we can choose $m = 1$ and $n = -1/3$, which then gives us the section s defined by $y_1 \mapsto \frac{1}{2}x_1 + \frac{1}{6}x_2$, $y_2 \mapsto \frac{7}{6}x_2 - \frac{1}{2}x_1$, $y_3 \mapsto x_1 - \frac{1}{3}x_2$. The image of s is an enlarged sharp monoid containing P: Notice that if we take dual cones of the above picture then $\mathbf{Cone}(S(Q))$ is a cone inside of $\mathbf{Cone}(P)$ that has a section back to $\mathbf{Cone}(Q)$.

A particularly useful observation is in order. Let Q_i and Q_j be any two monoids from a formally smooth covering family. Let $G = (Q_i \oplus_P Q_j)^{gp}$, the pushout $Q_1 \oplus_P Q_2$ being taken in the sharp and saturated category, and

$$H = Q_i^{gp} \oplus_{P^{gp}} Q_j^{gp} \cong Q_i^{gp} \oplus Q_j^{gp}/P^{gp},$$

the isomorphism coming from general abelian category theory nonsense (see e.g. [2]). Since $(Q_1 \oplus_P Q_2)^{gp} \cong Q_1^{gp} \oplus_{P^{gp}} Q_2^{gp}$ with the pushout being taken in the category of integral monoids, it follows that in the category of sharp and saturated monoids $(Q_1 \oplus_P Q_2)^{gp} \cong (Q_1^{gp} \oplus_{P^{gp}} Q_2^{gp})/((Q_1 \oplus_P Q_2)^{sat})^*$, which by Lemma (2.1.2) says that $G = H/H_{tor}$. As a corollary to the previous lemma, we observe that $H_{tor} = \{0\}$.

Corollary 5.1.18. *With the notation as above, $H_{tor} = \{0\}$.*

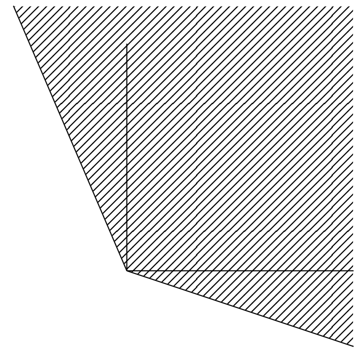


Figure 5.4: The shaded region is the monoid $s(Q)$ containing P.

Proof. Let $x \in H$ such that $nx = 0$ for some x. Let $\overline{x}$ be the image of x in the quotient group H/Q_i^{gp}. Then x is still a torsion element, but by Lemma (5.1.16) we have that $\overline{x} = 0$. And therefore $x \in Q_i^{gp}$. But Q_i^{gp} is torsion free since Q_i is saturated. Therefore $x = 0$, from which the result follows. □

We have another set of isomorphisms that follows immediately.

Corollary 5.1.19. *We have the following isomorphisms:*

$$(Q_i \oplus_P Q_j)^{gp} \cong Q_i^{gp} \oplus_{gp} Q_j^{gp} \cong Q_i^{gp} \oplus Q_j^{gp}/P^{gp}.$$

Proof. Apply the previous corollary to $G = H/H_{tor}$. □

This property of formally smooth covering families will prove to be quite useful when assessing properties of descent on the category of cones with the formally smooth topology.

5.2 Descent Properties for Formally Smooth Covering Families

This section will be devoted to studying the following diagram of sequences:

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d] & \prod_i Q_i \arrow[r] \arrow[d] & \prod_{i,j} Q_i \oplus_P Q_j \arrow[d] \\
0 \arrow[r] & P^{gp} \arrow[r] & \prod_i Q_i^{gp} \arrow[r] & \prod_{i,j} (Q_i \oplus_P Q_i)^{gp}.
\end{tikzcd} \tag{5.2.1}$$

It is worth noting that the pushouts in this diagram are being taken in the category of sharp and saturated integral monoids. In the next section, wherein we will show that taking formally smooth covering families to be covering forms a topology on the category of sharp saturated monoids with sharp morphisms, the exactness of the first row of this sequence will imply that the topology is subcanonical. Indeed, this will show that each functor **Cone**(P) represented by P is in fact a sheaf.

We will constantly be using our geometric intuition to guide our algebraic formulation. Geometrically, the intersection of cones has a non-empty relative interior precisely when the cones do not intersect in a face. We formulate this algebraically in terms of overlapping pairs.

Lemma 5.2.2. *Assume that P, Q, and Q' are objects in $(\mathfrak{Mon}^{sat})^\sharp$, $P \to Q$ and $P \to Q'$ are sharp, and $P^{gp} \cong Q^{gp} \cong Q'^{gp}$. Then the following are equivalent:*

1) *Q and Q' are an overlapping pair,*

2) *$(Q + Q')^* = 0$.*

Proof. If $P^{gp} \cong Q^{gp} \cong Q'^{gp}$ then $Q \oplus_P Q' = Q + Q'$, taken in P^{gp}. There is a nonzero element $q_1 + q_1' \in (Q + Q')^*$ if and only if there is some $q_2 + q_2' \in Q + Q'$ such that $q_1 + q_1' + q_2 + q_2' = 0$ and hence $q_1 + q_2$ is an element of Q that has an inverse in Q'. This is the case if and only if the map $Q \to Q \oplus_P Q'$ is not sharp, and hence if and only if Q and Q' are not an overlapping pair. $\square$

Given a formally smooth covering family $\{P \xrightarrow{f_i} Q_i\}_{i \in I}$, the collection of f_i that are injective is a formally smooth covering subfamily – this follows from Lemma (5.1.7). Moreover, by Lemma (5.1.16), for any of those f_i that are formally smooth at some point $Q \to M$, the quotient Q_i^{gp}/P^{gp}

is torsion free. Therefore we may assume from this point forward that every f_i is injective and that all of the quotients are torsion free. In particular, this implies that $P \to \prod_{i\in I} Q_i$ is injective. Therefore we only need to prove exactness of the sequence (5.2.1) at $\prod_{i\in I} Q_i$.

Theorem (5.2.6) contains the formal statement that utimately is the main player in this section. We will build up to this main theorem in stages. First, note that if we have a two element family $\{P \to Q_i\}_{i=1}^2$ with $P^{gp} \cong Q_1^{gp} \cong Q_2^{gp}$, then

$$(Q_1 \oplus_P Q_2)^{gp} \cong ((Q_1 + Q_2)^{sat})^{gp}/(Q_1 + Q_2)^* \cong P^{gp}/(Q_1 + Q_2)^*. \tag{5.2.3}$$

Indeed, the assumption $P^{gp} \cong Q_1^{gp} \cong Q_2^{gp}$ implies that $Q_1 \oplus_P Q_2$ is the sum $Q_1 + Q_2$ inside of P^{gp} taken in the integral category, and $(Q_1 + Q_2)^{gp} \cong P^{gp} + P^{gp} = P^{gp}$.

Proposition 5.2.4. *Let $\{P \to Q_i\}_{i\in I}$ be a formally smooth covering family with $P^{gp} \cong Q_i^{gp}$ for all i. Then* (5.2.1) *is exact.*

Proof. Applying the isomorphisms $Q_i^{gp} \cong Q_j^{gp} \cong P^{gp}$ for each i and j and equation (5.2.3) gives us the isomorphisms

$$(Q_i \oplus_P Q_j)^{gp} \cong P^{gp}/(Q_i + Q_j)^*$$

for all i and j. Now, the morphism $\prod_i P^{gp} \to \prod_{i,j} P^{gp}/(Q_i + Q_j)^*$, coming from the above isomorphisms and the bottom row of diagram (5.2.1), takes the i^{th} and j^{th} component of x to $x_i - x_j$. Suppose that the image of x under this morphism is 0, that is $x_i - x_j \in (Q_i + Q_j)^*$ for all i and j.

Consider the relation $\sim$ on the index set I of the covering family defined by $i \sim j$ if and only if $(Q_i + Q_j)^* = 0$ (Note: the relation is symmetric and reflexive but not transitive; it is not necessarily an equivalence relation on I). We build a graph Γ out of this relation on the index set as follows. Include a vertex v_i for every element Q_i of the covering family. Then include an edge between v_i and v_j whenever $i \sim j$. To see that (5.2.1) is exact, it will suffice to show that $\sim$ is connected, which amounts to showing that Γ is connected. We state this formally in the following lemma.

Lemma 5.2.5. *With the hypotheses of* (5.2.4)*, the graph* Γ *is connected.*

Proof. Let $x \in P^{gp}$ such that $x, -x \notin P$. Choose some monoid Q from the covering family. If for every $x \in Q$, there does not exist any Q' in the covering family for which $-x \in Q'$, then Q forms an overlapping pair with all Q'. If this is the case, then Γ is connected, from which we conclude that (5.2.1) is exact. Otherwise, there exists some Q' and some $x \in Q$ such that $-x \in Q'$.

Assume that Q' and x exist. Define the subgraphs Γ^+ and Γ^- of Γ as follows:

$$\Gamma^+ = \{v_i \mid -x \notin Q_i\}, \qquad \Gamma^- = \{v_i \mid x \notin Q_i\}.$$

To say that such Q, x, and Q' exist is to say that there is some $v \in \Gamma^+ \setminus \Gamma^-$ and $v' \in \Gamma^- \setminus \Gamma^+$. In the event that the graph only consists of one element, then there is nothing to show. Suppose that the results of (5.2.5) hold when Γ contains up to N vertices. Then suppose that there are $N+1$ vertices in Γ . As $x, -x \notin P$, it follows that each of $P \to P[x]$ and $P \to P[-x]$ is sharp, and hence both of the families

$$\{P[x] \xrightarrow{f_{i,x}} Q_i[x]\}_{v_i \in \Gamma^+} \qquad \text{and} \qquad \{P[-x] \xrightarrow{f_{i,-x}} Q_i[-x]\}_{v_i \in \Gamma^-}$$

are formally smooth covering by Lemma (5.1.12). Moreover, since each of Γ^+ and Γ^- is strictly contained in Γ, we may apply induction to deduce that each of Γ^+ and Γ^- is connected.

There exists a sharp valuative monoid M, and a sharp morphism $f_x : P[x, -x] \to M$. Since $x, -x \notin P$ by design, the morphism $f : P \to M$ is also sharp. Therefore there exists some Q_i such that $P \to Q_i$ has the lifting property at $Q_i \to M$. Let $P \to M^{gp} \times \epsilon P^{gp}$ be defined by $p \mapsto f(p) + \epsilon p$. Then choose any valuative monoid $M' \subseteq M \times \epsilon P^{gp}$ containing the image of $P[x]$. This naturally induces a commutative diagram

$$\begin{tikzcd} P[x] \arrow[r] \arrow[d] & M' \arrow[d] \\ Q_i[x] \arrow[r] \arrow[ur, dashed] & M, \end{tikzcd}$$

where under the dotted arrow, we will have $x \mapsto \epsilon x$. Therefore it must not be the case that $-x \in Q_i$ since otherwise $-x \mapsto -\epsilon x$ under the dotted arrow, but M' is sharp which forces the image of x

to actually be zero. It follows then that $-x \notin Q_i$. Likewise, if we repeat this argument with M' containing the image of $P[-x]$, then we may conclude that $x \notin Q_i$. It follows that $v_i \in \Gamma^+ \cap \Gamma^-$ and hence Γ is connected $\square$

It follows from the proof of the lemma that the sequence (5.2.1) is exact. $\square$

We may now remove the hypothesis that the ambient groups are isomorphic, and prove that an arbitrary smooth covering family satisfies descent.

Theorem 5.2.6. *Let $\{P \to Q_i\}_{i \in I}$ be a formally smooth covering family. Then the sequence*

$$0 \to P \to \prod_i Q_i \rightrightarrows \prod_{i,j} Q_i \oplus_P Q_j$$

is exact.

Proof. For each i, let $Q_i' = (f_i^{gp})^{-1}(Q_i)$. Then each $Q_i' \to Q_i$ is an exact morphism with $(Q_i')^{gp} \cong P^{gp}$ for all i, and $P \to Q_i$ factors through $P \to Q_i'$ for all i, from which it follows immediately that $\{P \to Q_i'\}_i$ is a formally smooth covering family. Therefore by Lemma (5.2.4), the sequence

$$0 \to P \to \prod_i Q_i' \rightrightarrows \prod_{i,j} Q_i' \oplus_P Q_j'$$

is exact. Furthermore, this gives us a morphism of sequences

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d] & \prod Q_i' \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,j} Q_i' \oplus_P Q_j' \arrow[d] \\
0 \arrow[r] & P \arrow[r] & \prod_i Q_i \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} Q_i \oplus_P Q_j.
\end{tikzcd} \tag{5.2.7}$$

We have isomorphisms $Q_i'^{gp} \cong P^{gp}$ for all i, and $(Q_i' \oplus_P Q_j')^{gp} \cong P^{gp}/(Q_i' + Q_j')^*$ for all i, j. For any formally smooth covering family, the subfamily of injective morphisms is still covering, and hence we can assume each $P \to Q_i$ is injective. It is then also the case that $\prod_i Q_i'^{gp} \to \prod_i Q_i^{gp}$ is injective. Now, applying the isomorphism $(Q_i' \oplus_P Q_j')^{gp} \cong P^{gp}/(Q_i' + Q_j')^*$ we get morphisms

$$P^{gp}/(Q_i' + Q_j')^* \to (Q_i \oplus_P Q_j)^{gp}$$

for all i and j. Applying the groupification functor to (5.2.7), we obtain the following diagram

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & P^{gp} \arrow[r] \arrow[d, equal] & \prod Q_i'^{gp} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "h"] & \prod_{i,j} P^{gp}/(Q_i' + Q_j')^* \arrow[d, "h'"] \\ 0 \arrow[r] & P^{gp} \arrow[r] \arrow[d] & \prod_i Q_i^{gp} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,j}(Q_i \oplus_P Q_j)^{gp} \\ & 0 & \prod_i (Q_i \oplus_{Q_i'} Q_i)^{gp} & . \end{tikzcd} \tag{5.2.8}$$

By Lemma (5.2.5), for each i there exists some j such that $(Q_i' + Q_j')^* = 0$ and hence we obtain morphisms $P^{gp} \to (Q_i \oplus_P Q_j)^{gp}$ for all such i and j. This will be useful in proving the following, the proof of which we postpone until after the proof of the theorem.

Lemma 5.2.9. *The morphism h' is injective.*

Let $x = (x_i)_i \in \prod_i Q_i$ such that $x_i - x_j = 0$ in each $(Q_i \oplus_P Q_j)^{gp}$. From Corollary (5.1.19), we have

$$(Q_i \oplus_P Q_i)^{gp} \cong Q_i^{gp} \oplus_{P^{gp}} Q_i^{gp} \cong Q_i^{gp} \oplus Q_i^{gp}/P^{gp}$$

for all i. Therefore, by the middle column of (5.2.8) and that $Q_i'^{gp} \cong P^{gp}$, for each i there is an exact sequence

$$0 \to P^{gp} \to Q_i^{gp} \to Q_i^{gp} \oplus Q_i^{gp}/P^{gp}.$$

Hence $x_i \mapsto 0 \in Q_i^{gp}/P^{gp}$ for each i, from which it follows that $x_i \in P^{gp}$ for all i. Then, since h' is injective by Lemma (5.2.9), the preimage $h'^{-1}(x_i - x_j)$ is also zero. As the top row is exact, we deduce that $(f_i^{gp})^{-1}(x_i) = x \in P^{gp}$ for all i. Finally, for each i the morphism $P \to Q_i$ factors through Q_i', from which we conclude that the bottom row of (5.2.8) is exact.

Consider now the diagram

$$\begin{tikzcd} 0 \arrow[r] & P \arrow[r] \arrow[d] & \prod_i Q_i \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & \prod_{i,j} Q_i \oplus_P Q_j \arrow[d] \\ 0 \arrow[r] & P^{gp} \arrow[r] & \prod_i Q_i^{gp} \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j}(Q_i \oplus_P Q_j)^{gp}. \end{tikzcd}$$

Take any monoid M that equalizes the top row. Then it will also equalize the bottom row producing a commutative diagram

$$\begin{tikzcd}
M \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& P \arrow[r] \arrow[d] & \prod_i Q_i \arrow[d] \\
& P^{gp} \arrow[r] & \prod_i Q_i^{gp}
\end{tikzcd}$$

of solid arrows. Since formally smooth covering families are exact by Lemma (5.1.11), the square is cartesian; hence by the universal property of the fibered product we get a unique dotted arrow showing that P is the equalizer of the top row, which completes the proof of the theorem. □

We now provide the proof of (5.2.9) that was omitted during the proof of Theorem (5.2.6).

Proof of Lemma (5.2.9). Let i and j be a indices such that $(Q'_i + Q'_j)^* = 0$. Then we have a commutative diagram of solid arrows

$$\begin{tikzcd}
P^{gp} \arrow[r, "\sim"] \arrow[d, "\wr"] & Q_i'^{gp} \arrow[r, hook] \arrow[d, "\wr"] & Q_i^{gp} \arrow[dd, hook] \\
Q_i'^{gp} \arrow[r, "\sim"] \arrow[d] & (Q'_i + Q'_j)^{gp} \arrow[dr, dashed, hook] & \\
Q_j^{gp} \arrow[rr] & & (Q_i \oplus_P Q_j)^{gp}
\end{tikzcd}$$

with a unique dotted arrow coming from the universal property of the pushout. By [2], Lemma 12.5.13, since the morphism $P^{gp} \to Q_j^{gp}$ is injective, then so is $Q_i^{gp} \to Q_i^{gp} \oplus_{P^{gp}} Q_j^{gp}$, and consequently so is $Q_i^{gp} \to (Q_i \oplus_P Q_j)^{gp}$. By commutativity of the diagram, the dotted arrow is the composition of $P^{gp} \xrightarrow{\sim} Q_i'^{gp} \hookrightarrow Q_i^{gp} \hookrightarrow (Q_i \oplus_P Q_j)^{gp}$, which is therefore injective. By Lemma (5.2.5), for every i there exists a j such that $(Q'_i + Q'_j)^* = 0$, from which it follows that h' is injective. □

We have effectively shown that taking covering families on $(\mathfrak{Mon}^{sat})^*$, with sharp morphisms, to be formally smooth families generates a subcanonical Grothendieck topology.

Theorem 5.2.10. *On the category $(\mathfrak{Mon}^{sat})^\sharp$ with sharp morphisms, taking covering families to be formally smooth covering families defines a Grothendieck topology.*

Proof. The category $(\mathfrak{Mon}^{sat})^\sharp$ has pushouts, and by (5.1.12) formally smooth covering families pushout. Furthermore, any isomorphism is a formally smooth covering family. Thus we need only show that cover compose. Let $\{P \to Q_i\}_{i\in I}$ be a formally smooth covering family, and suppose for each i we have a formally smooth covering family $\{Q_i \to R_{ij}\}_j$. Furthermore suppose that we have a surjection of valuative monoids $M' \to M$ and, for some i and j, a commutative diagram of solid arrows

$$\begin{tikzcd} P \arrow[r] \arrow[d] & M' \arrow[dd] \\ Q_i \arrow[d] \arrow[ur, dashed] & \\ R_{ij} \arrow[r] \arrow[uur, dashed] & M. \end{tikzcd}$$

□

Both dotted arrows exist making each of the respective diagrams commute. Therefore the lift $R_{ij} \to M'$ exists making the diagram commute, from which it follows that $\{P \to R_{ij}\}_{i,j}$ is a formally smooth covering family, completing the proof.

We will call this the formally smooth topology on $(\mathfrak{Mon}^{sat})^\sharp$, and the consequence of (5.2.6) is that the topology is subcanonical.

Theorem 5.2.11. *The formally smooth topology is subcanonical.*

Proof. This is the content of Theorem (5.2.6). □

5.3 Formally Étale Families of Sharp and Saturated Monoids

In analogy with the definitions of formally smooth and formally étale morphisms of schemes, if we ask for the lifts appearing in the definition of a formally smooth morphism of monoids to be unique, we arrive at the definition of a formally étale morphism. Let P and Q be monoids, given with morphisms $P \to Q$, and $P \to M$ for a valuative monoid M. We will say that $P \to Q$ is formally étale at $Q \to M$ if the diagram

$$\begin{tikzcd} P \arrow[d] \arrow[dr] & \\ Q \arrow[r] & M \end{tikzcd}$$

commutes and for any valuative monoid M' with a surjective morphism $M' \to M$ and any sharp morphism $P \to M'$ the following diagram of solid arrows

$$\begin{tikzcd} P \arrow[r] \arrow[d] & M' \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & M \end{tikzcd}$$

commutes, and there exists a unique dotted arrow making the diagram commute. Just as we did for formally smooth morphisms, we can extend this notion to a formally étale covering family of monoids.

Definition 5.3.1. A family of monoid morphisms $\{P \to Q_i\}_{i \in I}$ is said to be formally étale if for any sharp morphism $P \to M$ to a valuative monoid, there exists some i such that $P \to Q_i$ is formally étale at $Q_i \to M$.

Again, we may think of the associated group P^{gp} of a monoid P as the cotangent space to **Cone**(P). Recall from algebraic geometry that if a morphism of rings $A \to B$ is étale then it is the case that $B \otimes_A \Omega_A \cong \Omega_B$. We have the analogous statement in the case of monoids.

Lemma 5.3.2. *Let $P \to Q$ be formally étale at $Q \to M$. Then $P^{gp} \cong Q^{gp}$.*

Proof. Let M' be a valuative monoid with a surjective morphism $M' \to M$, and let $P \to M'$ be any sharp morphism such that the diagram of solid arrows

$$\begin{tikzcd} P \arrow[r] \arrow[d] & M' \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & M \end{tikzcd}$$

commutes. Denote by $\mathcal{L}(Q, M')$ the collection of dotted arrows, i.e. lifts, that make the whole diagram commute. Suppose there is some lift $h \in \mathcal{L}(Q, M')$. Let $K = \ker(M'^{gp} \to M^{gp})$. Given any lift $h' \in \mathcal{L}(Q, M')$ we may produce an element $\varphi_{h'} \in \mathrm{Hom}(Q^{gp}/P^{gp}, K)$ by $\varphi_{h'} = h' - h$. This morphism has a two sided inverse given by $\varphi_{h'} \mapsto \varphi_{h'} + h$. Therefore $\mathcal{L}(Q, M') \cong \mathrm{Hom}(Q^{gp}/P^{gp}, K)$ provided that $\mathcal{L}(Q, M') \neq \emptyset$. However by assumption that $P \to Q$ is formally étale at $Q \to M$ we have that $\mathcal{L}(Q, M') = 0$ and hence $\mathrm{Hom}(Q^{gp}/P^{gp}, K) = 0$, which implies that Q^{gp}/P^{gp} is a torsion group since M'^{gp} is torsion free (M' is valuative). By Lemma (5.1.16) Q^{gp}/P^{gp} is torsion free and hence $Q^{gp}/P^{gp} = 0$. □

Theorem 5.3.3. *Let $f : P \to Q$ be formally smooth at $h : Q \to M$ for some valuative monoid M, and assume that $g^{gp} : P^{gp} \to M^{gp}$ is surjective. Then there exists a monoid Q' such that $P \to Q'$ is formally étale at $Q' \to M$, $Q' \to Q$ has a retraction, and $Q \to M$ factors through $Q' \to M$.*

Proof. By (5.1.13) there exists a retraction $s : Q^{gp} \to P^{gp}$ at M^{gp}. Let $Q' = s(Q)$. Then Q' contains P since s is a retraction. Let $h' : Q' \to M$ be the morphism coming from the composition of $h : Q \to M$ with the inclusion $Q' \subseteq Q'$. Then since $h = g \circ s|_Q$, and h is sharp, it follows that Q' is sharp. It is clear that s is a section of $Q' \hookrightarrow Q$; thus we need only show that $P \to Q'$ is formally étale at $Q' \to M'$. We first note that $(Q')^{gp} \cong P^{gp}$. Let $M' \to M$ be surjective with M' valuative and such that we have a commutative diagram of solid arrows

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & M' \arrow[dd] \\
Q' \arrow[ur, dashed] \arrow[d] & \\
Q \arrow[r] \arrow[uur, dashed] & M.
\end{tikzcd}$$

The dotted arrow $Q \to M$ exists and makes the diagram commute by hypothesis, and hence there exists an arrow $Q' \to M'$ making the diagram commute by composition. However, the set $\mathcal{L}$ of lifts $Q' \to M$ is in bijection with the set $\mathrm{Hom}(Q'^{gp}/P^{gp}, K) = 0$, where $K = \ker(M'^{gp} \to M^{gp})$, and hence $\mathcal{L} = 0$ from which it follows that the morphism $P \to Q'$ is formally étale. □

Moreover, it follows immediately that formally étale families generate a subcanonical Grothendieck topology.

Theorem 5.3.4. *On the category* $(\mathfrak{Mon}^{sat})^{\sharp}$ *with sharp morphisms, taking covering families to be formally étale families generates a subcanonical Grothendieck topology.*

Proof. The formally étale topology is subordinate the formally smooth topology. □

5.4 Formal Infinitessimal Smoothness

We have restricted our attention thus far to sharp morphisms to valuative monoids for the lifting criterion that defines the formally smooth topology. Geometrically, this gives us a notion of covering the points that are in the relative interior of the cones. However, we have no way of addressing whether the faces of the cones are covered in this topology as we have defined it – another way to phrase this is that we have topologized sharply monoidal spaces, but not fans. We can move away from sharp morphisms to address the issue of covering the faces of the cones. However, this requires a bit of a tweak to the notion of an infinitessimal extension that we have been using to define our topologies thus far. This slightly modified lifting condition recovers the formally smooth topology if we restrict to sharp morphisms and valuative monoids.

Let M be a valuative monoid. Choose any extension of M^{gp} by an abelian group V. Let M' be a submonoid of V that contains the maximal submonoid M'' of V such that the morphism $M'' \to M$ is sharp. Let $f : P \to Q$, $h : P \to M$, and $g : Q \to M$ be any morphisms of monoids such that $g \circ f = h$ – we are not requiring the morphisms to be sharp anymore.

Definition 5.4.1. With the notation as above, we will say that $P \to Q$ is formally infinitessimally smooth at $Q \to M$ provided for a commutative diagram of solid arrows

$$\begin{tikzcd} P \arrow[r] \arrow[d] & M' \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & M \end{tikzcd}$$

the dotted arrow exists making the whole diagram commute.

The utility of this lifting criterion is that we can base change to a face of the cone and consider whether the family is still covering. This was precluded in the original defintion of a formally smooth family since we were working in the category of sharp and saturated monoids with sharp morphisms. However, if we restrict ourselves to the case that the morphisms are sharp, then the notion of being formally infinitessimally smooth coincides with that of being formally smooth.

Proposition 5.4.2. *Let $f : P \to Q$, $h : P \to M$, and $g : Q \to M$ be sharp morphisms in the category of sharp, saturated monoids with M valuative and such that $g \circ f = h$. Then $P \to Q$ is formally smooth at $Q \to M$ if and only if $P \to Q$ is formally infinitessimally smooth at $Q \to M$.*

Proof. Suppose first that $P \to Q$ is formally smooth at $Q \to M$. Let V be an extension of M with a morphism $P \to V$. Let M' be a valuative submonoid of V containing the image of P and such that the diagram of solid arrows

$$\begin{tikzcd} P \arrow[r] \arrow[d] & M' \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & M \end{tikzcd}$$

commutes. Then the lift $Q \to M'$ exists making the whole diagram commute. In particular, the lift is a sharp morphism and hence the image of Q factors through the maximal submonoid M'' of V such that the morphism $M'' \to M$ induced from $V \to M$ is sharp. Therefore $P \to Q$ is formally infinitessimally smooth at $Q \to M$.

Conversely suppose that $P \to Q$ is formally infinitessimally smooth at $Q \to M$. Let M' be any valuative monoid with a morphism $P \to M'$ and a surjection $M' \to M$ such that the diagram of solid arrows commutes

$$\begin{tikzcd} P \arrow[r] \arrow[d] & M' \arrow[d] \\ Q \arrow[r] \arrow[ur, dashed] & M. \end{tikzcd}$$

Then the dashed arrow exists because M'^{gp} is an extension of M^{gp} and M' contains the maximal submonoid such that the morhpism to M is sharp. It follows that $P \to Q$ is formally smooth at $Q \to M$. $\square$

Example 5.4.3. Let P be the submonoid of $\mathbb{Z}^2$ generated by $x_1 = (1, 0)$ and $x_2 = (0, 1)$, the first quadrant in $\mathbb{Z}^2$. Let $M = \mathbb{N}$, and M' the maximal submonoid of $\mathbb{N} + \epsilon\mathbb{Z}$ such that the projection to $\mathbb{N}$ is sharp; M' is the submonoid of $\mathbb{Z}^2$ generated by $1 + \epsilon n$ for all $n \in \mathbb{Z}$, where when viewed as embedded in $\mathbb{Z}^2$, we have that $1 \mapsto x_1 + x_2$. Let $P \to \mathbb{N}$ by $x_1 \mapsto 1$ and $x_2 \mapsto 1$, i.e. the sum map $(m, n) \mapsto m + n$. Then we naturally get a commutative diagram

$$\begin{tikzcd} P \arrow[r] \arrow[dr] & M' \arrow[d] \\ & \mathrm{M} \end{tikzcd}$$

where $P \to M'$ is just the inclusion. In order for the diagram to commute, we can envision M' as sitting inside of $\mathbb{Z}^2$, as shown in figure 5.5.

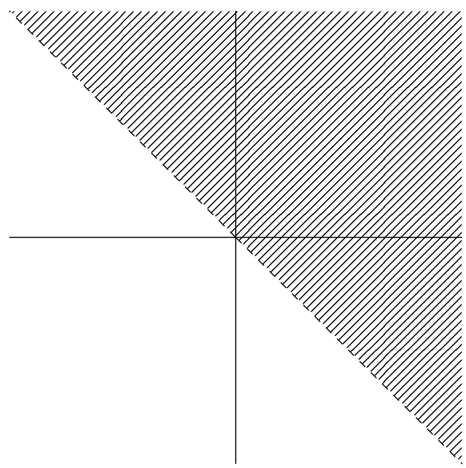


Figure 5.5: The lattice points in the shaded region comprise the maximal submonoid of $\mathbb{N} + \epsilon\mathbb{Z}$ such that the projection to $\mathbb{N}$ is sharp.

On the level of cones, we will have each of the generators $x_1 + x_2 + n\epsilon$ map to 1 in $\mathbb{N}$. Geometrically, the cone of $\mathbb{N}$ will be included inside of the cone of P as the ray through the point $x_1 + x_2 = (1, 1)$, and M' will sit inside of the cone of P. Each inequality $1 + n\epsilon \geq 0$ cuts out a half plane in the cone of P, and hence the cone of M' is the intersection of these half spaces. The picture of the situation on the level of cones is shown in figure 5.6.

Therefore, the infinitessimal lifting criterion can be viewed as lifting infinitessimal motion on

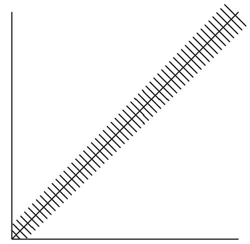


Figure 5.6: The "fuzz" extending from the ray passing through $(1, 1)$ is coming from both the e_1 and e_2 direction.

both sides of a ray inside the interior of the cone. If we base change to a face, i.e. along a morphism that is not sharp, then the infinitessimal motion will "flatten" out onto the face; we are not permitted to infinitessimally move from a face into the relative interior of the cones. Therefore the topology of the interior of the cone and topology of the faces are treated separately. This seems a promising direction for finding a topology that plays well with the algebraic topology (or topologies) that we use when studying, for example, log geometry. The "anticontiuity" between the algebraic and tropical perspectives has up until this point caused some problems with attempts at constructing a tropical topology that plays well with algebraic topologies. However, this infinitessimal lifting topology seems to take care of those issues. See [11] for a discussion on the logarithmic Picard group and its tropicalization. This very issue of not having a sufficient topology prevents us from being able to show that the tropical Picard group and tropical Jacobian of a curve are algebraic. To this point, we have used specific polyhedral subdivisions to show that certain tropical moduli problems are algebraic (e.g. in [3]), but this requires choosing "the right" polyhedral subdivision – effectively, we want to produce a universal curve, but this necessitates polyhedral complexes formed by gluing along face maps to pull back to polyhedral complexes formed by gluing along face maps, i.e. face maps should pull back to face maps. Thus it might be helpful in certain situations to be agnostic, in some sense, about a choice of subdivision, e.g. in the case that face maps pull back to subdivisions of faces. This infinitessimal version of the smooth and étale topologies seems to be promising in this regard.

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