

Scientific Report No. 3

ELECTROMAGNETIC CHARACTERISTICS OF A HORIZONTAL WIRE

ABOVE A DISSIPATIVE EARTH - PART I:

PROPAGATION OF TRANSMISSION-LINE AND FAST-WAVE MODES[†]

by

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October 1973

[†]This research was supported by the U.S. National Oceanic and Atmospheric Administration under Grant No. N22-126-72(G), entitled "Theory of Geological Remote Sensing Using Electric Dipole Impedance Data."

TABLE OF CONTENTS

	Page
1.1. Introduction	1
1.2. Modal characteristic equation.	3
1.3. Approximate expression of functions $P(\alpha)$ and $Q(\alpha)$	7
1.3.1 Some discussion on the approximation.	21
1.4. Discrete Solutions to the Modal Equation	23
1.5. Concluding Remarks	30
Appendix A	32
Appendix B	38
References	40
Acknowledgment	41

Abstract

The problem of propagating modes supported by a horizontal wire in the presence of a conducting earth, is investigated. In addition to the commonly-known transmission-line mode, a new solution is found which can be identified as a fast-wave mode and has a substantially less attenuation along the propagation direction. The existence of such a mode is shown to be a direct consequence of the well-known poles in the Sommerfeld integral representation. Characteristics of both modes are then presented.

1.1. Introduction

The problem of electromagnetic wave propagation on a horizontal wire of infinite length above an air-earth interface is of considerable interest. The earliest attempt to solve this problem was that of Carson in 1925.^[1] Carson solved the problem in terms of distributed parameters under several low frequency approximations. Later Kikuchi^[2,3] showed that as the frequency becomes very high the surface impedance of the wire becomes important and that the wave propagating along the wire is a surface wave bound to the wire. More recently Chang^[4], in solving the problem of an infinitely long horizontal antenna with a delta function voltage excitation, obtained an approximate modal equation valid for low frequencies and found an expression for the propagation constant of a single propagating mode. His results were consistent with those of Carson and Kikuchi^[1,2]. This mode reduces to the TEM mode of a two-wire transmission line when the conductivity of the earth becomes infinite. Thus, it can be referred to as the transmission-line mode. Recently, Wait^[5] has reformulated the problem and presented an exact modal equation in terms of the well-known integrals. However, only an approximate solution similar to the previous one was reported. Measurements of attenuation of waves propagating along the wire have been performed by dos Santos^[6] and Knight^[7].

In Carson's and Kikuchi's formulation, the propagation constant of the wire is obtained by using a perturbation technique.

In the perturbation term the phase velocity of wave propagation along the wire is assumed to be that of free space. This assumption is a valid one at the low frequencies considered by Carson. It leads to simple expressions for the distributed circuit parameters of the transmission system. For high frequencies however the perturbation term which can become dominant in some regions, is extremely sensitive to the nature of this approximation and incorrect results were obtained.

dos Santos^[5] realized the mathematical problems with the approximation used by Carson and Kikuchi. Using the exact modal equation with the thin wire approximation he argued, in addition to the transmission line mode mentioned previously, the existence of a so-called "improper" mode which violates the radiation condition. He then used this mode to explain the behavior of this wave propagating structure at high frequencies. It is shown in this work that the "improper" mode is not necessarily adequate to explain the high frequency behavior of the structure. Instead a new mode is found which does satisfy the radiation condition and can be shown to be important for the high frequency case. This new mode is characterized by a phase velocity greater than that of free space; a phenomenon which is in direct contrast to that of the transmission mode. Also it is attenuated at a substantially smaller rate than is the transmission mode. Finally, the propagation constant of the second mode is less sensitive to the radius of the wire than is the propagation constant of the transmission line mode.

The existence of the transmission line mode and the second mode can be shown directly from the exact modal equation. However, for ease of computation, an approximate modal equation valid over a wide range of earth parameters and structure dimensions, is derived. The resultant equation is particularly useful in demonstrating explicitly the existence of both modes, as well as providing some simple explanation as to why the previous approximation made by Carson and others does not yield correct results at high-frequencies.

1.2. Modal characteristic equation

Consider an infinitely long thin wire of radius a (meters) located at a height d (meters) over a plane interface between two half-spaces of electrical constants ϵ_1, μ_1 , and $\epsilon_2, \mu_2, \sigma_2$ respectively.* The current distribution on the wire is assumed to be of the form $I \exp(ik_1 \alpha x - i\omega t)$ where α is the yet undefined propagation constant relative to that of region 1 and ω is the angular frequency of the fields. Figure 1.1 illustrates the problem under study. Following Wait^[5], an equation can be written which represents the axial electric field in the region $z > 0$ for a filament of current at $z = d, y = 0$.

$$E_{1x} = k_1^2 \zeta^2 \Pi_1, \quad (1.1)$$

*The medium in the upper half-space usually represents the air, and the lower half-space, the earth.

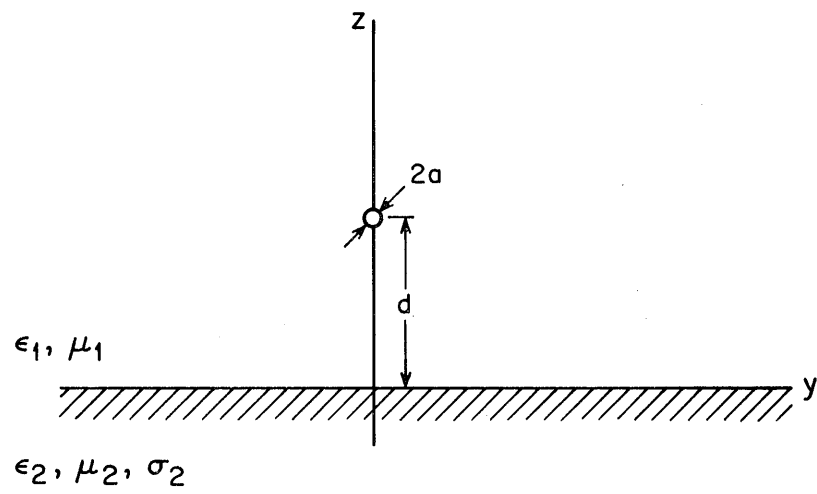


Figure 1.1

where $k_1^2 = \epsilon_1 \mu_1 \omega^2$ and $\zeta^2 = (1 - \alpha^2)$; Π_1 is the x-component of the electric Hertz potential and can be shown to be [5]

$$\begin{aligned} \Pi_1 = & \frac{-Z_0 I}{8k_1} \exp(i\alpha x) \{H_0^{(1)}(k_1 \zeta [(z-d)^2 + y^2]^{\frac{1}{2}}) - H_0^{(1)}(k_1 \zeta [(z+d)^2 + y^2]^{\frac{1}{2}}) \\ & + \frac{2}{i\pi \zeta^2} \int_{-\infty}^{\infty} \left[\frac{1}{u_1 + u_2} - \frac{\alpha^2}{u_2 + (k_2/k_1)^2 u_1} \exp[-(z+d)k_1 u_1] \exp(-i\lambda k_1 y) d\lambda \right] \end{aligned} \quad (1.2)$$

The argument of ζ is defined as $0 < \arg \zeta < \pi$ in order that the electromagnetic field satisfies the radiation condition at ∞ . It has been assumed that $\mu_1 = \mu_2 = \mu_0$; i.e. the permeability of free space. In addition,

$H_0^{(1)}(x)$ is the Hankel function of the first kind of order zero; $u_1 = (\lambda^2 - \zeta^2)^{\frac{1}{2}}$, $u_2 = (\lambda^2 + \alpha^2 - (k_2/k_1)^2)^{\frac{1}{2}}$ where $\text{Re}(u_1), \text{Re}(u_2) \geq 0$ so that the integral converges everywhere in the complex λ plane; $k_2^2 = k_1^2 n^2$ where n is the refractive index of medium 2, i.e. $n = [\epsilon_2(1+i\delta)/\epsilon_0]^{\frac{1}{2}}$ with δ (loss tangent) given as $\sigma_2/(\omega\epsilon_2)$, $0 < \arg n < \pi/4$; $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the intrinsic impedance of free space. The first term of eq. 1.2 is the direct contribution due to current on the wire in free-space. The second is the contribution of the image of the wire over a perfectly-conducting plane and the integral is the correction due to the finite conductivity of the lower half-space.

The other components of the field can be shown as

$$\begin{aligned}
 E_{1z} &= i\alpha k_1 \frac{\partial \pi_1}{\partial z} + i\mu_1 \omega \frac{\partial \pi_1^*}{\partial y} ; & E_{1y} &= -i\alpha k_1 \frac{\partial \pi_1}{\partial y} + i\mu_1 \omega \frac{\partial \pi_1^*}{\partial z} \\
 H_{1z} &= -i\alpha k_1 \frac{\partial \pi_1^*}{\partial z} + i\epsilon_1 \omega \frac{\partial \pi_1}{\partial y} ; & H_{1y} &= -i\alpha k_1 \frac{\partial \pi_1^*}{\partial y} - i\epsilon_1 \omega \frac{\partial \pi_1}{\partial z} \\
 H_{1x} &= k_1^2 \zeta^2 \pi_1^*
 \end{aligned} \tag{1.3}$$

where

$$\pi_1^* = \frac{I\alpha(n^2-1)}{2\pi k_1 \zeta^2} \exp(ik_1 \alpha z) \int_{-\infty}^{\infty} \frac{\lambda \exp(-i\lambda k_1 y) \exp[-u_1 k_1 (z+d)]}{(u_2 + n^2 u_1)(u_1 + u_2)} d\lambda . \tag{1.4}$$

A modal equation for this structure can be found by requiring the average axial electric field to be zero on the assumed perfectly conducting wire. By neglecting proximity effects around the wire, the following modal equation is obtained.

$$M(\alpha) = \zeta^2 \{ H_0^{(1)} [A\zeta] J_0 [A\zeta] - H_0^{(1)} [2D\zeta] \} + P(\alpha) - Q(\alpha) = 0 \tag{1.5}$$

where

$$P(\alpha) = \frac{2}{i\pi} \int_{-\infty}^{\infty} \frac{\exp(-2Du_1)}{u_1 + u_2} d\lambda \tag{1.5a}$$

$$Q(\alpha) = \frac{2\alpha^2}{i\pi} \int_{-\infty}^{\infty} \frac{\exp(-2Du_1)}{u_2 + n^2 u_1} d\lambda \tag{1.5b}$$

and $A = k_1 a$, $D = k_1 d$. $J_0(A\zeta)$ in the first term of (1.5) results from the fact that the current source is assumed to be distributed over the radius of the wire. It is apparent that

$J_0(A\zeta)$ is important only in the asymptotic behavior of $M(\alpha)$ where the primary term is dominant. For most applications however, we can approximate the term $J_0(A\zeta)$ by unity under a thin wire assumption that $k_1 a \ll 1$.

Solutions to the modal equation (1.5) represent the natural propagating modes of the structure. Note that it is only necessary to find the values of α for which $\text{Im } \alpha > 0$ since α enter the modal equation only as α^2 . As mentioned earlier two solutions, one representing the transmission line mode, and the other, a fast-wave mode can be found. Although the existence of these modes is independent of the subsequent approximations in the derivation, the location of the solutions in the complex α plane can be more conveniently described when certain approximations are made with the assumption of a conducting earth. Thus the question concerning the solution to the modal equation will be deferred and approximations to $P(\alpha)$ and $Q(\alpha)$ will be discussed.

1.3. Approximate expression of functions $P(\alpha)$ and $Q(\alpha)$

The integral function $P(\alpha)$ can be expressed in terms of known functions if the integral is approximated in the following manner. Multiplying the numerator of (1.5a) by $u_1 - u_2$, an alternate form of $P(\alpha)$ is obtained as

$$P(\alpha) = \frac{2}{i\pi(n^2-1)} \int_{-\infty}^{\infty} [u_1 - u_2] \exp(-2Du_1) d\lambda. \quad (1.6)$$

Due to the exponential-decaying behavior of the integrand,

it is clear that major contribution to the integral comes mainly from small λ . Specifically, we can write

$$\lambda < \lambda_c; \quad \lambda_c = \left[\left(\frac{1}{2D} \right)^2 + \zeta^2 \right]^{\frac{1}{2}}. \quad (1.7)$$

Thus, if $|n|$ is large, the term u_2 in (1.6) can be written as

$$u_2 \approx -in \left\{ 1 - \frac{\lambda_c^2 + \alpha^2}{2n^2} \right\} \quad (1.8)$$

The second term in the series expansion can be further neglected if

$$\left| \frac{\lambda_c^2 + \alpha^2}{2n^2} \right| \ll 1.$$

Under these conditions, u_2 can be approximated by $-in$ and consequently,

$$\begin{aligned} P(\alpha) &\approx \frac{2}{i\pi(n^2-1)} \int_{-\infty}^{\infty} (u_1 + in) \exp(-2Du_1) d\lambda \\ &= \frac{2}{i\pi(n^2-1)} \left\{ \left(\frac{\partial^2}{\partial w^2} - in \frac{\partial}{\partial w} \right) \int_{-\infty}^{\infty} \frac{e^{-wu_1}}{u_1} d\lambda \right\}_{w=2D} \\ &= \frac{2}{(n^2-1)} \{ \zeta H_1^{(1)}(2D\zeta) \left[\frac{1}{2D} - in \right] - \zeta^2 H_0^{(1)}(2D\zeta) \}. \end{aligned} \quad (1.9)$$

In deriving (1.9), the following identity has been used.

$$H_0^{(1)}(\sqrt{a^2 + b^2}\zeta) = \left(\frac{1}{i\pi} \right) \int_{-\infty}^{\infty} e^{-au_1} e^{-ib\lambda} \frac{d\lambda}{u_1} \quad (1.10)$$

It is worth noting that the above approximation is good for large values of the refractive index provided d is not too small or α too large. As it will become clear later in the derivation, the values of α of interest are not generally very large and d must be very small to be an important factor in the approximation^[9]. Thus, if $2D > 1$ this approximation appears to be valid for

$$\frac{1}{2|n|^2} \ll 1 \quad (1.11)$$

and if $2D < 1$ then the approximation is valid for

$$\left| \frac{1}{2(2D)^2 n^2} \right| \ll 1. \quad (1.12)$$

When the lower medium is highly conducting, eq. (1.12) is equivalent to the physical condition that the wire height must be larger than a skin depth in that medium.

To illustrate the degree of accuracy involved for small d , Fig. 1.2 is a comparison of a numerical integration of the exact form of $P(\alpha)$ with eq. 1.9 for points in a complex plane near the branch point $\alpha = 1$. In this example, the refractive index of the earth is $5.52 + i 4.53$. This corresponds to an earth with conductivity $2.7 \text{ m}\cdot\text{mhos/m}$ and a relative permittivity of 10 at 1 MHz. As expected, the approximation becomes less accurate as $2D|n|$ becomes small. A comparison of the exact and approximate forms of $P(\alpha)$ for other regions of the α plane of interest yields similar results.

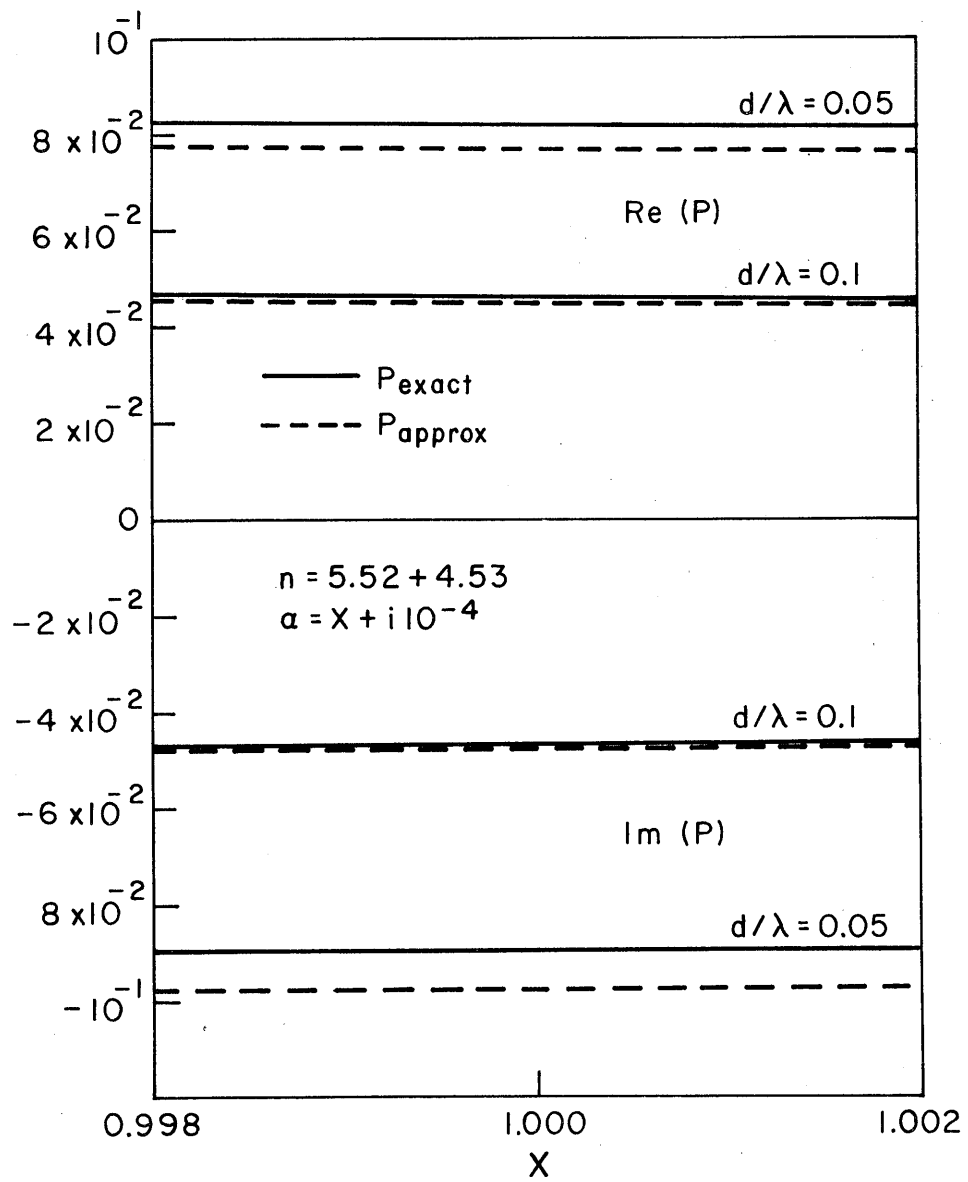


Figure 1.2

If now a similar approximation is used in $Q(\alpha)$, we obtain from (1.5) the following expression

$$\begin{aligned}
 Q(\alpha) &\approx \frac{2\alpha^2}{i\pi n^2} \int_{-\infty}^{\infty} \frac{\exp(-2Du_1)}{u_1 - i/n} d\lambda \quad (1.13) \\
 &= \frac{2\alpha^2}{i\pi n^2} \left\{ \int_{-\infty}^{\infty} \frac{\exp(-2Du_1)}{u_1} d\lambda + \frac{i}{n} \int_{-\infty}^{\infty} \frac{\exp(-2Du_1)}{u_1(u_1 - i/n)} d\lambda \right\} \\
 &= \frac{2\alpha^2}{n^2} H_0^{(1)}(2D\zeta) + \frac{2\alpha^2}{\pi n^3} W(\alpha) .
 \end{aligned}$$

Where $W(\alpha)$ can be written as

$$\begin{aligned}
 W(\alpha) &= \exp(-i2D/n) \left\{ \int_{-\infty}^{\infty} \frac{[\exp[-2D(u_1 - i/n)] - 1]}{u_1(u_1 - i/n)} d\lambda + \int_{-\infty}^{\infty} \frac{d\lambda}{u_1(u_1 - i/n)} \right\} \quad (1.14) \\
 &= [W_d(\alpha) + W_o(\alpha)] \exp[-i2D/n]
 \end{aligned}$$

It should be noted that although the replacement of u_2 by $-in$ can be shown to be valid over a wide range of n , the value of $Q(\alpha)$ could be more sensitive in some cases to this approximation than $P(\alpha)$ because of the presence of a pair of zeros in the denominator of the integrand. A comparison of (1.5b) and (1.13) clearly indicates that the location of the zeros has moved from $\pm[\zeta^2 - 1/(1+n^2)]^{1/2}$ in the exact expression, to $\pm[\zeta^2 - 1/n^2]^{1/2}$ in the approximate one. Thus, it appears that a better approximation to $Q(\alpha)$ is to replace u_2 by $in^2/(1+n^2)^{1/2}$, instead of in . However, when the lower half-space represents a conducting earth, the difference in the

two cases is negligible for all practical purposes. It is further noted that the approximate form of the integral can be obtained exactly by replacing the earth with an impedance surface and then deriving a modal equation in a manner identical to the one used here.[†][8,12]. Thus, the mathematical approximation presented here is indeed equivalent to the widely used surface-impedance concept.

The integral $W_0(\alpha)$ will be studied first and will be evaluated by deforming its contour of integration in the upper half of the complex λ plane. Thus it is necessary to study the singularities of the integrand of $W_0(\alpha)$ in the upper half λ -plane carefully for all α of interest. From (1.14), it is apparent that there is a pair of poles at the locations

$$\lambda_p = \pm (\zeta^2 - 1/n^2)^{1/2}. \quad (1.15)$$

These are "real" poles in the λ plane on the proper Riemann sheet defined by $\text{Re}(u_1) > 0$ for all values of α .

In addition to the poles there is a pair of branch points located at

$$\lambda_B = \pm \zeta. \quad (1.16)$$

As α moves, the locations of the poles and branch cuts in the λ plane change. Accordingly in figure 1.3 the motion of the poles (1 and 2) is traced as α traverses the paths $\alpha = 1 + S_\ell e^{i\phi}$, $-\pi < \phi < \pi$, $\ell = a, b$. (fig. 1.4).

[†]In reference [12], if we assign the transverse admittance Y_1 with a value equal to $n(\epsilon_0/\mu_0)^{1/2}$ and the transverse impedance Z_1 equal to $(1/n)(\mu_0/\epsilon_0)^{1/2}$ instead of the exact ones given by Wait, expressions similar to (1.9) and (1.13) for $P(\alpha)$ and $Q(\alpha)$ are obtained.

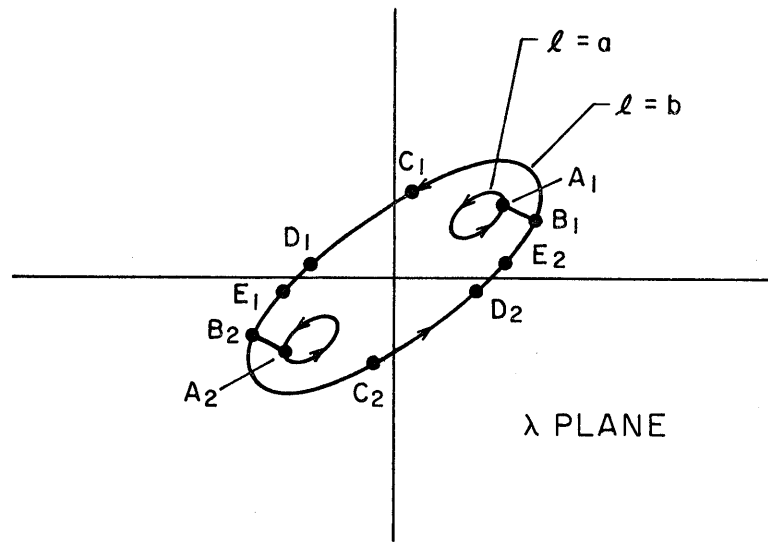


Figure 1.3

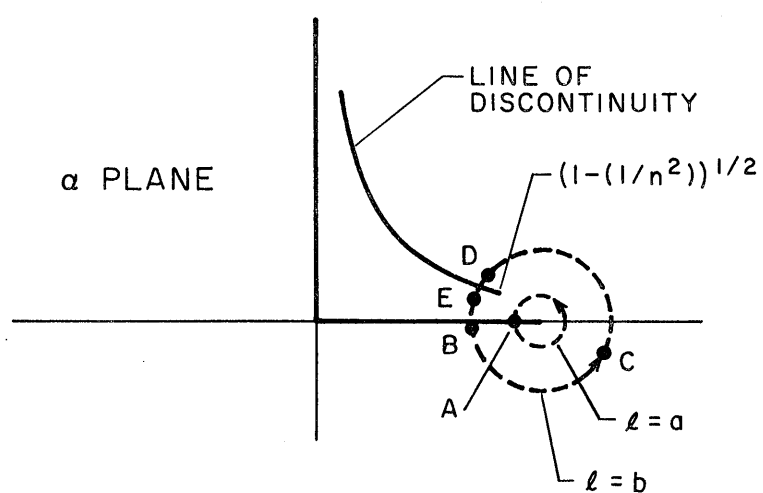


Figure 1.4

It is seen that on the path $\ell = a$ the two poles travel around the paths A_1, A_1 and A_2, A_2 and remain on their respective sides of the real λ axis throughout the traverse. However, on the path $\ell = b$, the poles traverse the paths B_1, C_1, D_1, E_1, B_2 and B_2, C_2, D_2, E_2, B_1 respectively. Thus, at point D on the traverse, pole 1 is in the upper half plane and is captured in the evaluation of $W_0(\alpha)$. At point E pole 2 is captured. Since the residues at point D_1 and at point E_2 in the λ plane are different, the function $W_0(\alpha)$ becomes discontinuous as α moves from D to E along path $\ell = b$. In general, the discontinuity in $W_0(\alpha)$ occurs for all those values of α such that the pole in the λ plane is located on the real axis. The locus of these points in the complex α -plane then can be shown to satisfy the following equation:

$$\alpha = \pm (1 - 1/n^2 - s^2)^{1/2}, \quad (1.17)$$

where s is a real number, varying from 0 to ∞ . This locus is indicated by the solid line in figure 4. Here, we should emphasize that our conclusion, i.e., the traverse in the α plane $\ell = a$ does not cross the real axis while the traverse $\ell = b$ does, is not dependent upon the nature of our approximation. Therefore a similar locus, slightly shifted, should also occur even if the exact form of $Q(\alpha)$ is retained.

Figure 1.5 is a sequence of plots showing the relative movement of the pole and branch point in the upper-half

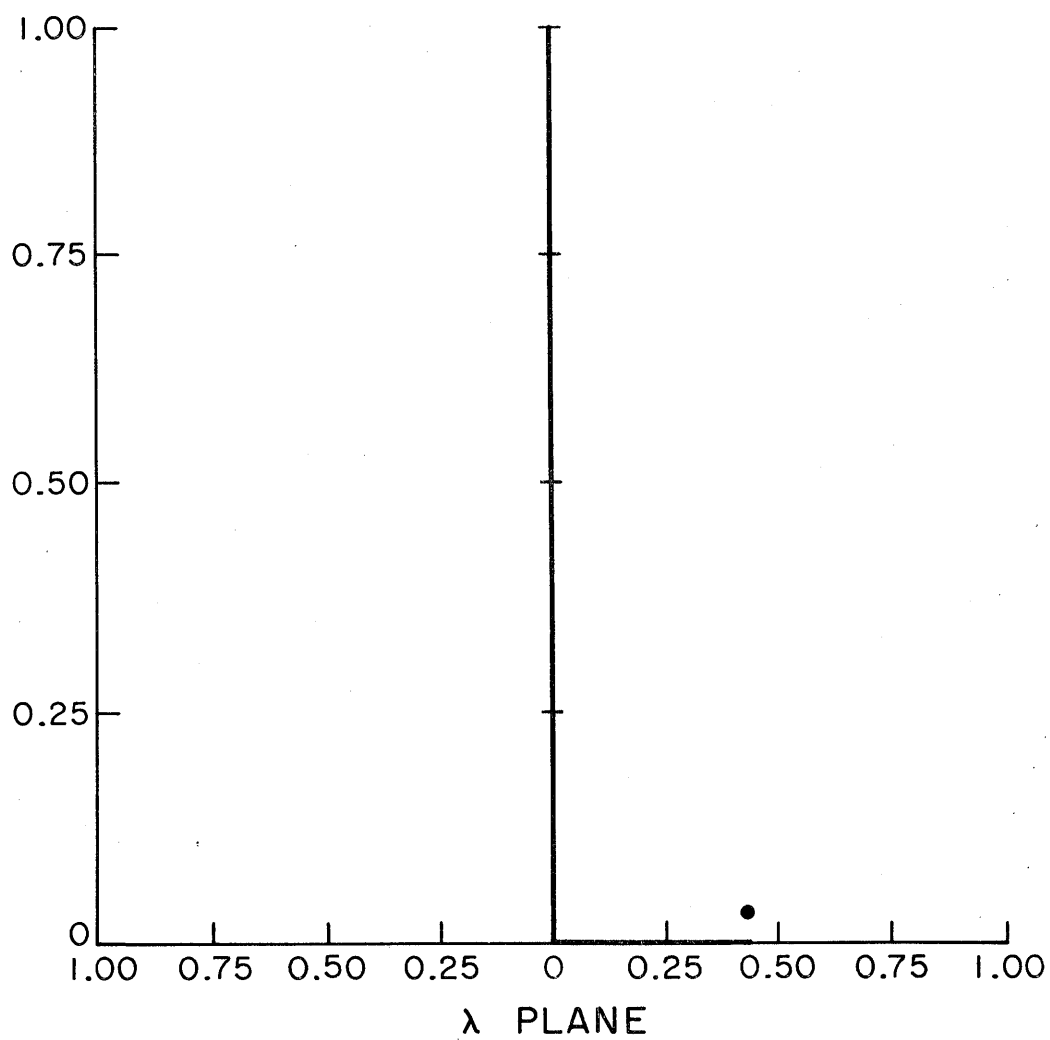


Figure 1.5(a)

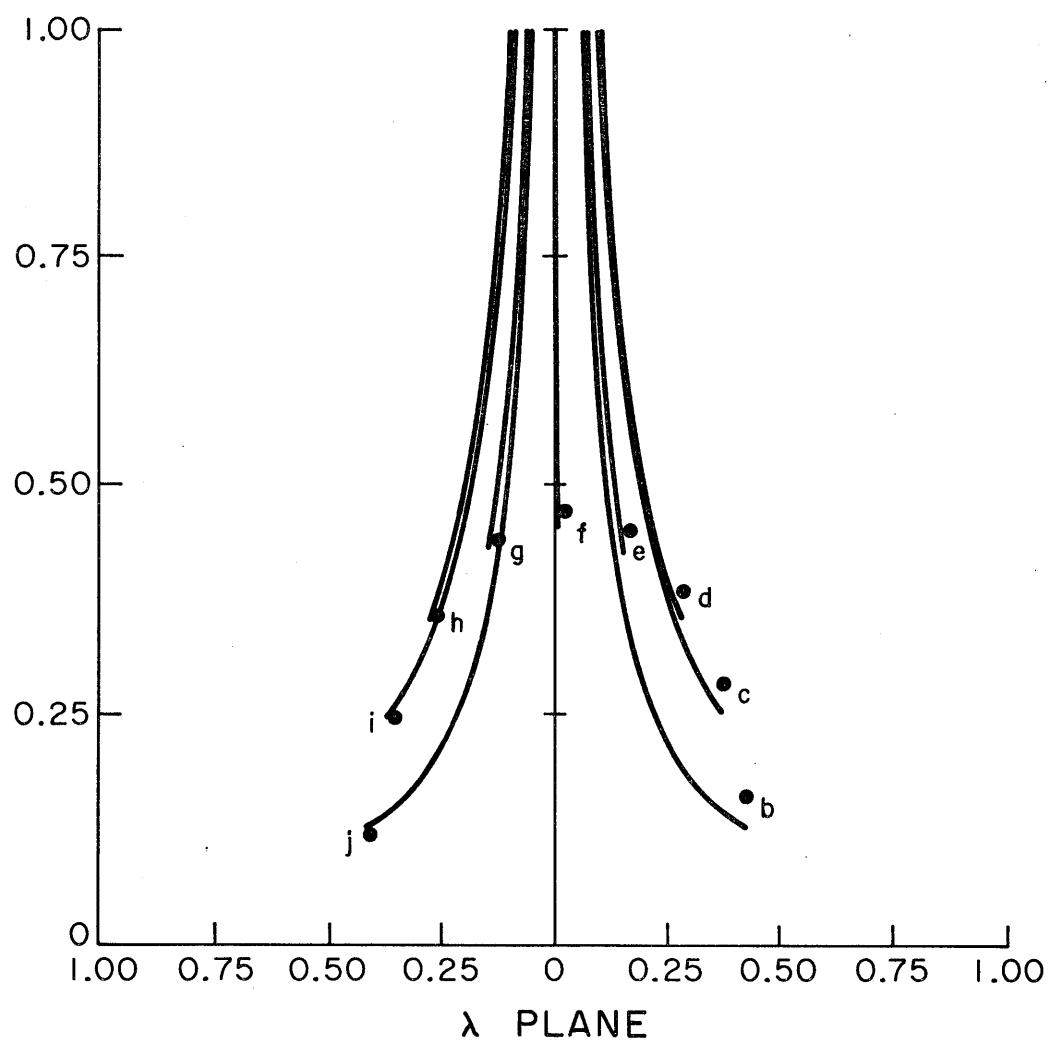


Figure 1.5(b-j)

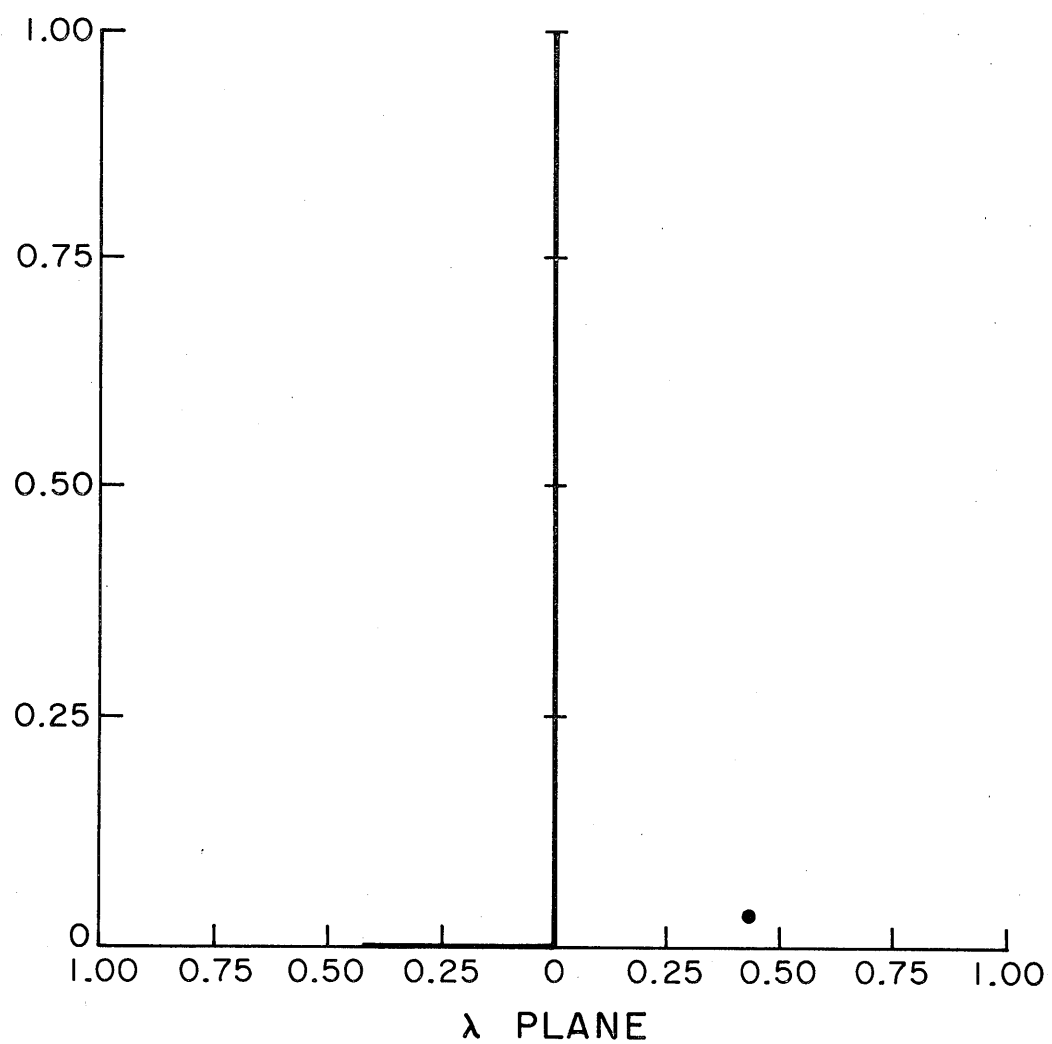


Fig. 1.5(k)

λ plane as α is varied along a path such as the $\ell = b$ path of fig. 1.4. This sequence illustrates the fact that the pole does not cross the branch cut and disappear. This is to be expected since the pole stays on the same Riemann sheet for all α .

An expression for $W_0(\alpha)$ which is valid over the entire α plane is now derived (Appendix A). The expression is

$$W_0(\alpha) = \frac{2\pi i}{\lambda_p} - \frac{2}{(\zeta^2 - 1/n^2)^{\frac{1}{2}}} \{ \ln[1/n - i(\zeta^2 - 1/n^2)^{\frac{1}{2}}] - \ln \zeta + i\pi \} \quad (1.18)$$

where $\text{Im } \lambda_p, \text{Im}(\zeta^2 - 1/n^2)^{\frac{1}{2}} \geq 0$ and the principal value in each of the two logarithmic terms is chosen. It follows that the first term has a square root singularity at the branch point $\alpha = (1 - 1/n^2)^{\frac{1}{2}} = \alpha_B$. The second term however has only a logarithmic singularity associated with the branch point $\alpha = 1$.

The finite integral $W_d(\alpha)$ is evaluated in Appendix B, and the result is repeated here as

$$W_d(\alpha) = -i\pi \sum_{m=0}^{\infty} \frac{i^m}{(n^m)_m!} I_m(2D, \zeta), \quad (1.19)$$

where $I_m(2D, s)$ can be expressed in terms of known functions. It is of interest to note that $W_d(\alpha)$ is an analytic function of α which has only a branch singularity of the kind $(1 - \alpha^2)^{\frac{1}{2}}$ at $\alpha = \pm 1$. Figure 1.6 is a plot of the exact value of Q based upon the numerical evaluation of (1.5b) and the approximate value of $Q(\alpha)$ based upon (1.14). An examination of the

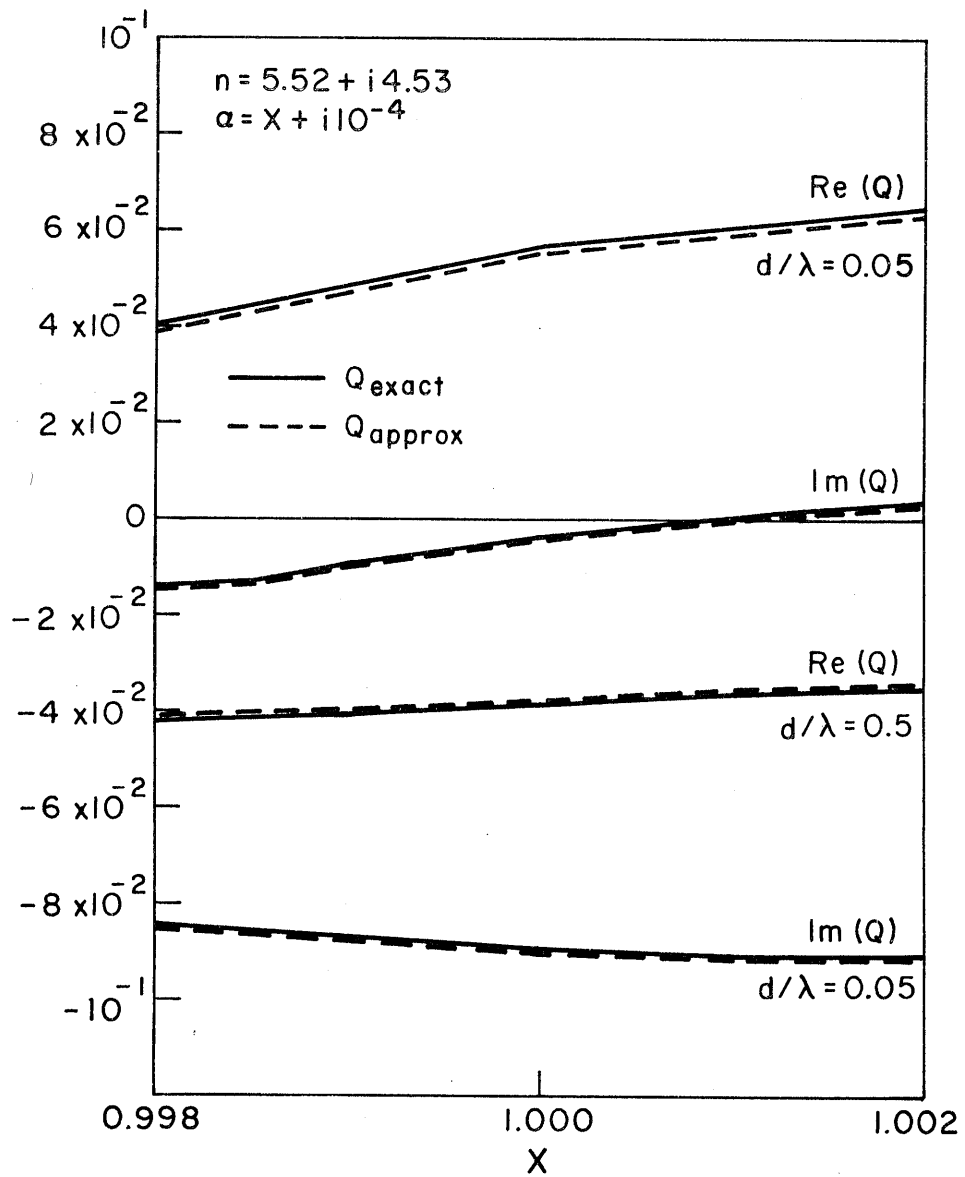


Figure 1.6

exact and approximate $Q(\alpha)$ for small values of d reveals that the agreement between the two is indeed satisfactory for the cases studied.

In the remainder of this work the approximate forms of the integrals are only used for values of the parameters such that accurate results can be obtained. In other cases a numerical integration of the exact form is used.

1.3.1 Some Discussion on The Approximation

In assessing the approximations made it is necessary to point out again that the problem concerned with the location of the zero in the denominator of the integrand of $Q(\alpha)$ is very important. Specifically, the denominator is of the form

$$f(\lambda) = u_2 + n^2 u_1 \quad (1.20)$$

This function has a zero in the λ plane at

$$\lambda_p = [1 - \alpha^2 - 1/(n^2 + 1)]^{1/2} \quad (1.21)$$

The branch point of $Q(\alpha)$ in the α plane occurs at the point where λ_p equals zero. At this point $Q(\alpha)$ exhibits an inverse square root singularity (in the α -plane) and is the dominant part in the evaluation of $M(\alpha)$. Thus a correct expression for λ_p as a function of α is quite critical to the evaluation of $Q(\alpha)$. In the approximation we made earlier, a term of the order $1/n^4$ is dropped so that

$$\lambda_p \approx (1 - \alpha^2 - 1/n^2)^{1/2}. \quad (1.22)$$

This has the effect of slightly moving the branch point (α_B) in the α plane ($\lambda_p(\alpha_B) = 0$). However, the inverse square root singularity at the branch point is preserved.

In the perturbation theory presented by Kikuchi^[2] the substitution of α by unity in $P(\alpha)$ and $Q(\alpha)$ and thus in the expression for $\lambda_p(\alpha)$ is made. Physically this appears to be a valid assumption but under this approximation the location of λ_p can be drastically altered as is evident from (1.20). Since λ_p is now not a function of α , $Q(\alpha)$ no longer exhibits the inverse square root singularity in the complex α -plane. Thus the approximation can lead to a total disappearance of discontinuous behavior in $W_0(\alpha)$ mentioned above.

It should also be mentioned that because of our substitution $u_2 = in$, $Q(\alpha)$ no longer exhibits a branch point at $\lambda = (n^2 - \alpha^2)^{1/2}$. Thus, the entire integration on both sides of the branch cut specified by the line $\text{Re } u_2 = 0$ in the upper half of the complex λ -plane has been ignored. However, for the practical situation of a conducting earth, such omission results in the region $|\alpha| \sim 1$ only an error term of the order $\exp(-2dn_i)$ where n_i is the imaginary part of the refractive index n .

1.4. Discrete Solutions to the Modal Equation

The modal equation for this wave guiding structure can now be written in its approximate form.

$$M(\alpha) = \zeta^2 [H_0^{(1)}(A\zeta)J_0(A\zeta) - H_0^{(1)}(2D\zeta)] + P(\alpha) - Q(\alpha) = 0 \quad (1.23)$$

where $P(\alpha)$ and $Q(\alpha)$ are defined in equations 1.9, 1.13, 1.14, 1.18, and B.3 - B.6. From the preceding analysis it is known that $M(\alpha)$ has branch points at $\alpha = 1$ and at $\alpha = (1 - 1/n^2)^{1/2} = \alpha_B$. The singularity at $\alpha = \alpha_B$ is due to $Q(\alpha)$ and is an inverse square root singularity. Thus, near this singularity, $Q(\alpha)$ is the dominant part of $M(\alpha)$.

The zero of $M(\alpha)$ will now be located and identified as propagating modes of this wave guiding structure. Two special cases which provide some insight into the locations of the zeros are presented. In the first case, a zero can be found approximately by assuming a large n and a small d , so that $Q(\alpha)$ can be neglected. Thus, if we retain only the dominant term of $P(\alpha)$, the following expression is obtained:

$$M(\alpha) = \frac{-2i}{\pi} \zeta^2 \ln 2D/A + \frac{4}{2D\pi n} = 0 \quad (1.24)$$

and

$$\alpha_{p1} \approx 1 + i [2Dn \ln 2D/A]^{-1} \quad (1.25)$$

It can be verified that at the location α_{p1} , the value of $Q(\alpha)$ is indeed very small compared to $P(\alpha)$. This zero is then identical to the one presented by Chang^[4] and by Carson^[1].

It is therefore not surprising that Carson's work leads to a correct result for low frequencies. We should further note that this zero represents a slow wave mode since it has a phase velocity smaller than the speed of light in free space. It is designated the transmission line mode or quasi-TEM mode since as $n \rightarrow \infty$ this mode becomes the TEM mode known to exist for a perfectly conducting wire over a perfectly conducting half space. Although (1.25) is an expression for the special case of small d and large n , the pole can be found under general conditions by using a numerical root search routine on the modal equation (1.5) where $P(\alpha)$ and $Q(\alpha)$ may either be evaluated numerically or expressed in their approximate forms. Figure 1.7 is a contour plot of constant energy density for this mode. For this highly conducting half-space it can be seen that the field structure approaches what would be expected of a TEM mode.

The existence of the second zero of $M(\alpha)$ is now discussed. As mentioned earlier, the value of $Q(\alpha)$ is generally very small except in the vicinity of the branch point $\alpha = \alpha_B$ where it has an inverse square root singularity. Since the transmission-line mode is located in a region where $Q(\alpha)$ is negligible, it is therefore stipulated that a new mode, different from the transmission-line mode, might exist in the region where the influence of $Q(\alpha)$ is more dominant. To investigate this possibility, we again take the special case of a large n and small d for consideration. Expansion of $M(\alpha)$ at $\alpha = \alpha_B$ yields the following

$$M(\alpha) \approx \zeta^2 \left(\frac{-2i}{\pi} \ln 2D/A \right) + \frac{2}{n\pi D} - \frac{4i}{n^3} (\zeta^2 - 1/n^2)^{-\frac{1}{2}}$$

By setting $n\zeta = -1 + \Delta$ or $\zeta^2 = (1-2\Delta)/n^2$ we now have an approximate modal equation of the form

$$M(\alpha) \approx \left(\frac{-2i}{n^2 \pi} \right) \ln 2D/A + \frac{2}{n\pi D} - \frac{4i}{n^3 R} = 0; \quad R = (-2\Delta)^{\frac{1}{2}}/n, \quad (1.26)$$

where $\text{Im}(R) > 0$. Thus, the location of the second zero of $M(\alpha)$, i.e. α_{p_2} , can be given as

$$\alpha_{p_2} \approx \alpha_B + \Delta/n^2,$$

where

$$\Delta = \frac{1}{2n^2 \left[\frac{1}{2D\pi} - \left(\frac{i}{2n\pi} \right) \ln 2D/A \right]^2} \quad (1.27)$$

Although for a more general case, a numerical root-finding subroutine is again needed to locate this zero, it is however apparent from the above derivation that the solution of this type is less sensitive to the radius of the wire than that of the transmission-line mode. It also has a phase velocity greater than the speed of light and is thus termed a fast-wave mode. To illustrate the difference between this mode and that of the transmission-line mode, the contours of constant energy density is shown in Fig. 1.8 using the same physical parameters.

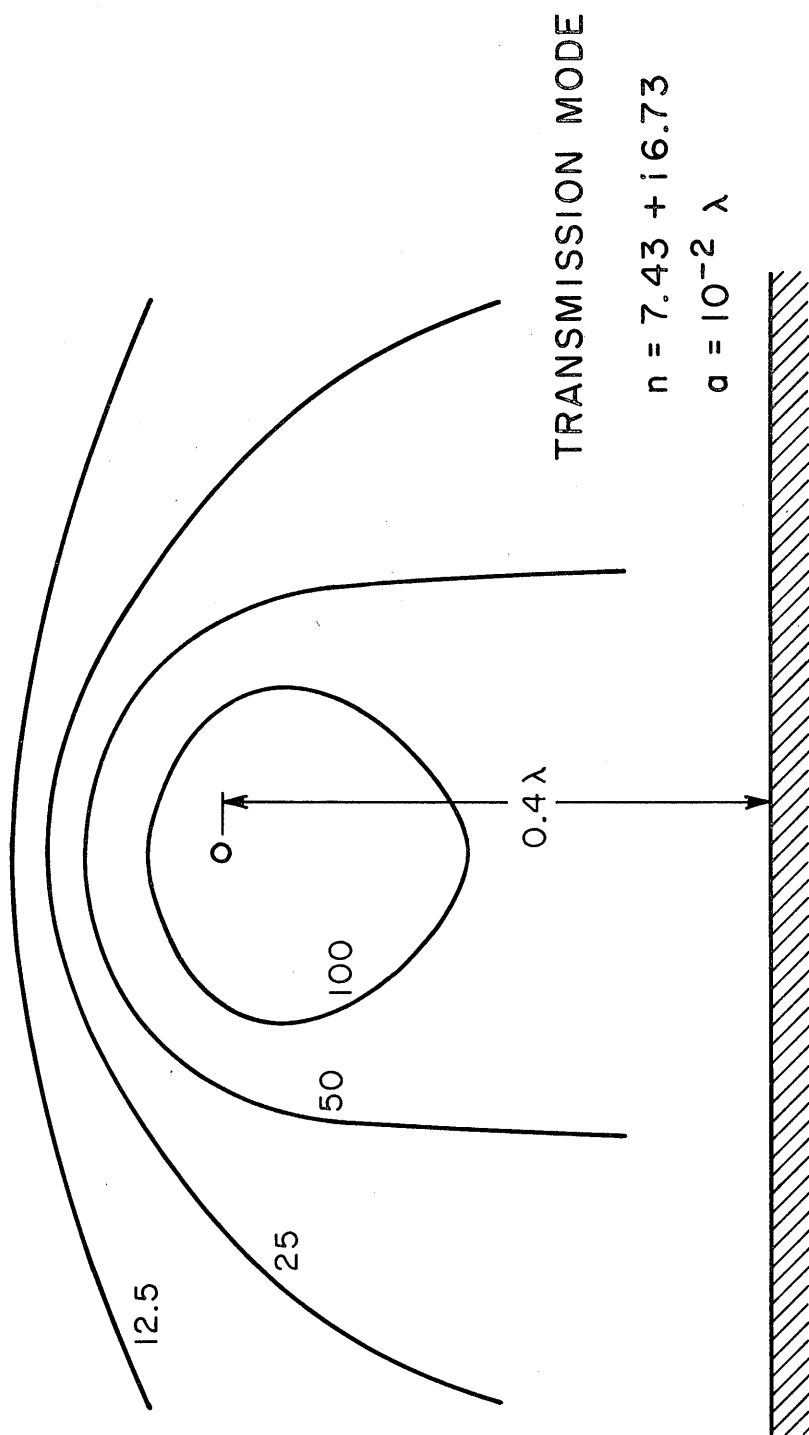


Figure 1.7

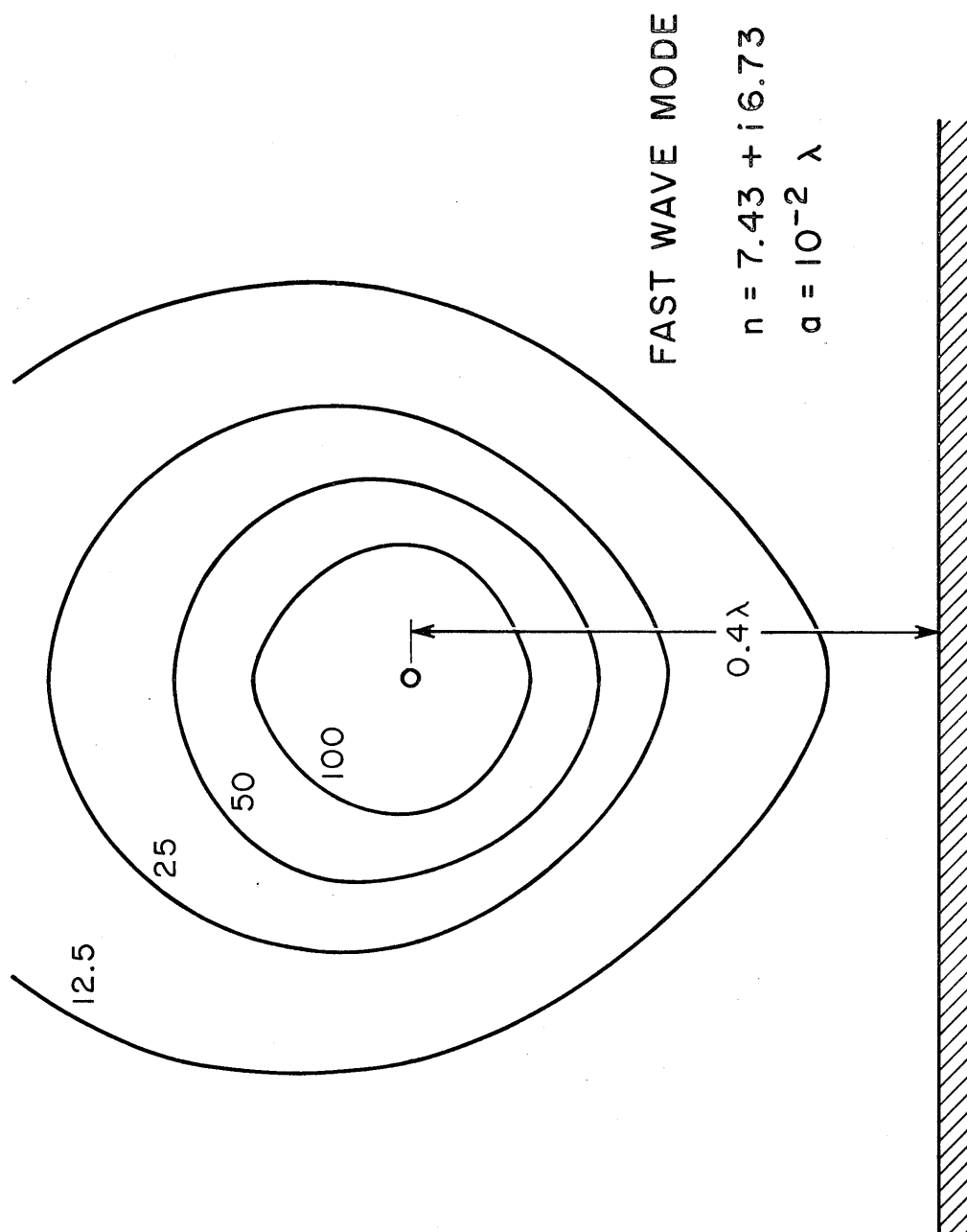


Figure 1.8

As mentioned earlier the locations of the two zeros cannot be obtained from 1.25 and 1.27 in general. Accurate results however can be obtained from the approximate modal equation given by 1.23, 1.9, 1.13, 1.14, 1.14, 1.18 and B.3-B.6. Only the first three terms in the expansion of $W_d(\alpha)$ were normally used. The numerical root searching scheme is based upon the winding-number of a function along a closed contour in the complex α -plane. The zeros were found to a tolerance of 10^{-5} . To indicate the kind of accuracy involved in the approximate modal equation given in (1.22), we have compared in Table 1.1, the computed zeros of (1.22) with those obtained from the exact modal equation, where the integrals $P(\alpha)$, $Q(\alpha)$ are evaluated numerically.

TABLE 1.1

Transmission line mode	numerical	$1.00109 + i 5.508 \times 10^{-3}$
	approximate	$1.00112 + i 5.537 \times 10^{-3}$
Fast wave mode	numerical	$.999072 + i 1.15 \times 10^{-3}$
	approximate	$.999062 + i 1.11 \times 10^{-3}$

$$n = 7.43 + i 6.73$$

$$d = .65\lambda$$

$$a = .01\lambda$$

Figure 1.9 shows the location of the zero corresponding to the transmission-line mode and that of the fast wave mode as a function of height d/λ . The refractive index in this case is chosen as $n = 7.43 + i6.73$. For a typical earth of

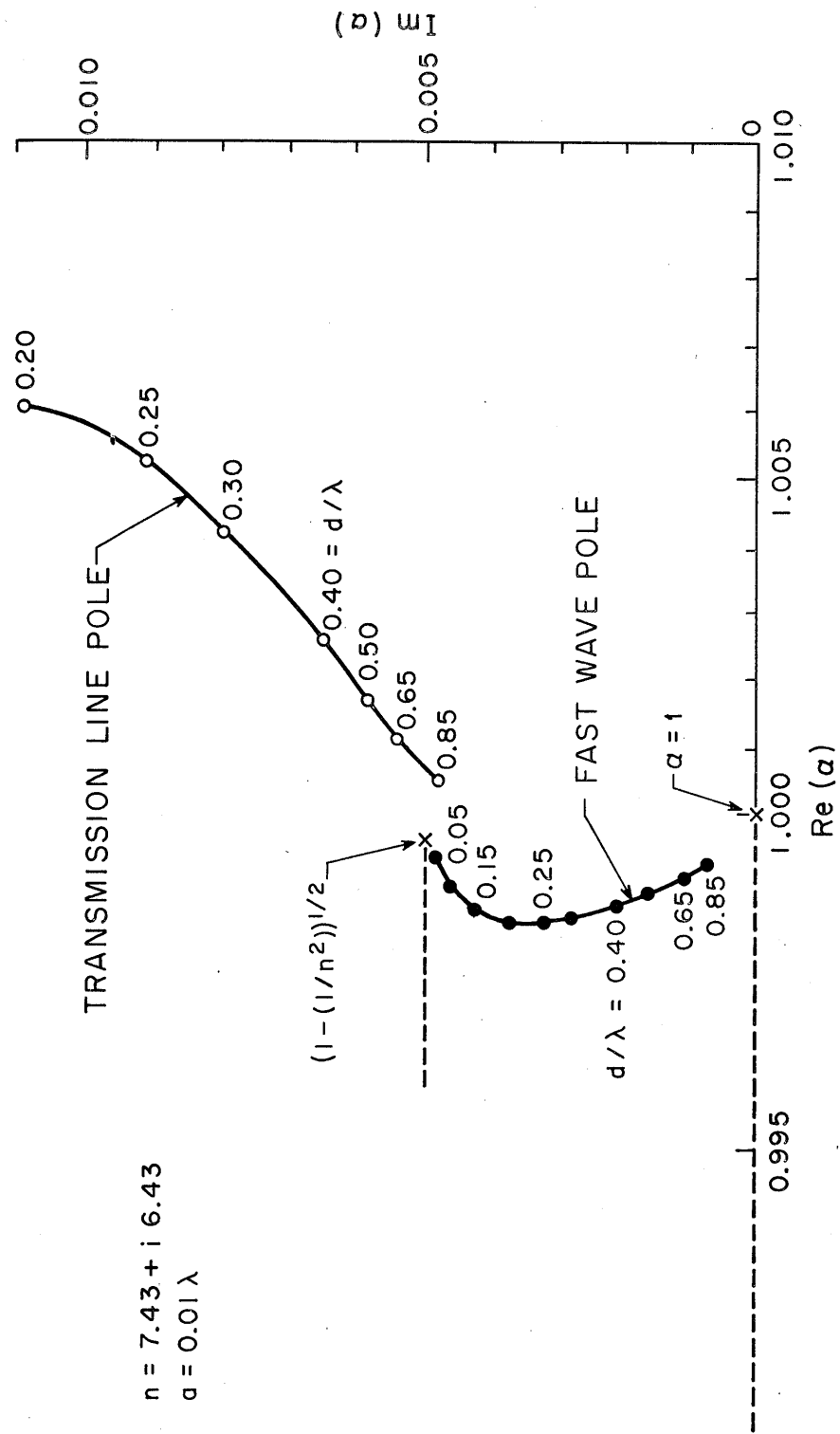


Figure 1.9

relative permittivity = 10 and conductivity = 10^{-2} mhos/m, this value of n corresponds to an operating frequency of 1.8 Mhz. It is of particular significance that the attenuation constant of the fast-wave mode is consistently less than that of the transmission-line mode. From all cases studied, it appears that the upper-bound of the attenuation constant for this new mode can be given by

$$\text{Im}(\alpha_{p2}) \leq \text{Im}\left(\frac{1}{2n^2}\right), \quad (1.28)$$

i.e. the one predicted by the Zenneck's fast-wave in the absence of the horizontal wire.^[13] It must be stressed however that our analysis has no bearing on the existence of a Zenneck's wave.

1.5. Concluding Remarks

The problem of propagating modes supported by a horizontal wire in the presence of a conducting earth is investigated. In addition to the commonly known transmission-line mode, a new solution which can be identified as a fast-wave mode, is found. The existence of such a mode is shown to be a direct consequence of the well-known pole in the Sommerfeld integral representation. Attenuation of this mode along the propagation direction appears to be substantially less than that of the transmission-line mode; an interesting feature which may have strong engineering implication. Since each of the two modes individually satisfies the boundary conditions at the air-earth interface and the radiation condition at infinity, it follows that the field distribution of two modes are mutually orthogonal, i.e.,

$$\iint_S \bar{a}_x \cdot (\bar{E}_1 \times \bar{H}_2 - \bar{E}_2 \times \bar{H}_1) ds = 0 , \quad (1.29)$$

where S is integrated over the entire transverse plane.

In principle, the new fast-wave mode can be excited independent of either the transmission-line mode or any other mode in the continuous spectrum. As it will be shown in Part II, the admittance of an infinitely-long horizontal antenna is strongly dictated by this mode at some heights above the earth at higher frequencies.

APPENDIX A

An expression will now be derived for $W_0(\alpha)$ which is valid over the entire α plane. Consider first the special case in which n has an imaginary part which is vanishingly small. In this case $W_0(\alpha)$ will adopt different forms for the four points shown in figure A.1. The four results will then be used to find a single expression which is valid over the entire α plane.

$$W_0(\alpha) = \int_{-\infty}^{\infty} \frac{d\lambda}{\lambda^2 - \lambda p^2} + \frac{i}{n} \int_{-\infty}^{\infty} \frac{d\lambda}{u_1(\lambda^2 - \lambda p^2)} \quad (A.1)$$

$$= W_{01}(\alpha) + \frac{i}{n} W_{02}(\alpha) .$$

W_{01} is found by deforming the contour and evaluating the residue.

$$W_{01}(\alpha) = \frac{\pi i}{\lambda p} \quad (A.2)$$

where λp is the pole which is found in the upper half of the λ plane.

Expressions will now be found for $W_{02}(\alpha)$ at the four points. At point A $\arg \zeta = \pi$ and the branch cuts in the λ plane are those of figure 1.5(k). The integral $W_{02}(\alpha)$ can be evaluated by deforming its contour around the branch cut and adding to that integration the residue of the pole at λ_p . Using the substitution $\eta = -i(\lambda^2 - \zeta^2)^{\frac{1}{2}}$ for the branch cut integration $W_{02}(\alpha)$ can be written as

$$W_{02}(\alpha) = B(\alpha) + R(\alpha) \quad (\text{A.3})$$

where

$$B(\alpha) = -2 \int_0^{\infty} \frac{d\eta}{(\eta^2 - \zeta^2)^{\frac{1}{2}} (\eta^2 - 1/n^2)} \quad (\text{A.4})$$

and R (the residue at $\lambda = \lambda_p$) is

$$R(\alpha) = \frac{\pi n}{\lambda_p} \quad (\text{A.5})$$

In these calculations $(\zeta^2 - \eta^2)^{\frac{1}{2}} = i(\eta^2 - \zeta^2)^{\frac{1}{2}} = \lambda$. Thus the sign of the square root is determined from the values of λ along the branch cut in the λ plane. The integral B can be treated as the sum of 3 parts ($B_1 + B_2 + B_3$) as suggested by the integration contour in fig. A.2. The pole at $1/n$ occurs to the right of $-\zeta$ since at the point A in the α plane $|n\zeta| < 1$.

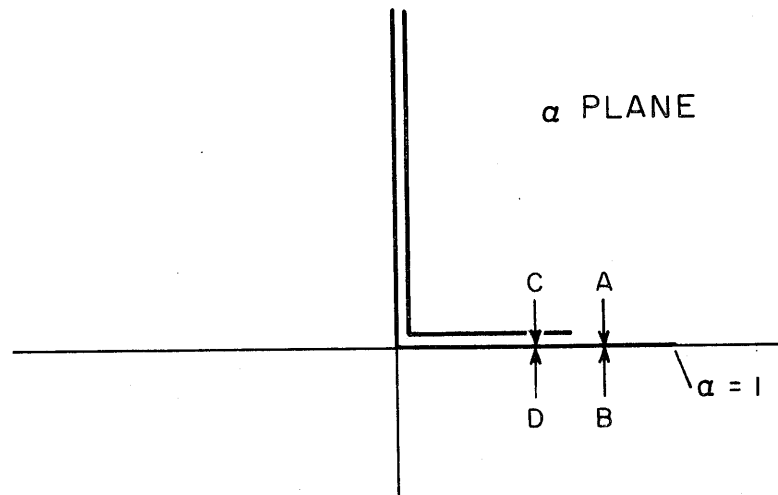


Figure A.1

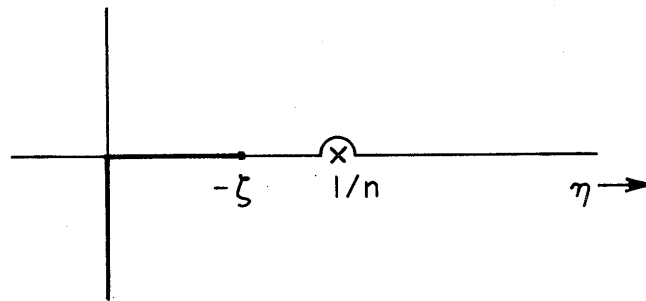


Figure A.2

The first part of B is the integration from 0 to $-\zeta$.

$$B_1 = -i2 \int_0^{-\zeta} \frac{d\eta}{(\zeta^2 - \eta^2)^{\frac{1}{2}} (\eta^2 - 1/n^2)} = \frac{-in\pi}{(1/n^2 - \zeta^2)^{\frac{1}{2}}} \quad (A.6)$$

where the square root is positive since the integrand is positive.

$(\text{Re}(1/n^2 - \zeta^2))^{\frac{1}{2}} \geq 0$ as a consequence of $\text{Im } \lambda \geq 0$.

The second part of B is the principal value of the integral from $-\zeta$ to ∞ . Setting $\eta = -\zeta \cosh \tau$ the integral becomes

$$B_2 = n^2 \left\{ \int_0^{\infty} \frac{d\tau}{n\zeta \cosh \tau + 1} + \int_0^{\infty} \frac{d\tau}{1 - n\zeta \cosh \tau} \right\} \quad (A.7)$$

Since the integrand is real, the integral must be real.

Using this fact, B_2 can be shown to be

$$B_2 = n \left\{ \frac{1}{(1/n^2 - \zeta^2)^{\frac{1}{2}}} \ln \frac{1/n + (1/n^2 - \zeta^2)^{\frac{1}{2}}}{1/n - (1/n^2 - \zeta^2)^{\frac{1}{2}}} \right\}, \quad (A.8)$$

where the principal value of the \ln is chosen. The last part of the integration is the half residue at $\eta = 1/n$ which can be shown to be

$$B_3 = \frac{in\pi}{(1/n^2 - \zeta^2)^{\frac{1}{2}}} \quad (A.9)$$

where the positive square root is again chosen in order to be consistent with the integration path in the λ plane. Thus, at point A in the α plane then $W_{0A}(\alpha)$ is obtained from (A.1-2, 6, 8-9).

$$W_{0A}(\alpha) = \frac{2\pi i}{\lambda_P} + \frac{i}{(1/n^2 - \zeta^2)^{\frac{1}{2}}} \ln \frac{1/n + (1/n^2 - \zeta^2)^{\frac{1}{2}}}{1/n - (1/n^2 - \zeta^2)^{\frac{1}{2}}} \quad (A.10)$$

The evaluation of $W_{02}(\alpha)$ at point B is different because the branch cuts in the λ plane are those of fig. 1.5(a). This difference appears only as a reversal of the sign of the term B_1 . The expression for $W_{0B}(\alpha)$ is then

$$W_{0B}(\alpha) = \frac{i}{(1/n^2 - \zeta^2)^{1/2}} \ln \frac{1/n + (1/n^2 - \zeta^2)^{1/2}}{1/n - (1/n^2 - \zeta^2)^{1/2}} \quad (A.11)$$

For points C and D the integration can be performed in a similar manner. In these two cases the pole at $1/n$ on the contour occurs between 0 and $-\zeta$ because at points C and D in the α plane $|n\zeta| > 1$. The result for points C and D can be written as

$$W_{0C/D}(\alpha) = \mp \frac{2\pi i}{|\lambda_p|} - \frac{2}{|(\zeta^2 - 1/n^2)|^{1/2}} \left\{ \ln [1/n - i |(\zeta^2 - 1/n^2)^{1/2}|] - \ln \zeta \right\}. \quad (A.12)$$

The absolute value signs emphasize that in the third part of the branch integration $\text{Im}(\zeta^2 - 1/n^2)^{1/2} > 0$ on the integration path for both C and D. W_{0A} and W_{0B} can be rewritten as

$$W_{0A} = \frac{2\pi i}{\lambda_p} + \frac{i2}{(1/n^2 - \zeta^2)^{1/2}} \ln \left[\frac{1/n + (1/n^2 - \zeta^2)^{1/2}}{|\zeta|} \right] \quad (A.13)$$

and

$$W_{0B} = \frac{i2}{(1/n^2 - \zeta^2)^{1/2}} \ln \left[\frac{1/n + (1/n^2 - \zeta^2)^{1/2}}{|\zeta|} \right] \quad (A.14)$$

A single expression valid at all four points is now written as

$$W_0(\alpha) = \frac{2\pi i}{\lambda_p} - \frac{2}{(\zeta^2 - 1/n^2)^{1/2}} \left\{ \ln [1/n - i(\zeta^2 - 1/n^2)^{1/2}] - \ln \zeta + i\pi \right\} \quad (A.15)$$

where $\text{Im } \lambda_p, \text{Im}(\zeta^2 - 1/n^2)^{\frac{1}{2}} \geq 0$ and the principal value of each of the \ln terms is chosen. Since there are no other singularities in the upper half of the α plane this expression is then valid for the entire half plane. Using physical continuity arguments it is also to be expected that (A.15) is valid for complex n . Further justification of these statements will come in a later section where a numerical comparison is made with the numerical evaluation of the exact integral form of the expression. The first term of (A.15) has a branch point in the α plane at $n\zeta = -1$. The branch cut associated with it corresponds to the line of discontinuity of figure 1. An examination of the integrations reveals that the first term is precisely the residue of the pole of the integrand of $W_0(\alpha)$. Thus the branch point at $n\zeta = -1$ is a direct consequence of the existence of the pole. In order to emphasize this, we have used λ_p in the first term of (A.15) instead of $(\zeta^2 - 1/n^2)^{\frac{1}{2}}$. The second term can be shown not to have a branch point at $n\zeta = -1$ and to be finite there. Thus at $n\zeta = -1$ the function $W_0(\alpha)$ only has an inverse square root singularity. However, $W_0(\alpha)$ also has branch points at $\alpha = \pm 1$ due to the $\ln \zeta$ in the 2nd term of (A.15).

APPENDIX B

The finite integral $W_d(\alpha)$ is now evaluated. The first significant fact is that the integrand has no pole at λ_p since the numerator has a zero which cancels the zero of the denominator at λ_p . Since in the case of $W_0(\alpha)$ the pole led to the branch point in the α plane at $n\zeta = -1$ it is expected that $W_d(\alpha)$ has no similar branch point. If now we recast $W_d(\alpha)$ in the form of

$$\begin{aligned} W_d(\alpha) &= - \int_0^{2D} \exp(iW/n) \int_{-\infty}^{\infty} \frac{e^{-Wu_1}}{u_1} d\lambda dW \\ &= -i\pi \int_0^{2D} \exp(iW/n) H_0^{(1)}(W\zeta) dW . \end{aligned} \quad (B.1)$$

When use is made of the identity given in (1.10) the integral may be analyzed as follows:

$$\exp(iW/n) = \sum_{M=0}^{\infty} \frac{(iW)^M}{(n)^M M!} \quad (B.2)$$

Thus,

$$W_d(\alpha) = -i\pi \sum_{M=0}^{\infty} \frac{(i)^M}{(n)^M M!} \int_0^{2D} W^M H_0^{(1)}(W\zeta) dW \quad (B.3)$$

Consider the general integral

$$I_M(2D, \zeta) = \int_0^{2D} W^M H_0^{(1)}(W\zeta) dW \quad (B.4)$$

It is possible to write a recursion relation as follows.

$$I_M(2D, \zeta) = \frac{(2D)^M}{\zeta} H_1^{(1)}(2D\zeta) + \frac{(M-1)}{\zeta^2} (2D)^{M-1} H_0^{(1)}(2D\zeta) - \frac{(M-1)^2}{\zeta^2} I_{M-2}(2D, \zeta) \quad (B.5)$$

where $I_0(2D, \zeta)$ and $I_1(2D, \zeta)$ can be expressed in closed form:

$$\begin{aligned} I_0(2D, \zeta) &= \int_0^{2D} H_0^{(1)}(W\zeta) dW = \\ &= 2D H_0^{(1)}(2D\zeta) + \frac{2D\pi}{2} \left[|S|_0(2D\zeta) H_1^{(1)}(2D\zeta) - |S|_1(2D\zeta) H_0^{(1)}(2D\zeta) \right] \end{aligned} \quad (B.6)$$

Here, $|S|_j(x)$ is the Struve function of order j ; [10]

$$I_1(2D, \zeta) = \frac{1}{\zeta^2} \{ 2D\zeta H_1^{(1)}(2D\zeta) + \frac{2i}{\pi} \} \quad (B.7)$$

For the case $(2D\zeta)^2 \ll 1$, an alternate approximation to $W_d(\alpha)$ is obtained. It is done by substituting the small argument expansion of the Hankel function

$$H_0^{(1)}(W\zeta) \approx \frac{i2}{\pi} \left\{ \frac{-i\pi}{2} + \ln \frac{n\zeta W}{2} + \gamma \right\}$$

then $W_d(\alpha)$ can be written as

$$W_d(\alpha) \approx -\pi n \exp(i2D/n) H_0^{(1)}(2D\zeta) - i2n [E_1(-i2D/n) - \ln n\zeta/2] \quad (B.8)$$

where $E_1(x)$ is the exponential integral of order 1.

Equation B.8 agrees with the one previously obtained by Chang. [4]

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Acknowledgment

We are indebted to Professors J.R. Wait and S.W. Maley for their continuous interests and stimulating discussions on this and other related projects. We also like to thank Mrs. Marie Kindgren for her preparation of the manuscript, Mr. E.K. Kuester for his technical assistance. The project is supported by the U.S. National Oceanic and Atmospheric Administration under Grant N22-126-72(G).