PARTIAL (SET) 2-STRUCTURES
PART 1:
REPRESENTATION PROBLEMS
(Prelininary version)

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ABSTRACT

The notion of a *partial 2-structure* (*p2s* for short) , is introduced ; it generalizes the notion of a *2-structure* discussed in [ER1] . Partial 2-structures may be considered to be edge-labeled graphs satisfying certain conditions . Then partial 2-structures on sets are considered where an edge between sets is labeled by the ordered symmetric difference of the sets. Such partial 2-structures arise in the study of state spaces of concurrent systems ; this connection is studied in more detail in Part 2 of this paper .

The main problem studied in this part of the paper is when a p2s structure can be represented as a partial set 2-structure.

INTRODUCTION

The notion of a 2-*structure* (2s for short) is a generalization of the notion of a (directed) graph ; it was introduced in [ER1] where also the basic theory of 2-*structures* was developed . We have demonstrated in [ER2] that each 2s is build-up in a *unique way* from 3 basic "building blocks" : *primitive* 2-structures, *linear* 2-structures and *complete* 2-structures . As an application one gets that each graph can be uniquely constructed from (decomposed into) *primitive* , *linear* and *complete* graphs .

In this paper we turn into a different area of applications of the theory of 2-structures . We consider state spaces of concurrent systems and in particular case graphs of various types of Petri nets , like , e. g. , condition/event systems and elementary net systems (see , e . e. [R] and [RT] , also in Part 2 of this paper we briefly recall some of these notions) .

A very basic assumption of the theory of Petri nets is that the extent of change caused by the occurrence of an event is independent of the (global) state at which it occurs (this assumption is often referred to as the *axiom of extensionality*) . As a matter of fact in condition/even systems or in elementary net systems this change is characterized by the *characteristic pair* of an event e : the set of conditions that cease to hold whenever e occurs (this set is denoted by $^\bullet e$ and the set of conditions that begin to hold whenever e occurs (this set is denoted by $e^\bullet$) . Thus given a global state C_1 (also referred as a *case*) of a system on occurrence of e leads to a state $C_2 = (C_1 - {}^{\bullet}e) \cup e^{\bullet}$. One may say that one gets a transition between C_1 and C_2 labeled by the ordered symmetric difference of C_1 and C_2 (i.e. , by $(C_1 - C_2 , C_2 - C_1) = ({}^{\bullet}e , e^{\bullet}))$ - where the label denotes the amount of change "caused by" the given transition . We refer the reader to [GLT] where some properties of edge-labeled graphs obtained in this way are discussed .

Edge-labeled graphs obtained in this way "almost" lead to 2-structures with sets as their domains . "Almost" , because one gets "partial graphs" in the sense that there does not have to be an edge between each pair of nodes .

Hence it is natural to consider *partial 2-structures* (where some edges may be omitted) and in particular to partial 2-structures with sets as their domains and with ordered symmetric differences as labels of the edges - such partial 2-structures are referred to as *partial set 2-structures* .

This is Part 1 of a paper consisting of two parts.

In Part 1 we investigate partial 2-structures and partial set 2-structures and in particular the main problem studied is when a partial 2-structure can be represented by ("isomorphically" mapped onto) a partial set 2-structure .

In Part 2 we will present applications of results from Part 1 to the study of state spaces of concurrent systems .

1. PRELIMINARIES

We assume the reader to be familiar with the rudiments of edge-labeled graphs .

$\emptyset$ denotes the empty set , and for a set X , $|X|$ denotes its cardinality and 2^X the set of all subsets of X . *In this paper we deal with finite sets only* .

A *total partition* of a set X is a family $\mathcal{P}$ of elements from $2^X - \emptyset$ such that $\bigcup_{P \in \mathcal{P}} P = X$ and $P_1 \cap P_2 = \emptyset$ for all $P_1 , P_2 \in \mathcal{P}$ such that $P_1 \neq P_2$. We will write $x \, \mathcal{P} \, y$ for $x , y \in X$ whenever there is a $P \in \mathcal{P}$ such that $x , y \in P$.

For a set X , $E_2(X) = \{ (x, y) : x, y \in X \text{ and } x \neq y \}$; elements of $E_2(X)$ are called 2-*edges of* X .

For sets X , Y the *ordered symmetric difference of* X and Y is the pair of sets $(X - Y, Y - X)$.

2. PARTIAL 2-STRUCTURES

In this section we introduce and illustrate by examples, the notions of a partial 2-structure and structural homomorphisms of them. Also, at the end of the section, an important notion of a region of a partial 2-structure is introduced.

Definition 2.1 A *partial* 2-*structure* (*p*2*s* for short) is a system $g = (D , F , P , L , \psi)$ where

D is a nonempty finite set (called the *domain* of g) ,

$F \subseteq E_2(D)$ (called the set *of* 2 *-edges* of g) ,

P is a total partition of F (called the *partition* of g) such that,

for all $x , y , z , t \in D$,

if $(x , y) , (z , t) , (y , x) , (t , z) \in F$ and $(x , y) P (z , t)$,

then $(y , x) P (t , z)$,

L is a finite set (called the *alphabet* of g) , and

ψ is a total injective function from P into L (called the *labeling function* of g) . □

For a p2s g we will use D_g , F_g , P_g , L_g , ψ_g to denote the domain, the set of 2-edges, the partition, the alphabet, and the labeling function of g , respectively . Also we use **P2S** to denote the class of all partial 2-structures .

Definition 2.2 . Let $g \in$ **P2S** .

(i) A label $A \in L_g$ is *applicable* iff there exists a $P \in P_g$ such that

$\psi_g(P) = A$.

(ii) g is *label minimal* *iff* every $A \in L_g$ is applicable . □

Remark 2.1 .

(1) If for a p2s g we have $F_g = E_2(D_g)$ - hence F_g consists of all 2-edges over D_g - then we may skip F_g from the specification of g and g becomes a 2-structure (2s for short) in the sense of [ER1] . In this way 2-structures are a special case of partial 2-structures ; the class of all 2-structures will be denoted by **2S** . Thus, one can say that each p2s is obtained from a 2s h by deleting some of the 2-edges of h .

(2) For a p2s structures g one may consider ψ_g to be the function labeling elements of F_g : simply every e belonging to a class P of $\mathcal{P}_g$ is labeled by $\psi_g(P)$. This is a very convenient way of specifying (the labeling functions of) p2s systems. As a matter of fact in this way one may view p2s systems as edge labeled graphs satisfying certain conditions (for a discussion of the relationship between 2s systems and edge labeled graphs the reader is referred to [ER1]) . In the sequel of this paper we will sometimes specify p2s systems through the usual graphical representation of edge labeled graphs .

(3) *In the view of the above discussed relationship between edge labeled* graphs and p2s systems, we will in the sequel replace the term "2-edge" by the term "edge" . □

By deleting some edges from a 2s one gets a p2s . By deleting some nodes and edges from a p2s h one gets a partial substructure of h .

Definition 2.3 . Let g, h $\in$ **P2S** . We say that g is a *partial substructure* of h iff

$D_g \subseteq D_h$,

$F_g \subseteq F_h$,

$P_g = \{ P \cap F_g : P \in P_h \text{ and } P \cap F_g \neq \emptyset \}$,

$L_g \subseteq L_h$, and

for every $P \in P_g$, $\psi_g(P) = \psi_h(P')$, where P' is the element of P_h such that $P \subseteq P'$. □

Remark 2.2 . The notion of a substructure of a 2s was discussed in [ER1] . In terms of this notion we can say that each p2s h is obtained from a 2s g by possibly removing some elements of D_g (together with all adjacent edges) obtaining a substructure g′ of g , and then by removing some edges from g′ one obtains h . □

Structural homomorphisms of partial 2-structures play the crucial role in this paper . They are formally defined as follows .

Definition 2.4 . Let g , h $\in$ **P2S** . A mapping $\alpha : D_g \longrightarrow D_h$ is a *structural homomorphism from* g *into* h iff for all x , y , z , t $\in D_g$ such that

$(x, y), (z, t) \in F_1$ and $(x, y) P_g (z, t)$,

(i) if $\alpha(x) = \alpha(y)$, then $\alpha(z) = \alpha(t)$, and

(ii) if $\alpha(x) \neq \alpha(y)$, then $(\alpha(x), \alpha(y)), (\alpha(z), \alpha(t)) \in F_h$ and $(\alpha(x), \alpha(y))\ P_h\ (\alpha(z), \alpha(t))$.

Moreover, if α is a bijection, then α is a *structural isomorphism from* g *onto* h . □

If g , h $\in$ **P2S** are such that there exists a structural homomorphism (structural isomorphism) from g into h , then we write g *shom* h (g *sisom* h , respectively) . If $L_g = L_h$ and there exists a structural isomorphism $\alpha : D_g \rightarrow D_h$ such that for all $(x, y) \in F_g$ with $\alpha(x) \neq \alpha(y)$ we have $\psi_g((x, y)) = \psi_h((\alpha(x), \alpha(y)))$, then we write g *isom* h and call α an *isomorphism from* g *onto* h ; in this case we may use unambiguously (although somewhat informally) the notation α(g) to denote h . (Note that in the above we have used the convention from the last remark allowing to specify the labeling function of a p2s system on its set of edges) .

Example 2.1 .

The following edge labeled graph :

Figure 2.1

does not represent a p2s because of the C-loop . (When we represent a p2s g by and edge-labeled graph , we assume that g is label minimal) .

By removing this loop we get an edge-labeled graph :

Figure 2.2

which *represents* a p2s but it *does not* represent a 2s , because we have pairs of different nodes with no edges between them .

By adding one labeled edge we get the following edge labeled graph :

Figure 2.3

This graph *does not* represent a p2s, because the label A does not have a unique "inverse" !

By modifying this edge-labeled graph as follows :

Figure 2.4

we get a representation of a p2s . □

Remark 2.3 . The above example illustrates some situations when an edge-labeled graph does not represent a p2s system. From the definition of a p2s system it directly follows that

(1) an edge-labeled graph g represents a p2s system iff

(i) g has no loops ,

(ii) between any two nodes of g there exists at most one edge, and

(iii) each label of g has a unique inverse if an inverse exits, meaning that : if v_1 , v_2 , v_3 , v_4 are nodes of g such that (v_1 , v_2) , (v_2 , v_1) , (v_3 , v_4) and (v_4 , v_3) are edges of g where (v_1 , v_2) and (v_3 , v_4) are labeled by a label A and (v_2 , v_1) is labeled by a label B , then (v_4 , v_3) is labeled by B .

It is also clear that :

(2) an edge-labeled graph g represents a 2s system iff conditions (i) and (iii) above are satisfied and the condition (ii) is changed into :

(ii′) there is precisely one edge between any two nodes of g . □

Example 2.2 .

Consider the following $g_1 \in$ **P2S** :

Figure 2.5

It is easily seen that g_1 is a partial substructure of the following $h_1 \in$ **2S** :

Figure 2.6

Now let $g_2 \in$ **P2S** be as follows:

Figure 2.7

Then g_1 *shom* g_2 , because the mapping $\alpha : D_{g_1} \longrightarrow D_{g_2}$ defined by : $\alpha(1) = \alpha(4) = 8$, $\alpha(3) = 7$, and $\alpha(2) = 9$, is a structural homomorphism of g_1 onto g_2 .

For the following $g_3 \in$ **2S** :

Figure 2.8

we have g_1 *shom* g_3 , because the mapping $\beta : D_{g_1} \longrightarrow D_{g_3}$ defined by $\beta(1) = \beta(3) = 1$ and $\beta(2) = \beta(4) = 2$, is a structural homomorphism of g_1 onto g_3 .

If we now change g_3 to the following $g_3' \in$ **2S** :

Figure 2.9

then this β is not a structural homomorphism of g_1 into g'_3 ; the reason is that $\beta(1) = 1$, $\beta(2) = 2$, and $\psi_{g_1}((1 , 2)) = \psi_{g_1}((2 , 1))$, while $\psi((\beta(1) , \beta(2))) \neq \psi((\beta(2) , \beta(1)))$. □

Example 2.3 .

Consider the following g $\in$ **2S** :

Figure 2.10

For the following h $\in$ **P2S** :

Figure 2.11

we have g *shom* h , because the mapping $\alpha : D_g \longrightarrow D_h$ defined by :
$\alpha(1) = \alpha(2) = 5$ and $\alpha(3) = \alpha(4) = 6$, is a structural homomorphism of g into h . □

The following notion will be very crucial in the proof of our main result in Section 4.

Definition 2.5 . Let g $\in$ **P2S** . A subset $R \subseteq D_g$ is a *region of* g) iff for all $(x , y) , (z , t) \in F_g$ such that $(x , y) \ P_g \ (z , t)$,

(i) if $x \in R$ and $y \notin R$, then $z \in R$ and $t \notin R$, and

(ii) if $x \notin R$ and $y \in R$, then $z \notin R$ and $t \in R$. □

We will use $\mathbf{R}_g$ to denote the set of all *nonempty* regions of g , and for an $x \in D_g$, $\mathbf{R}_g(x)$ denotes the set $\{ R \in \mathbf{R}_g : x \in R \}$. For an $R \in \mathbf{R}_g$ and an e $=$ $(x , y) \in F_g$, we say that e *is crossing R iff*

$(x \in R$ iff $y \in R)$.

Example 2.4 .

Consider the following p2s g :

Figure 2.12

Then $R_1 = \{1 , 3 , 6\}$ is a region of g : all A -labeled edges are leaving R , all B -labeled edges are coming into R , and all edges crossing R either way are labeled by either A or B.

On the other hand $R_2 = R_1 \cup \{5\}$ is not a region because the edge (1 , 4) labeled by A is crossing R_2 while the edge (3 , 5) labeled by A is inside R_2 . □

Remark 2.4 . It is instructive to notice that , for a p2s g , $\emptyset$ and D_g are regions of g . As a matter of fact, it is easily seen that if $R \in \mathbf{R}_g$ then $(D_g - R) \in \mathbf{R}_g$. □

3. PARTIAL SET 2-STRUCTURES

In this section we will consider partial 2-structures the nodes of which are sets (each of which is a subset of a certain common base set) and the edges of which are labeled by ordered symmetric differences of sets they connect. Such partial 2-structures have very natural applications in the theory of concurrent systems ; e.g. , state spaces of condition/event systems (see , e.g. , [R]) and state spaces of elementary net systems (see , e.g. , [RT]) are partial 2-structures of this kind . Moreover spaces of sets where a transition from a set to a set is labeled by the ordered symmetric difference of these sets are mathematically natural objects to consider .

Such partial 2-structures are defined as follows .

Definition 3.1 .

(i) Let X be a nonempty set .

The 2-structure $g = (2^X , F , P , L , \psi)$ such that

(1) for all $x , y , z , t \in 2^X$,

$(x , y) P (z , t)$ iff $P\ x - y = z - t$ and $y - x = t - z$,

(2) $L = \{ (y , z) : y , z \in 2^X$ and $y \cap z = \emptyset \}$, and

(3) for all $x , y \in 2^X$, $\psi((x , y)) = (x - y , y - x)$

is the 2-*structure* of X denoted by $S2S$ (X) .

(ii) A g $\in$ **2S** is called a *set* 2-*structure* (s2s for short) if g = $S2S$ (X) for a nonempty set X . A partial substructure of a set 2-structure is called a *partial* *set 2-structure* (ps2s for short) . □

Note that a ps2s g is *asymmetric* in the sense that :

if $(x , y) \in F_g$ and $(y , x) \in F_g$, then $\psi_g((x , y)) \neq \psi_g((y , x))$.

For a given nonempty set X we use $PS2S$ (X) to denote the set of all partial substructures of $S2S$ (X) . We use **S2S** and **PS2S** to denote the class of all set 2-structures and the class of all partial set 2-structures .

In this paper we will be especially interested in the class

$\{ g \in \mathbf{P2S} :$ there exists an $h \in \mathbf{PS2S}$ such that g *sisom* h } ;

this subclass of **P2S** is denoted by $\overline{\mathbf{P2S}}$.

Definition 3.2 . Let $g \in \mathbf{PS2S}$. The *base* of g , denoted *base*(g) , is the minimal (w.r.t. the set-theoretic inclusion) nonempty X such that

$g \in PS2S$ (X) . □

Remark 3.1 . Note that for each $g \in \mathbf{PS2S}$ there exists the unique minimal X such that $g \in PS2S$ (X) - thus the notion of the base of g is well defined. Note also that *base*($S2S$ (X)) = X . □

Often one may wish to remove certain elements from the base of a ps2s system - this leads to a ps2s system defined as follows .

Definition 3.3 . Let X be a nonempty set , $Y \subseteq X$ and let

$g \in PS2S$ (X) . The Y- *restriction of* g , denoted by $g \mid Y$, is the partial

2-structure (D, F, P, L, ψ) such that :

(i) $D = \{ z \cap Y : z \in D_g \}$,

(ii) $F = \{ (u \cap Y, z \cap Y) : (u, z) \in F_g \}$,

(iii) for all $x, u, z, t \in D$,

$(x, u)\ P\ (z, t)$ iff $(x - u) = (z - t)$ and $(u - x) = (t - z)$,

(iv) $L = \{ (u \cap Y, z \cap Y) : (u, z) \in L_g \}$,

(v) for all $u, z \in D$, $\psi((u, z)) = (u - z, z - u)$. □

Remark 3.2 . It is easily seen that , in the notation as above , $g \mid Y \in PS2S(X)$ and $base(g \mid Y) = base(g) \cap Y$. □

Each Y-restriction of a ps2s system g can be expressed through a structural homomorphism as follows .

Lemma 3.1 . Let X be a nonempty set , $Y \subseteq X$ and let $g \in PS2S(X)$. Then the mapping $\alpha : D_g \longrightarrow D_g \mid Y$ defined by : $\alpha(z) = z \cap Y$ for all $z \in D_g$, is a structural homomorphism of g onto $g \mid Y$. □

The structural homomorphism α above is referred to as the Y- *restricting mapping of* g and denoted by $rest_{g \mid Y}$.

Remark 3.3 . Note that if $g \in$ **P2S** , $h \in$ **PS2S** , and α is a structural homomorphism from g *onto* h , then h is uniquely determined by g and α . Hence in this case we will use the notation $\alpha(g) = h$. In particular in view of the above lemma we write (in the notation as above) $rest_{g \mid Y}(g)$ to denote

$g \mid Y$. □

Example 3.1.

Let $X = \{ 1 , 2 \}$.

Then $S2S$ (X) is as follows :

Figure 3.1

Let $g \in PS2S$ (X) be as follows:

Figure 3.2

Then, for $Y = \{ 2 \}$, $g \mid Y$ is as follows :

Figure 3.3

and $rest_{g \mid Y}$ is defined by :

$rest_{g \mid Y}(\{ 1 \}) = \emptyset$, $rest_{g \mid Y}(\{ 2 \}) = rest_{g \mid Y}(\{ 1 , 2 \}) = \{ 2 \}$. □

It may happen that the base of a ps2s g is very large w.r.t. the number of elements of D_g and F_g . We will demonstrate now that one can always find a subset Y of D_g which is of polynomial (actually quadratic) size in D_g and F_g

such that g *sisom* $g \mid Y$.

Theorem 3.1 . Let $g \in$ **PS2S** and let $X = base(g)$. There exists a $Y \subseteq X$ such that $|Y| \leq 8 |F_g|^2 + 2 |D_g|^2$ and g *sisom* $g \mid Y$.

Proof .

The idea in constructing $g \mid Y$ is to choose Y in such a way that it will contain elements from D_g which will guarantee that :

(i) whenever z , t are distinct elements of D_g , then $z \cap Y \neq t \cap Y$ (this will be our set Y_1) and

(ii) whenever (u , v) , (z , t) are distinct elements of F_g differently labeled by ψ_g , then $(u \cap Y , v \cap Y)$, $(z \cap Y , t \cap Y)$ are distinct elements of $F_g \mid Y$ differently labeled by $\psi_{g \mid Y}$ (this will be our set Y_2) .

Such a set Y is constructed as follows .

Let $(z , t) \in E_{D_g}$; clearly we have at most $| D_g |^2$ such elements .

If $z - t \neq \emptyset$, then we choose an arbitrary but fixed element of $z - t$, and if $t - z \neq \emptyset$, then we choose an arbitrary but fixed element of $t - z$.

The set of the so chosen (at most 2 elements) is denoted by $\gamma ((z , t))$.

$$\text{Let } Y_1 = \bigcup_{(z,t) \in E_{D_g}} \gamma ((z , t)) .$$

Clearly $|Y_1| \leq 2 |D_g|^2$.

Let $e = (u , v)$, $d = (z , t)$ be a pair of distinct edges from F_g ; clearly we have at most $|F_g|^2$ of such pairs (e , d) . Let

$W_1((e, d)) = \{ u - v, v - u \}$, $W_2((e, d)) = \{ z - t, t - z \}$ and let $Z((e, d)) = \{ (r, w) : r \in W_1 \text{ and } w \in W_2 \}$;

clearly $| Z((e, d)) | \leq 4$. Now for each $(r, w) \in Z((e, d))$ we determine $\gamma((r, w))$ in the same way that $\gamma((z, t))$ was determined above ; clearly each $\gamma((r, w))$ has at most 2 elements .

Let $$Y_2 = \bigcup_{(e,d) \in E_{D_g}} \bigcup_{(r,w) \in Z((e,d))} \gamma((r, w)) .$$

Clearly $| Y_2 | \leq 8 | F_g |^2$.

Now let $Y = Y_1 \cup Y_2$; hence $| Y | \leq 8 | F_g |^2 + 2 | D_g |^2$.

Consider now $rest_{g|Y}$.

Since $Y_1 \subseteq Y$, if $z \neq t$ for $z , t \in D_g$, then $z \cap Y \neq t \cap Y$ and so $rest_{g|Y}(z) \neq rest_{g|Y}(t)$.

Since $Y_2 \subseteq Y$, if (u, v) , (z, t) is a pair of distinct edges from F_g such that $\psi_g((u, v)) \neq \psi_g((z, t))$,

then $\psi_{g|Y}((u \cap Y, v \cap Y)) \neq \psi_{g|Y}((z \cap Y, v \cap Y))$.

Also by Lemma 3.1 , $rest_{g|Y}$ is onto .

Consequently g *sisom* $g | Y$. □

In this paper we are interested in the problem when an arbitrary p2s system is in $\overline{\mathbf{PS2S}}$ (we refer to this problem as "*the* ($\mathbf{P2S}$, $\overline{\mathbf{PS2S}}$) *-membership problem*") . As an immediate corollary of the above theorem we get the following result.

Corollary 3.1 . The (**P2S** , $\overline{\mathbf{PS2S}}$) -membership problem is decidable . □

In the next section of this paper we will discuss a specific procedure for solving the (**P2S** , $\overline{\mathbf{PS2S}}$) -membership problem .

It is convenient to deal with elements of **PS2S** which do not have "redundancies" in their bases . We will demonstrate that this is always possible ; first however we formalize the notion of a "nonredundant ps2 system."

Definition 3.4 . Let $g \in$ **PS2S** and let $X = base(g)$. We say that g is *tight* iff for all $a , b \in X$ the following holds :

if (for every $y \in D_g$, $a \in y$ iff $b \in y$) , then $a = b$. □

Example 3.2 .

The following $g \in$ **PS2S** :

Figure 3.4

is *not tight* because 2 and 3 are "indistinguishable" here. On the other hand , $g \mid \{ 2 , 5 \}$ which is as follows :

Figure 3.5

is tight . □

Lemma 3.2 . Let $g \in \mathbf{PS2S}$ and let $X = base(g)$. There exists a $Y \subseteq X$ such that $g \mid Y$ is tight and g *sisom* $g \mid Y$.

Proof .

Let $\sim_g$ be the equivalence relation on X defined by :

for all $a , b \in X$, $a \sim_g b$ iff for each $z \in D_g$, $a \in z$ iff $b \in z$. Then let Y to be a subset of X such that Y contains exactly one element from each equivalence class of $\sim_g$.

It is easily seen that $g \mid Y$ satisfies the conclusions of the lemma . □

We use *tight*(g) to denote the set of all $h \in \mathbf{PS2S}$ obtained from g by different choices of Y as above .

4. REPRESENTING P2S BY PS2S

In this section we will consider in more detail the (**P2S** , $\overline{\textbf{PS2S}}$) -membership problem . That is , we will be interested in the problem of when a g $\in$ **P2S** is in $\overline{\textbf{PS2S}}$ and if g $\in$ $\overline{\textbf{PS2S}}$ we will provide a "canonical" g′ in **PS2S** such that g *sisom* g′ .

The notion of a region of a ps2s system will play the crucial role in our considerations . We use this notion as follows .

Definition 4.1 . Let g $\in$ **PS2S** .

(i) The *regional* g-*mapping* , denoted reg_g, is the function from D_g into 2^{R_g} defined by :

for every z $\in$ D_g , $reg_g(z) = \mathbf{R}_g(z)$.

(ii) The *regional version of* g , denoted *rev*(g) , is the system (D , F , P , L , ψ) $\in$ **PS2S** such that

$D = \{ \mathbf{R}_g(x) : x \in D \}$,

for all X , Y $\in$ D ,

(X , Y) $\in$ F iff X $\neq$ Y and there exist (x , y) $\in$ F_g such that

$X = \mathbf{R}_g(x)$ and $Y = \mathbf{R}_g(y)$, and

$L = \{(u , v) :$ there exists a (X , Y) $\in$ F such that $u = X - Y$ and $v = Y - X \}$,

for all (X , Y) $\in$ F ,

$\psi((X , Y)) = (X - Y , Y - X)$. □

Remark 4.1 . It is instructive to notice that , for a ps2s g , *rev*(g) is label minimal . □

Example 4.1 .

Consider the following p2s g :

Figure 4.1

Here are all elements of $\mathbf{R}_g$:

$R_0 = \{ 1 , 2 , 3 , 4 , 5 , 6 , 7 , 8 \}$, $\overline{R}_0 = \emptyset$,

$R_1 = \{ 1 , 4 , 5 , 8 \}$, $\overline{R}_1 = \{ 2 , 3 , 6 , 7 \}$,

$R_2 = \{ 1 , 3 , 5 \}$, $\overline{R}_2 = \{ 2 , 4 , 6 , 7 , 8 \}$,

$R_3 = \{ 1 , 2 , 3 , 5 , 7 \}$, $\overline{R}_3 = \{ 4 , 6 , 8 \}$,

$R_4 = \{ 1 , 3 , 4 , 5 , 6 , 8 \}$, $\overline{R}_4 = \{ 2 , 7 \}$,

$R_5 = \{ 1 , 2 , 4 \}$, $\overline{R}_5 = \{ 3 , 5 , 6 , 7 , 8 \}$,

$R_6 = \{ 1 , 2 , 3 , 4 , 6 \}$, $\overline{R}_6 = \{ 5 , 7 , 8 \}$,

$R_7 = \{ 1 , 2 , 4 , 5 , 7 , 8 \}$, $\overline{R}_7 = \{ 3 , 6 \}$.

Then we have :

$\mathbf{R}_g(1) = \{ R_0 , R_1 , R_2 , R_3 , R_4 , R_5 , R_6 , R_7 \}$,

$\mathbf{R}_g(2) = \{ R_0 , \overline{R}_1 , \overline{R}_2 , R_3 , \overline{R}_4 , R_5 , R_6 , R_7 \}$,

$\mathbf{R}_g(3) = \{ R_0 , \overline{R}_1 , R_2 , \overline{R}_3 , R_4 , \overline{R}_5 , R_6 , \overline{R}_7 \}$,

$\mathbf{R}_g(4) = \{ R_0 , R_1 , \overline{R}_2 , \overline{R}_3 , R_4 , R_5 , R_6 , R_7 \}$,

$\mathbf{R}_g(5) = \{ R_0 , R_1 , R_2 , R_3 , R_4 , \overline{R}_5 , \overline{R}_6 , R_7 \}$,

$\mathbf{R}_g(6) = \{ R_0 , \overline{R}_1 , \overline{R}_2 , \overline{R}_3 , R_4 , \overline{R}_5 , R_6 , \overline{R}_7 \}$,

$\mathbf{R}_g(7) = \{ R_0 , \overline{R}_1 , \overline{R}_2 , R_3 , \overline{R}_4 , \overline{R}_5 , \overline{R}_6 , R_7 \}$, and

$\mathbf{R}_g(8) = \{ R_0 , R_1 , \overline{R}_2 , \overline{R}_3 , R_4 , \overline{R}_5 , \overline{R}_6 , R_7 \}$.

Consequently *rev*(g) is as follows :

Figure 4.2

where

$d_1 = (\{ R_1 , R_2 , R_4 \} , \{ \overline{R}_1 , \overline{R}_2 , \overline{R}_4 \})$,

$d_2 = (\{ R_1 , R_5 , R_7 \} , \{ \overline{R}_1 , \overline{R}_5 , \overline{R}_7 \})$,

$d_3 = (\{ \overline{R}_1 , \overline{R}_3 , \overline{R}_4 \} , \{ R_1 , \overline{R}_3 , R_4 \})$,

$d_4 = (\{ \overline{R}_1 , R_6 , \overline{R}_4 \} , \{ R_1 , \overline{R}_6 , R_4 \})$,

$d_5 = (\{ \overline{R}_2 , \overline{R}_3 \} , \{ R_2 , R_3 \})$,

$d_6 = (\{ \overline{R}_5 , \overline{R}_6 \} , \{ R_5 , R_6 \})$. □

Theorem 4.1 . Let g $\in$ **PS2S** . Then reg_g is a structural homomorphism from g onto *rev*(g) .

Proof .

Assume that $(x , y) , (z , t) \in F_g$ are such that $(x , y) P_g (z , t)$.

(1) Assume that there exists a $R \in (reg_g(x) - reg_g(y))$.

Hence $x \in R$ and $y \notin R$ and therefore , because $(x , y) P (z , t)$, the definition of a region implies that $z \in R$ and $t \notin R$.

Thus $R \in (reg_g(z) - reg_g(t))$.

(2) Similarly if we assume that there exists a $R \in (reg_g(y) - reg_g(x))$ then we arrive at the conclusion that $R \in (reg_g(t) - reg_g(z))$.

From (1) and (2) above it immediately follows that reg_g is a structural homomorphism from g into $rev(g)$. But , by the definition of $rev(g)$ it follows immediately that g is onto. □

We are ready now to prove the main result of this paper . First , however we need a definition and a lemma .

Definition 4.1 . Let $g \in$ **PS2S** . For each $x \in base(g)$,

$D_g(x) = \{ u \in D_g : x \in u \}$. □

Lemma 4.1 . Let $g \in$ **PS2S** and let $X = base(g)$. For each $x \in X$, $D_g(x) \in \mathbf{R}_g$.

Proof .

Consider $D_g(x)$ for an $x \in X$.

Let $(u , v) , (z , t) \in \mathbf{F}_g$ be such that $(u , v) P_g(z , t)$.

(1) If $u \in D_g(x)$ and $y \notin D_g(x)$, then $\psi_g((u , v)) = (A , B)$ is such that $x \in A$ and $x \notin B$. Since $(u , v) P_g (z , t)$, $\psi_g((z , t)) = (A , B)$ implying that $z \in D_g(x)$ and $t \notin D_g(x)$.

(2) Similarly we prove that if $u \notin D_g(x)$ and $y \in D_g(x)$, then $z \notin D_g(x)$ and $y \in D_g(x)$. □

Theorem 4.2 . For every $g \in \mathbf{P2S}$, $g \in \overline{\mathbf{PS2S}}$ iff reg_g is a structural isomorphism .

Proof .

If reg_g is a structural isomorphism , then for $h = rev(g)$ we have $h \in \mathbf{PS2S}$ and g *sisom* h , hence $g \in \overline{\mathbf{PS2S}}$.

To prove the reverse implication we proceed as follows .

Assume that there exists an $h \in \mathbf{PS2S}$ such that g *sisom* h ; let α be a structural isomorphism from g onto h . By Lemma 3.2 we may assume that h is tight . Let $h = (D , F , P , L , \psi)$ and let $X = base(h)$.

We will use the following notation : for each $x \in X$, $G_x = D_g(x)$, and $\mathbf{G} = \{ G_x : x \in X \}$.

Claim 4.1. For each $x \in X$, $G_x \in \mathbf{R}_h$ and moreover , for each $x , y \in X$, $G_x \neq G_y$ whenever $x \neq y$.

Proof .

Directly from Lemma 4.1 and from the fact that h is tight . □

The mapping α^{-1} is a structural isomorphism of h onto g and , for each $x \in X$, α^{-1} maps the region G_x of h into the region $\hat{G}_x$ of g .

Let $\hat{\mathbf{G}} = \{ \hat{G}_x : x \in X \}$.

The situation may be illustrated as follows :

Figure 4.3

Hence , for an a $\in \hat{G}_x$, the element $reg_g(a)$ contains $\hat{G}_x$, (among other regions) - see Claim 4.1 .

The situation may be illustrated as follows (where we set $g' = rev(g)$) :

Figure 4.4

Now we consider $\mathbf{g}' \mid \hat{\mathbf{G}}$. By Lemma 3.1 , $rest_{g' \mid \hat{\mathbf{G}}}$ is an onto structural homomorphism.

On the other hand if in $g' \mid \hat{\mathbf{G}}$ we replace each element $\hat{G}_x$ of the base set by x then we induce an obvious structural isomorphism β of $g' \mid \hat{\mathbf{G}}$ onto h .

Let γ be the composition of $rest_{g' \mid \hat{\mathbf{G}}}$ and β .

The situation my be illustrated as follows :

Figure 4.5

Since γ is a composition of an onto structural homomorphism $rest_{g' | \hat{G}}$ and a structural isomorphism β , γ is a structural homomorphism of g$'$ onto h . On the other hand the composition of reg_g and γ yields α^{-1} which is a structural isomorphism - hence both γ and reg_g must be structural isomorphisms .

This proves the reverse implication .

Hence the theorem holds . □

Actually studying the proof above we notice that there exists a very specific relationship between *rev*(g) and h . In order to express this relationship we need the following notion .

Definition 4.2. Let X be a nonempty set and let g $\in$ *PS2S*(X) . Let β be a total function on X . Let γ be the function on D_g defined by : for all z $\in D_g$, $\gamma(z) = \{ \beta(t) : t \in z \}$. We say that γ is *induced by* β , denoted by $\gamma = ind_\beta$ and we call γ a *renaming of* g ; if β is injective (bijective) then we call γ an *injective* (respectively *bijective*) *renaming* of g . □

For g , h $\in$ **PS2S** we write g *bren* h iff there exists a bijective renaming of g onto h .

From our proof of Theorem 4.2 it is clear that we were using a bijective renaming to get h from g$'$ | $\hat{\mathbf{G}}$. As a matter of fact Theorem 4.2 together with its proof yields the following corollary .

Corollary 4.1 . Let $g \in \overline{\mathbf{PS2S}}$ and let $h = rev(g)$. For every tight label minimal $g' \in \mathbf{PS2S}$, such that g *sisom* g' , there exists a $Z \subseteq base(h)$ and an injective $\beta : Z \longrightarrow base(g')$ such that the composition of $rest_{h\,|\,Z}$ and ind_{β} is an isomorphism of h onto g' . □

Theorem 4.2 gives us algorithm for deciding whether an arbitrary $g \in \mathbf{P2S}$ is in $\overline{\mathbf{PS2S}}$. Given a $g \in \mathbf{P2S}$ we construct $reg(g)$ and then check whether or not reg_g is a structural isomorphism . If it is then $g \in \mathbf{PS2S}$ and moreover $reg(g)$ is an element of $\mathbf{PS2S}$ such that g *sisom* $reg(g)$.

By Remark 2.3 we can extend this algorithm to an algorithm deciding whether or not an arbitrary edge-labeled graph represents a p2s system .

DISCUSSION

In this part of the paper we have presented the basic theory of p2s and ps2s systems centered around the problem of represents p2s systems in **PS2S** .

Our main theorem (Theorem 4.2) gives a characterization of $\overline{\mathbf{PS2S}}$, while Corollary 4.1 tells us more about the structure of mappings involved .

This part of the paper provides enough mathematical background for an attempt to solve the "state-space synthesis problem" for various kinds of Petri nets . The problem is : "given an edge-labeled graph , construct , whenever possible , a Petri net of a specific type the state space (case graph) of which is isomorphic to the given graph". Since elements of **P2S**(**PS2S**) corresponding to specific types of Petri nets will have to satisfy quite a number of specific conditions, there is still some work to be done before the problem is solved . Our solution is presented in Part 2 of this paper .

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