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A GENERAL THEORY ON SMALL, RADIATING APERTURES IN THE OUTER SHEATH OF A COAXIAL CABLE

by

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ABSTRACT

A pair of coupled, vector integral equations are formulated for finding the tangential electric field at the aperture. A subsequent use of a small aperture assumption allows us to reduce the integral equations to those of quasi-static type. Solution of these equations can be derived from three canonical integral equations; two correspond to the normal magnetic field distribution at the aperture when immersed in an incident tangential static magnetic field and one for the scalar potential at the aperture due to an incident static electric field normal to the aperture. Solutions to these canonical equations are well-known for the special cases of circular and elliptical apertures. The reflection and transmission coefficients in the coaxial-line caused by the aperture radiation are then expressed explicitly in terms of some moment functions associated with the solution of these canonical problems. Analytical expression for the equivalent circuit elements representing both the localized effect of the aperture and the radiation characteristics of the external surface of the outer cylinder are obtained. Our approach therefore requires no assumption on the validity of Bethe's small aperture theory as in a more conventional method.

1. Introduction

In the study of system electromagnetic compatibility (EMC), the need to describe the circuit characteristics of a small aperture in the outer sheath of a coaxial cable or cylinder arises from the concern of electromagnetic penetration into cylindrical enclosures, propagating along cables and leads and causing unwanted electromagnetic interferences with internal circuit elements [1-3]. Application of the same aperture problem is also found in the study of the shielding effectiveness of a braided-shielded cable [4-7], as well as the study of radiation properties of leaky lines used in tunnel communication system [8]. With the exception of [8] a typical procedure in dealing with this type of problem is first to replace the aperture by an equivalent electric and an equivalent magnetic dipole, the amplitude of these dipoles being proportional to the unperturbed electromagnetic field in the absence of the aperture, and then to determine the effect of these dipoles by solving the conventional source excitation problem in the coaxial region. While such a procedure usually can provide sufficiently accurate description of the scattered field both in the exterior and interior regions of the coaxial cable, it isyet to be determined whether it can also yield an adequate network representation of a radiating aperture in a common transmission-line analysis. To demonstrate this point, we use the result derived in this manner for a small aperture of electric polarizability α_e and dyadic magnetic polarizability $\bar{\bar{\alpha}}_m$ (geometry of the problem is depicted in Fig. 1.) Assuming the aperture is fed from the coaxial region so that there is no external field in the absence of the aperture, the equivalent network representation of the aperture in this case is shown in Fig. 2 as consisting of a series inductance L_b and a shunt capacitance C_b given, respectively, by [7]

$$L_b = \frac{\mu_o \alpha_m}{4\pi b^2} \quad ; \quad C_b = \frac{\varepsilon_o \alpha_e \eta^2}{4\pi^2 b^2 z_c^2} \tag{1}$$

where $\eta = (\mu_o/\varepsilon_o)^{\frac{1}{2}} = 120\pi$ ohms, is the characteristic impedance of a plane wave in air;

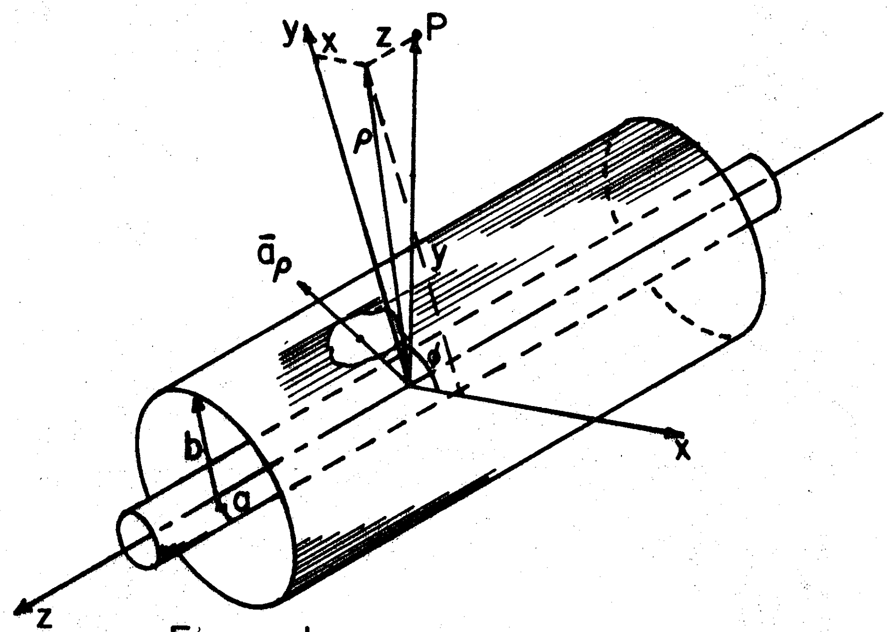


Figure 1a

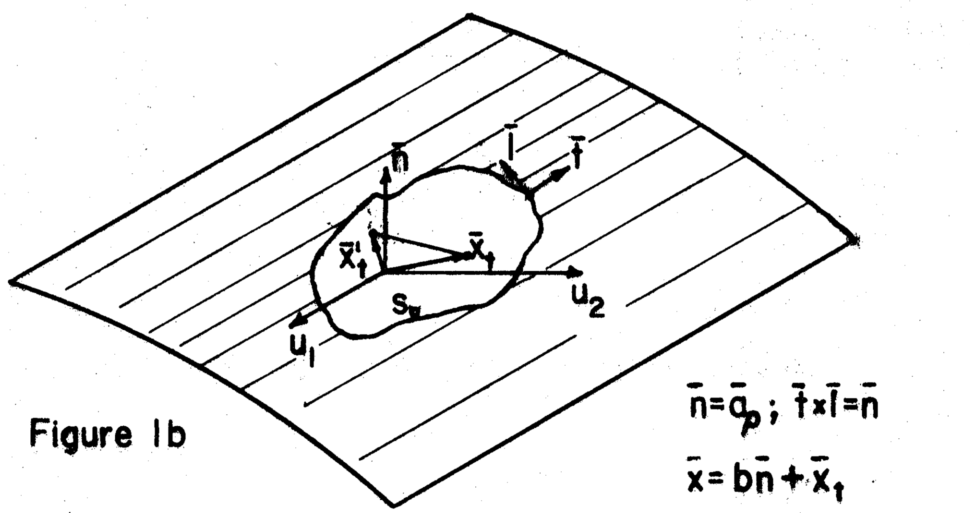
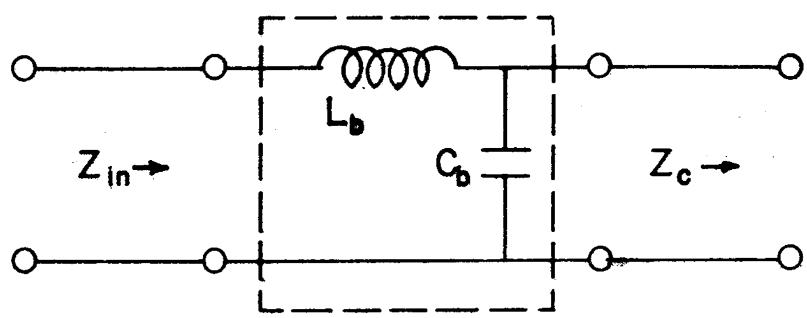


Figure 1b



$$L_b = \frac{\mu_0 \ \alpha_m}{4\pi^2 b^2}$$

$$C_b = \frac{\epsilon_0 \alpha_e \eta^2}{4\pi^2 b^2 Z_c^2}$$

(α_m, α_e) magnetic and electric polarizability

Figure 2

$Z_c = \eta \ \ell n \ b/a = (L/c)^{\frac{1}{2}}$, is the characteristic impedance of an air-filled coaxial line of outer radius b and inner radius a;

$L = (\mu_o/2\pi)\ell n(b/a)$; $C = 2n\varepsilon_o(\ell n \ b/a)^{-1}$, is respectively the distributed inductance and capacitance per unit length in the circuit representation of such a coaxial line;

(μ_o, ε_o) is the permeability and permittivity in air, and

α_m denotes the $\phi\phi$-component of $\bar{\bar{\alpha}}_m$.

We know that the outer surface of the outer conductor can be viewed as a cylindrical radiating antenna, fed by a coaxial-line through an aperture. That the radiation loss into the free-space has not been properly taken into account is immediately apparent in this representation, even though the effect of such a disruption to the line could be predominantly a reactive one.† The question then is how we can best incorporate the input impedance of this antenna, which describes its radiation property, into the conventional transmission-line analysis. Such an information could be of utmost importance for instance, in the calculation of the leakage coefficient of a leaky feeder consisting of periodic isolated holes and used in a tunnel communication system.

The above discussion points to the need of a more comprehensive theory on radiating apertures located on the outer sheath of a coaxial cable. From the viewpoint of boundary problems, a knowledge of the tangential electric or magnetic field in the aperture is all that is required for finding the fields everywhere both inside and outside of coaxial cable. In principle, this can be accomplished by first formulating and then sovling vector integral equations involving the unknown aperture field. We note that although this type of formulation generally can lend itself to numerical computation [9], a purely numerical approach to this problem however is not very useful in the characterization of apertures by their equivalent networks. The alternative then is to find analytical solutions in closed form under judiciously-chosen approximations.

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†The same comment is equally applicable to the treatment in [17] of coupling from one waveguide into another waveguide, or into a cavity resonator by a small aperture located at a suitable position on a common wall.

In practice, the kind of apertures we are interested is electrically much smaller than the free-space wavelength. For these apertures, coupling between the exterior and interior region of the coaxial cable is mainly supported by the fringing field existing in a very localized area near the aperture. Thus, one can, in principle study the effect of an aperture by first investigating the canonical problem associated with an infinite coaxial cable. In the case when the aperture in question is a long but narrow slot, it has been shown that the vector integral equation formulation reduces to a scalar one either exactly for a complete circumferential slot [8,10,11], or approximately for a sectional, transverse slot with the assumption that azimuthal variation of field in the aperture is not significant [12]. In either case, analytical solution to the integral equation can be obtained. The equivalent network representation of a circumferential slot thus derived is shown in Fig. 3a [10]. It is seen that the slot essentially provides a capacitive coupling which couples the input admittance of the external "antenna" Y_∞ into the coaxial transmission line circuit which has a characteristic impedance of Z_c. Application of this network is then extended to the case of a finite transmitting antenna of finite length, fed internally by a finite section of coaxial transmission-line as indicated in Fig. 3b. When compared with the network representation in Fig. 2, the significance of Y_∞ and Y_a is indeed apparent at least in the case of a circumferential slot.

For the more general case of a small aperture of arbitrary shape, the integral equation formulation of the tangential aperture field is much more involved because of the vector nature of the solution. Furthermore, as implied in the work of Bethe [13], Bouwkamp [14] and more recently, Van Bladel [15,16], auxiliary integral equation for the normal component of the aperture field is also needed in order to develop a complete, quasi-static solution, even though the exact solution requires no such additional equation. It is our purpose in this report to discuss the analytical solution of these integral equations, and subsequently to derive from the solution, the explicit network representation of small apertures consistent with the transmission line theory, thus enabling us to compensate the deficiency of the earlier analyses.

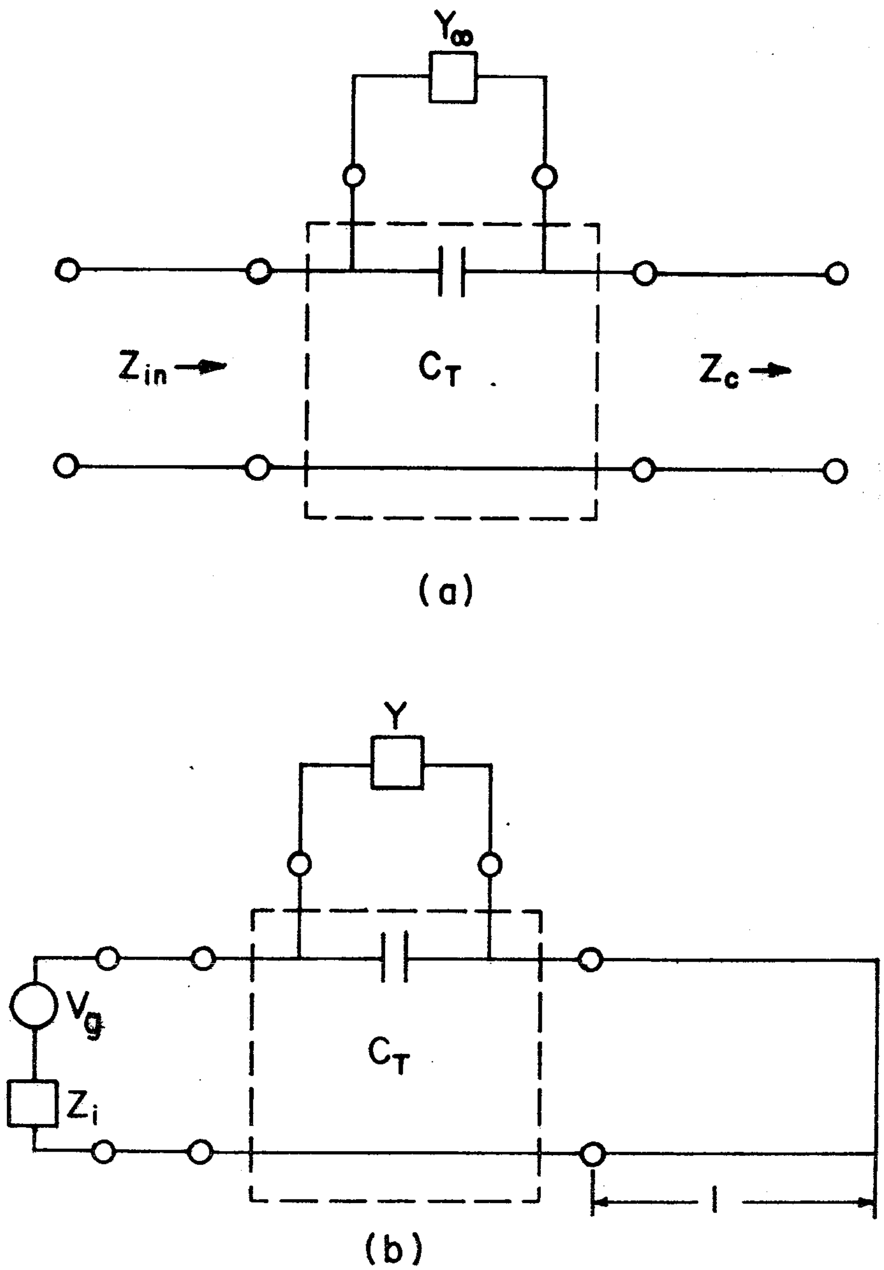

Figure 3

2. Integral equation formulation for a radiating aperture

Geometry of the problem under investigation is depicted in Fig. 1. It consists of a small aperture in the outer sheath of an otherwise perfectly conducting coaxial cable of infinite extent having a tubular outer conductor of radius b and vanishing thickness, and an inner conductor of radius a. As shown in the figure, two coordinate systems will be employed in the subsequent derivation; one is a cylindrical coordinate system (ρ,ϕ,z) with its axis coincides with the axis of the coaxial cable, and the other is a local coordinate system (u_1,u_2,u_3) whose origin is located somewhere in the aperture and is used to describe field points in the vicinity of the aperture. We also use a position vector $\overline{x}_t = u_1\overline{a}_1 + u_2\overline{a}_2$ tangent to the surface of the outer conductor to specify a point in the aperture with coordinates of (b,ϕ,z) in the global system and $(u_1,u_2,0)$ in the local system. Here, $\overline{a}_1$ and $\overline{a}_2$ are the unit vectors defined at each point in the direction of an increasing u_1 and u_2. The coaxial cable is excited by an incident current wave of unit amplitude in the form of $\exp(-i\omega t + ik_1 z)$ where ω is the angular operating frequency and $k_1 = \omega(\mu_1\varepsilon_1)^{\frac{1}{2}}$ is the wave number in the coaxial region.

In order to derive an appropriate integral equation for the aperture field, we first note that the fields in the exterior region where $\rho > b$ consist of only the scattered potentials π_{ez} and π_{hz}, representing essentially the TM- and TE-type of waves in this region. More specifically,

$$\overline{E} = \nabla\times\nabla\times(\pi_{ez}\overline{a}_z) + i\omega\mu_o\nabla\times(\pi_{hz}\overline{a}_z) \tag{2}$$

$$\overline{H} = \nabla\times\nabla\times(\pi_{hz}\overline{a}_z) - i\omega\varepsilon_o\nabla\times(\pi_{ez}\overline{a}_z) \tag{3}$$

where $\overline{a}_z$ is the unit vector in the z-direction. Since both π_{ez} and π_{hz} satisfy the scalar wave equation $(\nabla^2 + k^2)\pi_{e,hz} = 0$ where k is the wave number in air, it is not difficult to show that they can be expressed formally in terms of the value of π_{ez} and the normal derivative of π_{hz} on the boundary surface $\rho = b$ as

$$\pi_{ez}(\rho,\phi,z) = \sum_m \int d\alpha\, \hat{\pi}_{ez,m}(b;\alpha) H_m^{(1)}(\zeta\rho)[H_m^{(1)}(\zeta b)]^{-1} e^{i(m\phi+\alpha z)} \quad (4)$$

$$\pi_{hz}(\rho,\phi,z) = \sum_m \int d\alpha\, [\partial\rho' \hat{\pi}_{hz,m}(b;\alpha)] H_m^{(1)}(\zeta\rho)[\xi H_m'^{(1)}(\zeta b)]^{-1} e^{i(m\phi+\alpha z)} \quad (5)$$

where

$$\zeta = (k^2 - \alpha^2)^{\frac{1}{2}}, \quad \text{with the proper branch specified as } \mathrm{Im}(\alpha) > 0 \text{ for all } \alpha \quad (6)$$

and $H_m^{(1)}$ is the Hankel function of the first-kind and order m, $\hat{\pi}_{ez,m}$ and $\hat{\pi}_{hz,m}$ is the double Fourier transforms of π_{ez} and π_{hz}, respectively defined as

$$\hat{\pi}_{\substack{e\\h} z,m}(\rho;\alpha) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} dz \int_0^{2\pi} d\phi\, \pi_{\substack{e\\h} z}(\rho,\phi,z) \quad (7)$$

so that the summation in (4) and (5) is over integer m ranging from $-\infty$ to ∞, and the integration of α is continuously from $-\infty$ to $+\infty$ on the real axis of a complex α-plane with suitable deformations at $\alpha = \pm k$ in order to agree with the requirement of $\mathrm{Im}(\zeta) \geq 0$ given in (6). Consistent with this definition then, both $\hat{\pi}_{ez,m}(b;\alpha)$ and $\partial_\rho \pi_{hz,m}(b;\alpha)$ in (4) and (5) can be given in terms of the transform of the two tangential electric field components E_ϕ and E_z on the cylindrical surface as follows:

$$\hat{\pi}_{ez,m}(b;\alpha) = \zeta^{-2}\hat{E}_{zm}(b;\alpha); \quad (8)$$

$$\partial_\rho \hat{\pi}_{hz,m}(b;\alpha) = i(k\eta)^{-1}[E_{\phi m}(b;\alpha) + m\alpha(b\zeta^2)^{-1}E_{zm}(b;\alpha)] \quad (9)$$

Substitution of (8) and (9) into (4) and (5) now enables us to express π_{ez} and π_{hz}, and thus all field components, in terms of the two tangential electric field components in the aperture. Specifically, we can write down transforms of all three components of the magnetic field as

$$\hat{H}_{\rho m}(\rho;\alpha) = -\alpha(k\eta)^{-1}H_m'^{(1)}(\zeta\rho)[H_m'^{(1)}(\zeta b)]^{-1}\{E_{\phi m}(b;\alpha)+m\alpha(b\zeta^2)^{-1}E_{zm}(b;\alpha)\}$$

$$+ mk(\zeta^2\zeta\rho)^{-1}H_m^{(1)}(\zeta\rho)[H_m^{(1)}(\zeta b)]^{-1}E_{zm}(b;\alpha) \ ; \tag{10}$$

$$\hat{H}_{\phi m}(\rho;\alpha) = -im\alpha(k\eta\rho)^{-1}\ H_m^{(1)}(\zeta_\rho)[\zeta H_m'^{(1)}(\zeta b)]^{-1}\{E_{\phi m}(b;\alpha)+m\alpha(b\zeta^2)^{-1}E_{zm}(b;\alpha)\}$$

$$+ ik\eta^{-1}\ H_m'^{(1)}(\zeta_\rho)[\zeta H_m^{(1)}(\zeta b)]^{-1}\ E_{zm}(b;\alpha) \ ; \tag{11}$$

$$\hat{H}_{zm}(\rho;\alpha) = i(k\eta)^{-1}\zeta H_m^{(1)}(\zeta\eta)[H_m'^{(1)}(\zeta b)]^{-1}\ \{E_{\phi m}(b;\alpha) + m\alpha(b\zeta^2)^{-1}E_{zm}(b;\alpha)\} \tag{12}$$

Denoting the tangential electric and magnetic fields at the aperture as $\overline{E}_t(b,\phi,z)$, $\overline{H}_t(b,\phi,z)$ and its transform as $\hat{\overline{E}}_{tm}(b;\alpha)$, $\hat{\overline{H}}_{tm}(b;\alpha)$, we can show after some rearrangements that

$$\hat{\overline{H}}_{tm}(b;\alpha) = [\hat{H}_{\rho m}(b;\alpha)(im\ b^{-1}\overline{a}_\phi + i\alpha\overline{a}_z) + ik\eta^{-1}\overline{a}_\rho \times \hat{\overline{E}}_{tm}(b;\alpha)]\{H_m^{(1)}(b\zeta)[\zeta H_m'^{(1)}(b\zeta)]^{-1}\}$$

$$+ ik\eta^{-1}E_{zm}\overline{a}_\phi\{H_m^{(1)}(b\zeta)[\zeta H_m^{(1)}(b\zeta)]^{-1}[1-m^2(b\zeta)^{-2}]+H_m'^{(1)}(b\zeta)[\zeta H_m^{(1)}(b\zeta)]^{-1}\} \tag{13}$$

Now by recognizing that the tangential differential operator

$\nabla_t = (\overline{a}_\phi\ b^{-1}\partial_\phi + \overline{a}_z\partial_z)$ can be replaced by the multiplication factor of $(imb^{-1}\overline{a}_\phi + i\alpha\overline{a}_z)$ in the transform plane, we finally arrive at a suitable integral representation of the tangential magnetic field in the aperture as the following

$$-ik\eta(4\pi^2 b)\overline{H}_t(\overline{x}_t) = \int_{s_o} [-ik\eta H_\rho(\overline{x}_t')\nabla_t\Phi_1^{(0)}(\overline{x}_t-\overline{x}_t')$$

$$+ k^2\overline{a}_\rho \times \overline{E}_t(\overline{x}_t')\Phi_1^{(0)}(\overline{x}_t-\overline{x}_t') + k^2\overline{a}_\phi E_z(\overline{x}_t')\Phi_2^{(0)}(\overline{x}_t-\overline{x}_t')]ds' \ , \tag{14}$$

$$\text{for} \quad \overline{x}_t \in s_{o+}$$

where the two kernel function $\Phi_1^{(o)}$ and $\Phi_2^{(o)}$ are defined as

$$\Phi_1^{(o)}(\bar{x}_t) = \sum_m \int d\alpha \frac{H_m^{(1)}(\zeta b)}{\zeta H_m^{\prime(1)}(\zeta b)} e^{i(m\phi+\alpha z)} , \tag{15}$$

$$\Phi_2^{(o)}(\bar{x}_t) = \sum_m \int d\alpha \left[\frac{H_m^{\prime(1)}(\zeta b)}{\zeta H_m^{\prime(1)}(\zeta b)} + \left(1-\frac{m^2}{b^2\zeta^2}\right)\frac{H_m^{(1)}(\zeta b)}{\zeta H_m^{\prime(1)}(\zeta b)}\right] e^{i(m\phi+\alpha z)} \tag{16}$$

Here, the position vectors $\bar{x}_t$ and $\bar{x}_t'$ are used to denote the observation and source point in the aperture. The notation s_{o+} is used to describe an observation point in front of the aperture with $\rho = b+\delta$, as $\delta \to 0$. In deriving (14), use has also been made of the convolution theorem in both the ϕ- and z- integrations as well as the boundary condition of vanishing E_z, E_ϕ and H_ρ on the conducting portion of the cylindrical surface $\rho = b$. We further note that the integral representation of $\bar{H}_t(\bar{x}_t)$ in (14) involves both the normal component of the aperture magnetic field and the tangential electric field. This particular form is chosen in anticipation that major contribution to the integral would come from the term associated with H_ρ, rather than that of $\bar{a}_\rho \times \bar{E}_t$ or E_z under a small aperture approximation. However, any subsequent approximation by retaining only the term involving H_ρ would automatically eliminate one of the two linearly-independent boundary conditions regarding the continuity of the tangential magnetic field at the aperture, even though two conditions would be required in setting up the complete vector integral equation for the unknown aperture field. In order to overcome such a difficulty, we now introduce an auxiliary integral representation which relates the normal component of the aperture electric field with the two tangential electric field components in the aperture. One can show from (2),(4),(5),(8),(9) without much difficulty that the normal electric field at the aperture is given by

$$4\pi^2 b\, E_\rho(\bar{x}_t) = \int_{s_o} [-\bar{E}_t(\bar{x}_t')\cdot\nabla_t\Phi_1^{(0)}(\bar{x}_t-\bar{x}_t')+E_z(\bar{x}_t')\partial_z\Phi_2^{(0)}(\bar{x}_t-\bar{x}_t')]ds'$$

$$\text{for } \bar{x}_t \in s_{o+} \tag{17}$$

where the two kernel functions $\Phi_1^{(0)}$ and $\Phi_2^{(0)}$ are again defined in (15) and (16), respectively.

We now proceed to find the field components in the coaxial region where $a < \rho < b$. The total field in this region consists of an incident field and a scattered field due to the presence of the aperture. For an incident TEM current wave of unity in the coaxial cable, the expression for the corresponding incident field is of the following

$$\begin{aligned} \overline{E}^{inc} &= E_\rho^{inc}\,\overline{a}_\rho = \eta_1(2\pi\rho)^{-1}\exp(ik_1 z)\overline{a}_\rho \\ \overline{H}^{inc} &= H_\phi^{inc}\,\overline{a}_\phi = (2\pi\rho)^{-1}\exp(ik_1 z)\overline{a}_\phi \end{aligned} \tag{18}$$

where $k_1 = \omega(\mu_o\varepsilon_1)^{\frac{1}{2}} = n_r k$ is the wave number, and $\eta_1 = (\mu_o/\varepsilon_1)^{\frac{1}{2}} = n_r^{-1}\eta$ is the characteristic impedance of a plane wave in the coaxial region with a dielectric of permittivity ε_1. For an air-filled coaxial cable, $n_r=1$. Now since the incident electric field has no tangential component on the conducting surfaces of $\rho = a$ and $\rho = b$, the scattered potentials π_{ez} and π_{hz} in this region can be obtained in like manner as in the exterior region, with the only exception that the outgoing wave boundary condition at infinity has to be replaced by the vanishing tangential electric field condition on the surface of the inner conductor. The two scattered potentials are formally expressed in the form of

$$\pi_{ez}^s(\rho,\phi,z) = \sum_m \int d\alpha\ \pi_{ez,m}(b_-;\alpha)\Omega_{em}(a,\rho;\zeta_r)[\Omega_{em}(a,b;\zeta_r)]^{-1}e^{i(m\phi+\alpha z)} \tag{19}$$

$$\pi_{hz}^s(\rho,\phi,z) = \sum_m \int d\alpha[\partial_{\rho'}\ \pi_{hz,m}(b_-;\alpha)]\ \Omega_{hm}(a,\rho;\zeta_r)[\zeta_r\Omega'_{hm}(a,b;\zeta_r)]^{-1}e^{i(m\phi+\alpha z)} \tag{20}$$

where

$$\Omega_{em}(p,q;\zeta_r) = J_m(\zeta_r p)H_m^{(1)}(\zeta_r q)-J_m(\zeta_r q)H_m^{(1)}(\zeta_r p), \tag{21}$$

$$\Omega_{hm}(p,q;\zeta_r) = J'_m(\zeta_r p)H_m^{(1)}(\zeta_r q)-J_m(\zeta_r q)H_m'^{(1)}(\zeta_r p), \tag{22}$$

$\zeta_r = (k_1^2 - \alpha^2)^{\frac{1}{2}}$ with the proper branch specified by $I_m(\zeta_r) \geq 0$ for all α, and J_m is the Bessel function of order m. Here, the notation b_- is used to mean $\rho = b - \delta$ and $\delta \to 0$. Equations (19) and (20) are identical in form to the corresponding equations (4) and (5) in the exterior region, with the substitution of the Hankel function $H_m^{(1)}$ by Ω_{em} in the case of π_{ez}, and by Ω_{hm} in the case of π_{mz}. A similar manipulation like the one leading to (14) would therefore yield the scattered magnetic field in the aperture as

$$-ik\eta(4\pi^2 b)\overline{H}_t^s(\overline{x}_t) = \int_{s_o} [-ik\eta H_\rho(\overline{x}_t')\nabla_t\Phi_1^{(c)}(\overline{x}_t - \overline{x}_t')$$

$$+ k_1^2\overline{a}_\rho \times \overline{E}_t(x_t')\Phi_1^{(c)}(\overline{x}_t - \overline{x}_t') + k_1^2\overline{a}_\phi E_z(\overline{x}_t')\Phi_2^{(c)}(\overline{x}_t - \overline{x}_t')]ds',$$

$$\text{for } \overline{x}_t \in s_{o-}, \tag{24}$$

where the two kernel functions $\Phi_1^{(c)}$ and $\Phi_2^{(c)}$ are now given as

$$\Phi_1^{(c)}(\overline{x}_t) = \sum_m \int d\alpha \frac{\Omega_{hm}(a,b;\zeta_r)}{\zeta_r \Omega_{hm}'(a,b;\zeta_r)} e^{i(m\phi+\alpha z)} \tag{25}$$

$$\Phi_2^{(c)}(\overline{x}_t) = \sum_m \int d\alpha \left[\frac{\Omega_{em}'(a,b;\zeta_r)}{\zeta_r \Omega_{em}(a,b;\zeta_r)} + \left(1 - \frac{m^2}{b^2\zeta_r^2}\right)\frac{\Omega_{hm}(a,b;\zeta_r)}{\zeta_r \Omega_{hm}'(a,b;\zeta_r)}\right] e^{i(m\phi+\alpha z)} \tag{26}$$

Again, the notation s_{o-} in (24) is used to describe a cross-sectional area same as s_o but with $\rho = b - \delta$ and $\delta \to 0$. The total tangential magnetic field in the aperture is then given as

$$\overline{H}_t(\overline{x}_t) = \overline{H}_t^s(\overline{x}_t) + (2\pi b)^{-1} \exp(ik_1 z)\overline{a}_\phi \tag{27}$$

according to (18). We note that in the expression for $\Phi_1^{(c)}$ and $\Phi_2^{(c)}$, the integrands have poles at $\alpha = \pm p_{mn}$, $n = 1,2,\ldots$ where $\Omega_{em}(a,b;\zeta_r) = 0$, and at $\alpha = \pm q_{mn}$; $n = 1,2,\ldots$ where $\Omega_{hm}'(a,b;\zeta_r) = 0$, in addition to a pair of poles at $\alpha = \pm k_1$. Thus, one can in principle replace the integration over α by a sum of residue contributions from these poles. It can be shown that the terms associated with p_{mn} corresponds exactly to the contribution from the TM-modes, q_{mn} to the TE-modes and k_1 to the TEM mode.

In order to obtain a suitable integral equation for the aperture field, we invoke the continuity of all field components in the aperture. Thus, with the use of (14), (24) and (27), we finally arrive at the following expression

$$\int_{s_o} [-ik\eta H_\rho(\bar{x}_t')\nabla_t\Phi_1(\bar{x}_t-\bar{x}_t') + k^2\bar{a}_\rho\times \bar{E}_t(\bar{x}_t')\tilde{\Phi}_1(\bar{x}_t-\bar{x}_t')+k^2\bar{a}_\phi E_z(\bar{x}_t')\tilde{\Phi}_2(\bar{x}_t-\bar{x}_t')]ds'$$

$$= -ik\eta 2\pi\bar{a}_\phi \exp(ik_1 z) \; ; \qquad \text{for } \bar{x}_t \in s_o \, , \tag{28}$$

where

$$\Phi_j = \Phi_j^{(0)} - \Phi_j^{(c)}, \quad \tilde{\Phi}_j = \Phi_j^{(0)} - n_r^2\Phi_j^{(c)} \qquad \text{for } j = 1,2 \tag{29}$$

Equation (28) is then the vector integration equation for finding the tangential aperture electric field. Likewise one can also find from (18), (19), (20), the normal component of the electric field at the aperture $\rho = b_-$ as

$$4\pi^2 b\, E_\rho(\bar{x}_t) = \int_{s_o} [-\bar{E}_t(\bar{x}_t') \cdot \nabla_t\Phi_1^{(c)}(\bar{x}_t-\bar{x}_t') + E_z(\bar{x}_t')\partial_z\Phi_2^{(c)}(\bar{x}_t-\bar{x}_t')]ds'$$

$$+ 4\pi^2 b\, E_\rho^{inc}(\bar{x}_t) \, , \qquad \text{for } \bar{x}_t \in s_{o_-} \tag{30}$$

similar to the expression in (17) for the exterior region. An auxiliary integral equation is then obtained from (17) and (30) by invoking the continuity condition at the aperture,

$$\int_{s_o} [-\bar{E}_t(\bar{x}_t')\cdot\nabla_t\Phi_1(\bar{x}_t-\bar{x}_t')+ E_z(\bar{x}_t)\partial_z\Phi_2(\bar{x}_t-x_t')]ds'=-2\pi\eta_1\exp(ik_1 z),$$

$$\text{for } \bar{x}_t \in s_o \, , \tag{31}$$

where Φ_1 and Φ_2 are once again given by (29). As we mentioned earlier, the vector equation (28) alone in its exact form is sufficient to yield an unique solution to the tangential electric field at the aperture, and hence, fields everywhere. The auxiliary integral equation from this viewpoint indeed is only a redundant one and normally would not provide any more information which is not already contained in the solution of (28). However, as it will be shown later, the terms containing k^2 in (28) are only higher terms in the quasi-static limit and will not be retained in the homogeneous part of the integral equation. Thus , only H_ρ-component can be resolved from the approximate form (28), and one then needs to use the auxiliary integral equation (31) in order to obtain a complete solution to the problem.

3. Approximate integral equation: small-aperture approximation

Although the exact integral equation as appears in (28) is not immediately amendable to explicit solutions in close form, considerable simplification of the equation can be achieved under the so-called small aperture approximation in which we assume not only that the aperture is very small compared to the free-space wavelength $(k^2 s_o <<<< 1)$, but its characteristic length is also small compared with the cross-sectional dimension i.e. (b-a) of the coaxial cable. The consequence of this assumption is that local behavior of the fields would now act more like the static fields associated with the same aperture in a planar ground screen. This would allow us to adopt a quasi-static equivalence of the original problem, from which an analytic solution can be obtained. To achieve that, we first note that the integrals associated with the kernels Φ_j and $\tilde{\Phi}_j$, $j = 1,2$ actually diverge as the distance $r = |\bar{x}_t - \bar{x}'_t| \to 0$. Now since the singular behavior of these integrals can be retained by simply keeping the leading terms of the integrands as $\alpha \to \infty$, the proper procedure then is to evaluate the leading terms in the integrand exactly, while approximating terms which are smoothly varying with respect to r. Using the Hankel's asymptotic expansion for the Bessel and Neumann functions and their derivatives, we have from [18], after some manipulations, the

the following asymptotic expressions for the integrands of individual $\Phi_1^{(o)}$, $\Phi_2^{(o)}$ as defined in (15), (16), and $\Phi_1^{(c)}$, $\Phi_2^{(c)}$ defined in (25) and (26);

$$\frac{H_m^{(1)}(\zeta b)}{\zeta H_m'^{(1)}(\zeta b)} \simeq -ib(\zeta^2 b^2-m^2)^{-\frac{1}{2}} - (b/2)\zeta^2 b^2(\zeta^2 b^2-m^2)^{-2}$$

$$\frac{\Omega_{hm}(a,b;\zeta_r)}{\zeta\Omega'_{hm}(a,b;\zeta_r)} \simeq ib(\zeta_r^2 b^2-m^2)^{-\frac{1}{2}} - (b/2)\zeta_r^2 b^2(\zeta_r^2 b^2-m^2)^{-2}$$

$$(1-\frac{m^2}{b^2\zeta^2})\frac{H_m^{(1)}(\zeta b)}{\zeta H_m'^{(1)}(\zeta b)} + \frac{H_m'^{(1)}(\zeta b)}{\zeta H_m^{(1)}(\zeta b)} \simeq -b(\zeta^2 b^2-m^2)^{-1}$$

$$(1-\frac{m^2}{b^2\zeta^2})\frac{\Omega_{hm}(a,b;\zeta_r)}{\zeta\Omega'_{hm}(a,b;\zeta_r)} + \frac{\Omega'_{em}(a,b;\zeta_r)}{\zeta\Omega_{em}(a,b;\zeta_r)} \simeq -b(\zeta_r^2 b^2-m^2)^{-1}$$

Before we carry out the integration of involving these leading terms analytically, we note that only the difference of $\Phi_j^{(0)}$ and $\Phi_j^{(c)}$; $j = 1,2$ is actually involved in the integral equations (28) and (31). Mutual cancellation of many of these asymptotic terms occurs for an air-filled coaxial cable where $n_r = 1$ and $\zeta_r = \zeta$. Because of this reason, we shall restrict our discussion only to this case even though the procedure can be extended to the more general case. Thus, for the computation of $\Phi_1 = \Phi_1^{(o)} - \Phi_1^{(c)}$ we have from (15) and (25)

$$\Phi_1(\bar{x}_t) = \tilde{\Phi}_1(\bar{x}_t) = -8\pi^2 b\tilde{\phi}(\bar{x}_t) + \sum_m \int d\alpha\{\frac{H_m^{(1)}(\zeta b)}{\zeta H_m'^{(1)}(\zeta b)} - \frac{\Omega h_m(a,b;\zeta)}{\zeta_r\Omega'_{hm}(a,b;\zeta)}$$

$$+ i2b(\zeta^2 b^2-m^2)^{-\frac{1}{2}}\} e^{i(m\phi+\alpha z)} ; \tag{33}$$

where

$$\tilde{\Phi}(\bar{x}_t) = i(2\pi)^{-2} \sum_m \int d\alpha(\zeta^2 b^2 - m^2)^{-\frac{1}{2}} e^{i(m\phi+\alpha z)} \tag{34}$$

The integral in (33) now no longer diverges as $|\bar{x}_t| \to 0$. Consequently, a Taylor-series expansion of this term with respect to $k|\bar{x}_t|$, together with (34) yields a suitable form of Φ_1 as

$$\Phi_1(\bar{x}_t - \bar{x}'_t) \simeq -8\pi^2 b\phi(\bar{x}_t - \bar{x}'_t) + \psi^{(1)} + 0(kr) \tag{35}$$

where

$$\phi(\bar{x}_t - \bar{x}'_t) = \hat{\phi}(\bar{x}_t - \bar{x}_t) + (2\pi^2 b)^{-1} \ln(1 - e^{i2\pi kb}) \tag{36}$$

$$\psi^{(1)} = \sum_m \int d\alpha \left[\frac{H_m^{(1)}(\zeta b)}{\zeta H_m'^{(1)}(\zeta b)} - \frac{\Omega_{hm}(a,b;\zeta)}{\zeta\Omega_{hm}(a,b;\zeta)} + i2b(\zeta^2 b^2 - m^2)^{-\frac{1}{2}} \right] + 4\ln(1 - e^{i2\pi kb}) \tag{37}$$

For the evaluation of Φ_2, we note that the leading terms in $\Phi_2^{(0)}$ and $\Phi_2^{(c)}$ cancel completely, and hence, the integral for Φ_2 converges for all values of $|\bar{x}_t|$ including $|\bar{x}_t| \to 0$. A Taylor-series expansion of Φ_2 therefore yeilds

$$\Phi_2(\bar{x}_t) = \tilde{\Phi}_2(\bar{x}_t) = \psi^{(2)} + 0(kr) \tag{38}$$

where

$$\psi^{(2)} = \sum_m \int d\alpha \left\{ \left[\frac{H_m'^{(1)}(\zeta b)}{\zeta H_m^{(1)}(\zeta b)} - \frac{\Omega'_{em}(a,b;\zeta)}{\zeta\Omega_{em}(a,b;\zeta)} \right] + \left(1 - \frac{m^2}{b^2\zeta^2}\right) \left[\frac{H_m^{(1)}(\zeta b)}{\zeta H_m'^{(1)}(\zeta b)} - \frac{\Omega_{hm}(a,b;\zeta)}{\zeta\Omega'_{hm}(a,b;\zeta)} \right] \right\} \tag{39}$$

Substitution of the results given in (35) - (39) into (28) and (31) finally gives a pair of much simplified integral equations in the aperture $\overline{x}_t \varepsilon S_o$, valid to the order of (kr) as

$$\int_{s_o}[-ik\eta H_\rho(\overline{x}_t')\nabla_t\phi(\overline{x}_t-\overline{x}_t') + k^2\overline{a}_\rho \times \overline{E}_t(\overline{x}_t')\phi(\overline{x}_t-\overline{x}_t')]ds' = -ik\eta\overline{F}_o \tag{40}$$

$$-\int_{s_o}\overline{E}_t(\overline{x}_t') \cdot \nabla_t\phi(\overline{x}_t-\overline{x}_t')ds' = G_o \tag{41}$$

where

$$\overline{F}_o = -(4\pi b)^{-1}\overline{a}_\phi + ik(8\pi^2 b\eta)^{-1}\{\psi^{(1)}\int_{s_o}\overline{a}_\rho \times \overline{E}_t(\overline{x}_t')ds' + \psi^{(2)}\int_{s_o}\overline{a}_\phi E_z(\overline{x}_t')ds'\} \tag{42}$$

$$G_o = \eta(4\pi b)^{-1}, \tag{43}$$

The expression for $\phi(\overline{x}_t-\overline{x}_t')$ as given in (34) has been evaluated to the same order of accuracy in Appendix A,

$$\phi(\overline{x}_t-\overline{x}_t') = \frac{1}{2\pi r}\exp(ikr); \quad r = |\overline{x}_t - \overline{x}_t'| . \tag{44}$$

4. Formal solution to the approximate integral equation

In this section, we shall discuss the construction of a formal solution from known canonical problems. To achieve this, let us first define a vector function $\overline{F}_t$ in the direction tangent to the aperture.

$$\overline{F}_t(\overline{x}_t) = \int_{s_o}[\overline{E}_t(\overline{x}_t') \times \overline{n}]\phi(\overline{x}_t-\overline{x}_t')ds' \tag{45}$$

where $\overline{n} = \overline{a}_\rho$ is the unit normal and $\phi(\overline{x}_t-x_t')$ is the same kernel defined in (43). As shown in Appendix B, the function $\overline{F}_t$ possesses the following properties:

$$\nabla_t(\nabla_t \cdot \overline{F}_t) = ik\eta\int_{s_o}\overline{n}\cdot\overline{H}_t(\overline{x}_t')\nabla_t\phi(\overline{x}_t-\overline{x}_t')ds', \tag{46}$$

$$\overline{n} \cdot \nabla_t \times \overline{F}_t = -\int_{s_o}\overline{E}_t(\overline{x}_t')\cdot\nabla_t\phi(\overline{x}_t-\overline{x}_t')ds \tag{47}$$

where ∇_t denotes the usual differential operator taken in the plane of the aperture. As a consequence of (45)-(47), we can rewrite the two integral equations (39) and (41) in the form of

$$\nabla_t(\nabla_t \cdot \overline{F}_t) + k^2\overline{F}_t = ik\eta\overline{F}_o \; ; \quad \overline{n} \cdot (\nabla_t \times \overline{F}_t) = G_o \tag{48}$$

valid for $\overline{x}_t \, \varepsilon \, S_o$. Thus, the next step is to find the value of $\overline{F}_t$ from these simplified differential relationships. Once found, we can then find the unknown aperture field $\overline{E}_t \times \overline{n}$ from the solution of (44). Here it is of particular interest to note that the vector function $\overline{F}_t$ actually corresponds to the electric vector potential produced by an equivalent magnetic current source of $2\,\overline{E}_t \times \overline{n}$ over a surface area of S_o in air. In addition, one can also show that the governing equations for the case of a plane-wave oblique incident onto an aperture S_o on a planar conducting screen are essentially the same as (45) and (46), with the exception of the source terms $\overline{F}_o$ and G_o being replaced respectively by the tangential magnetic field and normal electric field of the incident plane wave [16].

To construct the formal solution to (46), we can first consider two canonical problems of magnetic type described by the following set of differential equations for $\overline{x}_t \, \varepsilon \, S_o$:

$$\nabla_t(\nabla_t \cdot \overline{f}_{t_j}) + k^2\overline{f}_{t_j} = \overline{a}_j \; ; \quad \overline{n} \cdot (\nabla_t \times \overline{f}_{t_j}) = 0 \tag{49}$$

for $j = 1,2$ and $\overline{a}_j$ are the two orthogonal unit vectors in the aperture surface, and another canonical problem of electric type described by

$$\nabla_t(\nabla_t \cdot \overline{f}_{t_3}) + k^2\overline{f}_{t_3} = 0 \; , \quad \overline{n} \cdot (\nabla_t \times \overline{f}_{t_3}) = 1 \tag{50}$$

Since the three canonical problems are linearly-independent from each other, we can expand the solution to the original equations (46) and (47) in the form of

$$\overline{F}_t = A_1\overline{f}_{t_1} + A_2\overline{f}_{t_2} + A_3\overline{f}_{t_3} \tag{51}$$

Substitution of (51) to (48) together with the use of (42) and (43), then provides us three independent equations for the determination of A_1, A_2 and A_3.

To demonstrate how this can be done, we first note that the scattered electric field in the aperture is related to the vector electric potential by $\overline{E}^S = \nabla \times \overline{F}_t$. Since the incident electric field is normal to the aperture, we have

$$\int_{s_o} \overline{n} \times \overline{E}_t ds = \int_{s_o} \overline{n} \times (\nabla_t \times \overline{F} + \overline{n} \times \partial_n \overline{F}_t) ds = -\int_{s_o} \partial_n \overline{F}_t ds \tag{52}$$

which can then be used to evaluate F_o. As a consequence, we have from (48) the following relationship,

$$A_1\overline{a}_1 + A_2\overline{a}_2 = ik\eta\{-(4\pi b)^{-1}\overline{a}_\phi - ik(8\pi^2 b\eta)^{-1}[\psi^{(1)} \sum_{\substack{i=1,2\\ j=1,2,3}} A_j M_{ij} \overline{a}_i$$

$$- \psi^{(2)} \sum_{\substack{i=1,2\\ j=1,2,3}} A_j M_{ij} (\overline{a}\phi \cdot \overline{a}_i)\overline{a}_i]\} \tag{53}$$

where the moment function M_{ij} is defined as

$$M_{ij} = \int_{s_o} \overline{a}_i \cdot \partial_n \overline{f}_{t_j} \, ds \tag{54}$$

Equating the like coefficients for $\overline{a}_1$ and $\overline{a}_2$ in (53) then allows us to obtain two of the three equations for the determination of A_j's:

$$A_i = -ik\eta(4\pi b)^{-1}\{ C_i + ik(2\pi\eta)^{-1}(\psi^{(1)} - C_i \psi^{(2)}) \sum_{j=1}^{3} M_{ij} A_j\} , \tag{55}$$

where

$$C_i = \overline{a}_\phi \cdot \overline{a}_i \; ; \quad i = 1,2 \tag{56}$$

Similarly, the application of (51) and the expression for G_o in (43) into the second equation in (48) yields immediately

$$A_3 = \eta(4\pi b)^{-1} \tag{57}$$

Equations (55)-(57) therefore uniquely determine the value of A_j for all j. The vector function $\overline{F}_t$ is now explicitly known in terms of solutions to the three canonical problems. Once again we note that since $\overline{F}_t$ as defined in (45) represents the electric vector potential due to the scattering from the aperture, the scattering electric field everywhere including those in the aperture, is then given by the expression $\nabla \times \overline{F}_t$, and the scattering magnetic field by $-i(k\eta)^{-1}[\nabla(\nabla_t \cdot F_t) + k^2 F_t]$ as a direct consequence of the Maxwell equation.

5. Discussions on the canonical problems

Before we proceed to derive the network equivalence of small apertures, it is important to discuss some of the features contained in the solution of the canonical problems. First of all, we note that the results of these canonical problems are connected to the coaxial-line problem directly from those moment functions M_{ij} in (54). As it will be shown later, the magnitude of those terms associated with M_{ij} is typically in the order of $(k^2 s_o)^{3/2}$ where s_o is the surface area of the aperture. Since the aperture is assumed to be electrically small, only a zero$\underline{\text{th}}$ order estimates of M_{ij} would be sufficient in the present problem. Thus, for the canonical problems of magnetic type, we have from (49) the approximate form of

$$\nabla_t (\nabla_t \cdot \overline{f}_{t_j}) = \overline{a}_j \quad ; \quad \overline{n} \cdot (\nabla_t \times \overline{f}_{t_j}) = 0 \quad \text{for } j = 1,2 \text{ and } \overline{x}_t \varepsilon s_p \tag{58}$$

which then has a same form as the case of an aperture immersed in a quasi-static magnetic field of unity in the $\overline{a}_j$ direction. It is shown in [15] and [16], via a derivation similar to the one leading to (48), that the associated integral equation is

$$\nabla_t \int_{s_o} \overline{n} \cdot \overline{h}_j (\overline{x}'_t)\, \phi_o(\overline{x}_t - \overline{x}'_t)\, ds' = \overline{a}_j \,, \quad \text{for } \overline{x}_t \varepsilon s_o \tag{59}$$

where

$$\phi_o(\overline{x}_t-\overline{x}_t') = (2\pi)^{-1}/|\overline{x}_t-\overline{x}_t'| \ , \tag{60}$$

and $n \cdot h_j = n \cdot \nabla\ (\nabla_t \cdot \overline{f}_{t_j})$ is the normal component of the magnetic field in the aperture.

One can further prove the following properties that [16]

i) the quality $\rho_j = n \cdot h_j$ is also proportional to the induce charge on an electric conducting disc having the same shape as s_o, immersed in an incident field $\overline{a}_j$;

ii) $\int_{s_o} \overline{\rho}(\overline{x}_t) ds = 0$ where $\rho(\overline{x}_t)$ as a vector is defined as $\overline{a}_1\rho_1(\overline{x}_t) + \overline{a}_2\rho_2(\overline{x}_t)$;

iii) $\overline{\rho}$ becomes infinity like $\ell^{-\frac{1}{2}}$ as the observation point approaches the rim of the disc, where ℓ is the distance measured from the rim.

Solution to the integral equation (59) is well-known for elliptical and circular apertures [13-17], and will not be repeated here. For an aperture of arbitrary shape, one needs to retort to numerical methods for the computation of the aperture field. Here, we merely mention that is is also customary to express the moment function M_{ij} in terms of $\overline{\rho}$. Thus, using the definition of M_{ij} in (54) and (52) and noting that $\overline{a}_i = \nabla_t u_i$, we have

$$\begin{aligned} M_{ij} &= \int_{s_o} \nabla_t u_i \cdot e_j(\overline{x}_t) \times \overline{n}\ ds \\ &= \int_{s_o} [\nabla_t \cdot (u_i \overline{e}_{t_j} \times \overline{n}) - u_i\ \nabla_t \cdot (\overline{e}_{t_j} \times \overline{n})] ds, \\ &\qquad\qquad \text{for } i,j = 1,2 \end{aligned} \tag{61}$$

where $\overline{e}_j$ is the aperture electric field in the canonical problem.

With the use of the divergence theorem and the boundary condition of a vanishing tangential electric field along the boundary contour ℓ that enclose s_o, one can readily show that the contribution from the first term in the square bracket is zero. The second term can be replaced by $-u_i(\bar{n}\cdot\nabla_t\times\bar{e}_{t_j}) = ik\eta\, u_i(\bar{n}\cdot\bar{h}_j)$ to yield

$$M_{ij} = ik\eta\int_{s_o} u_i(\bar{n}\cdot\bar{h}_j)ds = ik\eta\int_{s_o} u_i\rho_j(\bar{x}_t)ds, \quad \text{for } i,j=1,2 \tag{62}$$

As for the canonical problem of electric type, we have from (50) the approximate form

$$\nabla_t(\nabla_t\cdot\bar{f}_{t3}) = 0\ , \quad \bar{n}\cdot(\nabla_t\times\bar{f}_{t3}) = 1\ , \tag{63}$$

with the associated integral equation obtained from (41) as

$$\bar{n}\cdot\ \nabla_t\times\int_{s_o}[\bar{e}_{t3}(\bar{x}'_t)\times\bar{n}]\phi_o(\bar{x}_t-\bar{x}'_t)ds' = 1\ , \quad \text{for } \bar{x}_t\ \varepsilon\ s_o \tag{64}$$

where $\bar{e}_{t3}$ is the tangential aperture field and ϕ_o is defined in (60). Now since $\nabla_t\times(\bar{e}_{t3}\times\bar{n})\phi_o = -(\bar{e}_{t3}\cdot\nabla_t\phi_o)\bar{n}$, an alternative form of (63) can be given as

$$\int_{s_o}\bar{e}_{t3}(\bar{x}'_t)\cdot\nabla_t\phi_o(\bar{x}_t-\bar{x}'_t)ds' = -1, \tag{65}$$

which then has a same form as the case of an aperture immersed in a quasi-static electric field of unity in the direction normal to the aperture [15,16]. If now we define a scalar potential $\tau_3(\bar{x}_t)$ so that $\bar{e}_{t3}=-\nabla_t\tau_3$, it can be shown in portntial theory that [15,16] .

i) τ is positive and vanishes along the rim like $\ell^{\frac{1}{2}}$, where ℓ is the distance measured from the rim;

ii) the function $\tau(\bar{x}_t)$ also corresponds to half the potential difference across a flat magnetic conductor having the same shape as s_o, immersed in a unit field in the normal direction.

In addition, the moment function in this case is given by

$$M_{i3} = -\int_{s_o} \overline{a}_i \cdot (\nabla_t \tau_3 \times \overline{n}) ds = \oint_{\ell} \tau_3 \overline{a}_i \cdot d\overline{\ell} \ , \quad i = 1,2 \tag{66}$$

which according to (i), vanishes on the boundary contour ℓ, $i = 1,2$ enclosing s_o. Because of this, we can now define a dyadic function $\overline{\overline{M}}$ as

$$M_{ij} = \overline{a}_i \cdot \overline{\overline{M}} \cdot \overline{a}_j, \quad \text{for } i, j = 1,2,$$

and

$$\overline{\overline{M}} = \int_{s_o} \overline{x}_t \rho(\overline{x}_t) ds \tag{67}$$

Most often, the dyadic $\overline{\overline{M}}$ is referred to as a dyadic magnetic polarizability of an aperture. It is obvious that for an aperture with symmetry in both u_1 and u_2 directions (an elliptical aperture for instance), the diagonal element M_{ij} is identically zero for $i \neq j$. Using the result in (66), it is then not difficult to show from (55)-(57) that

$$\overline{A}_t = \frac{-ik\eta}{4\pi b} \left(\overline{\overline{I}} - \frac{k^2}{8\pi^2 b} \overline{\overline{\psi}}_c \overline{\overline{M}}\right)^{-1} \cdot \overline{C} \tag{68}$$

where

$$\overline{A}_t = A_1 \overline{a}_1 + A_2 \overline{a}_2 \ , \tag{69}$$

$$\overline{C} = (\overline{a}_\phi \cdot \overline{a}_1)\overline{a}_1 + (\overline{a}_\phi \cdot \overline{a}_2)\overline{a}_2 \ , \tag{70}$$

$$\overline{\overline{\psi}}_c = [\psi^{(1)} - (\overline{a}_\phi \cdot \overline{a}_1)\psi^{(2)}] \, \overline{a}_1 \overline{a}_1 + [\psi^{(1)} - (\overline{a}_\phi \cdot \overline{a}_2)\psi^{(2)}] \overline{a}_2 \overline{a}_2 \tag{71}$$

and $\overline{\overline{I}}$ is the unitary dyadic. For the case of symmetrical aperture where $M_{ij} = 0$ for $i \neq j$, solution of $\overline{A}$ takes a simpler form

$$A_i = \left(\frac{-ik\eta}{4\pi b}\right) \frac{(\overline{a}_\phi \cdot \overline{a}_i)}{1 - k^2 (8\pi^2 b)^{-1} [\psi^{(1)} - (\overline{a}_\phi \cdot \overline{a}_i)\psi^{(2)}] M_{ii}} \ ,$$

$$\text{for } i = 1,2 \tag{72}$$

together with $A_3 = \eta(4\pi b)^{-1}$ as given in (57). Provided that M_{ii} is known from the static solution, (72) together with (57) and (51) then yield the complete solution of the aperture field. Once again, we wish to emphasize that the fields associated with $\bar{f}_{t_j}$ for $j = 1,2$ describe the magnetic property of the aperture, while $\bar{f}_{t_j}$ for $j = 3$ describes the electric property. Now even though $\bar{f}_{t3}$ will not produce any contribution to the moment function M_{ij}, or the magnetic polarizability, its importance comes from the evaluation of the following integral

$$P = \int_{s_o} E_{z3}\, z ds , \tag{73}$$

which, as we shall see in the next section, is analogue to the magnetic polarizability. More specifically, if we now replace E_{z3} by $-a_z \cdot \nabla_t \tau_3$ so that $zE_{z3} = -\nabla_t \cdot (z\tau_3 \bar{a}_z) + \tau_3$, we have from (73)

$$P = -\oint_{\ell} z\tau_3(\bar{a}_z \cdot \bar{t})\, d\ell + \int_{s_o} \tau_3 ds .$$

The first integral vanishes because $\tau_3 = 0$ on the boundary. Consequently,

$$P = \int_{s_o} \tau_3 ds \tag{74}$$

which then is defined in the same way commonly referred to as the electric polarizability. We should also note here p and $\bar{\bar{M}}$ are identical to the so-called equivalent electric and magnetic dipoles in the application of Bethe's small aperture theory [13-16]. For an elliptical aperture, explicit expression of P and $\bar{\bar{M}}$ are known as

$$\begin{aligned} P &= -\frac{\pi \ell_1^3 (1-e^2)}{3E(e)} ; \\ \bar{\bar{M}} &= \bar{a}_1 \bar{a}_1 \frac{\pi \ell_1^3 e^2}{3[K(e)-E(e)]} + \bar{a}_2 \bar{a}_2 \frac{\pi \ell_1^3 e^2 (1-e)^2}{3[E(e)-K(e)(1-e^2)]} \end{aligned} \tag{75}$$

where $e = (1 - \ell_2^2/\ell_1^2)^{\frac{1}{2}}$ is the excentricity of the aperture; $\bar{a}_1, \bar{a}_2$ are the unit vectors along the major and minor axis; K, E are respectively elliptical integrals of the first and second kind with modules e. [17].

Different from most of the applications regarding the Bethe's Theory however is the fact that both the transmission-line and the antenna parameters are now explicitly contained in the form of $\psi^{(1)}$ and $\psi^{(2)}$ in the determination of the magnitudes of this dipole, and the direct usage of the dipole concept actually is not needed in the subsequent development in the network equivalence.

6. Reflection and transmission coefficient

The network representation of a small aperture can be determined from a knowledge of the reflected and the transmitted field in the transmission line. Using the formulation in (19) and (20) for the Hertzian potentials, one can write down the formal expression for the angular component of the magnetic field in the coaxial region as

$$H_\phi^s(\rho,\phi,z) = \sum_m \int d\alpha \left(\frac{i}{k\eta}\right)\left\{\left(\frac{-m\alpha}{\rho}\right) \frac{\Omega_{hm}(a,\rho;\zeta_r)}{\zeta\Omega'_{hm}(a,b;\zeta_r)} \left[\hat{E}_{\phi m}(b;\alpha) + \frac{m\alpha}{b\zeta_r^2}\hat{E}_{zm}(b;\alpha)\right] + k_1^2 \frac{\Omega'_{em}(a,\rho;\zeta_r)}{\zeta_r\Omega_{em}(a,b;\zeta_r)} \hat{E}_{zm}(b;\alpha)\right\} e^{i(m\phi+\alpha z)} \tag{76}$$

As we mentioned earlier, the integration over α can be expressed alternatively in terms of residue conbtibutions from the zeros of $\Omega_{em}(a,b;\zeta_r)$, $\Omega'_{hm}(a,b;\zeta_r)$, and a pair of poles as $\alpha = \pm k$, corresponding individually to the TM_{mn}, TE_{mn} and the TEM modes. Provided that the coaxial cable propagates only the TEM mode, the residue contribution from all other modes will decay exponentially from the aperture. Thus, the expression for the scattered magnetic field takes a much simpler form of

$$H_\phi^s(\rho,\phi,z) \simeq \pi(\eta_1\rho\ln b/a)^{-1}\hat{E}_{zo}(b;k_1)\exp(ik_1 z) \tag{77}$$

as the observation point $z \to \infty$. By substituting for the definition of the Fourier transform $\hat{E}_{zo}$, we have

$$H_\phi^s = V_o^+(4\pi\rho Z_c)^{-1}\, e^{ik_1 z} \tag{78}$$

where

$$V_o^+ = (2\pi b)^{-1} \int_{S_o} E_z(\bar{x}_t) e^{-ik_1 z}\, ds \tag{79}$$

and $Z_c = (\eta_1/2\pi)\ln b/a$ is the characteristic impedance of the coaxial transmission-line. Thus, the transmission coefficient, defined as $H_\phi = TH_\phi^{inc}$, is now known as

$$T = 1 + V_o^+/(2Z_c) \ . \tag{80}$$

To find an explicit expression for V_o^+, we can replace the exponential exp $(-ik_1 z)$ by the first two terms in its Taylor series to obtain

$$V_o^+ = (2\pi b)^{-1} \left[\int_{S_o} E_z(\bar{x}_t) ds - ik_1 \int_{S_o} zE_z(\bar{x}_t) ds \right] \tag{81}$$

The first term can therefore be identified with the magnetic polarizability while the second term with the electric polarizability .
If use is made of (52), (54) and (73), we can rewrite (81) as

$$\begin{aligned} V_o^+ &= (2\pi b)^{-1} \left[\sum_{i,j} (\bar{a}\phi \cdot \bar{a}_i) M_{ij} A_j - ik_1 P A_3 \right] \\ &= (2\pi b)^{-1} \left[\bar{C} \cdot \bar{\bar{M}} \cdot \bar{A}_t - ikPA_3 \right] \end{aligned} \tag{82}$$

where the expression for $\bar{C}$, $\bar{\bar{M}}$ and $\bar{A}_t$ are given in (69)-(71) for an air-filled line. Here, we note that the terms P and $\bar{\bar{M}}$ are the intrinsic properties of the aperture; $\bar{A}_t$ and A_3 are the excitation factors which involve both the transmission line and antenna parameters; $\bar{C}$ is a geometrical correction indicating the angle between the incident magnetic field and the aperture's axis.

In order to find a suitable expression for the reflection coefficient, we again retain only the TEM contribution in (76) by allowing the observation point to approach infinity in the negative z-direction,

$$H^{s}_{\phi}(\rho,\phi,z) \simeq \{(8\pi^2 Z_c b)^{-1} \int_{S_o} E_z(\bar{x}_t)e^{ik_1 z} ds\} \exp(-ik_1 z)$$

Thus, from the definition $H_\phi = H_\phi^{inc} + \Gamma(2\pi\rho)^{-1} \exp(-ik_1 z)$, we obtain an expression for the reflection coefficient Γ as

$$\Gamma = V_o^-/(2Z_c) \ , \tag{83}$$

where

$$V_o^- = (2\pi b)^{-1} \int_{S_o} E_z(\bar{x}_t)e^{ik_1 z} \tag{84}$$

which according to the discussion leading to (82), is known explicitly as

$$V_o^- = (2\pi b)^{-1} \ [\bar{C}\cdot\bar{\bar{M}}\cdot\bar{A}_t + ik_1 PA_3] \tag{85}$$

and $\bar{C}$, $\bar{\bar{M}}$, $\bar{A}_t$, P and A_3 again are given in the previous section for an air-filled coaxial-line. Equations (82) and (85) then comprise the complete information concerning the transmission and reflection of the current wave in the presence of an aperture of arbitrary shape. If however the aperture possesses the kind of axial symmetry as we mentioned earlier, expression for $V_o^{\pm}$ takes a simpler form according to (72),

$$V_o^{\pm} = -i\eta kb(8\pi^2)^{-1}\{\sum_{j=1,2} C_j^2[\frac{b^3}{M_{jj}} - (\frac{k^2b^2}{8\pi^2})(\psi^{(1)} - C_j\psi^{(2)})]^{-1} \pm \frac{P}{b^3}\}, \tag{86}$$

where $C_j = \bar{a}_\phi\cdot\bar{a}_j$ for $j=1,2$. For the special case where one of the axis, say $\bar{a}_1$ coincides with the direction of the incident magnetic field, i.e. $C_1 = 1$ and $C_2 = 0$, (86) further reduces to

$$V_o^{\pm} = -i\eta kb(8\pi^2)^{-1}\{[\frac{b^3}{M_{11}} - (\frac{k^2b^2}{8\pi^2})(\psi^{(1)} - \psi^{(2)})]^{-1} \pm \frac{P}{b^3}\}. \tag{87}$$

It is obvious that the term $\psi^{(1)} - \psi^{(2)}$ as given by (37) and (39), contains the only information regarding the radiation property of the aperture. Approximate expressions for this term is evaluated in Appendix C.

7. Explicit network representation

We now proceed to derive an explicit network equivalence of a small aperture by first defining a lumped impedance element Z, and a lumped admittance Y as follows:

$$Z^{-1} = \{\frac{i\eta kb}{8\pi^2} \sum_{j=1,2} C_j^2 [\frac{b^3}{M_{jj}} - (\frac{k^2b^2}{8\pi^2})(\psi^{(1)} - C_j\psi^{(2)})]^{-1}\}^{-1} - \frac{1}{2Z_c} \tag{88}$$

$$Y = \frac{i\eta kP}{8\pi^2 b^2 Z_c^2} \tag{89}$$

so that the expression for $V_o^{\pm}$ can be cast into a more compact form of

$$V_o^{\pm} = -(\frac{1}{Z} + \frac{1}{2Z_c})^{-1} \pm YZ_c^2 \tag{90}$$

where Z_c is the characteristic impedance of the coaxial transmission line. Here because $V_o^{\pm}$ is small, the terms $|YZ_c|$ and $|Z/Z_c|$ are usually much smaller than one. The apparent impedance as seen by the coaxial-line to the left of the aperture, is then obtained in terms of the reflection coefficient $\Gamma = V_o^-/2Z_c$ as

$$\begin{aligned} Z_{in} &= Z_o(1-\Gamma)/(1+\Gamma) \\ &= Z_c \frac{1 + (1+2Z_c/Z)^{-1} - YZ_c/2}{1 - (1+2Z_c/Z)^{-1} + YZ_c/2} \end{aligned} \tag{91}$$

A subsequent expansion of (91) with respect to $(YZ_c/2)$ therefore yields

$$Z_{in} = Z_s\{1 - YZ_c \frac{1}{1 - (1+2Z_c/Z)^{-2}}\} + 0(|YZ_c|^2); \tag{92}$$

where

$$Z_s = Z_c \frac{1 + (1+2Z_c/Z)^{-1}}{1 - (1+2Z_c/Z)^{-1}}$$

which can easily be shown as $(Z + Z_c)$. Thus, by retaining only the first-order terms in both Z/Z_c and YZ_c, we obtain the final expression of

$$Z_{in} \simeq Z + Z_c - YZ_c^2$$

or

$$Z_{in} \simeq Z + Z_c/(1 + YZ_c) \tag{93}$$

Equation (93) now gives a surprisingly simple network interpresentation in the form of a series Z and a shunt Y both located at the aperture in the transmission line circiut, as indicated in Fig. 4 When we compare this representation with the one derived from the equivalent-dipoles concept (i.e. Fig. 1), we find that the series inductance ($i\omega L_b$) should now be replaced by Z and the shunt capacitance, $i\omega C_b$ by Y However, we have not assumed a priori an equivalent electric and magnetic dipole to replace the aperture; as a consequence the radiation property of the external cylinder, as fed by the aperture, is now contained in the expression of Z, via the contribution of $\psi^{(1)} - C_j\psi^{(2)}$ in (88).

Let us now turn to the special case where one of the major axes of the aperture is oriented in the ϕ-direction. Using the expression for Y in (90), we have

$$Y = i\omega C_b;\ C_b = \frac{\varepsilon_o \alpha_e \eta^2}{8\pi^2 b^2 Z_c^2} \quad , \tag{94}$$

with the electric polarizability P now denoted as α_e . On the other hand, the use of (88) with $C_1 = 1$ and $C_2 = 0$ yields immediately

$$Z = \frac{1}{(i\omega L_b)^{-1} + Y_\Sigma} \tag{95}$$

where

$$L_b = \frac{\mu_o \alpha_m}{8\pi^2 b^2} \quad ; \tag{96}$$

$$Y_\Sigma = \frac{ikb}{\eta}\,[\psi^{(1)} - \psi^{(2)}] - \frac{1}{2Z_c} \tag{97}$$

and again we have changed the notation for the magnetic polarizability to α_m. We note that the expressions for the capacitance and inductance are identical to those given previously in (1) using the equivalent dipole concept (the factor of 2 arises apparently due to the different definition of the magnetic and electric dipoles; see the foot-note explanation in [7]). However, our expression contains an additional impedance element Y_Σ in parallel to the inductive element. In Appencix C we have shown that Y_Σ as given in (97) is explicitly known to be of the following form

$$Y_\Sigma = Y_{b\infty} + i\omega C_T ; \tag{98}$$

$$C_T = 2\varepsilon_o b\{\ell n[16\pi^3 b^2/d(b-a)] - 2\gamma + 1 + N_c(0;b/a)\} \tag{99}$$

$$Y_{b\infty} = i2kb\eta^{-1}[\ell n(kd/2) + \gamma - 1 + N_o(ka)] , \tag{100}$$

$$N_o(kb) = \int_0^\infty \{H_1^{(1)}(\zeta b)[H_o^{(1)}(\zeta b)]^{-1} + i\}\zeta^{-1} d\alpha - i\pi/2; \tag{101}$$

where $N_c(0;b/a)$ is a small quantity defined in (C.b) which depends only upon the ration b/a, and $\gamma = 0.577216$ is the Euler's constant. As pointed out in a previous work by this author [10], the term $Y_{b\infty}$ corresponds exactly to the input admittance of an infinite antenna of radius b, driven uniformly by a voltage source across a gap of finite width d. (The actual value chosen for d is of no consequence since its effect is cancelled completely by a same term in C_T). Physically, the electro-magnetic field, once escaped from the aperture will excite current on the external surface of the outer coaxial conductor which then acts like a transmitting antenna and radiates into the free-space. This then explains the appearance of $Y_{b\infty}$ in the formula. In addition to $Y_{b\infty}$, we also note from (98) the existence of a capacitive term C_T. Such a term is known to exist in the case of a circumfernetial slot [10] as is evident in Fig. 3 where the terms L_b and C_b associated with the electric and magnetic polarizabilities of a small aperture become insignificant.

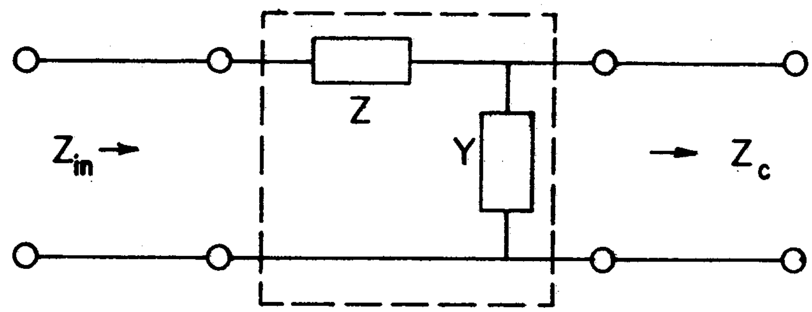

Figure 4(a)

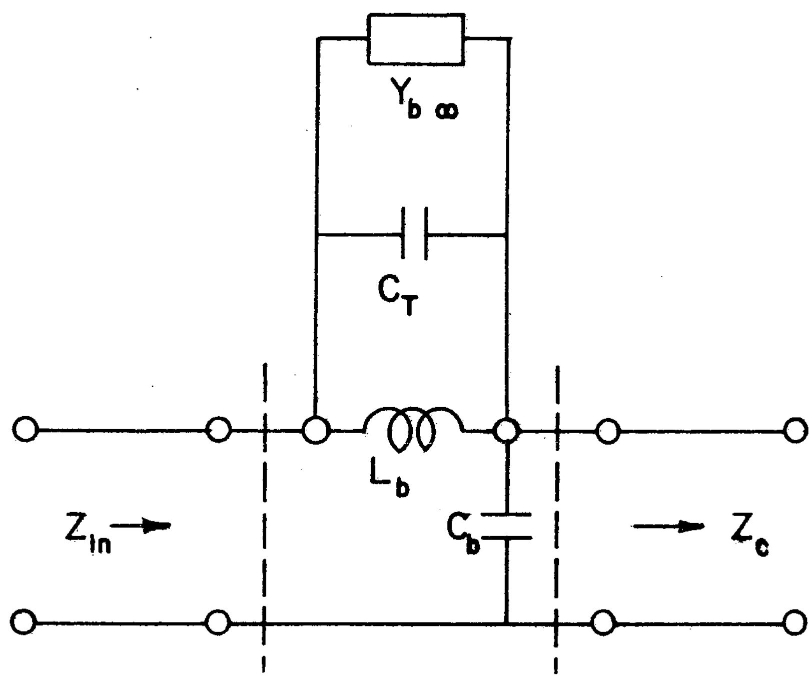

Figure 4(b)

8. Concluding Remarks

Analytical expression for the aperture field is obtained in this paper from an integral equation formulation of the problem and a subsequent use of the so-called quasi-static approach, in which the dominant part of the solution can be extracted from canonical problems, while retaining the basic features of the propagating mode in the coaxial region, and the radiation characteristic of the coaxial cylinder. Equivalent network representation of a small aperture is then derived without requiring the use of equivalent dipoles. The network equivalence of a symmetrical aperture with its axes oriented in the direction of the incident field, is shown in Fig. 4b. It consists of a lumped shunt capacitance C_b, and a series impedance Z composed of an inductive element L_b, a capacitive element C_T and the input admittance $Y_{b\infty}$ of the external antenna, all in parallel. Now since the fringe field exists only in close proximity of the aperture, the equivalent network excluding $Y_{b\infty}$ can be applied to truncated coaxial cylinders much in the same manner as in Fig.3b, so long as the cylinders are long compared with the characteristic length of the aperture. It is equally applicable to multiple or periodic apertures when the near-field reactive coupling of two adjacent apertures can be ignored.

Our analysis in principle can be extended to include the problem of electromagnetic penetration into a coaxial cable, or cylinder. This may be achieved either from the use of reciprocity, or a direct analysis involving an incident palne-wave field. The latter approach, however, can be very involved because the need to find the scattered field both in the presence and in the absence of the aperture. However, judging from the equivalent network obtained for a radiating aperture, and the particular way the antenna input admittance is coupled into the internal transmission-line circuit, it would appear that an equivalent voltage generator (determined by the product of the incident electric field strength and the effective length of the external coaxial cylinder acting as a receiving antenna) in series with $Y_{b\infty}$, or an equivalent current source in parallel with $Y_{b\infty}$, would be required. Investigation of this aspect of the problem will be reported later.

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APPENDIX A

In this appendix, we shall consider the evaluation of the kernel function $\phi(\bar{x}_t-\bar{x}_t')$ defined in (34) and (36) as

$$\phi = \tilde{\phi} + (2\pi^2 b)^{-1} \ln(1-\exp[i2\pi kb]) \tag{A.1}$$

$$\tilde{\phi} = i(2\pi)^{-2} \sum_m \int d\alpha (\zeta^2 b^2 - m^2)^{-\frac{1}{2}} \exp(im\phi + i\alpha z) \tag{A.2}$$

Recalling that $\zeta = (k^2-\alpha^2)^{\frac{1}{2}}$, the integrand is seen to have a pair of branch cuts with branch points located at $\alpha b = \pm\ (k^2b^2-m^2)^{\frac{1}{2}}$ for $k > m$ and $\pm i(m^2-k^2b^2)^{\frac{1}{2}}$ for $m > k$. Rewriting the integral in (A.2) with a parameter β defined as $(k^2-m^2/b^2)^{\frac{1}{2}}$ and $\mathrm{Im}(\beta) \geq 0$, we have

$$b^{-1} \int_{-\infty}^{\infty} d\alpha (\beta^2-\alpha^2)^{-\frac{1}{2}} \exp(i\alpha z)$$

$$= 2b^{-1} \int_0^{\beta} d\alpha (\beta^2-\alpha^2)^{-\frac{1}{2}} \cos \alpha z - i2b^{-1} \int_{\beta}^{\infty} d\alpha (\alpha^2-\beta^2)^{-\frac{1}{2}} \cos \alpha z$$

$$= 2b^{-1} \int_0^{\pi/2} d\theta \cos(\beta z \cos\theta) - i2b^{-1} \int_0^{\infty} d\theta \cos(\beta z \cosh\theta) \tag{A.3}$$

The two integrals in (A.3) are known exactly in the form of Bessel and Neumann functions, respectively [18]. Thus

$$\tilde{\phi} = i(4\pi b)^{-1} \sum_m H_0^{(1)}(\beta z) e^{im\phi} \quad ; \quad \beta = (k^2-m^2/b^2)^{\frac{1}{2}} \tag{A.4}$$

In order to evaluate the series in (A.4), we consider first the following contour integral

$$b \oint_c H_0^{(1)}(z[k^2-\nu^2]^{\frac{1}{2}}) \frac{e^{i\nu\phi b}}{e^{i2\pi\nu b}-1} d\nu$$

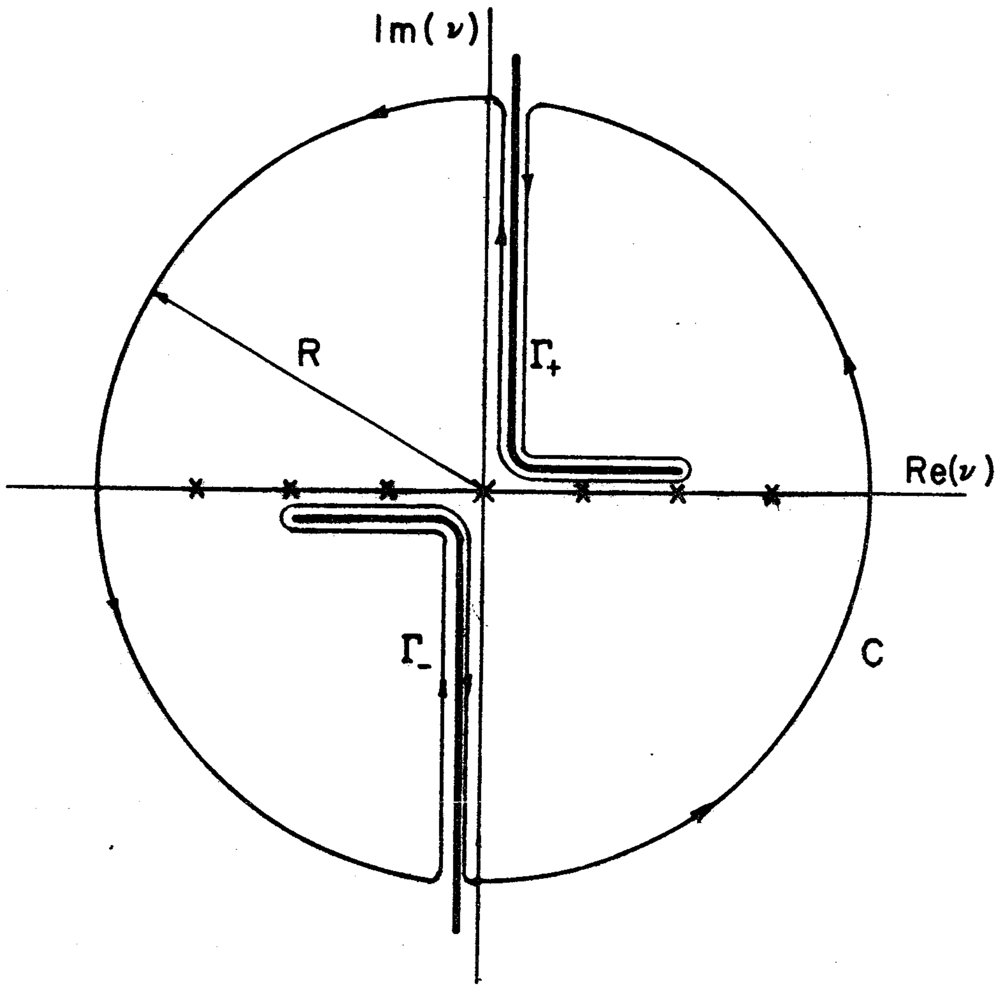

Figure A,I1

with the contour C specified in Fig. (A.1) which includes the integration along a circular path with radius R, and two branch cuts, Γ_+ and Γ_-. As $R \to \infty$, magnitude of the integrand behaves like

$$|\nu|^{-\frac{1}{2}} \exp[-\mathrm{Im}(k^2-\nu^2)^{\frac{1}{2}} z - (\mathrm{Im}\ \nu)\phi b]$$

in the upper half ν-plane where $\mathrm{Im}\ \nu > 0$, and like

$$|\nu|^{-\frac{1}{2}} \exp[-\mathrm{Im}(k^2-\nu^2)^{\frac{1}{2}} z + (\mathrm{Im}\ \nu)(2\pi-\phi)b]$$

in the lower half ν-plane where $\mathrm{Im}\ \nu<0$. Thus, the integration along the circular path vanishes as $R \to \infty$. The application of the residue theorem now yields,

$$\sum_m H_o^{(1)}(z[k^2-m^2/b^2]^{\frac{1}{2}})e^{im\phi} = b\left(\int_{\Gamma_+} + \int_{\Gamma_-}\right)H_o^{(1)}(z[k^2-\nu^2]^{\frac{1}{2}})\ \frac{e^{i\nu\phi b}}{e^{i2\pi\nu b}-1}\, d\nu$$

$$= 2b\int_k^{i\infty} J_o(z[k^2-\nu^2]^{\frac{1}{2}})\left\{\frac{e^{i\nu\phi b}}{e^{i2\pi b\nu}-1} - \frac{e^{-i\nu\phi b}}{e^{-i2\pi b\nu}-1}\right\}d\nu \tag{A.5}$$

In the derivation of (A.5), use is made of the relationship $H_o^{(1)}(z)+H_o^{(2)}(-Z) = 2J_o(Z)$. If now we rearrange the terms in the curly bracket as

$$-[e^{i\nu b\phi} + e^{i\nu b(2\pi-\phi)}] + \left[\frac{e^{i(2\pi+\phi)\nu b}}{e^{i2\pi b\nu}-1} - \frac{e^{i(2\pi-\phi)\nu b}}{e^{-i2\pi b\nu}-1}\right]$$

The integral concerning the terms in the first square bracket is known exactly [18]. On the other hand the terms in the second bracket decay rapidly whenever $2\pi|\nu|b > 1$ on the positive imaginary axis. Now since $b\phi << 1$ and $z << b$ under the small-aperture approximation, we can evaluate the second integral by replacing $J_o(z[k^2-\nu^2]^{\frac{1}{2}})$ and $\exp(i\phi b\nu)$ with their small-argument expansion. Consequently, we have from (A.5)

$$\sum_m H_o^{(1)}(z[k^2-m^2/b^2]^{\frac{1}{2}}) \simeq -i2b\left[\frac{e^{ik|\bar{x}_t|}}{|\bar{x}_t|} + \frac{e^{ikr_b}}{r_b}\right] + 2b\int_k^{i\infty} \frac{e^{i2\pi\nu b}+1}{e^{i2\pi\nu b}-1}\, e^{i2\pi\nu b}\, d\nu, \tag{A.6}$$

where $r_b = [(2\pi-\phi)^2 b^2 + z^2]^{\frac{1}{2}} \simeq 2\pi b$. The remaining integral in (A.5) is now known explicitly as $i(2\pi b)^{-1}\{e^{i2\pi kb} + 2\ln(1-\exp[i2\pi kb])\}$. Thus from (A.6), we have

$$\sum_m H_o^{(1)}(z[k^2-m^2/b^2])^{\frac{1}{2}} = -i2b[\frac{e^{ik|\bar{x}_t|}}{|\bar{x}_t|} - (2\pi b)^{-1} \ln(1-e^{i2\pi kb})] \ . \tag{A.7}$$

Substitution of (A.7) into (A.4) and (A.1) finally yields

$$\phi(\bar{x}_t) = (2\pi|\bar{x}_t|)^{-1} e^{ik|\bar{x}_t|} \tag{A.8}$$

Appendix B

In this Appendix, we shall derive appropriate expressions for $\nabla_t(\nabla_t \cdot F_t)$ and $\nabla_t \times \overline{F}_t$. According to (45), the vector potential function $\overline{F}_t(\overline{x}_t)$ is defined as

$$\overline{F}_t(\overline{x}_t) = \int_{s_o} [\overline{E}_t(\overline{x}_t') \times \overline{n}]\phi(\overline{x}_t - \overline{x}_t')ds' , \quad \text{for } \overline{x}_t \in s_o \tag{B.1}$$

The divergence of $\overline{F}_t$ is then given by

$$\nabla_t \cdot \overline{F}_t(\overline{x}_t) = \int_{s_o} \overline{E}_t(\overline{x}_t') \times \overline{n} \cdot \nabla_t \phi(\overline{x}_t - \overline{x}_t')ds' \tag{B.2}$$

since the differentiation of the source function $\overline{E}_t(\overline{x}_t') \times \overline{n}$, with respect to $\overline{x}_t$ is zero. Now because $\nabla_t \phi = -\nabla_t' \phi$, the integrand in (B.2) can be rewritten as $-\nabla_t' \cdot \{[\overline{E}_t(\overline{x}_t') \times \overline{n}]\phi\} + \phi\nabla_t' \cdot [\overline{E}_t(\overline{x}_t') \times \overline{n}]$ so that

$$\nabla_t \cdot \overline{F}_t(\overline{x}_t) = \oint_{\ell} \phi(\overline{x}_t - x_t')[\overline{E}_t(\overline{x}_t) \cdot \overline{\ell}]d\ell + \int_{s_o} \phi(\overline{x}_t - \overline{x}_t')[\nabla_t' \cdot \overline{E}_t(\overline{x}_t') \times \overline{n}]ds' \tag{B.3}$$

when the divergence theorem is applied, and $\overline{\ell}$ is the unit vector tangent to the live contour ℓ. The first integral is identically zero because of the vanishing tangential electric field at the boundary, i.e., $\overline{E}_t \cdot \overline{\ell} = 0$ on ℓ. The second integral, on the other hand, can be simplified according to $\nabla_t' \cdot [\overline{E}_t(\overline{x}_t') \times \overline{n}] = \overline{n} \cdot \nabla_t' \times \overline{E}_t(\overline{x}_t') = ik\eta\overline{n} \cdot \overline{H}(\overline{x}_t')$ for a small aperture where $\overline{n}$ is a constant vector. It follows from (B.3)

$$\nabla_t(\nabla_t \cdot \overline{F}_t) = ik\eta \int_{s_o} \overline{n} \cdot \overline{H}(\overline{x}_t')\nabla_t \phi(\overline{x}_t - \overline{x}_t') \tag{B.4}$$

which is the same as given in (46). Following a similar reasoning, it is then not difficult to show that

$$\nabla_t \times F_t = \int_{s_o} \nabla_t \phi(\overline{x}_t - \overline{x}_t') \times [\overline{E}_t(\overline{x}_t') \times \overline{n}]ds'$$

$$= -\int_{s_o} [\nabla_t \phi(\overline{x}_t - \overline{x}_t') \cdot \overline{E}_t(\overline{x}_t')]\overline{n}\, ds'$$

or alternatively we have,

$$\overline{n} \cdot \nabla_t \times \overline{F}_t = -\int_{s_o} \overline{E}_t(\overline{x}_t') \cdot \nabla_t \phi(\overline{x}_t - \overline{x}_t') ds' \qquad (B.5)$$

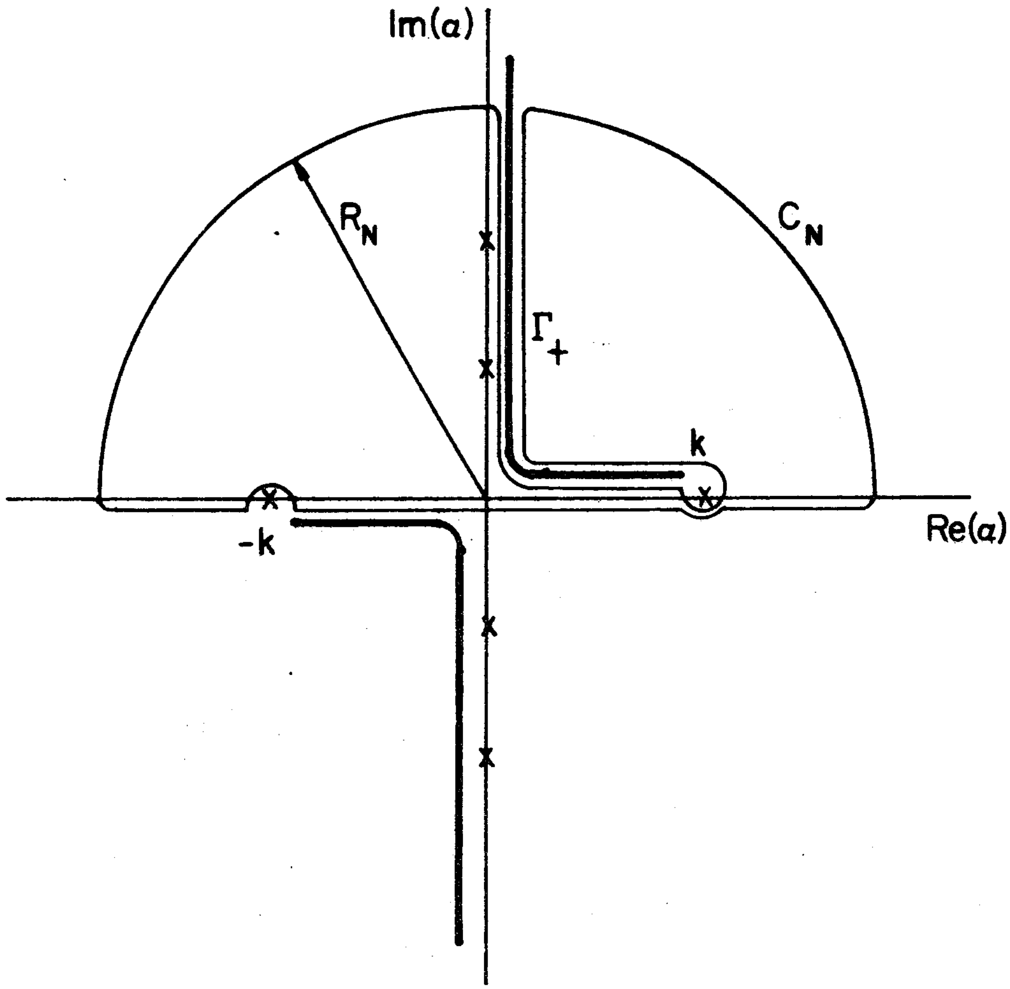

Figure C.1

APPENDIX C

The term $[\psi^{(1)} - \psi^{(2)}]$ defined in (37) and (39) can be rearranged as in the following,

$$[\psi^{(1)} - \psi^{(2)}] = -2\{2\,\ell n\, i\,(1-e^{i2\pi kb}) + N_o(kb) + N_1(kb;b/a) + N_2(kb;b/a)\} \tag{C.1}$$

where

$$N_o(kb) = \int_o^\infty d\alpha\left[-\frac{H_o'^{(1)}(\zeta b)}{\zeta H_o^{(1)}(\zeta b)} + \frac{i}{\zeta}\right] - i\,\pi/2\,, \tag{C.2}$$

$$N_1(kb;b/a) = \tfrac{1}{2}\int d\alpha\left[\frac{\Omega_{eo}'(a,b;\zeta)}{\zeta\Omega_{eo}(a,b;\zeta)} + \frac{i}{\zeta}\right] - i\,\pi/2 \tag{C.3}$$

$$N_2(kb;b/a) = -\frac{1}{2}\sum_{m=1}^{\infty}\int d\alpha\left\{\left[\frac{H_m'^{(1)}(\zeta b)}{\zeta H^{(1)}(\zeta b)} - \frac{\Omega_{em}'(a,b;\zeta)}{\zeta\Omega_{em}(a,b;\zeta)}\right] - \frac{m^2}{b^2\zeta^2}\left[\frac{H_m^{(1)}(\zeta b)}{H_m'^{(1)}(\zeta b)} - \frac{\Omega_{hm}(a,b;\zeta)}{\zeta\Omega_{hm}'(a,b;\zeta)}\right] - \frac{2ib}{(\zeta^2 b^2 - m^2)^{\frac{1}{2}}}\right\} \tag{C.4}$$

We first note that the term $N_o(kb)$ is only a function of the outer radius, b. As shown in [10], this terms is related to the input admittance $Y_{b\infty}$ of an infinite cylindrical antenna of the same radius, driven uniformly by a voltage source across a circumferential gap of width d, explicitly as follows

$$Y_{b\infty} = \frac{i2kb}{\eta}\,[\ell n\; kd/2 + \gamma + N_o(kb) - 1] \tag{C.5}$$

The term $N_1(kb;b/a)$, on the other hand, can be evaluated by deforming the contour of integration onto the upper half of a complex α-plane. Using the deformation detailed in Fig. C.1, it is not difficult to show that the integration on the real axis reduces to an integration over both sides of the upper branch cut, and a residue series due to the pole contribution at $\alpha = k$ and $\alpha = p_{on}$ where $p_{on} = i(\zeta_{on}^2 - k^2)^{\frac{1}{2}}$ and

$$\Omega_{eo}(a,b;\zeta_{on}) = J_o(\zeta_{on}a)H_o^{(1)}(\zeta_{on}b) - J_o(\zeta_{on}b)H_o^{(1)}(\zeta_{on}a) = 0 \ , \tag{C.6}$$

corresponding physically to the excitation of the TEM and the TM_{on}-modes.

$$N_1(kb;b/a) = \lim_{N\to\infty}\left\{\pi i \sum_{n=1}^{N} (\text{residue contribution at } \alpha = p_{on}) - \pi i(2kb\ell n\ b/a)^{-1} - i\pi/2 \right.$$
$$\left. + \ i\int_o^k (k^2-t^2)^{-\frac{1}{2}}dt + \int_k^{R_n}(t^2-k^2)^{-\frac{1}{2}}\right\} \tag{C.7}$$

where the upper limit of the second integral is defined as $R_N = N\pi/(b-a)+\delta$. and $\delta \to 0$. The two integrations in (C.7) are explicitly known as $i\pi/2$ and $\cosh^{-1}(R_N/k)$, respectively. Thus, upon the use of the approximation $\cosh^{-1}(R_N/k) \simeq \ell n\ 2\pi N/k(b-a)$, and the residue calculation, we have from (C.7)

$$N_1(kb;b/a) = -\frac{\pi i}{2kb\ \ell n\, b/a} - \ell n\frac{k(b-a)}{2\pi} + \lim_{N\to\infty}\left\{\ell n\ N - \sum_{n=1}^{N}\left(\frac{\pi i}{p_{on}b}\right)\left[1 - \frac{J_o^2(\zeta_n b)}{J_o^2(\zeta_n a)}\right]^{-1}\right\}, \tag{C.8}$$

In deriving (C.8), use has also been made of (C.6), and the Wronskian of two Bessel functions. In addition, we can replace $\ell n\, N$ by an algebraic series so that

$$N_1(kb;b/a) = \frac{-\pi i}{2kb\ \ell n\, b/a} - \ell n\ k(b-a)/2\pi - \gamma + W_1(kb;b/a), \tag{C.9}$$

and

$$W_1(kb;b/a) = \sum_{n=1}^{\infty}\left\{\frac{1}{n} - \left(\frac{\pi i}{p_{on}b}\right)\left[1 - \frac{J_o^2(\zeta_n b)}{J_o^2(\zeta_n a)}\right]^{-1}\right\} \tag{C.10}$$

In most cases, the radius b is usually small so that $p_{on} \simeq in\pi(b-a)^{-1}$ and $\zeta_n \simeq n\pi(b-a)^{-1}$. Thus, we have

$$W_1(kb;b/a) \simeq W_1(0;b/a) \tag{C.11}$$

We further note that for a sufficiently large n, the large argument expansion of the Bessel functions [18] yields immediately to the approximage expression of

$$\frac{1}{n} - \frac{1}{n}\left(\frac{b-a}{b}\right)\left\{1 - \frac{a}{b}\ \frac{\cos^2[n\pi b/(b-a)]}{\cos^2[n\pi a/(b-a)]}\right\}^{-1} \tag{C.12}$$

for each turn in the series. Now since $\cos[n\pi b/(b-a)] = (-1)^n \cos[n\pi a/(b-a)]$, the two terms in (C.12) start to cancel each other as soon as n becomes large . As shown in [10], this term is usually negligible in practice.

The only remaining integral is then $N_2(kb;b/a)$. For the range of small α where $|\zeta b|^2 << m^2$, a small-argument expansion gives rise to the following result.

$$\frac{H_m'^{(1)}}{\zeta H_m^{(1)}} - \frac{m^2}{b^2\zeta^2}\ \frac{H_m^{(1)}}{\zeta H_m'^{(1)}} \simeq \frac{b}{m}\ ;$$

$$\frac{\Omega'_{em}}{\zeta\Omega_{em}} - \frac{m^2}{b^2\zeta^2}\ \frac{\Omega_{hm}}{\zeta\Omega'_{hm}} \simeq -\frac{b}{m}\ \frac{1 + (a/b)^{2m}}{1 - (a/b)^{2m}}$$

Thus, the integrand becomes increasingly small as m increases, and is independent of k. Provided kb is small, we can therefore approximate ζ by $i\alpha$ to obtain

$$N_2(0;b/a) = \sum_{m=1}^{\infty} \int_0^{\infty} \frac{dx}{x}\left\{\left[\frac{K_m'(x)}{K_m(x)} - \frac{\Omega'_{em}}{\Omega_{em}}\right] - \frac{m^2}{x^2}\left[\frac{K_m(x)}{K_m'(x)} - \frac{\Omega_{hm}}{\Omega'_{hm}}\right] + \frac{2x}{(x^2+m^2)^{\frac{1}{2}}}\right\} \tag{C.13}$$

where $(\Omega'_{em}/\Omega_{em}) = [I_m(ax/b)K_m'(x) - I_m'(x)K_m(xa/b)]/[I_m(xa/b)K_m(x) - I_m(x)K_m(xa/b)]$ (C.14)

$$(\Omega_{hm}/\Omega'_{hm}) = [I_m'(ax/b)K_m(x) - I_m(x)K_m'(ax/b)]/[I_m'(ax/b)K_m'(x) - I_m'(x)K_m'(ax/b)] \tag{C.15}$$

and I_m, K_m are respectively the modified Bessel functions of the first and second kind. Now since both $N_2(o;b/z)$ and $W_1(0,b/a)$ are independent of kb, we can group them together and define a new function $N_c(0;b/a)$

$$N_c(0;b/a) = W_1(0;b/a) + N_2(0;b/a) \tag{C.16}$$

as a real function which varies only with the ratio of outer and inner radii of the coaxial-line. One further recalls from the devinition (C.4) and (C.10) that the value of N_C actually reduces to zero in the limit of a flat ground screen where $b \to \infty$ and $a/b \to 0$. Thus, one might view the function $N_c(0;b/a)$ as basically a correction term representing the change in stored energy as a result of placing the aperture in the outer sheath of a coaxial line instead of a planar screen environment. It is then apparent, at least intuitively, that this correction term is negligible when the size of the aperture is small compared with the overall dimension of the coaxial line.

Upon the substitution of (C.5), (C.9) and (C.16) into (C.7) we now have

$$\psi^{(1)} - \psi^{(2)} = \left(\frac{i\eta}{kb}\right)\left(Y_{b\infty} + \frac{1}{2Z_c} + i\omega C_T\right); \qquad (C.17)$$

where

$$C_T = 2\varepsilon b\{\ln 16\pi^3 b^2/[d(b-a)] - 2\gamma + 1 + N_C(0;b/a)\} \qquad (C.18)$$

and $Z_c = (\eta/2\pi) \ln b/a$ is the characteristic impedance of the coaxial-line. In deriving (C.17), use has also been made of the approximation that $1 - \exp(i2\pi kb) \simeq i2\pi kb$, for a small kb. Thus, from the definition in (97), we finally obtain a suitable expression for Y_Σ as

$$Y_\Sigma = Y_{b\infty} + i\omega C_T \; . \qquad (C.19)$$