Distribution and Properties of the Critical Values of Random Polynomials with Non-Independent and Non-Identically Distributed Roots

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Distribution and Properties of the Critical Values of Random Polynomials with Non-Independent and Non-Identically Distributed Roots

Thesis directed by Prof. Sean O'Rourke

This thesis considers the pairing between the distribution of the roots and the distribution of the critical values of random polynomials. The primary model of random polynomial considered in this thesis is a monic polynomial of degree n with a single complex variable given by

$$p_n(z) = \prod_{i=1}^{\alpha_n} (z - X_i)\left(z - \overline{X_i}\right) \prod_{j=1}^{\beta_n} (z - Y_j)$$

where $2\alpha_n + \beta_n = n$. In $p_n(z)$, both $(X_i)_{i=1}^{\alpha_n}$ and $(Y_j)_{j=1}^{\beta_n}$ are independent sequences of iid, complex valued random variables. In addition, this thesis will describe the relationship between the roots and critical values of similar models of random polynomials, specifically the model where $\beta_n = 0$.

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3 Distribution of Critical Values of a Random Polynomial with Non-Independent Roots 4

0.1 Introduction

This thesis discusses the relationship between the distribution of the roots and the distribution of the critical values of monic random polynomials with a single complex variable. Critical values are defined as being the zeros of the derivative of the polynomial and the roots are the zeros of the polynomial. The particular model of random polynomial considered in this thesis is

$$p_n(z) = \prod_{i=1}^{\alpha_n} (z - X_i) \left(z - \overline{X_i}\right) \prod_{j=1}^{\beta_n} (z - Y_j) \tag{1}$$

where $(X_i)_{i=1}^{\alpha_n}$ and $(Y_j)_{j=1}^{\beta_n}$ are independent sequences of independent and identically distributed (iid), complex valued random variables. A number of previous works have provided a background for the results of this thesis. The relationship between the roots and critical values of a polynomial whose roots are deterministic are explained in [13]. One of the most important results in the deterministic case is the Gauss–Lucas Theorem.

Theorem 1 (Gauss–Lucas Theorem, Theorem 2.1.1 in [20])**.** *For a non-constant polynomial, the critical points lie in the convex hull of the roots.*

Building on this case, Permantle and Rivin [19] showed under several assumptions that when the roots of a random polynomial are iid, then the empirical distribution of the critical values of the random polynomial converge weakly in probability to the distribution of the roots. This work was later refined by Subramanian in [23] and Kabluchko [11].

Theorem 2 (Kabluchko [11])**.** *Let $X_1, X_2, \ldots$ be an infinite sequence of iid, complex valued random variables and define $p_n : \mathbb{C} \to \mathbb{C}$ as a monic degree n polynomial given by $p_n(z) = \prod_{i=1}^{n}(z - X_i)$. Then for any bounded, continuous function $f : \mathbb{C} \to \mathbb{C}$,*

$$\frac{1}{n-1} \sum_{i=1}^{n-1} f(w_i^{(n)}) \to \mathbb{E}\left[f(X_1)\right]$$

in probability as $n \to \infty$ where $w_1^{(n)}, \ldots, w_{n-1}^{(n)}$ are the critical values of $p_n(z)$.

Kabluchko's work proved that when $X_1, \ldots, X_n$ are independent and identically distributed, complex valued random variables then the critical values behave like the roots since by the law of large numbers, $\frac{1}{n}\sum_{i=1}^{n} f(X_i) \to \mathbb{E}[f(X_1)]$ in probability. Building on the work of Kabluchko, O'Rourke [14] produced another version in which the distribution of the critical values converges to the distribution of the roots for random polynomials with dependent roots under several assumptions. O'Rourke and Williams [17] expanded on this work and Kabluchko's result to the case where p_n has $o(n)$ deterministic roots. Kabluchko's result has also been verified when there are both deterministic and random roots by Reddy in [21] and by Byun, Lee, and Reddy in [2]. The pairing of roots and critical values of random polynomials has also been studied on a more localized level by Hanin in [8], [9], and [10], O'Rourke and Williams in [17] and [16], O'Rourke and Wood in [18], Dennis and Hannay in [4], Kabluchko and Seidel in [12], and by Steinerberger in [22]. This thesis builds on results of Kabluchko [11], O'Rourke [14], and O'Rourke and Williams [17] and considers the case where the random polynomial $p_n(z)$ has roots that are not independent and identically distributed. Specifically, this model considers a monic random polynomial (1) where $(X_i)_{i=1}^{\alpha_n}$ and $(Y_j)_{j=1}^{\beta_n}$ are independent sequences of iid, complex valued random variables and $2\alpha_n + \beta_n = n$. As in previous results, this thesis will show that the distribution of the critical values of $p_n(z)$ converges in probability to the distribution of the roots of $p_n(z)$ as $n \to \infty$. This model assumes that the distributions of $(X_i)_{i=1}^{\alpha_n}$ and $(Y_j)_{j=1}^{\beta_n}$ are not identical and considers only the case where the dependence in the roots occurs by taking α_n of the roots of $p_n(z)$ to be the complex conjugates of the random variables in the sequence $(X_i)_{i=1}^{\alpha_n}$. Kabluchko's conclusion is supported by the pairing of the roots and critical values of the random polynomial shown in Fig. 1.

0.2 Main Results

This section introduces the main results of this thesis, specifically, a similar version of Theorem 2 for the model given by (1) as well as an analogous result when $\beta_n = 0$. Before providing the main results of this thesis, we will provide several helpful definitions. First, we say a random variable ξ is *degenerate* if ξ is constant almost surely. Then a random variable ξ is *non-degenerate* if ξ is not degenerate. Moreover, we define *almost every* or *almost all* $z \in \mathbb{C}$ as being all points $z \in \mathbb{C}$ except for a set with Lebesgue measure zero. We will now provide the version of Kabluchko's

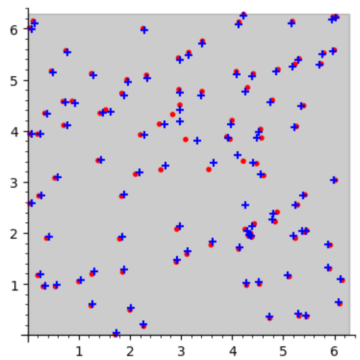

Figure 1: The roots (red dots) and critical values (blue crosses) of a random degree 100 polynomial, where all 100 roots are chosen independently and uniformly on the square $[0, 2\pi] \times [0, 2\pi]$.

result for (1).

Theorem 3. *Let $X_1, Y_1, X_2, Y_2, \ldots$ be an infinite sequence of independent, complex valued random variables with finite second moment such that $X_1, X_2, \ldots$ are identically distributed and $Y_1, Y_2, \ldots$ are identically distributed. Further let α_n, β_n be sequences of non-negative integers such that $2\alpha_n + \beta_n = n$ and $\frac{\alpha_n}{n} \to \alpha \in [0,1]$, $\frac{\beta_n}{n} \to \beta \in [0,1]$ as $n \to \infty$. Assume one of the following:*

(1) $\beta_n \to \infty$ and Y_1 is non-degenerate

(2) $\beta_n \not\to \infty$ and that for almost all $z \in \mathbb{C}$, $\frac{1}{z-X_1} + \frac{1}{z-\overline{X_1}}$ is non-degenerate.

For each $n \geq 1$, let $p_n : \mathbb{C} \to \mathbb{C}$ be a degree n polynomial given by

$p_n(z) = \prod_{i=1}^{\alpha_n} (z - X_i)\left(z - \overline{X_i}\right) \prod_{j=1}^{\beta_n} (z - Y_j)$. Then, for any bounded, continuous function

$f : \mathbb{C} \to \mathbb{C}$,

$$\frac{1}{n-1} \sum_{i=1}^{n-1} f\left(w_i^{(n)}\right) \to \alpha \mathbb{E}\left[f\left(X_1\right)\right] + \alpha \mathbb{E}\left[f\left(\overline{X_1}\right)\right] + \beta \mathbb{E}\left[f\left(Y_1\right)\right] \tag{2}$$

in probability as $n \to \infty$, where $w_1^{(n)}, \ldots, w_{n-1}^{(n)}$ are the critical values of $p_n(z)$.

The results of this theorem are corroborated by the pairing of the roots and critical values shown in Fig. 2 below. Taking $\beta_n = 0$ in (1), the below corollary follows immediately from Theorem 3. This result is supported by the pairing of the roots and critical values in Fig. 3, shown below.

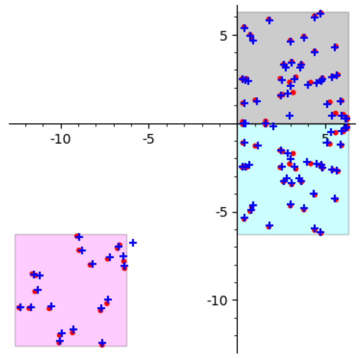

Figure 2: The roots (red dots) and critical values (blue crosses) of a random degree 100 polynomial, where 40 roots are chosen independently and uniformly on the square $[0, 2\pi] \times [0, 2\pi]$, another 40 are their complex conjugates, and the remaining 20 roots are chosen independently and uniformly on the square $[-4\pi, -2\pi] \times [-4\pi, -2\pi]$.

Corollary 4. *Let $X_1, X_2, \dots$ be an infinite sequence of iid, complex valued random variables which have a finite second moment. Assume that for almost every $z \in \mathbb{C}$, $\frac{1}{z-X_1} + \frac{1}{z-\overline{X_1}}$ is non-degenerate and let $\alpha_n = \lceil \frac{n}{2} \rceil$. For each $n \geq 1$, let $p_n : \mathbb{C} \to \mathbb{C}$ be a degree n polynomial given by $p_n(z) = \prod_{i=1}^{\alpha_n} (z - X_i)\left(z - \overline{X_i}\right)$. Then, for any bounded, continuous function $f : \mathbb{C} \to \mathbb{C}$,*

$$\frac{1}{n-1} \sum_{i=1}^{n-1} f\left(w_i^{(n)}\right) \to \frac{1}{2}\mathbb{E}\left[f\left(X_1\right)\right] + \frac{1}{2}\mathbb{E}\left[f\left(\overline{X_1}\right)\right] \tag{3}$$

in probability as $n \to \infty$, where $w_1^{(n)}, \dots, w_{n-1}^{(n)}$ are the critical values of $p_n(z)$.

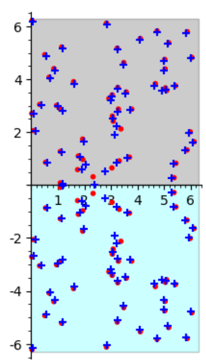

Figure 3: The roots (red dots) and critical values (blue crosses) of a random degree 100 polynomial, where 50 roots are chosen independently and uniformly on the square $[0, 2\pi] \times [0, 2\pi]$ and the remaining 50 roots are their complex conjugates.

The remainder of this thesis will provide the proof for Theorem 3. This proof will be divided into several sections, beginning with a section describing the notation used in the remaining sections of the thesis, a tools section which provides helpful theorems and lemmas which will be used in subsequent sections of the thesis, and several sections which provide smaller proofs that contribute to the proof of Theorem 3.

0.3 Notation

Here, we will define several important concepts which will be referenced in later theorems and proofs. First, the *ones vector*, denoted $\mathbb{1}_n$, is an $n \times 1$ vector given by $\begin{bmatrix} 1, & \ldots, & 1 \end{bmatrix}^T$. Similarly, the $n \times n$ *ones matrix* is denoted J_n and is given by $J_n = \mathbb{1}_n \mathbb{1}_n^T$. The $n \times n$ *identity matrix* is denoted I_n. Furthermore, the *set of* $n \times n$ *matrices with complex entries* is denoted by $M_n(\mathbb{C})$. We also let $||x||$ denote the *Euclidean norm of* x and $S^{n-1} = \{x \in \mathbb{R}^n : ||x|| = 1\}$ be the *unit sphere in* $\mathbb{R}^n$. We define the *ball centered at* $w \in \mathbb{C}$ *and with radius* $r > 0$ by $B(w, r) = \{z \in \mathbb{C} : |z - w| < r\}$ and we define $B(w, r)^c$ to be the *complement of* $B(w, r)$, meaning $B(w, r)^c = \{z \in \mathbb{C} : |z - w| \geq r\}$ be the set of points $z \in \mathbb{C}$ such that $z \notin B(w, r)$. Also, we let d^2z indicate that an *integral is over the complex plane with respect to* z. Furthermore, we define the *empirical spectral measure* of an $n \times n$ matrix A by $\mu_A = \frac{1}{n} \sum_{i=1}^{n} \delta_{\lambda_i}$ where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and δ_z is a point mass at z. In accordance with Definition 1.1.2 of [24], we write $X = o(Y)$ if $|X| \leq c(n)Y$ for some $c(n)$ that goes to zero as $n \to \infty$. Finally, we say that a sequence of random variables X_n is *bounded in probability* if for all $\varepsilon > 0$, there exists $C > 0$ such that $\mathbb{P}(| X_n |> C) < \varepsilon$ for all $n \geq 1$.

0.4 Tools

This section provides several lemmas and theorems that will be used in subsequent sections of this thesis to prove Theorem 3. The first lemma of this section provides the statement of Sylvester's Determinant Identity.

Lemma 5 (Sylvester's Determinant Identity [1])**.** *Let A be an $m \times n$ matrix and let B be an $n \times m$ matrix. Then* $\det(I_m + AB) = \det(I_n + BA)$ *where I_k is the identity matrix of order k.*

The following theorem from [3] describes the relationship between $p_n'(z)$ and $p_n(z)$ and is applicable to the model which will be considered in this thesis, Eq. (1).

Theorem 6. *[3] Let A be an $n \times n$ matrix with characteristic polynomial $p(z) = \prod_{j=1}^{n}(z - z_i)$ and $q(z)$ be a monic polynomial of degree $n-1$ given by $\frac{q(z)}{p(z)} = \sum_{j=1}^{n} \frac{\lambda_j}{z - z_j}$. There exists a rank one matrix H such that $H^2 = H$ and the characteristic polynomial of the matrix $A - AH$ is $zq(z)$. In particular, if A is the diagonal matrix D formed by $z_1, \dots, z_n$, then H can be chosen to be the matrix $\Lambda J_n = \begin{bmatrix} \lambda_1 & \dots & \lambda_1 \\ \vdots & & \vdots \\ \lambda_n & \dots & \lambda_n \end{bmatrix}$, where Λ is the diagonal matrix formed by $\lambda_1, \dots, \lambda_n$ and J_n is the $n \times n$ all one matrix.*

The below theorem from [25] provides conditions under which the empirical spectral measures of the eigenvalues of specific types of random matrices converge in probability and will be used in conjunction with the above theorem to make conclusions about Eq. (1).

Theorem 7 (Theorem 2.1 in [25])**.** *Suppose, for each n, that $A_n, B_n \in M_n(\mathbb{C})$ are ensembles of random matrices. Assume that (i) the expression*

$$\frac{1}{n^2}\,\mathrm{Trace}\,(A_n A_n^*) + \frac{1}{n^2}\,\mathrm{Trace}\,(B_n B_n^*)$$

is bounded in probability (resp., almost surely); (ii) for almost all complex numbers z,

$$\frac{1}{n}\log\left|\det\left(\frac{1}{\sqrt{n}}A_n - zI\right)\right| - \frac{1}{n}\log\left|\det\left(\frac{1}{\sqrt{n}}B_n - zI\right)\right|$$

converges in probability (resp., almost surely) to zero and, in particular, for each fixed z, these determinants are nonzero with probability $1 - o(1)$ for all n (resp., almost surely nonzero for all but finitely many n). Then, $\mu_{\frac{1}{\sqrt{n}}A_n} - \mu_{\frac{1}{\sqrt{n}}B_n}$ converges in probability (resp., almost surely) to zero where $\mu_{\frac{1}{\sqrt{n}}A_n}$ is the normalized empirical spectral measure of the eigenvalues of $\frac{A_n}{\sqrt{n}}$ and $\mu_{\frac{1}{\sqrt{n}}B_n}$ is the normalized empirical spectral measure of the eigenvalues of $\frac{B_n}{\sqrt{n}}$.

The following lemmas, Lemmas 8-12, provide short proofs about functions of non-degenerate random variables and will be referred to throughout the thesis.

Lemma 8. *If ξ is a non-degenerate, complex valued random variable, then* $\operatorname{Re}(\xi)$ *is non-degenerate or* $\operatorname{Im}(\xi)$ *is non-degenerate.*

Proof. Let ξ be a complex valued random variable. Assume that $\operatorname{Re}(\xi)$ and $\operatorname{Im}(\xi)$ are degenerate. That is, assume that $\operatorname{Re}(\xi) = a$ with probability 1 and $\operatorname{Im}(\xi) = b$ with probability 1 for some $a, b \in \mathbb{R}$. Then $\xi = \operatorname{Re}(\xi) + i \operatorname{Im}(\xi)$, so $\xi = a + ib$ with probability 1. Then ξ is degenerate with probability 1. Thus, if ξ is non-degenerate, then $\operatorname{Re}(\xi)$ is non-degenerate or $\operatorname{Im}(\xi_j)$ is non-degenerate. $\square$

Lemma 9. *If ξ is non-degenerate and $z \in \mathbb{C}$, then $z - \xi$ is non-degenerate.*

Proof. Let $z \in \mathbb{C}$ and suppose that $z - \xi$ is degenerate. Then $z - \xi = k$ for some $k \in \mathbb{C}$. This implies that $\xi = z - k$ where $z - k \in \mathbb{C}$. Then ξ is degenerate. Thus, if $z - \xi$ is degenerate for some $z \in \mathbb{C}$, then ξ is degenerate. $\square$

Lemma 10. *If X is a non-degenerate, complex valued random variable, then $\frac{1}{X}$ is non-degenerate provided that $X \neq 0$ with probability* 1*.*

Proof. Suppose that $\frac{1}{X}$ is degenerate and $X \neq 0$ with probability 1. Then $\frac{1}{X} = k$, where $k \in \mathbb{C}, k \neq 0$. Since $X \neq 0$ with probability 1, we have that $Xk = 1$. Dividing both sides by k since $k \neq 0$, we get $X = \frac{1}{k}$. This implies that X is degenerate. Hence, if $\frac{1}{X}$ is degenerate and $X \neq 0$ with probability 1, then X is degenerate. $\square$

Lemma 11. *If X is a non-degenerate complex valued random variable, then $\frac{1}{z-X}$ is non-degenerate for almost every $z \in \mathbb{C}$.*

Proof. Let X be a non-degenerate, complex valued random variable and let $z \in \mathbb{C}$. Then by Lemma 9, $z - X$ is non-degenerate. Notice that the set $\{z \in \mathbb{C} : \mathbb{P}(X = z) \geq \frac{1}{2}\}$ contains at most 2 values of z, that the set $\{z \in \mathbb{C} : \mathbb{P}(X = z) \geq \frac{1}{4}\}$ contains at most 4 values of z, and so forth. Since each of these sets must be finite and countable, then $\mathbb{P}(X = z) = 0$ for almost all $z \in \mathbb{C}$. Then for almost all $z \in \mathbb{C}$, $z - X \neq 0$ almost surely. Let z be one of the almost every $z \in \mathbb{C}$ such that $\mathbb{P}(z - X \neq 0) = 1$. Since $z - X$ is non-degenerate, by Lemma 10, then $\frac{1}{z-X}$ is non-degenerate. $\square$

Lemma 12. *If X is a non-degenerate, complex valued random variable, then $\overline{X}$ is non-degenerate.*

Proof. Suppose $\overline{X}$ is degenerate. Then $\overline{X} = k$ with probability 1 for some $k \in \mathbb{C}$ where k is constant. Observe that $X = \overline{\overline{X}}$. Then taking the complex conjugate we have that $\overline{\overline{X}} = \overline{k}$ with probability 1. This implies that $X = \overline{k}$ with probability 1, so X is degenerate. Hence, if X is non-degenerate, then $\overline{X}$ is non-degenerate. □

The following lemma from [6] will be used in the proof of the subsequent lemma, Lemma 14.

Lemma 13 (Proposition 2.20 in [6])**.** *If $f : \mathbb{C} \to \mathbb{C}$ is measurable and $\int_{B(0,M)} |f(z)|\, d^2z < \infty$ for each $M > 0$, then $|f(z)| < \infty$ for almost all $z \in \mathbb{C}$.*

This lemma and the following lemma, Lemma 15, show that two useful expectations are finite for almost all $z \in \mathbb{C}$.

Lemma 14. *If X_1 is a complex valued random variable, then $\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]$ is finite for almost every $z \in \mathbb{C}$.*

Proof. Let X_1 be a complex valued random variable. In order to show that $\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]$ is finite for almost every $z \in \mathbb{C}$, we will use Lemma 13. To do so, we will let $f(z) = \mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]$, $M > 10$, and consider $\int_{B(0,M)} |f(z)|\, d^2z$. Then

$$\begin{aligned}\int_{B(0,M)} |f(z)|\, d^2z &= \int_{B(0,M)} \left|\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]\right| d^2z \\ &= \int_{B(0,M)} \mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right] d^2z.\end{aligned}$$

By the Fubini-Tonelli theorem and since $\left|\frac{1}{z-X_1}\right| \geq 0$, then

$\int_{B(0,M)} |f(z)|\, d^2z = \mathbb{E}\left[\int_{B(0,M)} \left|\frac{1}{z-X_1}\right| d^2z\right]$. Then

$$\begin{aligned}\int_{B(0,M)} \left|\frac{1}{z-X_1}\right| d^2z &= \int_{B(0,M)\cap B(X_1,1)} \frac{1}{|z-X_1|} d^2z + \int_{B(0,M)\cap B(X_1,1)^c} \frac{1}{|z-X_1|} d^2z \\ &\leq \int_{B(X_1,1)} \frac{1}{|z-X_1|} d^2z + \int_{B(0,M)\cap B(X_1,1)^c} \frac{1}{|z-X_1|} d^2z \\ &\leq \int_{B(X_1,1)} \frac{1}{|z-X_1|} d^2z + \int_{B(0,M)} 1 d^2z \\ &\leq \int_{B(X_1,1)} \frac{1}{|z-X_1|} d^2z + \pi M^2.\end{aligned}$$

We will now consider $\int_{B(X_1,1)} \frac{1}{|z-X_1|} d^2z$. We will use a change of variables and let $w = z - X_1$. Then the region of integration becomes $B(0,1)$. Then $\int_{B(X_1,1)} \frac{1}{|z-X_1|} d^2z = \int_{B(0,1)} \frac{1}{|w|} d^2w$. Now, we will switch to polar coordinates to integrate. Observe that since $w \in \mathbb{C}$, then $|w|$ is the distance from the origin to w, which is r in polar coordinates. Then

$$\int_{B(0,1)} \frac{1}{|w|} d^2w = \int_0^{2\pi} \int_0^1 \frac{r}{r} dr d\theta = \int_0^{2\pi} \int_0^1 1 dr d\theta = \int_0^{2\pi} 1 d\theta = 2\pi.$$

Hence, $\int_{B(X_1,1)} \frac{1}{|z-X_1|} d^2z + \int_{B(0,M)\cap B(X_1,1)^c} \frac{1}{|z-X_1|} d^2z \leq 2\pi + \pi M^2$ and since $2\pi + \pi M^2$ is a constant, $\mathbb{E}\left[\int_{B(0,M)} \left|\frac{1}{z-X_1}\right| d^2z\right] \leq 2\pi + \pi M^2$. This implies that $\int_{B(0,M)} \mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right] d^2z \leq 2\pi + \pi M^2$. Then $\int_{B(0,M)} \mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right] < \infty$. Hence, by Lemma 13, $\left|\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]\right| < \infty$ for almost every $z \in \mathbb{C}$.

□

Lemma 15. *If X is a complex valued random variable, then $\mathbb{E}\left[\left|\frac{1}{z-X} + \frac{1}{z-\overline{X}}\right|\right]$ is finite for almost every $z \in \mathbb{C}$.*

Proof. Let X be a complex valued random variable. Notice that $\overline{X}$ is also a complex valued random variable. Then by Lemma 14, $\mathbb{E}\left[\left|\frac{1}{z-X}\right|\right]$ is finite for almost every $z \in \mathbb{C}$. Similarly, by Lemma 14, $\mathbb{E}\left[\left|\frac{1}{z-\overline{X}}\right|\right]$ is finite for almost every $z \in \mathbb{C}$. Let $z \in \mathbb{C}$ be one of the almost every $z \in \mathbb{C}$ such that $\mathbb{E}\left[\left|\frac{1}{z-X}\right|\right] < \infty$ and $\mathbb{E}\left[\left|\frac{1}{z-\overline{X}}\right|\right] < \infty$. We will now consider $\mathbb{E}\left[\left|\frac{1}{z-X} + \frac{1}{z-\overline{X}}\right|\right]$. Observe that by the triangle inequality, $\left|\frac{1}{z-X} + \frac{1}{z-\overline{X}}\right| \leq \left|\frac{1}{z-X}\right| + \left|\frac{1}{z-\overline{X}}\right|$. Then $\mathbb{E}\left[\left|\frac{1}{z-X} + \frac{1}{z-\overline{X}}\right|\right] \leq \mathbb{E}\left[\left|\frac{1}{z-X}\right| + \left|\frac{1}{z-\overline{X}}\right|\right]$. By the linearity of expectation, $\mathbb{E}\left[\left|\frac{1}{z-X}\right| + \left|\frac{1}{z-\overline{X}}\right|\right] = \mathbb{E}\left[\left|\frac{1}{z-X}\right|\right] + \mathbb{E}\left[\left|\frac{1}{z-\overline{X}}\right|\right]$. Since $\mathbb{E}\left[\left|\frac{1}{z-X}\right|\right] < \infty$ and $\mathbb{E}\left[\left|\frac{1}{z-\overline{X}}\right|\right] < \infty$, then $\mathbb{E}\left[\left|\frac{1}{z-X}\right|\right] + \mathbb{E}\left[\left|\frac{1}{z-\overline{X}}\right|\right]$ is finite. Hence, $\mathbb{E}\left[\left|\frac{1}{z-X} + \frac{1}{z-\overline{X}}\right|\right]$ is finite. □

Lemma 16 (see Theorem 11.1 in [7] and Exercise 3.2.13 in [5]). *Let ξ_n and ψ_n be sequences of random variables. If $\xi_n \to a$ in probability and $\psi_n \to b$ in probability where $a, b \in \mathbb{C}$ are constants, then $\xi_n + \psi_n \to a + b$ in probability.*

Lemma 17 (see Theorem 11.4 in [7] and Exercise 3.2.14 in [5]). *Let ξ_n be a sequence of random variables and b_n be a sequence of complex numbers. If $\xi_n \to a$ in probability and $b_n \to b$ where $a, b \in \mathbb{C}$ are constants, then $b_n \xi_n \to ba$ in probability.*

We will use the following version of the law of large numbers.

Lemma 18. *Let $X_1, X_2, \ldots$ be an infinite sequence of iid, complex valued random variables and let α_n be a sequence of non-negative integers such that $\frac{\alpha_n}{n} \to \alpha$, let $f : \mathbb{C} \to \mathbb{C}$ be continuous, and $\mathbb{E}\left[|f(X_1)|\right] < \infty$. Then $\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i) \to \alpha \mathbb{E}\left[f(X_1)\right]$ in probability.*

Proof. Let $X_1, X_2, \ldots$ be an infinite sequence of iid, complex valued random variables and let α_n be a sequence of non-negative integers such that $\frac{\alpha_n}{n} \to \alpha$. Further let $f : \mathbb{C} \to \mathbb{C}$ be continuous and $\mathbb{E}\left[|f(X_1)|\right]$ be finite. We will now consider two cases for the value of α.

Case 1. Suppose that $\alpha = 0$. Since $\frac{\alpha_n}{n} \to \alpha$, then $\frac{\alpha_n}{n} \to 0$. Let $\varepsilon > 0$ and consider $\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i)\right| \geq \varepsilon\right)$. Since $\left|\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i)\right|$ is non-negative, then by Markov's Inequality,

$$\mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i)\right| \geq \varepsilon\right) \leq \frac{1}{\varepsilon}\mathbb{E}\left[\left|\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i)\right|\right].$$

Then by the properties of absolute value,

$$\frac{1}{\varepsilon}\mathbb{E}\left[\left|\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i)\right|\right] = \frac{1}{\varepsilon}\mathbb{E}\left[\left|\frac{1}{n}\right|\left|\sum_{i=1}^{\alpha_n} f(X_i)\right|\right] \leq \frac{1}{\varepsilon}\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{\alpha_n} |f(X_i)|\right].$$

By the linearity of expectation,

$$\frac{1}{\varepsilon}\mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{\alpha_n} |f(X_i)|\right] = \frac{1}{\varepsilon n}\sum_{i=1}^{\alpha_n} \mathbb{E}\left[|f(X_i)|\right].$$

Since $X_1, \ldots, X_{\alpha_n}$ are identically distributed,

$$\frac{1}{\varepsilon n}\sum_{i=1}^{\alpha_n} \mathbb{E}\left[|f(X_i)|\right] \leq \frac{\alpha_n}{\varepsilon n}\mathbb{E}\left[|f(X_1)|\right].$$

Since $\frac{\alpha_n}{n} \to 0$, this implies that

$$\frac{\alpha_n}{\varepsilon n}\mathbb{E}\left[|f(X_1)|\right] \to 0.$$

Hence,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i)\right| \geq \varepsilon\right) = 0.$$

Thus,

$$\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i) \to 0$$

in probability.

Case 2. Suppose that $\alpha > 0$. Since $\frac{\alpha_n}{n} \to \alpha$, $\alpha_n \to \infty$. We will now consider $\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i)$. Multiplying by $\frac{\alpha_n}{\alpha_n}$, we get $\frac{\alpha_n}{n} \cdot \frac{1}{\alpha_n}\sum_{i=1}^{\alpha_n} f(X_i)$. Since $X_1, \ldots, X_{\alpha_n}$ are iid, then by the law of large numbers (see Theorem 2.4.1 in [5]), $\frac{1}{\alpha_n}\sum_{i=1}^{\alpha_n} f(X_i) \to \mathbb{E}\left[f(X_1)\right]$ in probability. Since $\frac{\alpha_n}{n} \to \alpha$ and $\frac{1}{\alpha_n}\sum_{i=1}^{\alpha_n} f(X_i) \to \mathbb{E}\left[f(X_1)\right]$ in probability, then by Lemma 17,

$$\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i) = \frac{\alpha_n}{n} \cdot \frac{1}{\alpha_n}\sum_{i=1}^{\alpha_n} f(X_i) \to \alpha\mathbb{E}\left[f(X_1)\right]$$

in probability. □

0.5 Proofs

This section provides the proof of Theorem 3. It is divided into several subsections that provide helper lemmas with accompanying proofs that will be used in the final proof of Theorem 3.

0.5.1 Lévy Concentration Lemma

This subsection provides necessary definitions, lemmas, and theorems involving the Lévy Concentration Lemma from [15] that will be used in the following subsection. The final two lemmas in this subsection focus on the properties of the real and imaginary components of random variables.

Proposition 19 (See Assumption 2.3 and Proposition 2.4 in [15])**.** *If ξ is a non-degenerate random variable, then there exist constants $\varepsilon_0, p_0, K_0 > 0$ such that ξ satisfies*

$$\mathbb{P}\left(\left|\xi - \xi'\right| \leq \varepsilon_0\right) \leq 1 - p_0, \quad \mathbb{P}\left(|\xi| \geq K_0\right) \leq \frac{p_0}{4} \tag{4}$$

where ξ' is an independent copy of ξ.

Definition 20 (Small ball probabilities, Definition 6.1 in [15])**.** Let Z be a random vector in $\mathbb{C}^n$. The *Lévy concentration function* of Z is defined as

$$\mathscr{L}(Z,t) := \sup_{u\in\mathbb{C}^n} \mathbb{P}\left(||Z-u|| \leq t\right)$$

for all $t \geq 0$.

Definition 21 (LCD, Definition 6.4 in [15])**.** Let $L \geq 1$. We define the *least common denominator* (LCD) of $x \in S^{n-1}$ as

$$D_L(x) := \inf\left\{\theta > 0 : \mathrm{dist}(\theta x, \mathbb{Z}^n) < L\sqrt{\log_+(\theta/L)}\right\},$$

where $\mathrm{dist}(v,T) := \inf_{u\in T}||v-u||$ is the distance from a vector $v \in \mathbb{R}^n$ to a set $T \subseteq \mathbb{R}^n$.

Lemma 22 (Simple lower bound for LCD, Lemma 6.5 in [15])**.** *For every $x \in S^{n-1}$ and every $L \geq 1$, one has*

$$D_L(x) \geq \frac{1}{2\,||x||_\infty},$$

where $||x||_\infty$ is the ℓ^∞-norm of the vector x.

Theorem 23 (Corollary 6.8 in [15])**.** *Let $\xi_1,\dots,\xi_n$ be non-degenerate, iid copies of a real random variable ξ. By Proposition 19, there exist constants $\varepsilon_0, p_0, K_0 > 0$ such that $\xi_1,\dots,\xi_n$ satisfy (4). Then there exists $C > 0$ (depending only on ε_0, p_0, and K_0) such that the following holds. Let $x = (x_1,\dots,x_n) \in S^{n-1}$ and consider the sum $S := \sum_{k=1}^n x_k\xi_k$. Then, for every $L \geq p_0^{-1/2}$ and $t \geq 0$, one has*

$$\mathscr{L}(S,t) \leq CL\left(t + \frac{1}{D_L(x)}\right).$$

Lemma 24. *If $\xi_1,\dots,\xi_n$ are complex valued random variables, then*

$$\mathscr{L}\left(\sum_{j=1}^n \xi_j, t\right) \leq \min\left\{\mathscr{L}\left(\sum_{j=1}^n \mathrm{Re}(\xi_j), t\right), \mathscr{L}\left(\sum_{j=1}^n \mathrm{Im}(\xi_j), t\right)\right\} \tag{5}$$

for all $t \geq 0$.

Proof. Let ξ_j be a complex valued random variable for $j = 1, 2, \ldots, n$ and let $t \geq 0$. We will now consider $\mathscr{L}\left(\sum_{j=1}^{n} \xi_j, t\right)$. Notice that by Definition 20, $\mathscr{L}\left(\sum_{j=1}^{n} \xi_j, t\right) = \sup_{u\in\mathbb{C}} \mathbb{P}\left(\left|\sum_{j=1}^{n} \xi_j - u\right| \leq t\right)$. Let $u \in \mathbb{C}$ and observe that $|\xi_j - u| \geq |\mathrm{Re}\,(\xi_j) - \mathrm{Re}\,(u)|$ and $|\xi_j - u| \geq |\mathrm{Im}\,(\xi_j) - \mathrm{Im}\,(u)|$. Then $\left|\sum_{j=1}^{n} \xi_j - u\right| \geq \left|\sum_{j=1}^{n} \mathrm{Re}\,(\xi_j) - \mathrm{Re}\,(u)\right|$ and $\left|\sum_{j=1}^{n} \xi_j - u\right| \geq \left|\sum_{j=1}^{n} \mathrm{Im}\,(\xi_j) - \mathrm{Im}\,(u)\right|$. This implies that $\mathbb{P}\left(\left|\sum_{j=1}^{n} \xi_j - u\right| \leq t\right) \leq \mathbb{P}\left(\left|\sum_{j=1}^{n} \mathrm{Re}\,(\xi_j) - \mathrm{Re}\,(u)\right| \leq t\right)$ and that $\mathbb{P}\left(\left|\sum_{j=1}^{n} \xi_j - u\right| \leq t\right) \leq \mathbb{P}\left(\left|\sum_{j=1}^{n} \mathrm{Im}\,(\xi_j) - \mathrm{Im}\,(u)\right| \leq t\right)$. Then $\sup_{u\in\mathbb{C}} \mathbb{P}\left(\left|\sum_{j=1}^{n} \xi_j - u\right| \leq t\right) \leq \sup_{u\in\mathbb{C}} \mathbb{P}\left(\left|\sum_{j=1}^{n} \mathrm{Re}\,(\xi_j) - \mathrm{Re}(u)\right| \leq t\right)$. Observe that the supremum is over all $u \in \mathbb{C}$ but since we are considering $\mathrm{Re}(u)$ in the probability, this is equivalent to considering the supremum over $u \in \mathbb{R}$ with u in the probability. Since the supremum will occur when $u \in \mathbb{R}$ and $\mathbb{R} \subset \mathbb{C}$, we can consider the supremum over $u \in \mathbb{C}$ with u in the probability because the supremum will occur when $u \in \mathbb{R}$, meaning that the imaginary portion of u is equal to zero. This implies that $\sup_{u\in\mathbb{C}} \mathbb{P}\left(\left|\sum_{j=1}^{n} \mathrm{Re}\,(\xi_j) - \mathrm{Re}(u)\right| \leq t\right) = \sup_{u\in\mathbb{C}} \mathbb{P}\left(\left|\sum_{j=1}^{n} \mathrm{Re}\,(\xi_j) - u\right| \leq t\right)$. Hence, $\sup_{u\in\mathbb{C}} \mathbb{P}\left(\left|\sum_{j=1}^{n} \xi_j - u\right| \leq t\right) \leq \sup_{u\in\mathbb{C}} \mathbb{P}\left(\left|\sum_{j=1}^{n} \mathrm{Re}\,(\xi_j) - u\right| \leq t\right)$. Using a similar argument, we see that $\sup_{u\in\mathbb{C}} \mathbb{P}\left(\left|\sum_{j=1}^{n} \xi_j - u\right| \leq t\right) \leq \sup_{u\in\mathbb{C}} \mathbb{P}\left(\left|\sum_{j=1}^{n} \mathrm{Im}\,(\xi_j) - u\right| \leq t\right)$. This proves (5). □

0.5.2 Lower Bound

This subsection provides a lower bound on the ratio of $p_n'(z)$ to $p_n(z)$.

Lemma 25. *Assume the same set-up as in Theorem 3. Then for almost every $z \in \mathbb{C}$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z - X_i} + \frac{1}{z - \overline{X_i}}\right) + \sum_{j=1}^{\beta_n} \frac{1}{z - Y_j}\right| \leq 1\right) = 0.$$

Proof. Assume the same set-up as in Theorem 3. We will now consider each assumption separately. *Case 1.* Assume that $\beta_n \to \infty$ and Y_1 is non-degenerate. Then by Lemma 14, $\mathbb{E}\left[\left|\frac{1}{z-Y_1}\right|\right] < \infty$ for almost every $z \in \mathbb{C}$. Let $z \in \mathbb{C}$ be one of the almost every $z \in \mathbb{C}$ such that $\mathbb{E}\left[\left|\frac{1}{z-Y_1}\right|\right] < \infty$ and such that $z - Y_j \neq 0$ with probability 1 for $j = 1, \ldots, \beta_n$, $z - X_i \neq 0$ with probability 1 for $i = 1, \ldots, \alpha_n$, and $z - \overline{X_i} \neq 0$ for $i = 1, \ldots, \alpha_n$. Since $\mathbb{E}\left[\left|\frac{1}{z-Y_1}\right|\right] < \infty$ and

$z - Y_j \neq 0$ with probability 1 for $j = 1, \ldots, \beta_n$, then $\frac{1}{z-Y_1}, \ldots, \frac{1}{z-Y_{\beta_n}}$ are finite with probability 1. Then by Lemma 11, $\frac{1}{z-Y_1}, \ldots, \frac{1}{z-Y_{\beta_n}}$ are non-degenerate. Since $\frac{1}{z-Y_1}, \ldots, \frac{1}{z-Y_{\beta_n}}$ are non-degenerate, then by Lemma 8, $\operatorname{Re}\left(\frac{1}{z-Y_j}\right)$ is non-degenerate or $\operatorname{Im}\left(\frac{1}{z-Y_j}\right)$ is non-degenerate for $j = 1, \ldots, \beta_n$. Without loss of generality, assume that $\operatorname{Re}\left(\frac{1}{z-Y_1}\right), \ldots, \operatorname{Re}\left(\frac{1}{z-Y_{\beta_n}}\right)$ are non-degenerate. Also, since $Y_1, \ldots, Y_{\beta_n}$ are iid, $\operatorname{Re}\left(\frac{1}{z-Y_1}\right), \ldots, \operatorname{Re}\left(\frac{1}{z-Y_{\beta_n}}\right)$ are iid. We will now consider $\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i} + \frac{1}{z-\overline{X_i}}\right) + \sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right| \leq 1\right)$. We can now condition on $\sum_{i=1}^{\alpha_n}\frac{1}{z-X_i} + \frac{1}{z-\overline{X_i}}$ since it is finite and absorb its contribution into $u \in \mathbb{C}$ in the Lévy Concentration function defined in Definition 20. We now want to bound $\mathscr{L}\left(\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}, 1\right)$. Then by Lemma 24,

$$\mathscr{L}\left(\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}, 1\right) \leq \min\left\{\mathscr{L}\left(\sum_{j=1}^{\beta_n}\operatorname{Re}\left(\frac{1}{z-Y_j}\right), 1\right), \mathscr{L}\left(\sum_{j=1}^{\beta_n}\operatorname{Im}\left(\frac{1}{z-Y_j}\right), 1\right)\right\}.$$

Since $\operatorname{Re}\left(\frac{1}{z-Y_1}\right)$ is non-degenerate, then

$$\mathscr{L}\left(\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}, 1\right) \leq \mathscr{L}\left(\sum_{j=1}^{\beta_n}\operatorname{Re}\left(\frac{1}{z-Y_j}\right), 1\right).$$

Rescaling by $\frac{1}{\sqrt{\beta_n}}$, we have that

$$\mathscr{L}\left(\sum_{j=1}^{\beta_n}\operatorname{Re}\left(\frac{1}{z-Y_j}\right), 1\right) = \mathscr{L}\left(\sum_{j=1}^{\beta_n}\frac{1}{\sqrt{\beta_n}}\operatorname{Re}\left(\frac{1}{z-Y_j}\right), \frac{1}{\sqrt{\beta_n}}\right).$$

Then by Theorem 23, there exists a constant $C_1 \geq 0$ which depends on the distribution of $\operatorname{Re}\left(\frac{1}{z-Y_1}\right)$ such that

$$\mathscr{L}\left(\sum_{j=1}^{\beta_n}\frac{1}{\sqrt{\beta_n}}\operatorname{Re}\left(\frac{1}{z-Y_j}\right), \frac{1}{\sqrt{\beta_n}}\right) \leq C_1\left(\frac{1}{\sqrt{\beta_n}} + \frac{1}{D_L(x)}\right)$$

where $x = \left(\frac{1}{\sqrt{\beta_n}}, \ldots, \frac{1}{\sqrt{\beta_n}}\right)$. Then by Lemma 22, $D_L(x) \geq \frac{\sqrt{\beta_n}}{2}$. Hence,

$$\mathscr{L}\left(\sum_{j=1}^{\beta_n}\frac{1}{\sqrt{\beta_n}}\operatorname{Re}\left(\frac{1}{z-Y_j}\right), \frac{1}{\sqrt{\beta_n}}\right) \leq C_1\left(\frac{1}{\sqrt{\beta_n}} + \frac{2}{\sqrt{\beta_n}}\right) = \frac{3C_1}{\sqrt{\beta_n}}.$$

Letting $C = 3C_1$, we have that

$$\mathscr{L}\left(\sum_{j=1}^{\beta_n}\frac{1}{\sqrt{\beta_n}}\operatorname{Re}\left(\frac{1}{z-Y_j}\right), \frac{1}{\sqrt{\beta_n}}\right) \leq \frac{C}{\sqrt{\beta_n}}.$$

This implies that

$$\mathscr{L}\left(\sum_{j=1}^{\beta_n} \frac{1}{z-Y_j}, 1\right) \leq \frac{C}{\sqrt{\beta_n}}.$$

Since this bound applies to the supremum over all $u \in \mathbb{C}$, this bound applies to the u which absorbed the contribution of $\sum_{i=1}^{\alpha_n} \frac{1}{z-X_i} + \frac{1}{z-\overline{X_i}}$. This implies that

$$\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i} + \frac{1}{z-\overline{X_i}}\right) + \sum_{j=1}^{\beta_n} \frac{1}{z-Y_j}\right| \leq 1\right) \leq \frac{C}{\sqrt{\beta_n}}.$$

Taking the limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i} + \frac{1}{z-\overline{X_i}}\right) + \sum_{j=1}^{\beta_n} \frac{1}{z-Y_j}\right| \leq 1\right) \leq \lim_{n\to\infty} \frac{C}{\sqrt{\beta_n}}.$$

Observe that since $\beta_n \to \infty$ and C is a constant, then $\lim_{n\to\infty} \frac{C}{\sqrt{\beta_n}} = 0$. Thus,

$$\lim_{n\to\infty} \mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i} + \frac{1}{z-\overline{X_i}}\right) + \sum_{j=1}^{\beta_n} \frac{1}{z-Y_j}\right| \leq 1\right) = 0.$$

Case 2. Assume that $\beta_n \not\to \infty$ and for almost every $z \in \mathbb{C}$, $\frac{1}{z-X_1} + \frac{1}{z-\overline{X_1}}$ is non-degenerate. Let $z \in \mathbb{C}$ be one of the almost every $z \in \mathbb{C}$ such that the above holds and such that $z - X_i \neq 0$, $z - \overline{X_i} \neq 0$ with probability 1 for $i = 1, \dots, \alpha_n$, and $z - Y_j \neq 0$ with probability 1 for $j = 1, \dots, \beta_n$. Then $\frac{1}{z-X_1} + \frac{1}{z-\overline{X_1}}$ is finite with probability 1 and $\frac{1}{z-Y_1}$ is finite with probability 1. Then by Lemma 8, $\mathrm{Re}\left(\frac{1}{z-X_1} + \frac{1}{z-\overline{X_1}}\right)$ is non-degenerate or $\mathrm{Im}\left(\frac{1}{z-X_1} + \frac{1}{z-\overline{X_1}}\right)$ is non-degenerate. Without loss of generality, suppose that $\mathrm{Re}\left(\frac{1}{z-X_1} + \frac{1}{z-\overline{X_1}}\right)$ is non-degenerate. Observe that since $X_1, X_2, \dots$ are iid, then $\overline{X_1}, \overline{X_2}, \dots$ are also iid which implies that $\mathrm{Re}\left(\frac{1}{z-X_1} + \frac{1}{z-\overline{X_1}}\right), \mathrm{Re}\left(\frac{1}{z-X_2} + \frac{1}{z-\overline{X_2}}\right), \dots$ are iid. Since $\beta_n \not\to \infty$, then $\alpha_n \to \infty$. Consider $\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i} + \frac{1}{z-\overline{X_i}}\right) + \sum_{j=1}^{\beta_n} \frac{1}{z-Y_j}\right| \leq 1\right)$. We will now condition on $\sum_{j=1}^{\beta_n} \frac{1}{z-Y_j}$ since it is finite and absorb its contribution into the vector $u \in \mathbb{C}$ in the Lévy Concentration function defined in Definition 20. We now want to bound $\mathscr{L}\left(\sum_{i=1}^{\alpha_n} \frac{1}{z-X_i} + \frac{1}{z-\overline{X_i}}, 1\right)$. Then by Lemma 24,

$$\mathscr{L}\left(\sum_{i=1}^{\alpha_n} \frac{1}{z-X_i} + \frac{1}{z-\overline{X_i}}, 1\right)$$
$$\leq \min\left\{\mathscr{L}\left(\sum_{i=1}^{\alpha_n} \mathrm{Re}\left(\frac{1}{z-X_i} + \frac{1}{z-\overline{X_i}}\right), 1\right), \mathscr{L}\left(\sum_{i=1}^{\alpha_n} \mathrm{Im}\left(\frac{1}{z-X_i} + \frac{1}{z-\overline{X_i}}\right), 1\right)\right\}. \quad (6)$$

Since $\operatorname{Re}\left(\frac{1}{z-X_1}+\frac{1}{z-\overline{X_1}}\right)$ is non-degenerate, then

$$\mathscr{L}\left(\sum_{i=1}^{\alpha_n}\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}},1\right)\le\mathscr{L}\left(\sum_{i=1}^{\alpha_n}\operatorname{Re}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right),1\right).$$

Rescaling by $\frac{1}{\sqrt{\alpha_n}}$, we have that

$$\mathscr{L}\left(\sum_{i=1}^{\alpha_n}\operatorname{Re}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right),1\right)=\mathscr{L}\left(\sum_{i=1}^{\alpha_n}\frac{1}{\sqrt{\alpha_n}}\operatorname{Re}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right),\frac{1}{\sqrt{\alpha_n}}\right).$$

Then by Theorem 23, there exists a constant $C_1\ge 0$ which depends on the distribution of $\operatorname{Re}\left(\frac{1}{z-X_1}+\frac{1}{z-\overline{X_1}}\right)$ such that

$$\mathscr{L}\left(\sum_{i=1}^{\alpha_n}\frac{1}{\sqrt{\alpha_n}}\operatorname{Re}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right),\frac{1}{\sqrt{\alpha_n}}\right)\le C_1\left(\frac{1}{\sqrt{\alpha_n}}+\frac{1}{D_L(x)}\right)$$

where $x=\left(\frac{1}{\sqrt{\alpha_n}},\dots,\frac{1}{\sqrt{\alpha_n}}\right)$. Then by Lemma 22, $D_L(x)\ge\frac{\sqrt{\alpha_n}}{2}$. Thus,

$$\mathscr{L}\left(\sum_{i=1}^{\alpha_n}\frac{1}{\sqrt{\alpha_n}}\operatorname{Re}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right),\frac{1}{\sqrt{\alpha_n}}\right)\le C_1\left(\frac{1}{\sqrt{\alpha_n}}+\frac{2}{\sqrt{\alpha_n}}\right)=\frac{3C_1}{\sqrt{\alpha_n}}.$$

Now, letting $C=3C_1$, we have

$$\mathscr{L}\left(\sum_{i=1}^{\alpha_n}\frac{1}{\sqrt{\alpha_n}}\operatorname{Re}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right),\frac{1}{\sqrt{\alpha_n}}\right)\le\frac{C}{\sqrt{\alpha_n}}.$$

Hence,

$$\mathscr{L}\left(\sum_{i=1}^{\alpha_n}\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}},1\right)\le\frac{C}{\sqrt{\alpha_n}}.$$

Since this bound applies to the supremum over all $u\in\mathbb{C}$, this bound applies to the u which absorbed the contribution of $\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}$. This implies that

$$\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\le 1\right)\le\frac{C}{\sqrt{\alpha_n}}.$$

Taking the limit as $n\to\infty$, we have

$$\lim_{n\to\infty}\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\le 1\right)\le\lim_{n\to\infty}\frac{C}{\sqrt{\alpha_n}}.$$

Observe that since $\alpha_n\to\infty$ and C is a constant, then $\lim_{n\to\infty}\frac{C}{\sqrt{\alpha_n}}=0$. Therefore,

$$\lim_{n\to\infty}\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\le 1\right)=0.$$

□

0.5.3 Upper Bound

This subsection provides an upper bound on the ratio of $p_n'(z)$ to $p_n(z)$.

Lemma 26. *Assume the same set-up as in Theorem 3. Then for almost every $z \in \mathbb{C}$ and any $c > 1$, there exists a constant $C > 0$ such that*

$$\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n} \frac{1}{z - X_i} + \sum_{i=1}^{\alpha_n} \frac{1}{z - \overline{X_i}} + \sum_{j=1}^{\beta_n} \frac{1}{z - Y_j}\right| \geq n^c\right) \leq \frac{C}{n^{c-1}}.$$

Proof. Assume the same set-up as in Theorem 3. Then by Lemma 14, $\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right], \mathbb{E}\left[\left|\frac{1}{z-\overline{X_1}}\right|\right]$, and $\mathbb{E}\left[\left|\frac{1}{z-Y_1}\right|\right]$ are finite for almost all $z \in \mathbb{C}$. Let $z \in \mathbb{C}$ be one of the almost every $z \in \mathbb{C}$ such that $\mathbb{E}\left[\left|\frac{1}{z-Y_1}\right|\right] < \infty$, $\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right] < \infty$, $\mathbb{E}\left[\left|\frac{1}{z-\overline{X_1}}\right|\right] < \infty$, and such that $z - Y_j \neq 0$, $z - X_i \neq 0$, and $z - \overline{X_i} \neq 0$ with probability 1 for $i = 1, \ldots, \alpha_n$ and $j = 1, \ldots, \beta_n$. Since $\mathbb{E}\left[\left|\frac{1}{z-Y_1}\right|\right] < \infty$ and $z - Y_j \neq 0$ with probability 1 for $j = 1, \ldots, \beta_n$, then $\frac{1}{z-Y_1}, \ldots, \frac{1}{z-Y_{\beta_n}}$ are finite with probability 1. Using similar arguments, $\frac{1}{z-X_1}, \ldots, \frac{1}{z-X_{\alpha_n}}$ are finite with probability 1 and $\frac{1}{z-\overline{X_1}}, \ldots, \frac{1}{z-\overline{X_{\alpha_n}}}$ are finite with probability 1. We now want to bound $\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n} \frac{1}{z-X_i} + \sum_{i=1}^{\alpha_n} \frac{1}{z-\overline{X_i}} + \sum_{j=1}^{\beta_n} \frac{1}{z-Y_j}\right| \geq n^c\right)$ from above, for some $c > 1$. To do so, we will use Markov's Inequality, which is valid since $\left|\sum_{i=1}^{\alpha_n} \frac{1}{z-X_i} + \sum_{i=1}^{\alpha_n} \frac{1}{z-\overline{X_i}} + \sum_{j=1}^{\beta_n} \frac{1}{z-Y_j}\right|$ is non-negative. Then by Markov's Inequality,

$$\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n} \frac{1}{z - X_i} + \sum_{i=1}^{\alpha_n} \frac{1}{z - \overline{X_i}} + \sum_{j=1}^{\beta_n} \frac{1}{z - Y_j}\right| \geq n^c\right) \leq \frac{\mathbb{E}\left[\left|\sum_{i=1}^{\alpha_n} \frac{1}{z-X_i} + \sum_{i=1}^{\alpha_n} \frac{1}{z-\overline{X_i}} + \sum_{j=1}^{\beta_n} \frac{1}{z-Y_j}\right|\right]}{n^c}. \tag{7}$$

We now want to show that $\mathbb{E}\left[\left|\sum_{i=1}^{\alpha_n} \frac{1}{z-X_i} + \sum_{i=1}^{\alpha_n} \frac{1}{z-\overline{X_i}} + \sum_{j=1}^{\beta_n} \frac{1}{z-Y_j}\right|\right]$ is bounded by nC where C is a constant. Observe that by the triangle inequality,

$$\mathbb{E}\left[\left|\sum_{i=1}^{\alpha_n} \frac{1}{z - X_i} + \sum_{i=1}^{\alpha_n} \frac{1}{z - \overline{X_i}} + \sum_{j=1}^{\beta_n} \frac{1}{z - Y_j}\right|\right] \leq \mathbb{E}\left[\left|\sum_{i=1}^{\alpha_n} \frac{1}{z - X_i}\right| + \left|\sum_{i=1}^{\alpha_n} \frac{1}{z - \overline{X_i}}\right| + \left|\sum_{j=1}^{\beta_n} \frac{1}{z - Y_j}\right|\right].$$

Then by the linearity of expectation, we have that

$$\mathbb{E}\left[\left|\sum_{i=1}^{\alpha_n} \frac{1}{z - X_i}\right| + \left|\sum_{i=1}^{\alpha_n} \frac{1}{z - \overline{X_i}}\right| + \left|\sum_{j=1}^{\beta_n} \frac{1}{z - Y_j}\right|\right] = \mathbb{E}\left[\left|\sum_{i=1}^{\alpha_n} \frac{1}{z - X_i}\right|\right] + \mathbb{E}\left[\left|\sum_{i=1}^{\alpha_n} \frac{1}{z - \overline{X_i}}\right|\right] + \mathbb{E}\left[\left|\sum_{j=1}^{\beta_n} \frac{1}{z - Y_j}\right|\right]. \tag{8}$$

Applying the triangle inequality to each sum, we have that

$$\mathbb{E}\left[\left|\sum_{i=1}^{\alpha_n}\frac{1}{z-X_i}\right|\right]+\mathbb{E}\left[\left|\sum_{i=1}^{\alpha_n}\frac{1}{z-\overline{X_i}}\right|\right]+\mathbb{E}\left[\left|\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\right]\leq$$

$$\mathbb{E}\left[\sum_{i=1}^{\alpha_n}\left|\frac{1}{z-X_i}\right|\right]+\mathbb{E}\left[\sum_{i=1}^{\alpha_n}\left|\frac{1}{z-\overline{X_i}}\right|\right]+\mathbb{E}\left[\sum_{j=1}^{\beta_n}\left|\frac{1}{z-Y_j}\right|\right]. \quad (9)$$

Then by the linearity of expectation,

$$\mathbb{E}\left[\sum_{i=1}^{\alpha_n}\left|\frac{1}{z-X_i}\right|\right]+\mathbb{E}\left[\sum_{i=1}^{\alpha_n}\left|\frac{1}{z-\overline{X_i}}\right|\right]+\mathbb{E}\left[\sum_{j=1}^{\beta_n}\left|\frac{1}{z-Y_j}\right|\right]=$$

$$\sum_{i=1}^{\alpha_n}\mathbb{E}\left[\left|\frac{1}{z-X_i}\right|\right]+\sum_{i=1}^{\alpha_n}\mathbb{E}\left[\left|\frac{1}{z-\overline{X_i}}\right|\right]+\sum_{j=1}^{\beta_n}\mathbb{E}\left[\left|\frac{1}{z-Y_j}\right|\right]. \quad (10)$$

Observe that since $X_1,\dots,X_{\alpha_n}$ are identically distributed, $\overline{X_1},\dots,\overline{X_{\alpha_n}}$ are identically distributed, and $Y_1,\dots,Y_{\beta_n}$ are identically distributed, then

$$\sum_{i=1}^{\alpha_n}\mathbb{E}\left[\left|\frac{1}{z-X_i}\right|\right]+\sum_{i=1}^{\alpha_n}\mathbb{E}\left[\left|\frac{1}{z-\overline{X_i}}\right|\right]+\sum_{j=1}^{\beta_n}\mathbb{E}\left[\left|\frac{1}{z-Y_j}\right|\right]\leq$$

$$\alpha_n\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]+\alpha_n\mathbb{E}\left[\left|\frac{1}{z-\overline{X_1}}\right|\right]+\beta_n\mathbb{E}\left[\left|\frac{1}{z-Y_1}\right|\right]. \quad (11)$$

Notice that since $\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]$ is finite, $\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]$ is bounded, meaning there exists a constant $C_1>0$ such that $\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]\leq C_1$. Similarly, there exists a constant $C_2>0$ such that $\mathbb{E}\left[\left|\frac{1}{z-\overline{X_1}}\right|\right]\leq C_2$ and there exists a constant $C_3>0$ such that $\mathbb{E}\left[\left|\frac{1}{z-Y_1}\right|\right]\leq C_3$. This implies that

$$\alpha_n\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]+\alpha_n\mathbb{E}\left[\left|\frac{1}{z-\overline{X_1}}\right|\right]+\beta_n\mathbb{E}\left[\left|\frac{1}{z-Y_1}\right|\right]\leq \alpha_nC_1+\alpha_nC_2+\beta_nC_3.$$

Observe that since $2\alpha_n+\beta_n=n$, then $\alpha_nC_1+\alpha_nC_2+\beta_nC_3\leq\max\{C_1,C_2,C_3\}n$. Let $C=\max\{C_1,C_2,C_3\}$. Then $\alpha_n\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]+\alpha_n\mathbb{E}\left[\left|\frac{1}{z-\overline{X_1}}\right|\right]+\beta_n\mathbb{E}\left[\left|\frac{1}{z-Y_1}\right|\right]\leq Cn$. Thus,

$$\mathbb{E}\left[\left|\sum_{i=1}^{\alpha_n}\frac{1}{z-X_i}+\sum_{i=1}^{\alpha_n}\frac{1}{z-\overline{X_i}}+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\right]\leq Cn.$$

Using (7) and simplifying, we have

$$\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\frac{1}{z-X_i}+\sum_{i=1}^{\alpha_n}\frac{1}{z-\overline{X_i}}+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\geq n^c\right)\leq\frac{nC}{n^c}=\frac{C}{n^{c-1}}.$$

□

0.5.4 Convergence of Roots and Critical Values

This subsection proves two lemmas, concerning the two assumptions which are required to prove Theorem 29.

Lemma 27. *Assume the same set-up as in Theorem 3. Further let* $C_n, D_n \in M_n(\mathbb{C})$ *where* $C_n = D_n\left(I_n - \frac{1}{n}J_n\right)$, $D_n = \operatorname{diag}\left(X_1,\dots,X_{\alpha_n},\overline{X_1},\dots,\overline{X_{\alpha_n}},Y_1,\dots,Y_{\beta_n}\right)$, *and* $J_n = \mathbb{1}_n\mathbb{1}_n^T$. *Then* $\frac{1}{n}\operatorname{Trace}\left(C_nC_n^*\right) + \frac{1}{n}\operatorname{Trace}\left(D_nD_n^*\right)$ *is bounded in probability.*

Proof. Assume the same set-up as in Theorem 3. Let $C_n = D_n\left(I_n - \frac{1}{n}J_n\right)$ and
$D_n = \operatorname{diag}\left(X_1,\dots,X_{\alpha_n},\overline{X_1},\dots,\overline{X_{\alpha_n}},Y_1,\dots,Y_{\beta_n}\right)$. Then
$D_n^* = \operatorname{diag}\left(\overline{X_1},\dots,\overline{X_{\alpha_n}},X_1,\dots,X_{\alpha_n},\overline{Y_1},\dots,\overline{Y_{\beta_n}}\right)$ and $C_n^* = \left(I_n - \frac{1}{n}J_n\right)\overline{D_n}$. Then
$\operatorname{Trace}\left(D_nD_n^*\right) = \sum_{i=1}^{\alpha_n}|X_i|^2 + \sum_{i=1}^{\alpha_n}|X_i|^2 + \sum_{j=1}^{\beta_n}|Y_j|^2$ and hence,

$$\frac{1}{n}\operatorname{Trace}\left(D_nD_n^*\right) = \frac{1}{n}\sum_{i=1}^{\alpha_n}|X_i|^2 + \frac{1}{n}\sum_{i=1}^{\alpha_n}|X_i|^2 + \frac{1}{n}\sum_{j=1}^{\beta_n}|Y_j|^2.$$

We now want to show that $\frac{1}{n}\operatorname{Trace}\left(D_nD_n^*\right)$ converges. Observe that the function $f(z) = |z|^2$ is continuous. Furthermore, since $X_1, \overline{X_1}$, and Y_1 have finite second moment, then $\mathbb{E}\left[|X_1|^2\right] = \mathbb{E}\left[\left|\overline{X_1}\right|^2\right]$ is finite, and $\mathbb{E}\left[|Y_1|^2\right]$ is finite. Since $X_1,\dots,X_{\alpha_n}$ are iid, complex valued random variables, $\frac{\alpha_n}{n}\to\alpha$, f is continuous, and $\mathbb{E}\left[|X_1|^2\right]$ is finite, then by Lemma 18 $\frac{1}{n}\sum_{i=1}^{\alpha_n}|X_i|^2 \to \alpha\mathbb{E}\left[|X_1|^2\right]$ in probability. Similarly, since $\overline{X_1},\dots,\overline{X_{\alpha_n}}$ are iid, complex valued random variables, $\frac{\alpha_n}{n}\to\alpha$, f is continuous, and $\mathbb{E}\left[\left|\overline{X_1}\right|^2\right]$ is finite, then by Lemma 18, $\frac{1}{n}\sum_{i=1}^{\alpha_n}\left|\overline{X_i}\right|^2 \to \alpha\mathbb{E}\left[\left|\overline{X_1}\right|^2\right]$ in probability. Also, since $Y_1,\dots,Y_{\beta_n}$ are iid, complex valued random variables, $\frac{\beta_n}{n}\to\beta$, f is continuous, and $\mathbb{E}\left[|Y_1|^2\right]$ is finite, then by Lemma 18, $\frac{1}{n}\sum_{j=1}^{\beta_n}|Y_j|^2 \to \beta\mathbb{E}\left[|Y_1|^2\right]$ in probability. Since $\frac{1}{n}\sum_{i=1}^{\alpha_n}|X_i|^2 \to \alpha\mathbb{E}\left[|X_1|^2\right]$ in probability, $\frac{1}{n}\sum_{i=1}^{\alpha_n}\left|\overline{X_i}\right|^2 \to \alpha\mathbb{E}\left[\left|\overline{X_1}\right|^2\right]$ in probability, and $\frac{1}{n}\sum_{j=1}^{\beta_n}|Y_j|^2 \to \beta\mathbb{E}\left[|Y_1|^2\right]$ in probability, then by applying Lemma 16 twice, we get $\frac{1}{n}\sum_{i=1}^{\alpha_n}|X_i|^2 + \frac{1}{n}\sum_{i=1}^{\alpha_n}|X_i|^2 + \frac{1}{n}\sum_{j=1}^{\beta_n}|Y_j|^2 \to \alpha\mathbb{E}\left[|X_1|^2\right] + \alpha\mathbb{E}\left[\left|\overline{X_1}\right|^2\right] + \beta\mathbb{E}\left[|Y_1|^2\right]$ in probability. This implies that

$$\frac{1}{n}\operatorname{Trace}\left(D_nD_n^*\right) \to \alpha\mathbb{E}\left[|X_1|^2\right] + \alpha\mathbb{E}\left[\left|\overline{X_1}\right|^2\right] + \beta\mathbb{E}\left[|Y_1|^2\right] \tag{12}$$

in probability. We will now consider $\frac{1}{n}\operatorname{Trace}(C_nC_n^*)$. Observe that

$$\frac{1}{n}\operatorname{Trace}(C_nC_n^*) = \frac{1}{n}\operatorname{Trace}\left(D_n\left(I_n - \frac{1}{n}J_n\right)\left(I_n - \frac{1}{n}J_n\right)\overline{D_n}\right).$$

Expanding the inner terms, we have that

$$\frac{1}{n}\operatorname{Trace}\left(D_n\left(I_n - \frac{2}{n}J_n + \frac{1}{n}J_n\right)\overline{D_n}\right) = \frac{1}{n}\operatorname{Trace}\left(D_n\left(I_n - \frac{1}{n}J_n\right)\overline{D_n}\right).$$

Then by cyclic rotation and the linearity of trace, we get

$$\begin{aligned}\frac{1}{n}\operatorname{Trace}\left(D_n\left(I_n - \frac{1}{n}J_n\right)\overline{D_n}\right) &= \\ \frac{1}{n}\operatorname{Trace}\left(D_n\overline{D_n} - \frac{1}{n}D_n\overline{D_n}J_n\right) &= \frac{1}{n}\operatorname{Trace}\left(D_n\overline{D_n}\right) + \frac{1}{n}\operatorname{Trace}\left(-\frac{1}{n}J_nD_n\overline{D_n}\right). \qquad (13)\end{aligned}$$

Expanding D_n and $\overline{D_n}$, we have that

$$\frac{1}{n}\operatorname{Trace}\left(D_n\overline{D_n}\right) = \frac{1}{n}\operatorname{Trace}\left(\operatorname{diag}\left(|X_1|^2,\ldots,|X_{\alpha_n}|^2,\left|\overline{X_1}\right|^2,\ldots,\left|\overline{X_{\alpha_n}}\right|^2,|Y_1|^2,\ldots,|Y_{\beta_n}|^2\right)\right)$$

and

$$\begin{aligned}\frac{1}{n}\operatorname{Trace}\left(-\frac{1}{n}J_nD_n\overline{D_n}\right) &= \\ \frac{1}{n}\operatorname{Trace}\left(-\frac{1}{n}J_n\operatorname{diag}\left(|X_1|^2,\ldots,|X_{\alpha_n}|^2,\left|\overline{X_1}\right|^2,\ldots,\left|\overline{X_{\alpha_n}}\right|^2,|Y_1|^2,\ldots,|Y_{\beta_n}|^2\right)\right)&. \qquad (14)\end{aligned}$$

Simplifying, we have

$$\begin{aligned}\frac{1}{n}\operatorname{Trace}(C_nC_n^*) &= \\ \frac{1}{n}\sum_{i=1}^{\alpha_n}|X_i|^2 + \frac{1}{n}\sum_{i=1}^{\alpha_n}\left|\overline{X_i}\right|^2 + \frac{1}{n}\sum_{j=1}^{\beta_n}|Y_j|^2 - \frac{1}{n^2}\sum_{i=1}^{\alpha_n}|X_i|^2 - \frac{1}{n^2}\sum_{i=1}^{\alpha_n}\left|\overline{X_i}\right|^2 - \frac{1}{n^2}\sum_{j=1}^{\beta_n}|Y_j|^2 & \qquad (15)\end{aligned}$$

Further simplifying, we see that

$$\begin{aligned}\frac{1}{n}\operatorname{Trace}(C_nC_n^*) &= \frac{n-1}{n^2}\sum_{i=1}^{\alpha_n}|X_i|^2 + \frac{n-1}{n^2}\sum_{i=1}^{\alpha_n}\left|\overline{X_i}\right|^2 + \frac{n-1}{n^2}\sum_{j=1}^{\beta_n}|Y_j|^2 \\ &= \frac{n-1}{n}\left(\frac{1}{n}\sum_{i=1}^{\alpha_n}|X_i|^2 + \frac{1}{n}\sum_{i=1}^{\alpha_n}\left|\overline{X_i}\right|^2 + \frac{1}{n}\sum_{j=1}^{\beta_n}|Y_j|^2\right) \\ &= \frac{n-1}{n}\cdot\frac{1}{n}\operatorname{Trace}(D_nD_n^*).\end{aligned}$$

By (12) and since $\frac{n-1}{n} \to 1$ then by Lemma 17,

$$\frac{n-1}{n} \cdot \frac{1}{n} \operatorname{Trace}(D_n D_n^*) \to$$
$$1\left(\alpha\mathbb{E}\left[|X_1|^2\right] + \alpha\mathbb{E}\left[|\overline{X_1}|^2\right] + \beta\mathbb{E}\left[|Y_1|^2\right]\right) = \alpha\mathbb{E}\left[|X_1|^2\right] + \alpha\mathbb{E}\left[|\overline{X_1}|^2\right] + \beta\mathbb{E}\left[|Y_1|^2\right] \quad (16)$$

in probability. Thus,

$$\frac{1}{n} \operatorname{Trace}(C_n C_n^*) \to \alpha\mathbb{E}\left[|X_1|^2\right] + \alpha\mathbb{E}\left[|\overline{X_1}|^2\right] + \beta\mathbb{E}\left[|Y_1|^2\right] \quad (17)$$

in probability. Finally, we will consider $\frac{1}{n} \operatorname{Trace}(C_n C_n^*) + \frac{1}{n} \operatorname{Trace}(D_n D_n^*)$. Observe that (17) and (12), then by Lemma 16,

$$\frac{1}{n} \operatorname{Trace}(C_n C_n^*) + \frac{1}{n} \operatorname{Trace}(D_n D_n^*) \to 2\alpha\mathbb{E}\left[|X_1|^2\right] + 2\alpha\mathbb{E}\left[|\overline{X_1}|^2\right] + 2\beta\mathbb{E}\left[|Y_1|^2\right]$$

in probability. Notice that since $\alpha \in [0,1]$, $\beta \in [0,1]$, and $X_1, \overline{X_1}$, and Y_1 have finite second moment, then $2\alpha\mathbb{E}\left[|X_1|^2\right] + 2\alpha\mathbb{E}\left[|\overline{X_1}|^2\right] + 2\beta\mathbb{E}\left[|Y_1|^2\right]$ is finite. Observe that since $\frac{1}{n} \operatorname{Trace}(C_n C_n^*) + \frac{1}{n} \operatorname{Trace}(D_n D_n^*)$ converges in probability then every subsequence of this sequence has a subsequence that converges and hence by Theorem 3.2.7 of [5], $\frac{1}{n} \operatorname{Trace}(C_n C_n^*) + \frac{1}{n} \operatorname{Trace}(D_n D_n^*)$ is tight, meaning it is bounded in probability. □

Lemma 28. *Under the assumptions of Theorem 3, let $C_n, D_n \in M_n(\mathbb{C})$ where $C_n = D_n\left(I_n - \frac{1}{n}J_n\right)$ and $D_n = \operatorname{diag}\left(X_1, \ldots, X_{\alpha_n}, \overline{X_1}, \ldots, \overline{X_{\alpha_n}}, Y_1, \ldots, Y_{\beta_n}\right)$. Then for almost all complex numbers z, $\frac{1}{n}\log|\det(C_n - zI_n)| - \frac{1}{n}\log|\det(D_n - zI_n)|$ converges in probability to zero and for almost all fixed z, these determinants are nonzero with probability $1 - o(1)$.*

Proof. Assume the same set-up as in Theorem 3. Then by Lemma 14, $\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]$ is finite, $\mathbb{E}\left[\left|\frac{1}{z-\overline{X_1}}\right|\right]$ is finite, and $\mathbb{E}\left[\left|\frac{1}{z-Y_1}\right|\right]$ is finite for almost every $z \in \mathbb{C}$. Let $z \in \mathbb{C}$ be one of the almost every $z \in \mathbb{C}$ such that the aforementioned condition holds and such that $z \neq 0$. Further let $C_n, D_n \in M_n(\mathbb{C})$ where $D_n = \operatorname{diag}\left(X_1, \ldots, X_{\alpha_n}, \overline{X_1}, \ldots, \overline{X_{\alpha_n}}, Y_1, \ldots, Y_{\beta_n}\right)$ and $C_n = D_n\left(I_n - \frac{1}{n}J_n\right)$. Later in the proof, we will show that $\det(D_n - zI_n)$ is nonzero with probability 1 and since the difference converges to 0, $\det(C_n - zI_n)$ is nonzero with probability $1 - o(1)$. Observe that by the properties of logarithms,

$$\frac{1}{n}\log\left|\det\left(C_n - zI_n\right)\right| - \frac{1}{n}\log\left|\det\left(D_n - zI_n\right)\right| = \frac{1}{n}\log\left|\frac{\det\left(C_n - zI_n\right)}{\det\left(D_n - zI_n\right)}\right|.$$

Using the properties of determinants, we have that

$$\frac{1}{n}\log\left|\frac{\det\left(C_n - zI_n\right)}{\det\left(D_n - zI_n\right)}\right| = \frac{1}{n}\log\left|\det\left(C_n - zI_n\right)\det\left(D_n - zI_n\right)^{-1}\right|$$

and notice that $D_n - zI_n$ is invertible because

$$\left(D_n - zI_n\right)^{-1} = \operatorname{diag}\left(\frac{1}{X_1 - z}, \ldots, \frac{1}{X_{\alpha_n} - z}, \frac{1}{\overline{X_1} - z}, \ldots, \frac{1}{\overline{X_{\alpha_n}} - z}, \frac{1}{Y_1 - z}, \ldots, \frac{1}{Y_{\beta_n} - z}\right)$$

and $\mathbb{E}\left[\left|\frac{1}{z-X_1}\right|\right]$ is finite, $\mathbb{E}\left[\left|\frac{1}{z-\overline{X_1}}\right|\right]$ is finite, and $\mathbb{E}\left[\left|\frac{1}{z-Y_1}\right|\right]$ is finite which implies that $\left|\frac{1}{z-X_1}\right|$, $\left|\frac{1}{z-\overline{X_1}}\right|$, and $\left|\frac{1}{z-Y_1}\right|$ are finite and hence, the terms of $\left(D_n - zI_n\right)^{-1}$ are finite. Since $\left(D_n - zI_n\right)^{-1}$ is a diagonal matrix with finite terms, then $\det\left(D_n - zI_n\right)^{-1}$ is nonzero. Expanding C_n, we have that

$$\frac{1}{n}\log\left|\det\left(C_n - zI_n\right)\det\left(D_n - zI_n\right)^{-1}\right| = \frac{1}{n}\log\left|\det\left(D_n - \frac{1}{n}D_nJ_n - zI_n\right)\det\left(D_n - zI_n\right)^{-1}\right|.$$

Using the properties of determinants, we have that

$$\frac{1}{n}\log\left|\det\left(D_n - \frac{1}{n}D_nJ_n - zI_n\right)\det\left(D_n - zI_n\right)^{-1}\right| = \frac{1}{n}\log\left|\det\left(\left(D_n - zI_n\right)^{-1}\left(\left(D_n - zI_n\right) - \frac{1}{n}D_nJ_n\right)\right)\right|. \quad (18)$$

Distributing and simplifying, we get

$$\frac{1}{n}\log\left|\det\left(\left(D_n - zI_n\right)^{-1}\left(\left(D_n - zI_n\right) - \frac{1}{n}D_nJ_n\right)\right)\right| = \frac{1}{n}\log\left|\det\left(I_n - \left(D_n - zI_n\right)^{-1}\frac{1}{n}D_nJ_n\right)\right|.$$

Since $J_n = \mathbb{1}\mathbb{1}^T$ and by Lemma 5, we have that

$$\frac{1}{n}\log\left|\det\left(I_n - \left(D_n - zI_n\right)^{-1}\frac{1}{n}D_nJ_n\right)\right| = \frac{1}{n}\log\left|\det\left(I_1 - \mathbb{1}^T\left(D_n - zI_n\right)^{-1}\frac{1}{n}D_n\mathbb{1}\right)\right|.$$

Distributing and applying the determinant, we have that

$$\frac{1}{n}\log\left|\det\left(I_1 - \mathbb{1}^T\left(D_n - zI_n\right)^{-1}\frac{1}{n}D_n\mathbb{1}\right)\right| = \frac{1}{n}\log\left|1 - \frac{1}{n}\operatorname{Trace}\left(\left(D_n - zI_n\right)^{-1}D_n\right)\right|.$$

Expanding the trace, we get

$$\frac{1}{n}\log\left|1-\frac{1}{n}\operatorname{Trace}\Big(\left(D_n-zI_n\right)^{-1}D_n\Big)\right| = \frac{1}{n}\log\left|1-\frac{1}{n}\left(\sum_{i=1}^{\alpha_n}\frac{X_i}{z-X_i}+\sum_{i=1}^{\alpha_n}\frac{\overline{X_i}}{z-\overline{X_i}}+\sum_{j=1}^{\beta_n}\frac{Y_j}{z-Y_j}\right)\right|.$$

Adding and subtracting z from the numerator of each fraction, we have

$$\frac{1}{n}\log\left|1-\frac{1}{n}\left(\sum_{i=1}^{\alpha_n}\frac{X_i+z-z}{X_i-z}+\sum_{i=1}^{\alpha_n}\frac{\overline{X_i}+z-z}{\overline{X_i}-z}+\sum_{j=1}^{\beta_n}\frac{Y_j+z-z}{Y_j-z}\right)\right|.$$

Splitting the fractions, we have that

$$\frac{1}{n}\log\left|1-\frac{1}{n}\left(\sum_{i=1}^{\alpha_n}\frac{X_i-z}{X_i-z}+\frac{z}{X_i-z}+\sum_{i=1}^{\alpha_n}\frac{\overline{X_i}-z}{\overline{X_i}-z}+\frac{z}{\overline{X_i}-z}+\sum_{j=1}^{\beta_n}\frac{Y_j-z}{Y_j-z}+\frac{z}{Y_j-z}\right)\right|.$$

Simplifying, we have

$$\frac{1}{n}\log\left|1-\frac{\alpha_n}{n}-\frac{z}{n}\sum_{i=1}^{\alpha_n}\frac{1}{X_i-z}-\frac{\alpha_n}{n}-\frac{z}{n}\sum_{i=1}^{\alpha_n}\frac{1}{\overline{X_i}-z}-\frac{\beta_n}{n}-\frac{z}{n}\sum_{j=1}^{\beta_n}\frac{1}{Y_j-z}\right|.$$

Combining like terms, we see that the above expression is equivalent to

$$\frac{1}{n}\log\left|1-\frac{2\alpha_n+\beta_n}{n}-\frac{z}{n}\left(\sum_{i=1}^{\alpha_n}\frac{1}{X_i-z}+\sum_{i=1}^{\alpha_n}\frac{1}{\overline{X_i}-z}+\sum_{j=1}^{\beta_n}\frac{1}{Y_j-z}\right)\right|.$$

Since $2\alpha_n+\beta_n=n$, $\frac{2\alpha_n+\beta_n}{n}=1$. Also, since there is an absolute value around all the sums, we can pull a negative out of each of the denominators and cancel it with the negative in front of the $\frac{z}{n}$ to get the equivalent expression

$$\frac{1}{n}\log\left|\frac{z}{n}\left(\sum_{i=1}^{\alpha_n}\frac{1}{z-X_i}+\sum_{i=1}^{\alpha_n}\frac{1}{z-\overline{X_i}}+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right)\right|.$$

Using the properties of logarithms, we have that the above expression is equivalent to

$$o(1)+\frac{1}{n}\log\left|\sum_{i=1}^{\alpha_n}\frac{1}{z-X_i}+\sum_{i=1}^{\alpha_n}\frac{1}{z-\overline{X_i}}+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|.$$

We now want to show that

$$\frac{1}{n}\log\left|\sum_{i=1}^{\alpha_n}\frac{1}{z-X_i}+\sum_{i=1}^{\alpha_n}\frac{1}{z-\overline{X_i}}+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|$$

converges in probability to zero. Observe that $\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)=\sum_{i=1}^{\alpha_n}\frac{1}{z-X_i}+\sum_{i=1}^{\alpha_n}\frac{1}{z-\overline{X_i}}$ and will hence be used interchangeably. Notice that by Lemma 25 and Lemma 26, on the complements of the events defined in those lemmas, we have that

$$1 \leq \left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right| \leq n^c \tag{19}$$

for some $c > 1$. This implies that

$$\frac{1}{n}\log|1| \leq \frac{1}{n}\log\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right| \leq \frac{1}{n}\log|n^c|.$$

Simplifying, we have

$$0 \leq \frac{1}{n}\log\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right| \leq \frac{c\log|n|}{n}.$$

Then

$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty}\frac{1}{n}\log\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right| \leq \lim_{n\to\infty}\frac{c\log|n|}{n}.$$

Since $c > 1$ is a constant, then $\lim_{n\to\infty}\frac{c\log|n|}{n}=0$. Then by the Squeeze Theorem,

$$\lim_{n\to\infty}\frac{1}{n}\log\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right| = 0.$$

We now want to show that the event in (19) holds with probability $1-o(1)$. To do so, we will consider

$$\mathbb{P}\left(1 \leq \left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right| \leq n^c\right).$$

Equivalently, we can consider the probability of

$$\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right| \geq 1\right)\bigcap \left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right| \leq n^c\right). \tag{20}$$

Here we will consider its complement, the probability of

$$\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\leq 1\right)\bigcup \left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\geq n^c\right). \quad (21)$$

Applying the union bound, we have that this probability is less than or equal to

$$\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\leq 1\right)+ \mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\geq n^c\right). \quad (22)$$

Observe that by Lemma 25,

$$\lim_{n\to\infty}\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\leq 1\right)=0$$

and by Lemma 26,

$$\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\frac{1}{z-X_i}+\sum_{i=1}^{\alpha_n}\frac{1}{z-\overline{X_i}}+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\geq n^c\right)\leq\frac{C}{n^{c-1}}$$

where $C>0$ is a constant and $c>1$. Taking the limit as $n\to\infty$, we have that

$$\lim_{n\to\infty}\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\frac{1}{z-X_i}+\sum_{i=1}^{\alpha_n}\frac{1}{z-\overline{X_i}}+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\geq n^c\right)\leq\lim_{n\to\infty}\frac{C}{n^{c-1}}.$$

Notice that since $C>0$ is a constant and $c>1$, then $\lim_{n\to\infty}\frac{C}{n^{c-1}}=0$. This implies that

$$\lim_{n\to\infty}\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\leq 1\right)+ \lim_{n\to\infty}\mathbb{P}\left(\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\geq n^c\right)=0. \quad (23)$$

Therefore,

$$\lim_{n\to\infty}\mathbb{P}\left(1\leq\left|\sum_{i=1}^{\alpha_n}\left(\frac{1}{z-X_i}+\frac{1}{z-\overline{X_i}}\right)+\sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}\right|\leq n^c\right)=1$$

which completes the proof. □

0.5.5 Main Result

This subsection uses the results of the previous subsections to prove Theorem 3 with the help of an additional theorem, Theorem 29.

Theorem 29. *Assume the same set-up as in Theorem 3. Notice that* $X_1, \ldots, X_{\alpha_n}, \overline{X_1}, \ldots, \overline{X_{\alpha_n}}$, $Y_1, \ldots, Y_{\beta_n}$ *are the roots of* $p_n(z)$ *and that* $w_1^{(n)}, \ldots, w_{n-1}^{(n)}$ *are the critical values of* $p_n(z)$ *and let* $f : \mathbb{C} \to \mathbb{C}$ *be a bounded and continuous function. Then, for each* n, *let* $C_n, D_n \in M_n(\mathbb{C})$ *where* $C_n = D_n\left(I_n - \frac{1}{n}J_n\right)$ *and* $D_n = \operatorname{diag}\left(X_1, \ldots, X_{\alpha_n}, \overline{X_1}, \ldots, \overline{X_{\alpha_n}}, Y_1, \ldots, Y_{\beta_n}\right)$. *Assume that (i) the expression*

$$\frac{1}{n}\operatorname{Trace}(C_n C_n^*) + \frac{1}{n}\operatorname{Trace}(D_n D_n^*)$$

is bounded in probability and (ii) for almost all complex numbers z,

$$\frac{1}{n}\log|\det(C_n - zI_n)| - \frac{1}{n}\log|\det(D_n - zI_n)|$$

converges in probability to zero and, in particular, for almost all fixed z, *these determinants are nonzero with probability* $1 - o(1)$. *Then,* $\frac{1}{n-1}\sum_{i=1}^{n-1} f(w_i^{(n)})$ *converges in probability to* $\alpha\mathbb{E}[f(X_1)] + \alpha\mathbb{E}\left[f(\overline{X_1})\right] + \beta\mathbb{E}[f(Y_1)]$.

Proof. Assume the same set-up as in Theorem 3. Notice that $X_1, \ldots, X_{\alpha_n}, \overline{X_1}, \ldots, \overline{X_{\alpha_n}}, Y_1, \ldots, Y_{\beta_n}$ are the roots of $p_n(z)$ and let $w_1^{(n)}, \ldots, w_{n-1}^{(n)}$ be the critical values of $p_n(z)$. For each n, let $C_n, D_n \in M_n(\mathbb{C})$ where $D_n = \operatorname{diag}\left(X_1, \ldots, X_{\alpha_n}, \overline{X_1}, \ldots, \overline{X_{\alpha_n}}, Y_1, \ldots, Y_{\beta_n}\right)$ and $C_n = D_n\left(I_n - \frac{1}{n}J_n\right)$. Suppose that $\frac{1}{n}\operatorname{Trace}(C_n C_n^*) + \frac{1}{n}\operatorname{Trace}(D_n D_n^*)$ is bounded in probability and for almost all complex numbers z, $\frac{1}{n}\log|\det(C_n - zI_n)| - \frac{1}{n}\log|\det(D_n - zI_n)|$ converges in probability to zero and, in particular, for almost all fixed z, these determinants are nonzero with probability $1 - o(1)$. Observe that the characteristic polynomial of D_n is (1) and that $\frac{p_n'(z)}{p_n(z)} = \sum_{i=1}^{\alpha_n}\frac{1}{z-X_i} + \sum_{i=1}^{\alpha_n}\frac{1}{z-\overline{X_i}} + \sum_{j=1}^{\beta_n}\frac{1}{z-Y_j}$. Further notice that $\frac{1}{n}p_n'(z)$ is a monic polynomial of degree $n-1$. Then by Theorem 6, there exists a rank one matrix $\frac{1}{n}J_n$ where $J_n = \mathbb{1}^T\mathbb{1}$ such that $\left(\frac{1}{n}J_n\right)^2 = \frac{1}{n}J_n$ and the characteristic polynomial of $C_n = D_n - \frac{1}{n}D_nJ_n$ is $\frac{z}{n}p_n'(z)$. Observe that since C_n is an $n \times n$ matrix, it must have n eigenvalues. From the characteristic polynomial given by Theorem 6, we see that the eigenvalues of C_n

are given by the critical values of $p_n(z)$, $w_1^{(n)}, \ldots, w_{n-1}^{(n)}$ and 0. Let μ_{C_n} be the empirical spectral measure of C_n and let μ_{D_n} be the empirical spectral measure of D_n. Then

$$\int f d\mu_{C_n} = \frac{1}{n}\sum_{i=1}^{n-1} f\left(w_i^{(n)}\right) + \frac{1}{n} f(0)$$

and

$$\int f d\mu_{D_n} = \frac{1}{n}\sum_{i=1}^{\alpha_n} f\left(X_i\right) + \frac{1}{n}\sum_{i=1}^{\alpha_n} f(\overline{X_i}) + \frac{1}{n}\sum_{j=1}^{\beta_n} f(Y_j).$$

Then by Theorem 7, $\mu_{C_n} - \mu_{D_n}$ converges in probability to zero. This implies that

$$\frac{1}{n}\sum_{i=1}^{n-1} f(w_i^{(n)}) + \frac{1}{n} f(0) - \frac{1}{n}\left(\sum_{i=1}^{\alpha_n} f(X_i) + \sum_{i=1}^{\alpha_n} f(\overline{X_i}) + \sum_{j=1}^{\beta_n} f(Y_j)\right) \to 0$$

in probability. We will now show that $\frac{1}{n} f(0)$ converges to zero. Since f is a bounded, continuous function, there exists a constant $M \in \mathbb{R}, M > 0$ such that

$$|f(z)| \leq M \tag{24}$$

for all $z \in \mathbb{C}$. Then $\frac{1}{n}|f(0)| \leq \frac{M}{n}$. This implies that

$$\lim_{n\to\infty} \frac{1}{n}|f(0)| \leq \lim_{n\to\infty} \frac{M}{n}.$$

Since M is a constant, $\lim_{n\to\infty} \frac{M}{n} = 0$. Then by the Squeeze Theorem, $\lim_{n\to\infty} \frac{1}{n}|f(0)| = 0$. Hence, $\frac{1}{n}|f(0)|$ converges to 0 which implies that $\frac{1}{n} f(0)$ converges to 0. Since $\frac{1}{n} f(0) \to 0$ and

$$\frac{1}{n} f(0) + \left(\frac{1}{n}\sum_{i=1}^{n-1} f(w_i^{(n)}) - \frac{1}{n}\left(\sum_{i=1}^{\alpha_n} f(X_i) + \sum_{i=1}^{\alpha_n} f(\overline{X_i}) + \sum_{j=1}^{\beta_n} f(Y_j)\right)\right) \to 0$$

in probability, then by Lemma 16

$$\frac{1}{n}\sum_{i=1}^{n-1} f(w_i^{(n)}) - \frac{1}{n}\left(\sum_{i=1}^{\alpha_n} f(X_i) + \sum_{i=1}^{\alpha_n} f(\overline{X_i}) + \sum_{j=1}^{\beta_n} f(Y_j)\right) \to 0 \tag{25}$$

in probability. Observe that by (24), $|f(X_1)| \leq M$ with probability 1, $\left|f(\overline{X_1})\right| \leq M$ with probability 1, and $|f(Y_1)| \leq M$ with probability 1. Then $\mathbb{E}\left[|f(X_1)|\right] \leq \mathbb{E}\left[M\right] = M$ so $\mathbb{E}\left[|f(X_1)|\right]$ is finite. Hence, $\mathbb{E}\left[f(X_1)\right]$ is finite. Using similar arguments, we get that $\mathbb{E}\left[f(\overline{X_1})\right]$ is finite and $\mathbb{E}\left[f(Y_1)\right]$ is finite. Since $X_1, \ldots, X_{\alpha_n}$ are iid, complex valued random variables, $\frac{\alpha_n}{n} \to \alpha$, f

is continuous, and $\mathbb{E}[f(X_1)]$ is finite, then by Lemma 18, $\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i) \to \alpha\mathbb{E}[f(X_1)]$ in probability. Similarly, since $\overline{X_1}, \ldots, \overline{X_{\alpha_n}}$ are iid, complex valued random variables, $\frac{\alpha_n}{n} \to \alpha$, f is continuous, and $\mathbb{E}\left[f(\overline{X_1})\right]$ is finite, then by Lemma 18, $\frac{1}{n}\sum_{i=1}^{\alpha_n} f(\overline{X_i}) \to \alpha\mathbb{E}\left[f(\overline{X_1})\right]$ in probability. Furthermore, since $Y_1, \ldots, Y_{\beta_n}$ are iid, complex valued random variables, $\frac{\beta_n}{n} \to \beta$, f is continuous, and $\mathbb{E}[f(Y_1)]$ is finite, then by Lemma 18, $\frac{1}{n}\sum_{j=1}^{\beta_n} f(Y_j) \to \beta\mathbb{E}[f(Y_1)]$ in probability. Since $\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i) \to \alpha\mathbb{E}[f(X_1)]$ in probability, $\frac{1}{n}\sum_{i=1}^{\alpha_n} f(\overline{X_i}) \to \alpha\mathbb{E}\left[f(\overline{X_1})\right]$ in probability, and $\frac{1}{n}\sum_{j=1}^{\beta_n} f(Y_j) \to \beta\mathbb{E}[f(Y_1)]$ in probability, then by applying Lemma 16 twice, we get $\frac{1}{n}\sum_{i=1}^{\alpha_n} f(X_i) + \frac{1}{n}\sum_{i=1}^{\alpha_n} f(\overline{X_i}) + \frac{1}{n}\sum_{j=1}^{\beta_n} f(Y_j) \to \alpha\mathbb{E}[f(X_1)] + \alpha\mathbb{E}\left[f(\overline{X_1})\right] + \beta\mathbb{E}[f(Y_1)]$ in probability. Thus,

$$\frac{1}{n}\left(\sum_{i=1}^{\alpha_n} f(X_i) + \sum_{i=1}^{\alpha_n} f(\overline{X_i}) + \sum_{j=1}^{\beta_n} f(Y_j)\right) \to \alpha\mathbb{E}[f(X_1)] + \alpha\mathbb{E}\left[f(\overline{X_1})\right] + \beta\mathbb{E}[f(Y_1)] \tag{26}$$

in probability. By (25) and (26), then by Lemma 16

$$\frac{1}{n}\sum_{i=1}^{n-1} f(w_i^{(n)}) \to \alpha\mathbb{E}[f(X_1)] + \alpha\mathbb{E}\left[f(\overline{X_1})\right] + \beta\mathbb{E}[f(Y_1)]$$

in probability. We will now consider $\frac{1}{n-1}\sum_{i=1}^{n-1} f(w_i^{(n)})$. Observe that

$$\frac{1}{n-1}\sum_{i=1}^{n-1} f(w_i^{(n)}) = \frac{n}{n-1}\cdot\frac{1}{n}\sum_{i=1}^{n-1} f(w_i^{(n)}).$$

Further notice that $\frac{n}{n-1}$ converges to 1 and

$$\frac{1}{n}\sum_{i=1}^{n-1} f(w_i^{(n)}) \to \alpha\mathbb{E}[f(X_1)] + \alpha\mathbb{E}\left[f(\overline{X_1})\right] + \beta\mathbb{E}[f(Y_1)]$$

in probability. Then by Lemma 17,

$$\frac{1}{n-1}\sum_{i=1}^{n-1} f(w_i^{(n)}) \to \alpha\mathbb{E}[f(X_1)] + \alpha\mathbb{E}\left[f(\overline{X_1})\right] + \beta\mathbb{E}[f(Y_1)]$$

in probability. □

Proof of Theorem 3. Let $X_1, Y_1, X_2, Y_2, \ldots$ be an infinite sequence of independent, complex valued random variables with finite second moment such that $X_1, X_2, \ldots$ are identically distributed and $Y_1, Y_2, \ldots$ are identically distributed. Further let α_n, β_n be sequences of non-negative integers such that $2\alpha_n + \beta_n = n$ and $\frac{\alpha_n}{n} \to \alpha$, $\frac{\beta_n}{n} \to \beta$. Assume one of the following:

(1) $\beta_n \to \infty$ and $Y_1, \dots, Y_{\beta_n}$ are non-degenerate

(2) $\beta_n \not\to \infty$ and that for almost all $z \in \mathbb{C}$, $\frac{1}{z-X_1} + \frac{1}{z-\overline{X_1}}$ is non-degenerate.

For each $n \geq 1$, let $p_n : \mathbb{C} \to \mathbb{C}$ be a degree n polynomial given by (1). We will also let $f : \mathbb{C} \to \mathbb{C}$ be a bounded and continuous function and for each $n \geq 1$, we define $D_n, C_n \in M_n(\mathbb{C})$ as $D_n = \operatorname{diag}\left(X_1, \dots, X_{\alpha_n}, \overline{X_1}, \dots, \overline{X_{\alpha_n}}, Y_1, \dots, Y_{\beta_n}\right)$ and $C_n = D_n\left(I_n - \frac{1}{n}J_n\right)$. Notice that by Lemma 27, $\frac{1}{n}\operatorname{Trace}(C_nC_n^*) + \frac{1}{n}\operatorname{Trace}(D_nD_n^*)$ is bounded in probability and by Lemma 28, $\frac{1}{n}\log|\det(C_n - zI_n)| - \frac{1}{n}\log|\det(D_n - zI_n)|$ converges in probability to zero for almost every $z \in \mathbb{C}$. Applying Theorem 29 completes this proof. $\square$

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