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STATIONARY PRINCIPLES FOR OPERATOR EQUATIONS
WITH APPLICATIONS TO ELECTROMAGNETIC THEORY

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1.0 Introduction

This report is devoted to an approach which, starting from differential or integral equations, leads to stationary forms of expressions for certain quantities such as the self-impedance of an antenna, the radiation pattern, etc. It is also shown that for certain simple geometries, not necessarily separable, it is possible to obtain an improvement for the unknown field itself. Although there exist a number of papers which discuss the formulation of the stationary forms for various quantities of interest, there does not seem to be available a unified approach leading one to a variational form for the desired quantity. In the literature frequently certain manipulations are carried out with an integral equation and then it is asserted that the resulting form is variational; occasionally it is demonstrated after lengthy differentiation that this may indeed be so. However, no clue is usually provided a priori as to how one should know how to proceed in a definite direction and arrive at the desired form.

In the following, a general approach will be outlined, starting from an inhomogeneous operator equation

\[ L\psi = a, \quad (1.1) \]

from which one may derive the stationary forms for a quantity related to the unknown \( \psi \) and for \( \psi \) itself.

2.0 Derivation of Stationary Forms from Error Analysis

In (1.1) let \( L \) be an integral or differential operator, let \( \psi \) be an unknown function, and let "a" be either a known function, or be equal to \( \lambda \psi \), in which case (1.1) becomes an eigenvalue equation.

The problem is to find a certain quantity \( g \), which is related to \( \psi \) through the equation

\[ g = < b, \psi >, \quad (2.1) \]

where \( < > \) implies a suitably defined scalar product. It will be found later that the quantity \( g \) can be made to correspond, for suitable forms of
b, to certain quantities such as impedance and the like. If a solution of (1.1) were available, certainly (2.1) would then provide the desired quantity $g$. Oftentimes, however, only an approximation to $\Psi$, say $\Psi_a$, is known. Through an error-analysis technique, a method is here to be developed which will yield answers correct to the second order for $g$, when $\Psi_a$ has a first-order deviation from the correct $\Psi$. The eigenvalue problem in which the desired quantity is $\lambda$, and $a = \lambda \Psi$, is also of interest. Here again it will be shown how one can construct stationary forms for $\lambda$ based on the method of error analysis; in fact, it will be instructive to work out the latter problem first.

So consider first a one-dimensional eigenvalue problem stated by the equation and boundary conditions*:

$$L\Psi(x) = \lambda \Psi(x); \text{where } \Psi(x) = 0 \text{ for } x = 0, \alpha$$  \hspace{1cm} (2.2)

where $L$ is a suitable differential or integral operator. It will be assumed that the operator $L$ is self-adjoint, and that any two functions $\eta_1(x), \eta_2(x)$, which satisfy the same boundary conditions as $\Psi$, satisfy the relations

$$< \eta_1, L \eta_2 > = < \eta_2, L \eta_1 >; \hspace{0.5cm} < \eta_1, \eta_2 > = < \eta_2, \eta_1 >$$  \hspace{1cm} (2.3)

where the scalar product is to be suitably defined. To this end, it is assumed that the definition of the scalar product is such that the following relation is satisfied for any three functions $x_1, x_2, x_3$:

$$< x_1 x_2 + x_3 > = < x_1 x_2 > + < x_1 x_3 >$$

(One of the common definitions of a scalar product for two functions $h_1, h_2$ is

$$< h_1, h_2 > = \int_0^\alpha h_1 h_2 \, dx.$$  \hspace{1cm} (2.4)

However, the considerations outlined here are valid for more general types of scalar products, so long as (2.3) is satisfied for the operator under consideration.)

Inspection of (2.2) shows that it would not do to calculate $\lambda$ using the equation

*It is not necessary to include the boundary conditions on $\Psi$ if $L$ is an integral operator, but they are necessary to define $\Psi$ uniquely when $L$ is a differential operator.
\[
\lambda = \frac{L\psi}{\psi} \quad (2.4)
\]

unless, of course, the correct \(\psi\) is already known. This is because any approximation \(\psi_a\) of \(\psi\) is not going to produce a constant \(\lambda\) when substituted in the right hand side of (2.4). It might be better to attempt to find two suitable constants, each related to \(\psi\), rather than use the functions \(L\psi\) and \(\psi\) themselves. One can, for instance, evaluate two constants \(\langle h, L\psi \rangle\) and \(\langle h, \psi \rangle\), then form the ratio (2.4), and write:

\[
\lambda = \frac{\langle h, L\psi \rangle}{\langle h, \psi \rangle}; \text{ where } h = 0 \text{ for } x = 0, \alpha \quad (2.5)
\]

So far nothing has been said about the function \(h(x)\). It is necessary to pick the function \(h\) suitably such that the right hand side of (2.5) is insensitive to first-order errors in \(\psi\). This is accomplished by following through an error analysis as is demonstrated in the following: let the approximation to \(\psi\) be given by

\[
\psi_a = \psi + \beta; \quad \beta = 0 \text{ for } x = 0, \alpha \quad (2.6)
\]

where \(\beta\) is a first-order variation from the correct form of \(\psi\). Then, if the function \(h\) satisfies the same boundary conditions as \(\psi\), the scalar product of the numerator of (2.5) becomes, from the application of properties (2.3)

\[
\langle h, L\psi_a \rangle = \langle h, L\psi \rangle + \langle h, L\beta \rangle = \lambda \langle h, \psi \rangle + \langle h, L\beta \rangle = \lambda \langle h, \psi \rangle + \beta \langle h, Lh \rangle \quad (2.7)
\]

One similarly has, for the approximation to the denominator of (2.5), that

\[
\langle h, \psi_a \rangle = \langle h, \psi \rangle + \langle \beta, \psi \rangle \quad (2.8)
\]

and hence,

\[
\frac{\langle h, L\psi_a \rangle}{\langle h, \psi_a \rangle} = \frac{\lambda \langle h, \psi \rangle + \beta \langle h, Lh \rangle}{\langle h, \psi \rangle + \langle \beta, \psi \rangle} \quad (2.9)
\]

Recall now that the function \(h\) is to be chosen such that the right hand side of (2.9) is \(\lambda + O(\beta^2)\). From the form of (2.9) it is seen that this
will be accomplished, if \( h \) is required to satisfy

\[
Lh = \lambda h,
\]

but the fly in the ointment is that this still does not help one determine \( h \). This is because if \( h \) does satisfy the above equation, it must, by the definition of \( \psi \), equal \( \psi \) itself. But note that a first-order error in \( h \) will produce only second order errors in \( \langle \beta, Lh \rangle \) and \( \langle \beta, h \rangle \), and therefore also only a second order error in the right hand side of (2.9). If, for instance, one substitutes in (2.9) an approximation for \( \psi \), i.e. \( \psi_a = (\psi + \beta) \) for \( h \), one has

\[
\frac{\langle \psi_a, L \psi_a \rangle}{\langle \psi_a, \psi_a \rangle} = \lambda \frac{\langle \psi_a, \psi \rangle + \langle \beta, L \psi \rangle + \langle \beta, L \beta \rangle}{\langle \psi_a, \psi \rangle + \langle \beta, \psi \rangle + \langle \beta, \beta \rangle}
\]

\[
= \lambda \left(1 + \frac{\langle \beta, L \beta \rangle}{\lambda \langle \psi_a, \psi \rangle + \langle \beta, \psi \rangle}\right)
\]

\[
= \lambda + O(\beta^2)
\]

Hence, it has been shown that

\[
\lambda = \frac{\langle \psi, L \psi \rangle}{\langle \psi, \psi \rangle}
\]

is stationary with respect to first-order variations about the correct form of \( \psi \).

Next, consider the inhomogeneous problem defined by the equation

\[
L \psi = a
\]

where "\( a \)" is a specified function of position in a specified range. Let it be required to find a stationary expression for \( g \) where

\[
g = \langle b, \psi \rangle
\]

and let an auxiliary function \( U \) be defined such that
\[ U = b \quad (2.13) \]

Then, alternative representations for \( g \) are

\[ g = \langle b, \psi \rangle = \langle LU, \psi \rangle = \langle U, L\psi \rangle = \langle U, a \rangle \quad (2.14) \]

Now, let it be assumed that approximations for \( U \) and \( \psi \), say \( U_a \) and \( \psi_a \), are available, and let

\[ \psi_a = \psi + \delta_1 \]

\[ U_a = U + \delta_2 \quad (2.15) \]

where \( \delta_1 \), \( \delta_2 \) are the errors involved. Then, using (2.13) and (2.14) one can derive the following relation between the approximation \( g_a \) and the function \( g \):

\[ g_a = \langle LU_a, \psi_a \rangle = \langle (LU + L\delta_2), (\psi + \delta_1) \rangle \]

\[ = \langle LU, \psi \rangle + \langle L\delta_2, \psi \rangle + \langle LU, \delta_1 \rangle + \langle L\delta_2, \delta_1 \rangle \]

\[ = g + \langle \delta_2, a \rangle + \langle b, \delta_1 \rangle + \langle L\delta_2, \delta_1 \rangle \quad (2.16) \]

error terms

From (2.16), it is clear that if approximations \( U_a \) and \( \psi_a \) are used in the representation for \( g \) given in (2.14) it would produce the error terms shown in the right hand side of (2.16). It should be noted that the first-order errors terms are \( \langle \delta_2, a \rangle \) and \( \langle \delta_1, b \rangle \), and the objective is to find an expression for \( g_a \) which would be free of these first-order errors.

It is easily seen that \( \langle \delta_2, a \rangle \) and \( \langle \delta_1, b \rangle \) are the error terms corresponding to \( \langle U, a \rangle \) and \( \langle \psi, b \rangle \). Furthermore, from (2.14), \( \langle U, a \rangle \) and \( \langle \psi, b \rangle \) are the alternate representations for \( g \). In other words one can write using (2.15),

\[ \langle U_a, a \rangle = \langle U, a \rangle + \langle \delta_2, a \rangle = g + \langle \delta_2, a \rangle \quad (2.17) \]

and

\[ \langle \psi_a, b \rangle = \langle \psi, b \rangle + \langle \delta_1, b \rangle = g + \langle \delta_1, b \rangle \quad (2.18) \]
It is now a straightforward step to combine (2.16), (2.17) and (2.18) and derive:

\[- \langle U_a, \psi_a \rangle + \langle U_a, a \rangle + \langle \psi_a, b \rangle = g \text{ + second order error terms} \tag{2.19}\]

which is equivalent to saying that

\[g = \langle U, a \rangle + \langle \psi, b \rangle - \langle LU, \psi \rangle, \tag{2.20}\]

which is an expression for \(g\) that is insensitive to first-order changes of \(U\) and \(\psi\) about their correct forms. A second form, which can be easily verified to be stationary by the use of (2.16), (2.17) and (2.18), is

\[g = \frac{\langle U, a \rangle \langle \psi, b \rangle}{\langle LU, \psi \rangle} \tag{2.21}\]

For most problems, (2.21) is more suitable than (2.20) because it is independent of the scale factor of the assumed forms of \(U\) or \(\psi\). It should be pointed out that it is also possible to devise other stationary forms of \(g\).

Another problem with which one is often confronted is that of finding a better approximation for \(\psi\) itself. It will be rather useful to have a stationary form of representation for the solution of (2.12) itself. A formula is easily devised by following the approach outlined above. The problem becomes similar to the one discussed immediately above if one sets \(g = \psi\). From the definition of \(g\) given by (2.12) or

\[g = \langle b, \psi \rangle\]

one has \(b = \delta(x - x_o)\), since

\[\int \psi(x_o) \delta(x - x_o) \, dx_o = \psi(x), \text{ for } x \text{ inside defined range,}\]

and where

\[\delta(x - x_o) = \text{Dirac delta function.}\]

One then has to find an approximate solution of the equation
\[ LG = \delta(x - x_0) \]

and use it in (2.20) or (2.21) which yields

\[ \psi(x_0) = \langle \psi, \delta(x - x_0) \rangle + \langle G, a \rangle - \langle G, L\psi \rangle \]  \hfill (2.22)

and

\[ \psi(x_0) = \frac{\langle \psi, \delta(x - x_0) \rangle}{\langle G, L\psi \rangle} \langle G, a \rangle \] \hfill (2.23)

One can use either of the above two equations, whichever form seems convenient for the problem.

Although the discussion above was directed toward one-dimensional operators and functions, it is easy to extend the ideas to two- or three-dimensional cases. The considerations also remain valid when \( \psi \) and \( a \) are vector functions of space. \( L \) also may, in general, be a dyadic operator. In order to apply the above formulas, it is only necessary to make sure that the scalar product is suitably defined and that (2.3) is satisfied.

The formulas in the above are more or less symbolic and mathematical. In order to get a better picture it may be worthwhile at this stage to discuss a few physical problems. The following discussion will be directed only toward the inhomogeneous equation, since it is the one that occurs in the problems which are the subject of discussion in the present report.

3.0 Illustrative Examples of Stationary Formulations

In this section, several examples will be given for the derivation of stationary forms for quantities such as the reflection coefficient, impedance, scattering cross section, etc. The first one discussed concerns a finite symmetrical septum in a parallel-plate waveguide.

3.1 Reflection coefficient for a finite septum in a parallel-plate guide.

The geometry of the problem is shown in Fig. 3.1 It is seen that the corresponding free-space problem is that of an array of metal plates which is used as an artificial dielectric lens at microwave frequencies.

Let the only component of the E-vector be a y-component, and let the incident field be given by
\[ E_1 = e^{-j \gamma_1 z} \sin \frac{\pi x}{a} \] (3.1)

where

\[ \gamma_1 = \left[ k^2 - \left( \frac{\pi}{a} \right)^2 \right]^{1/2} \]

---

\[ \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \]

---

\[ E_y \]

---

\[ x = \frac{a}{2} \]

---

\[ S \]

---

\[ x = a \]

---

\[ x = a \]

---

\[ x = a \]

---

\[ z = D \]

---

Fig. 3.1 Geometry of septum in parallel-plate waveguide.

Using the compensations theorem (given in the Appendix), and with the simplifications possible for only the y-component of \( E \) present, one can derive:

\[ E(x,z) = E_1(x,z) + \int_0^D B(z_o) \ G(x,z,z_o) \ dz_o \]

where

\[ B(z_o) = H_z(x_o,z_o) \bigg|_{x_o = \frac{a+}{2}} - H_z(x_o,z_o) \bigg|_{x = \frac{a-}{2}} \] (3.2)

and \( G(x,z,z_o) = \) Electric field produced at \( (x,z) \) by a unit current line-source oriented along the y-direction, and located at \( (a/2, z_o) \) in the absence of the septum.

The expression for \( G \) is constructed fairly easily. If \( \mathbf{A} = \mathbf{\bar{A}} - A_y \mathbf{y} \)

is the vector potential, then \( A_y \) satisfies the wave equation and boundary conditions:

\[ \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial z^2} + k^2 A_y = -\mu \ \delta(x - \frac{a}{2}) \ \delta(z - z_o) \] (3.3)

where

\[ A_y = 0 \ \text{for} \ x = D, a. \]
The solution of (3.3) is well known, and is given by

\[
A_y = -\frac{\mu}{a} \sum_{n=1}^{\infty} \frac{1}{\gamma_n} e^{-j \gamma_n |z - z_0|} \sin \frac{n\pi x}{a} \sin \frac{n\pi x_0}{a}, \quad x_0 = \frac{a}{2}
\]

\[
= -\frac{\mu}{a} \sum_{n=0}^{\infty} \frac{(-1)^n}{\gamma_{2n+1}} e^{-j \gamma_{2n+1} |z - z_0|} \sin (2n + 1) \frac{n\pi}{a}
\]

where

\[
\gamma_n = -j \left[ \left( \frac{n\pi}{a} \right)^2 - k^2 \right]^{1/2}, \quad n > 1
\]

so

\[
E_y = -j \omega A_y = G(x, z, z_0).
\]

(3.5)

Substituting the expression for \(E_y\) of (3.5) and \(E_z\) of (3.1) into (3.2), one has

\[
E(x, z) = e^{-j \gamma_1 z} \sin \frac{n\pi}{a} + \int_0^D P(z_0) \sum_{n=0}^{\infty} \frac{(-1)^n}{\gamma_{2n+1}} e^{-j \gamma_{2n+1} |z - z_0|} \sin (2n + 1) \frac{n\pi}{a} \, dz_0
\]

(3.6)

where let

\[
P(z_0) = -\frac{\omega}{a} B(z_0).
\]

Now let the quantity of primary interest be the reflection coefficient \(R\) in the guide. If the guide only allows the propagation of the dominant mode, then for large negative \(z\), one has
\[
\lim_{z \to -\infty} E(x,z) = e^{-j \gamma_1 z} \sin \frac{\pi x}{a} + \Re e^{j \gamma_1 z} \sin \frac{\pi x}{a} 
\] (3.7)

From (3.6)
\[
\lim_{z \to -\infty} E(x,z) = e^{-j \gamma_1 z} \sin \frac{\pi x}{a} + e^{j \gamma_1 z} \sin \frac{\pi x}{a} \int_{o}^{D} p(z_0) e^{-j \gamma_1 z_0} dz_0 
\] (3.8)

Equation (3.8) follows from the fact that the \( \gamma_n \) are negative imaginary quantities for \( n > 1 \). Comparing (3.7) and (3.8) one has then
\[
R = \int_{o}^{D} p(z_0) e^{-j \gamma_1 z_0} dz_0 
\] (3.9)

Now if one lets \( x = a/2 \) for \( 0 < z < D \) in (3.6) one obtains the integral equation
\[
e^{-j \gamma_1 z} = -\int_{o}^{D} p(z_0) \sum_{n=0}^{\infty} \frac{1}{\gamma_{2n+1}} e^{-j \gamma_{2n+1} |z-z_0|} dz_0 
\]
\[
= \int_{o}^{D} p(z_0) K(z,z_0) dz_0 
\] (3.10)

where
\[
K(z,z_0) = -\sum_{n=0}^{\infty} \frac{1}{\gamma_{2n+1}} e^{-j \gamma_{2n+1} |z-z_0|} dz_0 
\]
\[
= K(z_0,z) 
\]

Equation (3.10) may be written in a symbolic form as
\[
L \Psi = \eta 
\]
where $L$ is the integral operator with the kernel $K(z, z_o)$, $\psi = P(z_o)$, and $\eta = e^{-j\gamma_1 z}$. Because of the symmetry of $K(z, z_o)$ with respect to its arguments, it is seen that the operator $L$ is of the type for which (2.3) is satisfied.

Since the desired quantity is the reflection coefficient $R$, one has from (3.9) the scalar product

$$ R = \langle b, \psi \rangle = \int_0^D b(z) \psi(z) \, dz, $$

where $b(z) = \eta(z)$.

Hence it is obvious that the auxiliary function $U$, which is required to satisfy $LU = b$, is identically equal to $\psi$. Then, using (2.21), the stationary form for $R$ is immediately obtained as

$$ R = \frac{\langle \eta, \psi \rangle^2}{\langle \psi, L\psi \rangle} \tag{3.11} $$

or explicitly as

$$ R = \frac{\left[ \int_0^D e^{-j\gamma_1 z} P(z) \, dz \right]^2}{\int_0^D \int_0^D P(z_o) K(z, z_o) P(z) \, dz \, dz_o} \tag{3.12} $$

Equation (3.12) is stationary with respect to first-order changes in $P(z)$ about its correct form. It is also insensitive to any scale factor associated with $P(z)$. It is seen that the advantage of following the present method is that the stationary form for the reflection coefficient may be arrived at in a logical manner. Furthermore, it is unnecessary to go through lengthy differentiations to prove that (3.12) is indeed stationary. The proof carried out earlier in a symbolic form certainly holds in the present case.
(3.2) Self-impedance of an antenna in the presence of a circular ground screen.

Consider the problem of calculating the self-impedance of an antenna placed along the axis of a circular perfectly-conducting ground plane of finite radius.

Fig. 3.2 Geometry of antenna above circular ground screen.

The geometry is shown in Fig. 3.2.

From (5.1) in the Appendix, we have

$$
\int \int_S \left[ \vec{E}_A \times \vec{H}_B - \vec{E}_B \times \vec{H}_A \right] \cdot d\vec{S} = \int \int_R \vec{J}_B \cdot (\vec{E}_A - \vec{E}_A) \, dv
$$

(3.13)

where for this problem, magnetic current sources are absent, so

$$\vec{M}_B = 0.$$

Here, \(\vec{E}_A, \vec{H}_A\) and \(\vec{E}_B, \vec{H}_B\) define the fields caused separately by the antennas \(A\) and \(B\) in the presence of an infinite screen, while \(\vec{J}_B\) denotes the current density distribution in the antenna \(B\). The primed fields are those that exist in the presence of the finite screen, and \(S\) is the surface outside of the screen; i.e., let \(S\) be the region \(\rho > a\) for \(z = 0\). The left-hand side, when simplified and using the fact that the only non-zero fields are \(E_\rho\) and \(H_\phi\), reads thus:

$$
\int \left[ \vec{E}_A' \times \vec{H}_B - \vec{E}_B' \times \vec{H}_A \right] \cdot d\vec{S} = 2\pi \int_0^\infty \left[ \frac{E_{\rho A}'}{\rho A} \right] H_{\phi B} \left|_{z=0} \right. \, d\rho,
$$

(3.14)

since \(\vec{E}_B = 0\) at \(z = 0\). The right-hand side may also be simplified if
B is a dipole antenna fed by a pair of terminals, for if a terminal current $I_B$ produces the fields $\overline{E}_B$, $\overline{H}_B$, and if the open-circuit voltage at the terminals of B due to antenna A is $V_{BA}$, then one may show that

$$\int \overline{J}_B \cdot (\overline{E}_A' - \overline{E}_A) \, dv = (V_{BA} - V_{BA}') \, I_B$$

(3.15)

If the antenna A is fed by a current $I_A$ at its terminals when producing fields $E_A$ and $E_A'$, then $V_{BA}$ and $V_{BA}'$ may be simply expressed as

$$V_{BA} = -Z_{BA} \, I_A'$$

$$V_{BA}' = -Z_{BA}' \, I_A$$

where $Z_{BA} = Z_{AB}$ is the mutual impedance between A and B in the presence of an infinite ground screen, and $Z_{BA}'$ is the same with the finite screen. (3.15) may thus be rewritten as

$$Z_{BA}' - Z_{BA} = \frac{2\pi}{I_B I_A} \int_a^\infty E_{\rho A}'(\rho) \, H_\phi(\rho) \, d\rho.$$  \hspace{1cm} (3.16)

Now if antenna B is made to coincide with antenna A, one immediately obtains from (3.16) an expression for $\Delta Z_{AA}$, the change in the self-impedance $Z_{AA}$. The expression (3.16) then can be written:

$$(Z_{AA}' - Z_{AA}) = \Delta Z_{AA} = \frac{2\pi}{I_A} \int_0^a E_{\rho A}'(\rho) \, H_\phi(\rho) \, d\rho.$$ \hspace{1cm} (3.17)

where the subscripts A, B on the fields have been dropped.

The change in the self-impedance which is produced by the presence of a finite ground screen as compared to the infinite ground plane has thus been expressed in terms of the integral of the product of the E field at $z = 0$, $\rho < a$ in the presence of the finite screen times the known magnetic field which exists in the same region with an infinite screen. It should be mentioned that (3.17) is the same as given by Monteth\textsuperscript{1}.

To obtain a stationary form for $\Delta Z_{AA}'$, one must as a first step obtain an integral equation with $E_{\rho}'$ as the unknown. Furthermore, from the form
of (3.17) it is seen that it is also desirable to have \( H_\phi \) (unprimed) as the known function outside the integral, so that the equation can be written in the operator form:

\[
LE_\rho' = H_\phi ,
\]

(3.18)

where \( L \) is an integral operator in the range concerned. From the method outlined previously, it is evident that

\[
\Delta Z_{AA} = \frac{<E_\rho',H_\phi>^2}{<H_\phi,LE_\rho'>}
\]

(3.19)

will be stationary for small variations in \( E_\rho' \).

If it is not possible to obtain an integral equation for \( E_\rho' \) with \( H_\phi \) as the known function outside the integral, one then has to define an auxiliary equation of the type

\[
LU = H_\phi ,
\]

then find an approximate solution for \( U \), and use it in the stationary form. Through the use of the compensation theorem, however, it is rather straightforward to obtain the desired integral equation for \( E_\rho' \), as is shown in the following.

One notes, first of all, that the magnetic field at \( z = 0 \) in the presence of an infinite screen is just twice the free-space magnetic field; and furthermore, in the case of a finite screen, that the magnetic field at \( z = 0 \) in the region outside the screen is just the free space field itself. The latter is true because the radial currents on the screen do not produce any magnetic field in the plane \( z = 0 \). This can be summed up by saying

\[
H_\phi = 2H_{\phi f} \quad \text{for } z = 0;
\]

and

\[
H_\phi' = H_{\phi f} \quad \text{for } z = 0, \quad \rho > a
\]

(3.20)

where \( H_{\phi f} \) denotes the free-space magnetic field at \( z = 0 \) due to antenna \( A \). Since there is only one component of magnetic field, it is not really
necessary to go to the dyadic form of the compensation theorem; instead, one can start with (5.1), and set \( \bar{J}_2 = 0 \) and \( \bar{M}_2 = \bar{a}_\phi \delta(p - p_o) \delta(z - z_o)/\rho \). Realizing that \( \bar{E}_2 = 0 \) (because of the infinite screen), one readily obtains, after some simplification,

\[
2\pi \int_a^\infty H_\phi^2(p_o, z_o, \rho, 0) E_{\rho A}(\rho, 0) d\rho = H_{\phi A}^2(p_o, z_o) - H_{\rho A}(p_o, z_o).
\]  

(3.21)

\( H_{\phi 2} \) is the magnetic field at \( z = 0 \), due to the source \( \bar{M}_2 \) in the presence of an infinite screen. Setting \( z_o = 0 \), there is obtained through the use of (3.20), the desired integral equation:

\[
2\pi \int_a^\infty H_\phi^2(p_o, \rho) E_{\rho A}(\rho) d\rho = - \frac{1}{2} H_\phi(p_o).
\]  

(3.22)

By comparison with (3.18), (3.22) is seen to be of the desired form.

It is now a simple step to write the explicit form of the stationary form of \( \Delta Z_{AA} \) from the symbolic scalar-product form appearing in (3.19). Finally, note that since the geometry of a circularly symmetric loop of magnetic current lying in a plane is a separable one, the expression for \( H_{\phi 2} \) may be obtained using the well-known methods.

3.3 Radiation pattern of an antenna in the presence of a screen.

The configuration here considered is the same as the one discussed previously, but the interest now is in determining the change in the far-field pattern because of the presence of the screen.

In this case it is again convenient to consider the two antennas A and B of Fig. 3.2, but with the assumption that antenna B is a circularly symmetric loop of electric or magnetic current having its vector direction the same as the field component desired. This is done only with a view to simplifying matters by taking advantage of the circular symmetry of the fields. Also, it is obvious that it is sufficient to find the \( H_\phi \) field everywhere, since the components of the \( E \)-field in space are easily obtained from the knowledge of the \( \bar{H} \)-field.

Once more using (5.1) in the Appendix with \( \bar{J}_B = 0 \) and \( \bar{M}_B = \bar{a}_\phi \delta(p - p_o) \delta(z - z_o)/\rho \), one may obtain:
\[ 2\pi \int_{a}^{\infty} E_{\rho A}'(\rho) H_{\phi B}(\rho, \rho_{o}', z_{o}) \, d\rho = H_{\phi A}'(\rho_{o}', z_{o}) - H_{\phi A}'(\rho_{o}', z_{o}) = \Delta H_{\phi A} \] (3.23)

(3.23) is an integral representation for the change in the \( H_{\phi} \) field of antenna \( A \), in terms of the \( E_{\rho A} \) - field for \( \rho > a \), \( z = 0 \), and due to antenna \( A \), in the presence of a finite screen; and in terms of the \( H_{\phi B} \) - field in that same region, due to a circularly symmetric magnetic current loop \( B \) placed at \( \rho = \rho_{o}' \), \( z = z_{o} \) in the presence of an infinite ground plane.

Assuming \( H_{\phi B} \) may be obtained by the method of separation of variables, the only unknown in (3.23) is \( E_{\rho A}' \).

The symbolic form for (3.23) is evidently

\[ \Delta H_{\phi A} = 2\pi \left< E_{\rho A}', H_{\phi B} \right> \] (3.24)

where the arguments of the fields are taken to be the same as appearing in (4.47). It is recognized that the right-hand side of (3.23) is of the form of \( 2\pi <\psi, b> \) with

\[ \psi = E_{\rho A}' \]

\[ b = H_{\phi B} \]

One would therefore have to formulate, as a first step, an integral equation for the unknown \( E_{A}' \) which may be written symbolically as

\[ LE_{A}' = a, \]

where "a" is a known function. Also, an auxiliary equation has to be formulated for a function \( U \) satisfying

\[ LU = b. \]

Once these two equations are formulated it is a simple matter to obtain a stationary form for the increment \( \Delta H_{\phi A} \).

It is actually rather straightforward to derive the desired integral equations. Consider, first of all, another source \( C \) specified by
\[ H_C = \bar{a}_\phi \delta(r - \rho_1) \delta(z)/\rho, \] which is also circularly symmetric and oriented along unit vector \( \bar{a}_\phi \). Then using (5.1) it is possible to write, for the source A and B respectively, the equations:

\[ 2\pi \int_a^\infty E_{\rho A}'(\rho) H_{\phi C}(\rho, \rho_1) \, d\rho = H_{\phi A}'(\rho_1) - H_{\phi A}''(\rho_1) \quad \text{for } z = 0; \quad (3.25) \]

and

\[ 2\pi \int_a^\infty E_{\rho B}'(\rho, \rho_o' z_o') H_{\phi C}(\rho, \rho_1) \, d\rho = H_{\phi B}'(\rho_1, \rho_o' z_o') - H_{\phi B}''(\rho_1, \rho_o' z_o') \quad \text{for } l = 0. \quad (3.26) \]

But from the argument related to (3.20) for both the sources A and B we have the relations

\[ H_{\phi}'' = H_{\phi f} \quad \text{for } z = 0, \quad \rho > a; \]

and

\[ H_{\phi} = 2H_{\phi f} \quad \text{for } z = 0, \]

where \( H_{\phi f} \) again denotes the free-space magnetic field due to either source. Hence, (3.25) and (3.26) may be rewritten in the final desired forms:

\[ 2\pi \int_a^\infty E_{\rho A}'(\rho) H_{\phi C}(\rho, \rho_1) \, d\rho = \frac{1}{2} H_{\phi A}'(\rho_1) \quad (3.27) \]

\[ 2\pi \int_a^\infty E_{\rho B}'(\rho, \rho_o' z_o') H_{\phi C}(\rho, \rho_1) \, d\rho = \frac{1}{2} H_{\phi B}'(\rho_1, \rho_o' z_o'). \quad (3.28) \]

Symbolically, the above equations have the forms

\[ E_{\rho A}' = a \]
and

\[ \mathbf{L} \mathbf{E} = \mathbf{b}, \]

where either form is typically denoted by:

\[ \mathbf{L} \mathbf{E} = \int_{a}^{b} \mathbf{H} \mathbf{C} (\rho, \mathbf{r}) \, d\rho \]

Hence, using (2.21) and (3.24), there follows the stationary form

\[ \Delta \mathbf{H} (\rho, \mathbf{r}) = \int_{a}^{b} \mathbf{H} \mathbf{C} (\rho, \mathbf{r}) \, d\rho \]

\[ \frac{1}{2} \int_{a}^{b} \left[ \int_{a}^{b} \mathbf{E} \mathbf{A} (\rho, \mathbf{r}) \, d\rho \right] \, d\rho \]

Equation (3.29) is insensitive to first-order variations in the unknowns \( \mathbf{E} \mathbf{A} \) and \( \mathbf{E} \mathbf{B} \).

It should be pointed out that a stationary form could equally well have been developed for the range of integration \( 0 < \rho < a \), in terms of the discontinuity of the magnetic field across the screen. This would have been accomplished by following the same procedure as used above, except for taking \( S \) as the surface of the screen, rather than the region \( \rho > a \) outside of the screen.

(3.4) Transmission cross-section of an aperture in an infinite screen.

The problem of calculating the transmission cross-section has been discussed by Schwinger. It is felt, however, that there is some merit in discussing this problem using the methods presented in this report. The ideas involved here are useful in treating problems involving antennas backed by arbitrary screens or apertures as will be found later.

First, recall that the transmission cross-section \( \sigma \) of an aperture is defined as follows:

\[ \sigma = \frac{\text{Total power transmitted through the aperture}}{\text{Power density in the incident field}} \]
Fig. 3.3 Geometry of aperture in an infinite screen.

Now, let the incident plane-wave fields on the aperture in the x-y plane, as in Fig. 3.3, be given by

\[ \vec{E}_{\text{inc}} = E_o \, e^{-j\vec{N} \cdot \vec{R}}, \quad \vec{H}_{\text{inc}} = H_o \, e^{-j\vec{N} \cdot \vec{R}} \] \hspace{1cm} (3.31)

where: \( \vec{N} \) = direction of wave normal,
\[ \vec{R} = \vec{a}_x x + \vec{a}_y y + \vec{a}_z z \] = radius vector,

and \( \frac{|E_o|}{|H_o|} = \eta = \) free-space wave impedance.

Then if we let \( N[\sigma] \) denote the numerator of (3.29), and \( D[\sigma] \) the denominator, note that they may be expressed in terms of the tangential fields in the aperture and in terms of the incident fields as follows:

\[ N[\sigma] = \frac{1}{2} \text{Re} \int_{ap} \left( \vec{a}_z \times \vec{E}_A' \right) \cdot \left( \vec{a}_z \times \vec{H}_o \, e^{j\vec{N} \cdot \rho} \right) \, dx \, dy \] \hspace{1cm} (3.32)

where: \( \vec{E}_A', \vec{H}_A' \) = aperture fields due to the incident wave,
\[ \rho = \) radius vector in the plane \( z = 0, \]

and \( ap = \) aperture surface of integration.
In writing (3.32) use has been made of the fact that $\overline{H}_A' = \overline{H}_{inc}$. Also,

$$D[\sigma] = \frac{1}{2} \frac{|E_o|^2}{\eta}.$$  \hspace{1cm} (3.33)

It is evident that $\sigma$ may be written in the scalar-product symbolism of this report as

$$\sigma = C \Re(\Gamma)$$  \hspace{1cm} (3.34)

where:

$$\Gamma = \langle \overline{a} z \times \overline{E}_A', \overline{a} z \times \overline{H}_o e^{j\beta N \cdot R} \rangle$$

and

$$C = \frac{\eta}{|E_o|^2}.$$  

Since $\overline{E}_o$ and $\overline{H}_o$ are known, the only unknown in $\Gamma$ is $(\overline{a} z \times \overline{E}_A')$, or the tangential $E$-field in the aperture produced by the plane-wave source.

Note now that $\Gamma$ of (3.34) is of the form:

$$\Gamma = \langle \overline{\psi}, \overline{b} \rangle$$

where

$$\overline{b} = \overline{H}_o e^{j\beta N \cdot R}$$

and

$$\overline{\psi} = (\overline{a} z \times \overline{E}_A');$$

so in order to permit deriving a stationary form of $\Gamma$, one should again attempt to formulate two integral equations of the type

$$L \overline{\psi} = \overline{a}$$

$$L \overline{U} = \overline{b}.$$

Since $\overline{\psi}$ and $\overline{b}$ are vector functions, $L$ is going to be a dyadic operator.

To obtain the desired integral equations, one can utilize the integral equation based on the vector compensation theorem, and one finds that (5.3) is the most suitable relation to use. Denoting by $S$ the surface of the aperture, and choosing for the "unperturbed system" the infinite
screen having no aperture, yields, from (5.3):

\[ \bar{a}_z \times \left[ \overline{H}_A'\left(Q_0\right) - \overline{H}_A\left(Q_0\right) \right] = \bar{a}_z \times \int_{A} \overline{M}_B\left(Q_0, Q\right) \cdot \bar{a}_z \times \overline{E}_A'\left(Q\right) \, dx \, dy, \quad z_0 = 0 \]  

(3.35)

where \( Q \) and \( Q_0 \) imply \((x, y, 0)\) and \((x_0, y_0, 0)\) respectively. In deriving (3.35) from (5.3), note that use has been made of the fact that \( \bar{a}_z \times \overline{E}_A \) and \( \bar{a}_z \times \overline{E}_B' \) corresponding to the case when the screen is infinite, are identically zero at \( z = 0 \). Again since

\[ \overline{H}_A'\left(Q_0\right) = \overline{H}_{\text{inc}} = H_0 e^{-j\beta N \cdot \rho} \]

and

\[ \overline{H}_A\left(Q_0\right) = 2\overline{H}_{\text{inc}}, \]

then (3.35) simplifies to

\[ - \bar{a}_z \times H_0 e^{-j\beta N \cdot \rho} = \bar{a}_z \times \int_{A} \overline{M}_B\left(Q, Q_0\right) \cdot \bar{a}_z \times \overline{E}_A'\left(Q\right) \, dx \, dy. \]  

(3.36)

Equation (3.36) is of the form

\[ L \Psi = a \]

where for any vector function \( \overline{X} \), we imply

\[ L \overline{X} = a \bar{a}_z \times \int_{A} \overline{M}_B\left(Q, Q_0\right) \cdot \overline{X} \, dx \, dy \]

An auxiliary equation may be defined with the known function outside the integral as \( - \bar{a}_z \times H_0 e^{j\beta N \cdot \rho} \) which corresponds to vector \( \overline{B} \). Let the equation be

\[ - \bar{a}_z \times H_0 e^{j\beta N \cdot \rho} = \bar{a}_z \times \int_{A} \overline{M}_B\left(Q, Q_0\right) \cdot \bar{a}_z \times \overline{E}_C'\left(Q\right) \, dx \, dy, \]  

(3.37)

where \( \overline{E}_C'\left(Q\right) \) is the unknown E-field in the aperture corresponding to
\[ \tilde{H}_C'(Q_0) - \tilde{H}_C(Q_0) = -\tilde{a}_z \times \tilde{H}_o e^{i\beta \tilde{N} \cdot \rho}. \] By comparing (3.36) and (3.37), it is easily seen that the latter corresponds to an incident wave with the wave normal \(-\tilde{N}\) rather than \(\tilde{N}\).

The stationary form for \(\Gamma\) is now easily obtained on the basis of (2.21), (3.32), (3.34), and (3.35), and is given by

\[ \Gamma = \frac{\langle \tilde{a}_z \times \tilde{E}_A^0, \tilde{a}_z \times \tilde{H}_o e^{i\beta \tilde{N} \cdot R} \rangle}{\langle \tilde{a}_z \times \tilde{E}_C^0, \tilde{a}_z \times e^{-i\beta \tilde{N} \cdot R} \rangle} \quad (3.38) \]

Comparison shows that (3.36) turns out to be the same as the expression derived by Schwinger, but the approach used here appears to be more direct.

3.5 **Field distribution in a thin slot in an infinite screen, irradiated by a plane wave.**

![Diagram](image)

**Fig. 3.4** Thin, linear slot in an infinite plane conductor.

Let a thin slot of length \(\ell\) in a plane screen be illuminated by a plane wave with the \(\tilde{E}\)-vector polarized in the \(y\)-direction, as shown in Fig. 3.4. Let the problem be to find a stationary form for the field distribution in the slot.

Because the dimension of the slot along the \(y\)-direction is small,
it may be assumed that the $E_y$-field in the slot is uniform along $y$. Hence
the problem reduces to finding the $x$-variation of $E_y$ in the slot.

Taking $S$ as the slot region and realizing that under the present
assumption, the only $E$-field component in the slot is $E_y$, the following
integral equation is easily derived by simplifying (5.3).

\[ H_{xA}'(x_o) - H_{xA}(x_o) = \int_0^L H_{x_B}(x, x_o) E_{ya}'(x)dx, \]

or,

\[ -H_{xinc}(x_o) = -\frac{1}{2} H_{xA}(x_o) = \int_0^L H_{x_B}(x, x_o) E_{ya}'(x)dx, \text{ for any } x_o \text{ in the slot,} \]

(3.39)

where $H_{y_B}$ is the magnetic field produced by a magnetic current source
\[ \bar{M}_B = \bar{a}_y \delta(x - x_o) \text{ in the slot.} \]

Observe that (3.39) is of the form

\[ L\eta = a \]

where $\psi = E_{ya}'(x)$ and $a = -\frac{1}{2} H_{xA}(x_o)$ (known); and for any $\eta(x)$, $L\eta$ is given
by

\[ L\eta = \int_0^L \eta(x) H_{x_B}(x, x_o)dx. \]

Since the desired quantity is $E_{ya}'(x)$ itself one may write:

\[ E_{ya}'(x_1) = \int_0^L E_{ya}'(x) \delta(x - x_1)dx \text{ for } 0 < x_1 < L \]

\[ = \langle \psi, b \rangle \]

(3.40)

where $b = \delta(x - y_1)$.

Hence there will have to be defined an auxiliary equation
\[ \delta(x_0 - x_1) = \int_0^l H_{xB}(x,x_0) \ E'^{yC}(x,x_1) dx, \text{ for } x_0, x_1 \text{ in the slot.} \]

(3.41)

A physical interpretation of (3.41) and of \( E'^{yC} \) goes somewhat as follows. From (3.39) it is observed that the left-hand side is equal to \(-H_{xinc} \). Upon comparing (3.39) and (3.41), one infers that if \( E'^{yA} \) is the electric field in the slot due to plane-wave incidence, then \( E'^{yC} \) is the corresponding field when the source is in the slot itself, and it is characterized by a magnetic field given by the function \(-\delta(x_0 - x_1)\). An alternative way of putting this is to say that \( E'^{yC} \) is the field produced in the slot by a concentrated \( y \)-directed electric current in the slot at \( x = x_1 \).

![Diagram](image)

**Fig. 3.5** Showing unit current generator C in the slot.

A good approximation of the electric field \( E'^{yC} \), or for that matter the voltage distribution along \( x \), may be obtained from transmission-line theory for the situation shown in Fig. 3.5.

Now since (3.41) may be written symbolically as

\[ L\psi = b, \]

the stationary forms for \( E'^{xA} \) or \( \psi \) may be obtained using either (2.22) or (2.23). The expressions are

\[ E'^{xA}(x_0) = F(x_0) + \langle E'^{xC}(\xi, x_0), -H_{xinc}(\xi) \rangle \]

\[ -\langle E'^{xC}(\xi, x_0), \int F(x)H_{xB}(x, \xi) dx \rangle \]

(3.42)

where \( F(x_0) \) is an approximation to \( E'^{xA}(x_0) \). Also in an alternate form,
\[ E_{xA}'(x_o) = \frac{F(x_o)}{E_{xC}'(\xi, x_o), H_{xinc}(\xi)} \cdot \frac{<E_{xC}(\xi, x_o), \int F(x) H_{xB}(x, \xi) dx>}{<E_{xC}(\xi, x_o), \int F(x) H_{xB}(x, \xi) dx>}. \] (3.43)

The variable of integration in the scalar products is taken as \( \xi \).

So (3.42) and (3.43) are the desired expressions for \( E_{xA}' \), which yield a better approximation of \( E_{xA}' \) itself when good first-order approximations for \( E_{xA}' \) and \( E_{xC}' \) are available.

4.0 Conclusion

This chapter has been devoted to a method for developing stationary forms of expressions for quantities such as impedance, radiation pattern, etc. The method has been applied to several problems for the sake of illustration. It is proposed to apply the compensation theorem and the method of stationary formulation to a class of mixed-boundary-value problems. A report of the work on these problems will be included in a future report.
APPENDIX

5.1 Vector Compensations Theorem

Let the electric and magnetic current sources $J_1$ and $M_1$ produce the field denoted by $E_1$ and $H_1$; similarly let the sources $J_2$ and $M_2$ produce the field denoted by $E_2$ and $H_2$. Now consider a closed surface $S$ not containing any of the sources. If a change is made in the parameters of the medium within $S$, then the field due to sources $J_1$ and $M_1$ will take different values which we denote by $E_1'$ and $H_1'$; correspondingly the field due to sources $J_2$ and $M_2$ will take different values $E_2'$ and $H_2'$.

One form of the vector compensation theorem may be stated as follows:

\[
\int_S (E_1' \times H_2 - E_2 \times H_1') \cdot ds = \int_R (E_1' - E_1) \cdot dv - \int_R (M_1' - M_1) \cdot dv
\]  \hspace{1cm} (5.1)

where $R$ is a volume containing the sources $J_2$ and $M_2$, but not intersecting surface $S$.

The vector compensation theorem may be used to formulate the following integral equations for $E_1'$ and $H_1'$:

\[
\begin{align*}
\overline{n} \times (E_1'(Q) - E_1(Q)) &= \overline{n} \times \left[ \int_S H_2(Q_1, Q_0) \cdot (\overline{n} \times E_1'(Q)) \right. \\
&\left. + E_2(Q_1, Q_0) \cdot (\overline{n} \times H_1'(Q)) \right] ds,
\end{align*}
\]  \hspace{1cm} (5.2)

\[
\begin{align*}
\overline{n} \times (H_1'(Q) - H_1(Q)) &= \overline{n} \times \left[ \int_S M_2(Q_1, Q_0) \cdot (\overline{n} \times E_1'(Q)) \right. \\
&\left. + N_2(Q_1, Q_0) \cdot (\overline{n} \times H_1'(Q)) \right] ds.
\end{align*}
\]  \hspace{1cm} (5.3)
In (5.2), the symbols $\overline{H}_2(Q_0^1)$ and $\overline{E}_2(Q_0^1)$ represent dyadics defined in terms of an orthogonal system of unit vectors $\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{c}$ by

$$
\overline{H}_2(Q_0^1) = \frac{-\overline{E}_2(a)}{a} + \frac{-\overline{E}_2(b)}{b} + \frac{-\overline{E}_2(c)}{c}
+ \frac{-\overline{E}_2(a) - \overline{E}_2(b) - \overline{E}_2(c)}{2a + 2b + 2c} \tag{5.4}
$$

and

$$
\overline{E}_2(Q_0^1) = \frac{-\overline{H}_2(a)}{a} + \frac{-\overline{H}_2(b)}{b} + \frac{-\overline{H}_2(c)}{c}
+ \frac{-\overline{H}_2(a) - \overline{H}_2(b) - \overline{H}_2(c)}{2a + 2b + 2c} \tag{5.5}
$$

In (5.4) and (5.5):

$$
\overline{E}_2(a) = a \overrightarrow{E}_2
$$

$$
\overline{H}_2(a) = a \overrightarrow{H}_2
$$

$$
\overline{E}_2(a) = b \overrightarrow{E}_2
$$

$$
\overline{H}_2(a) = b \overrightarrow{H}_2
$$

$$
\overline{E}_2(a) = c \overrightarrow{E}_2
$$

$$
\overline{H}_2(a) = c \overrightarrow{H}_2
$$

where $\overrightarrow{E}_2$ and $\overrightarrow{H}_2$ are the E and H-field vectors at point $Q_0^1$ on the surface $S$, due to an electric current source $\overline{J}_2 = \overline{a} \delta (Q_0^1 - Q)$ at point $Q$. The symbols with superscripts (b) and (c) are similarly defined.

The dyadics $\overline{H}_2(Q_0^1)$ and $\overline{E}_2(Q_0^1)$ are defined by expressions similar to (5.5) and (5.4) respectively; but, for this case, the source is a magnetic current $\overline{M} = \overline{k} \delta (Q_0^1 - Q)$ where $\overline{k}$ takes on values $\overrightarrow{a}, \overrightarrow{b}$ and $\overrightarrow{c}$. 
BIBLIOGRAPHY

