Compositional Synthesis of not Necessarily Stabilizable Stochastic Systems via Finite Abstractions

Abolfazl Lavaei, Sadegh Soudjani, and Majid Zamani

***Abstract*— In this paper, we propose a compositional framework for the construction of finite abstractions (a.k.a. finite Markov decision processes (MDPs)) for networks of not necessarily stabilizable discrete-time stochastic control systems. The proposed scheme is based on a notion of *finite-step* stochastic simulation function, using which one can employ an abstract system as a substitution of the original one in the controller design process with guaranteed error bounds. In comparison with the existing notions of simulation functions, a *finite-step* stochastic simulation function needs to decay only after some *finite* numbers of steps instead of at each time step. In the first part of the paper, we develop a new type of small-gain conditions which are *less* conservative than the existing ones. The proposed condition compositionally quantifies the distance in probability between the interconnection of stochastic control subsystems and that of their (finite or infinite) abstractions. In particular, using this relaxation via finite-step stochastic simulation functions, it is possible to construct finite abstractions such that stabilizability of each subsystem is not necessarily required. In the second part of the paper, for the class of linear stochastic control systems, we construct finite MDPs together with their corresponding finite-step stochastic simulation functions. Finally, we demonstrate the effectiveness of the proposed results by compositionally constructing finite MDP of a network of four subsystems such that one of them is not stabilizable.**

I. Introduction

Controller design to achieve some complex specifications for large-scale interconnected stochastic systems has been inherently a challenging task due to the computational complexity. In order to overcome this challenge, one promising approach is to develop compositional frameworks using notions of stochastic simulation functions. Accordingly, one can first abstract the original system by a simpler one possibly finite or with a lower dimension, design a controller for it, and then refine it to a controller for the concrete system, by providing quantified errors for this detour process.

In order to reduce the complexity of controller synthesis problems, construction of *finite* abstractions was introduced over the last few years. In finite abstractions, each discrete state and input respectively correspond to an aggregate of continuous states and inputs of the original system. Since the abstractions are finite, the algorithmic machineries from computer science [1] are applicable to synthesize controllers for concrete systems.

There have been several results on the construction of (in)finite abstractions for *continuous-time* stochastic systems in the past few years. Infinite approximation techniques for jump-diffusion systems are investigated in [2]. Finite bisimilar abstractions for incrementally stable stochastic switched systems are proposed in [3]. Other existing results include construction of finite abstractions for stochastic control systems without discrete dynamics [4], and for randomly switched stochastic systems [5]. For jump-diffusion systems, compositional construction of infinite abstractions is recently discussed in [6] using small-gain type conditions.

For *discrete-time* stochastic models with continuous-state spaces, construction of finite abstractions for formal verification and synthesis is initially presented in [7]. Extension of such techniques to infinite horizon properties and improvement of the construction algorithms in terms of scalability are proposed in [8] and [9], respectively. Formal abstraction-based policy synthesis and compositional construction of finite abstractions using dynamic Bayesian networks are discussed in [10] and [11], respectively. Compositional construction of infinite abstractions (reduced order models) is proposed in [12] and [13] using small-gain type conditions and dissipativity-type properties of subsystems and their abstractions, respectively. Moreover, compositional finite bisimilar abstractions for networks of stochastic systems are presented in [14] using disturbance bisimulation relations. Recently, compositional synthesis of large-scale stochastic systems using a relaxed dissipativity approach is proposed in [15].

There have been also some results in the context of stability verification for non-stochastic systems. Nonconservative small-gain conditions based on finite-step Lyapunov functions were originally introduced in [16]. Moreover, nonconservative small-gain conditions for closed sets using finite-step ISS Lyapunov functions are presented in [17]. Recently, compositional construction of finite abstractions via relaxed small-gain conditions for discrete-time systems is discussed in [18]. Although the proposed results in [18] employ relaxed small-gain conditions via finite-step ISS Lyapunov functions, their setting is non-stochastic.

Our main contribution is to develop a compositional scheme using relaxed small-gain type conditions for the construction of finite MDPs for networks of discrete-time stochastic control systems. The proposed framework relies on a relation between each subsystem and its abstraction employing a new notion of simulation functions, called *finite-step* stochastic simulation functions. In comparison with the existing notions of simulation functions in which stability or stabilizability of each subsystem is required, a finite-step simulation function needs to decay only after some finite numbers of steps instead of at each time step. This relaxation results in a *less* conservative version of small-gain conditions, using which one can compositionally construct finite MDPs such that stabilizability of each subsystem is not necessarily

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A. Lavaei is with the Department of Electrical and Computer Engineering, Technical University of Munich, Germany. S. Soudjani is with the School of Computing, Newcastle University, United Kingdom. M. Zamani is with the Computer Science Department, University of Colorado Boulder, USA. M. Zamani is with the Computer Science Department, Ludwig Maximilian University of Munich, Germany. Email: lavaei@tum.de, Sadegh.Soudjani@newcastle.ac.uk, majid.zamani@colorado.edu.

required. Proofs of all statements are omitted due to space limitations.

Related literature. Compositional construction of finite MDPs for networks of discrete-time stochastic control systems is recently discussed in [19], [20] and [21], but by using a *classic* simulation function in where stabilizability of each subsystem is required. In general, the proposed compositional approach here is *less* conservative in the sense that the stabilizability of individual subsystems here is not necessarily required.

II. Discrete-Time Stochastic Control Systems

A. Notation

The sets of real numbers, nonnegative and positive integers are denoted by $\mathbb{R}$, $\mathbb{N} := \{0, 1, 2, \ldots\}$, and $\mathbb{N}_{\geq 1} := \{1, 2, 3, \ldots\}$, respectively. We employ $x = [x_1; \ldots; x_N]$ to denote the corresponding vector of dimension $\sum_i n_i$, given N vectors $x_i \in \mathbb{R}^{n_i}$, $n_i \in \mathbb{N}_{\geq 1}$, and $i \in \{1, \ldots, N\}$. We denote by $\|\cdot\|$ the infinity norm. The identity matrix in $\mathbb{R}^{n\times n}$ and the column vector in $\mathbb{R}^{n\times 1}$ with all elements equal to one are denoted by $\mathbb{I}_n$, and $\mathbb{1}_n$, respectively. We denote by $\mathcal{I}_d$ and symbol $\circ$, the identity function and composition of functions, respectively. Given functions $f_i : X_i \rightarrow Y_i$, for any $i \in \{1, \ldots, N\}$, their Cartesian product $\prod_{i=1}^N f_i : \prod_{i=1}^N X_i \rightarrow \prod_{i=1}^N Y_i$ is defined as $(\prod_{i=1}^N f_i)(x_1, \ldots, x_N) = [f_1(x_1); \ldots; f_N(x_N)]$. A function $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$, is said to be a class $\mathcal{K}$ function if it is continuous, strictly increasing, and $\gamma(0) = 0$. A class $\mathcal{K}$ function $\gamma(\cdot)$ is said to be a class $\mathcal{K}_\infty$ if $\gamma(r) \rightarrow \infty$ as $r \rightarrow \infty$.

B. Discrete-Time Stochastic Control Systems

In this work, we are interested in discrete-time stochastic control systems (dt-SCS) defined by the tuple

$$\Sigma = (X, W, U, \varsigma, f), \tag{1}$$

in which the state space of the system $X \subseteq \mathbb{R}^n$ is a Borel space. The measurable space with $\mathcal{B}(X)$, which is the Borel sigma-algebra on the state space, is denoted by $(X, \mathcal{B}(X))$. The *internal* and *external* input spaces of the system are presented by sets $W \subseteq \mathbb{R}^p$ and $U \subseteq \mathbb{R}^m$ which both are Borel spaces. A sequence of independent and identically distributed (i.i.d.) random variables on a set V_ς with sample space Ω is denoted by

$$\varsigma := \{\varsigma(k) : \Omega \rightarrow V_\varsigma,\ k \in \mathbb{N}\}.$$

Moreover, the map $f : X \times W \times U \times V_\varsigma \rightarrow X$ is a measurable function characterizing the state evolution of the system.

Remark 2.1: Since our main contribution is to develop a *compositional* framework, we are interested here in the interconnected discrete-time stochastic control systems without internal inputs with map $f : X \times U \times V_\varsigma \rightarrow X$.

Evolution of the state of dt-SCS Σ in (1), for given initial state $x(0) \in X$ and input sequences $\mathsf{w}(\cdot) : \mathbb{N} \rightarrow W$ and $\nu(\cdot) : \mathbb{N} \rightarrow U$, can be described by

$$\Sigma : \begin{cases} x(k+1) = f(x(k), \mathsf{w}(k), \nu(k), \varsigma(k)), \\ x(\cdot) \in X, \mathsf{w}(\cdot) \in W, \nu(\cdot) \in U, k \in \mathbb{N}. \end{cases} \tag{2}$$

The sets $\mathcal{W}$ and $\mathcal{U}$, respectively, associated to W and U, are collections of sequences $\{\mathsf{w}(k) : \Omega \rightarrow W,\ k \in \mathbb{N}\}$ and $\{\nu(k) : \Omega \rightarrow U,\ k \in \mathbb{N}\}$, in which $\mathsf{w}(k)$ and $\nu(k)$ are independent of $\varsigma(t)$ for any $k, t \in \mathbb{N}$ and $t \geq k$. For any initial state $a \in X$, $\mathsf{w}(\cdot) \in \mathcal{W}$, $\nu(\cdot) \in \mathcal{U}$, the random sequence $x_{a\mathsf{w}\nu} : \Omega \times \mathbb{N} \rightarrow X$, satisfying (2) is called the *solution process* of Σ under initial state a, *internal* input w, and *external* input ν. If X, W, U are finite sets, system Σ is called finite, and infinite otherwise.

For a dt-SCS Σ, we are interested in synthesizing control policies that are Markov.

Definition 2.2: Given the dt-SCS Σ in (2), a Markov policy is a sequence $\mu = (\mu_0, \mu_1, \mu_2, \ldots)$ of universally measurable stochastic kernels μ_n [22]. Given $X \times W$, each stochastic kernel μ_n is defined on the input space U such that for all $(x_n, \mathsf{w}_n) \in X \times W$, $\mu_n(U|(x_n, \mathsf{w}_n)) = 1$. We denote by Π_M the class of all such Markov policies.

In the following subsection, we define the M-sampled systems, based on which one can employ the proposed *finite-step* stochastic simulation function to quantify the mismatch between the interconnected dt-SCS and that of their (in)finite abstractions.

C. M-Sampled Systems

The existing methodologies for compositional (in)finite abstractions of interconnected discrete-time stochastic control systems [12], [13], [19], [20], [21], rely on the assumption of each subsystem to be individually stabilizable. This assumption does not hold in general even if the interconnected system is stabilizable. The main idea behind the *relaxed* small-gain condition proposed in this paper is as follows. We show that the individual stabilizability requirement can be relaxed by incorporating the stabilizing effect of the neighboring subsystems in a local unstabilizable subsystem. Once the stabilizing effect is appeared, we construct abstractions of subsystems and employ small-gain theory to provide compositionality results. Our approach relies on looking at the solution process of the system in future time instances while incorporating the interconnection of subsystems. The following example illustrates this idea.

Example 2.3: Consider two linear dt-SCS Σ_1, Σ_2 with dynamics

$$\begin{aligned} x_1(k+1) &= 1.01x_1(k) + 0.4\mathsf{w}_1(k) + \varsigma_1(k), \\ x_2(k+1) &= 0.55x_2(k) - 0.2\mathsf{w}_2(k) + \varsigma_2(k), \end{aligned} \tag{3}$$

that are connected with the constraint $\mathsf{w}_i = x_{3-i}$, for $i = \{1, 2\}$. For simplicity, these two dt-SCS do not have external inputs, i.e., $\nu_i \equiv 0$ for $i = \{1, 2\}$. Note that the first subsystem is not stable thus not stabilizable as well. By looking at the solution process two steps ahead and considering the interconnection, one can write

$$\begin{aligned} x_1(k+2) = &\,0.94x_1(k) + 0.62w_1(k) + 0.4\varsigma_2(k) \\ &+ 1.01\varsigma_1(k) + \varsigma_1(k+1), \\ x_2(k+2) = &\,0.22x_2(k) - 0.31w_2(k) - 0.2\varsigma_1(k) \\ &+ 0.55\varsigma_2(k) + \varsigma_2(k+1), \end{aligned} \tag{4}$$

which we denote them by $\Sigma_{\mathsf{aux1}}, \Sigma_{\mathsf{aux2}}$ in which $w_i = x_{3-i}$, for $i = \{1, 2\}$. These two subsystems in (4) are now stable. This motivates us to construct abstractions of original subsystems (3) based on auxiliary subsystems (4).

Remark 2.4: Note that after interconnecting the subsystems with each other and propagating the dynamics in the next M-steps, the interconnection topology may change (cf. case study). Then the *internal input* of the auxiliary system (w) may be different from the original one (w).

The main contribution of this paper is to provide a general methodology for compositional synthesis of interconnected dt-SCS with not necessarily stabilizable subsystems, by looking at the solution process M*-step* ahead. For this, we raise the following assumption on the input signal.

Assumption 1: The control input is nonzero only at time instances $\{(k+M-1),\ k=jM, j\in\mathbb{N}\}$.

This assumption helps us in decomposing the network after M transitions after which each subsystem depends only on its own external input. This is essential for decentralized controller design. On the other hand, this assumption restricts the external input to take values only at particular time instances making the controller synthesis problem more conservative.

Next lemma shows how dynamics of the M-sampled systems, call auxiliary system Σ_{aux}, can be acquired.

Lemma 2.5: Suppose we are given N dt-SCS Σ_i defined by

$$\Sigma_i:\begin{cases}x_i(k+1)=f_i(x_i(k),\mathsf{w}_i(k),\nu_i(k),\varsigma_i(k)),\\ x_i(\cdot)\in X_i,\mathsf{w}_i(\cdot)\in W_i,\nu_i(\cdot)\in U_i,k\in\mathbb{N},\end{cases}\tag{5}$$

which are connected in a network with constraints $\mathsf{w}_i=[x_1;\ldots;x_{i-1};x_{i+1};\ldots;x_N],\forall i\in\{1,\cdots,N\}$. Under Assumption 1, the M-sampled systems $\Sigma_{\mathsf{aux}i}$, which are the solutions of Σ_i at time instances $k=jM, j\in\mathbb{N}$, have the dynamics

$$\Sigma_{\mathsf{aux}i}:\begin{cases}x_i(k+M)=\tilde{f}_i(x_i(k),w_i(k),\nu_i(k+M-1),\tilde{\varsigma}_i(k)),\\ x_i(\cdot)\in X_i,w_i(\cdot)\in W_i,\nu_i(\cdot)\in U_i,k=jM,j\in\mathbb{N},\end{cases}\tag{6}$$

where $\tilde{\varsigma}_i(k)$ is a vector containing noise terms as follows:

$$\tilde{\varsigma}_i(k)=[\bar{\varsigma}_1(k);\ldots;\bar{\varsigma}_i^*(k);\ldots;\bar{\varsigma}_N(k)],\bar{\varsigma}_i^*(k)=[\varsigma_i(k);\ldots;\varsigma_i(k+M-1)],\bar{\varsigma}_j(k)=[\varsigma_j(k);\ldots;\varsigma_j(k+M-2)],\forall j\in\{1,\ldots N\},\ j\neq i.\tag{7}$$

Note that some of the noise term in $\tilde{\varsigma}_i(k)$ may be eliminated depending on the interconnection graph, but all the terms are present for a *fully interconnected* network. Proof of Lemma 2.5 is based on recursive application of vector field f_i and utilizing Assumption 1. Computation of vector field $\tilde{f}_i$ is illustrated in the next example on a network of two linear dt-SCS.

Example 2.6: Consider two linear dt-SCS Σ_i with dynamics

$$\begin{aligned}x_1(k+1)&=A_1x_1(k)+D_1\mathsf{w}_1(k)+B_1\nu_1(k)+R_1\varsigma_1(k),\\ x_2(k+1)&=A_2x_2(k)+D_2\mathsf{w}_2(k)+B_2\nu_2(k)+R_2\varsigma_2(k),\end{aligned}\tag{8}$$

connected with constraints $\mathsf{w}_i=x_{3-i}, i\in\{1,2\}$. Matrices $A_i,D_i,B_i,R_i,\ i\in\{1,2\}$, have appropriate dimensions. We can rewrite the given dynamics as

$$x(k+1)=\bar{A}x(k)+\bar{D}\mathsf{w}(k)+\bar{B}\nu(k)+\bar{R}\varsigma(k),$$

with $x=[x_1;x_2],\mathsf{w}=[\mathsf{w}_1;\mathsf{w}_2],\nu=[\nu_1;\nu_2],\varsigma=[\varsigma_1;\varsigma_2]$, where

$$\bar{A}=\mathsf{diag}(A_1,A_2),\bar{D}=\mathsf{diag}(D_1,D_2),\bar{B}=\mathsf{diag}(B_1,B_2),\bar{R}=\mathsf{diag}(R_1,R_2).$$

By applying the interconnection constraints $\mathsf{w}=[\mathsf{w}_1;\mathsf{w}_2]=[x_2;x_1]=C[x_1;x_2]$ with $C=\begin{bmatrix}0 & I\\ I & 0\end{bmatrix}$, we have

$$x(k+1)=(\bar{A}+\bar{D}C)x(k)+\bar{B}\nu(k)+\bar{R}\varsigma(k).$$

Now by looking at the solutions M steps ahead, one gets

$$x(k+M)=(\bar{A}+\bar{D}C)^Mx(k)+\sum_{n=0}^{M-1}(\bar{A}+\bar{D}C)^n\bar{B}\nu(k+M-n-1)+\sum_{n=0}^{M-1}(\bar{A}+\bar{D}C)^n\bar{R}\varsigma(k+M-n-1).$$

After applying Assumption 1 and by partitioning $(\bar{A}+\bar{D}C)^M$ as

$$(\bar{A}+\bar{D}C)^M=\left[\begin{array}{c|c}\tilde{A}_1 & \tilde{D}_1\\ \hline \tilde{A}_2 & \tilde{D}_2\end{array}\right],$$

one can decompose the network and obtain the auxiliary subsystems proposed in (6) as follows:

$$\begin{aligned}x_1(k+M)&=\tilde{A}_1x_1(k)+\tilde{D}_1w_1(k)+B_1\nu_1(k+M-1)+\tilde{R}_1\tilde{\varsigma}_1(k),\\ x_2(k+M)&=\tilde{A}_2x_2(k)+\tilde{D}_2w_2(k)+B_2\nu_2(k+M-1)+\tilde{R}_2\tilde{\varsigma}_2(k),\end{aligned}$$

where $w_i=x_{3-i}$, for $i=\{1,2\}$, are the new internal inputs, $\tilde{\varsigma}_1(k),\tilde{\varsigma}_2(k)$ are defined as in (7) with $N=2$, and $\tilde{R}_1$, $\tilde{R}_2$ are matrices of appropriate dimensions which can be computed based on the matrices in (8). As seen, $\tilde{A}_1$ and $\tilde{A}_2$ are now depend also on D_1,D_2, which may make the pairs $(\tilde{A}_1,B_1)$ and $(\tilde{A}_2,B_2)$ stabilizable.

Remark 2.7: The main idea behind the proposed approach is that we first look at the solutions of the unstabilizable subsystems, during which we connect the subsystems with each other based on their interconnection networks. We go ahead until all subsystems are stabilizable (if possible). Once the stabilizing effect is evident, we decompose the network such that each subsystem is only in terms of its own state, and external input. In contrast to the given original systems, the interconnection topology may change meaning that the internal input of auxiliary system may be different from the original one (cf. case study). Furthermore, external input of the auxiliary systems after doing the M-step analysis is given at instants $k+M-1,\ k=jM,\ j\in\mathbb{N}$. Finally, noise in the auxiliary systems is now a sequence of noises of other subsystems in different time steps depending on the type of interconnection.

In the next section, we first define the notions of *finite-step* stochastic pseudo-simulation and simulation functions to quantify the error in probability between two dt-SCS (with both internal and external inputs) and two interconnected dt-SCS (without internal inputs), respectively. Then we employ dynamical representation of $\widehat{\Sigma}_{\mathsf{aux}}$ to compare interconnections of dt-SCS and those of their abstract counterparts based on finite-step stochastic simulation functions.

III. Finite-Step Stochastic Pseudo-Simulation and Simulation Functions

In this section, we introduce the notion of *finite-step* stochastic pseudo-simulation function (FPSF) for dt-SCS with both internal and external inputs. We also define the notion of *finite-step* stochastic simulation function (FSF) for dt-SCS without internal inputs. We then quantify closeness of two interconnected dt-SCS based on FSF.

Remark 3.1: Simulation function is a *Lyapunov-like* function defined over the Cartesian product of the state spaces, which relates the state trajectory of the abstract system to the state trajectory of the original one such that the mismatch between two systems remains within some guaranteed error bounds. We employ here a notion of finite-step simulation function inspired by the notion of finite-step Lyapunov functions [23].

Definition 3.2: Consider dt-SCS Σ_i and $\widehat{\Sigma}_i$, where $\hat{W}_i \subseteq W_i$ and $\hat{X}_i \subseteq X_i$. A function $V_i : X_i \times \hat{X}_i \to \mathbb{R}_{\geq 0}$ is called a ***finite-step*** stochastic pseudo-simulation function (FPSF) from $\widehat{\Sigma}_i$ to Σ_i if there exist $M \in \mathbb{N}_{\geq 1}$, $\alpha_i, \kappa_i \in \mathcal{K}_\infty$, with $\kappa_i < \mathcal{I}_d$, $\rho_{\mathrm{int}i}, \rho_{\mathrm{ext}i} \in \mathcal{K}_\infty \cup \{0\}$, and constant $\psi_i \in \mathbb{R}_{\geq 0}$, such that for all $k = jM, j \in \mathbb{N}$, $x_i := x_i(k) \in X_i, \hat{x}_i := \hat{x}_i(k) \in \hat{X}_i$,

$$\alpha_i(\|x_i - \hat{x}_i\|) \leq V_i(x_i, \hat{x}_i), \tag{9}$$

and for any $\hat{\nu}_i := \hat{\nu}_i(k + M - 1) \in \hat{U}_i$, there exists $\nu_i := \nu_i(k + M - 1) \in U_i$ such that for any $w_i := w_i(k) \in W_i$ and $\hat{w}_i := \hat{w}_i(k) \in \hat{W}_i$,

$$\mathbb{E}\Big[V_i(x_i(k + M), \hat{x}_i(k + M)) \,\big|\, x_i, \hat{x}_i, w_i, \hat{w}_i, \nu_i, \hat{\nu}_i\Big] \tag{10}$$
$$\leq \max\Big\{\kappa_i(V_i(x_i, \hat{x}_i)), \rho_{\mathrm{int}i}(\|w_i - \hat{w}_i\|), \rho_{\mathrm{ext}i}(\|\hat{\nu}_i\|), \psi_i\Big\}.$$

We denote by $\widehat{\Sigma}_i \preceq_{\mathcal{FPS}} \Sigma_i$ if there exists an FPSF V_i from $\widehat{\Sigma}_i$ to Σ_i. We drop the term *finite-step* for the case $M = 1$, and instead call it a *classic* simulation function, which is identical to the ones defined in [19], [20], [21].

Remark 3.3: Note that κ_i defined in (10) depends on M and is required to be less than $\mathcal{I}_d$. FPSF V_i here is *less* conservative than the *classic* simulation function defined in [19], [20], [21]. In other words, condition (10) may not be satisfied for $M = 1$ but may hold for some $M \in \mathbb{N}_{>1}$. Such a dependency on M increases the class of systems for which the condition (10) is satisfiable. This relaxation allows some of the individual subsystems to be even unstabilizable.

Remark 3.4: In Definition 3.2, second condition implicitly implies existence of an *interface function* $\nu_i(k+M-1) = \nu_{\hat{\nu}_i}(x_i(k), \hat{x}_i(k), \hat{\nu}_i(k+M-1))$, for all $k = jM, j \in \mathbb{N}$, satisfying inequality (10). This function is employed to refine a synthesized policy $\hat{\nu}_i$ for $\widehat{\Sigma}_i$ to a policy ν_i for Σ_i.

Definition 3.2 can also be stated for systems without internal inputs by eliminating all the terms related to $w, \hat{w}$, as the next definition.

Definition 3.5: Consider two dt-SCS Σ and $\widehat{\Sigma}$ without internal input, where $\hat{X} \subseteq X$. A function $V : X \times \hat{X} \to \mathbb{R}_{\geq 0}$ is called a ***finite-step*** stochastic simulation function (FSF) from $\widehat{\Sigma}$ to Σ if there exists $M \in \mathbb{N}_{\geq 1}$, and $\alpha \in \mathcal{K}_\infty$ such that

$$\forall x(k) := x \in X, \hat{x}(k) := \hat{x} \in \hat{X},\ \alpha(\|x - \hat{x}\|) \leq V(x, \hat{x}), \tag{11}$$

and $\forall x(k) := x \in X$, $\forall \hat{x}(k) := \hat{x} \in \hat{X}$, $\forall \hat{\nu}(k+M-1) := \hat{\nu} \in \hat{U}$, $\exists \nu(k+M-1) := \nu \in U$ such that

$$\mathbb{E}\Big[V(x(k + M), \hat{x}(k + M)) \,\big|\, x, \hat{x}, \nu, \hat{\nu}\Big]$$
$$\leq \max\Big\{\kappa(V(x, \hat{x})), \rho_{\mathrm{ext}}(\|\hat{\nu}\|), \psi\Big\}, \tag{12}$$

for some $\kappa \in \mathcal{K}_\infty$ with $\kappa < \mathcal{I}_d$, $\rho_{\mathrm{ext}} \in \mathcal{K}_\infty \cup \{0\}$, $\psi \in \mathbb{R}_{\geq 0}$, and $k = jM, j \in \mathbb{N}$.
We call $\widehat{\Sigma}$ an abstraction of Σ, and denote by $\widehat{\Sigma} \preceq_{\mathcal{FSF}} \Sigma$ if there exists an FSF V from $\widehat{\Sigma}$ to Σ.

Next theorem is borrowed from [12, Theorem 3.3], and shows how FSF can be employed to compare the state trajectories of two dt-SCS (without internal inputs) in a probabilistic setting. The theorem also holds for our setting here since our max form implies the additive form employed in [12, Theorem 3.3].

Theorem 3.6: Let Σ and $\widehat{\Sigma}$ be two dt-SCS without internal input, where $\hat{X} \subseteq X$. Suppose V is an FSF from $\widehat{\Sigma}$ to Σ at the times $k = jM, j \in \mathbb{N}$, and there exists a constant $0 < \hat{\kappa} < 1$ such that the function $\kappa \in \mathcal{K}_\infty$ in (12) satisfies $\kappa(r) \geq \hat{\kappa} r$, $\forall r \in \mathbb{R}_{\geq 0}$. For any random variables a and $\hat{a}$ as the initial states of the two dt-SCS, and for any external input trajectory $\hat{\nu}(\cdot) \in \hat{\mathcal{U}}$ that preserves Markov property for the closed-loop $\widehat{\Sigma}$, there exists an input trajectory $\nu(\cdot) \in \mathcal{U}$ of Σ through the interface function associated with V such that the following inequality holds:

$$\mathbb{P}\left\{\sup_{k=jM,\, 0\leq j\leq T_d} \|x_{a\nu}(k) - \hat{x}_{\hat{a}\hat{\nu}}(k)\| \geq \varepsilon \,|\, [a;\hat{a}]\right\} \tag{13}$$
$$\leq \begin{cases} 1-(1-\frac{V(a,\hat{a})}{\alpha(\varepsilon)})(1-\frac{\widehat{\psi}}{\alpha(\varepsilon)})^{T_d} & \text{if } \alpha(\varepsilon)\geq\frac{\widehat{\psi}}{\hat{\kappa}}, \\ (\frac{V(a,\hat{a})}{\alpha(\varepsilon)})(1-\hat{\kappa})^{T_d}+(\frac{\widehat{\psi}}{\hat{\kappa}\alpha(\varepsilon)})(1-(1-\hat{\kappa})^{T_d}) & \text{if } \alpha(\varepsilon)<\frac{\widehat{\psi}}{\hat{\kappa}}, \end{cases}$$

where the constant $\widehat{\psi} \geq 0$ satisfies $\widehat{\psi} \geq \rho_{\mathrm{ext}}(\|\hat{\nu}\|_\infty) + \psi$.

IV. Compositional Abstractions for Interconnected Systems

In this section, we assume that we are given a complex stochastic control system Σ composed of $N \in \mathbb{N}_{\geq 1}$ discrete-time stochastic control subsystems Σ_i as in (5), where their internal inputs are partitioned as

$$\mathsf{w}_i = [\mathsf{w}_{i1}; \ldots; \mathsf{w}_{i(i-1)}; \mathsf{w}_{i(i+1)}; \ldots; \mathsf{w}_{iN}], \tag{14}$$

where $\mathsf{w}_{ij}, \forall i, j \in \{1, \ldots, N\}, i \neq j$, are the states of other subsystems as appeared in the interconnection constraint (15). Now we are ready to define the *interconnected* stochastic control systems.

Definition 4.1: Suppose we are given $N \in \mathbb{N}_{\geq 1}$ discrete-time stochastic control subsystems Σ_i, $i \in \{1, \ldots, N\}$, with the internal input configuration as in (14). The interconnection of Σ_i for any $i \in \{1, \ldots, N\}$, denoted by $\mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$, is the interconnected stochastic control system Σ, such that $X := \prod_{i=1}^N X_i$, $U := \prod_{i=1}^N U_i$, and function $f := \prod_{i=1}^N f_i$, subjected to the following constraint:

$$\forall i, j \in \{1, \ldots, N\}, i \neq j: \quad \mathsf{w}_{ij} = x_j, \quad X_j \subseteq W_{ij}. \tag{15}$$

Suppose we are given $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems (5) together with their corresponding finite abstractions $\widehat{\Sigma}_i$, where $\hat{W}_i \subseteq W_i$ and $\hat{X}_i \subseteq X_i$, in which V_i is an FPSF from $\widehat{\Sigma}_i$ to Σ_i with the corresponding functions and constant denoted by α_i, κ_i, $\rho_{\mathrm{int}i}$, $\rho_{\mathrm{ext}i}$, and ψ_i. Prior to presenting the next theorem, we raise the following small-gain assumption.

Assumption 2: Assume that $\mathcal{K}_\infty$ functions κ_{ij} defined as

$$\kappa_{ij}(r) := \begin{cases} \kappa_i(r) & \text{if } i = j \\ \rho_{\mathrm{int}i}(\alpha_j^{-1}(r)) & \text{if } i \neq j, \end{cases}$$

satisfy

$$\kappa_{i_1 i_2} \circ \kappa_{i_2 i_3} \circ \cdots \circ \kappa_{i_{r-1} i_r} \circ \kappa_{i_r i_1} < \mathcal{I}_d \tag{16}$$

for all sequences $(i_1, \ldots, i_r) \in \{1, \ldots, N\}^r$ and $r \in \{1, \ldots, N\}$. Note that small-gain condition (16) implies the existence of $\mathcal{K}_\infty$ functions $\sigma_i > 0$ [24, Theorem 5.5], satisfying

$$\max_{i,j}\Big\{\sigma_i^{-1} \circ \kappa_{ij} \circ \sigma_j\Big\} < \mathcal{I}_d, \qquad i, j = \{1, \ldots, N\}. \tag{17}$$

In the next theorem, we leverage small-gain Assumption 2 together with concavity assumption of $\max_i \sigma_i^{-1}$ to show the main compositionality result of the paper.

Theorem 4.2: Suppose we are given the interconnected dt-SCS $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ induced by $N \in \mathbb{N}_{\geq 1}$ stochastic control subsystems Σ_i. Let each Σ_i admits an abstraction $\widehat{\Sigma}_i$ with the corresponding FPSF V_i. If Assumption 2 holds and also

$$\forall i, j \in \{1, \ldots, N\}, i \neq j: \quad \hat{W}_{ij} = \hat{X}_j,$$

then function $V(x, \hat{x})$ defined as

$$V(x, \hat{x}) := \max_i \{\sigma_i^{-1}(V_i(x_i, \hat{x}_i)\}, \tag{18}$$

for σ_i as in (17), is an FSF function from $\widehat{\Sigma} = \mathcal{I}(\widehat{\Sigma}_1, \ldots, \widehat{\Sigma}_N)$ to $\Sigma = \mathcal{I}(\Sigma_1, \ldots, \Sigma_N)$ at the times $k = jM, j \in \mathbb{N}$ provided that $\max_i \sigma_i^{-1}$ is concave.

V. Construction of Finite Abstractions

In the previous sections, we considered Σ_i and $\widehat{\Sigma}_i$ as general discrete-time stochastic control systems without considering the cardinality of their state spaces. In this section, Σ_i and $\widehat{\Sigma}_i$ are considered as an infinite dt-SCS and its finite abstraction, respectively. We first discuss construction of finite MDPs as abstractions of dt-SCS, and then impose conditions on the infinite dt-SCS $\Sigma_{\mathsf{aux}i}$ in order to find an FPSF from $\widehat{\Sigma}_i$ to Σ_i. In particular, we focus on the construction of finite MDPs whose state trajectories are close to those of the concrete system at times $k = jM, j \in \mathbb{N}$, for some $M \in \mathbb{N}_{\geq 1}$.

A. Finite Abstractions of dt-SCS

Given a dt-SCS Σ_{aux} in (6), we construct its finite abstraction $\widehat{\Sigma}_{\mathsf{aux}}$ following the approach of [9]. The abstraction algorithm works based on selecting finite partitions of state and input sets as

$$X = \cup_{i=1}^{n_x} \mathsf{X}_i, \quad W = \cup_{i=1}^{n_w} \mathsf{W}_i, \quad U = \cup_{i=1}^{n_u} \mathsf{U}_i,$$

and selecting representative points $\bar{x}_i \in \mathsf{X}_i$, $\bar{w}_i \in \mathsf{W}_i$, and $\bar{\nu}_i \in \mathsf{U}_i$. It then constructs finite sets $\hat{X} = \{\bar{x}_i, i = 1, \ldots, n_x\}, \hat{W} = \{\bar{w}_i, i = 1, \ldots, n_w\}$, and $\hat{U} = \{\bar{u}_i, i = 1, \ldots, n_u\}$, that contain representative points of the partition sets as abstract states and inputs. Dynamics over the abstract states are defined via function $\hat{f} : \hat{X} \times \hat{W} \times \hat{U} \times V_\varsigma \to \hat{X}$ as

$$\hat{f}(\hat{x}(k), \hat{w}(k), \hat{\nu}(k{+}M{-}1), \tilde{\varsigma}(k)) = \Pi_x(\tilde{f}(\hat{x}(k), \hat{w}(k), \hat{\nu}(k{+}M{-}1), \tilde{\varsigma}(k))), \tag{19}$$

where $\Pi_x : X \to \hat{X}$ is the map that assigns to any $x \in X$, the representative point $\hat{x} \in \hat{X}$ of the corresponding partition set containing x. The initial state of $\widehat{\Sigma}_{\mathsf{aux}}$ is also selected according to $\hat{x}_0 := \Pi_x(x_0)$ with x_0 being the initial state of Σ_{aux}.

Abstraction map Π_x employed in (19) satisfies

$$\|\Pi_x(x) - x\| \leq \delta, \ \forall x \in X, \tag{20}$$

where δ is the *state* discretization parameter defined as $\delta := \sup\{\|x - x'\|, \ x, x' \in \mathsf{X}_i, \, i = 1, 2, \ldots, n_x\}$.

B. Construction of Finite Abstractions

In this subsection, we focus on the class of linear dt-SCS. Suppose we are given a network composed of N linear discrete-time stochastic control subsystems as follows:

$$\Sigma_i : x_i(k{+}1) = A_i x_i(k) {+} D_i \mathsf{w}_i(k) {+} B_i \nu_i(k) {+} R_i \varsigma_i(k), \tag{21}$$

where the additive noise $\varsigma_i(k)$ is a sequence of independent random vectors with multivariate standard normal distributions. Suppose w_i is partitioned as (14). Let $M \in \mathbb{N}_{\geq 1}$ be given. By employing the interconnection constraint (15) and Assumption 1, the dynamic of the sampled system at M*-step* forward can be written as

$$\Sigma_{\mathsf{aux}i} : x_i(k{+}M) = \tilde{A}_i x_i(k) + \tilde{D}_i w_i(k) + B_i \nu_i(k{+}M{-}1) + \tilde{R}_i \tilde{\varsigma}_i(k),$$

where $\tilde{\varsigma}_i(k)$ for the fully interconnected network is obtained as in (7). Although the pairs (A_i, B_i) may not be necessarily stabilizable, we assume that the pairs $(\tilde{A}_i, B_i)$ after M-*step* are stabilizable as discussed in Example 2.3. Therefore, we can construct the finite Markov decision processes as presented in Section V-A from the new auxiliary system. To do so, take the following simulation function candidate from $\widehat{\Sigma}_{\mathsf{aux}i}$ to $\Sigma_{\mathsf{aux}i}$

$$V_i(x_i, \hat{x}_i) = ((x_i - \hat{x}_i)^T \tilde{M}_i (x_i - \hat{x}_i))^{\frac{1}{2}}, \tag{22}$$

where $\tilde{M}_i$ is a positive-definite matrix of appropriate dimension. In order to show that V_i in (22) is an FPSF from $\widehat{\Sigma}_i$ to Σ_i, we require the following assumption on $\Sigma_{\mathsf{aux}i}$.

Assumption 3: Assume that there exist matrices $\tilde{M}_i \succ 0$, and K_i of appropriate dimensions such that the matrix inequality

$$(1 + 2\pi_i)(\tilde{A}_i + B_i K_i)^T \tilde{M}_i (\tilde{A}_i + B_i K_i) \preceq \hat{\kappa}_i \tilde{M}_i, \tag{23}$$

holds for some constant $0 < \hat{\kappa}_i < 1$ and $\pi_i > 0$.

Now, we raise the main result of this subsection.

Theorem 5.1: Assume system $\Sigma_{\mathsf{aux}i}$ satisfies Assumption 3. Let $\widehat{\Sigma}_{\mathsf{aux}i}$ be its finite abstraction as described in Subsection V-A with state discretization parameter δ_i. Then function V_i defined in (22) is an FPSF from $\widehat{\Sigma}_i$ to Σ_i.

VI. Case Study

In this section, we demonstrate the effectiveness of the proposed results by considering an interconnected system composed of four discrete-time linear stochastic control subsystems, i.e. $\Sigma = \mathcal{I}(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$, such that one of them is not stabilizable. The discrete-time linear stochastic control subsystems are given by

$$\Sigma_i : \begin{cases} x_1(k+1) = 1.001 x_1(k) + 0.4 \mathsf{w}_1(k) + \varsigma_1(k), \\ x_2(k+1) = -0.95 x_2(k) - 0.08 \mathsf{w}_2(k) + \nu_2(k) + \varsigma_2(k), \\ x_3(k+1) = -0.94 x_3(k) - 0.05 \mathsf{w}_3(k) + \nu_3(k) + \varsigma_3(k), \\ x_4(k+1) = 0.6 x_4(k) + 0.9 \mathsf{w}_4(k) + \nu_4(k) + \varsigma_4(k), \end{cases} \tag{24}$$

where

$$\mathsf{w}_1 = x_2 + x_3, \mathsf{w}_2 = x_1 + x_3, \mathsf{w}_3 = x_2, \mathsf{w}_4 = x_3.$$

As seen, the first subsystem is not stabilizable. Then we proceed with looking at the solution of Σ_i two steps ahead, i.e., $M = 2$,

$$\Sigma_{\mathsf{aux}i} : \begin{cases} x_1(k{+}2) = 0.97 x_1(k) + \tilde{D}_1 w_1(k) + \tilde{R}_1 \tilde{\varsigma}_1(k), \\ x_2(k{+}2) = 0.8745 x_2(k) + \tilde{D}_2 w_2(k) + \nu_2(k{+}1) \\ \qquad + \tilde{R}_2 \tilde{\varsigma}_2(k), \\ x_3(k{+}2) = 0.8876 x_3(k) + \tilde{D}_3 w_3(k) + \nu_3(k{+}1) \\ \qquad + \tilde{R}_3 \tilde{\varsigma}_3(k), \\ x_4(k{+}2) = 0.36 x_4(k) + \tilde{D}_4 w_4(k) + \nu_4(k{+}1) \\ \qquad + \tilde{R}_4 \tilde{\varsigma}_4(k), \end{cases} \tag{25}$$

where

$\tilde{D}_1=[-0.0004;-0.0076]^T, \tilde{D}_2=[-0.0041;0.1192]^T,$
$\tilde{D}_3=[0.004;0.0945]^T, \tilde{D}_4=[-0.045;-0.306]^T,$
$w_1{=}[x_2;x_3], w_2{=}[x_1;x_3], w_3{=}[x_1;x_2], w_4{=}[x_2;x_3],$
$\tilde{\varsigma}_1(k){=}[\varsigma_3(k);\varsigma_2(k);\varsigma_1(k);\varsigma_1(k{+}1)], \tilde{\varsigma}_3(k){=}[\varsigma_2(k);\varsigma_3(k);\varsigma_3(k{+}1)],$
$\tilde{\varsigma}_2(k){=}[\varsigma_3(k);\varsigma_1(k);\varsigma_2(k);\varsigma_2(k{+}1)], \tilde{\varsigma}_4(k){=}[\varsigma_3(k);\varsigma_4(k);\varsigma_4(k{+}1)].$

Moreover, $\tilde{R}_i=[\tilde{R}_{i1};\tilde{R}_{i2};\tilde{R}_{i3};\tilde{R}_{i4}]^T\,\forall i\in\{1,2\}$, where

$$\tilde{R}_{11}=0.4, \tilde{R}_{12}=0.4, \tilde{R}_{13}=1.001, \tilde{R}_{14}=1,$$
$$\tilde{R}_{21}=-0.08, \tilde{R}_{22}=-0.08, \tilde{R}_{23}=-0.95, \tilde{R}_{24}=1.$$

and $\tilde{R}_i=[\tilde{R}_{i1};\tilde{R}_{i2};\tilde{R}_{i3}]^T\,\forall i\in\{3,4\}$, where
$\tilde{R}_{31}{=}-0.05, \tilde{R}_{32}{=}-0.941, \tilde{R}_{33}{=}1, \tilde{R}_{41}{=}0.9, \tilde{R}_{42}{=}0.6, \tilde{R}_{43}{=}1.$

One can readily see that $\tilde{A}_1$ is stable. Now, we proceed with constructing the finite Markov decision processes from the M-sampled systems as acquired in (25). We fix FPSF as $V_i(x_i,\hat{x}_i)=(x_i-\hat{x}_i)^T\tilde{M}_i(x_i-\hat{x}_i)$. One can readily verify that condition (23) is satisfied with
$\hat{\kappa}_1=0.9597, \hat{\kappa}_2=0.588, \hat{\kappa}_3=0.7115, \hat{\kappa}_4=0.337,$
$K_2=-0.1745, K_3=-0.1176, K_4{=}0, \pi_1=0.01, \pi_2{=}0.1,$
$\pi_3=0.1, \pi_4=0.8, \tilde{M}_i=1, \quad \forall i\in\{1,2,3,4\}.$

Then function $V_i(x_i,\hat{x}_i)=(x_i-\hat{x}_i)^2$ is an FPSF from $\widehat{\Sigma}_i$ to Σ_i satisfying condition (9) with $\alpha_i(s)=s^2, \forall i\in\{1,2,3,4\}$ $\forall s\in\mathbb{R}_{\geq 0}$, and condition (10) with
$\kappa_i(s){=}0.99s, \rho_{\text{ext}i}(s){=}0, \forall i{\in}\{1,2,3,4\}, \rho_{\text{int}1}(s){=}0.8802s^2,$
$\rho_{\text{int}2}(s){=}0.8517s^2, \rho_{\text{int}3}(s){=}0.8344s^2, \rho_{\text{int}4}(s){=}0.9779s^2,$
$\psi_1=7409\,\delta^2, \psi_2=555\,\delta^2, \psi_3=433\,\delta^2, \psi_4=57.48\,\delta^2.$

Now we check small-gain condition (16) that is required for the compositionality result. By taking $\sigma_i(s)=s\ \forall i\in\{1,2,3,4\}$, one can readily verify that the small-gain condition (16) and as a result condition (17) are satisfied. Hence, $V(x,\hat{x})=\max_i(x_i-\hat{x}_i)^2$ is an FSF from $\widehat{\Sigma}$ to Σ satisfying conditions (11) and (12) with $\alpha(s)=s^2$, $\kappa(s)=0.99\,s$, $\rho_{\text{ext}}(s)=0$, $\forall s\in\mathbb{R}_{\geq 0}$, and $\psi=7409\,\delta^2$.

By taking the state set discretization parameter $\delta=0.001$, and starting the initial states of the interconnected systems Σ and $\widehat{\Sigma}$ from $\mathbb{1}_4$ and employing Theorem 3.6, we guarantee that the distance between states of Σ and of $\widehat{\Sigma}$ will not exceed $\varepsilon=1$ at the times $k=2j, j=\{0,\ldots,30\}$ with probability at least 90%, i.e.

$$\mathbb{P}(\|x_{a\nu}(k)-\hat{x}_{\hat{a}\hat{\nu}}(k)\|\leq 1,\ \forall k{=}2j, j{=}\{0,\ldots,30\})\geq 0.9.$$

Note that for the construction of finite abstractions, we have selected the center of partition sets as representative points.

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