Survey of the theory of Fourier transforms on $\mathbf{R}^n$, $\mathbf{H}^2$, and homogeneous spaces of semisimple Lie groups

by

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Abstract

The Fourier transform on $\mathbf{R}^n$ has many useful properties that prove to be useful in studying major problems arising in analysis—such as those arising in the study of differential equations. One can also develop the Fourier transform for abelian or compact locally compact Hausdorff groups, which shares many of the same remarkable properties of the Fourier transform on $\mathbf{R}^n$. Even further, the Fourier transform can be defined on homogeneous spaces $X = G/K$ where G is a connected noncompact semisimple Lie group with finite center and K is a maximal compact subgroup. The purpose of this paper is to survey the major properties and features of the Fourier transforms on these spaces, at the level of a student familiar with real analysis. We also explore how one can define and study pseudo-differential operators on such homogeneous spaces $X = G/K$.

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Contents

Chapter 1

Fourier transforms on $\mathbf{R}^n$ and on topological groups

The Fourier transform on the Euclidean space $\mathbf{R}^n$ is an immensely powerful tool used to study a variety of problems arising in areas of mathematical analysis, particularly in partial differential equations and harmonic analysis. In fact, there exists a generalization of the Fourier transform which can be defined for any locally compact group—and in the case that the locally compact group is abelian or compact the Fourier theory is well-behaved. The purpose of this chapter is to recall the basic properties of the Fourier transform on $\mathbf{R}^n$ as well as give a cursory discussion of the Fourier transform theory for locally compact groups.

Since the notion of a Fourier transform on a smooth manifold is not defined in general, our purpose in this chapter is to identify the so-called generic features of the Fourier transform on $\mathbf{R}^n$ (and on any locally compact group that is abelian or compact) which we ought to expect of any integral transform on a smooth manifold which generalizes or extends the Fourier transform to that manifold in a suitable sense. In Chapters 2 and 3 we will describe a type of integral transform which can be defined on a certain class of manifolds which captures these generic features very well and thus arguably provides a "correct" generalization of the Fourier transform to those particular manifolds.

In Section 1.1, we shall provide a non-exhaustive overview of the Fourier transform on $\mathbf{R}^n$; covering the theory of the Fourier transform on $L^1(\mathbf{R}^n)$, $L^2(\mathbf{R}^n)$, the space of test functions $C_c^\infty(\mathbf{R}^n)$, the Schwartz functions $\mathcal{S}(\mathbf{R}^n)$, and on the space of tempered distributions $\mathcal{S}'(\mathbf{R}^n)$. We shall for the most part provide all the necessary proofs for they are often concise and often they highlight techniques that are used later in this thesis—although in certain cases we omit a few proofs. In Section 1.2, we then discuss the Fourier transform on locally compact groups that are abelian or compact. Here we shall provide fewer proofs in the interest of length and readability. Finally, in Section 1.3 we very briefly highlight some properties of the Fourier transform on certain compact Lie groups, namely on the torus $\mathbf{T}$ and on compact semisimple Lie groups. There the Fourier transform interacts quite nicely with differential operators on the group as does the Fourier transform on $\mathbf{R}^n$.

Preliminaries and notation

The symbols $\mathbf{R}$, $\mathbf{C}$, $\mathbf{N}$, $\mathbf{N}_0$, and $\mathbf{Z}$ denote the space of real numbers, complex numbers, the positive integers, the nonnegative integers, and the integers respectively. We also write $\mathbf{R}^+ = \{x \in \mathbf{R} \colon x > 0\}$ for the multiplicative group of positive real numbers. By a multi-index α on $\mathbf{R}^n$ we mean an element $\alpha \in \mathbf{N}_0^n$. If α is a multi-index we employ the notation that $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ if $x \in \mathbf{R}^n$ and that the symbol ∂^α denotes the differential operator $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$. The *order* or *absolute value* of a multi-index α is the number $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$.

Recall that for $1 \leq p < \infty$ the space $L^p(\mathbf{R}^n)$ is the set of measurable functions f on $\mathbf{R}^n$ satisfying $\|f\|_p = \left[\int_{\mathbf{R}^n} |f(x)|^p \, dx\right]^{1/p} < \infty$ where dx is the Lebesgue measure. The $L^p(\mathbf{R}^n)$ spaces are Banach spaces and the norm on these spaces is denoted by $\|\cdot\|_p$. On $L^1(\mathbf{R}^n)$ we can define a multiplication called *convolution* where for each $f, g \in L^1(\mathbf{R}^n)$ their convolution $f * g$ is defined by

$$f * g(x) = \int_{\mathbf{R}^n} f(y)g(x-y)\,dy\,.$$

That this function is also integrable comes from the estimate $\|f * g\|_1 \leq \|f\|_1\|g\|_1$ which in fact shows that convolution on $L^1(\mathbf{R}^n)$ is continuous. This turns $L^1(\mathbf{R}^n)$ into a commutative Banach algebra under convolution. We can add an *involution* on $L^1(\mathbf{R}^n)$ where the involution of an integrable function f is defined by $f^*(x) = \overline{f(-x)}$. Together with these operations, $L^1(\mathbf{R}^n)$ becomes what we call a *Banach *-algebra*. We also have the space $L^\infty(\mathbf{R}^n)$ which is the space of essentially bounded functions on $\mathbf{R}^n$. This space is too a Banach space with norm denoted by $\|\cdot\|_\infty$.

The spaces $C_c^\infty(\mathbf{R}^n) = \mathcal{D}(\mathbf{R}^n)$ and $C^\infty(\mathbf{R}^n) = \mathcal{E}(\mathbf{R}^n)$ will denote the sets of compactly supported smooth functions and the smooth functions on $\mathbf{R}^n$, respectively. We shall use the notations $C_c^\infty(\mathbf{R}^n)$ and $\mathcal{D}(\mathbf{R}^n)$ (as well as $C^\infty(\mathbf{R}^n)$ and $\mathcal{E}(\mathbf{R}^n)$) which denote the same respective spaces and we shall use those notations interchangeably. The space $C^m(\mathbf{R}^n)$ will denote the space of functions with continuous derivatives up to order m and the space $C_0(\mathbf{R}^n)$ will denote the space of continuous functions which vanish at infinity.

1.1 Fourier transform on $\mathbf{R}^n$

The Fourier transform on $\mathbf{R}^n$ is defined for any integrable function by the following formula.

Definition 1.1.1. For $f \in L^1(\mathbf{R}^n)$ define the *Fourier transform* of f as the function $\widehat{f}$ on $\mathbf{R}^n$ which is given by the following formula

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-i\langle x,\xi\rangle}\,dx\,. \tag{1.1.1}$$

Here, the bracket $\langle\cdot,\cdot\rangle$ is the Euclidean inner product (dot product) on $\mathbf{R}^n$. We note that the integral in (1.1.1) converges everywhere since $f \in L^1(\mathbf{R}^n)$ and so that the Fourier transform of an integrable function is well-defined. We will sometimes write $\mathscr{F}$ to denote the map $f \mapsto \widehat{f}$. Correspondingly, we have the so-called inverse Fourier transform which is defined as follows.

Definition 1.1.2. For $f \in L^1(\mathbf{R}^n)$ define the *inverse Fourier transform* of f by

$$\mathscr{F}^{-1}f(x) = \check{f}(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} f(\xi)e^{i\langle x,\xi\rangle}\,d\xi = \frac{1}{(2\pi)^n}\mathscr{F}f(-x). \tag{1.1.2}$$

There are several properties of the Fourier transform that we will find quite useful which are contained in the following proposition.

Proposition 1.1.1. *Suppose that $f, g \in L^1(\mathbf{R}^n)$.*

1. $\|\widehat{f}\|_\infty \leq \|f\|_1$.
2. *If $f \in C^m(\mathbf{R}^n)$ and $\partial^\alpha f \in C_0(\mathbf{R}^n) \cap L^1(\mathbf{R}^n)$ for $|\alpha| \leq m-1$ and $\partial^\alpha f \in L^1(\mathbf{R}^n)$ for $|\alpha| = m$, then $\widehat{\partial^\alpha f} = (i\xi)^\alpha \widehat{f}(\xi)$.*
3. *If $x^\alpha f(x) \in L^1(\mathbf{R}^n)$, then $\partial^\alpha \widehat{f}(\xi) = \mathscr{F}((-ix)^\alpha f(x))(\xi)$.*
4. $\int_{\mathbf{R}^n} \widehat{f}(x)g(x)\,dx = \int_{\mathbf{R}^n} f(x)\widehat{g}(x)\,dx$.
5. *If $y \in \mathbf{R}^n$ and we put $\tau_y f(x) = f(x+y)$, then $\widehat{\tau_y f}(\xi) = e^{i\langle y,\xi\rangle}\widehat{f}(\xi)$. Also the Fourier transform of $x \mapsto e^{i\langle x,\eta\rangle}f(x)$ is $\widehat{f}(\xi-\eta)$.*

Proof. (1) follows from the elementary estimate $|\widehat{f}(\xi)| \leq \int_{\mathbf{R}^n} |f(x)|\, dx = \|f\|_1$. For (2) if $|\alpha| = 1$, then by integration by parts

$$\int_{\mathbf{R}^n} \partial^\alpha f(x) e^{-i\langle x,\xi\rangle}\, dx = f(x)e^{-i\langle x,\xi\rangle}\Big|_{-\infty}^{\infty} - \int_{\mathbf{R}^n} (-i\xi)^\alpha f(x) e^{-i\langle x,\xi\rangle}\, dx = (i\xi)^\alpha \widehat{f}(\xi).$$

By induction we obtain the result for $|\alpha| \leq m$ as well. For (3), this is a case of differentiation under the integral sign using the dominated convergence theorem. Finally for (4) note that both integrals are equal to

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(x)g(y)e^{-i\langle x,y\rangle}\, dx\, dy,$$

by Fubini's theorem. Lastly for (5), the first part follows easily by a standard change of variables in the integral defining the Fourier transform and second part is obvious. □

In essence, if $x^\alpha f \in L^1(\mathbf{R}^n)$ for large orders of $|\alpha|$ the more smooth the Fourier transform $\widehat{f}$ is. Conversely, the more smooth f is (along with a certain decay conditions on its derivatives at infinity), the quicker $\widehat{f}$ decays in ξ. Indeed, assuming (2) in the previous theorem we obtain that $|\widehat{f}(\xi)| \leq |\xi^\alpha|^{-1}\|\partial^\alpha f\|_1$ for ξ away from zero. Thus the greater the order of α, the faster f decays as $|\xi| \to \infty$. Next we shall prove the well-known Fourier inversion theorem but first we prove a statement about the Gaussian functions and Gaussian integrals on $\mathbf{R}^n$.

Lemma 1.1.2. *If $a > 0$ and $x \in \mathbf{R}^n$, then the Gaussian function $f(x) = e^{-a^2|x|^2}$ has integral: $\int_{\mathbf{R}^n} f(x)\, dx = (\pi/a^2)^{n/2}$. Moreover, $\widehat{f}(\xi) = (\pi/a^2)^{n/2}e^{-|\xi|^2/4a^2}$.*

Proof. The first assertion is a familiar proof from a course in advanced calculus. For the second proof first suppose that $n = 1$. Then

$$(\widehat{f})'(\xi) = \int_{\mathbf{R}^n} -ixe^{-a^2x^2}e^{-ix\xi}\, dx = \frac{i}{2a^2}\widehat{f'}(\xi) = -\frac{\xi}{2a^2}\widehat{f}(\xi).$$

In particular, we have that

$$\frac{d}{d\xi}\left(e^{\xi^2/4a^2}\widehat{f}(\xi)\right) = \frac{\xi}{2a^2}e^{\xi^2/4a^2}\widehat{f}(\xi) + e^{\xi^2/4a^2}(\widehat{f})'(\xi) = 0.$$

Thus the function $e^{\xi^2/4a^2}\widehat{f}(\xi)$ is constant. Evaluating at $\xi = 0$ we find that this constant is equal to $\widehat{f}(0) = (\pi/a^2)^{1/2}$. So $\widehat{f}(\xi) = (\pi/a^2)^{1/2}e^{-\xi^2/4a^2}$. For the arbitrary n-dimensional case we put $f_j(x) = e^{-a^2x_j^2}$ so that $f = \prod_{j=1}^n f_j$. Then we can verify

$$\widehat{f}(\xi) = \prod_{j=1}^n \widehat{f_j}(\xi_j) = \left(\frac{\pi}{a^2}\right)^{n/2} e^{-|\xi|^2/4a^2}.$$

□

Now if $g \in L^1(\mathbf{R}^n)$, put $g_t(x) = t^{-n}g(x/t)$ for $t > 0$ and set $c = \int g$. Then the convolution $\phi * g_t$ converges to $c\phi$ in L^p as $t \to 0$ for each $\phi \in L^p(\mathbf{R}^n)$ $(1 \leq p < \infty)$, see [8, Thm. 8.14] for a short proof of this fact. In particular for the function $g(x) = e^{-|x|^2/4}$ we have that the Fourier transform of the function

$$\psi_t(z) = \exp\big(i\langle x,z\rangle - t^2|z|^2\big)$$

is the function $\widehat{\psi_t}(y) = (\pi)^{n/2}g_t(x-y)$. We are now in the position to prove the celebrated Fourier inversion theorem.

Theorem 1.1.3 (Fourier inversion theorem)**.** *If $f, \widehat{f} \in L^1(\mathbf{R}^n)$, then $f = \mathscr{F}^{-1}\mathscr{F}f = \mathscr{F}\mathscr{F}^{-1}f$ almost everywhere.*

Proof. Let ψ_t and g_t be as above and observe that by Proposition 1.1.1. Property (4)

$$\int_{\mathbf{R}^n} \psi_t(\xi)\widehat{f}(\xi)\,d\xi = \int_{\mathbf{R}^n} \widehat{\psi_t}(y)f(y)\,dy = (\pi)^{n/2} f * g_t(x).$$

Now as $t \to 0$, by the dominated convergence theorem the left-most integral above converges to

$$\int_{\mathbf{R}^n} \widehat{f}(\xi)e^{i\langle x,\xi\rangle}\,d\xi\,.$$

While by Lemma 1.1.2 and the discussion preceding this theorem, the right-most integral converges to $(2\pi)^n f(x)$ for almost every $x \in \mathbf{R}^n$. Thus,

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} \widehat{f}(\xi)e^{i\langle x,\xi\rangle}\,d\xi$$

holds almost everywhere. Thus, $f = \mathscr{F}^{-1}\mathscr{F} f$ and that $f = \mathscr{F}\mathscr{F}^{-1} f$ follows from the relations $\mathscr{F}^{-1} f(\xi) = (2\pi)^{-n}\widehat{f}(-\xi)$ and $\mathscr{F} f(\xi) = (2\pi)^n\mathscr{F}^{-1}(-\xi)$. $\square$

Our primary concern will be to study how the Fourier transform behaves when applied to various function spaces—of which the primary spaces that we will be concerned with are $L^1(\mathbf{R}^n)$, $L^2(\mathbf{R}^n)$, $C_c^\infty(\mathbf{R}^n)$, and the "Schwartz space" $\mathcal{S}(\mathbf{R}^n)$ which we define later. We will also investigate the Fourier transform for the topological dual of $\mathcal{S}(\mathbf{R}^n)$.

1.1.1 Images of classical function spaces

We shall first begin our discussion with the characterization of the image of $\mathscr{F}$ on $L^1(\mathbf{R}^n)$.

Lemma 1.1.4 (Riemann-Lebesgue lemma)**.** *If $f \in L^1(\mathbf{R}^n)$, then $\widehat{f} \in C_0(\mathbf{R}^n)$.*

Proof. The smooth compactly supported functions are dense in $L^1(\mathbf{R}^n)$ and clearly by Proposition 1.1.1 Property (2), $\mathscr{F}(C_c^\infty(\mathbf{R}^n)) \subset C_0(\mathbf{R}^n)$. Since $C_0(\mathbf{R}^n)$ is complete and $\mathscr{F}$ is continuous (it is bounded) by Proposition 1.1.1 Property (1), the result is immediate since $C_c^\infty(\mathbf{R}^n) \subset L^1(\mathbf{R}^n)$ is dense in $L^1(\mathbf{R}^n)$. $\square$

Interestingly this is the best description of the image of the Fourier transform on $L^1(\mathbf{R}^n)$. In particular, the Fourier transform does not map $L^1(\mathbf{R}^n)$ surjectively onto $C_0(\mathbf{R}^n)$. There seems to be no other useful description of functions in $\mathscr{F}(L^1(\mathbf{R}^n))$ other than to say that they are elements of $\mathscr{F}(L^1(\mathbf{R}^n))$. Moving on, we combine our previous results into the following theorem.

Theorem 1.1.5. *The Fourier transform is a continuous $*$-homomorphism of $L^1(\mathbf{R}^n)$ into a dense subalgebra of $C_0(\mathbf{R}^n)$. That is the Fourier transform maps $L^1(\mathbf{R}^n)$ into $C_0(\mathbf{R}^n)$ and satisfies*

$$\mathscr{F} f^* = \overline{\mathscr{F} f},$$
$$\mathscr{F}(f * g) = (\mathscr{F} f)(\mathscr{F} g).$$

Proof. First we verify that $(f*g)\widehat{\ } = \widehat{f}\widehat{g}$ and that $\widehat{f^*} = \overline{\widehat{f}}$ for $f, g \in L^1(\mathbf{R}^n)$. So first note by a routine application of Fubini's theorem:

$$(f*g)\widehat{\ }(\xi) = \int_{\mathbf{R}^n} (f*g)(x)e^{-i\langle x,\xi\rangle}\,dx = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(y)g(x-y)e^{-i\langle x,\xi\rangle}\,dx\,dy$$
$$= \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(y)g(x)e^{-i\langle x+y,\xi\rangle}\,dx\,dy = \int_{\mathbf{R}^n} f(y)e^{-i\langle y,\xi\rangle}\,dy \cdot \int_{\mathbf{R}^n} g(x)e^{-i\langle x,\xi\rangle}\,dx = \widehat{f}(\xi)\widehat{g}(\xi).$$

It is also obvious $\widehat{f^*} = \overline{\widehat{f}}$. Hence $\mathscr{F}$ is a continuous $*$-homomorphism (continuity is given by Proposition 1.1.1 Property (1)).

To see that $\mathscr{F}(L^1(\mathbf{R}^n))$ is a subalgebra dense in $C_0(\mathbf{R}^n)$, we note that $\mathscr{F}(L^1(\mathbf{R}^n))$ is a subalgebra since if $\widehat{f},\widehat{g}\in\mathscr{F}(L^1(\mathbf{R}^n))$, then $\widehat{f}\widehat{g}=(f*g)\widehat{\ }\in\mathscr{F}(L^1(\mathbf{R}^n))$. It also separates points since if $\xi,\eta\in\mathbf{R}^n$ were such that $\widehat{f}(\xi)=\widehat{f}(\eta)$ for all $f\in L^1(\mathbf{R}^n)$, then $e^{-i\langle x,\xi\rangle}=e^{-i\langle x,\eta\rangle}$ for all $x\in\mathbf{R}^n$ so that $\xi=\eta$. As we have seen, $\mathscr{F}(L^1(\mathbf{R}^n))$ is closed under complex conjugation, therefore by the Stone-Weierstrass theorem $\mathscr{F}(L^1(\mathbf{R}^n))$ is a dense subalgebra of $C_0(\mathbf{R}^n)$.

□

Although the Fourier transform is only defined on $L^1(\mathbf{R}^n)$ we can extend its definition to $L^2(\mathbf{R}^n)$. In particular, we can extend $\mathscr{F}$ continuously to $L^2(\mathbf{R}^n)$ so that $\mathscr{F}$ is an isometry of $L^2(\mathbf{R}^n)$ onto the Hilbert space $L^2(\mathbf{R}^n;(2\pi)^{-n}dx)$. For this part of the section we will write $L^2(\mathbf{R}^n;dx)$ and $L^2(\mathbf{R}^n;(2\pi)^{-n}dx)$ corresponding to the L^2 spaces on $\mathbf{R}^n$ arising from the measures dx and $(2\pi)^{-n}dx$.

Theorem 1.1.6 (Plancherel theorem). *If $f\in L^1\cap L^2$, then $\widehat{f}\in L^2$ and $\mathscr{F}$ defined on $L^1\cap L^2$ extends continuously to a unitary isomorphism of $L^2(\mathbf{R}^n;dx)$ onto $L^2(\mathbf{R}^n;(2\pi)^{-n}dx)$.*

Proof. Put $W=\{f\in L^1(\mathbf{R}^n)\colon \widehat{f}\in L^1(\mathbf{R}^n)\}$. Since for $f\in W$ we have $\|f\|_\infty<\infty$ and thus by an interpolation inequality (cf. [8, Prop. 6.10]): $\|f\|_2\le\|f\|_1^{1/2}\|f\|_\infty^{1/2}$ so that $W\subset L^1(\mathbf{R}^n)\cap L^2(\mathbf{R}^n)$. In particular, since W contains $C_c^\infty(\mathbf{R}^n)$ we have that W is dense in $L^2(\mathbf{R}^n)$.

Now for $f,g\in W$ we then have by the inversion formula and Fubini's theorem

$$\int_{\mathbf{R}^n} f(x)\overline{g(x)}\,dx=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} f(x)\left\{\int_{\mathbf{R}^n}\overline{\widehat{g}(y)}e^{-i\langle x,y\rangle}\,dy\right\}dx=\frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\widehat{f}(y)\overline{\widehat{g}}(y)\,dy\,.$$

Thus, the restriction of $\mathscr{F}$ to W is an isometry of W into $L^2(\mathbf{R}^n;(2\pi)^{-n}dx)$. What this means is that if $X=L^2(\mathbf{R}^n;dx)$ and $Y=L^2(\mathbf{R}^n;(2\pi)^{-n}dx)$, then for $f,g\in W$ we have $\langle f,g\rangle_X=\langle\widehat{f},\widehat{g}\rangle_Y$. In particular we see that if $f\in W$, then also $\widehat{f}\in L^2(\mathbf{R}^n)$.

By the inversion theorem we have $\mathscr{F}(W)=W$. Thus if we extend $\mathscr{F}$ uniquely and continuously to $L^2(\mathbf{R}^n)$ we obtain a unitary isomorphism from $L^2(\mathbf{R}^n;dx)$ onto $L^2(\mathbf{R}^n;(2\pi)^{-n}dx)$. That is if $f\in L^2(\mathbf{R}^n)$ we define $\widehat{f}=\lim_{n\to\infty}\widehat{f_n}$ (a limit which is interpreted in the sense of L^2) for a sequence $\{f_n\}_{n\in\mathbf{N}}\subset W$ which converges to f in the L^2 norm. For the time being, denote this extension by $\mathscr{F}_e$.

We finally conclude the proof by showing that $\mathscr{F}_e=\mathscr{F}$ on $L^1(\mathbf{R}^n)\cap L^2(\mathbf{R}^n)$. To this end, if we put $g(x)=e^{-\pi|x|^2}$, then for $f\in L^1(\mathbf{R}^n)\cap L^2(\mathbf{R}^n)$ we have $f*g_t\in L^1(\mathbf{R}^n)$ by the Riesz-Thorin interpolation theorem and

$$(f*g_t)\widehat{\ }(\xi)=\widehat{f}(\xi)\widehat{g_t}(\xi)=e^{-t^2|\xi|^2/4\pi}\widehat{f}(\xi)\in L^1(\mathbf{R}^n).$$

Thus, $f*g_t\in W$ and as $t\to 0$ we have $f*g_t\to f$ in both $L^1(\mathbf{R}^n)$ and $L^2(\mathbf{R}^n)$. So $(f*g_t)\widehat{\ }\to\widehat{f}$ uniformly as continuous functions and $(f*g_t)\widehat{\ }\to\mathscr{F}_e(f)$ in L^2 by definition. Thus it follows that $\mathscr{F}_e(f)(\xi)=\lim_{t\to 0}(f*g_t)\widehat{\ }(\xi)=\widehat{f}(\xi)$ on $L^1(\mathbf{R}^n)\cap L^2(\mathbf{R}^n)$ as required. Moreover, by this conclusion we have proved that $\widehat{f}\in L^2(\mathbf{R}^n)$ whenever $f\in L^1(\mathbf{R}^n)\cap L^2(\mathbf{R}^n)$.

□

We observe that in the formula for the definition of the Fourier transform of a function $f\in L^1(\mathbf{R}^n)$ that it may sometimes make sense to define $\widehat{f}$ on $\mathbf{C}^n$ by complexification of the Fourier variable. Thus, for an integrable function f we define its so-called *complex Fourier transform* by the formula

$$\widehat{f}(z)=\int_{\mathbf{R}^n} f(x)e^{-i\langle x,z\rangle}\,dx$$

where $\langle x,z\rangle=\langle x,\operatorname{Re}z\rangle+i\langle x,\operatorname{Im}z\rangle$. There is however an issue of convergence since it is not clear whether the integrand above is integrable since $|e^{-i\langle x,z\rangle}|=e^{\langle x,\operatorname{Im}z\rangle}$ has exponential growth. Thus to study the complex Fourier transform it is necessary to restrict our attention to a smaller class of functions. For instance, if f is integrable and compactly supported, then the function $\widehat{f}(z)$ is well-defined for all $z\in\mathbf{C}^n$. In fact, by the dominated convergence theorem we can differentiate underneath the integral, and we determine that $\widehat{f}$ is *holomorphic*.

Typical function spaces that we may wish to study under the complex Fourier transform are $C_c(\mathbf{R}^n)$ or $C_c^m(\mathbf{R}^n)$ for $m \in \mathbf{N}$, since functions belonging to such spaces have well-defined complex Fourier transforms. For our purposes, we will study $C_c^\infty(\mathbf{R}^n)$ since the complex Fourier transform enjoys many features unique to this space. To characterize the image of $C_c^\infty(\mathbf{R}^n)$ under the complex Fourier transform we first require a definition.

Definition 1.1.3. For a holomorphic function f on $\mathbf{C}^n$, we say that f is of uniform exponential type $A > 0$ if for all $N \in \mathbf{N}$

$$\sup_{z \in \mathbf{C}^n} |f(z)(1+|z|)^N e^{-A|\operatorname{Im} z|}| < \infty.$$

For $A > 0$ let $\mathscr{H}_A(\mathbf{C}^n)$ denote the space of holomorphic functions of exponential type A and set $\mathscr{H}(\mathbf{C}^n) = \bigcup_{A>0} \mathscr{H}_A(\mathbf{C}^n)$.

Now we state our main theorem which links the spaces $C_c^\infty(\mathbf{R}^n)$ and $\mathscr{H}(\mathbf{C}^n)$ via the complex Fourier transform.

Theorem 1.1.7 (Paley-Wiener theorem)**.** *The complex Fourier transform is a bijection of $C_c^\infty(\mathbf{R}^n)$ onto $\mathscr{H}(\mathbf{C}^n)$.*

Proof. If $f \in C_c^\infty(\mathbf{R}^n)$ has its support in the ball of radius R, then elementary estimations on the integrand shows that $\widehat{f}$ is a holomorphic function of uniform exponential type R. Conversely, if $\psi \in \mathscr{H}(\mathbf{C}^n)$ is of uniform exponential type $A > 0$, we define the function

$$h(x) = \int_{\mathbf{R}^n} \psi(\xi) e^{i\langle x,\xi\rangle}\, d\xi\,.$$

Using the Cauchy integral formula we can shift the contour of integration so that

$$h(x) = \int_{\mathbf{R}^n} \psi(\xi + i\eta) e^{i\langle x,\xi+i\eta\rangle}\, d\xi$$

where $\eta \in \mathbf{R}^n$. Now if $|x| > A$,

$$|h(x)| \le \int_{\mathbf{R}^n} |\psi(\xi+i\eta)| e^{-\langle x,\eta\rangle}\, d\xi \le C_N e^{-\langle x,\eta\rangle + A|\eta|} \int_{\mathbf{R}^n} (1+|\xi|)^{-N}\, d\xi \le \widetilde{C}_N e^{-\langle x,\eta\rangle + A|\eta|},$$

where C_N and $\widetilde{C}_N$ are positive constants and N is chosen to be suitably large. If we put $\eta = tx$ where $t > 0$, then the right hand side becomes $\widetilde{C}_N e^{-t|x|^2 + At|x|} = \widetilde{C}_N e^{t|x|(A-|x|)}$. Taking $t \to \infty$ we have that $h(x) = 0$ for all $|x| > A$. Thus, $\operatorname{supp} h \subset B_A(0)$ and it is clear that h is a smooth function. By Fourier inversion $\widehat{h} = \psi$ and so we are done. $\square$

We note that the Fourier transform maps $C_c^\infty(\mathbf{R}^n)$ into a class of functions that *does not* contain $C_c^\infty(\mathbf{R}^n)$. Indeed, if $f \in C_c^\infty(\mathbf{R}^n)$ then $\widehat{f}$ is analytic and thus cannot be compactly supported unless $f \equiv 0$. A sensible question becomes whether or not there exists a class of functions on which the Fourier transform is a homeomorphism to itself. The answer is affirmative and is provided by the Schwartz space defined as follows.

Definition 1.1.4. Define the space of functions

$$\mathcal{S}(\mathbf{R}^n) = \left\{ f \in C^\infty(\mathbf{R}^n) \colon \nu_{\alpha,k}(f) = \sup_{x \in \mathbf{R}^n} |\partial^\alpha f(x)(1+|x|)^k| < \infty, k \in \mathbf{N}, \alpha \in \mathbf{N}^n \right\}.$$

We topologize $\mathcal{S}(\mathbf{R}^n)$ by means of the seminorms $\nu_{\alpha,k}$ from which $\mathcal{S}(\mathbf{R}^n)$ becomes a Fréchet space. The space $\mathcal{S}(\mathbf{R}^n)$ is called the *Schwartz space*.

Theorem 1.1.8. *The Fourier transform on $\mathcal{S}(\mathbf{R}^n)$ is a linear isomorphism onto itself.*

Proof. This is more or less trivial by Proposition 1.1.1 and the Fourier inversion formula. The only possible nontrivial fact is that the Fourier transform is continuous on $\mathcal{S}(\mathbf{R}^n)$. This can be proven as follows. If $f \in \mathcal{S}(\mathbf{R}^n)$, then there exists for each seminorm η on $\mathcal{S}(\mathbf{R}^n)$ another seminorm γ such that $\eta(\widehat{f}) \leq \gamma(f)$. This can be seen via the estimate

$$\begin{aligned}|(i\xi)^\alpha \partial^\beta \widehat{f}(\xi)| &\leq \int_{\mathbf{R}^n} |x^\beta \partial^\alpha f(x)|\, dx \leq \int_{\mathbf{R}^n} (1+|x|)^{|\beta|} |\partial^\alpha f(x)|\, dx \\ &= \int_{\mathbf{R}^n} (1+|x|)^{m-m} (1+|x|)^{|\beta|} |\partial^\alpha f(x)| \leq c_0 \nu_{\alpha, m+|\beta|}(f)\end{aligned}$$

where m is a positive integer and c_0 is the constant

$$c_0 = \int_{\mathbf{R}^n} (1+|x|)^{-m}\, dx\,.$$

By an appropriate choice of m, we can select $c_0 \leq 1$ so that the result follows. Conversely, the inverse Fourier transform is continuous essentially by the same argument whence $\mathscr{F}$ is a homeomorphism of $\mathcal{S}(\mathbf{R}^n)$ onto itself. $\square$

The Plancherel theorem, the Paley-Wiener theorem, and Theorem 1.1.5 are each important in various respects which we shall not endeavor to discuss in depth. Briefly, the Plancherel theorem plays a significant role in the theory of partial differential equations and is pivotal in defining the most important Sobolev spaces $H^s(\mathbf{R}^n)$. The Paley-Wiener theorem on the other hand gives us a crucial link between smooth compactly supported functions and the decay of the complex Fourier transforms. This characterization later becomes particularly important in the area of microlocal analysis and the definition of wave front sets of distributions (see [14, Ch. 8] for more details). Theorem 1.1.5 on the other hand is important since it is somewhat "generic" which we will discuss in Section 1.2.

We now turn our attention to the so-called distributions which can be envisaged as "generalized" functions. Our main goal will be to define the various spaces of distributions and most importantly extend the Fourier transform to these spaces of distributions in a suitable manner.

1.1.2 Distributions

We recall that if $U \subset \mathbf{R}^n$ is an open set of $\mathbf{R}^n$, then the space $C_c^\infty(U)$ is the space of smooth functions on $\mathbf{R}^n$ with compact support contained in U and the space $C^\infty(U)$ is the space of smooth functions defined on U. We shall write $\mathcal{D}(U) = C_c^\infty(U)$ and $\mathcal{E}(U) = C^\infty(U)$ in this subsection which is the notation used by Laurent Schwartz, and we shall call $\mathcal{D}(U)$ the space of *test functions* on U. Before we begin, let us discuss the topology typically imposed on $\mathcal{D}(U)$ and $\mathcal{E}(U)$.

The topology on $\mathcal{E}(U)$ is quite simple to describe. For $f \in \mathcal{E}(U)$ and each compact subset $K \subset U$, we define the seminorms

$$\nu_{\alpha,K}(f) = \sup_{x \in K} |\partial^\alpha f(x)|,$$

where α is a multi-index and ∂^α is the corresponding differential operator. Then, $\mathcal{E}(U)$ is topologized by the means of the seminorms $\nu_{\alpha,K}$ which is to say that a sequence $\{f_j\}_{j\in J} \subset \mathcal{E}(U)$ converges to $f \in \mathcal{E}(U)$ if and only if $\nu_{\alpha,K}(f_j - f) \to 0$ for each multi-index α and each compact set $K \subset U$. With this topology, $\mathcal{E}(U)$ is a Fréchet space. We define $\mathcal{E}'(U)$ to be the topological dual of $\mathcal{E}(U)$. We shall discuss the topology on $\mathcal{E}'(U)$ later.

The topology of $\mathcal{D}(U)$ is slightly more delicate however. First, if $K \subset U$ is a compact set then we put $\mathcal{D}(K; U)$ to denote the space of compactly supported smooth functions on U with support in K. We then topologize $\mathcal{D}(K;U)$ by the seminorms $\kappa_\alpha(f) = \sup_{x\in K} |\partial^\alpha f(x)|$ for $f \in \mathcal{D}(K;U)$ and α a multi-index as before. Then we say that a sequence $\{f_n\}_{n\in\mathbf{N}} \subset \mathcal{D}(U)$ converges to $f \in \mathcal{D}(U)$ if and only if

1. There is a compact set $K' \subset U$ such that $f_n, f \in \mathcal{D}(K'; U)$ for all $n \in \mathbf{N}$.
2. And $f_n \to f$ in the topology of $\mathcal{D}(K'; U)$.

With this topology $\mathcal{D}(U)$ is not a Fréchet space but is of course a locally convex topological vector space. In fact it is not even a sequential space but such sophisticated considerations of the topology of $\mathcal{D}(U)$ are unimportant for our purposes.

Definition 1.1.5. We say a linear functional $T \colon \mathcal{D}(U) \to \mathbf{C}$ is continuous if for each compact set $K \subset U$, the restriction $T|_{\mathcal{D}(K;U)} : \mathcal{D}(K;U) \to \mathbf{C}$ is continuous. We then define the space of distributions on U to be the space of all continuous linear functionals on $\mathcal{D}(U)$. We denote this space by $\mathcal{D}'(U)$.

If $T \in \mathcal{D}'(U)$ is a distribution, we shall often write $T(f) = \langle T, f\rangle$ for $f \in \mathcal{D}(U)$ and call this the pairing of T and f. There is a wealth of examples of distributions. Perhaps the most famous example is the *Dirac delta function* with point mass at $x \in U$, denoted δ_x, which defines a distribution by $\langle \delta_x, f\rangle = f(x)$ where $f \in \mathcal{D}(U)$. Any continuous function g on U defines a distribution by the rule

$$\langle g, f\rangle = \int_{\mathbf{R}^n} g(x) f(x)\, dx\,.$$

In fact, if $g \in L^p$ where $p \geq 1$, then $\langle g, f\rangle = \int_{\mathbf{R}^n} g(x)f(x)\,dx$ is well-defined by Hölder's inequality and is continuous by virtue of the fact that $|\langle g, f\rangle| \leq S_f \|g\|_p \|f\|_\infty$ where S_f is the measure of the support of f. More generally, if $g \in L^1_{\text{loc}}(U)$ is any locally integrable function the pairing, $\langle g, f\rangle = \int_{\mathbf{R}^n} g(x)f(x)\,dx$ defines a distribution and we thus have an inclusion $L^1_{\text{loc}}(U) \hookrightarrow \mathcal{D}'(U)$.

This large class of distributions which is given by integrating test functions against other functions motivates a useful, but not necessarily rigorous, notation

$$\langle T, f\rangle = \int_{\mathbf{R}^n} T(x) f(x)\, dx,$$

where in this equality we think of T as a *function* so that the pairing $\langle T, f\rangle$ is thought of as arising from integration against T as a "function." This perspective allows us to carry many of the familiar operations on functions, such as differentiation to distributions by so-called *duality*. Which heuristically means that operations defined for $\mathcal{D}(U)$ are extended to $\mathcal{D}'(U)$ by means of an adjoint with respect to the bilinear form $(f, g) = \int_{\mathbf{R}^n} f(x)g(x)\,dx$. To typify what we mean by this let us define the derivative of a distribution. If $f, g \in \mathcal{D}(U)$, then by integration by parts

$$\langle \partial^\alpha g, f\rangle = \int_{\mathbf{R}^n} \partial^\alpha g(x) f(x)\, dx = (-1)^{|\alpha|} \int_{\mathbf{R}^n} g(x) \partial^\alpha f(x)\, dx\,. \tag{1.1.3}$$

In particular $(-1)^{|\alpha|}\partial^\alpha$ is the adjoint of ∂^α with respect to the bilinear form (f, g) defined on $\mathcal{D}(U)$. Thus we extend the derivative "by duality" to distributions by the following definition.

Definition 1.1.6. If $T \in \mathcal{D}'(U)$ we define the derivative $\partial^\alpha T$ by $\langle \partial^\alpha T, f\rangle = (-1)^{|\alpha|}\langle T, \partial^\alpha f\rangle$.

Thus when T is a smooth function, then our definition agrees with the standard result by integration by parts. In this setup a distribution is *smooth* since any test function is smooth.

Now if we recall, the support of a function g is the complement of the largest open set on which g vanishes. So if $g \in L^1_{\text{loc}}(U)$ and $f \in \mathcal{D}(U)$ is such that $\operatorname{supp} g \cap \operatorname{supp} f = \emptyset$, then

$$\langle g, f\rangle = \int_{\mathbf{R}^n} f(x) g(x)\, dx = 0.$$

This motivates a way of defining the *support* of a distribution as a set of points in U rather than a set of functions in $\mathcal{D}(U)$ in the following way.

Definition 1.1.7. If $V \subset U$ is an open set, then we say a distribution $T \in \mathcal{D}'(U)$ vanishes on V if $\langle T, f\rangle = 0$ for each $f \in \mathcal{D}(U)$ with $\operatorname{supp} f \subset V$. We define the *support* of T, denoted by $\operatorname{supp} T$, to be the complement of the largest open set of U on which T vanishes.

Note that by the method of partition of unity, if T vanishes on open sets $U_\alpha \subset U$ for $\alpha \in A$, then T also vanishes on $\bigcup_{\alpha\in A} U_\alpha$. Thus, the concept of the "largest open set" on which T vanishes makes sense. If $\operatorname{supp} T$ is compact, in fact T defines an element $T' \in \mathcal{E}'(U)$ where we define T' by $T'(f) = T(\chi f)$ where $f \in \mathcal{E}(U)$ and $\chi \in \mathcal{D}(U)$ is any smooth function with $\chi = 1$ on $\operatorname{supp} T$. Clearly, T' does not depend on the choice of χ and so T' is well defined. Furthermore, T' is continuous on $\mathcal{E}'(U)$ since $T|_{\mathcal{D}(\operatorname{supp}\chi;U)}$ is continuous. Thus any compactly supported distribution defines an element of $\mathcal{E}'(U)$. Interestingly, if $S \in \mathcal{E}'(U)$, then S defines a distribution with compact support. This follows from the fact that since S is continuous on $\mathcal{E}(U)$ we can find finitely many seminorms ν_{α_j,K_j}, $j = 1, 2, \ldots, m$ such that

$$|S(f)| \le \sum_{j=1}^{m} c_j \nu_{\alpha_j,K_j}(f), \text{ for all } f \in \mathcal{E}(U)$$

for some choice of constants $c_j \ge 0$. If $f \in \mathcal{D}(U)$ is such that $\operatorname{supp} f \cap K_j = \emptyset$ for each j, then $S(f) = 0$ so that $\operatorname{supp} S \subset \bigcup_{j=1}^m K_j$. Thus S is a distribution of compact support. In summary, we have the following result.

Proposition 1.1.9. *We identify the space $\mathcal{E}'(U)$ with the space of compactly supported distributions in $\mathcal{D}'(U)$.*

Another operation that is defined for test functions is convolution. Recall that if $f, g \in \mathcal{D}(U)$, then their convolution $f * g$ is a compactly supported smooth function defined by

$$f * g(x) = \int_{\mathbf{R}^n} f(y)g(x-y)\,dy\,. \tag{1.1.4}$$

The support of $f * g$ is contained in $\operatorname{supp} f + \operatorname{supp} g$ and so $f * g \in \mathcal{D}(U + U)$ (here the sum of two subsets X and Y of $\mathbf{R}^n$ is the set $X + Y = \{x + y \colon x \in X, y \in Y\}$). Analogously if $x \in \mathbf{R}^n$, and if $T \in \mathcal{D}'(U)$ and $f \in \mathcal{D}(x - U)$, we define the convolution of T and f by $T * f(x) = \langle T_y, f(x-y)\rangle$. Here T_y denotes the distribution T acting on the y-variable of the function $y \mapsto f(x - y)$ on U. Using our integration notation, this is simply

$$T * f(x) = \int_{\mathbf{R}^n} T(y)f(x-y)\,dy\,.$$

If $T \in \mathcal{D}'(\mathbf{R}^n)$ and if $f \in \mathcal{D}(\mathbf{R}^n)$, then the convolution $T * f$ is defined on all of $\mathbf{R}^n$. In fact, $T * f$ is smooth as a function on $\mathbf{R}^n$ by the following proposition.

Proposition 1.1.10. *Let $f \in \mathcal{D}(\mathbf{R}^n)$ and $T \in \mathcal{D}'(\mathbf{R}^n)$, then the convolution $x \mapsto T * f(x)$ is a smooth function on $\mathbf{R}^n$ with $\partial^\alpha(T * f) = (\partial^\alpha T) * f = T * (\partial^\alpha f)$.*

Proof. Let $\{e_j\}_{j=1}^n$ denote the standard basis vectors of $\mathbf{R}^n$. Note that if $\phi \in \mathcal{D}(U)$, then for each j the difference quotient

$$\phi_h^j(x) = \frac{\phi(x + he_j) - \phi(x)}{h}$$

converges uniformly to $\partial_j\phi(x)$ as $h \to 0$ by Taylor's theorem. Thus, we have by continuity

$$\partial_j(T * f)(x) = \lim_{h\to 0}\langle T_y, f_h^j(x-y)\rangle = \langle T_y, \partial_j f(x-y)\rangle.$$

But also by definition and the chain rule $\langle T_y, \partial_j f(x-y)\rangle = \langle \partial_j T_y, f(x-y)\rangle$. Thus, the result holds for any differential operator ∂^α by induction. □

One can extend further the convolution between distributions and functions to the convolution of certain well-behaved distributions. First, if $T \in \mathcal{D}'(\mathbf{R}^n)$ and $f \in C_c^\infty(\mathbf{R}^n)$, then in view of the proof of Proposition 1.1.10, the function $\phi(x) = \langle T_y, f(x+y)\rangle$ is a smooth function on $\mathbf{R}^n$. It is not necessarily compactly supported and so does not define an element of $\mathcal{D}(\mathbf{R}^n)$ (but does define such an element if $T \in \mathcal{E}'(\mathbf{R}^n)$). However, for $S \in \mathcal{E}'(\mathbf{R}^n)$ the assignment $\langle S, \phi\rangle$ is well-defined. Now if $f, g, h \in \mathcal{D}(\mathbf{R}^n)$, then we observe an identity

$$\langle f * g, h\rangle = \int_{\mathbf{R}^n} f * g(y) h(y)\, dy = \int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(x)g(y)h(x+y)\, dy\, dx\,.$$

This motivates us to define the convolution of the distributions $T \in \mathcal{D}'(\mathbf{R}^n)$ and $S \in \mathcal{E}'(\mathbf{R}^n)$ by $\langle S * T, f\rangle = \langle S_y, \langle T_x, f(x+y)\rangle\rangle$ which defines a distribution on $\mathcal{D}(\mathbf{R}^n)$. Using Proposition 1.1.10, we also have $\partial^\alpha(S * T) = (\partial^\alpha S) * T = S * (\partial^\alpha T)$ for any differential operator ∂^α.

There are other operations on distributions one may be interested in, such as multiplication of distributions, however for our purposes the operations of differentiation and convolution highlight the most important operations on distributions for this thesis. Indeed, as in the situation for functions, the Fourier transform interacts nicely with the convolution and differentiation of functions in the same spirit as of Proposition 1.1.1. As we shall see shortly, we shall extend the notion of the Fourier transform to distributions (to a particular subset of distributions to be precise), and we shall study the familiar properties of the Fourier transform as it interacts with the convolution and differentiation of those distributions. But first, we shall make a short interlude into discussing the topologies on $\mathcal{D}'(U)$ that we have neglected in our study of the space of distributions $\mathcal{D}'(U)$.

Topologies on $\mathcal{E}'(U)$, $\mathcal{D}'(U)$, and $\mathcal{S}'(\mathbf{R}^n)$

We can consider the topological dual of $\mathcal{S}(\mathbf{R}^n)$ to obtain $\mathcal{S}'(\mathbf{R}^n)$ which we call the *tempered distributions*. As can be easily seen, these are in fact distributions in themselves and thus $\mathcal{S}'(\mathbf{R}^n) \subset \mathcal{D}'(U)$ for any open $U \subset \mathbf{R}^n$. Note that, however a tempered distribution is not necessarily compactly supported (nor do they even define a linear functional on $\mathcal{E}(U)$). So we have the following (strict!) inclusions $\mathcal{E}'(U) \hookrightarrow \mathcal{S}'(\mathbf{R}^n) \hookrightarrow \mathcal{D}'(U)$. Now, there are two topologies that we can endow on $\mathcal{E}'(U)$, $\mathcal{S}'(\mathbf{R}^n)$, and $\mathcal{D}'(U)$. These are the weak* topologies and the strong topologies which we define by convergence of nets.

Definition 1.1.8. Let X be $\mathcal{E}(U)$, $\mathcal{S}(\mathbf{R}^n)$, or $\mathcal{D}(U)$ and X' denote the respective space of distributions. The *weak*-topology* on X' is the topology of pointwise convergence. That is to say a net T_α in X' converges to T if for each f in X we have $T_\alpha(f) \to T(f)$ as a net.

Definition 1.1.9. Let X be $\mathcal{E}(U)$, $\mathcal{S}(\mathbf{R}^n)$, or $\mathcal{D}(U)$ and X' denote the respective space of distributions. The *strong topology* on X' is the topology of uniform convergence. That is to say a sequence T_α in X' converges to T if for each bounded set B in X we have $\sup_{f\in B} |T_\alpha(f) - T(f)| \to 0$ as a net.

The strong topology is quite literally stronger than the weak*-topology, which is to say that if τ_s and τ_* are the strong and weak* topologies, then $\tau_* \subset \tau_s$. However, when we speak of convergences of *sequences*, then a sequence converges strongly if and only if it converges weakly (see [25, Ch. 28]). There are many different reasons why one would choose one topology over the other. The weak*-topology is certainly much easier to understand and characterize. Under the weak*-topology we have the following density result.

Theorem 1.1.11. *Recall that we have the canonical inclusion $\mathcal{D}(U) \hookrightarrow \mathcal{D}'(U)$. This inclusion is dense in the weak*-topology.*

See [8, Prop. 9.5] for a proof. Considering sequences converge in the strong topology if and only if they converge in the weak*-topology we have that the above theorem provides a sequence of test functions $\phi_k \in \mathcal{D}(U)$ to each distribution $T \in \mathcal{D}'(U)$ such that $\phi_k \to T$ in the strong topology of $\mathcal{D}'(U)$. Characterizing convergence in $\mathcal{D}'(U)$ using sequences is certainly more preferable than using nets and so we shall often impose the strong topology on $\mathcal{D}'(U)$ (as well as on $\mathcal{E}'(U)$ and $\mathcal{S}'(\mathbf{R}^n)$). Since when using sequences we obtain essentially the same convergences as for the weak*-topology.

Using the strong topology is also preferable since when endowed with the strong topology we have the following theorem of Laurent Schwartz (see A.2.4).

Theorem 1.1.12 (Schwartz kernel theorem)**.** *Suppose that U and V are open subsets of $\mathbf{R}^n$ and let $L(\mathcal{D}(U), \mathcal{D}'(V))$ be the space of continuous linear operators from $\mathcal{D}(U)$ into $\mathcal{D}'(V)$ where $\mathcal{D}'(V)$ and $L(\mathcal{D}(U), \mathcal{D}'(V))$ are given the strong topology. Then $L(\mathcal{D}(U), \mathcal{D}'(V)) \cong \mathcal{D}'(U \times V)$ is an isomorphism of topological vector spaces where $\mathcal{D}'(U \times V)$ is given the strong topology.*

This isomorphism is given by the map $A \mapsto K_A$ where $K_A \in \mathcal{D}'(U \times V)$ is defined by

$$\langle Af, g\rangle = \langle K_A, f \otimes g\rangle$$

where $f \in \mathcal{D}(U)$, $g \in \mathcal{D}(V)$ and $f \otimes g(x, y) = f(x)g(y)$. We call K_A the Schwartz kernel of A.

Essentially, the Schwartz kernel theorem tells us that if $A \colon \mathcal{D}(U) \to \mathcal{D}(V)$ is a continuous linear operator, then we can find a unique distribution K_A on $\mathcal{D}(U \times V)$ which completely characterizes A. This is useful since we can bring the power of distribution theory to bear to study continuous linear operators which enlarges our toolkit of techniques to study such objects. The Schwartz kernel theorem also holds for the compactly supported and tempered distributions as well, meaning that we have the following topological isomorphisms:

$$L(\mathcal{S}(\mathbf{R}^n), \mathcal{S}'(\mathbf{R}^n)) \cong \mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^n),$$
$$L(\mathcal{E}(U), \mathcal{E}'(V)) \cong \mathcal{E}'(U \times V)$$

where each dual space is given the strong topology.

Fourier transform of tempered distributions

Much like how we used duality to define the differentiation and convolution of distributions we can similarly define the Fourier transform of distributions by duality. By Proposition 1.1.1 Property (4) we have that whenever $f, g \in L^1(\mathbf{R}^n)$ that

$$\int_{\mathbf{R}^n} f(x)\widehat{g}(x)\, dx = \int_{\mathbf{R}^n} \widehat{f}(x)g(x)\, dx\,.$$

Thus, if $T \in \mathcal{D}'(\mathbf{R}^n)$, we may be tempted to define the Fourier transform of T by $\langle \widehat{T}, f\rangle = \langle T, \widehat{f}\rangle$ where $f \in \mathcal{D}(\mathbf{R}^n)$. However, this in fact does not make sense since by the Paley-Wiener theorem $\widehat{f}$ is not even compactly supported. However, if we take $T \in \mathcal{S}'(\mathbf{R}^n)$, then defining $\widehat{T}$ in this way is well-defined by Theorem 1.1.8. Thus we propose the following definition.

Definition 1.1.10. If $T \in \mathcal{S}'(\mathbf{R}^n)$ is a tempered distribution we define the *Fourier transform* of T by duality: $\langle \widehat{T}, f\rangle = \langle T, \widehat{f}\rangle$.

We first have an important theorem which characterizes the Fourier transform of a compactly supported distribution.

Theorem 1.1.13. *If $T \in \mathcal{E}(\mathbf{R}^n)$, then $\widehat{T}$ is a holomorphic function given by $y \mapsto \langle T_x, e^{-i\langle x,y\rangle}\rangle$.*

Proof. What we mean by $\widehat{T}$ being a holomorphic function is that $\widehat{T}$ agrees with $y \mapsto \langle T_x, e^{-i\langle x,y\rangle}\rangle$ as distributions (elements of $\mathcal{D}'(\mathbf{R}^n)$). Thus, if $\phi \in \mathcal{D}(\mathbf{R}^n)$, then $\widehat{T}$ is a well-defined element of $\mathcal{D}'(\mathbf{R}^n)$ and

$$\langle \widehat{T}, \phi(x)\rangle = \langle T_x, \int_{\mathbf{R}^n} \phi(y)e^{-i\langle x,y\rangle}\, dy\rangle.$$

Now the function $y \mapsto \phi(y)e^{-i\langle x,y\rangle}$ is a compactly supported function from $\mathbf{R}^n$ into $\mathcal{E}(\mathbf{R}^n)$. In fact it is an instructive exercise to prove that this map is continuous from $\mathbf{R}^n$ to $\mathcal{E}(\mathbf{R}^n)$. Thus, by Theorem A.2.2 we can commute T_x with the integral to obtain

$$\langle \widehat{T}, \phi(x)\rangle = \langle T_x, \int_{\mathbf{R}^n} \phi(y)e^{-i\langle x,y\rangle}\, dy\rangle = \int_{\mathbf{R}^n} \phi(y)T_x(e^{-i\langle x,y\rangle})\, dy\,.$$

Thus $\widehat{T}$ and $y \mapsto \langle T_x, e^{-i\langle x,y\rangle}\rangle$ agree as distributions. That $y \mapsto \langle T_x, e^{-i\langle x,y\rangle}\rangle$ is smooth follows as in the proof of Proposition 1.1.10 and if we complexify y, then we see that the Cauchy-Riemann equations hold so that $y \mapsto \langle T_x, e^{-i\langle x,y\rangle}\rangle$ is a holomorphic function. $\square$

Now returning to the general study of Fourier transforms of distributions we have the following analogues of Proposition 1.1.1 Property (2) and the convolution theorem of Theorem 1.1.5 for distributions. First a definition.

Definition 1.1.11. If $T \in \mathcal{D}'(U)$ and $\phi \in \mathcal{E}(U)$, then we define $\phi T \in \mathcal{D}'(U)$ by $\langle \phi T, f\rangle = \langle T, \phi f\rangle$.

Now if $T \in \mathcal{S}'(\mathbf{R}^n)$ and $S \in \mathcal{E}'(\mathbf{R}^n)$, then in the previous section we have defined the convolution $S * T$. We can also define the convolution $T * S$ by the rule: $T * S(f) = \langle T_y, \langle S_x, f(x+y)\rangle\rangle$. That this indeed gives rise to a tempered distribution $T * S \in \mathcal{S}'(\mathbf{R}^n)$ stems from the fact that the function $y \mapsto \langle S_x, f(x+y)\rangle$ is also a Schwartz function. We leave the verification of this claim to the reader. Now for the result of our interest.

Theorem 1.1.14. *If $T \in \mathcal{S}'(\mathbf{R}^n)$, then $(\partial^\alpha T)\widehat{\ }(y) = (iy)^\alpha \partial^\alpha \widehat{T}(y)$. And if $S \in \mathcal{E}'(\mathbf{R}^n)$, then $T * S$ belongs to $\mathcal{S}'(\mathbf{R}^n)$ and the Fourier transform satisfies $(T * S)\widehat{\ } = \widehat{T}\widehat{S}$.*

Proof. By definition, the first statement means that $\langle (\partial^\alpha T)\widehat{\ }, f\rangle = \langle (iy)^\alpha \partial_y^\alpha T, f(y)\rangle = \langle \partial^\alpha T_y, (iy)^\alpha f(y)\rangle$. Thus the first statement follows immediately by Proposition 1.1.1. That $T * S \in \mathcal{S}'(\mathbf{R}^n)$ follows from the fact that if $\phi \in \mathcal{S}(\mathbf{R}^n)$, then $\langle S_x, \phi(x+y)\rangle$ is a Schwartz function as discussed above. Now if $f \in \mathcal{D}(\mathbf{R}^n)$, we write $\widehat{S}(y) = S_x(e^{-i\langle x,y\rangle})$ and observe

$$\langle (T * S)\widehat{\ }, f\rangle = \langle T_y, \langle S_x, \widehat{f}(x+y)\rangle\rangle.$$

Since

$$\langle S_x, \widehat{f}(x+y)\rangle = \int_{\mathbf{R}^n} f(z)\widehat{S}(z)e^{-i\langle y,z\rangle}\, dz$$

we have $\langle T_y, \langle S_x, \widehat{f}(x+y)\rangle\rangle = \langle T_y, (\widehat{S}f)\widehat{\ }(y)\rangle = \widehat{T}(\widehat{S}f)$. This proves the statement. $\square$

Similarly, we can define the *inverse Fourier transform* of a tempered distribution T by $\langle \check{T}, f\rangle = \langle T, \check{f}\rangle$ which once again stems from duality. We similarly have $(\widehat{T})\check{\ } = (\check{T})\widehat{\ }$ as in the case of ordinary functions. There are many more useful topics that we can explore by studying Fourier analysis on distributions more closely and carefully. However, for our purposes, particularly the convolution theorem shall prove sufficient motivation when we come to Chapter 3. The main point is that the standard Fourier analysis for functions extends in a natural way to distributions with the caveat that we must restrict ourselves to Schwartz functions and tempered distributions. This highlights another important feature of the Schwartz space which plays a pivotal role in our discussion in Chapter 3.

1.2 Extensions to topological groups

The Fourier transform on $\mathbf{R}^n$ can in fact be extended to any locally compact Hausdorff topological group. However it is only for the abelian and compact groups does there exist a "good" Fourier transform theory (at least at the moment). By good Fourier transform theory we mean inversion and Plancherel type theorems in analogy to the Euclidean case. To get a good idea of how we can generalize the Fourier transform let us recall that the Fourier transform of a function $f \in L^1(\mathbf{R}^n)$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-i\langle x,\xi\rangle}\, dx\,.$$

As we have seen the complex exponentials $e_\xi(x) = e^{i\langle x,\xi\rangle}$ have played a key role in our Fourier analysis on $\mathbf{R}^n$. For instance, they are eigenfunctions for all constant coefficient differential operators with $\partial_j e_\xi = i\xi_j e_\xi$. However these functions play a far more basic role as reflected in the following proposition.

Proposition 1.2.1. *Let $f\colon \mathbf{R}^n \to \mathbf{C}$ be a continuous function such that for each $x, y \in \mathbf{R}^n$ we have $f(x+y) = f(x)f(y)$ and $|f(x)| = 1$. Then $f \in C^\infty(\mathbf{R}^n)$ and there exists $\xi \in \mathbf{R}^n$ such that $f(x) = e^{i\langle x,\xi\rangle}$.*

Proof. Choose a function $g \in C_c^\infty(\mathbf{R}^n)$ with the property that $\int_{\mathbf{R}^n} f(y)g(y)\,dy = 1$. Then we have

$$f(x) = \int_{\mathbf{R}^n} f(x)f(y)g(y)\,dy = \int_{\mathbf{R}^n} f(x+y)g(y)\,dy = \int_{\mathbf{R}^n} f(y)g(x+y)\,dy\,.$$

By the dominated convergence theorem, $f \in C^\infty(\mathbf{R}^n)$ and $f(0) = 1$. Let e_j be the basis vector of $\mathbf{R}^n$ with zeros in all its entries except for a one in the jth position. Then computing the partial derivative of f:

$$\partial_j f(x) = \lim_{h\to 0} \frac{f(x+he_j) - f(x)}{h} = f(x) \lim_{h\to 0} \frac{f(he_j) - 1}{h} = f(x)\partial_j f(0).$$

Put $f_j(t) = f(te_j)$ for $t \in \mathbf{R}$, then $f(x) = \prod_{j=1}^n f_j(x_j)$. Moreover $f_j'(t) = f_j(t)\partial_j f(0)$ so that if we put $z_j = \partial_j f(0)$ we have $f_j(x_j) = e^{z_j x_j}$. Note that $|f_j| = 1$ so each $z_j = i\xi_j$ for some $\xi_j \in \mathbf{R}$. So if $\xi = (\xi_1, \ldots, \xi_n)$ we have $f(x) = e^{i\langle x,\xi\rangle}$. ☐

The importance of this fact is that the complex exponential functions determine all the continuous homomorphisms from $\mathbf{R}^n$ to the torus $\mathbf{T} = \{z \in \mathbf{C} \colon |z| = 1\}$. More importantly, the complex exponentials are examples of (one-dimensional) *representations* of $\mathbf{R}^n$. As we are hinting at, the representation theory of the group becomes enormously important in the theory of the Fourier transform.

We shall only give a survey of the Fourier transform theory for a locally compact Hausdorff (compact or abelian) group.

1.2.1 Representations of topological groups

Throughout this section we denote by G a locally compact Hausdorff topological group and we shall refer to such groups as locally compact (without the qualifier Hausdorff).

Definition 1.2.1. Let V be a topological vector space and let $\operatorname{End} V$ denote the set of continuous endomorphisms of V. A *representation* of G on V is a homomorphism $\pi\colon G \to \operatorname{End} V$ such that the mapping $(x, v) \mapsto \pi(x)v$ from $G \times V$ into V is continuous. We shall call the pair (π, V) a representation of G. The *dimension* of π, which we denote by d_π, is the dimension of V.

We shall be more or less be concerned with *unitary* representations. We say a representation π of G on a Hilbert space H is unitary if $\pi(x)$ is a unitary operator on H for all $x \in G$. To motivate an example of a unitary representation first we recall that if G is a locally compact group, we know that there exists a left Haar measure dx on G. That is a Radon measure on G characterized by

$$\int_G f(yx)\,dx = \int_G f(x)\,dx$$

for all $y \in G$.

Example 1.2.2. The most obvious representation of G on a Hilbert space H is when we take $H = L^2(G)$ and $[\pi(x)f](y) = f(x^{-1}y) = L_x f(y)$. We leave these details as an exercise to the reader (see [9, Ch. 3] for those details). This representation is called the *left-regular representation* of G.

Example 1.2.3. The exponential functions $e_\xi(x)\colon \mathbf{R}^n \to \mathbf{C}$, $x \mapsto e^{i\langle x,\xi\rangle}$ are unitary representations of $\mathbf{R}^n$ on $\mathbf{C}$.

As one may surmise, representations can be determined "up to an equivalence" in the following manner.

Definition 1.2.2. If (π, V) and (δ, W) are two unitary representations of G, an *intertwining operator* T of (π, V) and (δ, W) is a continuous linear map $T\colon V \to W$ such that $T\pi(x) = \delta(x)T$ for all $x \in G$. We say that π and δ are *unitarily equivalent* if there exists a unitary intertwining operator $U\colon V \to W$ of (π, V) and (δ, W).

If (π, V) is a representation and if $X \subset V$ is a subspace, we say that X is *invariant* under π if the orbit $\pi(G)X = \{\pi(x)v \colon x \in X, v \in X\}$ is contained in X. In the case that X is a closed invariant subspace, the restriction of π to X determines a *subrepresentation* $(\pi|_X, X)$. We say that the representation (π, V) is *irreducible* if π has no nontrivial subrepresentations (i.e. $X \neq V$ or $X \neq \{0\}$).

Definition 1.2.3. We denote $\widehat{G}$ to be the set of all unitary irreducible representations on G modulo unitary equivalence. We call $\widehat{G}$ the dual object of G.

If G is an abelian group, then it turns out by Schur's lemma that any irreducible unitary representation is one-dimensional by Schur's lemma [9, Thm. 3.5]. In particular if (π, V) is a unitary irreducible representation, then $V \cong \mathbf{C}$, and for $x \in G$ and $z \in \mathbf{C}$, we have that there is a complex number w_x such that $\pi(x)z = w_x z$ for all $z \in \mathbf{C}$. Thus we can identify π with a continuous complex valued function $\chi_\pi \in C(G)$. Since π is unitary it follows that $|\chi_\pi| = 1$, in particular χ_π is the *character* of π. In fact, since the elements of $\widehat{G}$ in this case are just functions of complex modulus 1, we can turn $\widehat{G}$ into an abelian group via the operation of pointwise multiplication of these characters. In this situation, we call $\widehat{G}$ the dual group of G. We can topologize the dual group with the topology of locally uniform convergence (the same topology as on $C(G)$), and with this topology, $\widehat{G}$ becomes a locally compact abelian group.

Example 1.2.4. If $G = \mathbf{R}^n$, then the unitary irreducible representations of $\mathbf{R}^n$ are the functions $\widehat{\mathbf{R}^n} = \{e_\xi \colon \mathbf{R}^n \to \mathbf{C} \colon e_\xi(x) = e^{i\langle x, \xi \rangle}, \xi \in \mathbf{R}^n\}$. In this case $\widehat{\mathbf{R}^n}$ is isomorphic to $\mathbf{R}^n$ itself.

Example 1.2.5. If $G = \mathbf{T}^n$, the n-dimensional torus, then the unitary irreducible representations are the functions $\widehat{\mathbf{T}^n} = \{e_\kappa \colon \mathbf{T}^n \colon \mathbf{C} \colon e_\kappa(x) = e^{i\langle x, \kappa \rangle},\ \kappa \in \mathbf{Z}^n\}$. In this situation we have the identification $\widehat{\mathbf{T}^n} \cong \mathbf{Z}^n$. Conversely, the integers $\mathbf{Z}^n$ is also a locally compact abelian group whose dual group is isomorphic to $\mathbf{T}^n$, here its characters are the functions of the form: $e^{i\theta}(\kappa) := e^{i\theta_1\kappa_1} \cdots e^{i\theta_n\kappa_n}$ where $(e^{i\theta_1}, \ldots, e^{i\theta_n}) \in \mathbf{T}^n$ and $\kappa \in \mathbf{Z}^n$.

Remark 1. Technically the elements of the dual groups above are equivalence classes. However the characters of irreducible unitary representations from the same equivalence class are the same.

If G is not abelian, then it becomes slightly more difficult to characterize $\widehat{G}$. However if G is compact, then any unitary irreducible representation is finite dimensional [9, Thm. 5.2]. So one can identify elements of $\widehat{G}$ with matrices (after a choice of basis), of varying dimension. In this situation there is no (at least obvious or canonical) way to turn $\widehat{G}$ into a group. Nevertheless, there exists a canonical topology on $\widehat{G}$ (here G need not be compact or abelian) called the *Fell topology* (see [9, Sec. 7.2] for a definition). If G is abelian, then with the Fell topology $\widehat{G}$ has the topology that we have described just earlier. If G is compact, then the Fell topology is the discrete topology on $\widehat{G}$. Conversely, if G is a discrete group, then with the Fell topology $\widehat{G}$ becomes a compact space.

Turning now to the Fourier transform we focus our attention to the space $L^1(G)$. We recall that $L^1(G)$ is a Banach *-algebra with multiplication given by the convolution

$$f * g(x) = \int_G f(y)g(y^{-1}x)\, dx$$

and involution given by $f(x) = \delta(x^{-1})\overline{f}(x^{-1})$ where $\delta \colon G \to \mathbf{R}^+$ is the modular function. We refer the reader to Section A.2.1 for the definition of a Banach *-algebra and to Section A.1 for the definitions of convolution and the modular function and their essential properties. It is interesting to note that G is abelian if and only if $L^1(G)$ is commutative and that if G is abelian, then $\delta \equiv 1$.

Now if (π, V) is a unitary representation on a Hilbert space of a locally compact group G, we obtain a representation on $L^1(G)$ defined by

$$\pi(f) = \int_G f(x)\pi(x)\, dx \in \operatorname{End} V,$$

where $f \in L^1(G)$ and dx is the left Haar measure on G. This is an "operator valued" integral where if $v \in V$, we interpret $\pi(f)v$ in the sense of weak integrals. That is to say

$$\langle \pi(f)v, u\rangle_V = \int_G f(x)\langle \pi(x)v, u\rangle_V \, dx$$

where $\langle \cdot, \cdot \rangle_V$ is the Hilbert space norm on V. Since $\langle \pi(f)v, u\rangle_V$ is a continuous linear functional on V, by the Riesz representation theorem $\pi(f)v$ exists. The operator norm satisfies $\|\pi(f)\|_{\mathrm{op}} \leq \|f\|_1$ and so we can see that $f \mapsto \pi(f)$ is a representation on $L^1(G)$. The basic properties of the mapping $f \mapsto \pi(f)$ are summarized by the following proposition.

Proposition 1.2.6. *If $f, g \in L^1(G)$, then $\pi(L_x f) = \pi(x)\pi(f)$ and $\pi(f * g) = \pi(f)\pi(g)$.*

Proof. The equality $\pi(L_x f) = \pi(x)\pi(f)$ is easily verified. And

$$\pi(f * g) = \int_G f * g(x)\pi(x)\, dx = \int_G \left\{ \int_G f(y)g(y^{-1}x)\, dy \right\} \pi(x)\, dx$$
$$= \int_G \int_G f(y)g(x)\pi(yx)\, dy\, dx = \int_G \int_G f(y)g(x)\pi(y)\pi(x)\, dx\, dy = \pi(f)\pi(g).$$

Of course, one must ask if our manipulations are valid since the above integrals are actually weak integrals. Indeed, one is really trying to check that for all $v \in V$ that $\pi(f * g)v = \pi(f)\pi(g)v$. To this end, we see

$$\langle \pi(f * g)u, v\rangle_V = \int_G f * g(x)\langle \pi(x)u, v\rangle_V \, dx = \int_G \int_G f(y)g(x)\langle \pi(y)\pi(x)u, v\rangle_V \, dx\, dy\,.$$

Since

$$\int_G f(y)g(x)\langle \pi(y)\pi(x)u, v\rangle_V \, dx = \left\langle f(y)\pi(y) \left\{ \int_G g(x)\pi(x)u\, dx \right\}, v \right\rangle_V = f(y)\langle \pi(y)\pi(g)u, v\rangle_V$$

we have

$$\langle \pi(f * g)u, v\rangle_V = \int_G f(y)\langle \pi(y)\pi(g)u, v\rangle_V \, dy = \langle \pi(f)\pi(g)u, v\rangle_V,$$

□

and we are done.

1.2.2 Fourier transform for abelian and compact groups

Abelian groups

If we assume G is an abelian group, we note that $\widehat{G}$ is an abelian group as well. And if $[\pi] \in \widehat{G}$ is an equivalence class, then the characters of each element in $[\pi]$ are all the same. In our following definition of the Fourier transform our definition does not depend on a particular choice of representative. So throughout this section we shall have already chosen one representative from each equivalence class in $\widehat{G}$ and shall use the shorthand π to denote the equivalence class $[\pi] \in \widehat{G}$.

Definition 1.2.4. If $f \in L^1(G)$, we define its *Fourier transform* to be the map $\widehat{f} \colon \widehat{G} \to \mathbf{C}$ defined by

$$\widehat{f}(\pi) = \int_G f(x)\chi_\pi(x^{-1})\, dx = \int_G f(x)\overline{\chi_\pi(x)}\, dx\,. \tag{1.2.1}$$

Here $\overline{\chi_\pi}$ is the complex conjugate of the character χ_π. We will also write $\widehat{f}(\chi_\pi) = \widehat{f}(\pi)$. In the Euclidean case where $G = \mathbf{R}^n$, this definition agrees precisely with the ordinary Fourier transform through the identification $\mathbf{R}^n \cong \widehat{\mathbf{R}^n}$. We shall now state some of the critical theorems which characterize the Fourier transform for locally compact abelian groups.

Theorem 1.2.7. *The Fourier transform is a continuous $*$-homomorphism from $L^1(G)$ into $C_0(\widehat{G})$.*

Proof. In view of Proposition 1.2.6, we have that $(f * g)\widehat{\ } = \widehat{f}\widehat{g}$ for $f, g \in L^1(G)$. Furthermore,

$$\widehat{f^*}(\chi_\pi) = \int_G \overline{f(x^{-1})\chi_\pi(x)}\, dx = \int_G \overline{f(x)}\chi_\pi(x)\, dx = \overline{\widehat{f}(\chi_\pi)}$$

so that the Fourier transform preserves the involution.

We can in fact prove the rest of this theorem all in one go via Gelfand theory which is a statement about commutative Banach algebras (see A.2.1 for the relevant theorems and definitions). Namely, we wish to show that the Fourier transform coincides with the Gelfand transform when G is abelian.

Our work reduces to showing that the spectrum of $L^1(G)$ is identified with the characters in $\widehat{G}$. If χ_π is a character on $\widehat{G}$, then it induces a nonzero multiplicative linear functional by $\chi_\pi(f) = \widehat{f}(\overline{\chi_\pi})$. Conversely, if T is a multiplicative linear functional, then we have by ordinary duality considerations that there exists $\phi \in L^\infty(G)$ such that $T(f) = \int_G f(x)\phi(x)\, dx$. Now for $f, g \in L^1(G)$

$$\begin{aligned}\int_G T(f)\phi(x)g(x)\, dx = T(f)T(g) = T(f * g) &= \int_G \int_G f(y)g(y^{-1}x)\phi(x)\, dy\, dx \\ &= \int_G \int_G f(y^{-1}x)g(y)\phi(x)\, dy\, dx = \int_G T(L_y f)g(y)\, dy\,.\end{aligned}$$

Therefore $T(f)\phi(x) = T(L_x f)$ almost everywhere on G. Since $y \mapsto L_y f$ is a continuous map from G into $L^1(G)$ we have that if we choose $T(f) \neq 0$, that ϕ is almost everywhere equal to a continuous function and so we may assume that ϕ is continuous. Also $T(f)\phi(xy) = T(L_{xy}f) = T(L_x L_y f) = \phi(x)T(L_y f) = \phi(x)\phi(y)T(f)$ so that $\phi(xy) = \phi(x)\phi(y)$. Lastly $\phi(x^n) = \phi(x)^n$ and $\phi(x^{-n}) = \phi(x)^{-n}$. Since ϕ is bounded on G we must have $|\phi(x)| = 1$ for all $x \in G$. Thus, ϕ is infact a character of G and can be regarded as an element of $\widehat{G}$. What we have shown is that the set of nonzero multiplicative linear functionals can be identified with $\widehat{G}$. Thus $\widehat{G} = \sigma(L^1(G))$ as sets. These in fact also coincide topologically by Theorem 3.31 of [9]. Thus it follows that $\widehat{G}$ coincides with $\sigma(L^1(G))$ as topological spaces.

Therefore the Gelfand transform coincides with the Fourier transform and so by Gelfand theory we have that the image of $L^1(G)$ under the Fourier transform is a subset of $C_0(\widehat{G}) \cong C_0(\sigma(L^1(G)))$ (which is in fact dense in $C_0(\widehat{G})$ by the Stone-Weierstrass theorem).

We can see the continuity of the Fourier transform without appealing to the Gelfand theory by virtue of the estimate $\|\widehat{f}\|_\infty \leq \|f\|_1$. ☐

Turning our attention to the issue of inversion of the Fourier transform we recall that on $\mathbf{R}^n$ we have that if $f, \widehat{f} \in L^1(\mathbf{R}^n)$, then the Fourier transform of f can be inverted by $f(x) = (2\pi)^{-n}\widehat{\widehat{f}}(x^{-1})$. We have the analogous theorem for locally compact abelian groups as well.

Theorem 1.2.8 (Fourier inversion formula)**.** *There exists a Haar measure $d\chi_\pi$ on $\widehat{G}$ so that if $f \in L^1(G)$ and $\widehat{f} \in L^1(\widehat{G}, d\chi_\pi)$, then*

$$f(x) = \int_{\widehat{G}} \widehat{f}(\chi_\pi)\chi_\pi(x)\, d\chi_\pi = \int_{\widehat{G}} f * \chi_\pi(x)\, d\chi_\pi, \tag{1.2.2}$$

for almost every $x \in G$. Since the right-hand side is a continuous function of x, it follows that if f is continuous, then the above formula holds for all $x \in G$.

A proof can be found in [9, Thm. 4.33]. The above theorem provides us a key statement that there is a Haar measure on $\widehat{G}$ so that the above formula holds (almost everywhere) for each $f \in L^1(G)$. This measure on $\widehat{G}$ is known as the *Plancherel measure* and is quite important for the harmonic analysis on locally compact groups. In the case of $\mathbf{R}^n$, if we choose our characters to be the functions $x \mapsto e^{i\langle x,\xi\rangle}$, then the corresponding Plancherel measure on $\widehat{\mathbf{R}^n} \cong \mathbf{R}^n$ is $(2\pi)^{-n}dx$. However if we choose our characters to be the functions $x \mapsto e^{2\pi i\langle x,\xi\rangle}$, then the corresponding Plancherel measure is merely dx.

Example 1.2.9. If $f\colon \mathbf{T} \to \mathbf{C}$ is a continuous periodic function, then its Fourier transform is defined by

$$\widehat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta$$

where $n \in \mathbf{Z}$. Since $\widehat{\mathbf{T}} = \mathbf{Z}$ is discrete, then a Haar measure $\mathbf{Z}$ is a multiple of the counting measure. In fact, in this case the Plancherel measure is the counting measure. In particular, for $g \in L^1(\mathbf{Z})$ we have

$$\int_{\mathbf{Z}} g(n)\, dn = \sum_{n=-\infty}^{\infty} g(n).$$

If we assume that $\sum_{n=-\infty}^{\infty} |\widehat{f}(n)| < \infty$, then by the Fourier inversion formula we have that f is almost everywhere equal to its Fourier series:

$$f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \widehat{f}(n) e^{in\theta}.$$

In our two examples of Fourier transforms on $\mathbf{R}^n$ and on $\mathbf{T}$ we note that the inversion formula also uses characters of the corresponding dual group which is to say, that the functions $\xi \mapsto e^{i\langle x,\xi\rangle}$ and $n \mapsto e^{in\theta}$ are also characters on $\widehat{\mathbf{R}^n}$ and $\mathbf{Z}$. In fact, if $\widehat{G}$ is the dual group of an abelian group G, then the function $\chi_\pi \mapsto \chi_\pi(x)$ for some fixed $x \in G$ is also a character of $\widehat{G}$. This phenomenon is encoded by the *Pontryagin duality* (see [9, Sec. 4.3]).

Theorem 1.2.10 (Pontryagin Duality Theorem)**.** *The mapping $\Lambda\colon G \mapsto \widehat{\widehat{G}}$ given by $x \mapsto (\chi_\pi \mapsto \chi_\pi(x))$ is an isomorphism of topological groups.*

Using this duality, we note that the Fourier transform on $\widehat{G}$ takes the form of

$$(\mathscr{F}_{\widehat{G}} f)(x) = \int_{\widehat{G}} f(\chi_\pi)(\chi_\pi)^{-1}(x)\, d\chi_\pi = \int_{\widehat{G}} f(\chi_\pi)\overline{\chi_\pi}(x)\, d\chi_\pi\,.$$

where $d\chi_\pi$ is the Haar measure on $\widehat{G}$. If we let $\mathscr{F}_G$ and $\mathscr{F}_{\widehat{G}}$ denote the Fourier transforms on G and $\widehat{G}$ respectively, then the Fourier inversion theorem states that there exists a constant c_0 so for all $f \in L^1(G)$ and $\mathscr{F}_G f \in L^1(\widehat{G})$ we have

$$f(x) = c_0 \mathscr{F}_{\widehat{G}} \mathscr{F}_G f(x^{-1})$$

almost everywhere.

The other important theorem for us is the *Plancherel theorem.* In the Euclidean case, the Plancherel theorem stated that the Fourier transform extends from $L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$ to a isometric isomorphism of $L^2(\mathbf{R}^n)$ and $L^2(\mathbf{R}^n, (2\pi)^{-n} dx)$. In the locally compact abelian group case we have the same result.

Theorem 1.2.11 (Plancherel Theorem)**.** *The Fourier transform on $L^1(G) \cap L^2(G)$ extends to a unitary isomorphism from $L^2(G)$ to $L^2(\widehat{G})$ where $\widehat{G}$ is endowed with the Plancherel measure.*

Theorem 1.2.7, the Fourier inversion formula, and the Plancherel theorem are the bread and butter of harmonic analysis and encapsulate all the theorems relating to the Fourier transform on $\mathbf{R}^n$ without any reference to a differential structure (i.e. Paley-Wiener theorem, Schwartz space, etc.). One could wonder what happens if G becomes an abelian Lie group and in this instance one may ask how does the Fourier transform interact with the differential operators on G. We shall carry out a short discussion of this for compact Lie groups however at the moment we shall briefly cover the abstract Fourier transform theory for a compact group.

Compact groups

For an abelian group G we have defined the Fourier transform as a function on $\widehat{G}$ which is the set of irreducible unitary representations which is identified with the set of all continuous characters of G. If G is not abelian, then its unitary irreducible representations can become quite complicated however when G is compact, all of its unitary irreducible representations are finite dimensional. We define the Fourier transform on a compact group G as follows.

Definition 1.2.5. If $f \in L^1(G)$ and π is the chosen representative of the equivalence class $[\pi] \in \widehat{G}$, then we define the π-Fourier coefficient by

$$\widehat{f}(\pi) = \int_G f(x)\pi(x^{-1})\, dx\,. \tag{1.2.3}$$

We note that $\widehat{f}(\pi) \in \operatorname{End} V_\pi$. If we choose to represent π as a $d_\pi \times d_\pi$ matrix over $\mathbf{C}$ where we identify $V_\pi \cong \mathbf{C}^{d_\pi}$, then we have that $\widehat{f}(\pi) \in M_{d_\pi}(\mathbf{C})$. Clearly a draw back to this definition of a Fourier transform is that it is now operator valued (or really matrix valued).

Since G is compact, we have by Hölder's inequality that $L^p(G) \subset L^1(G)$ for $p \geq 1$ where we are using the Haar measure dx which is normalized so that $\int_G 1\, dx = 1$. If $f \in L^2(G)$, then we have an inversion formula given by the Peter-Weyl theorem.

Theorem 1.2.12 (Peter-Weyl)**.** *If $f \in L^2(G)$, then*

$$f(x) = \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{Tr}\Big(\widehat{f}(\pi)\pi(x)\Big) \tag{1.2.4}$$

which converges to f in $L^2(G)$ and to f pointwise for almost every $x \in G$. If $\chi_\pi(x) = \operatorname{Tr}(\pi(x))$, then this can be rewritten as

$$f(x) = \sum_{[\pi]\in\widehat{G}} d_\pi f * \chi_\pi(x). \tag{1.2.5}$$

As usual, a proof is provided in Folland [9, Sec. 5.2]. Since G is compact, $\widehat{G}$ is discrete. So if we choose the weighted counting measure on $\widehat{G}$ defined by $\mu(A) = \sum_{[\pi]\in A} d_\pi$ where $A \subset \widehat{G}$, then we have the "integral" formula

$$f(x) = \int_{\widehat{G}} \operatorname{Tr}\Big(\widehat{f}(\pi)\pi(x)\Big)\, d\mu([\pi]),$$

for $f \in L^2(G)$. To state the Plancherel theorem we require a definition.

Definition 1.2.6. Define the space $L^2(\widehat{G})$ as the space of all functions f on $\widehat{G}$ which satisfy:

1. $f([\pi]) \in \operatorname{End} V_\pi$ whenever $[\pi] \in \widehat{G}$.
2. And
$$\|f\|^2_{L^2(\widehat{G})} = \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{Tr}(f([\pi])f([\pi])^*) < \infty.$$
Here, $f([\pi])^*$ denotes the adjoint operator of $f([\pi])$.

With the assignment $f \mapsto \|f\|_{L^2(\widehat{G})}$ defines a norm on $L^2(\widehat{G})$ under which $L^2(\widehat{G})$ becomes a complex Hilbert space with the respect to the inner product

$$\langle f, g\rangle_{L^2(\widehat{G})} = \sum_{[\pi]\in\widehat{G}} d_\pi \operatorname{Tr}(f([\pi])g([\pi])^*).$$

Then the Plancherel theorem reads as follows.

Theorem 1.2.13 ([20, Thm. 7.6.13]). *The Fourier transform defines a unitary isometric isomorphism from* $L^2(G)$ *onto* $L^2(\widehat{G})$.

There is some noticeable degeneracy to the Fourier transform theory on a compact group. Unlike in the abelian group case there is no easy analogue of a Riemann-Lebesgue lemma. In fact, such a lemma makes no sense since $C_0(\widehat{G}) = C(\widehat{G})$ consists of *scalar-valued* functions while the Fourier transform is operator valued. Thus any appropriate formulation of a Riemann-Lebesgue lemma must incorporate such types of functions and also ought to propose a logical topology on them.

To close our remarks on the abstract Fourier transform theory for locally compact abelian and compact groups we summarize the so-called "generic" results that we have obtained. If G is abelian or compact, we have that the Fourier transform diagonalizes convolution on $L^1(G)$, that is it turns the convolution of functions into products of functions (or composition of operators). We also have that there is a Plancherel theorem which establishes unitary isomorphisms between $L^2(G)$ and $L^2(\widehat{G})$ where the isomorphism is provided by an extension of the Fourier transform. We also have a Fourier inversion formula which holds for certain classes integrable functions in G. When G is abelian we can say much more. In particular, we obtain an analogue of the Riemann-Lebesgue.

We can also seek to provide analogues for the Schwartz space theory as in Theorem 1.1.8. In the case that G is abelian, this theory is provided by the Schwartz-Bruhat spaces which can be defined for any locally compact abelian group. In particular, if $\mathcal{S}_b(G)$ denotes the Schwartz-Bruhat space of an abelian group, then the Fourier transform establishes a homeomorphism between $\mathcal{S}_b(G)$ and $\mathcal{S}_b(\widehat{G})$. In the case that $G = \mathbf{R}^n$, then $\mathcal{S}_b(\mathbf{R}^n) = \mathcal{S}(\mathbf{R}^n)$. Schwartz-Bruhat spaces are a bit tricky to define so we provide a reference [19]. Intuitively, Schwartz-Bruhat spaces are characterized as satisfying a certain decay condition on functions and their so-called "generalized derivatives." In the following section however, we can provide an analogue of the Schwartz space theory rather explicitly.

1.3 Some compact Lie groups

1.3.1 The torus T

The torus $\mathbf{T}$ is of course a compact abelian Lie group. If we identify $\mathbf{T}$ with the unit circle in $\mathbf{C}$, we use the standard angular coordinates $\theta \mapsto e^{i\theta}$ for $\theta \in \mathbf{R}$. So in particular, the functions $\mathbf{T}$ are identified the 2π periodic functions on $\mathbf{R}$. As usual if $f \in L^1(\mathbf{T})$, we write

$$\mathscr{F}(f)(n) = \widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-in\theta}\, d\theta$$

and for $g \in L^1(\mathbf{Z})$ we put

$$\mathscr{F}^{-1}(g)(\theta) = \check{g}(\theta) = \sum_{n\in\mathbf{Z}} g(n)e^{in\theta}.$$

Using the angular coordinates one can show that any smooth differential operator D on $\mathbf{T}$ is of the form $D = \sum_{j=1}^n a_j(x)\frac{d^j}{d\theta^j}$ ($a_j \in C^\infty(\mathbf{T})$) defined globally on $\mathbf{T}$ (see [18, pp. 176]).

Now if $f \in C^k(\mathbf{T})$ and $j \leq k$, then putting $f^{(j)} = \frac{d^j}{d\theta^j}f$ we have that the Fourier transform on the circle satisfies a variant of Property (2) in Proposition 1.1.1 in the sense that

$$\widehat{f^{(j)}}(n) = (in)^j\widehat{f}(n)$$

by integration by parts and the periodicity of f and its derivatives. So in particular, the Fourier transform on $\mathbf{T}$ interacts just as nicely with constant coefficient differential operators as it does on $\mathbf{R}^n$. In particular, if $f \in C^k(\mathbf{T})$ for $k \geq 2$, then the Fourier series

$$\sum_{n=-\infty}^{\infty}\widehat{f}(n)e^{in\theta}$$

converges absolutely. By the Fourier inversion theorem for abelian groups, f is equal almost everywhere to its Fourier series but by the absolute convergence of its Fourier series it follows by the dominated convergence theorem that f equals its Fourier series everywhere. And by the dominated convergence theorem again we can differentiate f by differentiating the Fourier series of f term by term.

It is interesting to note that if $f \in C^\infty(\mathbf{T})$, then the Fourier transform of f satisfies a Schwartz-like estimate:

$$\sup_{n\in\mathbf{Z}} |(1+|n|)^k \widehat{f}(n)| < \infty$$

for all $k \in \mathbf{N}$. This motivates a definition of the *Schwartz space* $\mathcal{S}(\mathbf{Z})$ on $\mathbf{Z}$ defined to be the set of all functions $g\colon \mathbf{Z} \to \mathbf{C}$ which satisfy $\nu_k(g) = \sup_{n\in\mathbf{Z}} |(1+|n|)^k g(n)| < \infty$ for all $k \in \mathbf{N}$. The space $\mathcal{S}(\mathbf{Z})$ is topologized by the seminorms ν_k. We now have Schwartz's isomorphism theorem for the torus.

Theorem 1.3.1. *The Fourier transform $\mathscr{F}\colon C^\infty(\mathbf{T}) \to \mathcal{S}(\mathbf{Z})$ is a topological isomorphism of $C^\infty(\mathbf{T})$ onto $\mathcal{S}(\mathbf{Z})$.*

Proof. It is clear that $\mathscr{F}(C^\infty(\mathbf{T})) \subset \mathcal{S}(\mathbf{Z})$. If $g \in \mathcal{S}(\mathbf{Z})$, then the function $f(\theta) = \sum_{n\in\mathbf{Z}} g(n)e^{in\theta}$ is easily seen to be smooth by term by term differentiation which is justified by the dominated convergence theorem. And by the Fourier inversion theorem we determine that the Fourier transform is a bijection of $C^\infty(\mathbf{T})$ onto $\mathcal{S}(\mathbf{Z})$.

To show that the Fourier transform is a homeomorphism we first remark that the topology of $C^\infty(\mathbf{T})$ is induced by the seminorms, $\gamma_j(f) = \sup_{\theta\in\mathbf{T}} |f^{(j)}(\theta)|$, $f \in C^\infty(\mathbf{T})$, $j \in \mathbf{N}$. Now if $f \in C^\infty(\mathbf{T})$, then

$$\nu_k(\widehat{f}) = \sup_{n\in\mathbf{Z}} \frac{1}{2\pi}\left|\int_0^{2\pi} (1+|n|)^k f(\theta)e^{-in\theta}\, d\theta\right| \le \sum_{j=1}^{k} \binom{k}{j}\gamma_j(f).$$

So that $\mathscr{F}\colon C^\infty(\mathbf{T}) \to \mathcal{S}(\mathbf{Z})$ is continuous. The inverse $\mathscr{F}^{-1}\colon \mathcal{S}(\mathbf{Z}) \to C^\infty(\mathbf{T})$ is also continuous since for $g \in \mathcal{S}(\mathbf{Z})$ we have

$$\gamma_j(\check{g}) \le \sum_{n\in\mathbf{Z}} |n^j g(n)| \le \nu_{j+2}(g) \sum_{n\in\mathbf{Z}} \frac{|n|^j}{(1+|n|)^{j+2}} \le c_0\nu_{j+2}(g).$$

Here c_0 is a constant independent of g. Correspondingly, $\mathscr{F}^{-1}$ is continuous as well so that $\mathscr{F}$ is a homeomorphism.

□

The same type of results hold for the n-torus $\mathbf{T}^n$ with the associated Schwartz space on $\mathbf{Z}^n$ defined by

$$\mathcal{S}(\mathbf{Z}^n) = \left\{ g\colon \mathbf{Z}^n \to \mathbf{C}\colon \sup_{m\in\mathbf{Z}^n} (1+|m|)^k g(n)| < \infty,\ k \in \mathbf{N} \right\}$$

where $|m| = \sqrt{m_1^2 + \cdots + m_n^2}$ if $m = (m_1, \ldots, m_n) \in \mathbf{Z}^n$.

Remark 2. In allusion to the "Schwartz-Bruhat spaces" discussed in the previous section, the Schwartz-Bruhat spaces on $\mathbf{T}^n$ and on $\mathbf{Z}^n$ are $\mathcal{S}_b(\mathbf{T}^n) = C^\infty(\mathbf{T}^n)$ and $\mathcal{S}_b(\mathbf{Z}^n) = \mathcal{S}(\mathbf{Z}^n)$.

1.3.2 Compact semisimple Lie groups

In this subsection we summarize some analogous properties of the Fourier transform for compact semisimple Lie groups which hold for the torus. In particular we explain how the Fourier transform and irreducible unitary representations interact with differential operators on the group.

Suppose that $\pi\colon G \to \operatorname{End} V_\pi$ is a representation of a Lie group G on a Banach space V_π. We say that a vector $u \in V_\pi$ is smooth if the function $g \mapsto \pi(g)u$ is smooth as a function from G into V_π. Here the definition of the smoothness of vector valued functions is as follows.

Definition 1.3.1. Let $f\colon \mathbf{R} \to V$ where V is a Banach space. We say that f is differentiable if the difference quotient

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

exists in V for each $x \in \mathbf{R}^n$. Likewise, $f\colon \mathbf{R}^n \to V$ is partially differentiable in the jth coordinate if it is differentiable in the jth coordinate holding all other coordinates fixed. We say that $f\colon \mathbf{R}^n \to V$ is smooth if for all constant coefficient differential operators D, that Df exists in V.

Extending the definition of smoothness of Banach space valued functions f on a smooth manifold M is obvious: f is smooth if and only if the function $f \circ \phi^{-1}$ is smooth for all coordinate systems (U, ϕ) on M.

Now given a unitary representation (π, V_π) of a Lie group, we say that a vector $v \in V_\pi$ is smooth if the function $g \mapsto \pi(g)v$ is a smooth map. Denote the space of smooth vectors in V_π with respect to π by $C^\infty(\pi)$. We can identify a subspace $\mathcal{G}(\pi) \subset C^\infty(\pi)$ where $\mathcal{G}(\pi)$ is generated by all vectors of the form $\pi(f)u$ where $f \in C_c^\infty(G)$ and $u \in V_\pi$. We have the following result.

Proposition 1.3.2. *If $f \in C_c^\infty(G)$, then $\pi(f)u$ ($u \in V_\pi$) is a smooth vector. Moreover, $\mathcal{G}(\pi)$ is dense in V_π and $\mathcal{G}(\pi) = V_\pi$ if V_π is finite dimensional.*

Proof. Indeed, the vector $\pi(f)u$ ($f \in C_c^\infty(G)$, $u \in V_\pi$) is smooth since $\pi(x)\pi(f)u = \pi(L_x f)u$ where $L_x f(y) = f(x^{-1}y)$. From this observation it is not too difficult to verify that $\pi(L_x f)u$ is smooth (use local coordinates). In particular, $\mathcal{G}(\pi)$ is dense (and thus $C^\infty(\pi)$) in V_π since if $\{\psi_\alpha\}_{\alpha\in A}$ is any system of mollifiers (see Appendix A.1.4), then $\pi(\psi_\alpha)u \to u$ for any vector $u \in V_\pi$. Thus if V_π is finite-dimensional, then $C^\infty(\pi) = V_\pi$ (finite dimensional subspaces are closed). The significance of this statement is that if V_π is finite-dimensional and taking $V_\pi = \mathbf{C}^{d_\pi}$, the matrix components $\pi_{ij}\colon G \to \mathbf{C}$ of π are smooth functions. □

Remark 3. The space $\mathcal{G}(\pi)$ is called the *Gårding space* of π.

The Laplacian on a compact semisimple Lie group

Recall that if G is a Lie group, then the Lie algebra $\mathfrak{g}$ is identified with the tangent space at the identity of G. Concretely speaking, $\mathfrak{g}$ is the space of all left-invariant vector fields on G. If $X \in \mathfrak{g}$, then the associated vector field, also denoted by X, acts on $\mathcal{E}(G)$ by

$$Xf(g) = \left\{ \frac{d}{dt} f(g \exp tX) \right\}_{t=0}$$

where $\exp \mathfrak{g} \to G$ is the exponential map and $g \in G$. The universal enveloping algebra of G is the space $U(\mathfrak{g})$ which is roughly speaking the algebra generated by all vector fields $X \in \mathfrak{g}$ via composition (also see Definition 3.1.6). That is to say that an element of $D \in U(\mathfrak{g})$ is a finite sum of compositions of vector fields of the form $X_1 \circ X_2 \circ \cdots \circ X_m$ where $X_1, \ldots, X_m \in \mathfrak{g}$. Elements of $U(\mathfrak{g})$ act in the obvious way where if $X_1 \circ X_2 \circ \cdots \circ X_m$ is a finite composition of vector fields as before, then

$$(X_1 \circ X_2 \circ \cdots \circ X_m)f(g) = \left\{ \frac{\partial^m}{\partial t_1 \partial t_2 \ldots \partial t_m} f(g(\exp t_m X_m) \cdots (\exp t_2 X_2)(\exp t_1 X_1)) \right\}_{t_1=t_2=\cdots=t_m=0}$$

Intuitively speaking, $U(\mathfrak{g})$ gives all the higher-order left-invariant differential operators on G.

Definition 1.3.2. The *adjoint representation* $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{ad}(X)(Y) = [X, Y]$. The *Killing form* is given by $(X, Y) \mapsto B(X, Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$. The Killing form is bilinear and symmetric. We say that $\mathfrak{g}$ is *semisimple* if the Killing form B is non-degenerate which is to say that the mapping $X \mapsto (Y \mapsto B(X, Y))$ is an isomorphism of $\mathfrak{g}$ onto the dual space $\mathfrak{g}^*$.

We say a Lie group G is semisimple if $\mathfrak{g}$ is semisimple. We shall now restrict our attention to the semisimple compact groups. If $\{X_1, \ldots, X_n\} \subset \mathfrak{g}$ is a basis for $\mathfrak{g}$, then we denote $\{X^1, \ldots, X^n\}$ to be the dual basis with respect to B (i.e. $B(X_i, X^j) = \delta_{ij}$). This basis exists due to the semisimplicity of $\mathfrak{g}$ (i.e. the non-degeneracy of B).

Definition 1.3.3. The *Casimir operator* with respect to the Killing form is the element $\Omega \in U(\mathfrak{g})$ given by

$$\Omega = \sum_{j=1}^{n} X_j \circ X^j.$$

We list some basic facts about the Killing form Ω. First, our definition of Ω is independent of our choice of basis for $\mathfrak{g}$. Second, Ω lies within the center of $U(\mathfrak{g})$ which means that for each $D \in U(\mathfrak{g})$ we have $D \circ \Omega = \Omega \circ D$ (see [20, Thm. 8.3.43]). Third, Ω is in fact bi-invariant which is to say that it commutes with both left and right translation in the Lie group G. We shall write $\Delta_G := \Omega$ and call Δ_G the *Laplacian* of G. The reason for this naming convention is that when G is equipped with a bi-invariant Riemannian metric (which is just a constant multiple of the Killing form B in this case), then Δ_G coincides with the corresponding Laplace-Beltrami operator. Now we have an important theorem characterizing the relationship between irreducible unitary representations and the Laplacian Δ_G.

Theorem 1.3.3. *Suppose $[\pi] \in \widehat{G}$ and let π be a representation of $[\pi]$. Then if $\pi_{ij} : G \to \mathbf{C}$ are the matrix coefficients of π where we have represented π in some basis, then there is $\lambda \in \mathbf{C}$ such that $\Delta_G \pi_{ij} = \lambda \pi_{ij}$ for all $1 \leq i, j \leq d_\pi$.*

For the proof of Theorem 1.3.3 see [20, Th. 8.3.47]. This theorem provides us with the analogue of the fact that the exponential functions $x \mapsto e_\xi(x) = e^{i\langle x, \xi \rangle}$ are eigenfunctions of the Laplacian on $\mathbf{R}^n$ (with eigenvalue $-|\xi|^2$). Armed with this theorem we shall be able to state the main result on how the Fourier transform on G interacts with smooth functions on G.

Fourier transform and the Schwartz space $\mathcal{S}(\widehat{G})$

Once and for all we choose a representative π from each equivalence class $[\pi] \in \widehat{G}$ and simply refer to the equivalence class $[\pi]$ by π. We put $|\pi| = \sqrt{|\lambda_\pi|}$ where λ_π is the eigenvalue of Δ_G for a matrix component of π (which does not depend on the choice of matrix component by Theorem 1.3.3). Finally, if T is an endomorphism of a Hilbert space, then $\|T\|_{HS} = \sqrt{\mathrm{Tr}(T^*T)}$ is the *Hilbert-Schmidt norm* of T (here T^* is the adjoint of T).

The Schwartz space $\mathcal{S}(\widehat{G})$ is defined as follows.

Definition 1.3.4. Let $\mathcal{S}(\widehat{G})$ be the set of functions F satisfying the following conditions:

1. $F(\pi) \in \operatorname{End} V_\pi$.
2. $\nu_k(F) = \sup_{\pi \in \widehat{G}} (1 + |\pi|)^k \|F(\pi)\|_{HS} < \infty$ for each $k \in \mathbf{N}$.

We call $\mathcal{S}(\widehat{G})$ the *Schwartz space* of $\widehat{G}$ and we topologize $\mathcal{S}(\widehat{G})$ with respect to the seminorms ν_k. With these seminorms $\mathcal{S}(\widehat{G})$ is a Fréchet space.

The natural Schwartz space on G is quite clearly $C^\infty(G)$ itself. This is because any notion of decay at "infinity" in the group is moot since G is compact and $C^\infty(G)$ is topologized by the uniform seminorms:

$$\nu_D(f) = \sup_{x \in G} |Df(x)|$$

for each differential operator D. The analogy of Theorem 1.1.8 is the following.

Theorem 1.3.4. *The Fourier transform $\mathscr{F}$ on G is a topological isomorphism from $C^\infty(G)$ onto $\mathcal{S}(\widehat{G})$.*

We do not provide a proof here for it requires a bit more machinery than we are willing or in a position to furnish, although a complete proof can be found in Sugiura's paper [22, Thm. 4]. The first part of the theorem is quite simple to see however. Since by virtue of integration by parts we have $\widehat{\Delta_G f}(\pi) = \lambda_\pi \widehat{f}(\pi)$. So it follows that for each $n \in \mathbf{N}$, there is a constant $C_n \geq 0$ such that $\|\widehat{f}(\pi)\|_{HS} \leq C_n(1 + |\pi|^2)^{-n}$ for all

$\pi \in \widehat{G}$. So we see $\widehat{f}$ satisfies a form of rapid decay. Of course the more trickier direction is proving that the inverse Fourier transform maps $\mathcal{S}(\widehat{G})$ onto $C^\infty(G)$.

There is an alternative proof which can be obtained in Ruzhansky-Turunen [20, Sec. 10.3]. There, the Schwartz space $\mathcal{S}(\widehat{G})$ is defined differently than the definition given by Definition 1.3.4 which is a definition provided by Sugiura [22, pp. 44].

1.4 Remarks

The main purpose of this chapter was to provide us with a sufficient motivation and background for the Fourier transform theory on $\mathbf{R}^n$ as well as for a locally compact topological group. In Section 1.1 we highlighted the major theorems, particularly the Plancherel and Fourier inversion theorems—as well as characterized the images of the standard function spaces which arise in real analysis. As we have hoped to highlight in our short survey on the harmonic analysis of abelian and compact topological groups, the Plancherel and Fourier inversion theorems are in fact quite generic features and are valid for those groups as well under suitable formulations.

On $\mathbf{R}^n$, we have also studied the Fourier transform as it acts on various spaces of smooth functions. The main result provided by Theorem 1.1.8 states that the Fourier transform is a topological isomorphism of $\mathcal{S}(\mathbf{R}^n)$ onto itself. In the extension of the Fourier transform to Lie groups, Theorem 1.3.4 suggests that a "Schwartz-type isomorphism" theorem is also a generic feature of the Fourier transform and should also exist naturally for the Fourier transforms on Lie groups as well.

Furthermore, although we have not stated any results for the extension of the Fourier transform to distributions on Lie groups one can imagine that the same type of results should hold in a similar sense as to the theory developed in Section 1.1. The main point is that we would wish to extend the Fourier transform and its related results to "arbitrary" Lie groups (not necessarily abelian or compact). However, the Fourier transform theory for noncompact nonabelian groups is far more difficult to study. Firstly, the representation theory of a nonabelian noncompact group is already quite complicated for there are irreducible unitary representations of infinite dimension—which for instance creates complications in conjuring up a hypothetical inversion formula.

Nevertheless, our discussions in this chapter suggest that a reasonable formulation of the Fourier transform for a nonabelian and noncompact group G should also feature analogues to the Plancherel and Fourier inversion theorems. Moreover, if G is a Lie group, there should be some description of how the Fourier transform interacts with differential operators such as the Laplacian in view of Theorem 1.3.3. Additionally, analogues of the Schwartz spaces should exist for this group G under which the analogues of Theorems 1.1.8 and 1.3.4 hold as well.

In the next chapter these analogies shall be established in a particular setting for the group $G = \mathbf{SU}(1, 1)$, which is a nonabelian noncompact group. As we will see, the Fourier transform theory which we shall formulate behaves beautifully for those functions f which are invariant under right translation by the subgroup $K = \mathbf{SO}(2)$. In particular, the presentation of the Fourier transform theory for G can be done without explicit reference to its representation theory and even more importantly the Fourier transform which we shall present takes scalar-valued functions to scalar-valued functions unlike the operator-valued Fourier transform of nonabelian compact groups.

Chapter 2

Harmonic analysis on $\mathbf{H}^2$

In this chapter we shall extend our Fourier transform theory to the hyperbolic plane $\mathbf{H}^2$. Here the Fourier transform enjoys analogous generic properties which we have discussed at length in Chapter 1; such as the Riemann-Lebesgue lemma, Plancherel, and Paley-Wiener theorems. The style of proofs of these theorems is quite analogous to the proofs presented in Chapter 1 yet they are not entirely the same. In particular, the techniques that we shall use in Chapter 2 are largely tied to the unique geometric structure of $\mathbf{H}^2$.

We finally remark that in developing our Fourier transform we must take care since $\mathbf{H}^2$ is not a group but rather a homogeneous space of the nonabelian noncompact Lie group $\mathbf{SU}(1,1)$. Thus applying the representation theoretic approach of Section 1.2 to developing the Fourier transform theory on $\mathbf{H}^2$ becomes much more difficult. So in order to motivate our definition of the Fourier transform on $\mathbf{H}^2$ we take the approach of Helgason which melds aspects of the geometry of $\mathbf{H}^2$ and the group structure of $\mathbf{SU}(1,1)$.

2.1 Preliminaries

Consider the open unit disk $D = \{x \in \mathbf{C} \colon |x| < 1\}$ which is naturally an open two dimensional smooth submanifold of $\mathbf{R}^2$. That is a complex number $x \in D$ is regarded as an element of $\mathbf{R}^2$ by the pair $(\operatorname{Re} x, \operatorname{Im} x)$. The reason for regarding D as a subset of $\mathbf{C}$ is so that we can take advantage of some of the geometric properties afforded by complex multiplication.

Definition 2.1.1. We define a Riemannian metric $\mathbf{g}$ on D by

$$\mathbf{g}_x(u,v) = \langle u,v\rangle_x = \frac{u\cdot v}{(1-|x|^2)^2} \tag{2.1.1}$$

where $x \in D$, $u, v \in T_xD$ are tangent vectors, and $u \cdot v$ denotes the standard Euclidean inner product (again regarding $u, v \in \mathbf{R}^2$).

The *norm* of a tangent vector $v \in T_pD$ is denoted by $|v| = \mathbf{g}_p(v,v)^{1/2}$. We shall not verify it here, however the Riemannian curvature determined by the above metric $\mathbf{g}$ is constant and equal to -4.

Definition 2.1.2. We call D with the metric $\mathbf{g}$ the *hyperbolic plane* and it is notated as $\mathbf{H}^2$ and this particular definition of $\mathbf{H}^2$ is called the Poincaré disk model for the hyperbolic plane. Moreover, we denote the origin in D by o.

Let $\gamma\colon [a,b] \to \mathbf{H}^2$ $(a \le b)$ be a differentiable curve, then the arc length of γ is defined as

$$L(\gamma) = \int_a^b |\gamma'(t)|\,dt\,. \tag{2.1.2}$$

We define a metric on $\mathbf{H}^2$ (as on any Riemannian manifold) by $d(x,y) = \inf_\gamma L(\gamma)$ where the infimum is taken over all differentiable curves γ connecting x to y. Let x be a point on the x-axis in D. If γ is a differentiable curve on $[a,b]$ connecting o to x, then if we write γ in the standard coordinates $\gamma(t) = (x(t), y(t))$, we note that in D

$$\frac{x'(t)^2}{(1-x(t)^2)^2} \le \frac{x'(t)^2 + y'(t)^2}{(1-x(t)^2-y(t)^2)^2} \tag{2.1.3}$$

so that the arc length of any curve connecting o to x is bounded below by the length of any straight line segment connecting o to x. Thus if $x(t) = tx$ $(0 \le t \le 1)$, then

$$d(o,x) = L(x) = \int_0^1 \frac{|x|}{1-t^2x^2}\,dt = \frac{1}{2}\log\frac{1+|x|}{1-|x|} = \operatorname{atanh}|x| \tag{2.1.4}$$

where atanh is the inverse hyperbolic tangent.

Definition 2.1.3. For $x \in \mathbf{H}^2$ will write $x = \tanh re^{i\theta}$ where $r = d(o,(|x|,0)) = \operatorname{atanh}|x|$. These coordinates on $\mathbf{H}^2$ are called the *geodesic coordinates.*

The geodesic coordinates yield a kind of polar coordinate representation. We will show in a moment that indeed for arbitrary $x \in \mathbf{H}^2$, that $d(o,x) = r$. By elementary complex analysis the subgroup G of the group of Möbius transformations on $\mathbf{C}$ defined by

$$G = \left\{ \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1,\ a,b \in \mathbf{C} \right\} = \mathbf{SU}(1,1) \tag{2.1.5}$$

acts on $\mathbf{H}^2$ by means of the transformations

$$g \cdot x = \frac{ax+b}{\bar{b}x+\bar{a}} \quad \text{where } x \in \mathbf{H}^2,\ g = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}. \tag{2.1.6}$$

This action is a transitive group action and is obviously smooth. The subgroup fixing o is the subgroup

$$K = \left\{ \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix} : |a|^2 = 1,\ a \in \mathbf{C} \right\} = \mathbf{SO}(2) \tag{2.1.7}$$

which is compact in G. Thus we can regard $\mathbf{H}^2$ as the homogeneous space $\mathbf{H}^2 = G/K$ and by [18], $\mathbf{H}^2$ is diffeomorphic to G/K given its canonical smooth structure. Furthermore, G leaves the Riemannian metric on $\mathbf{H}^2$ invariant as evidenced by the following proposition.

Proposition 2.1.1. *For $g \in G$, we let $\tau_g\colon \mathbf{H}^2 \to \mathbf{H}^2$ be the map $\tau_g(x) = g \cdot x$. Then for tangent vectors $u, v \in T_x\mathbf{H}^2$ we have $\langle d(\tau_g)_x u, d(\tau_g)_x v\rangle_{g\cdot x} = \langle u,v\rangle_x$.*

Proof. To see this let g be as in (2.1.6). Put $u,v \in T_x\mathbf{H}^2$ with corresponding smooth curves $\gamma_u, \gamma_v\colon I \to \mathbf{R}$ (I an open interval containing 0) where $\gamma_u(0) = x = \gamma_v(0)$ and $\gamma_u'(0) = u$, $\gamma_v'(0) = v$, then the differential of τ_g applied to u is $d(\tau_g)_x u = (g \cdot \gamma_u)'(0) = u(\bar{b}x + \bar{a})^{-2}$ and likewise for $(d\tau_g)_x v$. Although these vectors appear to be complex, we are interpreting them as points in $\mathbf{R}^2$ in the obvious way. Now we have

$$\langle d(\tau_g)_x u, d(\tau_g)_x v\rangle_{g\cdot x} = (u\cdot v)|\bar{b}x+\bar{a}|^{-4}(1-|g\cdot x|^2)^{-2} = (u\cdot v)(|\bar{b}x-\bar{a}|^2 - |ax+b|^2)^{-2}.$$

We leave it to the reader to verify that $|\bar{b}x-\bar{a}|^2 - |ax+b|^2 = 1-|x|^2$ (compute it in polar coordinates), and so we find $\langle d(\tau_g)_x u, d(\tau_g)_x v\rangle_{g\cdot x} = (u\cdot v)(1-|x|^2)^{-2} = \langle u,v\rangle_x$. ☐

It follows that for each $g \in G$, τ_g is a isometry of $\mathbf{H}^2$ for all $g \in G$. Consequently if $x \in \mathbf{H}^2$ and $k \in K$ is a rotation sending x to $(|x|,0)$, then $d(o,x) = d(o,(|x|,0))$ which fully justifies the coordinate representation $x = \tanh re^{i\theta}$ as discussed earlier.

From (2.1.1), we find that in standard coordinates the matricial entries of the Riemannian metric are given by $\mathbf{g}_{ij} = (1 - |x|^2)^{-2}\delta_{ij}$ so that the Riemannian volume form becomes

$$dx = \sqrt{\det \mathbf{g}}\, dx_1\, dx_2 = (1 - |x|^2)^{-2}\, dx_1\, dx_2 \tag{2.1.8}$$

where $x = (x_1, x_2) = x_1 + ix_2$ and the Laplace-Beltrami operator becomes

$$\Delta_{\mathbf{H}^2} = \frac{1}{\sqrt{\det \mathbf{g}}}\partial_j(\mathbf{g}^{ij}\sqrt{\det \mathbf{g}}\partial_i) = (1 - x_1^2 - x_2^2)^2(\partial_{x_1}^2 + \partial_{x_2}^2). \tag{2.1.9}$$

Here we use Einstein's summation convention and have let $\mathbf{g}^{ij}$ to denote the components of the inverse matrix of $\mathbf{g}$. These objects are invariant under isometries and so are invariant under the action by G, that is dx is a G-invariant volume form which gives rise to a G-invariant Riemannian measure and $\Delta_{\mathbf{H}^2} \circ \tau_g = \tau_g \circ \Delta_{\mathbf{H}^2}$ for all $g \in G$. By Theorem A.1.5, this gives rise to a Haar measure dg on G such that

$$\int_{\mathbf{H}^2} f(x)\, dx = \int_G f(g \cdot o)\, dg \quad (f \in L^1(\mathbf{H}^2)) \tag{2.1.10}$$

which will become indispensable to our analysis in the future.

Horocycles and plane waves

To motivate our candidate for the Fourier transform on $\mathbf{H}^2$ it is useful to consider the Fourier transform on $\mathbf{R}^n$. Recall that the Fourier transform defined for functions in $L^1(\mathbf{R}^n)$ is given by

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x)e^{-i\langle x,\xi\rangle}\, dx,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. The exponentials appearing above $x \mapsto e^{i\langle x,\xi\rangle}$ ($\xi \in \mathbf{R}^n$) can be considered in the polar form where we write $\xi = r \cdot \omega$ ($r \geq 0$, $\omega \in \mathbf{S}^{n-1}$) so that $e^{i\langle x,\xi\rangle} = e^{ir\langle x,\omega\rangle} = e_{r,\omega}(x)$. In this form two facts become readily apparent. First, we have that for the Laplacian Δ that $\Delta e_{r,\omega} = -r^2 e_{r,\omega}$ so that $e_{r,\omega}$ is an eigenfunction of Δ with eigenvalue $-r^2$. Secondly, $e_{r,\omega}$ is constant on every hyperplane with normal vector ω, that is, it is a *plane wave*. Furthermore, writing $\widehat{f}$ in these polar coordinates and using the hyperplanar measure from Theorem A.3.1:

$$\widehat{f}(r \cdot \omega) = \int_{\mathbf{R}} \int_{\langle x,\omega\rangle = p} f(x)e^{-irp}\, d\sigma(x)\, dp \tag{2.1.11}$$

which is simply the one-dimensional Fourier transform of the Radon transform of f at the angle ω. Analogously for the hyperbolic plane $\mathbf{H}^2$ a candidate for a Fourier transform of a reasonable function f could be a certain one-dimensional Fourier transform of a Radon transform of f about some generalization of an angle. Here a Radon transform on $\mathbf{H}^2$ should integrate functions on some analogue of a hyperplane (say a one dimensional submanifold) and a one-dimensional Fourier transform should integrate the Radon transform against an eigenfunction of $\Delta_{\mathbf{H}^2}$ which is constant on each "hyperplane."

The correct notion of a hyperplane for $\mathbf{H}^2$ in this case is a horocycle.

Definition 2.1.4. Let $B = \{x \in \mathbf{C} \colon |x| = 1\}$ so that $\overline{D} = D \cup B$. Then a *horocycle* $\xi \subset D$ in $\mathbf{H}^2$ with *normal* $b \in B$ is the set of points such that $\xi \cup \{b\}$ is a circle in $\overline{D}$.

We will denote the space of horocycles on $\mathbf{H}^2$ by Ξ. We state without proof that if $x \in \mathbf{H}^2$ and $b \in B$, then there is a unique horocycle, denoted by $\xi(x, b)$, which contains x and has normal b. Clearly a horocycle in $\mathbf{H}^2 = D$ together with its normal on B determines a circle contained within the disk $\overline{D}$ which intersects the boundary B tangentially (see Fig. 2.1).

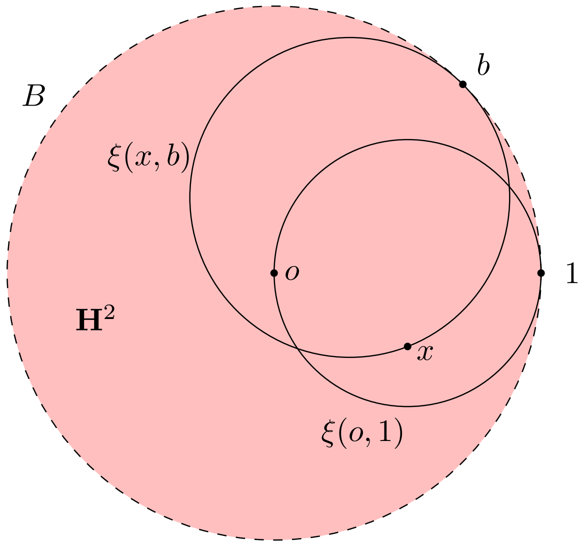

Figure 2.1: The unique horocycle through x and b.

Definition 2.1.5. The distance of a horocycle ξ to the origin o is the shortest distance from o to a point $x \in \xi$. If $p \geq 0$ is the distance between a horocycle ξ and o, then the *signed distance* between ξ and o is the quantity $\sigma(\xi)p$ where $\sigma(\xi) = 1$ if o lies outside the horocycle ξ (regarding ξ as a circle in $\overline{D}$) and is -1 otherwise.

In analogy to the inner product on $\mathbf{R}^n$ giving the signed distance between the hyperplane containing a point and a given normal we have the following definition of the non-Euclidean product.

Definition 2.1.6. Define the bracket $\langle \cdot, \cdot \rangle \colon \mathbf{H}^2 \times B \to \mathbf{R}$ by $\langle x, b \rangle =$ "signed distance between o and the horocycle $\xi(x, b)$".

We can in fact compute $\langle x, b \rangle$ explicitly.

Lemma 2.1.2. *For $x \in \mathbf{H}^2$ and $b \in B$ we have that*

$$\langle x, b \rangle = \frac{1}{2} \log \frac{1 - |x|^2}{|x - b|^2}. \tag{2.1.12}$$

Proof. Suppose that o is outside the horocycle. Let y be the point on the horocycle through x and b closest to o so that $\langle x, b \rangle = \operatorname{atanh} |y|$. Thus it suffices to compute $|y|$ which can be found via the law of cosines. Let (x, o, b) denote the triangle with vertices x, o, and b, similarly we have the triangle (x, c, b) where c is the center of the horocycle. Let xob denote the angle formed by the line segments adjoining xo and ob; and likewise for xoc. Then by the law of cosines

$$\cos(xob) = \frac{|x|^2 + 1 - |x - b|^2}{2|x|} = \frac{|x|^2 + (\frac{1}{2}(1 + |y|))^2 - (\frac{1}{2}(1 - |y|))^2}{|x|(1 + |y|)} = \cos(xoc)$$

we find

$$|y| = \frac{|x|^2 + |x - b|^2 - 1}{|x|^2 - |x - b|^2 - 1}$$

so that we obtain the desired relation by using (2.1.4). If o is inside the horocycle, then the above becomes

$$\cos(xob) = \frac{|x|^2 + 1 - |x - b|^2}{2|x|} = \frac{|x|^2 + (\frac{1}{2}(1 - |y|))^2 - (\frac{1}{2}(1 + |y|))^2}{|x|(1 - |y|)} = \cos(xoc)$$

so that

$$|y| = -\frac{|x|^2 + |x-b|^2 - 1}{|x|^2 - |x-b|^2 - 1}.$$

Hence using (2.1.4) again we obtain the above relation as well. $\square$

By Lemma 1.1 we have that for the plane wave $x \mapsto e^{w\langle x,b\rangle}$ (where $w \in \mathbf{C}$), then

$$e^{w\langle x,b\rangle} = \left(\frac{1-|x|^2}{|x-b|^2}\right)^{\frac{w}{2}}. \tag{2.1.13}$$

Now using (2.1.9) we can easily compute

$$\Delta_{\mathbf{H}^2} e^{w\langle x,b\rangle} = w(w-2)e^{w\langle x,b\rangle} \tag{2.1.14}$$

so that the functions $x \mapsto e^{w\langle x,b\rangle}$ are eigenfunctions of $\Delta_{\mathbf{H}^2}$ with eigenvalue $w(w-2)$. These "plane waves" will become the building blocks of our Fourier transform on $\mathbf{H}^2$ in the same way the complex exponential functions are the underpinnings of the Fourier transform on $\mathbf{R}^n$.

We note that for $g \in G$ and a horocycle ξ with normal b, then $g \cdot \xi$ also determines a horocycle with normal $g \cdot b$. This is because for a circle C in the disk $\overline{D}$, then $g \cdot C$ is also a circle in $\overline{D}$ and G maps the circle B to itself. Thus, $g \cdot \xi$ is a circle in $\overline{D}$ with one point of intersection at the boundary B at the point $g \cdot b$. Now consider the subgroups A and N of G defined by

$$A = \left\{ a_t = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} : t \in \mathbf{R} \right\}, \tag{2.1.15}$$

$$N = \left\{ n_s = \begin{pmatrix} 1+is & -is \\ is & 1-is \end{pmatrix} : s \in \mathbf{R} \right\}. \tag{2.1.16}$$

Geometrically speaking if we consider the orbit $A \cdot o$, then this corresponds to the geodesic through $b_0 = 1$ and o (i.e. the straight line). Additionally, the orbit $N \cdot o$ corresponds to the horocycle $\xi(o, 1)$. This can be seen by virtue of the fact that the equations

$$\sin(\theta) = \frac{-2s}{1+s^2}, \quad \cos(\theta) = \frac{s^2-1}{1+s^2}$$

has a solution for each $\theta \in (0, 2\pi)$.

Now, for $a_t \in A$ we have that the horocycle $a_t N \cdot o$ has signed distance t and normal b_0. Since every point $x \in \mathbf{H}^2$ lies on some horocycle of the form $aN \cdot o$ we have that for $g \in G$, the point $g \cdot o$ lies on some horocycle $aN \cdot o$ ($a \in A$). Since A normalizes N we obtain that $gK = naK$. Therefore $g = nak$ for some $n \in N$, $a \in A$, and $k \in K$ which exhibits G in an *Iwasawa decomposition* $G = NAK = KAN$. We also have the *Cartan decomposition* of $G = K\overline{A^+}K$ where $\overline{A^+} = \{a_t \colon t \geq 0\}$. This decomposition is easy to see geometrically since any point $x \in \mathbf{H}^2$ can be parametrized by its geodesic distance from o and an angle θ with the positive x-axis. Thus, $x = ka_tK$ where $t = d(o, x)$ and for some unique $k \in K$. We can introduce the coordinates $(s, t) \mapsto n_s a_t \cdot o$ on $\mathbf{H}^2$ so that

$$x_1 + ix_2 = \frac{\sinh t - ise^{-t}}{\cosh t - ise^{-t}}. \tag{2.1.17}$$

Thus the Riemannian measure dx with respect to the element $ds\, dt$ becomes

$$dx = e^{-2t}\, dt\, ds\,.$$

Therefore, integration of functions on $\mathbf{H}^2$ takes the form

$$\int_{\mathbf{H}^2} f(x)\,dx = \int_{\mathbf{R}}\int_{\mathbf{R}} f(n_s a_t \cdot o)e^{-2t}\,dt\,ds\,.$$

In analogy to the measure (A.3.1), we put $d\sigma_t = e^{-2t}ds$ as the measure on the horocycle $Na_t \cdot o$. If we put $\xi_t(b_0) = \xi(a_t \cdot o, b_0)$, then the above integral becomes

$$\int_{\mathbf{H}^2} f(x)\,dx = \int_{\mathbf{R}}\int_{\xi_t(b_0)} f(x)\,d\sigma_t(x)\,dt$$

where we integrate f over the space of horocycles $\xi_t(b_0)$ with normal b_0. Since the measure dx is K-invariant we can rotate by an angle θ so that by our above analysis the integral finally becomes

$$\int_{\mathbf{H}^2} f(x)\,dx = \int_{\mathbf{R}}\int_{\xi(\tanh te^{i\theta}, e^{i\theta})} f(x)\,d\sigma_t(x)\,dt\,. \tag{2.1.18}$$

This exhibits the integral of f as the integral of its Radon transform on the space of "parallel" horocycles through the point $e^{i\theta} \in B$. The Radon transform of a function $f \in C_c(\mathbf{H}^2)$ is defined as follows.

Definition 2.1.7. For $f \in C_c(\mathbf{H}^2)$ the *Radon transform* of f is the function $\widehat{f}\colon \Xi \to \mathbf{C}$ defined by

$$\widehat{f}(\xi(\tanh te^{i\theta}, e^{i\theta})) = \int_{\xi(\tanh te^{i\theta}, e^{i\theta})} f(x)\,d\sigma_t(x)\,.$$

We can extend the Radon transform to $L^1(\mathbf{H}^2)$ by the same formula however we must take care since by (2.1.18) we only obtain that for integrable f the Radon transform $\widehat{f}(\xi(\tanh te^{i\theta}, e^{i\theta}))$ exists for almost every $t \in \mathbf{R}$.

Remark 4. In fact, the measures dt and ds determine Haar measures da and dn on A and N respectively such that for $f \in L^1(A)$ and $g \in L^1(N)$ we have

$$\int_A f(a)\,da = \int_{\mathbf{R}} f(a_t)\,dt,$$
$$\int_N g(n)\,dn = \int_{\mathbf{R}} g(n_s)\,ds\,.$$

Thus, we now have three ways of integrating a function $f \in L^1(\mathbf{H}^2)$ illustrated below:

$$\int_{\mathbf{H}^2} f(x)\,dx = \int_G f(g\cdot o)\,dg = \int_{AN} f(na\cdot o)e^{-2\log a}\,da\,dn\,.$$

Where we have defined log by:

Definition 2.1.8. Define the map $\log\colon A \to \mathbf{R}$ by $\log a_t = t$. This is a homomorphic homeomorphism (in fact a diffeomorphism) of abelian Lie groups.

Integral formulas

Before we close this subsection we shall discuss some integral formulas which will simplify our work and give us some new integration techniques. Recall that for any $f \in C_c(\mathbf{H}^2)$, that we have the following integration scheme:

$$\int_{\mathbf{H}^2} f(x)\,dx = \int_G f(g\cdot o)\,dg = \int_N\int_A f(na\cdot o)e^{-2\log a}\,da\,dn\,.$$

It is a fact that G is a unimodular group, so put $F(g) = f(g^{-1}\cdot o)$, then by the unimodularity of dg we have

$$\int_G f(g\cdot o)\,dg = \int_G F(g)\,dg = \int_{\mathbf{H}^2}\int_K F(hk)\,dk\,d(hK)\,.$$

Here $d(hK) = dx$ is the Riemannian measure on $\mathbf{H}^2$ (see Section A.1.2). However by our previous formula:

$$\int_N \int_A \int_K F(nak) e^{-2\log a}\, dk\, da\, dn = \int_N \int_A \int_K f(k^{-1}a^{-1}n^{-1} \cdot o) e^{-2\log a}\, dk\, da\, dn\,.$$

Now K is unimodular because it is compact, A is unimodular because it is abelian, and N is unimodular because it is nilpotent. Thus, making the transformations $k \mapsto k^{-1}$, $a \mapsto a^{-1}$, and $n \mapsto n^{-1}$ we have

$$\int_G f(g \cdot o)\, dg = \int_N \int_A \int_K f(kan \cdot o) e^{2\log a}\, dk\, da\, dn = \int_{K\times A\times N} f(kan \cdot o) e^{2\log a}\, dk\, da\, dn\,.$$

Also, we can write

$$\int_N f(na \cdot o)\, dn = e^{2\log a} \int_N f(an \cdot o)\, dn\,.$$

That this can be done is due to the fact that

$$\int_N f(na \cdot o)\, dn = \int_{\mathbf{R}} f(n_s a_t \cdot o)\, ds = \int_{\mathbf{R}} f(a_t(a_t^{-1} n_s a_t) a_t^{-1} a_t \cdot o)\, ds = \int_{\mathbf{R}} f(a_t n_{se^{-2t}} \cdot o)\, ds\,.$$

Making the transformation $s \mapsto se^{2t}$, which has Jacobian $\Phi(s) = e^{2t}$, we have

$$\int_{\mathbf{R}} f(n_s a_t \cdot o)\, ds = \int_{\mathbf{R}} f(a_t n_s \cdot o) e^{2t}\, ds = e^{2\log a} \int_N f(an \cdot o)\, dn\,.$$

Hence we have obtained the following integral formulas for $f \in C_c(\mathbf{H}^2)$:

$$\int_{\mathbf{H}^2} f(x)\, dx = \int_G f(g \cdot o)\, dg = \int_{K\times A\times N} f(kan \cdot o) e^{2\log a}\, dk\, da\, dn = \int_{K\times A\times N} f(kna \cdot o)\, dk\, da\, dn \quad (2.1.19)$$

$$= \int_{N\times A} f(na \cdot o) e^{-2\log a}\, dn\, da = \int_{A\times N} f(an \cdot o)\, dn\, da\,. \quad (2.1.20)$$

The same statements hold for $f \in L^1(\mathbf{H}^2)$ as well by Fubini's theorem. Thus, we can write the Radon transform of Definition 2.1.7 in the following group theoretic formulation:

$$\widehat{f}(\xi(\tanh t e^{i\theta}, e^{i\theta}) = \int_N f(kan \cdot o)\, dn = \widehat{f}(ka \cdot \xi_0)$$

where $k \in K$ corresponds to the rotation by $e^{i\theta}$, $a \in A$ corresponds to the dilation by $\tanh t$, and $\xi_0 = \xi(o, b_0)$. So in particular, we have that the integral of f over $\mathbf{H}^2$ can be expressed as

$$\int_{\mathbf{H}^2} f(x)\, dx = \int_A \widehat{f}(ka \cdot \xi_0)\, da$$

for each $k \in K$. Compare this formula to the integral formula of Theorem A.3.1. This group theoretic version of the Radon transform will become very useful later. We also define the *modified Radon transform* as follows.

Definition 2.1.9. For $f \in C_c(\mathbf{H}^2)$, define the *modified Radon transform* of f by

$$\mathcal{R}f(ka \cdot \xi_0) = e^{\log a} \int_N f(kan \cdot o)\, dn\,.$$

That is $\mathcal{R}f(ka \cdot \xi_0) = e^{\log a} \widehat{f}(ka \cdot \xi_0)$.

2.2 Spherical functions and the spherical transform

We shall first consider the Fourier transform of *radial* functions on $\mathbf{H}^2$. That is functions that depend only on the geodesic distance from o which is equivalent to being K-invariant, i.e. a function is K-invariant if $f(k\cdot x) = f(x)$ for all $k \in K$ and $x \in \mathbf{H}^2$. On $\mathbf{R}^2$ using the polar coordinates $(x, y) = (\lambda\cos\phi, \lambda\sin\phi)$ and formula (A.3.3), the Fourier transform of a radial function f is

$$\widehat{f}(\lambda,\phi) = \widehat{f}((\lambda\cos\phi,\lambda\sin\phi)) = \int_{\mathbf{R}^2} f(x)e^{-i\lambda\langle x,(\cos\phi,\sin\phi)\rangle}\,dx = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi} f(r)e^{-ir\lambda\cos(\theta-\phi)}r\,dr\,d\theta\,.$$

Recall that the Bessel function $J_0(r)$ is the solution to the differential equation

$$r^2\frac{d^2f}{dr^2} + r\frac{df}{dr} + r^2 f = 0. \tag{2.2.1}$$

In other words, it is a radial function on $\mathbf{R}^n$ that satisfies $\Delta J_0(r) = -J_0(r)$ and if we put $J_0(\lambda r)$, then it satisfies $\Delta J_0(\lambda r) = -\lambda^2 J_0(\lambda r)$. Furthermore J_0 has the following integral representation

$$J_0(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{ir\cos\theta}\,d\theta\,.$$

Hence the Euclidean Fourier transform of a radial function is also a radial function and is written

$$\widehat{f}(\lambda) = \int_0^\infty f(r)J_0(-r\lambda)r\,dr\,. \tag{2.2.2}$$

Thus the Fourier transform of a radial function is given by integration against radial eigenfunctions of the Laplacian and it satisfies $(\Delta f)\widehat{\ }(\lambda) = -\lambda^2\widehat{f}(\lambda)$. This situation for $\mathbf{R}^2$ motivates a Fourier transform of radial functions on $\mathbf{H}^2$ which practically should be given by integration against radial eigenfunctions of $\Delta^2_{\mathbf{H}}$.

Definition 2.2.1. A *spherical function* on $\mathbf{H}^2$ is a radial eigenfunction of $\Delta_{\mathbf{H}^2}$.

If we write a point $x = \tanh re^{i\theta}$ in geodesic coordinates, then we can rewrite the Laplacian $\Delta_{\mathbf{H}^2}$ in these coordinates as

$$\Delta_{\mathbf{H}^2} = \partial_r^2 + 2\coth 2r\partial_r + 4\sinh^{-2}(2r)\partial_\theta^2. \tag{2.2.3}$$

Therefore a spherical function ϕ satisfies the differential equation

$$\partial_r^2\phi + 2\coth 2r\partial_r\phi = -\lambda^2\phi. \tag{2.2.4}$$

The choice of the eigenvalue above is for technical reasons. It is a fact that any two radial eigenfunctions of $\Delta_{\mathbf{H}^2}$ with the same eigenvalue are proportional as illustrated by the following proposition.

Proposition 2.2.1. *The radial eigenfunctions of $\Delta_{\mathbf{H}^2}$ with the same eigenvalue are proportional.*

Proof. The differential equation

$$\partial_r^2\phi + 2\coth 2r\partial_r\phi = -\lambda\phi$$

is solvable by Frobenius' method by using the substitution of $w = \sinh 2r$. So expanding ϕ in a power series of $\sinh 2r$, $\phi(r) = \sum_{m=0}^\infty a_m(\sinh 2r)^m$ we have

$$\sum_{m=1}^\infty 4a_m m(\sinh 2r)^m + \cosh^2(2r)\sum_{m=2}^\infty 4a_m m(m-1)(\sinh 2r)^{m-2} + \cosh^2(2r)\sum_{m=1}^\infty 4a_m m(\sinh 2r)^{m-2}$$
$$+\lambda\sum_{m=0}^\infty a_m(\sinh 2r)^m = 0.$$

Using $\cosh^2(2r) = 1+\sinh^2(2r)$ and comparing the coefficients we find that the coefficients are completely determined by the value of $a_0 \in \mathbf{C}$. Hence all radial eigenfunctions are proportional. $\square$

Remark 5. That ϕ can be expanded into a power series of the type described above follows from the fact that ϕ is an analytic function on $\mathbf{H}^2$. That ϕ is analytic follows from the fact that since ϕ satisfies (2.2.4) which is a second-order differential equation with analytic coefficients.

For technical reasons, we will be interested in radial eigenfunctions with eigenvalue $-(\lambda^2+1)$. Since the function $e_{\lambda,b}(x) = e^{(i\lambda+1)\langle x,b\rangle}$ is an eigenfunction with eigenvalue $-(\lambda^2+1)$, then integrating $e_{\lambda,b}$ over b we have that the function

$$\phi_\lambda(x) = \int_B e_{\lambda,b}(x)\,db \tag{2.2.5}$$

is a radial eigenfunction of $\Delta_{\mathbf{H}^2}$ with eigenvalue $-(\lambda^2+1)$ (db is the standard circular measure on $B = \mathbf{S}^1$). In particular, using formula (2.1.13) and putting $x = \tanh r e^{i\phi}$ and $b = e^{i\theta}$, then

$$\phi_\lambda(a_r\cdot o) = \phi_\lambda(\tanh r) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\cosh 2r - \sinh 2r\cos\theta)^{-(i\lambda+1)/2}\,d\theta\,. \tag{2.2.6}$$

Now if $g\in G$, $k\in K$, we have that the function

$$F(x) = \int_K \phi_\lambda(gk\cdot x)\,dk$$

is also a radial eigenfunction of $\Delta_{\mathbf{H}^2}$ with eigenvalue $-(\lambda^2+1)$ by passing the derivative under the integral via the dominated convergence theorem. Hence is proportional to ϕ_λ and the constant of proportionality is $\phi_\lambda(g\cdot o)$. Thus the spherical function satisfies an analogue of the mean value formula

$$\int_K \phi_\lambda(gk\cdot x)\,dk = \phi_\lambda(g\cdot o)\phi_\lambda(x). \tag{2.2.7}$$

Definition 2.2.2. For a radial function $f\in C_c(\mathbf{H}^2)$ we define its *spherical transform* by

$$\widetilde{f}(\lambda) = \int_{\mathbf{H}^2} f(x)\phi_{-\lambda}(x)\,dx\,. \tag{2.2.8}$$

Let $\mathcal{D}_K(\mathbf{H}^2)$ denote the smooth compactly supported K-invariant functions. We now state our main results for the spherical transform.

Theorem 2.2.2 (Inversion formula)**.** *For $f\in\mathcal{D}_K(\mathbf{H}^2)$, the spherical transform $\widetilde{f}$ is inverted by the formula*

$$f(x) = \int_{\mathbf{R}} \widetilde{f}(\lambda)\phi_\lambda(x)\,d\mu(\lambda) = \int_{\mathbf{R}} f*\phi_\lambda(x)\,d\mu(\lambda)\,. \tag{2.2.9}$$

for some Radon measure $d\mu(\lambda)$ on $\mathbf{R}$ called the Plancherel measure.

Theorem 2.2.3 (Plancherel identity)**.** *Furthermore for $f\in\mathcal{D}_K(\mathbf{H}^2)$ we have*

$$\int_{\mathbf{H}^2}|f(x)|^2\,dx = \int_{\mathbf{R}}|\widetilde{f}(\lambda)|^2\,d\mu(\lambda)\,. \tag{2.2.10}$$

Now, to prove Theorem 1.3 we must first investigate the Plancherel measure $d\mu(\lambda)$. The work of Harish-Chandra showed that the measure $d\mu(\lambda)$ is given by $c_0|\mathbf{c}(\lambda)|^{-2}\,d\lambda$ where $\mathbf{c}$ is a certain meromorphic function on $\mathbf{C}$, c_0 a constant, and $d\lambda$ is the ordinary Lebesgue measure on $\mathbf{R}$. The function $\mathbf{c}$ is called *Harish-Chandra's* $\mathbf{c}$*-function.*

Theorem 2.2.4. *If* $\mathrm{Re}(i\lambda) > 0$, *then*

$$\mathbf{c}(\lambda) = \lim_{r\to\infty} e^{(i\lambda+1)r}\phi_\lambda(a_r\cdot o) \tag{2.2.11}$$

exists and equals

$$\mathbf{c}(\lambda) = \pi^{-1/2}\frac{\Gamma(\frac{1}{2}i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+1))}.$$

Proof. Note that $\phi_\lambda = \phi_{-\lambda}$ by Proposition 2.2.1. Since $d(o, g\cdot o) = d(o, g^{-1}\cdot o)$ as the action of G on $\mathbf{H}^2$ are isometries and $a_r^{-1} = a_{-r}$ we have that $\phi_\lambda(a_r\cdot o) = \phi_\lambda(a_{-r}\cdot o)$. Thus combining these facts and using the substitution $u = \tanh\theta/2$ we have that the integral in (2.2.6) becomes

$$\phi_\lambda(a_r\cdot o) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\cosh 2r - (\sinh 2r)\frac{1-u^2}{1+u^2}\right)^{(i\lambda-1)/2}(1+u^2)^{-1}\,du\,.$$

Using the identities $\cosh t + \sinh t = e^t$ and $\cosh t - \sinh t = e^{-t}$ the integrand simplifies to

$$\phi_\lambda(a_r\cdot o) = \frac{e^{(i\lambda-1)r}}{\pi}\int_{-\infty}^{\infty}(1+e^{-4r}u^2)^{(i\lambda-1)/2}(1+u^2)^{-(i\lambda+1)/2}\,du\,.$$

Assume that $\mathrm{Re}(i\lambda) > 0$. Set $\lambda = \xi + i\eta$ and choose $\epsilon < 1/2$ small enough so that $1 + 2\epsilon\eta > 0$. Then estimating the integrand leads to

$$\begin{aligned}(1+e^{-4r}u^2)^{-(\eta+1)/2}(1+u^2)^{(\eta-1)/2} &\le (1+u^2)^{-\eta/2+\epsilon\eta}(1+u^2)^{(\eta-1)/2}\\ &= (1+u^2)^{\epsilon\eta-1/2}.\end{aligned}$$

The last expression is integrable on $\mathbf{R}$ since $\eta < 0$, therefore by the dominated convergence theorem we can commute the limit and the integral:

$$\lim_{r\to\infty} e^{-(i\lambda-1)r}\phi_\lambda(a_r\cdot o) = \frac{2}{\pi}\int_0^{\infty}(1+u^2)^{-(i\lambda+1)/2}\,du = \mathbf{c}(\lambda).$$

Using the substitution $t = (1+u^2)^{-1}$ and the formula for the beta function:

$$\mathbf{c}(\lambda) = \frac{1}{\pi}\int_0^1 t^{(i\lambda+1)/2}t^{-3/2}(1-t)^{-1/2}\,dt = \pi^{-1/2}\frac{\Gamma(\frac{1}{2}i\lambda)}{\Gamma(\frac{1}{2}(i\lambda+1))}$$

and we are done. □

We are now in a position to prove Theorem 1.3. Suppose that f is a radial function and set $x = \tanh s e^{i\theta}$ so that $f(x) = f(\tanh s)$. Define a function $F\colon [1,\infty) \to \mathbf{C}$ by $F((\cosh s)^2) = f(\tanh s)$. Then by the chain rule, $F'((\cosh s)^2) = f'(\tanh s)(2(\sinh s)(\cosh^3 s))^{-1}$. A radial function g on $\mathbf{R}$ has derivative $g'(w) = -g'(-w)$ and so it must follow that $g'(0) = 0$. So in particular, $f'(o) = 0$ and thus the limit $\lim_{u\to 1} F'(u)$ exists by L'Hospital's theorem. Now by our integral formulas

$$\begin{aligned}\widetilde{f}(\lambda) = \int_{\mathbf{R}} e^{-i\lambda t - t}\int_{\mathbf{R}} f(n_p a_t\cdot o)\,dp\,dt &= \int_{\mathbf{R}} e^{-i\lambda t-t}\int_{\mathbf{R}} F((\cosh t)^2 + p^2e^{-2t})\,dp\,dt\\ &= \int_{\mathbf{R}} e^{-i\lambda t}\int_{\mathbf{R}} F((\cosh t)^2 + y^2)\,dy\,dt\end{aligned}$$

where we have made the substitution $y = pe^{-t}$. For $u \ge 1$, the equation

$$\phi(u) = \int_{\mathbf{R}} F(u+y^2)\,dy$$

can be solved in the following manner:

$$\begin{aligned}\int_{\mathbf{R}}\phi'(u+x^2)\,dx = \int_{\mathbf{R}}\int_{\mathbf{R}} F'(u+x^2+y^2)\,dx\,dy &= 2\pi\int_0^{\infty} F'(u+r^2)r\,dr\\ &= \pi\int_0^{\infty} F'(u+q)\,dq\,.\end{aligned}$$

By the fundamental theorem of calculus we conclude that

$$-\pi F(u) = \int_{\mathbf{R}} \phi'(u + x^2)\, dx$$

since $\lim_{q\to 0} F(u+q)$ exists and $\lim_{q\to\infty} F(u+q) = 0$ as $f \in \mathcal{D}(\mathbf{H}^2)$. Therefore we have that

$$f(o) = F(1) = -\frac{1}{\pi}\int_{\mathbf{R}} \phi'(1 + x^2)\, dx = -\frac{1}{\pi}\int_{\mathbf{R}} \phi'(\cosh^2 x)\cosh x\, dx\,.$$

Since

$$\widetilde{f}(\lambda) = \int_{\mathbf{R}} \phi(\cosh^2 t) e^{-i\lambda t}\, dt$$

we have by the ordinary Fourier inversion formula on $\mathbf{R}$

$$\phi(\cosh^2 t) = \frac{1}{2\pi}\int_{\mathbf{R}} \widetilde{f}(\lambda) e^{i\lambda t}\, d\lambda = \int_{\mathbf{R}} \widetilde{f}(\lambda)\cos(\lambda t)\, d\lambda$$

since $\widetilde{f}$ is symmetric in λ due to the fact that $\phi_\lambda = \phi_{-\lambda}$. Since $\widetilde{f}$ decays faster than any polynomial in λ at infinity (since $f \in \mathcal{D}(\mathbf{H}^2)$), we can differentiate ϕ, and under the integral sign to get

$$-\phi'(\cosh^2 t) 2\cosh(t)\sinh(t) = \frac{1}{2\pi}\int_{\mathbf{R}} \widetilde{f}(\lambda)\lambda\sin\lambda t\, d\lambda\,. \tag{2.2.12}$$

Dividing through by $\sinh t$ and using the formula

$$\int_0^\infty \frac{\sin\lambda t}{\sinh t}\, dt = \frac{\pi}{2}\tanh\left(\frac{\pi\lambda}{2}\right)$$

and integrating (2.2.12) in t over $[0,\infty)$ we obtain

$$f(o) = \frac{1}{2\pi^2}\int_{\mathbf{R}} \widetilde{f}(\lambda)\frac{\lambda\pi}{2}\tanh\left(\frac{\pi\lambda}{2}\right) d\lambda\,. \tag{2.2.13}$$

But if $\lambda \in \mathbf{R}$ we have that $|\mathbf{c}(\lambda)|^{-2} = (\lambda\pi/2)\tanh(\lambda\pi/2)$ whence

$$f(o) = \frac{1}{2\pi^2}\int_{\mathbf{R}} \widetilde{f}(\lambda)|\mathbf{c}(\lambda)|^{-2}\, d\lambda\,. \tag{2.2.14}$$

Now let $g \in G$ and set

$$F(x) = \int_K f(gk\cdot x)\, dk\,.$$

Using elementary properties of the measures dk and dx and the spherical function ϕ_λ:

$$\begin{aligned}\widetilde{F}(\lambda) &= \int_K\int_{\mathbf{H}^2} f(gk\cdot x)\phi_{-\lambda}(x)\, dx\, dk = \int_K\int_{\mathbf{H}^2} f(g\cdot x)\phi_{-\lambda}(k^{-1}\cdot x)\, dx\, dk \\ &= \int_{\mathbf{H}^2} f(g\cdot x)\phi_{-\lambda}(x)\, dx = \int_{\mathbf{H}^2} f(x)\phi_{-\lambda}(g^{-1}\cdot x)\, dx = \int_K\int_{\mathbf{H}} f(k\cdot x)\phi_{-\lambda}(g^{-1}k\cdot x)\, dx\, dk \\ &= \int_{\mathbf{H}^2} f(x)\left(\int_K \phi_{-\lambda}(g^{-1}k\cdot o)\, dk\right) dx = \phi_{-\lambda}(g^{-1}\cdot o)\int_{\mathbf{H}^2} f(x)\phi_{-\lambda}(x)\, dx = \phi_\lambda(g\cdot o)\widetilde{f}(\lambda).\end{aligned}$$

(The fact that $\phi_{-\lambda}(g^{-1}\cdot o) = \phi_\lambda(g\cdot o)$ is a consequence of formula (2.3.4) which we prove later). Thus applying the result of (2.2.14) to F we obtain

$$f(g\cdot o) = F(o) = \frac{1}{2\pi^2}\int_{\mathbf{R}} \widetilde{F}(\lambda)|\mathbf{c}(\lambda)|^{-2}\,d\lambda = \frac{1}{2\pi^2}\int_{\mathbf{R}} \widetilde{f}(\lambda)\phi_\lambda(g\cdot o)|\mathbf{c}(\lambda|^{-2}\,d\lambda\,. \tag{2.2.15}$$

Thus Theorem 1.3 is proved and the measure $d\mu(\lambda)$ is equal to $(2\pi^2)^{-1}|\mathbf{c}(\lambda)|^{-2}\,d\lambda$. Additionally $f*\phi_\lambda$ is radial and so integrating it over K we obtain

$$\begin{aligned} f*\phi_\lambda(g\cdot o) = \int_K\int_G f(h\cdot o)\phi_\lambda(h^{-1}kg\cdot o)\,dh\,dk &= \int_G f(h\cdot o)\phi_{-\lambda}(h\cdot o)\phi_\lambda(g\cdot o)\,dh \\ &= \widetilde{f}(\lambda)\phi_\lambda(g\cdot o) \end{aligned}$$

so that

$$f(x) = \frac{1}{2\pi^2}\int_{\mathbf{R}} f*\phi_\lambda(x)|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,.$$

To prove Theorem 1.4 we apply (2.2.14) to the function

$$F(g\cdot o) = \int_{\mathbf{H}^2} f(g\cdot x)\overline{f(x)}\,dx\,.$$

So the spherical transform of F is

$$\begin{aligned} \widetilde{F}(\lambda) = \int_G\int_G f(gh\cdot o)\overline{f(h\cdot o)}\phi_{-\lambda}(g\cdot o)\,dg\,dh &= \int_G\int_G f(g\cdot o)\overline{f(h\cdot o)}\phi_{-\lambda}(gh\cdot o)\,dg\,dh \\ &= \int_G f(g\cdot o)\overline{f(h\cdot o)}\phi_{-\lambda}(g\cdot o)\phi_{-\lambda}(h\cdot o)\,dg\,dh = |\widetilde{f}(\lambda)|^2. \end{aligned}$$

So we conclude

$$F(o) = \int_{\mathbf{H}^2} |f(x)|^2\,dx = \frac{1}{2\pi^2}\int_{\mathbf{R}} |\widetilde{f}(\lambda)|^2|\mathbf{c}(\lambda)|^{-2}\,d\lambda\,.$$

Thus we have obtained the Plancherel identity for radial functions $f\in\mathcal{D}_K(\mathbf{H}^2)$.

2.3 The Fourier transform

In this section we will extend our spherical transform from K-invariant functions to more arbitrary functions. The extension of the spherical transform will of course be the Fourier transform on $\mathbf{H}^2$. Before we define our more general Fourier transform, we start with two preliminary lemmas.

Lemma 2.3.1. *For $g\in G$ and $x\in\mathbf{H}^2$ we have the fundamental identity*

$$\langle g\cdot x, g\cdot b\rangle = \langle x,b\rangle + \langle g\cdot o, g\cdot b\rangle. \tag{2.3.1}$$

Proof. We provide a more general proof in Proposition 3.1.7. Geometrically speaking since G acts on $\mathbf{H}^2$ by isometries the distance between the horocycles $\xi(x,b)$ and $\xi(o,b_0)$ is the same as the distance between the horocycles $\xi(g\cdot o, g\cdot b)$ and $\xi(g\cdot x, g\cdot b)$ (here $b_0 = 1$).

So writing the horocycle $\xi(x,b) = kaN\cdot o$ for some $k\in K$ and $a\in A$ we can multiply this horocycle by k^{-1} to get $\xi(k^{-1}\cdot x, k^{-1}b) = aN\cdot o$ where $k^{-1}b = b_0$. Writing $g = k_1a_1n_1$ for $k_1\in K$, $a_1\in A$ and $n_1\in N$ using the Iwasawa decomposition $G = KAN$ we see that

$$g\xi(k^{-1}\cdot x, k^{-1}b) = \xi(gk^{-1}\cdot ox, gk^{-1}b) = k_1a_1n_1aN\cdot o = k_1a_1aN\cdot o$$

since A normalizes N. Thus, $\langle gk^{-1}\cdot x, gk^{-1}b\rangle = \log(a_1 a) = \log(a_1) + \log(a)$. Now we easily see that $\log(a) = \langle k^{-1}\cdot x, k^{-1}b\rangle$ and $\log(a_1) = \langle g\cdot o, g(k^{-1}b)\rangle$. Thus,

$$\langle gk^{-1}\cdot x, gk^{-1}b\rangle = \langle k^{-1}\cdot x, k^{-1}b\rangle + \langle g\cdot o, g(k^{-1}b)\rangle.$$

Replacing g by gk we obtain

$$\langle g\cdot x, g\cdot b\rangle = \langle k^{-1}\cdot x, k^{-1}b\rangle + \langle g\cdot o, g\cdot b\rangle.$$

On the other hand $\xi(k^{-1}\cdot x, k^{-1}b) = aN\cdot o$ and $\xi(x,b) = kaN\cdot o$. So $\langle k^{-1}\cdot x, k^{-1}b\rangle = \log(a) = \langle x, b\rangle$. $\square$

This geometric identity is our analogue of the well known inner product identities on $\mathbf{R}^n$. Next, we have a lemma which relates how the Haar measure db transforms under translation by elements in G.

Lemma 2.3.2. *Consider the Haar measure db on B. Then if $g \in G$, we have $d(g\cdot b) = e^{-2\langle g\cdot o, g\cdot b\rangle}db$.*

Proof. Note that if $x \in \mathbf{H}^2$, we can write the action of $g \in G$ on x in the form $g\cdot x = k\cdot(x+z)/(\bar{z}x+1)$ where $z \in \mathbf{C}$ and $|z| < 1$ and $k \in K$ is a rotation. In fact if

$$g = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix},$$

then $z = \beta/\alpha$ and $k = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$ where $\alpha = |\alpha|e^{it}$. Let $db = d\theta$ where $d\theta$ is usual angular measure on B induced by the Lebesgue measure on $\mathbf{R}$ and write

$$e^{i\psi(\theta)} = e^{2it}\left(\frac{e^{i\theta}+z}{\bar{z}e^{i\theta}+1}\right),$$

then the measure of interest is $d\psi = \psi'(\theta)d\theta$. However a straightforward calculation shows that

$$\psi'(\theta) = \frac{1-|z|^2}{|e^{i\theta}-z|^2} = e^{2\langle g^{-1}\cdot o, e^{i\theta}\rangle} = e^{-2\langle g\cdot o, g\cdot e^{i\theta}\rangle}$$

since $g^{-1}\cdot o = -z$ and by (2.1.13). $\square$

Remark 6. Thus if dk is the Haar measure on K, then $d(g\cdot k) = e^{-2\langle g\cdot o, gk\cdot b_0\rangle}dk$ for $g \in G$. If $g \in K$, then naturally $d(g\cdot k) = dk$ since $\langle g\cdot o, gk\cdot b_0\rangle = 0$ which agrees with the fact that dk is a Haar measure on K.

Definition 2.3.1. We define the *Fourier transform* of a function $f \in C_c(\mathbf{H}^2)$ by the formula

$$\widetilde{f}(\lambda, b) = \int_{\mathbf{H}^2} f(x)e^{(-i\lambda+1)\langle x,b\rangle}\,dx\,. \tag{2.3.2}$$

To prove a type of inversion formula for the Fourier transform we require another lemma.

Lemma 2.3.3. *If $f \in C_c(\mathbf{H}^2)$, then*

$$f * \phi_\lambda(g\cdot o) = \int_B \widetilde{f}(\lambda, b)e^{(i\lambda+1)\langle g\cdot o, b\rangle}\,db\,.$$

Proof. Since f has compact support we can appeal to Fubini's theorem to obtain:

$$\begin{aligned} f * \phi_\lambda(g\cdot o) &= \int_G f(h\cdot o)\phi_\lambda(h^{-1}g\cdot o)\,dh = \int_B\int_G f(h\cdot o)e^{(i\lambda+1)\langle h^{-1}g\cdot o, b\rangle}\,dh\,db \\ &= \int_B\int_G f(h\cdot o)e^{(-i\lambda+1)\langle h\cdot o,b\rangle}e^{(i\lambda+1)\langle g\cdot o,b\rangle}\,dh\,db = \int_B \widetilde{f}(\lambda,b)e^{(i\lambda+1)\langle g\cdot o,b\rangle}\,db\,. \end{aligned}$$

Here we have used the following identity by applying the geometric identity (2.3.1):

$$\langle h^{-1}g \cdot o, b\rangle = \langle g \cdot o, h \cdot b\rangle + \langle h^{-1} \cdot o, b\rangle = \langle g \cdot o, h \cdot b\rangle - \langle h \cdot o, h \cdot b\rangle. \tag{2.3.3}$$

So that

$$\phi_\lambda(h^{-1}g \cdot o) = \int_B e^{(i\lambda+1)\langle h^{-1}g\cdot o, b\rangle}\, db = \int_B e^{(i\lambda+1)(\langle g\cdot o, h\cdot b\rangle - \langle h\cdot o, h\cdot b\rangle)}\, db = \tag{2.3.4}$$

$$\int_B e^{(i\lambda+1)\langle g\cdot o, b\rangle} e^{-(i\lambda+1)\langle h\cdot o, b\rangle} e^{2\langle h\cdot o, b\rangle}\, db = \int_B e^{(i\lambda+1)\langle g\cdot o, b\rangle} e^{(-i\lambda+1)\langle h\cdot o, b\rangle}\, db\,. \tag{2.3.5}$$

In the second line above we have made the substitution $b \mapsto h^{-1}b$ and used the previous lemma. □

Theorem 2.3.4. *If $f \in C_c(\mathbf{H}^2)$, then we have the following inversion formula:*

$$f(x) = \int_{\mathbf{R}} \int_B \widetilde{f}(\lambda, b) e^{(i\lambda+1)\langle x, b\rangle}\, d\mu(\lambda)\, db \tag{2.3.6}$$

Proof. To prove the inversion formula for the Fourier transform for $f \in C_c(\mathbf{H}^2)$, let $g \in G$ and consider the K-invariant function: $F(x) = \int_K f(gk \cdot x)\, dk$. By the inversion formula for the spherical transform we have $F(x) = \int_{\mathbf{R}} \widetilde{F}(\lambda)\phi_\lambda(x)\, d\mu(\lambda)$ and

$$\begin{aligned}\widetilde{F}(\lambda) = \int_G \int_K f(gkh \cdot o)\phi_{-\lambda}(h \cdot o)\, dk\, dh &= \int_G f(gh \cdot o)\phi_{-\lambda}(h \cdot o)\, dh \\ &= \int_G f(h \cdot o)\phi_{-\lambda}(g^{-1}h \cdot o)\, dh = f * \phi_\lambda(g \cdot o).\end{aligned}$$

Hence,

$$\begin{aligned}f(g \cdot o) = F(o) = \int_{\mathbf{R}} \widetilde{F}(\lambda)\, d\mu(\lambda) &= \int_{\mathbf{R}} f * \phi_\lambda(g \cdot o)\, d\mu(\lambda) \\ &= \int_{\mathbf{R}} \int_B \widetilde{f}(\lambda, b) e^{(i\lambda+1)\langle g\cdot o, b\rangle}\, d\mu(\lambda)\, db\,.\end{aligned}$$

Thus we have established the Fourier inversion theorem for $C_c(\mathbf{H}^2)$. □

It is easy to check that the restriction of the Fourier transform to $C_c(\mathbf{H}^2)_K$ (the K-invariant compactly supported continuous functions) reduces to the spherical transform defined in the previous section. So in fact, the Fourier transform is an extension of the spherical transform to functions lacking the condition of K-invariance on the left. If $f, g \in C_c(\mathbf{H}^2)_K$, then we have $(f * g)\widetilde{\ } = \widetilde{f} \cdot \widetilde{g}$. However, this is not true for the Fourier transform for arbitrary functions in $C_c(\mathbf{H}^2)$. Indeed, the convolution algebra $C_c(\mathbf{H}^2)$ is noncommutative (which we will take for granted) which automatically prohibits the Fourier transform from converting convolution on $\mathbf{H}^2$ to multiplication of functions on the space $\mathbf{R} \times B$ due to considerations of taking the inverse Fourier transform.

This does lead us to conclude that the convolution algebra $C_c(\mathbf{H}^2)_K$ is indeed commutative. This is in fact a special case of a result of Harish-Chandra. However, we do have a suitable statement of the convolution theorem for the general Fourier transform on $\mathbf{H}^2$.

Theorem 2.3.5 (Convolution theorem)**.** *Suppose that $f, g \in C_c(\mathbf{H}^2)$ and that g is K-invariant, then $(f * g)\widetilde{\ } = \widetilde{f} \cdot \widetilde{g}$.*

Proof. First note that $\widetilde{g}(\lambda, b) = \widetilde{g}(\lambda)$ since the Fourier transform of g is constant in b by K-invariance of g. This fact follows by the observation that if $k \in K$ and $b \in B$, then

$$\widetilde{g}(\lambda, k \cdot b) = \int_G g(x \cdot o) e^{(-i\lambda+1)\langle x\cdot o, k\cdot b\rangle}\, dx = \int_G g(kx \cdot o) e^{(-i\lambda+1)\langle x\cdot o, b\rangle}\, dx = \widetilde{g}(\lambda, b).$$

Thus $\widetilde{g}$ is constant in b and integrating the Fourier transform of g over B yields the spherical transform of g. Now a simple calculation shows (in what follows dx and dy are the same Haar measure on G):

$$\int_G \left(\int_G f(y \cdot o) g(y^{-1}x \cdot o)\, dy \right) e^{(-i\lambda+1)\langle x\cdot o, b\rangle}\, dx = \int_G \int_G f(y\cdot o) g(y^{-1}x\cdot o) e^{(-i\lambda+1)\langle x\cdot o,b\rangle}\, dy\, dx$$
$$= \int_G \int_G f(y\cdot o) g(x\cdot o) e^{(-i\lambda+1)\langle yx\cdot o, b\rangle}\, dy\, dx = \int_G \int_G f(y\cdot o) g(x\cdot o) e^{(-i\lambda+1)(\langle x, y^{-1}b\rangle + \langle y,b\rangle)}\, dy\, dx$$
$$= \int_G f(y\cdot o)\widetilde{g}(\lambda, y^{-1}b) e^{(-i\lambda+1)\langle y,b\rangle}\, dy = \widetilde{f}(\lambda, b)\widetilde{g}(\lambda).$$

□

In analogy to the relationship of the Radon transform to the Fourier transform on $\mathbf{R}^n$ on $\mathbf{H}^2$ we have the following connection. Since

$$\int_{\mathbf{H}^2} f(x)\, dx = \int_{AN} f(an\cdot o)\, da\, dn$$

we have

$$\widetilde{f}(\lambda, k\cdot b_0) = \int_{\mathbf{H}^2} f(x) e^{(-i\lambda+1)\langle x, k\cdot b_0\rangle}\, dx = \int_{\mathbf{H}^2} f(k\cdot x) e^{(-i\lambda+1)\langle x, b_0\rangle}\, dx$$
$$= \int_{AN} f(kan\cdot o) e^{(-i\lambda+1)\log a}\, da\, dn = \int_A \widehat{f}(ka\cdot \xi_0) e^{(-i\lambda+1)\log a}\, da\,.$$

Thus the Fourier transform of $f \in C_c(\mathbf{H}^2)$ takes the form of the *Euclidean Fourier transform* of the Radon transform of f (note that A is group theoretically isomorphic to the Euclidean space $\mathbf{R}$ with $\log\colon A \to \mathbf{R}$ being the group isomorphism). In fact, from this representation we can deduce many of the same theorems that hold for the Fourier transform on $\mathbf{R}^n$ to $\mathbf{H}^2$ using elements of the real variable Fourier transform theory. We discuss these topics in the next section.

2.3.1 Plancherel and Paley-Wiener theorems

Definition 2.3.2 (Simplicity criterion)**.** We say that $\lambda \in \mathbf{C}$ is *simple* if the map $T\colon L^2(B) \to C^\infty(\mathbf{H}^2)$ given by

$$Tf(x) = \int_B e^{(i\lambda+1)\langle x,b\rangle} f(b)\, db$$

is injective, i.e. $\ker T = \{0\}$.

Proposition 2.3.6. *The nonsimple points in* $\mathbf{C}$ *are the elements of the form* $\lambda = i(1+2k)$, $k \in \mathbf{Z}^+$.

Proof. Represent $x \in \mathbf{H}^2$ in the geodesic coordinates $x = (\tanh r)e^{i\phi}$ and put $b = e^{i\theta}$, then using formula (2.1.13) and the standard angular measure $db = (2\pi)^{-1} d\theta$ we have

$$Tf(x) = \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1-\tanh^2 r}{|(\tanh r)e^{i\phi} - e^{i\theta}|^2}\right]^{-(i\lambda+1)/2} f(e^{i\theta})\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1-\tanh^2 r}{|\tanh r - e^{i\theta}|^2}\right]^{-(i\lambda+1)/2} f(e^{i(\theta+\phi)})\, d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{\operatorname{sech}^2 r}{(\tanh r - \cos\theta)^2 + \sin^2\theta}\right]^{-(i\lambda+1)/2} f(e^{i(\theta+\phi)})\, d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \left[\frac{1}{\sinh^2 r - \sinh r\cosh r\cos\theta + 1}\right]^{-(i\lambda+1)/2} f(e^{i(\theta+\phi)})\, d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi} (\cosh 2r - \sinh 2r\cos\theta)^{-(i\lambda+1)/2} f(e^{i(\theta+\phi)})\, d\theta\,.$$

Thus, if $\lambda = i(1+2k)$, then letting $f_k(e^{i\theta}) = e^{i(k+1)\theta}$, then one can verify that

$$Tf_k(x) = \frac{1}{2\pi}\int_0^{2\pi} (\cosh 2r - \sinh 2r\cos\theta)^k e^{i(k+1)(\theta+\phi)}\, d\theta = 0.$$

To see this, note that by a symmetry argument $\operatorname{Im} Tf_k = 0$. Investigating $\operatorname{Re} Tf_k$ we expand $(\cosh 2r - \sinh 2r\cos\theta)^k$ into a sum by the binomial theorem and we obtain a finite some of integrals of the form $c_n \int_0^{2\pi} \cos^n\theta\cos(k+1)\theta\, d\theta$ where $0 \le n \le k$ and $c_n \in \mathbf{R}$. By the product to sum formula we obtain that the integral

$$\int_0^{2\pi} \cos^n(\theta)\cos(k+1)\theta\, d\theta = \frac{1}{2^{n+1}} \sum_{\alpha\in\{1,-1\}^{n+1}} \int_0^{2\pi} \cos\left[\theta(\sum_{j=1}^n \alpha_j + \alpha_{n+1}(k+1))\right] d\theta\,.$$

Here $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$. The sum $\sum_{j=1}^n \alpha_j + \alpha_n(k+1) \neq 0$ for $n \le k$, thus each integral appearing in the sum above vanishes. Hence $Tf_k \equiv 0$ and so λ is not simple. To study those λ such that $\lambda \neq i(1+2k)$ where $k \in \mathbf{Z}^+$ we require a lemma.

Lemma 2.3.7. *Let $f \in L^1([0,2\pi])$ and put*

$$H(t) = \int_0^{2\pi} (\cosh t - \sinh t\cos\theta)^{-s} f(\theta)\, d\theta\,.$$

Then if $-s \notin \mathbf{Z}^+$, then $H = 0$ implies that $\int_0^{2\pi} \cos^n(\theta) f(\theta)\, d\theta = 0$ for each $n \in \mathbf{N}$.

Proof. We proceed by induction on n. Observe

$$\frac{d^n}{dt^n}(\cosh t - \sinh t\cos\theta)^{-s} = \sum_{j=1}^n \alpha_j^n (\cosh t - \sinh t\cos\theta)^{-s-j}(\sinh t - \cosh t\cos\theta)^j$$

for some choice of constants $\alpha_j^n \in \mathbf{N}$. In particular $\alpha_n^n = (-1)^n s(s+1)\cdots(s+n-1)$ which is always nonzero if s is not a negative integer. So if $H = 0$, then $H^{(n)} = 0$ and so by commuting the derivative with the integral and setting $t = 0$ we conclude that for each $n \in \mathbf{N}$ that $\int_0^{2\pi} \cos^n(\theta) f(\theta)\, d\theta = 0$. ☐

Returning to the proof of the proposition we have that if $\lambda \neq i(1+2k)$ for $k \in \mathbf{Z}^+$, then $-s = -1/2(i\lambda+1) \notin \mathbf{Z}^+$ and so by the conclusion of our lemma we have that if $Tf = 0$, then for all $n \in \mathbf{N}$,

$$\int_0^{2\pi} \cos^n(\theta) f(e^{i(\theta+\phi)})\, d\theta = 0. \tag{2.3.7}$$

Expanding f into its L^2 Fourier series we have $f(e^{i(\theta+\phi)}) = \sum_{m\in\mathbf{Z}} a_m e^{im\theta} e^{im\phi}$, then by (2.3.7) and by the product to sum formula we have $a_n e^{in\phi} = -a_{-n} e^{-in\phi}$ for all $\phi \in [0, 2\pi]$. Thus the terms of the Fourier series for f cancel and so $f = 0$. This concludes the proof. ☐

Proposition 2.3.8. *If $-\lambda$ is simple, then the space of functions $\{\widetilde{f}(\lambda,\cdot)\colon f \in \mathcal{D}(\mathbf{H}^2)\}$ is dense in $L^2(B)$.*

Proof. If this were not the case, then there exists some $g \in L^2(B)$ such that $\langle \widetilde{f}(\lambda,\cdot), g\rangle = 0$ for all $f \in \mathcal{D}(\mathbf{H}^2)$ by Proposition A.3.1. Hence we have

$$\int_B \widetilde{f}(\lambda, b) g(b)\, db = \int_B \int_{\mathbf{H}^2} f(x) g(b) e^{(-i\lambda+1)\langle x,b\rangle}\, dx\, db = \int_{\mathbf{H}^2} f(x) \int_B g(b) e^{(-i\lambda+1)\langle x,b\rangle}\, db\, dx = 0.$$

By the density of $\mathcal{D}(\mathbf{H}^2)$ in $L^1(\mathbf{H}^2)$ we conclude that

$$\int_B g(b) e^{(-i\lambda+1)\langle x,b\rangle}\, db = 0$$

for all $x \in \mathbf{H}^2$ which contradicts the simplicity of $-\lambda$. ☐

Now we arrive at the Plancherel theorem.

Theorem 2.3.9 (Plancherel Theorem). *The Fourier transform* $\mathscr{F}\colon f \mapsto \widetilde{f}$ *extends to an isometry of* $L^2(\mathbf{H}^2)$ *onto* $L^2(B \times \mathbf{R}^+; 2\, d\mu(\lambda, b))$. *Here* $d\mu(\lambda, b) = d\mu(\lambda)\, db$.

Proof. We first show that $\|f\|_{L^2(\mathbf{H}^2)} = \|\widetilde{f}\|_{L^2(B\times\mathbf{R}^+)}$ for $f \in \mathcal{D}(\mathbf{H}^2)$. Then extending the Fourier transform by continuity to all of $L^2(\mathbf{H}^2)$ we obtain that the Fourier transform is an isometry of $L^2(\mathbf{H}^2)$ into $L^2(B \times \mathbf{R}^+; 2\, d\mu(\lambda, b))$. Afterwards we prove surjectivity.

So if $f \in \mathcal{D}(\mathbf{H}^2)$ and we have that if $\lambda \in \mathbf{R}$, then λ is simple. Furthermore we have the elementary identity

$$\int_{\mathbf{H}^2} f * \phi_\lambda(x) \overline{f}(x)\, dx = \int_B |\widetilde{f}(\lambda, b)|^2\, db$$

so that after integrating this identity over $\mathbf{R}^+$ with the measure $(\pi^2)^{-1}|\mathbf{c}(\lambda)|^{-2} d\lambda$ we have

$$\begin{aligned}\|\widetilde{f}\|^2_{L^2(B\times\mathbf{R}^+)} = \frac{1}{2\pi^2}\int_{\mathbf{R}}\int_{\mathbf{H}^2} f * \phi_\lambda(x)\overline{f}(x)\, dx\, |\mathbf{c}(\lambda)|^{-2}\, d\lambda &= \frac{1}{2\pi^2}\int_{\mathbf{H}^2} \overline{f}(x)\left\{\int_{\mathbf{R}} f * \phi_\lambda(x)|\mathbf{c}(\lambda)|^{-2}\, d\lambda\right\} dx \\ &= \int_{\mathbf{H}^2} |f(x)|^2\, dx = \|f\|^2_{L^2(\mathbf{H}^2)}.\end{aligned}$$

Note that in the transition from the third to the fourth equality we have used the Fourier inversion formula and we have used the symmetry of $f * \phi_\lambda$ over $\mathbf{R}$. We extend the Fourier transform to all of $L^2(\mathbf{H}^2)$ by setting $\widetilde{f} = \lim_{n\to\infty} \widetilde{f}_n$ where $f_n \to f$ in $L^2(\mathbf{H}^2)$ and each $f_n \in \mathcal{D}(\mathbf{H}^2)$. Although it is elementary, we remark that this extension is well-defined since the Fourier transform is an isometry on a dense subspace of $L^2(\mathbf{H}^2)$.

To prove surjectivity of the Fourier transform we note that it is sufficient to prove that the image of the subspace $X = \mathcal{D}(\mathbf{H}^2) \hookrightarrow L^2(\mathbf{H}^2)$ under the Fourier transform is dense in $L^2(B \times \mathbf{R}^+)$. This is because the range of a linear isometry between Banach spaces is always closed. By way of contradiction suppose that there is $F \in L^2(B \times \mathbf{R}^+)$ so that $F \in \widetilde{X}^\perp$.

Consider the subspace $\tilde{X}^K = \mathcal{D}_K(\mathbf{H}^2) \hookrightarrow L^2(\mathbf{H}^2)$. Then we have that $\widetilde{X}^K$ forms an algebra under multiplication (since if $f_1, f_2 \in \mathcal{D}_K(\mathbf{H}^2)$, we have $(f_1 * f_2)^\sim(\lambda) = \widetilde{f}_1(\lambda) \cdot \widetilde{f}_2(\lambda)$) and is closed under complex conjugation since $\overline{\widetilde{f}}(\lambda) = \widetilde{\overline{f}}(-\lambda) = \widetilde{\overline{f}}(\lambda)$ since $\phi_\lambda = \phi_{-\lambda}$. Next, the elements of $\widetilde{X}^K$, regarded as continuous functions on $\mathbf{R}$, vanish at infinity since we note that

$$\widetilde{f}(\lambda) = \int_A \widehat{f}(a \cdot \xi_0) e^{(-i\lambda+1)\log a}\, da$$

and thus by the standard Riemann-Lebesgue Lemma on $\mathbf{R}$, $\widetilde{f}(\lambda) \to 0$ as $|\lambda| \to \infty$. Finally, the algebra $\widetilde{X}^K$ separates points on $\mathbf{R}^+$ since if $\widetilde{f}(\lambda_1) = \widetilde{f}(\lambda_2)$ for all $f \in \mathcal{D}_K(\mathbf{H}^2)$, then by density we have $\phi_{\lambda_1} = \phi_{\lambda_2}$. But applying the Laplacian $\Delta_{\mathbf{H}^2}$ to ϕ_{λ_j} for $j = 1, 2$ we find $\lambda_1^2 = \lambda_2^2$ so that $\lambda_1 = \lambda_2$. Thus, by the Stone-Weierstrass theorem for locally compact Hausdorff spaces we find that $\widetilde{X}^K$ is a dense subalgebra of $C_0(\mathbf{R}^+)$.

So if $F \in \widetilde{X}^\perp$, then for a fixed $f \in \mathcal{D}(\mathbf{H}^2)$ and $\psi \in \mathcal{D}_K(\mathbf{H}^2)$ we have

$$\int_{\mathbf{R}^+} \widetilde{\psi}(\lambda)\left\{\int_B F(\lambda, b)\widetilde{f}(\lambda, b)\, db\right\} d\mu(\lambda) = \int_{B\times\mathbf{R}^+} (f * \psi)^\sim(\lambda, b) F(\lambda, b)\, d\mu(\lambda, b) = 0.$$

Thus by density considerations we have that the function

$$P_f\colon \lambda \mapsto \int_B F(\lambda, b)\widetilde{f}(\lambda, b)\, db$$

vanishes almost everywhere. Let N_f be the nullset for which P_f is nonzero. For each $n \in \mathbf{N}$, let $\phi_n \in \mathcal{D}(\mathbf{H}^2)$ be a smooth characteristic function of the ball $B_n(o) = \{x \in \mathbf{H}^2 \colon d(o, x) < n\}$. Writing $x = a + ib \in \mathbf{H}^2$, let

V denote the vector space generated by all functions of the form $\phi_n(x)P(a,b)$ where P is a polynomial of a and b with rational coefficients. Thus, we remark that V is a countable subspace of $\mathcal{D}(\mathbf{H}^2)$.

So in particular, if we put $N = \bigcup_{f\in V} N_f$, then N being the countable union of nullsets is a Lebesgue nullset of $\mathbf{R}^+$. Choosing a sequence $f_k \in V$ which converges to f uniformly, we conclude that

$$\lim_{k\to\infty} \int_B F(\lambda,b)\widetilde{f}_k(\lambda,b)\,db = \int_B F(\lambda,b)\widetilde{f}(\lambda,b)\,db = 0,$$

for all $\lambda \in \mathbf{R}^+ \setminus N$. Thus, for each $\lambda \in \mathbf{R}^+ \setminus N$, $f \in \mathcal{D}(\mathbf{H}^2)$, we have $P_f(\lambda) = 0$. Thus by the simplicity of $\lambda \in \mathbf{R}^+ \setminus N$, and Proposition 2.3.8, the function $b \mapsto F(\lambda,b) = 0$ for all $\lambda \in \mathbf{R}^+ \setminus N$. Hence $F(\lambda,b) = 0$ almost everywhere. So by Proposition A.2.1, $\widetilde{X}$ is dense in $L^2(\mathbf{R}^+ \times B)$. Whence the Fourier transform $\mathscr{F}$ is surjective on $L^2(\mathbf{H}^2)$ which finishes the proof. □

Definition 2.3.3. Let $N > 0$ be a real number. We say a smooth function $F\colon \mathbf{C} \times B \to \mathbf{C}$, is of uniform exponential type N if F is holomorphic in the variable λ and for each $m \in \mathbf{N}$

$$\sup_{\lambda\in\mathbf{C}, b\in B} |F(\lambda,b)(1+|\lambda|)^m e^{-N|\operatorname{Im}\lambda|}| < \infty. \tag{2.3.8}$$

Denote the set of functions on $\mathbf{C} \times B$ of uniform exponential type N by $\mathscr{H}_N$ and put $\mathscr{H}_0 = \bigcup_{N>0} \mathscr{H}_N$. Finally, let $\mathscr{H}$ denote the subspace of $\mathscr{H}_0$ to be the set of elements which satisfy the following symmetry condition

$$\int_B \psi(\lambda,b)e^{(i\lambda+1)\langle x,b\rangle}\,db = \int_B \psi(-\lambda,b)e^{(-i\lambda+1)\langle x,b\rangle}\,db$$

for all $x \in \mathbf{H}^2$.

Theorem 2.3.10 (Paley-Wiener theorem)**.** *The Fourier transform $\mathscr{F}$ is a bijection of $\mathcal{D}(\mathbf{H}^2)$ onto $\mathscr{H}$.*

The proof of this theorem is quite long for the purposes of this thesis and we shall not furnish the entire proof. However, we find it more fruitful to cover the basic idea of the proof instead. First, it is clear that if $f \in \mathcal{D}(\mathbf{H}^2)$ has support in the ball $B_N(o)$, then for the Fourier transform

$$\widetilde{f}(\lambda,b) = \int_{\mathbf{H}^2} f(x)e^{(-i\lambda+1)\langle x,b\rangle}\,dx$$

we can complexify λ and differentiate in λ directly on the integral. Furthermore, since $b \mapsto \langle x,b\rangle$ is smooth we determine that $\widetilde{f}(\lambda,b)$ is a smooth function that is holomorphic in λ. Furthermore by elementary estimates we observe

$$|(1+|\lambda|^2)^m \widetilde{f}(\lambda,b)| \le \|\Delta^m f\|_1 \sup_{x\in\operatorname{supp} f} e^{|\operatorname{Im}\lambda|\langle x,b\rangle} \le \|\Delta^m f\|_1 e^{N|\operatorname{Im}\lambda|},$$

which leads us to conclude that $\widetilde{f} \in \mathscr{H}_N$. To see that $\widetilde{f}$ satisfies the aforementioned symmetry condition we note that the spherical functions satisfy $\phi_\lambda = \phi_{-\lambda}$, thus for $x \in \mathbf{H}^2$

$$f * \phi_\lambda(x) = \int_B \widetilde{f}(\lambda,b)e^{(i\lambda+1)\langle x,b\rangle}\,db = \int_B \widetilde{f}(-\lambda,b)e^{(-i\lambda+1)\langle x,b\rangle}\,db = f * \phi_{-\lambda}(x)$$

as needed. The converse is far more complicated and contains the bulk of the proof. Essentially, the major difficulty is proving the inverse Fourier transform $\mathscr{F}^{-1}$ maps $\mathscr{H}$ into $\mathcal{D}(\mathbf{H}^2)$.

Recall that for the torus $\mathbf{T}$ and any function $f \in C^\infty(\mathbf{T})$ we have the Fourier series expansion

$$f(e^{i\theta}) = \sum_{n\in\mathbf{Z}} \widehat{f}(n)e^{in\theta}$$

which converges to f absolutely on $\mathbf{T}$. In particular, for $K = \mathbf{SO}(2)$, the irreducible unitary representations are precisely the functions $\chi_n(e^{i\theta}) = e^{in\theta}$ where $n \in \mathbf{Z}$, i.e. $\mathbf{T} = K$. So for $\psi \in \mathscr{H}$ if we consider the smooth function

$$\Psi(x) = \int_{\mathbf{R}\times B} \psi(\lambda, b) e^{(i\lambda+1)\langle x,b\rangle}\, d\mu(\lambda, b) \tag{2.3.9}$$

we can consider the (absolutely convergent) Fourier series expansion of the function $\gamma \mapsto \Psi(e^{i\gamma}x)$ evaluated at $\gamma = 0$, to obtain

$$\Psi(x) = \frac{1}{2\pi} \sum_{m\in\mathbf{Z}} \int_0^{2\pi} \Psi(e^{i\theta}x)\chi_m(e^{-i\theta})\, d\theta\,. \tag{2.3.10}$$

Inserting the formula for ψ into the sum above we obtain:

$$\begin{aligned}
\Psi(x) &= \sum_{m\in\mathbf{Z}} \frac{1}{2\pi} \int_{\mathbf{R}\times B} \psi(\lambda, b) \left\{ \int_0^{2\pi} e^{(i\lambda+1)\langle x, e^{-i\theta}b\rangle} \chi_m(e^{-i\theta})\, d\theta \right\} d\mu(\lambda, b) \\
&= \sum_{m\in\mathbf{Z}} \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_{\mathbf{R}} \psi(\lambda, e^{i\phi}) \left\{ \int_0^{2\pi} e^{(i\lambda+1)\langle x, e^{i(\phi-\theta)}\rangle} \chi_m(e^{-i\theta})\, d\theta \right\} |\mathbf{c}(\lambda)|^{-2}(2\pi^2)^{-1}\, d\lambda\, d\phi \\
&= \sum_{m\in\mathbf{Z}} \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_{\mathbf{R}} \psi(\lambda, e^{i\phi}) e^{-im\phi} \left\{ \int_0^{2\pi} e^{(i\lambda+1)\langle x, e^{i\theta}\rangle} \chi_m(e^{i\theta})\, d\theta \right\} |\mathbf{c}(\lambda)|^{-2}(2\pi^2)^{-1}\, d\lambda\, d\phi\,.
\end{aligned}$$

We now expand $\psi(\lambda, e^{i\phi})$ into its own absolutely convergent Fourier series $\psi(\lambda, e^{i\phi}) = \sum_{k\in\mathbf{Z}} \psi_k(\lambda)e^{ik\phi}$ where the Fourier coefficients $\psi_k(\lambda)$ are given by

$$\psi_k(\lambda) = \frac{1}{2\pi} \int_0^{2\pi} \psi(\lambda, e^{i\phi})\chi_k(e^{-i\phi})\, d\phi\,.$$

Substituting the Fourier series for $\psi(\lambda, e^{i\phi})$ into our integrals we obtain

$$\begin{aligned}
\Psi(x) &= \sum_{m\in\mathbf{Z}} \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_{\mathbf{R}} \psi(\lambda, e^{i\phi}) e^{-im\phi} \left\{ \int_0^{2\pi} e^{(i\lambda+1)\langle x, e^{i\theta}\rangle} \chi_m(e^{i\theta})\, d\theta \right\} |\mathbf{c}(\lambda)|^{-2}(2\pi^2)^{-1}\, d\lambda\, d\phi \\
&= \sum_{m\in\mathbf{Z}} \frac{1}{2\pi} \int_{\mathbf{R}} \psi_m(\lambda) \left\{ \int_0^{2\pi} e^{(i\lambda+1)\langle x, e^{i\theta}\rangle} \chi_m(e^{i\theta})\, d\theta \right\} |\mathbf{c}(\lambda)|^{-2}(2\pi^2)^{-1}\, d\lambda\,.
\end{aligned}$$

Definition 2.3.4. For $m \in \mathbf{Z}$ and $\lambda \in \mathbf{C}$, define the *Eisenstein integral* (also called the *generalized spherical function*) by

$$\Phi_{\lambda,m}(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{(i\lambda+1)\langle x, e^{i\theta}\rangle} \chi_m(e^{i\theta})\, d\theta\,. \tag{2.3.11}$$

Clearly $\Phi_{\lambda,m}$ is smooth on $\mathbf{H}^2$.

Thus, our final expression for Ψ becomes

$$\Psi(x) = \frac{1}{2\pi^2} \sum_{m\in\mathbf{Z}} \int_{\mathbf{R}} \psi_m(\lambda)\Phi_{\lambda,m}(x)\, d\mu(\lambda)\,. \tag{2.3.12}$$

We trust the reader to verify that all the exchanges of the integrals that we have made are valid due to the rapid decay of ψ and by the absolute convergence of the respective Fourier series expansions. To prove the Paley-Wiener theorem we shall need to show that Ψ has compact support and that $\widetilde{\Psi} = \psi$ so that the assignment $\psi \mapsto \Psi$ is an inverse to the Fourier transform.

As one may expect, if $\psi \in \mathscr{H}_N$, then $\operatorname{supp} \Psi \subset B_N(o)$. This will be done by showing that $\int_{\mathbf{R}} \psi_m(\lambda)\Phi_{\lambda,m}(x)\,d\mu(\lambda)$ vanishes for $x \notin B_N(o)$. This is not necessarily any easier to do from our present position. We will outline those steps.

Step 1: Let $\Psi_m(x) = \int_{\mathbf{R}} \psi_m(\lambda)\Phi_{\lambda,m}(x)\,d\mu(\lambda)$. If $d(o,x) > N$, then we wish to show that $\Psi_m(x) = 0$. In fact it is actually sufficient to prove that $\Psi_m(\tanh r) = 0$ for $r > N$ since $\Psi_m(e^{i\theta}x) = e^{im\theta}\Psi(x)$.

To this end, we must carefully study the Eisenstein integral. In order to simplify our study, we determine a special series expansion for the Eisenstein integrals for which the series' terms have a simpler asymptotic behavior. The particular series expression for the Eisenstein integrals is given by the following proposition.

Proposition 2.3.11. *For $i\lambda \notin 2\mathbf{Z}$, the Eisenstein integral $\Phi_{\lambda,m}$ has the following series expansion:*

$$\Phi_{\lambda,m}(\tanh r) = \mathbf{c}(\lambda)\sum_{n=0}^{\infty}\Gamma_n(\lambda)e^{(i\lambda-1-2n)r} + \mathbf{c}(-\lambda)\frac{p_m(i\lambda)}{p_m(-i\lambda)}\sum_{n=0}^{\infty}\Gamma_n(-\lambda)e^{(-i\lambda-1-2n)r} \tag{2.3.13}$$

where the gamma coefficients satisfy for each $\epsilon > 0$ the estimate $\sup_{n\in\mathbf{N}}|\Gamma_n(\lambda)|e^{\epsilon n} < \infty$ and p_m is the polynomial

$$p_m(y) = \prod_{j=0}^{|m|-1}\left(\frac{1}{2}(y+1)+j\right).$$

Step 2: Using the series expansion for the Eisenstein integrals, in the integral defining Ψ_m we expand $\Phi_{\lambda,m}$ into its series and study the integral by integrating termwise. In order to do this we must be certain that we can commute the sum and integral. This guarantee is provided by application of the following two lemmas.

Lemma 2.3.12. *For $\lambda \in \mathbf{R}$ and $n \in \mathbf{N}$, there exists uniform constants $c, d > 0$ (independent of λ and n) such that $|\Gamma_n(\lambda)| \le c(1+n^d)$.*

Lemma 2.3.13. *The following relation holds: $\psi_m(\lambda) = \psi_m(-\lambda)p_m(-i\lambda)p_m(i\lambda)^{-1}$.*

Thus if $r > N$, then inserting the series representation for $\Phi_{\lambda,m}$ we obtain two integrals

$$\int_{\mathbf{R}} \mathbf{c}(\lambda)|\mathbf{c}(\lambda)|^{-2}\sum_{n=0}^{\infty}\Gamma_n(\lambda)e^{(i\lambda-1-2n)r}\psi_m(\lambda)\,d\lambda$$

and

$$\int_{\mathbf{R}} \mathbf{c}(-\lambda)\frac{p_m(i\lambda)}{p_m(-i\lambda)}|\mathbf{c}(\lambda)|^{-2}\sum_{n=0}^{\infty}\Gamma_n(-\lambda)e^{(-i\lambda-1-2n)r}\psi_m(\lambda)\,d\lambda\,.$$

Thus using Lemma 2.3.12 we can estimate on the integrand and determine that the series $\sum_{n=0}^{\infty}|\Gamma_n(\pm\lambda)|e^{(-1-2n)r}$ converges. So the first integral is dominated by

$$I = A\int_{\mathbf{R}}|\mathbf{c}(\lambda)|^{-1}|\psi_m(\lambda)|\,d\lambda \tag{2.3.14}$$

for some finite constant $A > 0$ while using Lemma 2.3.13 and the relation $\mathbf{c}(\lambda)\mathbf{c}(-\lambda) = |\mathbf{c}(\lambda)|^2$, the second integral is also dominated by I. The integral I is finite due to the fact that $|\mathbf{c}(\lambda)|^{-1}$ has polynomial growth and the rapid decay of ψ_m.

Therefore by Fubini's theorem we can interchange the sum and integral to obtain for Ψ_m and using Lemma 2.3.12 again we get

$$\Psi_m(\tanh r) = (2\pi^2)^{-1}2\sum_{n=0}^{\infty}e^{-(1+2n)r}\int_{\mathbf{R}}\mathbf{c}(-\lambda)^{-1}\Gamma_n(\lambda)\psi_m(\lambda)e^{i\lambda r}\,d\lambda\,.$$

Step 3: The last few steps are to show that

$$Q(r) = \int_{\mathbf{R}} \mathbf{c}(-\lambda)^{-1}\Gamma_n(\lambda)\psi_m(\lambda)e^{i\lambda r}\,d\lambda = 0$$

when $r > N$. This is proved in a similar fashion to the classical Paley-Wiener theorem for $\mathbf{R}^n$ by using the Cauchy integral theorem. The necessary augmentation is that when shifting the contour of integration, we must be mindful since Γ_n and $\mathbf{c}$ both have poles. However, there exists a region of the half plane where the contour of integration may shifted while maintaining a holomorphic integrand. From this, we apply a modified argument of the classical Paley-Wiener theorem to obtain that $Q(r) = 0$ when $r > N$. Thus $\Psi_m \equiv 0$ and $\Psi \equiv 0$ when $r > N$.

Therefore, the assignment $\psi \mapsto \Psi$ maps $\mathscr{H}$ into $\mathcal{D}(\mathbf{H}^2)$. The final step is to prove that $\widetilde{\Psi} = \psi$. Proving this will give both the surjectivity and injectivity of the Fourier transform. To this end put $F = \widetilde{\Psi} - \psi$, then by the Fourier inversion formula

$$Y(x) = \int_{\mathbf{R}} \left\{ \int_B F(\lambda, b)e^{(i\lambda+1)\langle x,b\rangle}\,db \right\} d\mu(\lambda, b) = 0.$$

In particular we see that by using the symmetry condition that F satisfies

$$\int_{\mathbf{R}^+} \left\{ \int_B F(-\lambda, b)e^{(-i\lambda+1)\langle x,b\rangle}\,db \right\} d\mu(\lambda, b) = 0.$$

If we let $f \in \mathcal{D}(\mathbf{H}^2)$ and we multiply by the above integral by f, integrate over $\mathbf{H}^2$, and use Fubini's theorem we have

$$\int_{\mathbf{R}^+ \times B} F(-\lambda, b)\widetilde{f}(\lambda, b)\,d\mu(\lambda, b) = 0.$$

Recall that as the functions $b \mapsto \widetilde{f}(\lambda, b)$ ($f \in \mathcal{D}(\mathbf{H}^2)$) are a dense subset of $L^2(B)$ we must conclude that $F(-\lambda, b) \equiv 0$ on $\mathbf{R}^+ \times B$. But again since

$$\int_B F(\lambda, b)e^{(i\lambda+1)\langle x,b\rangle}\,db = \int_B F(-\lambda, b)e^{(-i\lambda+1)\langle x,b\rangle}\,db = 0, \quad \lambda \in \mathbf{R}^+$$

we find that by the simplicity of λ that $F(\lambda, b) \equiv 0$ on $\mathbf{R}^+ \times B$ as well. Thus $F \equiv 0$ or in other words, $\widetilde{\Psi} = \psi$ and so we are done.

The Fourier transform on $L^1(\mathbf{H}^2)$

So far we have treated the Fourier transform on the classical function spaces $\mathcal{D}(\mathbf{H}^2)$ and $L^2(\mathbf{H}^2)$. In the case of $\mathcal{D}(\mathbf{H}^2)$, the Fourier transform is defined explicitly whereas for $L^2(\mathbf{H}^2)$ the Fourier transform is defined indirectly by extending the Fourier transform by continuity from $\mathcal{D}(\mathbf{H}^2) \hookrightarrow L^2(\mathbf{H}^2)$ to all of $L^2(\mathbf{H}^2)$. The obvious question is if whether one can define the Fourier transform for $L^1(\mathbf{H}^2)$ since the classical Fourier transform theory for a locally compact (abelian or compact) group G permits the Fourier transform to be well-defined on the group algebra $L^1(G)$.

The major obstacle for us it would seem is that if $f \in L^1(\mathbf{H}^2)$, then the integral

$$\widetilde{f}(\lambda, b) = \int_{\mathbf{H}^2} f(x)e^{(-i\lambda+1)\langle x,b\rangle}\,dx \tag{2.3.15}$$

may not converge due to the exponential growth of $e^{\langle x,b\rangle}$ as $x \to b$. As it turns out we can in fact obviate this issue quite satisfactorily by the following lemma.

Lemma 2.3.14 (Riemann-Lebesgue Lemma)**.** *Let $f \in L^1(\mathbf{H}^2)$. Then there exists a subset $B' \subset B$ with $B \setminus B'$ having db-measure zero such that for $b \in B'$,*

1. *The Fourier transform $\widetilde{f}(\lambda, b)$ defined by formula (2.3.15) exists for each λ in the tube $T = \{\lambda \in \mathbf{C} \colon |\operatorname{Im}\lambda| \leq 1\}$ and is holomorphic in its interior.*

2. $\lim_{|\operatorname{Re}\lambda|\to\infty} \widetilde{f}(\lambda, b) = 0$ *uniformly for* $|\operatorname{Im}\lambda| \leq 1$.

To prove the Riemann-Lebesgue lemma we require an intermediate lemma.

Lemma 2.3.15. *The spherical function ϕ_λ is bounded if $|\operatorname{Im}\lambda| \leq 1$.*

Proof. Let $f \in L^1_K(\mathbf{H}^2)$ (the K-invariant integrable functions) and suppose $|\operatorname{Im}\lambda| \leq 1$, then since f is radial, the modified Radon transform $\mathcal{R}f(a_t \cdot \xi_0)$ is even in t. Thus

$$\int_{\mathbf{H}^2} |f(x)||\phi_\lambda(x)|\,dx \leq \int_{\mathbf{H}^2} |f(x)|\phi_{i\operatorname{Im}\lambda}(x)\,dx = \int_A \mathcal{R}|f|(a\cdot\xi_0)e^{(\operatorname{Im}\lambda)\log a}\,da$$

$$= \int_{\mathbf{R}} \mathcal{R}|f|(a_t\cdot\xi_0)e^{(\operatorname{Im}\lambda)t}\,dt = \int_0^\infty (e^{-(\operatorname{Im}\lambda)t} + e^{(\operatorname{Im}\lambda)t})\mathcal{R}|f|(a_t\cdot\xi_0)\,dt \leq 2\int_0^\infty e^{-t}\mathcal{R}|f|(a_t\cdot\xi_0)\,dt \leq \|f\|_1.$$

Thus the pairing $\langle f, \phi_\lambda\rangle$ is continuous, so $\phi_\lambda \in L^\infty_K(\mathbf{H}^2)$ by duality considerations. $\square$

We remark that a stronger result exists which states that ϕ_λ is bounded if and only if $|\operatorname{Im}\lambda| \leq 1$. However we will not require the converse for the proof of the Riemann-Lebesgue lemma on the hyperbolic plane.

Proof of the Riemann-Lebesgue lemma. Let $f \in L^1(\mathbf{H}^2)$, then

$$\int_B |\widetilde{f}(\lambda, b)|\,db \leq \int_B\int_{\mathbf{H}^2} |f(x)|e^{(\operatorname{Im}\lambda+1)\langle x,b\rangle}\,dx\,db = \int_{\mathbf{H}^2} |f(x)|\phi_{i\operatorname{Im}\lambda}(x)\,dx \leq \|f\|_1.$$

Thus $\widetilde{f}(\lambda, b)$ exists for almost every b and so there is a nullset B'_λ for which $\widetilde{f}(\lambda, b)$ exists on $B_\lambda = B \setminus B'_\lambda$. Put $B' = B_i \cap B_{-i}$ and write $\lambda = \xi + i\eta$ ($|\eta| \leq 1$). For $b \in B'$ we have that the Fourier integral

$$\widetilde{f}(\xi + i\eta, b) = \int_{\mathbf{H}^2} f(x)e^{-i\xi\langle x,b\rangle}e^{(\eta+1)\langle x,b\rangle}\,dx,$$

can be majorized uniformly in $\lambda = \xi + i\eta$. In particular, set $\mathbf{H}^2_+ = \{x \in \mathbf{H}^2 \colon \langle x, b\rangle \geq 0\}$ and put $\mathbf{H}^2_- = \mathbf{H}^2 \setminus \mathbf{H}^2_+$. Then the following majorization holds:

$$|\widetilde{f}(\xi + i\eta, b)| \leq \int_{\mathbf{H}^2_+} |f(x)|e^{2\langle x,b\rangle}\,dx + \int_{\mathbf{H}^2_-} |f(x)|\,dx \leq \widetilde{|f|}(i, b) + \widetilde{|f|}(-i, b).$$

So for fixed $b \in B'$, $\widetilde{f}(\lambda, b)$ is uniformly bounded on T and is continuous in λ by the dominated convergence theorem. Hence if γ is a closed C^1 curve on the interior of T, then using Fubini's theorem

$$\int_\gamma \widetilde{f}(\lambda, b)\,d\lambda = \int_\gamma\int_{\mathbf{H}^2} f(x)e^{(-i\lambda+1)\langle x,b\rangle}\,dx\,d\lambda = \int_{\mathbf{H}^2} f(x)\left\{\int_\gamma e^{(-i\lambda+1)\langle x,b\rangle}\,d\lambda\right\}dx = 0.$$

So $\widetilde{f}(\lambda, b)$ is holomorphic on the interior of T by Morera's theorem (note $\widetilde{f}(\lambda, b)$ is continuous in λ by the dominated convergence theorem). Since $B \setminus B'$ has measure zero we obtain the result of (1).

To prove (2) we note that if $f \in L^1(\mathbf{H}^2)$, then its Radon transform

$$\widehat{f}(ka\cdot\xi_0) = \int_N f(ka\cdot n)\,dn$$

exists for almost all $k \in K$ and $a \in A$. Indeed, this fact is true because

$$\int_{\mathbf{H}^2} |f(x)|\,dx = \int_G |f(g\cdot o)|\,dg = \int_{K\times A\times N} |f(kan)|e^{2\log a}\,dk\,da\,dn = \int_K\int_A \widehat{|f|}(ka\cdot\xi_0)e^{2\log a}\,dk\,da < \infty.$$

Now we can shrink B' so that $\widehat{f}(ka \cdot \xi_0)$ exists for almost every $a \in A$ for each $k \cdot b_0 \in B'$ and so that B' still has complement measure zero. Now note that if we put $\lambda = \xi + i\eta$, then

$$\widetilde{|f|}(\lambda, k \cdot b_0) = \int_X |f(k \cdot x)| e^{(-i\lambda+1)\langle x, b_0 \rangle} \, dx = \int_{AN} |f(kan \cdot o)| e^{(-i\lambda+1)\log a} \, da \, dn$$
$$= \int_A \left\{ \widehat{|f|}(ka \cdot \xi_0) e^{(\eta+1)\log a} \right\} e^{-i\xi \log a} \, da \, .$$

So if $\xi = 0$, then we see that for each $|\eta| \leq 1$ that $\widehat{f}(ka_t \cdot \xi_0) e^{(\eta+1)t} \in L^1(\mathbf{R})$ and so has a Euclidean Fourier transform. Moreover,

$$\widetilde{f}(\xi + i\eta, k \cdot b_0) = \int_A \left\{ \widehat{f}(ka \cdot \xi_0) e^{(\eta+1)\log a} \right\} e^{-i\xi \log a} \, da = \int_{\mathbf{R}} \left\{ \widehat{f}(ka_t \cdot \xi_0) e^{(\eta+1)t} \right\} e^{-i\xi t} \, dt$$

which is the Fourier transform of $\widehat{f}(ka_t \cdot \xi_0) e^{(\eta+1)t}$ at ξ. So by the ordinary Riemann-Lebesgue lemma on $\mathbf{R}^n$, we see that $\widetilde{f}(\xi + i\eta, b) \to 0$ as $|\xi| \to \infty$. To show that this holds uniformly for $|\eta| \leq 1$ we consider the following function on $\mathbf{H}^2$ defined by $\Psi(an \cdot o) = e^{2\log a} + 1$. And for fixed $k \cdot b_0 \in B'$ we put $\Psi'(x) = \Psi(k^{-1} \cdot x)$ (and observe $f \cdot \Psi' \in L^1(\mathbf{H}^2)$ as we have previously shown). Let $f_N \in \mathcal{D}(\mathbf{H}^2)$ so that

$$\int_{\mathbf{H}^2} |f(x) - f_N(x)| \Psi'(x) \, dx \to 0$$

as $N \to \infty$. Put $g_N = f - f_N$ and we now have

$$|\widetilde{g_N}(\xi + i\eta, k \cdot b_0)| \leq \int_{AN} |f(kan \cdot o) - f_N(kan \cdot o)| e^{(\eta+1)\log a} \, da \, dn$$
$$\leq \int_N \int_0^\infty |f(ka_t n \cdot o) - f_N(ka_t n \cdot o)| e^{2t} \, dt \, dn + \int_N \int_{-\infty}^0 |f(ka_t n \cdot o) - f_N(ka_t n \cdot o)| \, dt \, dn$$
$$\leq \int_{\mathbf{H}^2} |f(k \cdot x) - f_N(k \cdot x)| \Psi(x) \, dx = \int_{\mathbf{H}^2} |f(x) - f_N(x)| \Psi'(x) \, dx \, .$$

So choosing N large enough we have $|\widetilde{g}(\xi + i\eta, k \cdot b_0)| < \epsilon/2$ for all $|\eta| \leq 1$. Now by the Paley-Wiener theorem $|\widetilde{f}_N(\xi + i\eta, k \cdot b_0)| \leq \epsilon/2$ for $|\xi| > M$ where M is some positive constant (and here $|\eta| \leq 1$). Since $\widetilde{f} = \widetilde{g}_N + \widetilde{f}_N$ we obtain $|\widetilde{f}(\xi + i\eta, b)| \leq \epsilon$ for all $|\xi| > M$ and $|\eta| \leq 1$. Thus we have (2). $\square$

We can now prove a type of inversion formula for functions in $L^1(\mathbf{H}^2)$ with absolutely integrable Fourier transforms.

Theorem 2.3.16. *The Fourier transform is injective on $L^1(\mathbf{H}^2)$. Furthermore, if $f \in L^1(\mathbf{H}^2)$ and $\widetilde{f} \in L^1(\mathbf{R} \times B)$, then*

$$f(x) = \int_{\mathbf{R} \times B} \widetilde{f}(\lambda, b) e^{(i\lambda+1)\langle x, b \rangle} \, d\mu(\lambda, b)$$

almost everywhere on $\mathbf{H}^2$.

Proof. Let ψ_r be a family of K bi-invariant mollifiers on $L^1(G)$ (see A.1.4) and let $f \in L^1(\mathbf{H}^2)$. Now we have by the Fubini theorem

$$\int_{\mathbf{R} \times B} \widetilde{f}(\lambda, b) \widetilde{\psi}_r(\lambda) e^{(i\lambda+1)\langle h \cdot o, b \rangle} \, d\mu(\lambda, b) = \int_{\mathbf{R}} f * \phi_\lambda(h \cdot o) \widetilde{\psi}_r(\lambda) \, d\mu(\lambda)$$
$$= \int_{\mathbf{R}} \left\{ \int_G f(g \cdot o) \phi_\lambda(g^{-1}h \cdot o) \, dg \right\} \widetilde{\psi}_r(\lambda) \, d\mu(\lambda) = \int_G \left\{ \int_{\mathbf{R}} \widetilde{\psi}_r(\lambda) \phi_\lambda(g^{-1}h \cdot o) \, d\mu(\lambda) \right\} f(g \cdot o) \, dg$$
$$= f * \psi_r(h \cdot o).$$

Thus if $\widetilde{f} \equiv 0$, then $f * \psi_r \equiv 0$ and as $r \to 0$ we have $f * \psi_r \to f$ in L^1 so that $f \equiv 0$. Thus the kernel of the Fourier transform is trivial and so the Fourier transform is injective on $L^1(\mathbf{H}^2)$. If we assume that $\widetilde{f} \in L^1(\mathbf{R} \times B, d\mu(\lambda)\, db)$, then the function $(\lambda, b) \mapsto \widetilde{f}(\lambda, b) e^{(i\lambda+1)\langle x,b\rangle}$ is also in $L^1(\mathbf{R} \times B, d\mu(\lambda)\, db)$. Moreover, we have

$$\lim_{r\to\infty} \widetilde{\psi}_r(\lambda) = \lim_{r\to\infty} \int_{\mathbf{H}^2} \psi_r(x)\phi_{-\lambda}(x)\, dx = 1$$

since

$$\left| \int_{\mathbf{H}^2} \psi_r(x)[\phi_{-\lambda}(x) - 1]\, dx \right| \leq \sup_{x \in \operatorname{supp} \psi_r} |\phi_{-\lambda}(x) - 1| \to 0$$

as $r \to \infty$. Thus, since $f * \psi_r \to f$ in L^1 we can choose a subsequence so that $f * \psi_{r_j} \to f$ almost everywhere and applying the dominated convergence theorem:

$$f(x) = \lim_{j\to\infty} f * \psi_{r_j}(x) = \lim_{j\to\infty} \int_{\mathbf{R}\times B} \widetilde{f}(\lambda, b)\widetilde{\psi}_{r_j}(\lambda) e^{(i\lambda+1)\langle x,b\rangle}\, d\mu(\lambda, b) = \int_{\mathbf{R}\times B} \widetilde{f}(\lambda, b) e^{(i\lambda+1)\langle x,b\rangle}\, d\mu(\lambda, b)$$

for almost every $x \in \mathbf{H}^2$. $\square$

2.4 Remarks

In summary, the Fourier transform that we have defined on the hyperbolic plane $\mathbf{H}^2$ shares many of the same useful properties of the Fourier transforms on $\mathbf{R}^n$ and on locally compact groups. The keynote results: the Plancherel theorem, Fourier inversion formula, and the Riemann-Lebesgue Lemma hold analogously for the Fourier transform on $\mathbf{H}^2$ just as they do for locally compact abelian groups. What is more is that this Fourier transform also interacts nicely with the smooth structure of $\mathbf{H}^2$ much like how the Euclidean Fourier transform behaves on $\mathbf{R}^n$ or the Fourier transform behaves on $\mathbf{T}$.

For instance if $\Delta_{\mathbf{H}^2}$ is Laplacian on $\mathbf{H}^2$, then by Green's identities

$$(\Delta_{\mathbf{H}^2} f)\widetilde{\ }(\lambda, b) = \int_{\mathbf{H}^2} \{\Delta_{\mathbf{H}^2} f(x)\}\, e^{(-i\lambda+1)\langle x,b\rangle}\, dx = \int_{\mathbf{H}^2} f(x) \left\{ \Delta_{\mathbf{H}^2} e^{(-i\lambda+1)\langle x,b\rangle} \right\} dx = -(\lambda^2+1)\widetilde{f}(\lambda, b),$$

which is an instance of Proposition 1.1.1 Property (2). This type of estimate led us to the Paley-Wiener theorem for $C_c^\infty(\mathbf{H}^2)$. What is lacking from our discussion in this chapter is a discussion of the hypothesized Schwartz space $\mathcal{S}(\mathbf{H}^2)$ for which a version of Theorem 1.1.8 holds. We shall discuss the Schwartz spaces in the next chapter.

There are unfortunately a few defects to this Fourier transform which we have encountered in our discussion. One of the issues that we are faced with is that since $L^1(\mathbf{H}^2)$ as a convolution is noncommutative, then the Fourier transform does not satisfy $(f * g)\widetilde{\ } = \widetilde{f}\widetilde{g}$ in general if g is not K-invariant. This seemingly minor obstacle will actually cause several large problems that we will encounter in Chapter 3. Another issue is that the Fourier transform does not behave nicely with respect to translation by G on $\mathbf{H}^2$. In particular if $\tau_g f(x) = f(g \cdot x)$ for $g \in G$, then it is not necessarily true that

$$(\tau_g f)\widetilde{\ }(\lambda, b) = e^{(-i\lambda+1)\langle g\cdot o,b\rangle}\widetilde{f}(\lambda, b)$$

on $\mathbf{H}^2$ in comparison to the result of Proposition 1.2.6. What is true is that

$$(\tau_g f)\widetilde{\ }(\lambda, b) = e^{(-i\lambda+\rho)\langle g^{-1}\cdot o,b\rangle}\widetilde{f}(\lambda, g\cdot b)$$

which complicates things slightly due to the term $\widetilde{f}(\lambda, g \cdot b)$ which depends also on g. However for the spherical transform we have that if f is K-invariant, then

$$\begin{aligned}(\tau_g f)\widetilde{\ }(\lambda) = \int_{\mathbf{H}^2} f(x)\phi_{-\lambda}(g^{-1}x)\, dx = \int_K \int_{\mathbf{H}^2} f(kx)\phi_{-\lambda}(g^{-1}kx)\, dx &= \int_{\mathbf{H}^2} f(x)\phi_{-\lambda}(g^{-1}\cdot o)\phi_{-\lambda}(x)\, dx \\ &= \phi_{-\lambda}(g^{-1}\cdot o)\widetilde{f}(\lambda) = \phi_\lambda(g\cdot o)\widetilde{f}.\end{aligned}$$

where we have applied (2.2.7). Again, while this will seem minor, this type of obstacle provides us technical difficulties which we shall encounter as we attempt to study objects called pseudo-differential operators—and in many situations we will find that the spherical transform is much more well-behaved and more convenient to work with.

Chapter 3

Fourier transforms for homogeneous spaces of connected semisimple Lie groups

3.1 Preliminaries

3.1.1 Cartan and Iwasawa decompositions of semisimple Lie groups

Suppose that $\mathfrak{g}$ is a finite-dimensional real Lie algebra. Recall that the Killing form $B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbf{R}$ is the map defined by $B(X, Y) = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$ where for each $X \in \mathfrak{g}$ the map $\mathrm{ad}(X)\colon \mathfrak{g} \to \mathfrak{g}$ is defined by $\mathrm{ad}(X)(Y) = [X, Y]$ where $[\cdot, \cdot]$ is the Lie bracket on $\mathfrak{g}$. We say that $\mathfrak{g}$ is semisimple if the Killing form B is non-degenerate, which is to say that the map from $\mathfrak{g}$ to $\mathfrak{g}^*$ given by $Y \mapsto (X \mapsto B(X, Y))$ is an isomorphism. From now on, we shall assume that $\mathfrak{g}$ is semisimple.

If $\theta\colon \mathfrak{g} \to \mathfrak{g}$ is an involutive Lie algebra automorphism of $\mathfrak{g}$ (i.e. θ is an automorphism, $\theta[X, Y] = [\theta X, \theta Y]$ for all $X, Y \in \mathfrak{g}$, and $\theta^2 = \mathrm{id}$), then we say that θ is a *Cartan involution* of $\mathfrak{g}$ if the bilinear form $B_\theta(X, Y) = -B(X, \theta Y)$ is positive definite. If θ is a Cartan involution of $\mathfrak{g}$, then being an involution over $\mathbf{R}$, it is diagonalizable and so we can obtain an eigenspace decomposition of $\mathfrak{g}$ according to θ given by

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

where $\mathfrak{k}$ is the eigenspace corresponding to the eigenvalue $+1$ of θ, and $\mathfrak{p}$ is the eigenspace corresponding to the eigenvalue -1 of θ. This type of decomposition is called the *Cartan decomposition* of $\mathfrak{g}$ and it is a fact that any semisimple Lie algebra has a Cartan involution and thus has a Cartan decomposition (see [16, Prop. 6.14]). In this decomposition $\mathfrak{k}$ is a Lie subalgebra (since $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$) whereas $\mathfrak{p}$ is an ordinary vector space. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace of $\mathfrak{p}$. All maximal abelian subspaces in $\mathfrak{p}$ have the same dimension [16, Thm. 6.51]. For each $\alpha \in \mathfrak{a}^*$ we put

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}\colon\ \mathrm{ad}(H)(X) = \alpha(H)X,\ \text{for all } H \in \mathfrak{a}\}.$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha$ is nontrivial, then we call α a *root* (or restricted root) of the pair $(\mathfrak{g}, \mathfrak{a})$ and $\mathfrak{g}_\alpha$ the *root space* of α. Let Σ denote the set of roots corresponding to the pair $(\mathfrak{g}, \mathfrak{a})$. It is known that for each $X \in \mathfrak{p}$, that $\mathrm{ad}(X)\colon \mathfrak{g} \to \mathfrak{g}$ is diagonalizable (see [12, Ch. VI, Lemma 1.2]). In fact by the Jacobi identity the set of maps $\{\mathrm{ad}\, H\colon H \in \mathfrak{a}\}$ form a commuting set of linear transformations of $\mathfrak{g}$ and so this family can be simultaneously diagonalized. This then yields a joint eigenspace decomposition of $\mathfrak{g}$ by

$$\mathfrak{g} = \mathfrak{g}_0 + \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_0 = \mathfrak{a} + \mathfrak{m}$ and where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. In particular there are only finitely many distinct roots for $(\mathfrak{g}, \mathfrak{a})$.

Turning our attention to the subspace $\mathfrak{a}$ we shall call a point $H \in \mathfrak{a}$ *regular* if $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$, otherwise it is called a *singular* point. We put $H_\alpha = \{H \in \mathfrak{a} \colon \alpha(H) = 0\}$ and call H_α the hyperplane at α. The subset of regular elements $\mathfrak{a}' \subset \mathfrak{a}$ is thus the complement $\mathfrak{a}' = \mathfrak{a} \setminus \bigcup_{\alpha \in \Sigma} H_\alpha$. Imposing upon $\mathfrak{a}$ the standard Euclidean topology, the hyperplanes H_α divide $\mathfrak{a}$ into finitely many connected components and we shall thus call a *Weyl chamber* in $\mathfrak{a}$ to be a connected component of $\mathfrak{a}'$.

Fix a Weyl chamber $\mathfrak{a}^+$ of $\mathfrak{a}$. We shall say a root α is *positive* if $\alpha|_{\mathfrak{a}^+}$ is positive valued. And we say a positive root is *simple* if it is not the sum of two positive roots. Then we can describe the Weyl chamber $\mathfrak{a}^+$ by $\mathfrak{a}^+ = \{H \in \mathfrak{a} \colon \alpha(H) > 0,\ \alpha \in \Sigma'\}$ where $\Sigma' \subset \Sigma$ is the set of simple roots. Let Σ^+ denote the set of positive roots (corresponding to $\mathfrak{a}^+$) and put $\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$.

Then $\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{g}$ by the following simple observation that if $\alpha, \beta \in \Sigma^+$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$ which follows by using the Jacobi identity of the Lie bracket. Since the sum of two positive roots is also positive, it follows that $\mathfrak{n}$ is a Lie subalgebra. In fact $\mathfrak{n}$ is nilpotent (its lower central series terminates) due to the fact that the set of positive roots is finite. We thus obtain the so-called *Iwasawa decomposition* of $\mathfrak{g}$ via the following theorem.

Theorem 3.1.1. *If $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$ are as given above, then we have the direct sum decomposition*

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n},$$

which is called the Iwasawa decomposition of $\mathfrak{g}$.

Now we turn our attention to Lie groups; recall that we say a Lie group G is semisimple if its Lie algebra $\mathfrak{g}$ is semisimple. Suppose that G is a connected semisimple Lie group. Given a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and a corresponding Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ of its Lie algebra, then there exists unique connected Lie subgroups K, A, and N corresponding to the subalgebras $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$. Let $\overline{\mathfrak{a}^+}$ denote the closure of $\mathfrak{a}^+$ in $\mathfrak{a}$ and put $\overline{A^+} = \exp \overline{\mathfrak{a}^+}$, then we have the following group theoretic decompositions of G.

Theorem 3.1.2. *If G and K, A, and N are as above, then we have the Cartan decomposition*

$$G = K\overline{A^+}K. \tag{3.1.1}$$

Here, each $g \in G$ can be uniquely decomposed as $g = k_1(\exp H)k_2$ for unique $k_1, k_2 \in K$ and $H \in \overline{\mathfrak{a}^+}$. We also have the Iwasawa decomposition

$$G = KAN. \tag{3.1.2}$$

The connected subgroups A, and N are abelian and nilpotent respectively. Moreover the map $(k, a, n) \mapsto kan$ is a diffeomorphism from $K \times A \times N$ onto G. Furthermore, K contains the center of G and is compact if and only if the center of G is finite. Finally, the groups K, A, and N are all unimodular and G is unimodular.

The proofs of these theorems can be found in [12, Ch. VI and Ch. IX]. The *real rank* of G is then defined to be the dimension of $\mathfrak{a}$. Now given a Lie subgroup $K \subset G$ of a semisimple Lie group G one would be interested to know whether or not the corresponding decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition. Since for the rest of this chapter we shall only be interested in semisimple Lie groups with finite center we shall answer this question for these groups only.

Theorem 3.1.3. *Let G be a connected semisimple Lie group with finite center and $K \subset G$ be a maximal compact subgroup. Then, there exists a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ such that the Lie algebra of K is $\mathfrak{k}$.*

This theorem is a consequence of combining the results of [16, Thm. 6.31] and Theorems 1.1 and 2.2 of Chapter VI [12]. Finishing up this section, we note that if G is a connected semisimple Lie group with Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ so that $G = KAN$, then by Theorem 3.1.2 for each $g \in G$ we can write g uniquely as $g = k_1a_1n_1 = n_2a_2k_2$ for unique $k_1, k_2 \in K$, $a_1, a_2 \in A$, and $n_1, n_2 \in N$. Using this decomposition we have the following definition.

Definition 3.1.1. Let G be a connected semisimple Lie group with corresponding Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ so that $G = KAN = NAK$. We define the functions $A\colon G \to \mathfrak{a}$ and $H\colon G \to \mathfrak{a}$ by

$$g = k_1 \exp H(g) n_1 = n_2 \exp A(g) k_2, \tag{3.1.3}$$

where $k_1, k_2 \in K$ and $n_1, n_2 \in N$. Since the Iwasawa decomposition is unique, the functions $g \mapsto A(g)$ and $g \mapsto H(g)$ are well defined and are smooth from G onto $\mathfrak{a}$. The definitions of A and H imply that $H(g) = -A(g^{-1})$.

Remark 7. We will often be explicit enough so that there is no confusion between the *function* $A\colon G \to \mathfrak{a}$ and the *subgroup* $A = \exp \mathfrak{a}$. The choice of notation seems to be more or less standard and it is the notation used by Helgason [11–13].

Since $\mathfrak{a}$ is an abelian subalgebra, the connected abelian subgroup $A = \exp \mathfrak{a}$ is diffeomorphic to $\mathfrak{a}$ via the exponential map $\exp\colon \mathfrak{a} \to A$. We let $\log\colon A \to \mathfrak{a}$ be the inverse of $\exp\colon \mathfrak{a} \to A$. In terms of the functions A and H above we have $\log a = A(a) = H(a)$ for all $a \in A$.

Example 3.1.4 (The group $\mathbf{SU}(1,1)$)**.** In Chapter 2, we considered the group $\mathbf{SU}(1,1)$ which we will shall take for granted as being semisimple. If we put

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

then we can consider the function $\Phi\colon \mathrm{GL}(2,\mathbf{C}) \to M_2(\mathbf{C})$ where $\Phi(A) = A^*LA$ and A^* is the conjugate transpose. If we consider the group $G = \Phi^{-1}(L)$, the Lie subgroup $\mathbf{SU}(1,1) \subset G$ is the set of matrices $\mathbf{SU}(1,1) = \{A \in G\colon \det A = 1\}$. The Lie algebra of $\mathbf{SU}(1,1)$, denoted $\mathfrak{su}(1,1)$, is then given by $\mathfrak{su}(1,1) = \{X \in \mathfrak{g}\colon \exp tX \in \mathbf{SU}(1,1),\ \text{for all } t \in \mathbf{R}\}$. For $\exp tX \in \mathbf{SU}(1,1)$ for all $t \in \mathbf{R}$, we require that $\det \exp tX = 1$ which is equivalent to $\exp t \operatorname{Tr} X = 1$. This only happens when $\operatorname{Tr} X = 0$. Furthermore, the Lie algebra of G is calculated as $\mathfrak{g} = \ker d_I\Phi = \{X \in M_2(\mathbf{C})\colon X^*L + LX = 0\}$. Thus, $\mathfrak{su}(1,1) = \{X \in M_2(\mathbf{C})\colon X^*L + LX = 0, \operatorname{Tr} X = 0\}$. This has a more concrete description as the collection of matrices

$$\mathfrak{su}(1,1) = \left\{ \begin{pmatrix} ia & b \\ \bar{b} & -ia \end{pmatrix} : a \in \mathbf{R}, b \in \mathbf{C} \right\}.$$

Correspondingly, the group $\mathbf{SO}(2)$ is a maximal compact subgroup of $\mathbf{SU}(1,1)$ and has the Lie algebra

$$\mathfrak{so}(2) = \mathbf{R} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

We thus have the Cartan decomposition $\mathfrak{su}(1,1) = \mathfrak{so}(2) + \mathfrak{p}$. Moreover, the vector space $\mathfrak{p}$ admits the decomposition $\mathfrak{p} = \mathfrak{a} + \mathfrak{n}$ where

$$\mathfrak{a} = \mathbf{R} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathfrak{n} = \mathbf{R} \begin{pmatrix} i & -i \\ i & -i \end{pmatrix}.$$

Since $\mathfrak{a} \subset \mathfrak{p}$ is a maximal abelian subspace, this determines an Iwasawa decomposition of $\mathfrak{g}$. Since $\mathfrak{n}$ is a one-dimensional vector space there is only one positive root $\alpha \in \mathfrak{a}^*$ corresponding to the Weyl chamber $\mathfrak{a}^+ = \{H \in \mathfrak{a}\colon \alpha(H) > 0\}$ and this root can be verified to be determined by $\alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2$. If $X = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} \in \mathfrak{a}$ for some $x \in \mathbf{R}$, then we have $\exp X = \begin{pmatrix} \sum_{n=0}^\infty \frac{x^{2n}}{(2n)!} & \sum_{n=0}^\infty \frac{x^{2n+1}}{(2n+1)!} \\ \sum_{n=0}^\infty \frac{x^{2n+1}}{(2n+1)!} & \sum_{n=0}^\infty \frac{x^{2n}}{(2n)!} \end{pmatrix} = \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix}$. Thus, $\exp \mathfrak{a} = A$ as in 2.1.15. Finally if $Y = \begin{pmatrix} iy & -iy \\ iy & -iy \end{pmatrix} \in \mathfrak{n}$ for $y \in \mathbf{R}$, then we easily compute $\exp Y = \begin{pmatrix} 1+iy & -iy \\ iy & 1-iy \end{pmatrix}$. Thus, $\exp \mathfrak{n} = N$ as well just as in 2.1.16. Theorem 3.1.2 then gives us that $\mathbf{SU}(1,1) = KAN$, the Iwasawa decomposition described in Chapter 2. Using the Weyl chamber $\mathfrak{a}^+$ the set $\overline{A^+}$ is then $\overline{A^+} = \left\{ \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} : x \geq 0 \right\}$. So we also have the Cartan decomposition $\mathbf{SU}(1,1) = K\overline{A^+}K$ as well.

We conclude this subsection by introducing a norm on the dual space $\mathfrak{a}^*$. The Killing form restricted to $\mathfrak{a} \times \mathfrak{a}$ is known to be positive definite and so defines a non-degenerate bilinear form on $\mathfrak{a}$. If we complexify the vector spaces $\mathfrak{a}$ and $\mathfrak{a}^*$, obtaining $\mathfrak{a}_c$ and $\mathfrak{a}_c^*$, we also obtain an extension of B to $\mathfrak{a}_c$ to a complex non-degenerate bilinear form which we also denote by B. Thus it follows that to each $\lambda \in \mathfrak{a}_c^*$ there exists a unique $H_\lambda \in \mathfrak{a}_c$ such that $\lambda(H) = B(H_\lambda, H)$ for all $H \in \mathfrak{a}_c$.

Definition 3.1.2. For $\lambda, \mu \in \mathfrak{a}_c^*$ define their inner product by $\langle \lambda, \mu \rangle = B(H_\lambda, H_{\overline{\mu}})$ where H_λ and H_μ are defined above. Also put $|\lambda| = \langle \lambda, \lambda \rangle^{1/2}$. With this complex bilinear form $\langle \cdot, \cdot \rangle$, $\mathfrak{a}_c^*$ becomes a complex Hilbert space.

3.1.2 The homogeneous space $X = G/K$ and the horocycle space $\Xi = G/MN$

Let G be a connected semisimple Lie group with finite center and let $K \subset G$ be a maximal compact subgroup. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n} = \mathfrak{k} + \mathfrak{p}$ be Iwasawa decompositions and Cartan decompositions of $\mathfrak{g}$ respectively so that we can write $G = KAN$. Finally let $M = \{k \in K : ka = ak, \text{ for all } a \in A\}$ be the centralizer of A in K.

We can form the homogeneous space $X = G/K$ which we endow with the quotient topology via the quotient map $q \colon G \to G/K$, $g \mapsto gK$. The homogeneous space X carries a unique smooth structure such that q is a smooth submersion. We let $o = eK \in X$ and call o the *origin* of X. The quotient map q has the property that $d_e q \colon \mathfrak{g} \mapsto T_o X$ maps $\mathfrak{g}$ onto $T_o X$ and that $\mathfrak{k} \subset \ker d_e q$ and $d_e q$ maps $\mathfrak{p}$ bijectively to $T_o X$. We can further give X a Riemannian structure invariant under G (see [12, Ch. II, Section 4] or [18, Thm. 21.17] for further details and constructions). This G-invariant Riemannian metric then gives rise to a G-invariant Radon measure, dx, induced from the Riemannian volume form (also see A.1.2).

Henceforth, we assume that X is given the quotient topology and the unique smooth structure discussed above; in addition to a G-invariant Riemannian structure and denote its corresponding G-invariant Riemannian measure by dx.

Definition 3.1.3. Let $X = G/K$ be the homogeneous space discussed above. A *horocycle* ξ in X is a set of points in X of the form $\xi = gNg^{-1} \cdot o$, where $g \in G$. We denote the space of horocycles in X by Ξ.

Here we summarize by a theorem a few useful facts about horocycles which we will need.

Theorem 3.1.5 ([13, Ch. II, Prop. 1.4] and [13, Ch. II, Thm. 1.1])**.** *The map $(kM, a) \mapsto kaMN \cdot o$ is a bijection from $K/M \times A$ onto Ξ, hence any horocycle $\xi \in \Xi$ is of the form $\xi = kaMN \cdot o$ for some unique $kM \in K/M$ and $a \in A$. Moreover, the group G acts transitively on the space of horocycles Ξ via the map $g \mapsto g \cdot \xi$ where $\xi \in \Xi$, and the subgroup fixing the horocycle $MN \cdot o$ is MN.*

This result then implies that Ξ can then be identified with the homogeneous space G/MN; so there is no ambiguity in writing $\Xi = G/MN$. We shall thus call a point $kM \in K/M$ the *normal* of the horocycle $\xi = kaMN \cdot o$.

Definition 3.1.4. The *composite distance* of a horocycle $\xi = kaMN \cdot o \in \Xi$ to the origin is defined to be the vector $\log a \in \mathfrak{a}$.

Proposition 3.1.6. *For each $x \in X$ and $kM \in K/M$, there exists a unique horocycle containing x with normal kM. We denote this horocycle by $\xi(x, kM)$.*

Proof. Write $x = gK$ for some $g \in G$. Using the Iwasawa decomposition $G = NAK$ write

$$k^{-1}g = n(\exp A(k^{-1}g))k,$$

then $k^{-1}gK \in MN(\exp A(k^{-1}g)) \cdot o = (\exp A(k^{-1}g))MN \cdot o$ (the group A normalizes N and remember M centralizes A). So $k(\exp A(k^{-1}g))MN \cdot o$ is the unique horocycle containing $x = gK$ with normal kM. $\square$

If $x \in X$ and $kM \in K/M$, then we denote the composite distance of $\xi(x, kM)$ to o by $A(x, kM)$. By the proof of Proposition 3.1.6 we see that $A(x, kM) = A(k^{-1}g)$ where $x = gK$. Let $\kappa \colon G \to K$ and $u \colon G \to K$ be the functions defined by $g = \kappa(g) \exp H(g) n_1 = n_2 \exp A(g) u(g)$ where $n_1, n_2 \in N$. We then put $k_g = \kappa(gk)$ and $g(kM) = k_g M$—which defines a group action of G on K/M.

Proposition 3.1.7. *For $g \in G$, $x \in X$, and $kM \in K/M$ we have*

$$A(g \cdot x, g(kM)) = A(x, kM) + A(g \cdot o, g(kM)). \tag{3.1.4}$$

Proof. We shall prove $A(g \cdot x, kM) = A(x, g^{-1}(kM)) + A(g \cdot o, kM)$. Writing $x = hK$ put

$$k^{-1}gh = n_1 \exp A(k^{-1}g)u(k^{-1}g)h = n_2 \exp A(k^{-1}gh)k_2,$$

where of course $n_1, n_2 \in N$ and $k_1, k_2 \in K$. As A normalizes N we have for some $n_1', n_2' \in N$ that

$$n_1' u(k^{-1}g)h = n_2' \exp\big(A(k^{-1}gh) - A(k^{-1}g)\big)k_2.$$

Thus $A(u(k^{-1}g)h) = A(k^{-1}gh) - A(k^{-1}g)$. Since $u(k^{-1}g) = \kappa(g^{-1}k)^{-1}$ we obtain the result. $\square$

Example 3.1.8. Recall that the group $\mathbf{SU}(1,1)$ acts on the hyperbolic plane $\mathbf{H}^2$ by means of the Möbius transformations

$$g \cdot x \mapsto \frac{ax+b}{\overline{b}x+\overline{a}}, \quad g = \begin{pmatrix} a & b \\ \overline{b} & \overline{a} \end{pmatrix} \in \mathbf{SU}(1,1), \quad x \in \mathbf{H}^2. \tag{3.1.5}$$

As elaborated on in Chapter 2, the hyperbolic plane is the homogeneous space $\mathbf{H}^2 = \mathbf{SU}(1,1)/\mathbf{SO}(2)$. Let $K = \mathbf{SO}(2)$, A, and N be as defined as in Example 3.1.4 (or Chapter 2). The center of $\mathbf{SU}(1,1)$ is of course readily computed to be $Z = \{I, -I\}$ and the centralizer of A in K is $M = Z$. According to the theory we have developed, the horocycles in $\mathbf{H}^2$ are given by $kaMN \cdot o$ for $k \in K$ and $a \in A$. We also have the identification

$$K/M = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, \pi] \right\}. \tag{3.1.6}$$

Now we remark that outright $K \cong B$ where $B = \{|x|^2 = 1 \colon x \in \mathbf{C}\}$ as *topological groups.* However in Chapter 2, K acts on the closed disk $\overline{D} = \{x \in \mathbf{C} \colon |x| \leq 1\}$ by (using the action prescribed by Möbius transformations)

$$k \cdot x \mapsto e^{2i\theta}x, \quad k = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \ \theta \in [0, 2\pi]. \tag{3.1.7}$$

So in particular the action defined in (3.1.7) when restricted to B is a transitive group action and the isotropy group of any point with respect to this action is M. We thus have, with respect to the action of (3.1.7), the *homogeneous space identification* $K/M \cong B$. Finally if $b = kM$, then we have that the "inner product" in Definition 2.1.6 is precisely $\langle x, b \rangle = A(x, kM)$.

Remark 8. It might be confusing to the reader because one would expect that the group $K = \mathbf{SO}(2)$ to act on B by means of the maps

$$k \cdot x \mapsto e^{i\theta}x, \quad k = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \ \theta \in [0, 2\pi]$$

where $x \in B$, whence the isotropy group of any point with respect to this action is trivial. But in actuality the action of K on B employed in Chapter 2 is defined by means of the Möbius transformations which is consistent with the action defined in (3.1.5) in which case we obtain the group action of (3.1.7).

We now fix a bit of notation. Since G acts on functions on X by the map $\tau(g)f(x) = f(g \cdot x)$, then if F is some function space on X and $H \subset G$ is a set, we let F_H denote the space of elements $f \in F$ which satisfy $\tau(h)f = f$ for all $h \in H$. We call the elements of F_H the H-invariant F functions. Note that $F_H = F_{\langle H \rangle}$ where $\langle H \rangle$ is the subgroup generated by H. For instance, we let $L^1_K(X)$, $\mathcal{E}_K(X)$, and $\mathcal{D}_K(X)$ denote the K-invariant integrable functions, smooth functions, and compactly supported smooth functions respectively.

Also, if $\phi, \psi \in L^1(X)$, then the convolution $(\phi * \psi)(x)$ is defined as $(\phi^q * \psi^q)(g \cdot o)$ (and likewise for ψ^q) where $\phi^q = \phi \circ q$ and $x = gK$. Thus to convolve on X we lift to functions on $L^1(G)$ and convolve in G.

3.1.3 Some integral formulas

Here we introduce several integration formulas, similar to the ones derived in Section 2.1, for a connected semisimple Lie group. Let G be a connected semisimple Lie group, and $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ an Iwasawa decomposition of $\mathfrak{g}$. For a predetermined Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, let Σ^+ be the corresponding system of positive roots and set $\rho = \frac{1}{2}\sum_{\alpha \in \Sigma^+}(\dim \mathfrak{g}_\alpha)\alpha$. As usual, let K, A, and N denote the Lie subgroups corresponding to the Lie subalgebras $\mathfrak{k}$, $\mathfrak{a}$, and $\mathfrak{n}$ so that we have the Iwasawa decomposition $G = KAN$ and the Cartan decomposition $G = K\overline{A^+}K$. If dg is a fixed Haar measure on G, we would like to decompose dg into the product of Haar measures dk, da, and dn on K, A, and N respectively.

For $f \in L^1(G)$ and given Haar measures dk, da, and dn on K, A, and N respectively we can normalize the Haar measure dg on G so that we have the following integration formulas:

$$\int_G f(g)\,dg = \int_{K\times A\times N} f(kan)e^{2\rho(\log a)}\,dk\,da\,dn\,. \tag{3.1.8}$$

Hence, we have the decomposition $dg = e^{2\rho \log a}\,dk\,da\,dn$. For suitable f defined on AN, say $f \in C_c(AN)$, we have

$$\int_N f(an)\,dn = e^{-2\rho(\log a)}\int_N f(na)\,dn\,. \tag{3.1.9}$$

Thus we also have that for $f \in L^1(G)$ that

$$\int_G f(g)\,dg = \int_{K\times A\times N} f(kna)\,dk\,da\,dn = \int_{K\times A\times N} f(ank)\,dk\,da\,dn\,. \tag{3.1.10}$$

Now considering $X = G/K$ the homogeneous space of the previous section where dx is the Riemannian measure on x, then Theorem A.1.5 implies that the Haar measure dg can be normalized so that for any $f \in L^1(X)$ we have $\int_X f(x)\,dx = \int_G f(gK)\,dg$.

3.1.4 Differential operators

Let V be a finite-dimensional vector space over a field $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$. The *symmetric algebra* of V, denoted by $S(V)$, is the unital associative commutative $\mathbf{K}$-algebra given by the quotient $T(V)/I$ where $T(V)$ is the tensor algebra over V and I is the two-sided ideal generated by elements of the form $v \otimes u - u \otimes v$. If $\lambda\colon V \to A$ is a linear map between V and a unital associative commutative $\mathbf{C}$-algebra A and $\iota\colon V \to S(V)$ is the canonical inclusion map, then there exists a unique unital algebra homomorphism $\Lambda\colon S(V) \to A$ so that the diagram

$$\begin{tikzcd} S(V) \arrow[dr, "\Lambda"] & \\ V \arrow[u, "\iota"] \arrow[r, "\lambda"] & A \end{tikzcd} \tag{3.1.11}$$

commutes.

When $\mathbf{K} = \mathbf{R}$, then clearly V carries the structure of a smooth manifold diffeomorphic to $\mathbf{R}^{\dim V}$. An element $v \in V$ determines an element $v \in S(V)$ which then determines a differential operator $\partial(v)$ acting on smooth functions. In this case if $f\colon V \to \mathbf{C}$ is a smooth function, then the operator $\partial(v)$ is defined by

$$\partial(v)f(x) = \frac{d}{dt}(f(x+tv))|_{t=0},\ t \in \mathbf{R}. \tag{3.1.12}$$

We then extend this definition in the obvious way so that each $u \in S(V)$ defines a differential operator which we denote by $\partial(u)$. For example if $u = v_1 \cdots v_n \in S(V)$ for $v_i \in V$ then $\partial(u) = \partial(v_1)\cdots\partial(v_n)$; and $c \in S(V)$ is a constant, then $\partial(c)f = cf$ if $f \in C^\infty(V)$. The assignment $u \mapsto \partial(u)$ is then a well-defined algebra homomorphism by the chain and product rules from calculus. With this understanding, if $\mathbf{D}_r(V)$ is the algebra of constant *real* coefficient differential operators on V, then it is canonically algebraically isomorphic to $S(V)$. We can consider the complexification of the symmetric algebra $S(V)$, defined by $S_c(V) = \mathbf{C} \otimes S(V)$. In this case, if $f\colon V \to \mathbf{C}$ is smooth, then for $z \in \mathbf{C}$ and $u \in S(V)$, we define the element $\partial(z \otimes u) := \partial(z)\partial(u)$ where $\partial(z)f = zf$. This sets up an identification between $S_c(V)$ and $\mathbf{D}(V)$, the algebra of all constant complex coefficient differential operators on V.

Consequently, if λ is as in (3.1.11), then there exists a unique complex linear extension $\Lambda'\colon S_c(V) \to \mathbf{C}$ of Λ so that $\lambda = \Lambda' \circ \iota$. Here Λ' is of course defined by $\Lambda'(z \otimes u) = \mathrm{Re}(z)\Lambda(u) + i\,\mathrm{Im}(z)\Lambda(u)$, $z \in \mathbf{C}$, $u \in S(V)$. We shall write Λ for Λ' without any confusion.

Definition 3.1.5. Let $D \in \mathbf{D}(V)$ be a differential operator on V and regard $D \in S_c(V)$. For λ a complex-valued linear map define $D \mapsto D(\lambda)$ by $D(\lambda) = \Lambda(D)$ where $\Lambda\colon S_c(V) \to \mathbf{C}$ is the unique algebra homomorphism described above.

Remark 9. If V is a complex finite-dimensional vector space, then the elements $u \in S(V)$ define holomorphic differential operators acting on holomorphic functions; where of course we define the holomorphic derivative

$$\partial(v)f(x) = \frac{d}{dz}(f(x + zv))|_{z=0}$$

where $z \in \mathbf{C}$, $v \in V$, and f a holomorphic function—and then extend this definition to all of $S(V)$.

Example 3.1.9. The symmetric algebra $S(\mathbf{R}^n)$ is the space of polynomials with real coefficients in n real variables. If λ is a linear complex-valued function on $\mathbf{R}^n$ and $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ is a differential operator, then

$$\partial^\alpha e^{\lambda(x)} = \partial^\alpha(\lambda) e^{\lambda(x)}.$$

Turning our attention to Lie groups, let G be a Lie group and let $\mathfrak{g}$ denote its Lie algebra. For each $g \in G$ there corresponds left and right translation operators $l(g)$ and $r(g)$ which act on smooth functions. These operators act on smooth functions f by the formulas $l(g)f(x) = f(g^{-1}x)$ and $r(g)f(x) = f(xg)$. A differential operator $D\colon C^\infty(G) \to C^\infty(G)$ is defined to be a linear map which does not enlarge the support of a smooth function, i.e. $\operatorname{supp} Df \subset \operatorname{supp} f$. We say that a differential operator D is left (resp. right) invariant if $l(g)D = Dl(g)$ (resp. $r(g)D = Dr(g)$) for all $g \in G$ and say that D is invariant or G-invariant if it is both left and right invariant.

Definition 3.1.6. The *universal enveloping algebra* of a Lie group G with Lie algebra $\mathfrak{g}$ is the object $U(\mathfrak{g}) = T(\mathfrak{g})/J$ where J is the two-sided ideal generated by the elements $X \otimes Y - Y \otimes X - [X, Y]$ for $X, Y \in \mathfrak{g}$.

The elements of the universal enveloping algebra $U(\mathfrak{g})$ correspond to both the algebras of left and right invariant differential operators with real coefficients via the identifications

$$\partial_l(X)f(g) = \frac{d}{dt}(f(g \exp tX))|_{t=0}, \tag{3.1.13}$$

$$\partial_r(X)f(g) = \frac{d}{dt}(f((\exp tX)g))|_{t=0} \tag{3.1.14}$$

where $X \in \mathfrak{g}$. Thus $\partial_l(X)$ and $\partial_r(X)$ define the corresponding left and right invariant differential operators corresponding to X. We shall typically regard $U(\mathfrak{g})$ as corresponding to the algebra of left-invariant differential operators with real coefficients. The center $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ is then identified with the algebra of the G-invariant differential operators with real coefficients. We let $\mathbf{D}(G)$ denote the algebra of left-invariant differential operators on G (with complex coefficients) which we then identify with $U(\mathfrak{g}_c)$ (the universal enveloping algebra of the complexification of $\mathfrak{g}$). In the particular case that $\mathfrak{g}$ is abelian (which happens for instance when its underlying Lie group is abelian), then $U(\mathfrak{g}) = S(\mathfrak{g})$. Nevertheless, we can identify the $S(\mathfrak{g})$ with $U(\mathfrak{g})$ by the following theorem below.

Theorem 3.1.10 (cf. [11, Ch. II, Thm. 4.3] or [16, Prop. 3.23])**.** *Let G be any Lie group with Lie algebra $\mathfrak{g}$. The map $\sigma\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ where σ is defined by*

$$\sigma(X_1 \cdots X_n) = \frac{1}{n!} \sum_{s \in S_n} X_{s(1)} \cdots X_{s(n)},$$

is an isomorphism of vector spaces. Here S_n is the symmetric group on n letters.

The map $\sigma\colon S(\mathfrak{g}) \to U(\mathfrak{g})$ is called the *symmetrization map*. Finally, if G is a Lie group and $H \subset G$ is a closed subgroup, then regarding a G/H as a smooth manifold we say a differential operator D on G/H is G-invariant if for all $g \in G$ we have $\tau(g)D = D\tau(g)$ where $\tau(g)f(x) = f(g \cdot x)$, $f \in C^\infty(G/H)$. We denote the algebra of all G-invariant differential operators by $\mathbf{D}(G/H)$.

3.2 The Fourier transform on $X = G/K$

Let G be a connected noncompact semisimple Lie group with finite center and let $K \subset G$ be a maximal compact subgroup. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$, $G = KAN$, be an Iwasawa decomposition of G according to a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. Let Σ^+ be the set of positive roots and put $\rho = \frac{1}{2}\sum_{\alpha\in\Sigma^+}(\dim \mathfrak{g}_\alpha)\alpha$.

Definition 3.2.1. Let M and M' denote respectively the centralizer and normalizer of A in K. Then the coset space $W = M'/M$ is a group and we call W the *Weyl group* of G.

The Weyl group is known to be finite. The Weyl group acts on $\mathfrak{a}$ via the linear transformations $(kM)H = \mathrm{Ad}(k)H$ ($kM \in W$, $H \in \mathfrak{a}$, with $k \in M'$) and acts on the linear dual space $\mathfrak{a}^*$ via the linear transformations $(kM)\lambda = \lambda \circ \mathrm{Ad}(k^{-1})$ (where $\lambda \in \mathfrak{a}^*$). We shall thus regard the Weyl group as acting naturally on $\mathfrak{a}$ and on $\mathfrak{a}^*$, and consequently W acts on the complexifications $\mathfrak{a}_c$ and $\mathfrak{a}_c^*$ in the obvious way.

Let $\mathbf{D}(A)$ denote the algebra of left-invariant differential operators on A which we naturally identify with $U(\mathfrak{a}_c)$. Then using symmetrization map σ of Theorem 3.1.10 we have that $\mathbf{D}(A)$ is identified with the (complexified) symmetric algebra $S_c(\mathfrak{a})$. Since W acts on $\mathfrak{a}_c$, we can extend this to an action of W on $S_c(\mathfrak{a})$ defined by $s(v_1 \cdots v_n) = s(v_1)\cdots s(v_n)$ where $v_i \in \mathfrak{a}_c$, $s \in W$. The subalgebra $I(\mathfrak{a}) \subset S_c(\mathfrak{a})$ consists of the W-invariant polynomial members of $S_c(\mathfrak{a})$. Under the symmetrization map the subalgebra $I(\mathfrak{a})$ corresponds to a subalgebra of $U(\mathfrak{g}_c)$ which we then identify with a subalgebra $\mathbf{D}_W(A) \subset \mathbf{D}(A)$. Let $X = G/K$ be the corresponding homogeneous space and let $\mathbf{D}(X)$ be the algebra of G-invariant differential operators.

Theorem 3.2.1 (cf. [11, Ch. II, Thm. 3.6] and [11, pp. 266])**.** *The submanifolds Ny and $A \cdot o$ satisfy for each $y \in A \cdot o$ the transversality condition*

$$T_y X = T_y(Ny) \oplus T_y(A \cdot o).$$

Moreover, for each D on X there is a unique differential operator $\Delta_N(D)$ on A such that for each N-invariant function $f \in C^\infty(X)$,

$$Df(a \cdot o) = \Delta_N(D) f^\dagger(a)$$

where $f^\dagger = f|_{A\cdot o}$.

The differential operator $\Delta_N(D)$ on A, which is defined for each differential operator D on X, is called the *radial part* or the *transversal part* of D. We finally can state the following theorem.

Theorem 3.2.2 (cf. [11, Ch. II, Cor. 5.19])**.** *Let e^ρ denote the smooth function on A defined by $e^\rho(a) = e^{\rho \log a}$; we also regard e^ρ as an operator on $C^\infty(A)$ by the assignment $e^\rho\colon f \mapsto e^\rho f$. Then the mapping*

$$\Gamma\colon D \mapsto e^{-\rho}\Delta_N(D) \circ e^\rho \tag{3.2.1}$$

is an isomorphism from $\mathbf{D}(X)$ onto $\mathbf{D}_W(A)$.

We shall now introduce the *plane waves* for the space X. In Chapter 2, we defined the function $x \mapsto e^{w\langle x,b\rangle}$ where $x \in \mathbf{H}^2$, $b \in B$, and $w \in \mathbf{C}$. This function was seen to be constant on each horocycle with normal b and a joint eigenfunction of G-invariant differential operators on $\mathbf{H}^2$ (the only invariant differential operators were operators that are polynomials in the Laplace-Beltrami operator on $\mathbf{H}^2$). In the case for $X = G/K$, the plane waves take the form $x \mapsto e^{wA(x,kM)}$ where $x \in X$, $kM \in K/M$, and $w \in \mathfrak{a}_c^*$. These functions are smooth since we can express each plane wave as a function on G, by $g \mapsto e^{wA(k^{-1}g)}$ which is smooth since $g \mapsto k^{-1}g$ is smooth, and the function $h \mapsto A(h)$ is a smooth map from G onto $\mathfrak{a}$. Furthermore, these functions are constant on each horocycle with normal kM. The plane waves are eigenfunctions of all operators in $\mathbf{D}(X)$ which is provided by the following lemma.

Proposition 3.2.3. *For $w \in \mathfrak{a}_c^*$ and each normal $kM \in K/M$, the functions $x \mapsto e^{wA(x,kM)}$ are joint eigenfunctions of each differential operator $D \in \mathbf{D}(X)$.*

Proof. First consider the function $e_w(x) = e^{wA(x,eM)}$. Using the fact that A normalizes N and Theorem 3.1.5 we have that e_w is N-invariant. Thus, if $D \in \mathbf{D}(X)$, then by Theorems 3.2.1 and 3.2.2

$$\begin{aligned} De_w(a \cdot o) &= \Delta_N(D)e_w(a \cdot o) = (e^{\rho}\Gamma(D) \circ e^{-\rho})e^{w \log a} \\ &= e^{\rho}\Gamma(D)e^{(w-\rho)\log a} = \Gamma(D)(w-\rho)e_w(a \cdot o) \end{aligned}$$

(cf. Example 3.2.4). Since the operator D and the functions e_w are N-invariant it thus follows that $De_w(x) = \Gamma(D)(w-\rho)e_w(x)$ for all $x \in X$. Now if we consider the function $e^{wA(x,kM)}$ for arbitrary $kM \in K/M$, then we have $e_w(k^{-1} \cdot x) = e^{wA(x,kM)}$. So by the G-invariance of D we also have $De_w(k^{-1} \cdot x) = \Gamma(D)(w-\rho)e_w(k^{-1} \cdot x)$ and so we are done. $\square$

Example 3.2.4. Let $X = \mathbf{SU}(1,1)/\mathbf{SO}(2) = \mathbf{H}^2$ and L be the Laplace-Beltrami operator on X given by (2.1.9) which is an element of $\mathbf{D}(X)$. In this case the element ρ is identified with the constant 1 (since by Example 3.1.4, the positive root $\alpha = 2$ so $\rho = \frac{1}{2}\alpha = 1$). Regarding the submanifold $A \cdot o$ as being diffeomorphic to $\mathbf{R}$ and with the aid of [11, Ch. II, Prop. 3.8] (and after a normalization coefficient) we find that the radial part of L is given by

$$\Delta_N(L) = e^t \frac{d^2}{dt^2} \circ e^{-t} - 1,$$

where the Laplacian L_A on the manifold $A \cdot o$ is d^2/dt^2. Then $\Gamma(L) = d^2/dt^2 - 1$. In particular, since we can identify $\mathfrak{a}_c^* \cong \mathbf{C}$ we have that for $\lambda \in \mathbf{C}$, that

$$Le^{(i\lambda+1)\langle x,b \rangle} = \Gamma(L)(i\lambda)e^{(i\lambda+1)\langle x,b \rangle}.$$

Then $\Gamma(L)(i\lambda) = (d^2/dt^2 - 1)(i\lambda) = (i\lambda)^2 - 1 = -(\lambda^2 + 1)$, (cf. formula (2.1.14)).

Spherical functions and the spherical transform

Now, in much the same way that we have defined spherical functions on $\mathbf{H}^2$ as K-invariant joint eigenfunctions of $\mathbf{D}(\mathbf{H}^2)$ we define spherical functions on the space X as the following:

Definition 3.2.2. A *spherical function* on X is a function $\phi \in C^\infty(X)$ such that it is K-invariant (i.e. $\phi(k \cdot x) = \phi(x)$ for all $k \in K$), and is an eigenfunction for each element of $\mathbf{D}(X)$.

Developing the properties of the spherical functions on the homogeneous space X is not necessarily easy and so we shall not provide the details. The important facts are summarized by the following result.

Theorem 3.2.5. *Any two spherical functions which have the same eigenvalues for each operator $D \in \mathbf{D}(X)$ are equal. Each spherical function is a constant multiple of a function of the form*

$$\phi_\lambda(x) = \int_K e^{(i\lambda+\rho)A(x,kM)}\, dk, \tag{3.2.2}$$

where $\lambda \in \mathfrak{a}_c^*$. Also if $\nu, \lambda \in \mathfrak{a}_c^*$, then $\phi_\nu = \phi_\lambda$ if and only if $\nu = s\lambda$ for some $s \in W$. The spherical function ϕ_λ is bounded if and only if λ lies in $\mathfrak{a}^* + iC(\rho)$ where $C(\rho)$ is the convex hull of the points $\{s\rho\colon s \in W\}$. Finally, the spherical functions ϕ_λ satisfy the following mean value property,

$$\phi_\lambda(h \cdot o)\phi_\lambda(g \cdot o) = \int_K \phi_\lambda(hkg \cdot o)\, dk, \quad g, h \in G. \tag{3.2.3}$$

The proof of Theorem 3.2.5 is contained in Corollary 2.3 and Theorems 4.3 and 8.1 of Chapter IV in [11]. Consider the following example for the hyperbolic plane.

Example 3.2.6. We have $\mathbf{H}^2 = \mathbf{SU}(1,1)/\mathbf{SO}(2)$. Using the standard Iwasawa decomposition $\mathbf{SU}(1,1) = KAN$ where K, A, and N are as in Example (3.1.4) we have that the centralizer of A in K is $M = \{I, -I\}$ while the normalizer of A in K is the group $M' = \{I, -I, S, -S\}$ where $S = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. The Weyl group is then $W = M'/M \cong \mathbf{Z}_2$. If $sM \in W$ is the non-identity element, then we have that

$$s \begin{pmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{pmatrix} s^{-1} = \begin{pmatrix} \cosh x & -\sinh x \\ -\sinh x & \cosh x \end{pmatrix}.$$

It then follows that the Weyl group acts on the Lie algebra $\mathfrak{a}$ via the two maps $H \mapsto H$ and $H \mapsto -H$, and consequently acts on $\mathfrak{a}_c^*$ via the maps $\lambda \mapsto \lambda$ and $\lambda \mapsto -\lambda$. Theorem 3.2.5 then implies that for the spherical functions on the hyperbolic plane that $\phi_\lambda = \phi_{-\lambda}$. Furthermore, the spherical functions ϕ_λ are bounded if and only if $\lambda \in \{x + iy \colon x \in \mathbf{R},\ |y| \leq 1\}$. Compare this to the results of Proposition 2.2.1 and Lemma 2.3.15.

Another important formula is Harish-Chandra's formula.

Lemma 3.2.7 ([11, Ch. IV, Lemma 4.4]). *The spherical functions ϕ_λ satisfy*

$$\phi_\lambda(h^{-1}g \cdot o) = \int_K e^{(-i\lambda+\rho)A(k^{-1}h)} e^{(i\lambda+\rho)A(k^{-1}g)}\, dk\,. \tag{3.2.4}$$

In particular the spherical functions satisfy $\phi_\lambda(g^{-1} \cdot o) = \phi_{-\lambda}(g \cdot o)$.

Definition 3.2.3. For a K-invariant function f we define its *spherical transform* by

$$\widetilde{f}(\lambda) = \int_X f(x)\phi_{-\lambda}(x)\, dx \tag{3.2.5}$$

for all $\lambda \in \mathfrak{a}_c^*$ for which (3.2.5) exists.

By Theorem 3.2.5 we have that the spherical transform is defined on $L^1_K(X)$ for each $\lambda \in \mathfrak{a}^*$. If $\lambda_n \to \lambda$ in the Hilbert space topology of $\mathfrak{a}_c^*$, then the functions $\phi_{\lambda_n} \to \phi_\lambda$ pointwise. In particular, by an application of the dominated convergence theorem (and an augmentation of the argument of the proof Riemann-Lebesgue Lemma of Chapter 2) we see that the spherical transforms of functions in $L^1_K(X)$ are continuous on $\mathfrak{a}^* + iC(\rho)$ (and for functions in $C_c(X)$ they are continuous on $\mathfrak{a}_c^*$). In view of the boundedness of the spherical functions due to Theorem 3.2.5 we have the following result.

Proposition 3.2.8. *For $f \in L^1_K(X)$, the function $\widetilde{f}$ is holomorphic on the interior of $\mathfrak{a}^* + iC(\rho)$.*

Proof. The proof is completely analogous to the proof of the Riemann-Lebesgue Lemma of Chapter 2. Here, we note that for $f \in L^1_K(X)$, we have $|\widetilde{f}(\lambda)| \leq \widetilde{|f|}(-i\operatorname{Im}\lambda)$ and so the integral $\int_X |f(x)\phi_{-\lambda}(x)|\, dx \leq \widetilde{|f|}(-i\operatorname{Im}\lambda)$ is uniformly bounded on compact sets on the interior of $\mathfrak{a}^* + iC(\rho)$. Hence we can see that we can apply Morera's theorem and use the holomorphicity of the function $\lambda \mapsto \phi_\lambda(x)$ to determine that $\widetilde{f}$ is holomorphic on the interior of $\mathfrak{a}^* + iC(\rho)$. ☐

To state the familiar inversion formula of Theorem 2.2.2 we shall need to introduce Harish-Chandra's $\mathbf{c}$-function in our present setting.

Definition 3.2.4. Harish-Chandra's $\mathbf{c}$-function is defined on $\mathfrak{a}^*$ by

$$\lambda \mapsto \mathbf{c}(\lambda) = \lim_{t\to\infty} e^{(-i\lambda+\rho)(tH)}\phi_\lambda(\exp tH \cdot o), \tag{3.2.6}$$

where H is an arbitrary element of the Weyl chamber $\mathfrak{a}^+$.

The $\mathbf{c}$-function is actually defined by formula (3.2.6) on a connected open dense set in $\mathfrak{a}_c^*$ and is holomorphic there (see [11, Ch. IV, Sec. 6]), and it is meromorphic on all of $\mathfrak{a}_c^*$. The $\mathbf{c}$-function is well-behaved on $\mathfrak{a}^*$ with all the derivatives of the function $\mathbf{c}(\lambda)^{-1}$ being bounded by polynomials in λ.

Finally, the inversion formula—stated for functions which are K-invariant, compactly supported, and smooth for the moment, is given as follows.

Theorem 3.2.9. *For functions $f \in \mathcal{D}_K(X)$, the spherical transform is inverted by*

$$f(x) = c_0 \int_{\mathfrak{a}^*} \widetilde{f}(\lambda)\phi_\lambda(x)|\mathbf{c}(\lambda)|^{-2}\, d\lambda \tag{3.2.7}$$

where c_0 is a constant independent of f and $d\lambda$ is the Lebesgue measure on $\mathfrak{a}^*$.

We shall write $d\mu(\lambda) = c_0|\mathbf{c}(\lambda)|^{-2}\, d\lambda$ henceforth. The spherical transform will of course be generalized to functions that are not K-invariant by Helgason's Fourier transform which we investigate next.

The Fourier transform

Definition 3.2.5. For a function f we define its *Fourier transform* by

$$\widetilde{f}(kM, \lambda) = \int_X f(x)e^{(-i\lambda+\rho)A(x,kM)}\, dx \tag{3.2.8}$$

for all $kM \in K/M$ and $\lambda \in \mathfrak{a}_c^*$ for which (3.2.8) exists.

Since the subgroup A is diffeomorphic to its Lie algebra, we can regard it as a vector space over $\mathbf{R}$. Thus, for functions $f \in L^1(A)$ where the Haar measure on A is induced from the Lebesgue measure on $\mathfrak{a}$ we have that its Fourier transform from Definition 1.1.1 is given by

$$\widehat{f}(\lambda) = \int_A f(x)e^{-i\lambda \log a}\, da\,.$$

Remark 10. Here the Fourier transform on A is defined on the dual space $\mathfrak{a}^*$ but is of course defined on $\mathfrak{a}$ since the linear functionals on $\mathfrak{a}^*$ is parametrized by $\mathfrak{a}$ via the isomorphism $(Y \mapsto (X \mapsto B(X,Y))$ from $\mathfrak{a}$ to $\mathfrak{a}^*$.

The Fourier transform of a function f on X is related to the standard Fourier transform on A as follows.

Definition 3.2.6. For a function f we define its *modified Radon transform* by

$$\mathcal{R}f(kaMN \cdot o) = e^{\rho \log a} \int_N f(kan \cdot o)\, dn,$$

whenever this integral exists.

Then using formula (3.1.10) we have the expression

$$\begin{aligned}\widetilde{f}(kM, \lambda) = \int_X f(k \cdot x)e^{(-i\lambda+\rho)A(x,eM)}\, dx &= \int_{AN} f(kan \cdot o)e^{(-i\lambda+\rho)\log a}\, da\, dn \\ &= \int_A \mathcal{R}f(kaMN \cdot o)e^{-i\lambda \log a}\, da\,.\end{aligned}$$

Hence the Fourier transform of a function on X is given by the standard (Euclidean) Fourier transform over A of its modified Radon transform. Of course, if f is a K-invariant function, then integrating the function $\widetilde{f}(kM, \lambda)$ over K—which is necessarily constant for each $kM \in K/M$, yields the spherical transform of f. We summarize the significant properties of the Fourier transform in the following proposition.

Proposition 3.2.10. *Suppose that $f_1 \in C_c(X)$ and that $f_2 \in C_c(X)$ is K-invariant.*

1. *We have $(f_1 * f_2)\widetilde{\ } = \widetilde{f}_1\widetilde{f}_2$.*
2. *Let $f_1^{\tau(g)}(x) = f_1(g \cdot x)$, then*

$$(f_1^{\tau(g)})\widetilde{\ }(kM, \lambda) = e^{(-i\lambda+\rho)A(g^{-1}\cdot o,kM)}\widetilde{f}(g(kM), \lambda). \tag{3.2.9}$$

3. *If f_1 is smooth, then for each operator $D \in \mathbf{D}(X)$, we have $(Df)\widetilde{\ } = \Gamma(D)(i\lambda)\widetilde{f}$.*

4. *The Fourier transform of f obeys the following symmetry condition for all $x \in X$ and $s \in W$:*

$$\int_K \widetilde{f}(kM,\lambda)e^{(i\lambda+\rho)A(x,kM)}\,dk = \int_K \widetilde{f}(kM,s\lambda)e^{(is\lambda+\rho)A(x,kM)}\,dk\,. \tag{3.2.10}$$

Proof. To prove (1) simply adapt the argument of Theorem 2.3.5. For (2), we have using Proposition 3.1.7

$$\begin{aligned}(f_1^{\tau(g)})\widetilde{\ }(kM,\lambda) = \int_X f(x)e^{(-i\lambda+\rho)A(g^{-1}x,kM)}\,dx &= \int_X f(x)e^{(-i\lambda+\rho)[A(x,g(kM))+A(g^{-1}\cdot o,kM)]}\,dx \\ &= e^{(-i\lambda+\rho)A(g^{-1}\cdot o,kM)}\widetilde{f}(g(kM),\lambda).\end{aligned}$$

In particular, if f_1 was K-invariant, then integrating the previous formula yields $(f_1^{\tau(g)})\widetilde{\ }(\lambda) = \phi_\lambda(g\cdot o)\widetilde{f}(\lambda)$. For (3), we write

$$\begin{aligned}(Df)\widetilde{\ }(kM,\lambda) = \int_X Df(x)e^{(-i\lambda+\rho)A(x,kM)}\,dx &= \int_X f(x)(D^*e^{(-i\lambda+\rho)A(x,kM)})\,dx \\ &= \Gamma(D^*)(-i\lambda)\widetilde{f}(kM,\lambda).\end{aligned}$$

Here D^* is the adjoint of D with respect to the bilinear form $(u,v) \mapsto \int_X u(x)v(x)\,dx$ (cf. the discussion of Section 3.3.1 in the preliminary subsection). It is actually the case that $\Gamma(D^*) = \Gamma(D)^*$ (cf. [11, Ch. II, Lemma 5.21]), then using Example 3.3.3 one calculates $\Gamma(D^*)(-i\lambda) = \Gamma(D)(i\lambda)$. Finally for (4) we have with $g,h \in G$ and using Harish-Chandra's formula

$$f*\phi_\lambda(g\cdot o) = \int_G f(h\cdot o)\phi_\lambda(h^{-1}g\cdot o)\,dh = \int_G\int_K f(h\cdot o)e^{(-i\lambda+\rho)A(k^{-1}h)}e^{(i\lambda+\rho)A(k^{-1}g)}\,dk\,dh \tag{3.2.11}$$

$$= \int_K \widetilde{f}(kM,\lambda)e^{(i\lambda+\rho)A(k^{-1}g)}\,dk\,. \tag{3.2.12}$$

The result then follows from the fact that $\phi_{s\lambda} = \phi_\lambda$ for all $s \in W$.

□

Remark 11. It should be noted that the convolution result in (1) of the above proposition only behaves nicely due to the same reasons outlined in Chapter 2. In particular, the convolution algebra $L^1(X)$ is noncommutative and by the Fourier inversion result which we will state later, the Fourier transform cannot be a homomorphism from $C_c(X)$ onto its image, since otherwise the inverse Fourier transform will also be a homomorphism and thus $C_c(X)$ will be a commutative convolution algebra, whence $L^1(X)$ is a commutative convolution algebra—a contradiction. However, when restricting to $L^1_K(X)$, this convolution algebra *is* commutative and so the result may follow through.

The result of (3) gives the familiar result from $\mathbf{H}^2$ and $\mathbf{R}^n$, whereby the action of invariant differential operators against functions on X corresponds to multiplication by polynomials in λ on the Fourier transform side.

The symmetry condition in (4) is also special since it imposes certain restrictions on the image of the Fourier transform. In the most simplest case for $G = \mathbf{SU}(1,1)$ we recall that the image of $\mathcal{D}(\mathbf{H}^2)$ is the space $\mathscr{H}$ consisting of the entire functions of uniform exponential type on $K/M \times \mathfrak{a}_c^* = B \times \mathbf{C}$ satisfying the relation (4) from above. This is a strict subset of the space $\mathscr{H}'$ which are the entire functions of uniform exponential type (not necessarily satisfying (4)). For instance, if $(kM,\lambda) \mapsto F(kM,\lambda)$ is an entire function of uniform exponential type constant in b, then $F \in \mathscr{H}$ if and only if F is an even function of λ.

We shall now state the familiar Paley-Wiener, Plancherel, and inversion theorems as well as the Riemann-Lebesgue Lemma in the setting for the homogeneous space $X = G/K$. To begin, note that K/M is a compact homogeneous space so we let dk_M denote the unique K-invariant Radon measure normalized so that the measure of K/M against this measure is 1. We then set $d\mu(kM,\lambda) = d\mu(\lambda)\,dk_M$.

Definition 3.2.7. For a measurable function ψ on $K/M \times \mathfrak{a}^*$ (with respect to the measure $d\mu(kM,\lambda)$), we shall define the *inverse Fourier transform* of ψ by

$$\mathscr{F}^{-1}\psi(x) = \int_{K/M}\int_{\mathfrak{a}^*} \psi(kM,\lambda)e^{(i\lambda+\rho)A(x,kM)}\, d\mu(kM,\lambda), \tag{3.2.13}$$

for each $x \in X$ for which this integral converges.

Theorem 3.2.11 (Inversion theorem)**.** *For $f \in \mathcal{D}(X)$, the Fourier transform is inverted by*

$$f(x) = (\mathscr{F}^{-1}\widetilde{f})(x) = \int_{K/M}\int_{\mathfrak{a}^*} \widetilde{f}(kM,\lambda)e^{(i\lambda+\rho)A(x,kM)}\, d\mu(kM,\lambda), \tag{3.2.14}$$

which holds for all $x \in X$. More generally, if $f \in L^1(X)$, then there exists a subset $B \subset K/M$ such that $(K/M) \setminus B$ has measure 0 and that the Fourier transform of f exists for all $kM \in B$ and $\lambda \in \mathfrak{a}^* + iC(\rho)$ where $C(\rho)$ is the convex hull of $\{s\rho \colon s \in W\}$. The assignment $\lambda \mapsto \widetilde{f}(kM,\lambda)$ (for $kM \in B$ fixed) defines a holomorphic function on the interior of $\mathfrak{a}^* + iC(\rho)$ and (3.2.14) holds for almost every $x \in X$ when $\widetilde{f} \in L^1(K/M \times \mathfrak{a}^*)$.

The proof of the inversion theorem for functions of class $f \in \mathcal{D}(X)$ is proved using similar arguments to those developed in Chapter 2. In particular, if $f \in \mathcal{D}(X)$, then if we fix $g \in G$, then the function

$$F(h) = \int_K f(gkh\cdot o)\, dk \tag{3.2.15}$$

defines an element of $\mathcal{D}_K(X)$. The spherical transform of F is then given by

$$\widetilde{F}(\lambda) = \int_G\int_K f(gkh\cdot o)\phi_{-\lambda}(h\cdot o)\, dk\, dh = \int_G f(gh\cdot o)\phi_{-\lambda}(h\cdot o)\, dh \tag{3.2.16}$$

$$= \int_G f(h\cdot o)\phi_\lambda(h^{-1}g\cdot o)\, dh = f * \phi_\lambda(g\cdot o). \tag{3.2.17}$$

Then using (3.2.12) and the spherical inversion formula on F:

$$\begin{aligned} f(g\cdot o) = F(e) = \int_{\mathfrak{a}^*} \widetilde{F}(\lambda)\, d\mu(\lambda) &= \int_{\mathfrak{a}^*}\int_K \widetilde{f}(kM,\lambda)e^{(i\lambda+\rho)A(k^{-1}g)}\, dk\, d\mu(\lambda) \\ &= \int_{K/M}\int_{\mathfrak{a}^*} \widetilde{f}(kM,\lambda)e^{(i\lambda+\rho)A(g\cdot o,kM)}\, d\mu(kM,\lambda)\,. \end{aligned}$$

To prove the inversion formula for $f \in L^1(X)$ one requires techniques similar to the ones employed in Chapter 2, (see [13, Ch. III, Thm. 1.8 and Thm. 1.9] for a proof). Now the Riemann-Lebesgue Lemma formulated for $f \in L^1(X)$ is given as follows.

Theorem 3.2.12 (Riemann-Lebesgue Lemma)**.** *Let $f \in L^1(X)$ and $B \subset K/M$ be the subset as in the statement of Theorem 3.2.11. Then we have*

$$\lim_{|\xi|\to\infty} \widetilde{f}(kM,\xi+i\eta) = 0 \tag{3.2.18}$$

uniformly in η where $\xi \in \mathfrak{a}^*$ and $\eta \in C(\rho)$.

Moving on to the Plancherel theorem put $\mathfrak{a}^*_+ = \{\lambda \in \mathfrak{a}^* \colon H_\lambda \in \mathfrak{a}^+\}$ where H_λ is the unique vector such that $\lambda(H) = B(H_\lambda, H)$ for all $H \in \mathfrak{a}^*$. In the case of the group $G = \mathbf{SU}(1,1)$ with the Weyl chamber being defined in Example 3.1.4, then $\mathfrak{a}^*_+ = \mathbf{R}^+$. Since the Fourier transform is defined on $C_c(X)$, we can extend it continuously to all of $L^2(X)$. Then the Plancherel theorem is formulated as follows.

Theorem 3.2.13 (Plancherel Theorem)**.** *The Fourier transform extends continuously to an isometric isomorphism of $L^2(X; dx)$ onto $L^2(K/M \times \mathfrak{a}_+^*; |W|\, d\mu(\lambda)\, dk_M)$.*

Here $|W|$ denotes the order of the Weyl group. The proof the Plancherel theorem is similar (and is just as complicated) to the proof of the Plancherel theorem specialized in Chapter 2—the proof of the above result is contained in [13, Ch. III, Thm. 1.5]. Specifically, one needs to work out the so-called *simplicity criterion* for $\lambda \in \mathfrak{a}_c^*$ which generalizes the statement of Definition 2.3.2 to work towards a complete proof.

Finally, we say a smooth function ψ on $K/M \times \mathfrak{a}_c^*$ is a holomorphic of uniform exponential type $A > 0$ if

1. The mapping $\lambda \to \psi(kM, \lambda)$ is holomorphic for each $kM \in K/M$.
2. And F satisfies the estimates

$$\sup_{(kM,\lambda)\in K/M\times\mathfrak{a}_c^*} |(1 + |\lambda|)^N e^{-A|\operatorname{Im}\lambda|}\psi(kM, \lambda)| < \infty,$$

for all $N \in \mathbf{N}$.

The space of holomorphic functions of uniform exponential type $A > 0$ will then be denoted by $\mathscr{H}_A$ and we put $\mathscr{H} = \bigcup_{A>0} \mathscr{H}_A$. We then intersect $\mathscr{H}$ with the space of all smooth functions ψ satisfying the symmetry condition:

$$\int_K \psi(kM, \lambda)e^{(i\lambda+\rho)A(x,kM)}\, dk = \int_K \psi(kM, s\lambda)e^{(is\lambda+\rho)A(x,kM)}\, dk \tag{3.2.19}$$

for all $x \in X$ and $s \in W$. We let $\mathscr{H}_W$ denote the space of functions obtained from this intersection. Then the Paley-Wiener theorem is stated as follows.

Theorem 3.2.14 (Paley-Wiener)**.** *The Fourier transform is a bijection of $\mathcal{D}(X)$ onto $\mathscr{H}_W$.*

The proof is quite long and can be found in [13, Ch. III, Sec. 3], although we remark that the style of the proof for the general homogeneous space X is quite analogous to the proof of the Paley-Wiener theorem for the hyperbolic plane.

3.3 The Harish-Chandra Schwartz space

We now come to the promised discussion of the Schwartz space on $X = G/K$ and the Fourier transform on this space. First, we recall that G admits the Cartan decomposition $G = K\overline{A^+}K = KAK$ so that each $g \in G$ can be written as $g = k_1ak_2$ for some $k_1, k_2 \in K$ and $a \in A$. We introduce a "norm" on G by defining $|g| = |\log a|$. This norm is K-bi-invariant and also satisfies $|gh| \leq |g||h|$ for all $g, h \in G$ which is merely a consequence of the Ad-invariance of the Killing form on $\mathfrak{g}$.

Let $D_1, D_2 \in U(\mathfrak{g})$, then these elements give rise to left-invariant and right-invariant differential operators $\partial_l(D_1)$ and $\partial_r(D_2)$ on G (cf. 3.1.13). We put $\Xi(g) = \phi_0(g \cdot o)$, where $\phi_0(x)$ is the spherical function with $\lambda = 0$. If $f \in C^\infty(G)$, then we define for $N \in \mathbf{N}$, $D_1, D_2 \in U(\mathfrak{g}_c)$

$$\nu_{N,D_1,D_2}(f) = \sup_{g\in G} |(1 + |g|)^N \Xi(g)^{-1}\partial_l(D_1)\partial_r(D_2)f(g)|.$$

Definition 3.3.1. The *Harish-Chandra Schwartz space* on G is the space

$$\mathcal{S}(G) = \{f \in C^\infty(G)\colon \nu_{N,D_1,D_2}(f) < \infty, \text{ for all } N \in \mathbf{N},\ D_1, D_2 \in U(\mathfrak{g})\}. \tag{3.3.1}$$

The space $\mathcal{S}(G)$ is topologized by the means of the seminorms ν_{N,D_1,D_2} which turns $\mathcal{S}(G)$ into a Fréchet space. The Schwartz space on X, denoted by $\mathcal{S}(X)$, is the closed subspace of $\mathcal{S}(G)$ consisting of those functions which are right invariant under K. The Schwartz space $\mathcal{J}(X)$ is the subspace of K-bi-invariant members of $\mathcal{S}(G)$.

Remark 12. One can define equivalently define the space $\mathcal{J}(X)$ as the space of smooth K-bi-invariant functions such that for each operator $D \in \mathbf{D}(G)$, the seminorms

$$\omega_{N,D}(f) = \sup_{g \in G} |(1+|g|)^N \Xi(g)^{-1} Df(g)|$$

where $N \in \mathbf{N}$, are finite. Under this definition we still have $\mathcal{J}(X) \subset \mathcal{S}(X)$ and the topology induced by these seminorms is the same as the topology subspace topology which $\mathcal{J}(X)$ inherits from $\mathcal{S}(X)$.

The definition of the Schwartz space on X is quite familiar to the ordinary definition of the Schwartz space on $\mathbf{R}^n$ with the key difference that we introduce the factor $\Xi(g)^{-1}$ in the definitions of the seminorms ν_{N,D_1,D_2}. The reason for this is so that the Fourier transform on $\mathcal{S}(X)$ is well-behaved (i.e. exists on $K/M \times \mathfrak{a}^*$)—we shall not go deeper into this however. From the estimates, alone for a function $f \in \mathcal{S}(X)$ it need not be true that $f \in L^1(X)$ (although it is true that the space $\mathcal{S}(X) \cap L^1(X)$ is dense in $L^1(X)$ for it contains $\mathcal{D}(X)$). What is true is that $\mathcal{S}(X) \subset L^2(X)$ and is dense there. Furthermore, $\mathcal{D}(X)$ is dense in $\mathcal{S}(X)$ with respect to the Schwartz topology (see [10, Sec. 9]).

We shall also define the Schwartz spaces $\mathcal{S}(K/M \times \mathfrak{a}^*)$ and $\mathcal{S}(\mathfrak{a}^*)$ as follows.

Definition 3.3.2. For a smooth function $f \in C^\infty(K/M \times \mathfrak{a}^*)$ let $N \in \mathbf{N}$, $v \in S(\mathfrak{a}^*)$, and $p \in U(\mathfrak{k})$, where $S(\mathfrak{a}^*)$ is the symmetric algebra of $\mathfrak{a}^*$ and $\mathfrak{k}$ is the Lie algebra of K. Then we define the function

$$\eta_{N,v,p}(f) = \sup_{(kM,\lambda) \in K/M \times \mathfrak{a}^*} |(1+|\lambda|)^N \partial_r(p)\partial(v) f(kM,\lambda)|.$$

Then $\mathcal{S}(K/M \times \mathfrak{a}^*) = \{f \in C^\infty(K/M \times \mathfrak{a}^*) \colon \eta_{N,v,p}(f) < \infty, \text{ for all } N \in \mathbf{N}, v \in S(\mathfrak{a}^*), p \in U(\mathfrak{k})\}$. This space is then topologized by the means of the seminorms $\eta_{N,v,p}$. Since $\mathfrak{a}^*$ is a normed finite-dimensional vector space, $\mathcal{S}(\mathfrak{a}^*)$ is then the ordinary Schwartz space on $\mathfrak{a}^*$ which is identified as a subspace of $\mathcal{S}(K/M \times \mathfrak{a}^*)$.

Finally, we let $\mathcal{S}_W(K/M \times \mathfrak{a}^*)$ denote the subspace of functions satisfying the symmetry condition (3.2.19) and $\mathcal{S}_W(\mathfrak{a}^*)$ is the subspace of W-invariant Schwartz functions in $\mathcal{S}(\mathfrak{a}^*)$. We can now state the version of the Schwartz isomorphism theorem for the homogeneous space X.

Theorem 3.3.1. *The Fourier transform of a function $f \in \mathcal{S}(X)$ exists everywhere on $K/M \times \mathfrak{a}^*$. The Fourier transform is a homeomorphism from $\mathcal{S}(X)$ onto $\mathcal{S}_W(K/M \times \mathfrak{a}^*)$, while the spherical transform is a homeomorphism from $\mathcal{J}(X)$ onto $\mathcal{S}_W(\mathfrak{a}^*)$.*

Proofs of the fact that the spherical transform is a homeomorphism of $\mathcal{J}(X)$ onto $\mathcal{S}_W(\mathfrak{a}^*)$ have been simplified over the years. Notably, Anker produced an elementary and elegant argument for this claim in [1] (the proof is also found in [13, Ch. III, Thm. 1.17]). The proof of the fact that the Fourier transform is a homeomorphism from $\mathcal{S}(X)$ onto $\mathcal{S}_W(K/M \times \mathfrak{a}^*)$ is far more difficult and is the topic of Eguchi's paper [6]. It appears that adapting Anker's simplified proof to Eguchi's result for the general Schwartz space $\mathcal{S}(X)$ is not possible (see [3]). One of the first consequences of this theorem is the following result.

Proposition 3.3.2. *The spaces $\mathcal{J}(X)$ and $\mathcal{S}(X)$ are nuclear.*

Proof. Since $\mathcal{S}_W(\mathfrak{a}^*)$ is a subspace of a nuclear space (i.e. the Schwartz space $\mathcal{S}(\mathfrak{a}^*)$, it is nuclear. Since $\mathcal{J}(X)$ is homeomorphic to $\mathcal{S}_W(\mathfrak{a}^*)$, then $\mathcal{J}(X)$ is also nuclear.

Now the space $\mathcal{S}(K/M \times \mathfrak{a}^*)$ is a nuclear space since it may be realized as the injective tensor product $\mathcal{S}(K/M \times \mathfrak{a}^*) = C^\infty(K/M)\widehat{\otimes}\mathcal{S}(\mathfrak{a}^*)$ which can be proved in a similar fashion by adapting the proof of [25, Thm. 51.6]. So $\mathcal{S}(K/M \times \mathfrak{a}^*)$ is nuclear as it is the tensor product of nuclear spaces. Since $\mathcal{S}(X)$ is homeomorphic to a subspace of $\mathcal{S}(K/M \times \mathfrak{a}^*)$ it also follows that $\mathcal{S}(X)$ is nuclear. ☐

The definition of nuclear spaces is a bit involved (see [25, Sec. 39]). However, all that we have used here in this paper is that any subspace of a nuclear space is nuclear and the injective tensor product (or projective tensor product) of nuclear spaces is nuclear as well.

3.3.1 Tempered distributions

Preliminaries

If $S \subset \mathbf{R}^n$ is an open set, then we recall that the topology on $C^\infty(S) = \mathcal{E}(S)$ is the topology induced by the seminorms $\sigma_{\alpha,K}(f) = \sup_{x\in K} |\partial^\alpha f(x)|$ as α runs through all multi-indices of length n and K runs through all compact subsets of S.

If M is a smooth manifold of dimension n, then if (U, ϕ) is a local coordinate system, then we define a differential operator on U to be a linear map $D\colon \mathcal{E}(U) \to \mathcal{E}(U)$ which does not enlarge supports, that is $\operatorname{supp} Df \subset \operatorname{supp} f$. By Peetre's theorem, each differential operator on U can be written, for some $N \in \mathbf{N}$, as

$$Df(x) = \sum_{|\alpha|\leq N} a_\alpha(x)\partial^\alpha(f \circ \phi^{-1})(\phi(x)) \tag{3.3.2}$$

for each $x \in V$ where V is a relatively compact open subset of U and where the operators ∂^α are the standard differential operators on $\mathbf{R}^n$ and $a_\alpha \in \mathcal{E}(V)$ (cf. [11, Ch. II, Lemma 1.5] for a proof). Then we define the topology of $\mathcal{E}(U)$ to be the topology induced by the seminorms $\sigma_{D,K}(f) = \sup_{x\in K} |Df(x)|$ as D runs through all differential operators on U and K runs through all compact subsets of U. In particular, a sequence $\{f_n\} \subset \mathcal{E}(U)$ converges to $f \in \mathcal{E}(U)$ if and only if $f_n \circ \phi^{-1} \to f \circ \phi^{-1}$ in $\mathcal{E}(\phi(U))$.

The topology of $\mathcal{E}(M)$ is then defined to be the weakest topology so that the restriction maps $\rho_U \colon \mathcal{E}(M) \to \mathcal{E}(U)$, $\rho_U \colon f \mapsto f|_U$ are continuous as (U, ϕ) runs through all local coordinate systems on M. In particular a sequence $\{f_n\} \subset \mathcal{E}(M)$ converges to f if and only if for each differential operator D on M, the sequence Df_n converges locally uniformly to Df. With this topology $\mathcal{E}(M)$ is a Fréchet space.

Now if $K \subset M$ is a compact set, then $\mathcal{D}(K)$ denotes the space of all smooth functions with support contained in K. The space $\mathcal{D}(K)$ is given the subspace topology when regarded as a subspace of $\mathcal{E}(M)$. The space $\mathcal{D}(M)$ of all compactly supported smooth functions is then given the inductive limit topology corresponding to the family of spaces $\mathcal{D}(K)$ as K varies over the compact subsets of M. In particular a sequence $\{f_n\} \subset \mathcal{D}(M)$ converges to $f \in \mathcal{D}(M)$ if and only if for some $m > 0$ there is a compact set K' such that $f, f_n \in \mathcal{D}(K')$ for all $n \geq m$ and $\{f_n\}_{n=m}^\infty$ converges to f in $\mathcal{D}(K')$. Since M is σ-compact, we can write M as a countable union of increasing compact sets and we thus see that $\mathcal{D}(M)$ is dense in $\mathcal{E}(M)$.

We then let $\mathcal{E}'(M)$ and $\mathcal{D}'(M)$ denote the linear topological duals of $\mathcal{E}(M)$ and $\mathcal{D}(M)$, the latter of which is called the space of *distributions*, and impose upon these spaces the strong topology (namely so that Schwartz's kernel theorem holds). The support of a distribution is defined in precisely the same way as in Definition 1.1.7, namely a distribution T vanishes on an open set V if $T(f) = 0$ for all smooth functions f with compact support contained in V—whence we define the *support* of a distribution T to be the complement of the largest open set on which T vanishes.

One important fact is that we have the continuous inclusion $\mathcal{E}'(M) \hookrightarrow \mathcal{D}'(M)$ and identify $\mathcal{E}'(M)$ with the distributions of *compact support* (cf. [11, Ch. II, Prop. 2.1]).

We shall now restrict our attention to unimodular Lie groups although we remark that some of our constructions that follow can be appropriately generalized to any manifold. Let G be a unimodular Lie group with Haar measure dx induced from a left-invariant n-form and let $T \in \mathcal{D}(G)$ be a distribution. If $f \in \mathcal{D}(G)$, we will use the notation

$$T(f) = \langle T, f\rangle = \int_G T(x)f(x)\, dx\,. \tag{3.3.3}$$

If D is a differential operator, then we let D^* denote the adjoint of D with respect to the bilinear form $(u, v) = \int uv$ on $\mathcal{D}(G)$ with respect to the Haar measure. That is $(Du, v) = (u, D^*v)$ and it is known that the adjoint D^* exists and defines a differential operator (cf. [11, Ch. II, Sec. 2.3]). Therefore if D is a differential operator we define the distribution DT by $DT(f) = T(D^*f)$. One can easily verify that the adjoint of a left (resp. right) invariant differential operator is also a left (resp. right) invariant differential operator.

Example 3.3.3. In the particular case that $D = X$ is a left-invariant (or right-invariant) vector field, then the adjoint can be calculated quite easily. In particular, we have if $X \in \mathfrak{g}$ and if we let X also denote the corresponding vector field, then for $f \in \mathcal{D}(G)$

$$\int_G \frac{d}{dt} f(y \exp tX)\, dy = \frac{d}{dt} \int_G f(y \exp tX)\, dy = \frac{d}{dt} \int_G f(y)\, dy = 0. \tag{3.3.4}$$

So $\int_G Xf = 0$. Then by the product rule, $0 = \int_G X(uv) = \int_G (Xu)v + \int_G u(Xv)$ for $u, v \in \mathcal{D}(G)$ so $X^* = -X$.

If T is a distribution and f is a smooth function, and at least one of T or f has compact support, then we define the convolutions $T * f$ and $f * T$ by

$$T * f(x) = \int_G T(y) f(y^{-1}x)\, dy = \langle T_y, f(y^{-1}x) \rangle, \tag{3.3.5}$$

$$f * T(x) = \int_G T(y) f(xy^{-1})\, dy = \langle T_y, f(xy^{-1}) \rangle. \tag{3.3.6}$$

Here T_y denotes the distribution acting on the variable y which we shall also sometimes write as $T(y)$. Under the given hypotheses for T and f, the convolutions $T * f$ and $f * T$ both define smooth functions which can be proved by similar methods to the proof used in Proposition 1.1.10. Finally, the convolution of distributions T and S, one of which has compact support, is the distribution defined by

$$(T * S)(f) = \int_G \int_G T(x) S(y) f(xy)\, dx\, dy = \langle S_y, \langle T_x, f(xy) \rangle \rangle. \tag{3.3.7}$$

Remark 13. There is a result known as the Fubini theorem for distributions (cf. [17, Thm. B.20]) which states that if X and Y are smooth manifolds, and if $T \in \mathcal{E}'(X)$ and $S \in \mathcal{E}'(Y)$, we can define a unique distribution $T \otimes S \in \mathcal{E}'(X \times Y)$ which satisfies:

1. If $f \in \mathcal{E}(X)$ and $g \in \mathcal{E}(Y)$, then for $(f \otimes g)(x, y) = f(x)g(y)$, we have $(T \otimes S)(f \otimes g) = T(f)S(g)$.
2. If $f \in \mathcal{E}(X \times Y)$, then $T \otimes S$ has the property

$$(T \otimes S)(f) = \langle T_x, \langle S_y, f(x, y) \rangle \rangle = \langle S_y, \langle T_x, f(x, y) \rangle \rangle.$$

We call $T \otimes S$ the *tensor product* of T and S.

Tempered distributions

We return to the case for the homogeneous space $X = G/K$ where G is a connected semisimple Lie group with finite center and K is a maximal compact subroup.

Definition 3.3.3. A *tempered distribution* on X is a continuous linear functional $u \in \mathcal{S}'(X)$. We impose the strong topology upon $\mathcal{S}'(X)$.

Note that we naturally have the chain of inclusions $\mathcal{E}'(X) \hookrightarrow \mathcal{S}'(X) \hookrightarrow \mathcal{D}(X)$. On $\mathbf{R}^n$, we defined the Fourier transform of a tempered distribution by the duality $\widehat{u}(f) = u(\widehat{f})$ using the observation that for $f, g \in \mathcal{S}(\mathbf{R}^n)$ that the integral identity

$$\int_{\mathbf{R}^n} \widehat{f}(x) g(x)\, dx = \int_{\mathbf{R}^n} f(x) \widehat{g}(x)\, dx \tag{3.3.8}$$

holds (cf. Proposition 1.1.1). To motivate the Fourier transform of tempered distributions $u \in \mathcal{S}'(X)$ we state a similar integral identity.

Proposition 3.3.4. *Let $\phi \in \mathcal{S}(X)$ and $\psi \in \mathcal{S}_W(K/M \times \mathfrak{a}^*)$. We have the identity*

$$\int_{K/M \times \mathfrak{a}^*} \widetilde{\phi}(kM, \lambda) \psi(kM, -\lambda)\, d\mu(kM, \lambda) = \int_X \phi(x) (\mathscr{F}^{-1}\psi)(x)\, dx\,. \tag{3.3.9}$$

Conveyed using bracket notation for the pairing of distributions this is merely

$$\langle \mathscr{F}\phi, \psi \rangle_{K/M \times \mathfrak{a}^*} = \langle \phi, \mathscr{F}^{-1}(\iota \circ \psi) \rangle_X, \tag{3.3.10}$$

where $(\iota \circ \psi)(kM, \lambda) = \psi(kM, -\lambda)$.

Proof. The fact that $\phi \in \mathcal{S}(X)$ and $\psi \in \mathcal{S}_W(K/M \times \mathfrak{a}^*)$ permits one to interchange the integral in the variable (kM, λ) with the integral defining the Fourier transform of ϕ by Fubini's theorem. Thus, checking the expression under the double integral leads to the result. $\square$

Remark 14. In the expression (3.3.10), the notations $\langle \cdot, \cdot \rangle_{K/M\times\mathfrak{a}^*}$ and $\langle \cdot, \cdot \rangle_X$ denote the bilinear forms $(u, v) \mapsto \int_{K/M\times\mathfrak{a}^*} uv$ and $(f, g) \mapsto \int_X fg$ on $\mathcal{S}_W(K/M \times \mathfrak{a}^*)$ and $\mathcal{S}(X)$ respectively, with respect to integration against the measures $d\mu(kM, \lambda)$ and dx.

It seems to us to be unclear that if $\psi \in \mathcal{S}_W(K/M \times \mathfrak{a}^*)$, then $\mathscr{F}^{-1}(\iota \circ \psi)$ defines an element of the Schwartz space $\mathcal{S}(X)$. However it is easy to see that it defines a smooth function. Another point is that if $f \in \mathcal{S}_W(K/M \times \mathfrak{a}^*)$, then $\iota \circ f \in \mathcal{S}_W(K/M \times \mathfrak{a}^*)$ if and only if $\overline{f} \in \mathcal{S}_W(K/M \times \mathfrak{a}^*)$. However, it is true that if $f \in \mathcal{S}_W(\mathfrak{a}^*)$, then $\iota \circ f \in \mathcal{S}_W(\mathfrak{a}^*)$.

Thus using (3.3.10) we define the Fourier transform of a compactly supported distribution as follows.

Definition 3.3.4. For $u \in \mathcal{E}'(X)$ define the distribution the *Fourier transform* $\widetilde{u} \in \mathcal{S}'_W(K/M \times \mathfrak{a}^*)$ by $\widetilde{u}(\psi) = u(\mathscr{F}^{-1}(\iota \circ \psi))$.

Definition 3.3.5. It is clear that if $v \in \mathcal{S}'(K/M \times \mathfrak{a}^*)$, then we can define the *inverse Fourier transform* of v as the tempered distribution $\mathscr{F}^{-1}v \in \mathcal{S}'(X)$ by $\mathscr{F}^{-1}v(\phi) = v(\iota \circ (\mathscr{F}\phi))$ where $\phi \in \mathcal{S}(X)$.

Let $\mathcal{E}'_K(X) \subset \mathcal{E}'(X)$ denote the subspace of compactly supported distributions satisfying $T(\tau(k)f) = T(f)$ for all $k \in K$ and $f \in \mathcal{E}(X)$. We call $\mathcal{E}'_K(X)$ the space of K-invariant compactly supported distributions.

Theorem 3.3.5. *For $u \in \mathcal{E}'(X)$, we can identify $\widetilde{u}$ with the smooth function*

$$\widetilde{u}(kM, \lambda) = \langle u, e^{(-i\lambda+\rho)A(x,kM)} \rangle \tag{3.3.11}$$

in the sense of distributions, which is to say that for $\phi \in \mathcal{D}(K/M \times \mathfrak{a}^*)$ we have

$$\widetilde{u}(\phi) = \int_{K/M\times\mathfrak{a}^*} \phi(kM, \lambda)\widetilde{u}(kM, \lambda)\, d\mu(kM, \lambda)\,.$$

Proof. We shall give two different proofs of fact (1). If $\phi \in \mathcal{D}(K/M \times \mathfrak{a}^*)$, then the integral

$$\int_{K/M\times\mathfrak{a}^*} \phi(kM, \lambda)e^{(i\lambda+\rho)A(x,kM)}\, d\mu(kM, \lambda)$$

exists as a $\mathcal{E}(X)$ valued integral and the maps $x \mapsto \phi(kM, \lambda)e^{(i\lambda+\rho)A(k^{-1}x)}$ are obviously smooth and so the above integral can also be interpreted as a $\mathcal{E}(X)$ valued *weak integral*. In fact, the map $F\colon K/M \times \mathfrak{a}^* \to \mathcal{E}(X)$ defined by $(kM, \lambda) \mapsto \phi(kM, \lambda)e^{(i\lambda+\rho)A(k^{-1}x)}$ is a compactly supported continuous function from $K/M \times \mathfrak{a}^*$ to the Fréchet space $\mathcal{E}(X)$ so that Theorem A.2.2 can be applied. Therefore for $\phi \in \mathcal{D}(K/M \times \mathfrak{a}^*)$ we have

$$\langle \widetilde{u}, \phi \rangle = \langle u, \mathscr{F}^{-1}(\iota \circ \phi) \rangle = \langle u, \int_{K/M\times\mathfrak{a}^*} \phi(kM, \lambda)e^{(-i\lambda+\rho)A(k^{-1}x)}\, d\mu(kM, \lambda) \rangle \tag{3.3.12}$$

$$= \int_{K/M\times\mathfrak{a}^*} \phi(kM, \lambda)\langle u, e^{(-i\lambda+\rho)A(k^{-1}x)} \rangle\, d\mu(kM, \lambda)\,. \tag{3.3.13}$$

So the distributions $\widetilde{u}$ and $\langle u, e^{(-i\lambda+\rho)A(k^{-1}x)} \rangle$ agree as distributions.

Alternatively, this can be proven by noting that $\langle \widetilde{u}, \phi \rangle = (u_x \otimes \phi_{kM,\lambda})(e^{(-i\lambda+\rho)A(k^{-1}x)})$ regarding $u = u_x \in \mathcal{E}'(X)$, $\phi = \phi_{kM,\lambda} \in \mathcal{E}'(K/M \times \lambda)$, and $e^{(-i\lambda+\rho)A(k^{-1}x)} \in \mathcal{E}(K/M \times \mathfrak{a}^* \times X)$. By the Fubini theorem for distributions we have $(u_x \otimes \phi_{kM,\lambda})(e^{(-i\lambda+\rho)A(k^{-1}x)}) = (\phi_{kM,\lambda} \otimes u_x)(e^{(-i\lambda+\rho)A(k^{-1}x)})$. This implies formula (3.3.13).

$\square$

We let $\mathcal{J}'(X)$ denote the space of tempered distributions on K-invariant Schwartz functions. We shall simply call these functionals tempered distributions when there is no ambiguity. Since $\mathcal{S}_W(\mathfrak{a}^*)$ is closed under the involution ι we can define the spherical transform (Fourier transform) on all $\mathcal{J}'(X)$ as follows.

Definition 3.3.6. If $u \in \mathcal{J}'(X)$, then we define the spherical transform by $\widetilde{u}(\psi) = u(\mathscr{F}^{-1}(\iota \circ \psi)$ where $\psi \in \mathcal{S}_W(\mathfrak{a}^*)$. This determines a distribution in $\mathcal{S}'_W(\mathfrak{a}^*)$.

Correspondingly, the inverse Fourier transform of a distribution $v \in \mathcal{S}'_W(\mathfrak{a}^*)$ is determined by $\mathscr{F}^{-1}v(\phi) = v(\iota \circ (\mathscr{F}\phi))$ for $\phi \in \mathcal{J}(X)$. This determines a distribution in $\mathcal{J}'(X)$.

Since the maps $\iota \colon \mathcal{S}_W(\mathfrak{a}^*) \to \mathcal{S}_W(\mathfrak{a}^*)$ and $\mathscr{F} : \mathcal{J}(X) \to \mathcal{S}_W(\mathfrak{a}^*)$ are homeomorphisms we easily obtain the following result.

Proposition 3.3.6. *The spherical transform $\mathscr{F}$ is a homeomorphism of $\mathcal{J}'(X)$ onto $\mathcal{S}'_W(\mathfrak{a}^*)$ when these spaces are given the weak*-topology.*

The spherical transform of a compactly supported distribution u can be verified along the lines of Theorem 3.3.5 to be given by the function

$$\widetilde{u}(\lambda) = u(\phi_{-\lambda}(x)). \tag{3.3.14}$$

Now if $u \in \mathcal{E}'_K(X)$ is regarded as a distribution $u \in \mathcal{E}'(X)$, then the Fourier transform of u coincides with the spherical transform of u. Indeed first note the following lemma.

Lemma 3.3.7 ([7, Prop. 4] or [10, Sec. 5])**.** *If $f \in \mathcal{E}(X)$, then the mapping $k \mapsto \int_K f(k \cdot x)\, dk$ is a continuous mapping of K into $\mathcal{E}(X)$.*

Now using the K-invariance of u we have that by Theorem A.2.2 and the above lemma:

$$\widetilde{u}(kM, \lambda) = \int_K \widetilde{u}(kM, \lambda)\, dk = \langle u, \int_K e^{(-i\lambda+\rho)A(x,kM)}\, dk\rangle = \langle u, \phi_{-\lambda}(x)\rangle = \widetilde{u}(\lambda). \tag{3.3.15}$$

Remark 15. We note that the way we have defined the Fourier transform of distributions in $\mathcal{E}'(X)$ coincides with [13, Ch. III]. And although we have defined the spherical transform for all of $\mathcal{J}'(X)$ we have not defined the Fourier transform on all of $\mathcal{S}'(X)$. One can suggest a definition for a Fourier transform on all of $\mathcal{S}'(X)$ by understanding that the homeomorphism $\mathscr{F} \colon \mathcal{S}(X) \to \mathcal{S}_W(K/M \times \mathfrak{a}^*)$ gives rise to the transpose $\mathscr{F}^* \colon \mathcal{S}'_W(K/M \times \mathfrak{a}^*) \to \mathcal{S}'(X)$ which is a homeomorphism when both $\mathcal{S}'(X)$ and $\mathcal{S}'_W(K/M \times \mathfrak{a}^*)$ are given the weak*-topologies. Here $\mathscr{F}^* v = v \circ \mathscr{F}$, $v \in \mathcal{S}_W(K/M \times \mathfrak{a}^*)$.

Thus one could define the Fourier transform of a distribution $u \in \mathcal{S}(X)$ as $\widetilde{u} = (\mathscr{F}^*)^{-1}u = (\mathscr{F}^{-1})^* u$. However in this case one finds that the Fourier transform of a distribution $u \in \mathcal{E}'(X)$ is the function $\widetilde{u}(kM, \lambda) = \langle u, e^{(i\lambda+\rho)A(x,kM)}\rangle$ which disagrees with our original definition by $-\lambda$. Even worse, this definition does not respect the integral identity established in Proposition 3.3.4 when $u \in \mathcal{S}(X)$.

If $u \in \mathcal{E}'(X)$, then we can characterize the Fourier transform $\widetilde{u}(kM, \lambda)$ in the following way. We say a smooth function F on $K/M \times \mathfrak{a}^*_c$ is a *slowly increasing holomorphic function of uniform exponential type* $A \geq 0$ if F is holomorphic for $\lambda \in \mathfrak{a}^*_c$ and if for all $N \in \mathbf{N}$, we have

$$\sup_{(kM,\lambda) \in K/M \times \mathfrak{a}^*_c} |(1+|\lambda|)^{-N} e^{-A|\operatorname{Im}\lambda|} F(kM, \lambda)| < \infty. \tag{3.3.16}$$

We let $\mathscr{K}^A(K/M \times \mathfrak{a}^*_c)$ denote the space of slowly increasing holomorphic functions of uniform exponential type $A \geq 0$ and put $\mathscr{K}(K/M \times \mathfrak{a}^*_c) = \bigcup_{A\geq 0} \mathscr{K}^A(K/M \times \mathfrak{a}^*_c)$. We then let $\mathscr{K}_W(K/M \times \mathfrak{a}^*_c)$ denote the space of elements in $\mathscr{K}(K/M \times \mathfrak{a}^*_c)$ satisfying the symmetry condition of (3.2.19). We also let $\mathscr{K}_W(\mathfrak{a}^*_c)$ denote the subspace of functions in $\mathscr{K}_W(K/M \times \mathfrak{a}^*_c)$ which are W-invariant and constant in kM. Then we have the Paley-Wiener theorem for compactly supported distributions.

Theorem 3.3.8 ([13, Ch. III, Cor. 5.9])**.** *The Fourier transform $\mathscr{F} : \mathcal{E}'(X) \to \mathscr{K}_W(K/M \times \mathfrak{a}^*_c)$ is a bijection. In particular, if $B_A(o)$ is the open ball centered at o with radius $A > 0$, then $\operatorname{supp} u \subset \overline{B_A(o)}$ if and only if $\widetilde{u} \in \mathscr{K}^A(K/M \times \mathfrak{a}^*_c)$.*

One also has that the spherical transform is a bijection of $\mathcal{E}'_K(X)$ onto $\mathscr{K}_W(\mathfrak{a}^*_c)$.

In analogy to the result that if $u \in \mathcal{E}'(X)$, then $\operatorname{supp} u \subset \overline{B_A(o)}$ if and only if $\widetilde{u} \in \mathscr{K}^A(K/M \times \mathfrak{a}_c^*)$, there is a similar statement which holds for the *singular support* of a compactly supported distribution under certain conditions (see [5]). We now close this section with the following theorem.

Theorem 3.3.9. *Let $T, S \in \mathcal{E}'(X)$ and suppose that S is K-invariant, then the following statements hold.*

1. *The Fourier transform of $T * S$ satisfies $(T * S)\tilde{}(kM, \lambda) = \widetilde{T}(kM, \lambda)\widetilde{S}(\lambda)$.*
2. *If $f \in \mathcal{D}_K(X)$, then $(S * f)\tilde{} = \widetilde{S}\widetilde{f}$. More generally if $f \in \mathcal{J}(X)$, then $(S * f)\tilde{} = \widetilde{S}\widetilde{f}$ and*

$$S * f(x) = \int_{\mathfrak{a}^*} \widetilde{f}(\lambda)\widetilde{S}(\lambda)\phi_\lambda(x)\, d\mu(\lambda)\,. \tag{3.3.17}$$

Proof. We first prove (1). We can extend a distribution U on X to a distribution U' on G by the formula

$$U'(f) = \langle U_{gK}, \int_K f(gk)\, dk \rangle,$$

where $f \in \mathcal{D}(G)$ (or $f \in \mathcal{E}(G)$)—the assignment $U \mapsto U'$ is injective. Now, we extend the convolution of distributions on X by defining $(T * S)(f) = T' * S'(f^q)$ where $f^q(g) = f \circ q(g) = f(g \cdot o)$. Here $q \colon G \to G/K$ is the projection map. In particular, we have

$$\begin{aligned}
(T * S)\tilde{}(kM, \lambda) &= \int_G \int_G T'(g)S'(h)e^{(-i\lambda+\rho)A(gh\cdot o, kM)}\, dg\, dh \\
&= \int_G \int_G T'(g)S'(h)e^{(-i\lambda+\rho)(A(h\cdot o, g^{-1}(kM)) + A(g\cdot o, kM))}\, dg\, dh \\
&= \int_G T'(g)S_x(e^{(-i\lambda+\rho)A(x, g^{-1}(kM))})e^{(-i\lambda+\rho)A(g\cdot o, kM)}\, dg \\
&= \widetilde{T}(kM, \lambda)\widetilde{S}(\lambda).
\end{aligned}$$

If we had considered the convolution $S * T$, then this defines a K-invariant distribution thus we can not have $(S * T)\tilde{} = \widetilde{S}\widetilde{T}$ as T is not K-invariant. We now prove (2). Since S and f are both compactly supported, their convolution $S * f$ is also a compactly supported smooth function. Let $B = (\operatorname{supp} S * f) \cup \operatorname{supp} f$ and for each $\lambda \in \mathfrak{a}^*$ define the distribution Φ_λ by

$$\Phi_\lambda(\psi) = \int_B \psi(x)\phi_{-\lambda}(x)\, dx\,.$$

Then the spherical transform of $S * f$ is given by

$$(S * f)\tilde{}(\lambda) = \int_X S * f(x)\phi_{-\lambda}(x)\, dx = \langle \Phi'_\lambda(g) \otimes S'(h), f(h^{-1}g \cdot o) \rangle.$$

Using Fubini's theorem for distributions on manifolds, we have

$$\langle \Phi'_\lambda(g) \otimes S'(h), f(h^{-1}g \cdot o) \rangle = \langle S'(h) \otimes \Phi'_\lambda(g), f(h^{-1}g \cdot o) \rangle.$$

Thus,

$$\langle \Phi'_\lambda(g), f(h^{-1}g \cdot o) \rangle = \int_G f(g \cdot o)\phi_{-\lambda}(hg \cdot o)\, dg = \phi_{-\lambda}(h \cdot o)\widetilde{f}(\lambda).$$

Thus, we have $\langle \Phi'_\lambda(g) \otimes S'(h), f(h^{-1}g \cdot o) \rangle = \langle S, \phi_{-\lambda} \rangle \widetilde{f}(\lambda) = \widetilde{S}(\lambda)\widetilde{f}(\lambda)$. Now if $f \in \mathcal{J}(X)$ and $S * f \in \mathcal{J}(X)$ we note that if $\{f_n\} \subset \mathcal{J}(X)$ is a sequence of compactly supported functions which converge to f (cf. [13, Ch.

III, Lemma 1.21]), then we have $S * f_n \to S * f$ pointwise and so by the inversion formula

$$S * f(x) = \lim_{n\to\infty} S * f_n(x) = \lim_{n\to\infty} \int_{\mathfrak{a}^*} \widetilde{f_n}(\lambda)\widetilde{S}(\lambda)\phi_\lambda(x)\, d\mu(\lambda) = \int_{\mathfrak{a}^*} \lim_{n\to\infty} \widetilde{f_n}(\lambda)\widetilde{S}(\lambda)\phi_\lambda(x)\, d\mu(\lambda)$$
$$= \int_{\mathfrak{a}^*} \widetilde{f}(\lambda)\widetilde{S}(\lambda)\phi_\lambda(x)\, d\mu(\lambda)\,.$$

The fact that we could commute the limit with the integral follows from the dominated convergence theorem in which we used the slow growth of $\widetilde{S}$ and the fact that $\widetilde{f_n} \to \widetilde{f}$ in the topology of $\mathcal{S}_W(\mathfrak{a}^*)$. □

3.4 Towards pseudo-differential operators

We shall now attempt to mimic a theory of "pseudo-differential operators" on the homogeneous space $X = G/K$, reminiscent to the theory of pseudo-differential operators on $\mathbf{R}^n$, using the Fourier transform. Unfortunately, due to some technical limitations this theory does not appear to yield a symbol calculus, although we investigate some alternatives.

3.4.1 Pseudo-differential operators on $\mathbf{R}^n$

If D is a constant coefficient differential operator and $f \in \mathcal{S}(\mathbf{R}^n)$, then using the Fourier inversion formula one can represent the function $Df(x)$ as the Fourier integral:

$$Df(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \widehat{f}(\xi) D(i\xi) e^{i\langle x,\xi\rangle}\, d\xi\,. \tag{3.4.1}$$

Here $D(i\xi)$ is defined as in Definition 3.1.5 where we identify $i\xi$ with the linear functional $x \mapsto i\langle x, \xi\rangle$. Many other types of operators can be represented as Fourier integrals as in (3.4.1) such as translation operators or convolution operators which are types of Fourier multipliers. We can generalize this a bit further by realizing that for a nicely behaved smooth function $a(x,\xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ (say of compact support or of at most polynomial growth), then the operator $Af(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} \widehat{f}(\xi) a(x,\xi) e^{i\langle x,\xi\rangle}\, d\xi$ is continuous from $\mathcal{S}(\mathbf{R}^n)$ to itself. We shall call a the *symbol* of A. We restrict our focus to those operators whose symbols belong to the so-called symbol classes $S^m(\mathbf{R}^n \times \mathbf{R}^n)$.

Definition 3.4.1. Let $m \in \mathbf{R}^n$. We define the symbol class $S^m(\mathbf{R}^n \times \mathbf{R}^n)$ to be the space of all smooth functions a which satisfies the estimate

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le C_{\alpha,\beta}(1 + |\xi|)^m \tag{3.4.2}$$

for some finite constant $C_{\alpha,\beta} > 0$ and for all multi-indices $\alpha, \beta \in \mathbf{N}^n$. If $a \in S^m(\mathbf{R}^n \times \mathbf{R}^n)$, then we associate to it an operator, denoted by $\mathrm{Op}(a)$ (and also commonly denoted by $a(x, D)$), by

$$\mathrm{Op}(a)f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \widehat{f}(\xi) a(x,\xi) e^{i\langle x,\xi\rangle}\, d\xi, \tag{3.4.3}$$

where $f \in \mathcal{S}(\mathbf{R}^n)$. As a shorthand we will write $S^m = S^m(\mathbf{R}^n \times \mathbf{R}^n)$.

Clearly, if $a \in S^m$, then $\mathrm{Op}(a)$ defines a continuous endomorphism of $\mathcal{S}(\mathbf{R}^n)$.

Example 3.4.1. If D is a constant coefficient differential operator and r is its order, then the symbol $D(i\xi)$ is a polynomial of order r. In particular, one can easily see that $D(i\xi) \in S^r(\mathbf{R}^n \times \mathbf{R}^n)$.

Example 3.4.2. If $g \in \mathcal{S}(\mathbf{R}^n)$, then the convolution operator $Gf(x) = f * g(x)$ has symbol $\widehat{g}(\xi)$. The symbol of the operator G is then an element of $S^m(\mathbf{R}^n \times \mathbf{R}^n)$ for all $m \in \mathbf{R}$.

We shall thus say that an operator $A\colon \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$ is a *pseudo-differential operator* if it is realized as $A = \mathrm{Op}(a)$ for some a belonging to one of the symbol classes. One can interpret pseudo-differential operators in the sense of distributions as follows. Let $a \in S^m$ and we formally write

$$\begin{aligned}\mathrm{Op}(a)f(x) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\left\{\int_{\mathbf{R}^n} f(y)e^{-i\langle y,\xi\rangle}\,dy\right\}a(x,\xi)e^{i\langle x,\xi\rangle}\,d\xi &= \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\int_{\mathbf{R}^n} f(y)a(x,\xi)e^{i\langle x-y,\xi\rangle}\,dy\,d\xi \\ &= f * \check{a}_x(x).\end{aligned}$$

Here $\check{a}_z(x)$ is the object $(2\pi)^{-n}\int_{\mathbf{R}^n} a(z,\xi)e^{i\langle x,\xi\rangle}\,d\xi$ which we interpret as the inverse Fourier transform of the symbol regarded as a distribution $a(z,\xi) = a_z(\xi) \in \mathcal{S}'(\mathbf{R}^n)$. Thus, interpreting $f * \check{a}_x(x)$ in the distributional sense leads us to

$$\begin{aligned}f * \check{a}_x(x) = \langle(\check{a}_x)(y), f(x-y)\rangle &= \left\langle a_x(\xi), \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} f(x-y)e^{i\langle y,\xi\rangle}\,dy\right\rangle \\ &= \langle a_x(\xi), (2\pi)^{-n}e^{i\langle x,\xi\rangle}\widehat{f}(\xi)\rangle = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n}\widehat{f}(\xi)a(x,\xi)e^{i\langle x,\xi\rangle}\,d\xi \\ &= \mathrm{Op}(a)f(x).\end{aligned}$$

Hence, the operator $\mathrm{Op}(a)$ can be realized as a particular convolution along a family of distributions $\check{a}_x$ parametrized by $x \in \mathbf{R}^n$. In fact, all that we have done here is write $\mathrm{Op}(a)$ in terms of its Schwartz kernel $K_a(x,y) \in \mathcal{S}'(\mathbf{R}^n \times \mathbf{R}^n)$ where

$$\mathrm{Op}(a)f(x) = \int_{\mathbf{R}^n} K_a(x,y)f(y)\,dy = \int_{\mathbf{R}^n}\check{a}_x(x-y)f(y)\,dy, \tag{3.4.4}$$

so that $K_a(x,y) = \check{a}_x(x-y)$. Moreover, the Fourier transform of $\check{a}_x$ is trivially equal to $a(x,\xi)$ in the sense of distributions. This realization of a pseudo-differential operator as a particular convolution operator will become very important in Section 3.4.3.

Our primary object of study is to answer the question of what happens when one composes two pseudo-differential operators. One would like that given two pseudo-differential operators, that their composition is also a pseudo-differential operator. The answer to this question is "yes" and we shall outline the proof for the cases when the symbols of the pseudo-differential operators are of compact support.

Suppose that a and b are symbols of compact support and let f be a Schwartz function. We put

$$\mathrm{Op}(b)f(y) = \frac{1}{(2\pi)^n}\int \widehat{f}(\xi)b(y,\xi)e^{i\langle y,\xi\rangle}\,d\xi$$

and then calculate the Fourier transform of $\mathrm{Op}(b)f$ as

$$(\mathrm{Op}(b)f)\widehat{\ }(\omega) = \frac{1}{(2\pi)^n}\int\int \widehat{f}(\xi)b(y,\xi)e^{i\langle y,\xi-\omega\rangle}\,d\xi\,dy\,.$$

We note that the double integral above can be integrated in any order since b is compactly supported and f is a Schwartz function. Then we calculate $\mathrm{Op}(a)\,\mathrm{Op}(b)f(x)$ as

$$\begin{aligned}\mathrm{Op}(a)\,\mathrm{Op}(b)f(x) &= \frac{1}{(2\pi)^n}\int(\mathrm{Op}(b)f)\widehat{\ }(\omega)a(x,\omega)e^{i\langle x,\omega\rangle}\,d\omega \\ &= \frac{1}{(2\pi)^{2n}}\int\int\int \widehat{f}(\xi)b(y,\xi)a(x,\omega)e^{i\langle y,\xi-\omega\rangle}e^{i\langle x,\omega\rangle}\,d\xi\,dy\,d\omega\,.\end{aligned}$$

Then multiplying the integrand by $e^{i\langle x,\xi\rangle}e^{-i\langle x,\xi\rangle} = 1$ we have

$$\mathrm{Op}(a)\,\mathrm{Op}(b)f(x) = \frac{1}{(2\pi)^n}\int \widehat{f}(\xi)e^{i\langle x,\xi\rangle}\left\{\frac{1}{(2\pi)^n}\int\int b(y,\xi)a(x,\omega)e^{i\langle y-x,\xi-\omega\rangle}\,dy\,d\omega\right\}d\xi\,.$$

Again the interchange of all these integrals is justified by using Fubini's theorem and the fact that a and b are compactly supported. Hence the symbol of $\operatorname{Op}(a)\operatorname{Op}(b)$ is the smooth function

$$\begin{aligned} c(x,\xi) &= \frac{1}{(2\pi)^n}\int\int b(y,\xi)a(x,\omega)e^{i\langle y-x,\xi-\omega\rangle}\,dy\,d\omega \\ &= \frac{1}{(2\pi)^n}\int \widehat{b}(\omega,\xi)a(x,\omega+\xi)e^{i\langle x,\omega\rangle}\,d\omega\,. \end{aligned}$$

From here it is clear that c defines a symbol belonging to one of the symbol classes since a is of compact support and $|\partial_\xi^\alpha \widehat{b}(\omega,\xi)| < \infty$ for all differential operators ∂_ξ^α. So $\operatorname{Op}(c) = \operatorname{Op}(a)\operatorname{Op}(b)$. Now, one can expand $a(x,\omega+\xi)$ into the Taylor polynomial

$$a(x,\omega+\xi) = \sum_{|\alpha|\le N}\frac{\omega^\alpha}{\alpha!}\partial_\xi^\alpha a(x,\xi) + r_N(x,\omega,\xi)$$

where $r_N(x,\omega,\xi)$ is the Taylor remainder term. Thus, we have

$$c(x,\xi) = \sum_{|\alpha|\le N}\frac{1}{(2\pi)^n\alpha!}\int \widehat{b}(\omega,\xi)\omega^\alpha\partial_\xi^\alpha a(x,\xi)e^{i\langle x,\omega\rangle}\,d\omega + R_N(x,\xi) \tag{3.4.5}$$

$$= \frac{1}{(2\pi)^n}\sum_{|\alpha|\le N}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi) + R_N(x,\xi), \tag{3.4.6}$$

here

$$R_N(x,\xi) = \frac{1}{(2\pi)^n}\int \widehat{b}(\omega,\xi)r_N(x,\omega,\xi)e^{i\langle x,\omega\rangle}\,d\omega\,.$$

Naïvely, this suggests that the symbol of the composition of $\operatorname{Op}(a)$ and $\operatorname{Op}(b)$ is the symbol c which has the asymptotic "formula"

$$c(x,\xi) \sim \frac{1}{(2\pi)^n}\sum_{\alpha}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi). \tag{3.4.7}$$

This observation holds analogously for symbols not necessarily of compact support by the following theorem.

Theorem 3.4.3 ([21, Ch. VI, Sec. 3]). *Let $a \in S^{m_1}$ and $b \in S^{m_2}$, then there is a symbol $c \in S^{m_1+m_2}$ such that* $\operatorname{Op}(c) = \operatorname{Op}(a)\operatorname{Op}(b)$. *Moreover, c is given by the formula*

$$c(x,\xi) = \sum_{|\alpha|<N}\frac{i^{-|\alpha|}}{\alpha!}\partial_\xi^\alpha a(x,\xi)\partial_x^\alpha b(x,\xi) \mod S^{m_1+m_2-N}, \tag{3.4.8}$$

for all $N \in \mathbf{N}$.

The composition formula of the theorem defines what is commonly referred to as the "symbol calculus." There are certainly many other types of operations on pseudo-differential operators one can study and [15,21,23] are very good references for further information. However, we shall restrict our attention to investigating the issue of a "composition formula" of (3.4.8) in the next two sections.

Taylor expansions of distributions

If $T \in \mathcal{D}'(\mathbf{R}^n)$ and $f \in \mathcal{D}(\mathbf{R}^n)$, then the modified convolution $T \times f$ defined by

$$T \times f(x) = \int_{\mathbf{R}^n} T(y)f(x+y)\,dy = \int_{\mathbf{R}^n} T(x+y)f(y)\,dy\,.$$

is a smooth function on $\mathbf{R}^n$. We can expand $f(x+y)$ into its Taylor polynomial to obtain

$$f(x+y) = \sum_{|\alpha|<N} \frac{1}{\alpha!} x^\alpha (\partial^\alpha f)(y) + f_N(x,y)$$

where $f_N(x,y)$ is the Taylor remainder. Then inserting the formula for the Taylor polynomial into the expression defining $T \times f$ we have

$$T \times f(x) = \sum_{|\alpha|<N} \frac{1}{\alpha!} x^\alpha \int_{\mathbf{R}^n} T(y) \partial^\alpha f(y)\, dy + \int_{\mathbf{R}^n} T(y) f_N(x,y)\, dy\,. \tag{3.4.9}$$

The assignment $f(y) \mapsto f_N(\cdot, y)$ is continuous on $\mathcal{D}(\mathbf{R}^n)$ and so this gives rise to a distribution

$$f \mapsto \int_{\mathbf{R}^n} T_N(x,y) f(y)\, dy = \int_{\mathbf{R}^n} T(y) f_N(x,y)\, dy,$$

which also defines a smooth function in x. Thus, we can interpret (3.4.9) as giving the Taylor formula of $T(x+y)$ by

$$T(x+y) = \sum_{|\alpha|<N} \frac{(-1)^{|\alpha|}}{\alpha!} x^\alpha \partial^\alpha T(y) + T_N(x,y), \tag{3.4.10}$$

where $T_N(x,y)$ is a distribution that plays the role of the Taylor remainder in the Taylor expansion of $T(x+y)$. More generally, if $A\colon \mathcal{D}(\mathbf{R}^n) \to \mathcal{D}(\mathbf{R}^n)$ is a continuous operator, then by the Schwartz kernel theorem there is a distribution $K \in \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ such that for $f, g \in \mathcal{D}(\mathbf{R}^n)$ we have $\langle Af(x), g(x)\rangle = \langle K(x,y), (f \otimes g)(x,y)\rangle$ where $(f \otimes g)(x,y) = f(y)g(x)$. Informally we can write

$$Af(x) = \int_{\mathbf{R}^n} K(x,y) f(y)\, dy\,. \tag{3.4.11}$$

If D is a differential operator with constant coefficients and D^* is its adjoint, then

$$\langle DAf(x), g(x)\rangle = \langle Af(x), D^* g(x)\rangle = \langle K(x,y), D^*_x (f \otimes g)(x,y)\rangle = \langle D_x K(x,y), (f \otimes g)(x,y)\rangle.$$

Thus, we are justified in writing

$$DAf(x) = \int_{\mathbf{R}^n} D_x K(x,y) f(y)\, dy\,.$$

Using this and the preceding discussion we can expand $K(x+z,y)$ into its Taylor polynomial

$$K(x+z,y) = \sum_{|\alpha|<N} \frac{1}{\alpha!} z^\alpha \partial_x^\alpha K(x,y) + K_N(x,z,y). \tag{3.4.12}$$

Here, the distribution $K_N(x,z,y) \in \mathcal{D}'(\mathbf{R}^n \times \mathbf{R}^n)$ is defined by

$$\int_{\mathbf{R}^n \times \mathbf{R}^n} K_N(x,z,y) f(y) g(x)\, dy\, dx = \langle (Af)_N(z,x), g(x)\rangle.$$

One can also consider Taylor expansions of tempered or of compactly supported distributions (or of Schwartz kernels). Also one can consider Taylor polynomials of Schwartz kernels of continuous operators $A\colon F \to F'$ where F is $\mathcal{D}(\mathbf{R}^n)$, $\mathcal{E}(\mathbf{R}^n)$, or $\mathcal{S}(\mathbf{R}^n)$. We leave it to the reader to work out the details.

3.4.2 As symbols in the classes $S^m(K/M \times \mathfrak{a}^* \times X)$ and $S^m_K(\mathfrak{a}^* \times X)$

We shall now attempt to investigate analogues of pseudo-differential operators on the homogeneous space $X = G/K$ via the Fourier transform. We shall follow two approaches, each with varying success. Namely, we shall use the method of studying pseudo-differential operators as particular convolution operators in Section 3.4.3 as well as pseudo-differential operators defined in terms of symbols in this section.

Mimicking the formula defining a pseudo-differential operator on $\mathbf{R}^n$, one would expect a pseudo-differential operator on $\mathcal{S}(X)$ to be a continuous endomorphism of $\mathcal{S}(X)$ of the form

$$Af(x) = \int_{K/M \times \mathfrak{a}^*} \widetilde{f}(kM, \lambda) a(kM, \lambda, x) e^{(i\lambda+\rho)A(x,kM)}\, d\mu(kM, \lambda)\,. \tag{3.4.13}$$

where $(kM, \lambda, x) \mapsto a(kM, \lambda, x)$ is a nicely behaved smooth function. In this case one would call a the *symbol* of A. Also, if $f \in \mathcal{J}(X)$, one would like to consider symbols that are invariant in kM and K-invariant in x so that we can obtain operators $B\colon \mathcal{J}(X) \to \mathcal{J}(X)$ such that

$$Bf(x) = \int_{\mathfrak{a}^*} \widetilde{f}(\lambda) b(\lambda, x) \phi_\lambda(x)\, d\mu(\lambda)\,. \tag{3.4.14}$$

We define the symbol classes as follows.

Definition 3.4.2. Let $m \in \mathbf{R}$, we define the symbol class $S^m(K/M \times \mathfrak{a}^* \times X)$ as the space of smooth functions a on $K/M \times \mathfrak{a}^* \times X$ such that for each $p \in U(\mathfrak{k})$, $v \in S(\mathfrak{a}^*)$, $D_1, D_2 \in U(\mathfrak{g})$ we have

$$|\partial_r(p)\partial_l(D_1)\partial_r(D_2)\partial(v)a(kM, \lambda, x)| \le C(1 + |\lambda|)^m \tag{3.4.15}$$

where C is some finite constant, perhaps depending on the elements p, v, D_1, and D_2. Furthermore, we impose that $\int_K a(kM, \lambda, x) e^{(i\lambda+\rho)A(y,kM)}\, dk = \int_K a(kM, s\lambda, x) e^{(is\lambda+\rho)A(y,kM)}\, dk$ for all $y \in X$ and $s \in W$. We also define the symbol class $S^m_K(\mathfrak{a}^* \times X)$ as the space of all smooth functions on $\mathfrak{a}^* \times X$ that are K-invariant which satisfy the estimates

$$|\partial_l(D_1)\partial_r(D_2)\partial(v)a(kM, \lambda, x)| \le C(1 + |\lambda|)^m$$

where as usual C is some constant and $v \in S(\mathfrak{a}^*)$ and $D_1, D_2 \in U(\mathfrak{g})$. Furthermore, we impose that the elements of $S^m_K(\mathfrak{a}^* \times X)$ are W-invariant in λ.

If $a \in S^m(K/M \times \mathfrak{a}^* \times X)$, then we define the operator $\mathrm{Op}(a)$ acting on $\mathcal{S}(X)$ by formula (3.4.13) where we take $\mathrm{Op}(a) = A$. Similarly if $b \in S^m_K(\mathfrak{a}^* \times X)$, then $\mathrm{Op}(b)$ is defined by formula (3.4.14) where of course we take $\mathrm{Op}(b) = B$.

Theorem 3.4.4. *If $b \in S^m_K(\mathfrak{a}^* \times X)$, then $\mathrm{Op}(b)$ is a continuous endomorphism of the Schwartz space of K-invariant functions $\mathcal{J}(X)$.*

Proof sketch. If $b \in S^m_K(\mathfrak{a}^* \times X)$ and $f \in \mathcal{J}(X)$, then if $D \in \mathbf{D}(G)$ we obtain

$$D\,\mathrm{Op}(b)f(x) = \int_{\mathfrak{a}^*} \widetilde{f}(\lambda) D(b(\lambda, x)\phi_\lambda(x))\, d\mu(\lambda)\,.$$

Using the product rule we write $D(b(\lambda, x)\phi_\lambda(x)) = \sum_{j=1}^n D_j a(\lambda, x) E_j \phi_\lambda(x)$ where $D_j, E_j \in \mathbf{D}(G)$ are suitable differential operators. Now for all j we can estimate $E_j \phi_\lambda(x)$ by

$$|E_j \phi_\lambda(x)| \le c_0(1 + |\lambda|)^s \phi_0(x)$$

where s and c_0 are some positive constants (cf. [13, Ch. III, Lemma 1.18]). Given $N \in \mathbf{N}$, there is some suitable constant s'

$$|(1 + |g|)^N \Xi(g)^{-1} D\,\mathrm{Op}(b)f(g \cdot o)| \le c_0(1 + |g|)^N \int_{\mathfrak{a}^*} |\widetilde{f}(\lambda)|(1 + |\lambda|)^{s'}\, d\mu(\lambda)\,. \tag{3.4.16}$$

Now following the argument of [13, Ch. III, Lemma 1.20] we can show that the integral of (3.4.16) is dominated (in g) by $c_1\sigma(f)$ where $c_1 > 0$ is a constant and σ is a seminorm on $\mathcal{J}(X)$. Thus, we see that $\mathrm{Op}(b)f$ is a Schwartz function that is K-invariant and defines a continuous endomorphism of $\mathcal{J}(X)$. □

We expect that if $a \in S^m(K/M \times \mathfrak{a}^* \times X)$, then $\mathrm{Op}(a)$ defines a continuous endomorphism of $\mathcal{S}(X)$. However to prove this one needs to understand the details of proving the continuity of the inverse Fourier transform on $\mathcal{S}_W(K/M \times \mathfrak{a}^*)$ into $\mathcal{S}(X)$—the details of which can be found [6]. This is hard (for the author at least) and we are not so confident on the details themselves to warrant stating it as a theorem outright.

Example 3.4.5. If $D \in \mathbf{D}(X)$, then it has the symbol $\xi(\lambda) = D(i\lambda)$ and $\xi \in S^m(K/M \times \mathfrak{a}^* \times X)$ where m is the order of D.

Example 3.4.6. If $\psi \in \mathcal{J}(X)$ and $f \in \mathcal{S}(X)$, then the convolution $f * \psi(g \cdot o)$ exists. Indeed, crude estimates show that

$$\int_G |f(h \cdot o)||\psi(h^{-1}g \cdot o)|\, dh \le c_0 \int_G (1 + |h|)^{-N} \Xi(h)^2\, dh$$

which is known to converge for N large enough (cf. [13, pp. 214-215]). Note that the existence of $f * \psi$ does not require ψ to be K-invariant. One can then consider the Fourier transform $(f * \psi)\tilde{}$ and verify that one is permitted to use Fubini's theorem in the integral defining the Fourier transform of $f * \psi$ to obtain $(f * \psi)\tilde{} = \widetilde{f}\widetilde{\psi}$. Thus, $(f * \psi)\tilde{} \in \mathcal{S}_W(K/M \times \mathfrak{a}^*)$ and so by Fourier inversion we see that $f * \psi \in \mathcal{S}(X)$ and the symbol of the convolution operator $\Psi(f) = f * \psi$ is $\widetilde{\psi}$. This symbol belongs to $S^m(K/M \times \mathfrak{a}^* \times X)$ for all $m \in \mathbf{R}$.

If $a \in S^m_K(\mathfrak{a}^* \times X)$, then for fixed x the function $\lambda \to a(x, \lambda)$, which we denote by $a_x(\lambda)$, defines a tempered distribution in $\mathcal{S}'(\mathfrak{a}^*)$. We can represent the associated operator $\mathrm{Op}(a)\colon \mathcal{J}(X) \to \mathcal{J}(X)$ as the convolution operator $f \mapsto f * (\mathscr{F}^{-1} a_x)(x)$. Indeed we have

$$\begin{aligned}\langle (\mathscr{F}^{-1}a_x)(y), f(xy^{-1})\rangle_X = \langle a_x, \iota \circ (\mathscr{F}_y f(xy^{-1})\rangle_{\mathfrak{a}^*} &= \langle a_x, f * \phi_\lambda(x)\rangle_{\mathfrak{a}^*} \\ &= \int_{\mathfrak{a}^*} \widetilde{f}(\lambda) a(x,\lambda)\phi_\lambda(x)\, d\mu(\lambda)\,.\end{aligned}$$

So that $\mathrm{Op}(a)f(x) = f * (\mathscr{F}^{-1}a_x)(x)$—this observation will be explored further in the next section. Now, one would like to obtain a composition formula for operators defined from their symbols from $S^m_K(\mathfrak{a}^* \times X)$ or $S^m(K/M \times \mathfrak{a}^* \times X)$. If we try to mimic our approach of Section 3.4.1, we can see that we encounter some difficulties. Specifically, suppose that $a, b \in S^m_K(\mathfrak{a}^* \times X)$ are compactly supported. Then we wish to study the composition $\mathrm{Op}(a)\,\mathrm{Op}(b)$ on $\mathcal{J}(X)$. So we calculate

$$(\mathrm{Op}(b)f)\tilde{}(\omega) = \int_X \int_{\mathfrak{a}^*} \widetilde{f}(\lambda) b(\lambda, y)\phi_\lambda(y)\phi_{-\omega}(y)\, d\mu(\lambda)\, dy\,.$$

Then,

$$\begin{aligned}\mathrm{Op}(a)\,\mathrm{Op}(b)f(x) &= \int_{\mathfrak{a}^*}\int_X\int_{\mathfrak{a}^*} \widetilde{f}(\lambda)b(\lambda,y)a(\omega,x)\phi_\lambda(y)\phi_{-\omega}(y)\phi_\omega(x)\, d\mu(\lambda)\, dy\, d\mu(\omega) \\ = \int_{\mathfrak{a}^*} \widetilde{f}(\lambda)\phi_\lambda(x) &\left\{\int_X\int_{\mathfrak{a}^*} b(\lambda,y)a(\omega,x)\phi_\lambda(y)\phi_{-\omega}(y)\phi_\omega(x)\phi_\lambda(x)^{-1}\, dy\, d\mu(\omega)\right\} d\mu(\lambda)\,.\end{aligned}$$

The interchange of all these integrals is valid due to the property that a and b are of compact support. Now, arguably we have that the symbol of $\mathrm{Op}(a)\,\mathrm{Op}(b)$ is given by

$$c(\lambda, x) = \int_X\int_{\mathfrak{a}^*} b(\lambda,y)a(\omega,x)\phi_\lambda(y)\phi_{-\omega}(y)\phi_\omega(x)\phi_\lambda(x)^{-1}\, dy\, d\mu(\omega)\,.$$

So clearly $c(\lambda, x)$ is a smooth function on $\mathfrak{a}^* \times X$. What is not clear is how a composition formula can be extracted from c such as in Theorem 3.4.8. We can simplify our formula defining c by using that for $g, h \in G$

$$\begin{aligned}\int_K \phi_\omega(g^{-1}kh \cdot o)\, dk = \phi_\omega(g^{-1} \cdot o)\phi_\omega(h \cdot o), &\quad \text{and} \\ \phi_{-\omega}(g \cdot o) = \phi_\omega(g^{-1} \cdot o).&\end{aligned}$$

Then we have

$$c(\lambda, h\cdot o) = \int_K \int_G \int_{\mathfrak{a}^*} b(\lambda, g\cdot o)a(\omega, h\cdot o)\phi_\omega(g^{-1}kh\cdot o)\phi_\omega(h\cdot o)\phi_\lambda(h\cdot o)^{-1}\,dg\,d\mu(\omega)$$
$$= \int_G \int_{\mathfrak{a}^*} b(\lambda, g\cdot o)a(\omega, h\cdot o)\phi_\omega(g^{-1}h\cdot o)\phi_\omega(h\cdot o)\phi_\lambda(h\cdot o)^{-1}\,dg\,d\mu(\omega)\,.$$

This takes care of the factor $\phi_\lambda(y)\phi_{-\omega}(y)$. However the factor involving $\phi_\omega(x)\phi_\lambda(x)^{-1}$ does not appear to be easily handled. One issue is that $\phi_\lambda(x)^{-1}$ is no longer a spherical function. Moreover, there is no useful formula describing the product of two spherical functions $\phi_\omega\phi_\lambda$ where $\lambda \neq \omega$. Thus the techniques used in Section 3.4.1 seem to be inapplicable to our situation. Perhaps a more serious problem is the fact that there is no intrinsic notion of a Taylor expansion on X (or G). In the theory for studying the composition of pseudo-differential operators on $\mathbf{R}^n$ (cf. [15, 21, 23]) or on a compact Lie group (cf. [20]) one makes heavy use of Taylor series expansions to obtain composition formulas of symbols arising from the composition of pseudo-differential operators. It is therefore quite unclear how one proceeds to develop a "symbol calculus" from the symbol classes we have given in Definition 3.4.2. So *if* there is a symbolic calculus using the symbol classes that we have defined, then new techniques are required.

In the next section we shall take an alternative approach of considering operators that arise as a convolution of functions in $\mathcal{J}(X)$ with a family of distributions. There we shall make the assumption that we *can* in fact perform Taylor expansions of smooth functions which we shall outline next.

3.4.3 As convolution operators with K-invariant kernels

A word on Taylor expansions

Let X be a smooth manifold (without boundary) and let $f \in C^\infty(X)$. We shall be interested in a type of Taylor expansion of f such that for a given point $x_0 \in X$ we have the approximation

$$f(x) = \sum_{|\alpha|\le n} \frac{1}{\alpha!}\gamma_\alpha(x)\partial^\alpha f(x_0) + r_n(x, x_0), \tag{3.4.17}$$

where ∂^α is a particular derivative of f, γ_α are smooth functions taking the role of the monomials $x \mapsto x^\alpha$ on $\mathbf{R}^n$, and r_n is a remainder term satisfying some estimates. This is achieved by using a system of local coordinates and then transporting back to the manifold to obtain the desired approximation. To this end, let (U, ϕ) be a coordinate system about a point $x_0 \in U$. Then the representation function $\widehat{f} = f \circ \phi^{-1}$ defines a smooth function $\widehat{f}\colon \phi(U) \to \mathbf{C}$. Consequently, about any point y_0 in the domain of $\phi(U)$ we can perform a Taylor expansion

$$\widehat{f}(y) = \sum_{|\alpha|\le n} \frac{1}{\alpha!} y^\alpha \partial^\alpha \widehat{f}(y_0) + \widehat{r_n}(y, y_0). \tag{3.4.18}$$

Now since there is an isomorphism between $C^\infty(U)$ and $C^\infty(\phi(U))$ (given by the pullback of ϕ) we can define smooth functions $\gamma_\alpha\colon U \to \mathbf{R}$ such that $\widehat{\gamma_\alpha}(y) = y^\alpha$ on $\phi(U)$. Thus, we can transport terms of the expression (3.4.18) to smooth functions on U to obtain expression (3.4.17). This gives, at least locally, a Taylor expansion of a smooth function on each coordinate system. A bit of a drawback to this approach is that this Taylor formula depends on a choice of local coordinates in addition to the fact it is a local expression.

Obtaining Taylor expansions for which expressions such as (3.4.17) hold globally is an interesting proposition. In Ruzhansky and Turunen's text [20, Sec. 10.6] they show that Taylor expansions can be obtained for any compact Lie group. There, the approach used relies on the fact that any compact Lie group G is isomorphic to some subgroup of $\mathbf{U}(n)$ for some $n \in \mathbf{N}$. Using this identification one can embed G as a closed submanifold of $\mathbf{R}^{n\times n}$. Then using a particular open neighborhood $U \supset G$ in $\mathbf{R}^n$ one "extends" smooth functions on G to smooth functions on U where one can perform Taylor expansions on U and then restrict these expansions back to G. The end result is that one can express smooth functions in a Taylor polynomial as in (3.4.17) globally on the Lie group G, although the Taylor expansion that is obtained will depend on the embedding of G into $\mathbf{R}^{n\times n}$ as well as the coordinates used. Global Taylor expansions can also be obtained on compact manifolds without boundary (see [4, Ch. 3]) using techniques that generalize the approach of Ruzhansky and Turunen.

The question of global Taylor expansions on *noncompact* manifolds or Lie groups seems to be more nebulous—and we do not consider how one can obtain such as expansions in this thesis. However we will be forthright and say that in the next section that we assume that global Taylor expansions exist on noncompact semisimple Lie groups G.

Convolution calculus

Let G be a Lie group and let $T\colon \mathcal{E}(G) \to \mathcal{E}(G)$ be a linear continuous operator. By the Schwartz kernel theorem we can associate to T the Schwartz kernel $K_T \in \mathcal{E}'(G\times G)$. We can thus informally write

$$Tf(x) = \int_G K_T(x,y) f(y)\, dy$$

for precisely the same reasons as in (3.4.11). We define the *right-convolution kernel* of T to be the distribution $R_T \in \mathcal{E}'(G\times G)$ which is defined by the relation $K_T(x,y) = R_T(x, y^{-1}x)$ so that using the standard distributional interpretation

$$\int_G K_T(x,y) f(y)\, dy = \int_G R_T(x,y) f(xy^{-1})\, dx\,.$$

In particular by a change of variables we have $R_T(x,y) = K_T(x, xy^{-1})$.

Returning to our study of pseudo-differential operators on $\mathcal{J}(X)$ we shall restrict ourselves to a family of operators $\mathfrak{A}$, where for each $A \in \mathfrak{A}$ the operator $A\colon \mathcal{J}(G) \to \mathcal{J}(G)$ is such that the left convolution kernel of A, satisfies $R_A \in \mathcal{E}'_K(G\times G)$ where we lift R_A from $\mathcal{E}'_K(X\times X)$ to $\mathcal{E}'_K(G\times G)$ as in Theorem 3.3.1. That is the kernel $R_A(x,y)$ is assumed to be K-bi-invariant in the variables $x, y \in G$. In our work that ensues we closely mirror the techniques used in [20, 24].

Since for each y, the distribution $R_A(x,y)$ is K-bi-invariant we shall define the *symbol* of the operator A by

$$\sigma_A(x,\lambda) = \int_G R_A(x,y)\phi_{-\lambda}(y)\, dy \tag{3.4.19}$$

where of course ϕ_λ is the spherical function and this has a distributional interpretation. For any concerns arising from our manipulation of distributions we refer the reader to [20, Ch. 10].

Theorem 3.4.7 (Quantization of operators)**.** *If $A \in \mathfrak{A}$, let σ_A denote its symbol. Then*

$$Af(x) = \int_{\mathfrak{a}^*} \widetilde{f}(\lambda)\sigma_A(x,\lambda)\phi_\lambda(x)\, d\mu(\lambda), \tag{3.4.20}$$

for all $f \in \mathcal{J}(G)$ and $x \in G$.

Proof. First suppose that $f \in \mathcal{D}_K(X)$ and fix $x_0 \in G$. Then write $A_{x_0}f(x) = f * R_A^{x_0}(x)$ where $R_A^{x_0}$ is the distribution defined by

$$f * R_A^{x_0}(x) = \int_G R_A(x_0, y) f(xy^{-1})\, dy\,.$$

Now adapting the proof of Theorem 3.3.1 we find that $(f * R_A^{x_0})\widetilde{\ }(\lambda) = \sigma_A(x_0,\lambda)\widetilde{f}(\lambda)$. Thus as for each fixed x we have $Af(x) = A_x f(x)$ so that we have obtained formula (3.4.20) for functions $f \in \mathcal{D}_K(X)$. If $f \in \mathcal{J}(G)$

and $\{f_n\}_{n\in\mathbf{N}} \subset \mathcal{D}_K(X)$ is a sequence of functions such that $f_n \to f$ in the topology of $\mathcal{J}(G)$ (cf. [13, Ch. III, Lemma 1.21]) we have by the continuity of A and the Fourier transform so that pointwise

$$Af(x) = \lim_{n\to\infty} Af_n(x) = \lim_{n\to\infty} \int_{\mathfrak{a}^*} \widetilde{f_n}(\lambda)\sigma_A(x,\lambda)\phi_\lambda(x)\,d\mu(\lambda) = \int_G \lim_{n\to\infty} \widetilde{f_n}(\lambda)\sigma_A(x,\lambda)\phi_\lambda(x)\,d\mu(x) \quad (3.4.21)$$

$$= \int_{\mathfrak{a}^*} \widetilde{f}(\lambda)\sigma_A(x,\lambda)\phi_\lambda(x)\,d\mu(\lambda)\,. \quad (3.4.22)$$

So we are done. □

The proof given above is essentially the same as the proof contained in [20, Thm. 10.44]. Thus by formula (3.4.20) we are at least legitimate to think that these also realize a class of “pseudo-differential operators” in analogy to the Euclidean case.

Example 3.4.8. If $g \in \mathcal{D}_K(X)$, then the operator $G : f \mapsto f * g$ is an element of $\mathfrak{A}$. The symbol of G is of course $\sigma_G(x,\lambda) = \widetilde{g}(\lambda)$ since in particular $R_G(x,y) = g(y)$.

Example 3.4.9. We have that $\mathbf{D}(X) \subset \mathfrak{A}$. Indeed, these operators can be expressed as derivatives of the Dirac delta distribution δ with point mass at the origin o by $Df = (Df) * \delta = f * (D\delta)$ (cf. [11, Ch. II, Sec. 5]). So in particular, the right convolution kernel of D is $R_D(x,y) = D\delta(y)$. The symbol of D is then

$$\sigma_D(x,\lambda) = \int_G \delta(y)D^*\phi_{-\lambda}(y)\,dy = \Gamma(D)(i\lambda).$$

Combining this fact and using Theorem 3.4.7 we have the expression

$$Df(x) = \int_G \widetilde{f}(\lambda)\Gamma(D)(i\lambda)\phi_\lambda(x)\,dx, \quad (3.4.23)$$

which would have been obtained ordinarily by commuting the operator D with the Fourier integral.

To understand the symbol of a composition of operators $A, B \in \mathfrak{A}$ we first obviously should verify that $AB \in \mathfrak{A}$. This is afforded by the following lemma.

Lemma 3.4.10. *The space of operators $\mathfrak{A}$ is closed under composition and thus forms an algebra.*

Proof. If $A, B \in \mathfrak{A}$, this amounts to showing that the right convolution kernel of $R_{AB}(x,y)$ of AB also lies in $\mathcal{E}'_K(G \times G)$ in which it is only necessary to verify that $R_{AB}(x,y)$ is K-bi-invariant. We can calculate the convolution kernel explicitly as follows:

$$ABf(x) = \int_G R_{AB}(x,y)f(xy^{-1})\,dy = \int_G R_A(x,y)Bf(xy^{-1})\,dy$$
$$= \int_G \left\{\int_G R_A(x,y^{-1})R_B(xy,zy)\,dy\right\} f(xz^{-1})\,dz\,.$$

Thus the right convolution kernel of AB is given by (in the distributional sense)

$$R_{AB}(x,z) = \int_G R_A(x,y^{-1})R_B(xy,zy)\,dy\,. \quad (3.4.24)$$

In particular, this expression of the kernel gives us that the kernel R_{AB} is also K-bi-invariant in each variable as a distribution restricted to $\mathcal{J}(X)$. □

Turning our attention to the question of the symbol of the composition we shall assume that there is a given *global* Taylor expansion with remainder valid for a smooth function on $C^\infty(G)$. In this case, we shall assume that for a function $f \in C^\infty(G)$ that for each point $x \in G$ it admits a Taylor expansion about a point x as

$$f(y) = \sum_{|\alpha| \le N} \frac{1}{\alpha!} \gamma_\alpha(x^{-1}y) \partial^\alpha f(x) + r_N(x,y), \tag{3.4.25}$$

for all $x, y \in G$, where the objects ∂^α correspond to certain differential operators on G, and the functions $\gamma_\alpha \colon G \to \mathbf{R}$ are smooth. We shall put $\zeta_\alpha(y) = (\alpha!)^{-1}\gamma_\alpha(y^{-1})$. Using formula (3.4.24) we compute the symbol $\sigma_{AB}(x,\lambda)$ as follows. The idea is to use the Taylor expansion to expand the convolution kernel R_{AB} in such a way so that

$$ABf(x) = \sum_{1 \le i \le n} f * (a_i^x * b_i^x)(x) + f * (\mathcal{R}_n)^x(x) = f * (R_{AB}^x)(x). \tag{3.4.26}$$

Then in view of Theorem 3.3.1 the symbol of AB is of the form

$$\sigma_{AB}(x,\lambda) = \sum_{1 \le i \le} \sigma_{b_i}(x,\lambda)\sigma_{a_i}(x,\lambda) + \sigma_{\mathcal{R}_n}(x,\lambda).$$

In this way, $\mathcal{R}_n$ should be a "remainder term" while the distributions a_i^x and b_i^x should be related to the right convolution kernels $R_A(x,y)$ and $R_B(x,y)$, respectively, in some way. So first, we note that

$$ABf(x) = \int_G \int_G R_A(x, y^{-1}x) R_B(y, z^{-1}y) f(z)\, dz\, dy\,. \tag{3.4.27}$$

On the other hand, if we fix $R_A^x(y) = R_A(x,y)$ and $R_B^x(z) = R_B(x,z)$, then by definition of the convolution of distributions

$$\begin{aligned} f * (R_B^x * R_A^x)(x) &= \int_G \int_G R_A(x,y) R_B(x,z) f(xy^{-1}z^{-1})\, dz\, dy \\ &= \int_G \int_G R_A(x, y^{-1}x) R_B(x,z) f(yz^{-1})\, dz\, dy \\ &= \int_G \int_G R_A(x, y^{-1}x) R_B(x, z^{-1}y) f(z)\, dz\, dy\,. \end{aligned}$$

In particular, if we take the Taylor expansion of $R_B(y, z^{-1}y)$ with remainder as

$$R_B(y, z^{-1}y) = \sum_{|\alpha| \le N} \zeta_\alpha(y^{-1}x) \partial_x^\alpha R_B(x, z^{-1}y) + \mathscr{R}_N(x, y, z^{-1}y). \tag{3.4.28}$$

Confer to Section 3.4.1 for the analogous meaning of the Taylor expansion of a distribution; also see [24, Ch. I, No. 1.34] and [20, Thm. 10.7.8]. Now we can rewrite (3.4.27) as

$$ABf(x) = \sum_{|\alpha| \le N} f * (b_\alpha^x * a_x^\alpha)(x) + f * (\mathcal{R}_N^x)(x). \tag{3.4.29}$$

Here, b_α^s, a_s^α, and $\mathcal{R}_N^s$ are respectively the distributions which are determined by

$$u * b_\alpha^s(x) = \int_G \partial^\alpha R_B(s, z^{-1}x) u(z)\, dz, \tag{3.4.30}$$

$$u * a_x^\alpha(x) = \int_G R_A(x, y^{-1}x) \zeta_\alpha(y^{-1}s) u(y)\, dy, \tag{3.4.31}$$

$$u * (\mathcal{R}_N^s)(x) = \int_G \int_G R_A(x, y^{-1}x) \mathscr{R}_N(s, y, z^{-1}y) u(z)\, dz\, dy, \tag{3.4.32}$$

where u is a smooth function. Of course, the distributions b_α^x, a_x^α, and $\mathcal{R}_N^x$ all define operators A^α, B_α, and $\mathcal{R}'_N$ respectively in $\mathfrak{A}$ since the convolution operators R_A and R_B are K-bi-invariant in each variable. Hence, the symbol of AB is then given by

$$\sigma_{AB}(x,\lambda) = \sum_{|\alpha|\le N} \sigma_{A^\alpha}(x,\lambda)\sigma_{B_\alpha}(x,\lambda) + \sigma_{\mathcal{R}'_N}(x,\lambda).$$

To summarize our result, we have the "composition formula".

Theorem 3.4.11. *If $A, B \in \mathfrak{A}$, then the symbol of AB can be expanded as*

$$\sigma_{AB}(x,\lambda) = \sum_{|\alpha|\le N} \sigma_{A^\alpha}(x,\lambda)\sigma_{B_\alpha}(x,\lambda) + \sigma_{\mathcal{R}'_N}(x,\lambda). \tag{3.4.33}$$

Here, the operators A^α, B_α and $\mathcal{R}'_N$ are defined by the relations (3.4.30-3.4.32).

3.5 Remarks

As we have surveyed, the Fourier transform on the homogeneous space $X = G/K$ is certainly a good generalization of the classical Fourier transform on $\mathbf{R}^n$ or on topological groups. It is has the advantage of being a *scalar-valued* function, rather than an operator-valued function as are the Fourier transforms on nonabelian groups. This feature allows the Fourier transform on X to serve as a useful tool to study certain differential equations on X. For an insight into how this can be done see [13, Ch. V]. In particular, Helgason proves that for any $D \in \mathbf{D}(X)$, then if $f \in C^\infty(X)$ there exists $u \in C^\infty(X)$ such that $Du = f$ on all of X.

The pseudo-differential operator theory that we have developed closely follows the work of [20, 24]. Most of the content of Sections 3.4.2 and 3.4.3 was investigated by myself and Dr. Mitsuru Wilson. In particular, M. Wilson introduced me to the symbolic calculus discussed in [20]. There we realized that the "symbol calculus" works best when pseudo-differential operators are realized as convolution operators and one obtains a "convolution calculus". In [20], pseudo-differential operators on compact Lie groups are obtained from Fourier transforms of the right-convolution kernels of continuous linear operators. Indeed, if $A\colon C^\infty(G) \to C^\infty(G)$ is a continuous linear operator, then it admits a right convolution kernel $R_A \in \mathcal{E}'(G \times G)$ by the Schwartz kernel theorem. One then defines the *symbol* of A to be

$$\sigma_A(x,\xi) = \int_G R_A(x,y)\xi(y^{-1})\,dy$$

where $\xi \in \widehat{G}$. That is, the symbol of the operator A is the Fourier transform of the distribution $R_A(x,y)$ "frozen" at the variable x. In this case, the symbol $\sigma(x,\xi) \in \operatorname{End} V_\xi$ where V_ξ is the representation space of ξ. By using the Peter-Weyl theorem, one can show that the operator A can be expressed as

$$Af(x) = \sum_{\xi\in\widehat{G}} d_\xi \operatorname{Tr}\Big(\xi(x)\sigma_A(x,\xi)\widehat{f}(\xi)\Big),$$

which holds for all $x \in G$ and $f \in C^\infty(G)$ [20, Thm. 10.4.4] which is a result that is analogous to Theorem 3.4.7. In this way one thinks of any continuous linear endomorphism A of $C^\infty(G)$ as being a pseudo-differential operators via their symbols. Questions such as the composition formula are settled by appealing to Taylor expansions on G in precisely the same way that we have used Taylor expansions in the previous section.

In our case, we had to be more careful with the operators that we have considered for developing a "symbol calculus." In particular, one of the cruxes in the proof of the composition formula in [20, Thm. 10.7.8] is that when $k_1, k_2 \in \mathcal{E}'(G)$ are distributions, one has $(k_1 * k_2)\widehat{\ } = \widehat{k_2}\widehat{k_1}$ (note the reversed order). Then by means of the Taylor expansion one can write the composition of two pseudo-differential operators in the form of (3.4.26), and using that $(k_1 * k_2)\widehat{\ } = \widehat{k_2}\widehat{k_1}$ one obtains a composition formula similar to the one obtained in Theorem 3.4.11. Hence in following this approach we required that the Fourier transform of compactly supported distributions on $X = G/K$ satisfies $(T * S)^{\sim} = \widetilde{T}\widetilde{S}$, however by Theorem 3.3.9 this really only happens if both T and S are K-invariant. In addition to the fact that $(T * f)^{\sim} = \widetilde{T}\widetilde{f}$ when $f \in \mathcal{J}(X)$ and $T \in \mathcal{E}'_K(X)$, we are naturally led to consider operators $A\colon \mathcal{J}(X) \to \mathcal{J}(X)$ whose Schwartz kernels are elements of $\mathcal{E}'_K(X \times X)$. Thus the lack of a good convolution theorem for distributions of compact support that are not necessarily K-invariant, imposes a limitation on what kinds of "pseudo-differential operators" we can easily study—at least when we try to follow the techniques of [20, 24].

Appendix A

Topological groups and functional analysis

A.1 Topological groups and Homogeneous spaces

A topological group G is a group with a topology such that multiplication and inversion in G are continuous operations. By a locally compact group we mean a topological group that is locally compact and Hausdorff. A Lie group is a topological group (and a topological manifold) equipped with a smooth structure making multiplication and inversion in G smooth operations. Since all topological manifolds are locally compact and Hausdorff, every Lie group is locally compact.

A.1.1 Topological groups

An important property of locally compact groups and Lie groups is that they all possess an essentially unique left-invariant measure. In the case for arbitrary locally compact groups a left-invariant measure is a Radon measure μ with the property that for any Borel set $E \subset G$ we have $\mu(xE) = \mu(E)$ (a right-invariant measure is defined analogously). We have the following theorem for locally compact groups.

Theorem A.1.1. *Let G be a locally compact group. There is a left (resp. right) invariant Radon measure on G and any two left (resp. right) invariant measures are positive multiples of each other.*

We call such left (resp. right) invariant Radon measures a left (resp. right) *Haar measure*. We often work with left Haar measures and so we simply call them Haar measures. Fix a left Haar measure μ and an element $x \in G$. An important question is what we can say about the measure $\mu_x(E) = \mu(Ex)$. The measure μ_x is also a left Haar measure and so is equal to μ differing by a positive factor $\delta(x)$. So we can define the function $\delta\colon G \to \mathbf{R}^+$ where $x \mapsto \delta(x)$ which in view of Theorem A.1.1 is independent of our choice of Haar measure. The function δ is called the *modular function* of G and is a continuous group homomorphism from G into the multiplicative group $\mathbf{R}^+ = \{x \in \mathbf{R}\colon x > 0\}$. A group is said be *unimodular* if $\delta \equiv 1$, that is $\mu_x = \mu$ for all $x \in G$ and so the left Haar measure is also a right Haar measure. For instance, any compact group is unimodular since $\mathbf{R}^+$ has only one compact subgroup and it is the singleton $\{1\}$. In this case, for compact groups the *normalized Haar measure* is the unique Haar measure dx such that $\int_G dx = 1$. If G a topological group and H is a subgroup of G, then δ_G and δ_H denote the modular functions on G and H respectively (in general it is not true that $\delta_G|_H = \delta_H$).

Our primary focus will be on Lie groups. As Lie groups are topological manifolds, they are σ-compact and thus when equipped with a Haar measure (or any Radon measure), they become σ-finite measure spaces. These spaces are particularly important since several of the heavy lifting theorems of integration theory, such as Fubini's theorem, apply to such spaces.

The existence of a Haar measure and the modular function have importance for the integration theory that we will need. The σ-algebra of measurable sets on G will always be taken to be $\mathcal{B}_G$, the Borel sets of G. If dx is a Haar measure and $y \in G$ and f an integrable function, then we have the following results:

$$\int_G f(yx)\,dx = \int_G f(x)\,dx, \tag{A.1.1}$$

$$\int_G f(xy)\,dx = \int_G f(x)\delta(y^{-1})\,dx, . \tag{A.1.2}$$

$$\int_G f(x^{-1})\,dx = \int_G f(x)\delta(x^{-1})\,dx\,. \tag{A.1.3}$$

We remark that (A.1.2) hints at a proof of the continuity of the modular function for if $f \in C_c(G)$, the function $y \to f(xy)$ is continuous from G to $C_c(G)$ and so $y \mapsto \int_G f(xy)\,dx$ is continuous and since inversion is continuous we quickly establish the continuity of δ (cf. [9, pp. 38]). Equation (A.1.1) follows immediately from the G-invariance of the Haar measure and for (A.1.3) confer to [9, Prop. 2.31].

Since in this thesis we are interested in connected semisimple Lie groups, we provide the following useful result.

Proposition A.1.2. *If G is a connected semisimple Lie group, then it is unimodular.*

Proof. As G is semisimple and connected, then it is known that $G = [G, G]$. So in particular $G/[G, G]$ is a compact space. More generally, if G is a topological group with the property that $G/[G, G]$ is compact, then G is unimodular. To see this, let $D\colon G/[G, G] \to \mathbf{R}^\times$ be the map $D(x[G, G]) = \delta(x)$ which is well-defined since $\delta|_{[G,G]} = 1$ since $\mathbf{R}^\times$ is abelian. By the characteristic property of quotient maps, D is continuous and $\delta(G) = D(G/[G, G]) = 1$ since $D(G/[G, G])$ is compact. □

We close this section with an interesting result that determines when a Haar measure on a group decomposes into a product of Haar measures on its subgroups.

Proposition A.1.3. *Let G be a locally compact topological group and let $H, K \subset G$ be topological subgroups of G such that $G = KH$ and that K normalizes H, i.e. $kH = Hk$ for each $k \in K$. Additionally suppose that H and K are both second countable. Then if dk and dh are Haar measures on K and H respectively, then the Haar measure on G is the product measure $dh\,dk$.*

Proof. The second countability condition on H and K means that the product measure $dh\,dk$ is a Radon measure. Let dk and dh be as above and define the positive linear functional ϕ on $C_c(G)$ by

$$\phi(f) = \int_K \int_H f(kh)\,dh\,dk\,.$$

Now, let $g \in G$ and let $g = k_1h_1 = h_2k_2$ and observe

$$\begin{aligned}\phi(\tau_g f) = \int_K \int_H f(k_1h_1kh)\,dh\,dk &= \int_K \int_H f(h_2k_2kh)\,dh\,dk = \int_K \int_H f(h_2kh)\,dh\,dk \\ &= \int_K \int_H f(kh_3h)\,dh\,dk = \int_K \int_H f(kh)\,dh\,dk = \phi(f)\end{aligned}$$

where $h_2k = kh_3$ and $h_3 \in H$. By the Riesz representation theorem $dh\,dk$ defines a left invariant Haar measure on G. □

In the event that H and K are not second countable it may be the case that $dh\,dk$ does not completely determine the Haar measure on H and K. But rather, the product measure $dh\,dk$ and the Haar measure on G agree when integrating $C_c(G)$ functions.

A.1.2 Homogeneous spaces

For our purposes a *homogeneous space* X is a locally compact Hausdorff topological space equipped with a transitive, continuous action by a locally compact group G. If we fix a point $o \in X$, then X is the orbit of o under the action of G. If we let K be the stabilizer of o, then we can identify X with the coset space G/K by a continuous bijective map $\Phi\colon G/K \to X$ where $\Phi(gK) = g \cdot o$. The selection of the point o is arbitrary, but when such a choice has been made and the corresponding identifications of X with G/K have been fixed, we call o the *origin* of X. The topology on G/K is the quotient topology derived from the natural map $q\colon G \to G/K$.

When G is σ-compact the map Φ is in fact a homeomorphism and so this identification is topological. In fact if X is a smooth manifold, G a Lie group and the action of G on X is smooth and transitive, then the map Φ so defined is a diffeomorphism (cf. [18, Thm. 21.18]). In these situations there is no difference, topological or differentiable, in doing our analysis on the space X or the space G/K and so we make no distinction between the two spaces structurally speaking, but often keep some level of distinction in mind out of notational convenience.

Remark 16. In particular if $\phi\colon G \to H$ is a surjective continuous group homomorphism between topological groups, then it is not always true that $G/\ker\phi \cong H$ as *topological spaces*.

Now suppose that X possesses a G-invariant Radon measure dx. An interesting question is if whether there is a way to normalize the Haar measure dg on G in such a way so that

$$\int_X f(x)\,dx = \int_G f(g\cdot o)\,dg, \quad f \in C_c(X) \tag{A.1.4}$$

so that integration on X becomes integration on G which in certain situations is more convenient. If we assume for a minute that $X = G/K$, then by [9, Thm. 2.51] all the G-invariant Radon measures on G/K are positive multiples of each other. For $f \in C_c(G)$ define the average of f over the orbit gK by

$$f^\flat(gK) = \int_K f(gk)\,dk\,.$$

The assignment $\flat$ maps $C_c(G)$ onto $C_c(G/K)$ see [2, pp. 30] or [9, Prop. 2.50]. By the Riesz representation theorem, we can find a unique Haar measure on G which satisfies

$$\int_G f(g)\,dg = \int_{G/K} f^\flat(x)\,dx\,.$$

Furthermore if we assume that for $f \in C_c(G/K)$ that $f \circ q \in C_c(G)$ with $\int_K 1\,dk = 1$, then we have the desired relation in (A.1.4). A sufficient condition to ensure that when $f \in C_c(G/K)$ that $f \circ q \in C_c(G)$ is that K is compact since for $F \subset G/K$ compact there exists $E \subset G$ compact such that $q(E) = F$, and that the saturation of E is the set $q^{-1}(q(E)) = q^{-1}(F) = EK$ which is compact if K is compact. Even more bluntly, assuming $\int_K 1\,dk = 1$ automatically makes K compact. In the more general situation where we distinguish X and G/K it is natural to try and identify functions in $C_c(G/K)$ with functions in $C_c(X)$ and vice versa. The most logical identification is the map $\xi\colon f \mapsto f \circ \Phi$, $f \in C_c(X)$ which maps $C_c(X)$ into $C_c(G/K)$ when Φ is proper and this map is a bijection when Φ^{-1} is continuous, that is when Φ is a homeomorphism. Truly the most convenient setup is that X is a homeomorphic to G/K in which case there is a natural correspondence between their spaces of Radon measures and their functions of compact support. In summary we have the following result.

Proposition A.1.4. *Let X be a locally compact Hausdorff space that is a homogeneous space for the locally compact group G and identify X with the coset space G/K. Suppose that dx is a G-invariant Radon measure on X, then if K is compact and the map $\Phi\colon G/K \to X$ is a homeomorphism, then there exists a unique Haar measure dg on G such that*

$$\int_X f(x)\,dx = \int_G f(g\cdot o)\,dg, \quad (f \in C_c(X)).$$

Owing to the fact the measure dx in the proposition is a Radon measure we have that $C_c(X)$ is dense in $L^1(X)$ from which it follows the above proposition holds for functions of class $L^1(X)$ as well. The content of this is of the following theorem, the proof is straightforward.

Theorem A.1.5. *With the assumptions of the above proposition we have for $f \in L^1(X)$ that*

$$\int_X f(x)\,dx = \int_G f(g \cdot o)\,dg\,. \tag{A.1.5}$$

Proof. Let $f \in L^1(X)$ and let $f_n \in C_c(X)$ be such that $f_n \to f$ in L^1. Passing to a subsequence f_m that converges to f almost everywhere we have $\int_X f(x)\,dx = \lim \int_X f_m(x)\,dx = \lim \int_G f_m(g \cdot o)\,dg$. The functions $\{f_m \circ q\}$ are a Cauchy sequence in $L^1(G)$ and so converges in L^1 to a function $\phi \in L^1(G)$ (and we may assume this convergence is almost everywhere, cf. [8, Thm. 2.30]). Since this convergence is almost everywhere, we have $\phi = f \circ q$ almost everywhere and thus

$$\int_X f(x)\,dx = \lim \int_X f_m(x)\,dx = \lim \int_G f_m(g \cdot o)\,dg = \int_G \phi(g)\,dg = \int_G f(g \cdot o)\,dg$$

as needed. □

In particular if $d\mu = dx$ and $d\lambda = dg$, then if the above is satisfied we have that for any Borel set $E \subset X$ that $\mu(E) = \lambda(q^{-1}(E)) = \lambda\{g \in G \colon g \cdot o \in E\}$. Thus Theorem A.1.5 provides conditions for when the Haar measure on G can be induced from a G-invariant measure on X via the quotient map. The case when K is compact and dk is the normalized Haar measure on K, the function $f^\flat$ for $f \in C_c(G)$ is called the *orbital mean* of f. It is useful to note Theorem A.1.5 is often applicable to Lie groups since all topological manifolds are σ-compact and thus Φ is always a homeomorphism. From now on, we will assume that X is homeomorphic (or diffeomorphic when needed) to G/K and that K is compact and we will simply write $X = G/K$. For convenience, we will sometimes write $f^q = f \circ q$ when f is a complex valued function on X.

A.1.3 Convolutions

Let G be a locally compact group with a Haar measure dy. For complex valued functions f and g on G, their *convolution* is defined as

$$(f * g)(x) = \int_G f(y) g(y^{-1}x)\,dy\,. \tag{A.1.6}$$

Convolution is not commutative in general although it is commutative if and only if G is abelian, for instance it is commutative on $\mathbf{R}^n$ under addition. The support of $f * g$ is contained in the closure of $(\operatorname{supp} f)(\operatorname{supp} g)$ since if $x \notin (\operatorname{supp} f)(\operatorname{supp} g)$, then $y^{-1}x \notin \operatorname{supp} g$ for $y \in \operatorname{supp} f$. Hence if f and g are of compact support, then so is their convolution.

If $X = G/K$ is a homogeneous space and $o \in X$ is the origin, then the convolution for functions f and g on X is defined as

$$(f * g)(x) = \int_G f(h \cdot o) g(h^{-1} \cdot x)\,dh,\ x \in X \tag{A.1.7}$$

where dh is the Haar measure on G as in Theorem A.1.5. As would be expected the convolution of two $L^1(X)$ functions is defined almost everywhere, is associative, but is not in general commutative, for these are consequences of convolution in $L^1(G)$. With this convolution product $L^1(X)$ becomes a Banach algebra. Furthermore, it is clear that $f * g$ are supported in the set $q(\overline{AB})$ where $A = \operatorname{supp} f^q$ and $B = \operatorname{supp} g^q$ (note that when K is compact, q is a closed map for the product of any closed set with a compact set is closed).

Proposition A.1.6. *If $f \in L^1(X)$ and $g \in L^\infty(X)$, then $f * g \in C(X)$. More generally if $f \in L^p(X)$ and $g \in L^q(X)$ where q is the conjugate exponent to p, then $f * g \in C(X)$.*

Proof. Note that as a consequence of Theorem A.1.5, we have $g \in L^\infty(X)$ implies $g^q \in L^\infty(G)$. By the characteristic property of surjective quotient maps we only need to prove that $(f * g)^q$ is continuous but this readily follows from the observation that for $z \in G$:

$$\|L_z(f * g)^q - (f * g)^q\|_{\sup} \le \|L_z f^q - f^q\|_1 \|g^q\|_\infty, \tag{A.1.8}$$

so that $(f * g)^q$ is left uniformly continuous (here $L_z f(x) = f(z^{-1}x)$). The second assertion follows from Hölder's inequality. $\square$

Now suppose that X is a smooth manifold and G acts smoothly on $X = G/K$.

Proposition A.1.7. *Let $f \in L^p(X)$, $\psi \in C_c^\infty(X)$, and D be a differential operator, then*

$$D(f * \psi)(x) = \int_G f(g \cdot o) D(L_g \psi(x))\, dg\,.$$

In particular if D is a G-invariant differential operator, then $D(f * \psi) = f * (D\psi)$.

Proof. For a fixed $x \in X$ let C be a compact neighborhood of x which is contained in a coordinate system (U, ϕ). Suppose that $D = \partial_j$ on a neighborhood of C contained in U where ∂_j is a coordinate vector corresponding to the local coordinates induced by (U, ϕ). Note that the function $(x, g) \mapsto (D_x L_g \psi)(x)$ is smooth on $X \times G$ since it is the composition of the mappings $(x, g) \mapsto g^{-1}x \mapsto \psi(g^{-1}x)$ which is then composed with the differential operator D_x (D_x is the operator D acting on the variable x). By an application of Peetre's theorem, there exists a compact set $C' \subset G$ such that for each $x \in C$ the function $g \mapsto D_x L_g \psi(x)$ has support in C'. It follows that there is a positive number M such that $|D_x L_g \psi(x)| \le M$ uniformly for all $x \in C$ and $g \in G$. Therefore by the dominated convergence theorem we can commute the operator D with the integral to get

$$D(f * \psi)(x) = D \int_G f(g \cdot o)\psi(g^{-1}x)\, dg = \int_G f(g \cdot o) D(L_g \psi(x))\, dg\,. \tag{A.1.9}$$

If D is any coefficient differential operator, then it can be written locally as a composition of first order differential operators. So the result follows by applying the result for first order differential operators obtained above. Finally if D is G-invariant, then $D(L_g \psi(x)) = L_g D\psi(x)$ for all $g \in G$ so that $D(f * \psi) = f * D\psi$. $\square$

If X is a Riemannian manifold on which G acts by isometries and dx a G-invariant measure on X, then when we convolve a function f with a radial function $\phi(x) = \psi(d(o, x))$ (and suppose that f and ϕ are sufficiently regular for $f * \phi$ to exist) the convolution becomes

$$f * \phi(y) = \int_X f(x)\psi(d(x, y))\, dx \tag{A.1.10}$$

analogous to the convolution of radial functions in the Euclidean setting.

A.1.4 Mollifiers on Lie groups

In this subsection G always denotes a Lie group. We say a function $f\colon G \to \mathbf{C}$ is inversion invariant if $f \circ \mathrm{inv} = f$. Inversion invariant functions are quite easy to come by since one can take any function f on G and set $f + f \circ \mathrm{inv}$ which is an inversion invariant function. In particular, since each neighborhood U of the identity contains a compact symmetric neighborhood we have that the inversion invariant functions in $C_c^\infty(U)$ is nonempty.

Proposition A.1.8. *Let $\{U_\alpha\}$ be a system of neighborhoods of the identity in G such that $U_\alpha \to \{e\}$. Let $\{\phi_\alpha\}$ be a family of smooth functions satisfying the following properties:*

1. $\phi_\alpha \in C_c^\infty(U_\alpha)$.

2. $\phi_\alpha \geq 0$ *and* $\int \phi_\alpha = 1$.

3. ϕ_α *is inversion invariant.*

Then if $f \in L^p(G)$*, then* $\|f * \phi_\alpha - f\|_p \to 0$ *and* $f * \phi_\alpha \in C^\infty(G)$*. If in addition* $f \in C(G)$*, then* $f * \phi_\alpha \to f$ *uniformly on compact subsets.*

Proof. The statement and proof is the content of Proposition 2.44 in Folland. The smoothness of the convolution $f * \psi_\alpha$ is trivial when one checks the claim in local coordinate systems. $\square$

Mollifiers on homogeneous spaces

We now turn our attention to homogeneous spaces. We first prove a few results concerning when mollifiers on G extend to mollifiers on the homogeneous space $X = G/K$.

Proposition A.1.9. *Let* $\phi \in C^\infty(G)$*, then* $\phi^\flat \in C^\infty(X)$.

Proof. By working in local coordinate systems in G it is clear that $\phi^\flat \circ q$ is smooth in G. Since $q\colon G \to G/K$ is a surjective smooth submersion by [18, Thm. 21.17] it follows by the characteristic property of surjective smooth submersions [18, Thm. 4.29] that the function $\phi^\flat$ is also smooth. $\square$

Lemma A.1.10. *The map* $^\flat$ *extends to a surjective bounded linear operator from* $L^p(G)$ *to* $L^p(X)$.

Proof. Evidently $^\flat$ is linear from its definition and is surjective by Theorem A.1.5. Now for $f \in C_c(G)$ using [9, Thm. 2.51] and Hölder's inequality:

$$\|f^\flat\|^p_{L^p(X)} = \int_{G/K} |f^\flat(x)|^p \, dx \leq \int_{G/K} \int_K |f(gk)|^p \, dk \, d(gK) = \|f\|^p_{L^p(G)} \tag{A.1.11}$$

so that $^\flat$ is bounded on $C_c(G)$. We extend $^\flat$ to all of $L^p(G)$ in the usual way by setting $f^\flat = \lim_{f_n \to f} f_n^\flat$ (for $f_n \in C_c(G)$ convergent to f in $L^1(G)$) which is well defined by (A.1.11). So in particular $^\flat$ is continuous on $L^p(G)$ and one can easily check that the operator norm satisfies $\|^\flat\|_{\text{op}} \leq 1$. $\square$

We now prove that $L^p(X)$ has a set of mollifiers $\{\psi_\alpha\} \subset C_c^\infty(X)$. We find a set of right approximate identities, for this sake let ϕ_α be the mollifiers defined earlier. Then we have $\psi_\alpha := \phi_\alpha^\flat$ is smooth and that for $f \in L^p(X)$ by Fubini's theorem

$$\begin{aligned}(f * \psi_\alpha)(h \cdot o) &= \int_G f(g \cdot o)\psi_\alpha(g^{-1}h \cdot o) \, dg = \int_G f(g \cdot o) \int_K \phi_\alpha(g^{-1}hk) \, dk \, dg \\ &= \int_K \int_G f(g \cdot o)\phi_\alpha(g^{-1}hk) \, dg \, dk = (f^q * \phi_\alpha)^\flat(h \cdot o)\end{aligned}$$

Hence

$$\|f * \psi_\alpha - f\|_{L^p(X)} = \|(f^q * \phi_\alpha)^\flat - (f^q)^\flat\|_{L^p(X)} = \|(f^q * \phi_\alpha - f^q)^\flat\|_{L^p(X)} \leq \|f^q * \phi_\alpha - f^q\|_{L^p(G)}. \tag{A.1.12}$$

Choosing α appropriately we find that $L^p(X)$ has a right approximate identity of consisting of smooth functions which we call mollifiers. The functions $f * \psi_\alpha$ are smooth so that one finds that $C_c^\infty(X)$ is dense in $L^p(X)$.

One can also require that the functions ψ_α be radial functions on X. Let $C_c(X)_K$ be the space the space of K-invariant continuous complex-valued functions of compact support on X and we write $\mathcal{D}_K(X)$ for $C_c^\infty(X) \cap C_c(X)_K$. We can emulate the map $^\flat$ above by introducing the map $^\natural\colon C_c(G) \to C_c(X)_K$ defined by

$$f^\natural(gK) = \int_K \int_K f(k'gk) \, dk \, dk' \, . \tag{A.1.13}$$

Then using the fact that $\delta_G|_K = 1$ we observe:

$$\begin{aligned}(f * \phi_\alpha^\natural)(h \cdot o) &= \int_G f(g \cdot o) \int_K \int_K \phi_\alpha(k' g^{-1} hk)\, dk\, dk'\, dg = \int_K \int_K \int_G f(g \cdot o)\phi_\alpha(k' g^{-1} hk)\, dg\, dk\, dk' \\ &= \int_K \int_K \int_G f(gk' \cdot o)\phi_\alpha(g^{-1}hk)\delta_G(k')^{-1}\, dg\, dk\, dk' = \int_K \int_G f(g \cdot o)\phi_\alpha(g^{-1}hk)\, dg\, dk \\ &= (f^q * \phi_\alpha)^\flat(h \cdot o)\end{aligned}$$

and we thus obtain that the functions $\{\phi_\alpha^\natural\}_{\alpha\in A} \in \mathcal{D}^\natural(X)$ yield the same approximation properties as the family $\{\psi_\alpha\}_{\alpha\in A} \subset C_c^\infty(X)$. The above theorems provide a generic way of defining mollifiers on any Lie group and any homogeneous space of the form $X = G/K$, however in these next few examples we will be explicit.

Example A.1.11. Suppose $X = G/K$ is a connected complete Riemannian manifold with distance function d compatible with its quotient topology for which the translation operators $\tau_g \colon x \mapsto g \cdot x$ $(g \in G)$ act as isometries. Then for $r > 0$ we can consider the family of smooth functions $\{\psi_r\}_{r>0}$ defined by $\psi_r(x) = n_r \cdot \eta(d(o,x)^2/r) = \eta_r(x)$. Here we choose r sufficiently small so that ψ_r is smooth and $\eta \in C^\infty(\mathbf{R})$ is the function

$$\eta(x) = \begin{cases} C \exp\left(\frac{1}{|x|^2-1}\right) \text{ if } x < 1, \\ 0 \text{ if } x \geq 1, \end{cases} \tag{A.1.14}$$

with C and n_r chosen so that $\int_X \eta_r = \int_{\mathbf{R}} \eta = 1$. We see that $\eta_r \in C_c^\infty(X)$ and by property of the metric d the function η_r^q is inversion invariant. Now observe that the sets $q^{-1}(B_{r\geq 0}(o))$ may be factored into the form $A_r K$ where $A_r \to \{e\}$ as $r \to 0$. Then for $f \in L^p(X)$ we have by the Minkowski integral inequality:

$$\|f * \eta_r - f\|_{L^p(X)} \leq \sup_{y\in q^{-1}(B_r(o))} \|R_y f^q - f^q\|_{L^p(G)} = \sup_{ak\in A_r K} \|R_{ak} f^q - f^q\|_{L^p(G)} \tag{A.1.15}$$

$$= \sup_{a\in A_r} \|R_a f^q - f^q\|_{L^p(G)}. \tag{A.1.16}$$

So that we readily see the functions η_r are approximations to the identity in $L^p(X)$ and it is easy enough to check that the convolution $f * \eta_r$ is smooth for any $f \in L^p(X)$.

Example A.1.12. More generally, if $\{\psi_\alpha\}_{\alpha\in A} \subset C_c^\infty(X)$ is any family of functions corresponding to the system of neighborhoods of the origin $\{U_\alpha\}_{\alpha\in A}$ (e.g. $\operatorname{supp}\psi_\alpha \subset U_\alpha$) with the property that ψ_α^q is inversion invariant, then this family defines a set of mollifiers for X.

Remark 17. As a quick remark we roughly sketch how one can extend the theory of convolutions to Riemannian manifolds. If M is a Riemannian manifold, then it is well-known that the set of isometries of M, denoted $\operatorname{Iso}(M)$, is a Lie group. If $G = \operatorname{Iso}(M)$ acts on M transitively, then so does the connected component G_o of the identity (which is necessarily a subgroup) act transitively on M. If we assume that G_o acts on M transitively and properly, then by picking a point $x \in M$ we have that M is diffeomorphic to the homogeneous space G_o/G_x where G_x is the stabilizer of x in G_o. Thus M can be regarded as a homogeneous space for a Lie group G and the convolution in G descends to convolution on M.

A.2 Functional analysis

We state a simple result about Hilbert spaces which we reference once in this thesis.

Proposition A.2.1. *Let $\mathcal{H}$ be a Hilbert space and $E \subset \mathcal{H}$ a subspace, then E is dense in $\mathcal{H}$ if and only if $E^\perp = \{0\}$.*

Proof. If E is dense, then continuity of the inner product implies that $E^\perp = \{0\}$; and if $E^\perp = \{0\}$, then $\overline{E} = (E^\perp)^\perp = \mathcal{H}$. ☐

Vector-valued integration

Definition A.2.1. Let V be a topological vector space. We say that V separates points if for each $u, v \in V$ there exists a linear functional $\varphi \in V^*$ such that $\varphi(u) \neq \varphi(v)$.

For example, every normed vector space separates points by the Hahn-Banach theorem.

Definition A.2.2 (Weak integration)**.** See [9, Thm. A.20]. Let $(X, \mathcal{M}, \mu)$ be a measure space and V be a topological vector space that separates points. Let $f\colon X \to V$ be a function. We say that f is *weakly integrable* if for each $\varphi \in V^*$ the function $\varphi \circ f$ is integrable and that there exists $v \in V$ such that for all $\varphi \in V^*$ we have

$$\varphi(v) = \int_X \varphi \circ f(x)\, dx\,. \tag{A.2.1}$$

We define the weak integral of f to be

$$\int_X f(x)\, dx = v.$$

Since V separates points, the definition of the weak integral of a vector valued function is well-defined. Unlike in the classical criterion for the integration of a scalar valued function it is far more difficult to verify the weak integrability for a vector valued function. However, we do have a sufficient condition for the integrability of a compactly supported continuous function into a Fréchet space.

Theorem A.2.2. *If V is a Fréchet space and μ a Radon measure on the locally compact Hausdorff space X, and if $F\colon X \to V$ is continuous and compactly supported, then $\int F\, d\mu$ exists and belongs to the closed linear span of the range F. If V is a Banach space, then*

$$\| \int F\, d\mu \| \leq \int \|F\|\, d\mu\,.$$

Proposition A.2.3. *Let $(W, \mathcal{M}, \mu)$ be a measure space and X and Y be two topological vector spaces which separate points and let $T\colon X \to Y$ be a continuous linear map. Suppose that $f\colon W \to X$ is a weakly integrable function, then $T \circ f$ is weakly integrable and*

$$T \int_W f(w)\, d\mu(w) = \int_W T \circ f(w)\, d\mu(w)$$

Proof. Let $\varphi \in Y^*$, then $\varphi \circ T \in X^*$ and so it follows that $T \circ f$ is weakly integrable. From this same observation we have that if $v = \int_W f(w)\, d\mu(w)$, then we have the obvious equalities

$$\varphi(T(v)) = (\varphi \circ T)(v) = \int_W (\varphi \circ T)(f)(w)\, d\mu(w) = \int_W \varphi(T \circ f)(w)\, d\mu(w)\,.$$

Thus the weak integral of $T \circ f$ is $T(v)$. ☐

The Schwartz kernel theorem

Defining topological tensor products and the condition for a locally convex vector space to be nuclear is too involved so we refer the reader to [25, Sec. 39]. If X and Y are

Theorem A.2.4 (Abstract kernel theorem)**.** *Let X and Y be two Fréchet spaces with at least one of which is nuclear and let X^* and Y^* be their strong duals. Then we have the following isomorphism*

$$X^* \widehat{\otimes} Y^* \cong (X \widehat{\otimes} Y)^* \cong L(X, Y^*). \tag{A.2.2}$$

Here $X \widehat{\otimes} Y$ denotes the topological tensor product of X and Y.

The proof is contained in [25, Prop. 50.1], although the proof that Trèves supplies only covers the case where X and Y are both nuclear as the argument there is easier.

Corollary A.2.5 (Classical Schwartz kernel theorem)**.** *Let X and Y be open sets in $\mathbf{R}^n$, then for each continuous operator $A\colon \mathcal{D}(X) \to \mathcal{D}'(Y)$, there exists a unique distribution $K_A \in \mathcal{D}'(X \times Y)$ such that for $\psi \in \mathcal{D}(X)$ and $\phi \in \mathcal{D}(Y)$ we have*

$$\langle A\psi, \phi\rangle = \langle K_A, \psi \otimes \phi\rangle. \tag{A.2.3}$$

We call K_A the Schwartz kernel *of the operator A.*

The utility of nuclear spaces is that these are the vector spaces for which the analogue of the Schwartz kernel theorem holds. Nuclear spaces were introduced by Grothendieck as he investigated generalizations to Schwartz's 1952 result on the Schwartz kernel theorem originally formulated for test functions on $\mathbf{R}^n$.

A.2.1 Gelfand theory

Gelfand theory roughly relates the structure of commutative unital C^*-algebras to the space of continuous functions on its spectrum, given a canonical topology. We shall quickly define what all these concepts mean in this section.

Definition A.2.3. A complex Banach algebra X is a complex Banach space that is also an algebra over the complex numbers which satisfies $\|xy\| \leq \|x\|\|y\|$ so that multiplication is continuous in X. Here $\|\cdot\|$ is the norm on X.

We say that X is *unital* if as an algebra X is unital and that X is *commutative* if as an algebra X is commutative. An *involution* on X is a map $*\colon X \to X$ (we write $*(x) = x^*$) which is antilinear and satisfies $(xy)^* = y^*x^*$ and $x^{**} = x$ for all $x, y \in X$. If X is a Banach algebra with an involution which also satisfies $\|x^*x\| = \|x\|^2$, then we say that X is a *C^*-algebra*.

We have the usual notion of homomorphism for Banach algebras and C^*-algebras. A homomorphism $\phi\colon X \to Y$ of Banach algebras X and Y is a continuous linear map which satisfies $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in X$. If X and Y are Banach algebras with involution, then a $*$-involution $\phi\colon X \to Y$ is a homomorphism of Banach algebras which also satisfies $*_Y \circ \phi = \phi \circ *_X$ (where $*_Y$ and $*_X$ are the involutions on X and Y respectively).

If X is a Banach algebra by a *multiplicative functional* $\phi\colon X \to \mathbf{C}$, we mean a continuous function $\phi \in X^*$ which as expected must satisfy $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in X$. From now on we shall assume that X is a commutative Banach algebra.

Definition A.2.4. Let X be a commutative Banach algebra. The *spectrum* of X, denoted by $\sigma(X)$, is the set of all nonzero multiplicative functionals on X.

The spectrum $\sigma(X)$ of a Banach algebra X is a subset of the closed unit ball of X^* in the weak*-topology. We topologize $\sigma(X)$ by imposing the weak*-topology when regarded as a subset of X^* (the topology of pointwise convergence). For each $x \in X$ we can define a continuous function $\widehat{x}\colon \sigma(X) \to \mathbf{C}$ given by $\widehat{x}(\phi) = \phi(x)$.

Definition A.2.5. The *Gelfand transform* or *Gelfand representation* Γ on X is the map $\Gamma\colon X \to C(\sigma(X))$ given by $\Gamma(x) = \widehat{x}$.

We have now the terminology to state the essential theorem, the proof of which can be in [9, Sec. 1.3].

Theorem A.2.6. *Suppose that X is a nonunital commutative Banach algebra. Then the following facts hold.*

1. *If $x \in X$, then the function $\widehat{x}$ vanishes at infinity so that $\widehat{x} \in C_0(\sigma(X))$.*

2. *The Gelfand transform $\Gamma\colon X \to C_0(\sigma(X))$ is a continuous homomorphism of Banach algebras.*

A.3 The Radon transform

For fun, we shall explain how one can construct the Radon transform on $\mathbf{R}^n$. Let m_n be the Lebesgue measure on $\mathbf{R}^n$ with the convention $m = m_1$. Integrating with respect to the polar coordinate system on $\mathbf{R}^n$ is simple to understand. Essentially, to integrate a function f on $\mathbf{R}^n$ we can instead integrate f over the spheres $S_r(0) = \{x \in \mathbf{R}^n : |x| = r\}$ $(r > 0)$ with respect to a specific measure on the spheres S_r. Intuitively this is a reasonable idea since the spheres are all disjoint and "fill out" $\mathbf{R}^n$ in a nice way. This same idea works for hyperplanes, in that given $\omega \in \mathbf{S}^{n-1}$ to integrate a function on $\mathbf{R}^n$ we can instead integrate f over the hyperplanes $\xi(p, \omega)$ $(p \in \mathbf{R})$ with respect to a particular "hyperplanar measure." To define the hyperplanar measure we essentially mirror the construction of the spherical measure.

Fix $\omega \in \mathbf{S}^{n-1}$ and let $\xi_d = \xi(\omega, d)$. We can map $\mathbf{R}^n$ continuously and bijectively into the space $\mathbf{R} \times \xi_d$ via the map $\Phi(x) = (\langle x, \omega\rangle - d, x + (d - \langle x, \omega\rangle)\omega)$. Φ merely records the signed distance of the hyperplane H_x containing x parallel to ξ_d and records the point x' which is the intersection point on ξ_d of the line perpendicular to H_x passing through x. Clearly, this map is continuous and so is its inverse $\Phi^{-1}(p, z) = z + (p - d)\omega$. Hence we define the measure on $\mathbf{R} \times \xi_d$ to be the induced measure $\mu(E) = m_n(\Phi^{-1}(E))$ $(E \subset \mathbf{R} \times \xi_d$ Borel). It is easy enough to check that we may decompose μ into the product measure $\mu = m \times \sigma$ where σ is a unique surface measure (called the hyperplanar measure) on ξ_d. And from this we can deduce the following theorem.

Theorem A.3.1. *There is a unique Borel measure σ on ξ_d such that $\mu = m \times \sigma$. Moreover if f is Lebesgue measurable on $\mathbf{R}^n$ and integrable and or positive, we have*

$$\int_{\mathbf{R}^n} f(x)\, dx = \int_{\mathbf{R}} \int_{\xi_d} f(x + (p - d)\omega)\, d\sigma(x)\, dp\,. \tag{A.3.1}$$

Proof sketch. The above theorem is proved in precisely the same way as that of [8, Thm. 2.49] and thus we refer to the reader to retrace the necessary steps. The only necessary modification is that for $E \subset \xi_d$ Borel we define $\sigma(E) = m(E_1)$ where $E_1 = \Phi^{-1}([0, 1] \times E)$ and the map $E \mapsto E_1$ takes Borel sets to Borel sets since Φ is a homeomorphism and commutes with unions, intersections, and complements so that σ is a Borel measure on ξ_d. Moreover let $T(\omega) = o$ be an orthogonal transformation where o is the north pole of $\mathbf{S}^{n-1}$ then

$$\begin{aligned}\mu([a, b] \times E) = m_n(\Phi^{-1}([a, b] \times E)) = m_n(T \circ \Phi^{-1}([a, b] \times E)) = (b - a)m_{n-1}(T(E)) &= (b - a)\sigma(E)\\ &= (m \times \sigma)([a, b] \times E).\end{aligned}$$

What we mean by $T(E)$ is the factor A in the Borel set $T(E_1) = [a, b] \times A$ and thus the section $T(E) = A$ is Borel, hence m_{n-1}-measurable. The rest of the proof follows as in [8, Thm. 2.49]. Completing μ and σ gives the desired measures satisfying property (A.3.1) for all Lebesgue measurable sets $E \in \mathcal{L}^n$. $\square$

We will also write the integral

$$\int_{\mathbf{R}} \int_{\xi(p,\omega)} f(x)\, d\sigma(x)\, dp = \int_{\mathbf{R}} \int_{\xi_d} f(x + (p - d)\omega)\, d\sigma(x)\, dp\,. \tag{A.3.2}$$

It is also easy to check that if $E \subset \xi_d$ is measurable then

$$\int_{\xi_d} \chi_E(x)\, d\sigma(x) = \int_{\xi_0} \chi_E(x + d\omega)\, d\sigma(x),$$

and thus the above holds for general integrable functions f by ordinary approximation arguments. Let Ξ be the set of all hyperplanes in $\mathbf{R}^n$. The Radon transform of a function that is integrable on each hyperplane in Ξ is the function $Rf : \Xi \to \mathbf{C}$ defined by

$$Rf(\xi) = \int_{\xi} f(x)\, d\sigma(x)\,.$$

Since any hyperplane is of the form $\xi(p, \omega)$ we can simply consider Rf to be a function on $\mathbf{R} \times \mathbf{S}^{n-1}$ instead. Note that integrals over $\mathbf{R}^n$ are simply integrals of the Radon transform over $\mathbf{R}$ at a fixed angle ω. If $f \in L^1(\mathbf{R}^n)$, we can write the Fourier transform of f in polar coordinates and using formula (A.3.1) we find

$$\widehat{f}(\lambda\omega) = \int_{\mathbf{R}^n} f(x) e^{-i\lambda\langle x,\omega\rangle}\, dx = \int_{\mathbf{R}} e^{-i\lambda p} \left[\int_{\xi(\omega,p)} f(x)\, d\sigma(x) \right] dp = (R_\omega f)\widehat{\ }(\lambda) \tag{A.3.3}$$

where $R_\omega f(p) = Rf(p, \omega)$.

Remark 18. The use of orthogonal transformations in the sketch of the proof in Theorem A.3.1 leads to a straightforward check of the intuitive fact that

$$\int_{\xi(\omega,p)} f(x)\, d\sigma(x) = \int_{\mathbf{R}^{n-1}} f \circ T(y, p)\, dy$$

where T is any orthogonal transformation mapping the hyperplane $\mathbf{R}^{n-1} \times \{p\}$ into $\xi(\omega, p)$, i.e. $T(o) = \omega$. Using this representation one finds for instance that if $f \in C_c^\infty(\mathbf{R}^n)$, that $R_\omega f \in C_c^\infty(\mathbf{R})$ for each $\omega \in \mathbf{S}^{n-1}$.

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