The Real Homotopy Type of Singular Spaces via The Whitney-deRham Complex

by

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This thesis studies certain invariants associated to a stratified space. These invariants are the Whitney-de Rham cohomology, it is the cohomology of a chain complex called the Whitney-de Rham complex of differential forms. At first glance this chain complex, and its cohomology appear to depend on several choices. The purpose of this thesis is twofold. First, to show that these invariants only depend on the homotopy type of the stratified space. Second, to show that the Whitney-de Rham Complex determines the real homotopy type of the space.

Contents

Chapter 1

Preliminaries

1.1 Notation and Prerequisites

This section establishes some standard notation and terminology that will be use throughout this paper. The natural numbers $\mathbb{N}$ consist of all non-negative integers $\{0, 1, 2, \ldots\}$. The abbreviation CDGA, DGA, CGA will be used in place of commutative differential graded algebra, differential graded algebra, commutative graded algebra respectively.

Some basic knowledge is assumed for this paper. It will be assumed that the reader is familiar with the basics of homological algebra[20],[27], Sheaf Theory[1],[3], smooth manifolds[10], and Algebraic Topology[7],[2]. Most of the necessary definition will be provided where needed, but some such as manifold, sheaf, singular homology, etc. will not be.

1.2 Introduction

The goal of this paper to better study certain homological invariants attached to a stratified space X, as defined by Pflaum[18], called the Whitney-deRham cohomology. This cohomology is defined as the cohomology of a commutative differential graded algebra, $\Omega^*_{\mathrm{W}}(X)$, associated to X in a non-canonical way. The main result is to show that though the definition of $\Omega^*_{\mathrm{W}}(X)$ depends on several choices, when certain conditions are imposed on X, the Whitney-deRham cohomology only depends on the homotopy type of X. This is achieved by showing that $\Omega^*_{\mathrm{W}}(X)$ is in fact weakly equivalent to the singular cochain complex $C^*(X, \mathbb{R})$ on X, as differential graded algebras. A restatement of the main results is as follows:

Theorem 1.2.1 Let X be a semi-analytic subset of $\mathbb{R}^n$, or X is a smooth cone space with global singular chart, then the sheaf of complexes

$$0 \longrightarrow \mathbb{R}_X \longrightarrow \mathcal{E}_X^\infty \longrightarrow \Omega^1_{W,X} \longrightarrow \cdots.$$

is a fine resolution of the locally constant sheaf $\mathbb{R}_X$ on X. Thus the Whitney-deRham cohomology of X is isomorphic to the real singular cohomology of X

$$\mathrm{H}^*_W(X) = H^*\left(\Gamma\left(X, \Omega^*_{W,X}\right)\right) \cong \mathrm{H}^*(X, \mathbb{R}).$$

In the case that X is semi-analytic the above result relies on Hironaka's embedded resolution of singularities[9]. The smooth cone space however does not rely on this. It should be noted that the class of spaces encompassed by the above theorem is quite large. It includes all smooth manifolds, smooth manifolds with boundary, smooth manifolds with corners, smooth manifolds with isolated singularities, real algebraic sets, real analytic sets, real semi-algebraic sets, complex algebraic sets, and complex analytic sets, and more.

This result, while being not quite as general as a theorem by Brasselet and Pflaum[17], greatly simplifies the proof in the case when X is semi-analytic. Their theorem works for sub-analytic sets, which are slightly more general than semi-analytic sets.

The complex $\Omega^*_{\mathrm{W}}(X) = \Gamma\left(X, \Omega^*_{W,X}\right)$ is called the Whitney-deRham complex. It is called this because of its similarity to the deRham complex $\Omega^*_{\mathrm{dR}}(X)$ on a smooth manifold[25],[24], which it generalizes. The main thing that is generalized is that $\Omega^*_{\mathrm{W}}(X)$ can be defined on spaces with singularities. However if it is defined on a smooth manifold then it is in general not isomorphic to the deRham complex as a chain complex, but only quasi-isomorphic, meaning both complexes have isomorphic cohomology and the isomorphism is induced from a chain homomorphism between the two complexes.

The complex $\Omega^*_{\mathrm{W}}(X)$ is a sort of hybridization between the classical deRham complex of differential forms, and the algebraic deRham complex as studied by Grothendieck[4],[5], Hartshorne[6], and Herrera-Lieberman[11]. In the later approach one embeds a variety in a smooth ambient variety and algebraicly completes it with respect to its defining ideal. Then one proceeds to define the complex of algebraic differential forms on the variety via the sheaf of Kähler differentials. In this work one proceeds similarly only the starting place is different. Instead one starts with a locally closed subset X of some Euclidean space $\mathbb{R}^n$ and considers the smooth functions on the ambients space modulo the ideal of functions J that vanish on X. Then one forms a similar completion procedure with respect to the ideal J. In this case one quotients by the ideal of smooth functions which vanish to all orders on X. Then one forms the Whitney-deRham complex on X in the same why that the algebraic deRham complex is formed.

The Whitney-deRham complex is a commutative differential graded algebra, and this extra algebra structure descents to the Whitney-deRham cohomology making it into a graded ring. It is shown in this paper that under certain hypothesis on the space X that this ring is isomorphic to the real singular cohomology ring. Thus proving a stronger result that just that the Whitney-deRham cohomology is isomorphic to the real singular cohomology.

This slightly stronger result allows for an application in the area of homotopy theory. If one has a commutative differential graded algebra A that is quasi-isomorphic to a specific commutative differential graded algebra, $A_{PL}(X)$ defined on X, then one can determine the real homotopy type of X directly from A. This allows for the conclusion that the Whitney-deRham complex determines the real homotopy type of X, and in fact the converse is true as well. The Whitney-deRham complex is determined by the real homotopy type of X, up to quasi-isomorphism of commutative differential graded algebras. The result is as follows:

Theorem 1.2.2 Let X be a semi-analytic subset of $\mathbb{R}^n$ or a smooth cone space with a global singular chart, such that X is simply connected and of finite type, then the Whitney-deRham complex $\Omega^*_{\mathrm{W}}(X)$ determines the real homotopy type, $\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{R}$, of X, and vice versa it is determined by the real homotopy type of X up to quasi-isomorphism.

This should be compared to the work by Hardt, et al.[26], which gives a way of computing the real homotopy type of semi-algebraic sets using the so call piecewise semi-algebraic cochains. The Whitney-deRham complex has the advantage that it is much easier to define than their chain complex. However it has the disadvantage that the chain complex of piecewise semi-algebraic cochains has been used in several places in the literature already, for example Kontsevich and Soibelman use it in their proof of the formality of the little disks operad[19].

1.3 Organization of Paper

This paper is organized into four sections and an appendix. The first introduces the types of spaces that are of interest. The second introduces the algebras of functions on these spaces that will be considered. The third defines the particular resolutions of the aforementioned algebras and contains most of the main results. The fourth section concerns a particular application, and develops the background necessary to understand it. Finally in the appendix a more classical approach to some of the definitions is taken.

Chapter 2 introduces the basic definitions of stratified spaces and morphisms between them along with the terminology to go along with them. Such as depth, dimension, and skeleton. Singular charts are then defined. Theses are like charts on manifolds except they respects the give stratifications of a space. Next the spaces that are of particular interest in this paper are defined. Namely smooth cone spaces and semi-analytic sets are defined. After this a several properties of such spaces are developed.

Chapter 3 briefly defines the sheaf of smooth functions so that one may next define the sheaf of Whitney functions. This sheaf is first defined locally. The local definition is needed to give a proper definition of the sheaf globally. Next it is show that the sheaf of Whitney functions is independent of the choice of open set used to define it. This is followed by exploring what these sheaves look like, locally, on smooth cone spaces, and on semi-analytic sets.

Chapter 4 defines the complex of differential forms as well as the Whitney-deRham complex, and the the relationship between them. In order to provide the necessary background the tangent bundle is introduced. Then the main results of the paper are shown. These results consist of proving that the Whitney-deRham cohomology is isomorphic to the real singular cohomology in the case of smooth cone spaces and semi-analytic sets. This is done simply by proving a Poincaré lemma for the complex of sheaves of Whitney-deRham differential forms.

Chapter 5 defines the basics of real homotopy theory. This theory is outlined enough so that the main points of the theory can be stated. After these results are stated the above work is then applied to obtained the desire theorem, that the Whitney-deRham complex on X determines the real homotopy type of the space X.

Chapter 2

Stratified Spaces

This first chapter introduces the types of spaces that are of interest in this paper. It starts with the basics of stratified spaces, and morphisms of stratified spaces. Then a few special types of spaces are introduced; namely smooth cone spaces, analytic spaces, and semi-analytic spaces.

Much of this chapter is taken from various parts of the book by Pflaum[18], for more details the reader should see this book.

2.1 Decomposed Spaces

This section introduces decomposed spaces and some associated terminology including the notions of depth, dimension, and the k-skeleton of such a space. One needs the concept of a decomposed spaces in order to define a stratified spaces, which are of much great interest in modern mathematics.

Let X be a paracompact Hausdorff space with countable topology and $\mathcal{Z}$ a locally finite partition of X into locally closed subspaces $S \subset X$. Each element $S \in \mathcal{Z}$ is called a piece of the partition, or just a piece.

Before coming to the definition of a decomposed space the above statement should be explained. $\mathcal{Z}$ is a partition of X means that

$$X = \bigsqcup_{S \in \mathcal{Z}} S.$$

The partition is considered to be locally finite if for each $x \in X$ there is an open set U in X containing x such that the number of pieces $S \in \mathcal{Z}$ for which $U \cap S \neq \emptyset$ is finite. Finally $S \subset X$ is relatively closed if there exists and open set V, and a closed set C, in X with $S = C \cap U$, i.e. S is closed in U in the subspace topology. For any piece $S \in \mathcal{Z}$ the topological closure of S in X will be denoted $\overline{S}$.

Definition 2.1.1 ([18]1.1.1) The pair $(X, \mathcal{Z})$ is called a decomposed space if the following conditions hold:

- Every piece $S \in \mathcal{Z}$ is a smooth manifold in the subspace topology induced from the topology on X.
- If $R \cap \overline{S} \neq \emptyset$ for any pair of pieces $R, S \in \mathcal{Z}$ then $R \subset \overline{S}$. In this case one writes $R \leq S$, and says that R is incident to S, or that R is a boundary piece of S.

One calls $\mathcal{Z}$ a decomposition of X.

It is clear that the above incidence relation $R \leq S$ defines a partial ordering on the set $\mathcal{Z}$.

Definition 2.1.2 ([18]1.1.5) The dimension of a decomposed space $(X, \mathcal{Z})$ is defined to be:

$$\dim X = \sup \{\dim S \mid S \in \mathcal{Z}\},$$

Where $\dim S$ is the dimension of the piece S as a smooth manifold.

Definition 2.1.3 ([18]1.1.5) For each $k \in \mathbb{N}$ the k-skeleton of the decomposed space $(X, \mathcal{Z})$ is defined to be:

$$X^k = \bigcup_{S \in \mathcal{Z}, \dim S \leq k} S$$

with the subspace topology induced from the topology on X.

Definition 2.1.4 ([18]1.1.5) For every point $x \in X$ define the depth of x to be:

$$\mathrm{dp}_{\mathcal{Z}}(x) = \sup \{k \in \mathbb{N} \mid \exists S_0, S_1, \ldots S_k \in \mathcal{Z} \mid x \in S_0 < S_1 < \cdots < S_k\}.$$

It is clear from the definition if $x, y \in S \in \mathcal{Z}$ then $\mathrm{dp}_{\mathcal{Z}}(x) = \mathrm{dp}_{\mathcal{Z}}(y)$. Therefore the depth of a piece is well-defined as $\mathrm{dp}_{\mathcal{Z}}(S) = \mathrm{dp}_{\mathcal{Z}}(x)$ for any $x \in S$. This leads to the definition of the depth of the space X

Definition 2.1.5 ([18]1.1.5) The depth of a decomposed space $(X, \mathcal{Z})$ is defined to be

$$\mathrm{dp}_{\mathcal{Z}}(X) = \sup \{\mathrm{dp}_{\mathcal{Z}}(S) \mid S \in \mathcal{Z}\}.$$

2.2 Morphisms of Decomposed Spaces

This section introduces morphisms of decomposed spaces, which will be needed to define morphisms of stratified spaces.

Let $(X, \mathcal{Z})$ and $(Y, \mathcal{Y})$ be decomposed spaces.

Definition 2.2.1 ([18]1.1.6) A continuous map $f : X \to Y$ is a morphism of decomposed spaces if for every piece $S \in \mathcal{Z}$ there is a piece $R_S \in \mathcal{Y}$ such that the following holds:

- $f(S) \subset R_S$,
- the restriction $f|_S : S \to R_S$ is smooth map of manifolds.

It is clear that the composition of two morphisms of decomposed spaces is again a morphism of decomposed spaces. If it is necessary to avoid confusion the map f will be denoted

$$f : (X, \mathcal{Z}) \to (Y, \mathcal{Y}).$$

Proposition 2.2.1 Let $f : (X, \mathcal{Z}) \to (Y, \mathcal{Y})$ be a morphism of decomposed spaces. Let $S, S' \in \mathcal{Z}$ such that $S \leq S'$, and let $R_S, R_{S'} \in \mathcal{Y}$ such that $f(S) \subset R_S$, and $f(S') \subset R_{S'}$, then $R_S \leq R_{S'}$.

Proof: The map $f : X \to Y$ is continuous, so f^{-1} commutes with set operations such as intersection and closure. By the fact that f is a morphisms of decomposed spaces, and $S \leq S'$ it is clear that

$$\emptyset \neq S \cap \overline{S'} \subset f^{-1}(R_S) \cap \overline{f^{-1}(R_{S'})} = f^{-1}\left(R_S \cap \overline{R_{S'}}\right).$$

This implies that $R_S \cap \overline{R_{S'}} \neq \emptyset$. Hence $R_S \leq R_{S'}$. $\square$

If the paracompact space X has two decompositions $\mathcal{Z}$ and $\mathcal{Y}$ it is said that $\mathcal{Z}$ is coarser that $\mathcal{Y}$ or that $\mathcal{Y}$ is finer that $\mathcal{Z}$ if the identity map is a morphisms of stratified spaces from $(X, \mathcal{Y})$ to $(X, \mathcal{Z})$.

2.3 Examples

This section introduces some basic examples of which will be used later.

Example 2.3.1 ([18]1.1.8) Every smooth manifold M is a decomposed space with one piece M.

Example 2.3.2 ([18]1.1.10) If M is a smooth manifold with boundary set $S_1 = \partial M$, and $S_2 = M \setminus \partial M$. This makes M into a decomposed space with two pieces.

Let X be a topological space, the cone over X is defined to be the quotient space

$$CX = ([0,1) \times X) \,/\, (\{0\} \times X)\,.$$

If $X = \emptyset$ then for notational convenience define the cone $C\emptyset = \{\star\}$ to be the space with one point.

Example 2.3.3 ([18]1.1.11) If M is a smooth manifold the cone CM is a decomposed space with pieces given by the cusp, $o = [\{0\} \times M]$, and the set $(0,1) \times M$.

Example 2.3.4 ([18]1.1.11) Similar to the previous example if $(X, \mathcal{Z})$ is a decomposed space then CS is also a decomposed space with pieces given by the cusp o and the sets $(0,1) \times S$ for each $S \in \mathcal{Z}$. For the decomposed space $(X, \mathcal{Z})$ it is easy to see

$$\dim CX = \dim X + 1,$$

$$\mathrm{dp}_{\mathcal{Z}}\,(CX) = \mathrm{dp}_{\mathcal{Z}}\,(X) + 1.$$

Where the decomposition of CX is denoted by $\mathcal{Z}$ also.

Example 2.3.5 ([18]1.1.12) Consider the space that is the union of two pieces $X = S_1 \cup S_2 \subset \mathbb{R}^2$. Where $S_1 = \{0\} \times (0,1)$, and $S_2 = \left\{(x,y) \in \mathbb{R}^2 \,\middle|\, x \geq 0, y = \sin\left(\frac{1}{x}\right)\right\}$. This space has $S_1 \leq S_2$, but $\dim S_1 = \dim S_2$. This type of space is considered to be pathological and should not be considered. There are several conditions that one may impose in order to rule out such spaces. The most widely known are Whitney's condition (A), and condition (B), but there are several others. See [18]§1.4 for more details.

2.4 Stratifications

This section begins with the notion of germs of sets, which are used to distinguish stratified spaces from decomposed spaces. This is followed by the definition of a stratified space. A stratified space can be thought of as an equivalence class of decomposed spaces. By doing this one is allowed to pass to a maximal decomposition in a specified equivalence class, and thereby each stratified space has a canonical decomposition.

Let X be a topological space with $x \in X$. Two subsets $A, B \subset X$ are said to be equivalent at x if there is an open set $U \subset X$ such that $x \in U$, and $A \cap U = B \cap U$. This clearly defines an equivalence relation on the power set of X.

Definition 2.4.1 ([18]1.2.1) The equivalence class of the set A under this equivalence relation is denoted $[A]_x$ and is called the germ of the set A at x. If $x \in A \subset B \subset X$ then $[A]_x$ is said to be a subgerm of $[B]_x$. This is denoted $[A]_x \subset [B]_x$.

Definition 2.4.2 ([18]1.2.2) A stratification of a topological space is a function $\mathcal{S} : x \mapsto [S_x]$, that assigns to each $x \in X$ the germ of a closed set S_x, such that the following axiom is satisfied:

- For every $x \in X$ there is an open set $U \subset X$ with $x \in U$, and there is a decomposition $\mathcal{Z}$ of U such that for every $y \in U$ the set germ $[S_y]$ coincides with the set germ of the piece of $\mathcal{Z}$ of which y is a element.

The pair $(X, \mathcal{S})$ is called a stratified space.

What the definition is saying is that, for each $x \in X$, there is a decomposition $\mathcal{Z}$ of a neighborhood, U, of x, such that for $y \in U$ with $y \in T \in \mathcal{Z}$, there is an open set V containing y such that $S_y \cap V = T \cap V$, i.e $[S_y] = [T]$.

For every decomposition $\mathcal{Z}$ of X there is an induced stratification $\mathcal{S}$ of X that sends x to $[C]$, where $x \in S = U \cap C$ with U open in X, C closed in X, and $S \in \mathcal{Z}$. Two decompositions $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of X are said to be equivalent if the stratification induced by them are the same. That is to say that the induced maps $\mathcal{S}_1$ and $\mathcal{S}_2$ are equal.

One can see that a stratified spaces is induces locally by a decomposition. This will allow one, locally, to refer to pieces of the space. This will be necessary when defining morphisms of stratified spaces.

A morphism between stratified spaces should be a continuous map that locally respects the decomposition that is defining the stratification.

Definition 2.4.3 ([18]1.2.2) A continuous function $f : X \to Y$ between stratified spaces $(X, \mathcal{S})$ and $(Y, \mathcal{R})$ is a morphism of stratified spaces, or for brevity a stratified morphism, if for every $x \in X$ there is an open neighborhood V of $f(x)$ in Y, and an open neighborhood U of x in X, together with decomposition $\mathcal{Z}$ of U, and $\mathcal{Y}$ of V inducing $\mathcal{S}|_U$ respectively $\mathcal{R}|_V$ such that the following axioms are satisfied:

- $f(U) \subset V$
- $f|_U : (U, \mathcal{Z}) \to (Y, \mathcal{Y})$ is a morphism of decomposed spaces.

If it is necessary to avoid confusion the map $f : X \to Y$ between stratified spaces will be written $f : (X, \mathcal{S}) \to (Y, \mathcal{R})$.

Theorem 2.4.1 ([18]1.2.7) Every stratified space $(X, \mathcal{S})$ has a decomposition $\mathcal{Z}_{\mathcal{S}}$ which satisfies the following maximality property:

- For every open set $U \subset X$ and every decomposition $\mathcal{Y}$ of U that induces $S|_U$ the decomposition $\mathcal{Z}_{\mathcal{S}}$, when restricted to U, is coarser than $\mathcal{Y}$.

Recall that $\mathcal{Z}_{\mathcal{S}}$ is coarser that $\mathcal{Y}$ if the identity map is a morphisms of decomposed spaces from $(U, \mathcal{Y})$ to $(U, \mathcal{Z}_{\mathcal{S}}|_U)$.

In this way one can see that for each stratification $\mathcal{S}$ of X there is a canonical decomposition $\mathcal{Z}_{\mathcal{S}}$ of X. The pieces of $\mathcal{Z}_{\mathcal{S}}$ are called the strata of X. To avoid cumbersome notation it will be said that a stratum S of the decomposition $\mathcal{Z}_{\mathcal{S}}$ is a stratum of $\mathcal{S}$, and be written $S \in \mathcal{S}$ rather than $S \in \mathcal{Z}_{\mathcal{S}}$.

Definition 2.4.4 A map $f : (X, \mathcal{S}) \to (Y, \mathcal{R})$ between stratified spaces is said to be a stratified diffeomorphism if f is a homeomorphism such that for each stratum $S \in \mathcal{S}$, there is a stratum $R_S \in \mathcal{R}$ with $f|_S : S \to R_S$ a diffeomorphism of smooth manifolds.

2.5 Singular Charts

As with manifolds, it is useful to have a notion of local coordinate charts on a stratified space. These charts will allow one to define various types of functions on the space by pulling them back from $\mathbb{R}^n$.

Definition 2.5.1 ([18]1.3.1) Let $(X, \mathcal{S})$ be a stratified space. A smooth singular chart on X, with chart domain $U \subset X$, is a homeomorphism $\phi : U \to \phi(U) \subset \mathbb{R}^n$ from an open set $U \subset X$ to a locally closed subset, $\phi(U)$, of $\mathbb{R}^n$, such that for every stratum $S \in \mathcal{S}$ the image $\phi(U \cap S)$ is a smooth submanifold of $\mathbb{R}^n$, and the restriction $\phi|_{U \cap S} : U \cap S \to \phi(U \cap S)$ is a smooth diffeomorphism.

For convenience the notation for singular charts $\phi : U \to O \subset \mathbb{R}^n$ will be used to express that $O \subset \mathbb{R}^n$ is open and $\phi(U) \subset O$ is a closed subset of U in the subspace topology.

To keep the technicalities to a minimum the only stratified spaces considered in this paper are those $(X, \mathcal{S})$ that come equipped with a global smooth singular chart $\phi_X : X \to O_X \subset \mathbb{R}^{n_X}$. When the context is clear this will be denoted simply as $\phi : X \to O \subset \mathbb{R}^n$.

The following definition will be used for the remainder of this paper as the working definition of a stratified space.

Definition 2.5.2 A stratified space $(X, \mathcal{S}, n)$ is a closed subset of $\mathbb{R}^n$ that is the image of a stratified space $(Y, \mathcal{Y})$ under a global singular chart $\phi : Y \to O \subset \mathbb{R}^n$, i.e. $X = \phi(Y) \subset O$. The stratification on X is given by $\mathcal{S} = \phi(\mathcal{Y})$. This means that the strata $S \in \mathcal{S}$ are smooth submanifolds of $\mathbb{R}^n$ which form a decomposition of X.

Hence to define a stratified space $(X, \mathcal{S}, n)$ one only needs to specify the set $X \subset \mathbb{R}^n$, and the strata $S \in \mathcal{S}$, as smooth submanifolds of $\mathbb{R}^n$.

2.6 Whitney Conditions (A), and (B)

The Whitney conditions are designed to be strong enough to rule out pathological stratified spaces, but be weak enough that most stratified spaces on interest satisfy them. Each condition imposes restrictions on how a stratum behaves near its boundary strata. They are phrased in terms the tangent space to the higher dimensional stratum and how it behaves with respect to the tangent space of a boundary stratum.

Let M be a smooth manifold, and $R, S \subset M$ smooth submanifolds. Let $s = \dim S$, $r = \dim R$, and $n = \dim M$. For any $y \in M$ let T_yR, T_yS, and T_yM denote the tangent space of R, S, and M respectively, where T_yR and T_yS are considered to be the zero vector space if $y \notin R$, $y \notin S$ respectively. Note that $T_yR, T_yS \subset T_yM$ are linear subspaces. For each $y \in M$ identify T_yM with a fixed n-dimensional vector space TM, and identify T_yS, T_yR with subsets of TM. Then as y varies in M, the spaces T_yS and T_yR can be though of as points in the Grassmanian manifold of r, or s-dimensional subspaces of TM respectively.

Definition 2.6.1 ([18]1.4.3) The pair (R, S) is said to satisfy the Whitney condition (A) at a point $y \in R$ if the following axiom is satisfied

- Let $(y_k)_{k\in\mathbb{N}}$ be a sequence of points in S converging to $y \in R$, such that the sequence of tangent spaces $T_{y_k}S$ converges in the Grassmannian of s-dimensional subspaces of TM to a subspace τ, then $T_yR \subset \tau$.

Let $U \subset M$ be an open neighborhood of $y \in R \subset M$ and $\phi : U \to \mathbb{R}^n$ be a smooth coordinate chart on M centered at y.

Definition 2.6.2 ([18]1.4.3) The pair (R, S) is said to satisfy the Whitney condition (B) at the point $y \in R$ with respect to the chart ϕ if the following conditions are satisfied

- Let $(x_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ be two sequence of points with $x_k \in R \cap U$, and $y_k \in S \cap U$ fulfilling the following three conditions

(1) $x_k \neq y_k$, and $\lim_{k\to\infty} x_k = \lim_{k\to\infty} y_k = y$.

(2) The sequence of connecting lines $\overline{\phi(x_k)\,\phi(y_k)} \subset \mathbb{R}^n$ converges in projective space to a line l.

(3) The sequence of tangent spaces $T_{y_k}S$ converges in the Grasmanian of s-dimensional subspaces of TM to a subspace τ.

Then $(T_y\phi)^{-1}(l) \subset \tau$.

One can show that this condition does not depend of the choice of ϕ.

Lemma 2.6.1 ([18]1.4.4) The Whitney condition (B) is independent of the choice of coordinate chart ϕ on M near $y \in R \subset M$. Thus, it is well defined to say that the pair (R, S) satisfy the Whitney condition (B) at $y \in R$.

The condition (B) is stronger than the condition (A), as is seen in the following lemma.

Lemma 2.6.2 ([18]1.4.5) If the pair (R, S) satisfies the Whitney condition (B) at $y \in R$, the it also satisfies the Whitney condition (A) at $y \in R$.

Definition 2.6.3 The stratified space $(X, \mathcal{S}, n)$ is said to be a Whitney (A) stratified space if for every pair of strata $R, S \in \mathcal{S}$ and every $y \in R$, the pair (R, S) satisfy the Whitney condition (A) at $y \in R$.

Similarly, the stratified space $(X, \mathcal{S}, n)$ is said to be a Whitney (B) stratified space if for every pair of strata $R, S \in \mathcal{S}$ and every $y \in R$, the pair (R, S) satisfy the Whitney condition (B) at $y \in R$.

2.7 Smooth Cone Space

Cone spaces will be one of two types of stratified spaces studied in this paper. The definition of a cone space requires a few comments.

A cone space is defined inductively via depth. Hence one defines a cone space of depth zero simply to be a manifold or a disjoint collection of manifolds. One then proceeds recursively to define a cone space of depth d to locally look like the cone over a cone space of depth $d-1$. A very easy case to visualize is when $d = 1$. Then a cone space of depth 1 looks locally like the produce of a manifold with the cone over one or more disjoint manifolds. A good example of a cone space of depth 1 to keep in mind is a manifold with isolated singularities.

Definition 2.7.1 ([18]3.10.1) A smooth cone space of depth 0 is a stratified space $(C, \mathcal{S}, m)$, such that C consists of countably many smooth connected disjoint submanifolds of $\mathbb{R}^m$. The strata of $\mathcal{S}$ are given by the union of connected components of C of equal dimension.

Definition 2.7.2 ([18]3.10.1) A smooth cone space of depth d is a stratified space $(C, \mathcal{S}, m)$, such that for all $x \in C$, the following data exists:

- a stratum $S_x \in \mathcal{S}$ containing x,
- a connected open set $A_x \subset C$ containing x,
- a compact smooth cone space $(L_x, \mathcal{L})$ of depth $d-1$, with L_x embedded in the unit sphere S^k, of $\mathbb{R}^{k+1}$
- a stratified diffeomorphism

$$k_x : A_x \to (S_x \cap A_x) \times CL_x \subset \mathbb{R}^m \times \mathbb{R}^{k+1}.$$

Here the CL_x is the cone over L_x in $\mathbb{R}^{k+1}$ with vertex at 0. A point in CL_x is given by $tx \in \mathbb{R}^{k+1}$, where $0 \leq t < 1$ and $x \in L_x \subset S^k$. Furthermore, the map k_x is required to satisfy:

- There exists an open set $B_x \subset \mathbb{R}^m$, such that $A_x = B_x \cap C$, a smooth diffeomorphism $K_x : B_x \to (S_x \cap A_x) \times B^{k+1}$ with $K_x|_{A_x} = k_x$. Here $B^{k+1} \subset \mathbb{R}^{k+1}$ is the open unit ball.

This last condition is imposed for technical purposes, and it will prove to be exactly the condition needed to prove the desired result about such spaces.

One can now say what is meant by a smooth cone space. It is a stratified space that locally looks like a cone space of depth d for some $d \geq 0$.

Definition 2.7.3 ([18]3.10.1) Given a stratified space $(X, \mathcal{S}, n)$ a smooth cone chart of depth d on X is a smooth singular chart $\phi : U \to V \subset \mathbb{R}^m$, with chart domain U, open in X, and image $C = \phi(U) \subset V$, with $(C, \mathcal{C}, m)$ a smooth cone space of depth d, and V open in $\mathbb{R}^m$.

One should notice that, in this case, X is a subset of $\mathbb{R}^n$, and C is a subset of $\mathbb{R}^m$ for possibly different m and n.

Definition 2.7.4 ([18]3.10.1) A stratified space $(X, \mathcal{S}, n)$ is said to be a smooth cone space, if for every $x \in X$, there exists an open neighborhood $U \subset X$ of x, and an integer d, such that U is the domain of a smooth cone chart

$$\psi : U \to V \subset \mathbb{R}^m$$

of depth d, as defined in the previous definition. The map ψ is required to satisfy the following condition:

- There exists an open set $W \subset \mathbb{R}^n$, with $U = W \cap X$, and a smooth diffeomorphism $\Psi : W \to V$ such that $\Psi|_U = \psi$

It should be noted that the above definition contains the adjective 'smooth.' This is so because of the condition that the map k_x and ψ_x are required to be the restriction of a smooth map on an open neighborhood of their respective domains. This condition can be weekend to require only that they be r-times differentiable for any $0 \leq r \leq \infty$. If this is done then the space is called a C^r-cone space. Much of the same theory developed in this paper for smooth cone spaces works for C^r-cone spaces once the definitions have been suitably adjusted. This is left for later work.

2.8 Semi-Analytic Space

The second main type of space that will be studied in this paper is semi-analytic sets. To define a semi-analytic set it is first necessary to define a real-analytic space, which is simply a space which locally looks like the zero set of a finite number of real-analytic functions. These definition come from [9]§1.

Definition 2.8.1 ([9]§1) An $\mathbb{R}$-ringed spaces $X = (|X|, \mathcal{O}_X)$ consists of a topological spaces $|X|$, and a sheaf of $\mathbb{R}$-algebras $\mathcal{O}_X$ on $|X|$.

Definition 2.8.2 ([9]§1) A local model of real analytic space is a $\mathbb{R}$-ringed space $(|S|, \mathcal{O}_S)$ which is obtained as follows:

$$|S| = \{x \in U \mid f_1(x) = \cdots = f_m(x)\}$$

and

$$\mathcal{O}_S = (\mathcal{A}_U / (f_1, \cdots, f_m) \mathcal{A}_U)|_S$$

where U is an open subset of some $\mathbb{R}^n$, $f_1, \ldots, f_n$ are real analytic functions on U, and $\mathcal{A}_U$ is the sheaf of germs of real-analytic functions on U.

Definition 2.8.3 ([9]§1) A real analytic space X is an $\mathbb{R}$-ringed space $X = (|X|, \mathcal{O}_X)$ which is everywhere locally isomorphic to some local model of real-analytic space.

Now that an analytic space has been defined one can move to the definition of a semi-analytic space. Semi-analytic spaces subsets of real-analytic spaces that are locally defined by analytic inequalities instead of equality.

Definition 2.8.4 ([9]2.1) A subset $A \subset X$ of a real analytic space X is said to be semi-analytic at a point $x \in X$ there exists an open neighborhood V of x in X and a finite number of of real analytic functions g_{ij}, f_{ij} on V such that

$$A \cap V = \bigcup_i \{x \in V \mid g_{ij}(x) = 0, f_{ij} > 0, \forall j\}.$$

The set $A \subset X$ is said to be semi-analytic in X if it is semi-analytic at every point $x \in A$.

It is easy to see that semi-analytic sets in X are preserved under: finite union, finite intersection, and by taking the set difference of any two.

The main fact about semi-analytic sets that will be used is Hironaka's embedded desingularization theorem.[9] Before stating the theorem there is some less familiar terminology that should be reviewed.

Definition 2.8.5 ([9]Note after 2.4) For a given coordinate system $(z_1, \cdots, z_n)$ a quadrant is a set defined by a system of some equalities $z_i = 0$ and some inequalities $\epsilon_j z_j > 0$ with $\epsilon = \pm 1$. Thus a union of quadrants is a union of such sets.

A union of quadrants is the semi-analytic version of normal crossings. The difference is that semi-analytic sets are defined via inequalities, and hence one gets quadrants in the place of hyperplanes.

Definition 2.8.6 ([9]1.14) A real-analytic map $\pi : \widehat{X} \to X$ between real analytic spaces is said to be almost everywhere an isomorphism if there is a closed real analytic subsapce S of X, such that S is nowhere dense in X, $\pi^{-1}(S)$ is nowhere dense in $\widehat{X}$, and π induces an isomorphism $\widehat{X} \setminus \pi^{-1}(S) \xrightarrow{\sim} X \setminus S$.

A map that is almost everywhere an isomorphism is quite similar to a bi-meromorphic or bi-rational equivalence.

Theorem 2.8.1 (Hironaka[9]2.4) Let X be a real analytic space. Let A be a globally defined semi-analytic set in X. Then there exists a smooth analytic space $\widehat{X}$ and a proper surjective real analytic map $\pi : \widehat{X} \to X$, such that for every point $y \in \widehat{X}$, there exists a local coordinate system $(z_1, \cdots, z_n)$ centered at y for which the following is true:

- within some neighborhood of y in $\widehat{X}$, $\pi^{-1}(A)$ is a union of quadrants with respect the to the coordinates $(z_1, \cdots, z_n)$.

Furthermore when X is smooth to begin with then π is almost everywhere an isomorphism.

It is important to note that in above theorem the set S can be chosen to be completely contained within X. Therefore π induces an isomorphism $\widehat{X} \setminus \pi^{-1}(A) \xrightarrow{\sim} X \setminus A$. Finally notice that $\pi^{-1}(A)$ is semi-analytic since it is locally a union of quadrants.

A final fact about semi-analytic sets is that they can be given the structure of a stratified space. This stratification is particularly nice, in that it satisfies Whitney's condition (B).

The theorem is stated in terms of sub-analytic sets. This is a more general class of sets that semi-analytic, however every semi-analytic set is also sub-analytic set.

Theorem 2.8.2 (Hironaka[9]4.8) Let A be a sub-analytic subset of a smooth analytic manifold $X = \mathbb{R}^n$. Then A admits a Whintey (B) stratification. To be precise, there exists a decomposition $A = \cup_\alpha A_\alpha$ inducing a stratification $\mathcal{A}$, which satisfies the following property:

- The family A_α form a decomposition of A by real-analytic submainfolds of X, each of which is sub-analytic in X.
- $(A, \mathcal{A}, n)$ is a Whitney (B) stratified space.

This theorem holds for any smooth analytic manifold X, but the condition that $X = \mathbb{R}^n$ is all that will be needed in this paper.

There is one more fact about semi-analytic spaces that will prove to be quite useful.

Theorem 2.8.3 For every Stratified Space $(X, \mathcal{S}, n)$ that satisfies Whitney's condition (B), there exists an open subset $U \subset \mathbb{R}^n$, with $X \subset U$, such that X is a deformation retract of U.

This is a direct result of the work by Mather on the control theory of stratified spaces. The main reason the above theorem is true is that every Whitney (B) stratified space, that can be embedded in some euclidean space $\mathbb{R}^n$, possesses smooth normal control data that is compatible with the embedding[18]:Theorem 3.6.9, [14]:Propostion 7.1. Because the technical details would only be a distraction and are not needed elsewhere in this paper they will be omitted. One should consult Pflaum[18] Chapter 3, or Mather[14] for the full technical constructions.

Corollary 2.8.1 For every semi-analytic subset X of $\mathbb{R}^n$ there is an open set U in $\mathbb{R}^n$ such that X is a deformation retract of U.

It should be noted that this open set U can be chosen so that there is a smaller open set $V \subset U$ with $X \subset V$ such that the closure $\overline{V}$ of V in $\mathbb{R}^n$ is contained in U, and if X is compact, then so is $\overline{V}$. Therefore since X is locally compact one can always choose this data locally.

Chapter 3

The Algebra of Whitney Functions

This chapter defines the sheaf of smooth functions, and the sheaf of Whitney functions. After these definitions are given the sheaf of Whitney functions is described locally for certain examples.

3.1 The Sheaf of Smooth Functions

This section begins by defining the sheaf of smooth functions on an open subset of $\mathbb{R}^n$. After this the stalks of this sheaf are described in terms of germs of smooth functions at a point.

Let $O \subset \mathbb{R}^n$ be an open set.

Definition 3.1.1 For each open set $U \subset O$ define the $\mathbb{R}$-algebra $\mathcal{C}^\infty(U)$ of smooth functions on U to consist of all those continuous functions on U whose partial derivatives exist and are continuous to all orders.

For open sets $V \subset U \subset O$ define the restriction map $\rho_{VU} : \mathcal{C}^\infty(U) \to \mathcal{C}^\infty(V)$ by $\rho_{VU}(f) = f|_V$ for each $f \in \mathcal{C}^\infty(U)$. It is clear for open sets $W \subset V \subset U$ that $\rho_{WV}\rho_{VU} = \rho_{WU}$. This implies that the assignment $U \mapsto \mathcal{C}^\infty(U)$ defines a presheaf of $\mathbb{R}$-algebras. It is a standard exercise to prove that this actually defines a sheaf.

Definition 3.1.2 The sheaf of smooth functions on an open set O in $\mathbb{R}^n$ will be denoted $\mathcal{C}^\infty_O$. When the context is clear the O will be omitted and the space of sections of this sheaf over an open set $U \subset O$ will be denoted $\mathcal{C}^{\infty}(U)$. The image of the restriction maps $\rho_{VU}(f)$ will be denoted $f|_V$ for any $f \in \mathcal{C}^{\infty}(U)$.

Definition 3.1.3 For any $x \in O$ the stalk of the sheaf $\mathcal{C}^{\infty}_O$ at x is denoted $\mathcal{C}^{\infty}_{O,x}$ and is defined to be equivalence classes of pairs (f, V) where $V \subset O$ is an open set containing x and $f \in \mathcal{C}^{\infty}(V)$. The equivalence relation is defined by $(f, V) \sim (g, W)$, if there exists an open set $Z \subset V \cap W$ containing x, such that $f|_Z = g|_Z$. The equivalence class of (f, V) under this relation is denoted $[f]_x \in \mathcal{C}^{\infty}_{O,x}$.

Notice that this is the same as defining $\mathcal{C}^{\infty}_{O,x}$ to be the direct limit

$$\mathcal{C}^{\infty}_{O,x} = \varinjlim_{\substack{x \in V \subset O \\ V \text{ open}}} \mathcal{C}^{\infty}(V),$$

where the limit is taken over the directed system of open sets V containing x, with the appropriate restriction maps.

Notice that for each $x \in O$ the stalk $\mathcal{C}^{\infty}_{O,x}$ is a local $\mathbb{R}$-algebra with multiplication given by $[f]_x [g]_x = [fg]_x$, and maximal ideal, $\mathfrak{m}_x$, given by those germs $[f]_x$ such that $f(x) = 0$.

3.2 Local Definition of Whitney Functions

Consider a locally closed subset C in $\mathbb{R}^n$. This means that there is an open set $U \subset \mathbb{R}^n$ with $C \subset U$ closed. Let $\mathcal{C}^{\infty}(U)$ be the $\mathbb{R}$-algebra of smooth functions on U.

Definition 3.2.1 The Ideal, $\mathcal{J}^{\infty}(C, U) \subset \mathcal{C}^{\infty}(U)$, is defined to be the collection of all $f \in \mathcal{C}^{\infty}(U)$, such that $Df|_C = 0$, for every differential operator D on $\mathcal{C}^{\infty}(U)$. It is called the ideal of smooth functions on U flat on C.

It should be noted that, since U is an open subset of $\mathbb{R}^n$, such a differential operator D can be written locally as a polynomial in the variables $\frac{d}{dx_1}, \cdots, \frac{d}{dx_n}$, where the x_i's are the local variables. Furthermore, it is easy to see that $\mathcal{J}^{\infty}(C, U)$ is an ideal by the chain rule.

The Whitney functions on C are defined by taking the quotient of smooth function on U by this ideal.

Definition 3.2.2 Define the algebra of Whitney functions on C to be the quotient

$$\mathcal{E}^{\infty}(C) = \mathcal{C}^{\infty}(U) / \mathcal{J}^{\infty}(C, U).$$

Let $J : \mathcal{C}^{\infty}(U) \to \mathcal{C}^{\infty}(U) / \mathcal{J}^{\infty}(C, U)$ be the quotient map.

Let $f \in \mathcal{C}^{\infty}(U)$ when the context is clear the coset $J(f)$ in $\mathcal{E}^{\infty}(C)$ will be represented either as $J(f)$ or $f + \mathcal{J}^{\infty}$, rather than $f + \mathcal{J}^{\infty}(C, U)$.

The following lemma will prove to be useful in the construction of the Whitney-deRham complex, and can be found as lemma C.3.3 in [18].

Lemma 3.2.1 The ideal $\mathcal{J}^{\infty}(C, U)$ is idempotent in the sense that the following equality holds:

$$(\mathcal{J}^{\infty}(C, U))^2 = \mathcal{J}^{\infty}(C, U).$$

The main way the previous lemma is used is

Corollary 3.2.1

$$\mathcal{J}^{\infty}(C, U) / (\mathcal{J}^{\infty}(C, U))^2 = 0.$$

3.3 Whitney Functions: A different perspective

In this section the algebra of Whitney functions is defined in a different way from how it was defined above. This is the way that this algebra is originally defined in Malgrange[13], and most other sources. It this section the classical definition is given, and then it is shown how this definition and the previous one are equivalent.

3.3.1 Compact Subsets of $\mathbb{R}^n$

Fix a positive integer n, and let $X \subset \mathbb{R}^n$ be a compact subset, with $y_1, y_2, \ldots, y_n$ being coordinates on $\mathbb{R}^n$. For any positive integer m let

$$\mathcal{N}(m, n) = \{\alpha \in \mathbb{N}^n \mid |\alpha| \le m\}.$$

Define the space of m-jets on X to be

$$\mathrm{J}^m(X) = \bigoplus_{\alpha\in\mathcal{N}(m,n)} \mathcal{C}(X).$$

An element $(f_\alpha)_{\alpha\in\mathcal{N}(m,n)} \in \mathrm{J}^m(X)$ will be denoted by $F = (f_\alpha)_{\alpha\in\mathcal{N}(m,n)}$ and for brevity referred to as F. For $0 \le k \le m$, $\beta \in \mathcal{N}(k,n)$, and $x \in X$ define the following functions:

$$T_x^k : \mathrm{J}^m(X) \to \mathcal{C}^\infty(\mathbb{R}^n)$$
$$J^k : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathrm{J}^k(X)$$
$$D^\beta : \mathrm{J}^k(X) \to \mathrm{J}^{k-|\beta|}(X)$$
$$\widetilde{T}_x^k : \mathrm{J}^k(X) \to \mathrm{J}^k(X)$$
$$R_x^k : \mathrm{J}^k(X) \to \mathrm{J}^k(X)$$

as follows:

For $y \in \mathbb{R}^n$ define

$$T_x^k F = \sum_{\alpha\in\mathcal{N}(k,n)} f_\alpha(x) \frac{(x-y)^\alpha}{\alpha!}.$$

For $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ define

$$J^k(f) = \left(\frac{d^\alpha f}{dx_\alpha}|_X\right)_{\alpha\in\mathcal{N}(k,n)}.$$

This notation means that each partial derivative of f has been restricted to the subset X.

For $F = (f_\alpha)_{\alpha\in\mathcal{N}(m,n)} \in \mathrm{J}^m(X)$ define

$$D^\beta(F) = (f_{\beta+\alpha})_{\alpha\in\mathcal{N}(m-|\beta|,n)}.$$

The last two are much easier to define as they are combinations of the above functions. Define

$$\widetilde{T}_x^k = J^k T_x^k$$
$$R^k x = 1 - T_x^k$$

Here 1 stands for the identity automorphism on $\mathrm{J}^k(X)$.

For any $\beta \in \mathcal{N}(m,n)$ denote the projection of $\mathrm{J}^m(X)$ onto the β-th factor when applied to an element F simply to be $F^\beta = f_\beta$.

Fix a positive integer m and a jet $F \in \mathrm{J}^m(X)$. For any $\beta \in \mathcal{N}(m,n)$ note that $\left(R_x^k(F)\right)^\beta(y)$ is a function from $X \times \mathbb{R}^n$ into $\mathbb{R}$. Let

$$r_{F,\beta} : X \times X \to \mathbb{R}$$

denote the restriction of this function to $X \times X$. This function will be called the β-th remainder term for F.

Definition 3.3.1 A jet $F \in \mathrm{J}^m(F)$ is called a Whitney function of class m if

$$r_{F,\beta}(x,y) = o\left(|x-y|^{m-|\beta|}\right)$$

for all $\beta \in \mathcal{N}(m,n)$. The collection of all Whitney functions of class m will be denoted by $\mathcal{E}^m(X)$.

The o notation in the above definition means that the β-th remainder term vanishes to order $m - |\beta|$ along the diagonal of $X \times X$.

Defined $|\bullet|_m$ on $J^m(X)$ by

$$|F|_m = \sup\{|f_\beta(x)|\,|\, x \in X, \beta \in \mathcal{N}(m,n)\},$$

and define $\|\bullet\|_m^X$ on $\mathcal{E}^m(X)$ by

$$\|F\|_m^X = |F|_m + \sup\{|r_{F,\beta}(x,y)|\,|\,(x,y) \in X \times X, x \neq y, \beta \in \mathcal{N}(m,n)\}.$$

It can be shown that $\|\bullet\|_m^X$ defines a norm, which makes $\left(\mathcal{E}^m(X), \|\bullet\|_m^X\right)$ a Banach space.

3.3.2 Closed subsets of $\mathbb{R}^n$

Let $X_1 \subset X_2 \subset \mathbb{R}^n$ be compact sets. Restriction of continuous functions on X_2 to X_1 induces a restriction of jets on X_2 to jets on X_1 by $F|_{X_1} = (f_\alpha|_{X_1})_{\alpha \in \mathcal{N}(m,n)}$ for all $F \in \mathrm{J}^m(X_2)$. Furthermore it is clear that $r_{F|_{X_1},\beta} = r_{F,\beta}|_{X_1 \times X_1}$, and thus the restriction of a Whitney function of class m is again a Whitney function of class m. Denote the restriction function by

$$\rho_{1,2} : \mathcal{E}^m(X_2) \to \mathcal{E}^m(X_1).$$

The fact that the remainder terms of $F|_X$ are the restriction of the remainder terms for F means that this restriction is a continuous map of Banach spaces.

For $X \subset \mathbb{R}^n$ a closed subset define $X_i = X \cap B(0,i)$, where $B(0,i)$ denotes the ball of radius i centered at 0 in $\mathbb{R}^n$. Then $X = \cup_{i\geq 0} X_i$ with each X_i being compact and $X_i \subset X_{i+1}$. The above paragraph says that for each pair i and j with $i \leq j$ there is a restriction map $\rho_{i,j} : \mathcal{E}^m(X_j) \to \mathcal{E}^m(X_i)$ such that $\rho_{k,l}\rho_{l,j} = \rho_{k,j}$ for $k \leq l \leq j$. This data forms an inverse system of Banach spaces.

Definition 3.3.2 The space of Whitney functions of class m on X is defined to be the inverse limit of the above inverse system of Banach spaces. That is $\mathcal{E}^m(X) = \lim \mathcal{E}^m(X_i)$. In particular $\mathcal{E}^m(X)$ is a Fréchet space.

An element $F \in \mathcal{E}^m(X)$ is of the form $F = (F_i)_{i\geq 0}$ where each $F_i \in \mathcal{E}^m(X_i)$ such that $F_i|_{X_j} = F_j$ for all $j \leq i$. Thus one can write $F = \left((f_{\alpha,i})_{i\geq 0}\right)$, but since $\mathcal{C}(X)$ can be thought of as the inverse limit of the $\mathcal{C}(X_i)$'s then F can actually be expressed $F = (f_\alpha)$ where each $f_\alpha \in \mathcal{C}(X)$. The condition that F must satisfy now is slightly different. Each $F \in \mathcal{E}^m(X)$ satisfies $r_{F,\beta}|_{K\times K}(x,y) = o\left(|x-y|^{m-|\beta|}\right)$ for every compact set $K \subset X$ and every $\beta \in \mathcal{N}(m,n)$.

3.3.3 Whitney functions of class infinity

Let $X \subset \mathbb{R}^n$ be a closed set. For each $i \leq j$ define the projection

$$j_{i,j} : \mathcal{E}^j(X) \to \mathcal{E}^j(X)$$

given by

$$j_{i,j}\left((f_\alpha)_{\alpha\in\mathcal{N}(j,n)}\right) = (f_\alpha)_{\alpha\in\mathcal{N}(i,n)}.$$

Note that $j_{i,j}j_{j,k} = j_{i,k}$ for all $i \leq j \leq k$, and hence the data of the $\mathcal{E}^m(X)$'s along with the projections $j_{i,j}$'s form an inverse system.

Definition 3.3.3 The space of Whitney functions of class infinity $\mathcal{E}^\infty(X)$ is defined to be the inverse limit of the above direct system.

$$\mathcal{E}^\infty(X) = \lim_m \mathcal{E}^m(X).$$

Each $\mathcal{E}^m(X)$ is a Fréchet space so $\mathcal{E}^\infty(X)$ is what some authors call an (LF)-space.

One can write an element of $\mathcal{E}^\infty(X)$ as $F = (f_\alpha)_{\alpha\in\mathbb{N}^n}$. Such an F must satisfy the condition that for every $m \geq 0$, every $\beta \in \mathcal{N}(m,n)$ and every compact set $K \subset X$, it is true that

$$r_{j_m(F),\beta}|_{K\times K}(x,y) = o\left(|x-y|^{m-|\beta|}\right).$$

Where the map $j_m : \mathcal{E}^\infty(X) \to \mathcal{E}^m(X)$ is the canonical map determined by the fact that $\mathcal{E}^\infty(X)$ is the limit of the above inverse system.

3.3.4 Algebra structure

Let X as above consider $F, G \in \mathrm{J}^m(X)$. Express $F = (f_\alpha)_{\alpha\in\mathcal{N}(m,n)}$ and $G = (g_\alpha)_{\alpha\in\mathcal{N}(m,n)}$. Define the product of F and G to be $FG = (h_\alpha)_{\alpha\in\mathcal{N}(m,n)}$, where the function

$$h_\alpha = \sum_{\beta+\gamma=\alpha} f_\beta g_\gamma.$$

One can see that if $F, G \in \mathcal{E}^m(X)$ then $FG \in \mathcal{E}^m(X)$. Thus it is clear that $\mathcal{E}^m(X)$ is a commutative $\mathbb{R}$-algebra. Furthermore it is clear that this product extends to the limit making $\mathcal{E}^\infty(X)$ into a commutative $\mathbb{R}$-algebra. In fact this product makes each $\mathcal{E}^m(X)$ into a Fréchet algebra, and hence $\mathcal{E}^\infty(X)$ becomes a (LF)-algebra. That is to say $\mathcal{E}^\infty(X)$ is the inverse limit of Fréchet algebras.

3.3.5 Whitney Extension Lemma

This section makes the connection between the approach developed in this appendix and the approach taken in the main paper. This connection is made through the Whitney extension lemma. Before stating the lemma some setup is required.

For each $m \in \mathbb{N}$ consider the map defined above as

$$J^m : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathrm{J}^m(X) : f \mapsto \left(\frac{d^\alpha f}{dx^\alpha}|_X\right)_{\alpha\in\mathcal{N}(m,n)}.$$

By Taylors theorem each jet $\left(\frac{d^\alpha f}{dx^\alpha}|_X\right)_{\alpha\in\mathcal{N}(m,n)}$ satisfies the condition required to be a Whitney function of order m. Thus this map has image contained in $\mathcal{E}^m(X)$ so it can be thought of as

$$J^m : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathcal{E}^m(X).$$

By repeatedly applying the product rule, one sees that J^m is in fact an algebra homomorphism. It is clear that the kernel of J^m is exactly the ideal $\mathcal{J}^m(X,\mathbb{R}^n)$ of smooth functions on $\mathbb{R}^n$ whose partial derivatives vanish to order m when restricted to X. Now the questions is whether or not J^m is surjective onto $\mathcal{E}^m(X)$. This is the content of the Whitney extension lemma.

Lemma 3.3.1 ([13] 3.2) The map J^m is surjective. Thus given a jet $(f_\alpha)_{\alpha\in\mathcal{N}(m,n)} \in \mathcal{E}^m(X)$, there is a smooth function $g\in\mathcal{C}^\infty(\mathbb{R}^n)$ such that $J^m(g) = (f_\alpha)_{\alpha\in\mathcal{N}(m,n)} \in \mathcal{E}^m(X)$. This says that $f_\alpha = \frac{d^\alpha g}{dx^\alpha}|_X$.

Now the connection with the definition in chapter 3 is clear. For each $m\in\mathbb{N}$ there is a short exact sequence

$$0 \longrightarrow \mathcal{J}^m(X,\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n) \xrightarrow{J^m} \mathcal{E}^m(X) \longrightarrow 0\cdot$$

Furthermore this exact sequence fits into an inverse system of short exact sequences. The maps in this inverse system are induced from the above defined maps $j_{i,j} : \mathcal{E}^j(X) \to \mathcal{E}^i(X)$. The inverse limit of this short exact sequence is again a short exact sequence, because each J^k is surjective[6],[12]. This limit is exactly the short exact sequence used above to define $\mathcal{E}^\infty(X)$

$$0 \longrightarrow \mathcal{J}^\infty(X,\mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^n) \longrightarrow \mathcal{E}^\infty(X) \longrightarrow 0.$$

In this way it is easy to see that the definition in this section and the definition given previously in the paper coincide. This also justifies the comments in the introductions that suggest that the Whitney-deRham complex is a completion with respect to the defining ideal of X. Thus making the connection between this theory and the algebraic deRham theory more clear.

3.4 Whitney Functions on a Stratified Space With a Global Singular Chart

Let $(X, \mathcal{S}, n)$ be a stratified space. There is an open set O in $\mathbb{R}^n$ such that X is closed in O. Every open subset U in X is locally closed in O. Assign to the open set U the $\mathbb{R}$-algebra $\mathcal{E}^\infty(U)$. If $V \subset U \subset X$ are open sets then there is an obvious containment $\mathcal{J}^\infty(U, O) \subset \mathcal{J}^\infty(V, O)$. Thus there is a homomorphism of $\mathbb{R}$-algebras $r_V^U : \mathcal{E}^\infty(U) \to \mathcal{E}^\infty(V)$ that clearly satisfied $r_V^U r_W^V = r_W^U$ for all open $W \subset V \subset U \subset O$. Therefore the assignment:

$$U \mapsto \mathcal{E}^\infty(U)$$

Defines a presheaf of $\mathbb{R}$-algebras $\mathcal{E}_X^\infty$ on X.

Lemma 3.4.1 $\mathcal{E}_X^\infty$ is a sheaf.

One can find the proof in section 1.5 of Pflaum[18]. The proof relies on the Whitney extension lemma, which one can find in the appendix to Plaum[18].

Definition 3.4.1 The sheaf of Whitney functions on the stratified space $(X, \mathcal{S}, n)$ is defined to be the sheaf $\mathcal{E}_X^\infty$. The algebra of Whitney functions on X is the algebra of global sections of this sheaf

$$\mathcal{E}^\infty(X) = \Gamma(X, \mathcal{E}_X^\infty)$$

.

For future reference it will be useful to have the following result:

Lemma 3.4.2 The sheaf of Whitney function $\mathcal{E}_X^\infty$ on $(X, \mathcal{S}, n)$ is a fine(flasque) sheaf.

Proof: This is proposition 1.5.4 in Pflaum[18]. □

3.5 Invariance of open set

The last section defined the sheaf of Whitney functions on a stratified space $(X, \mathcal{S}, n)$ in terms of smooth functions on an open set O containing X. In this section it is shown that this sheaf does not depend on the open set O.

Let $O_1, O_2 \subset \mathbb{R}^n$ be open subsets such that X is closed in each O_1 and O_2. Let $P = O_1 \cap O_2$. Define $\mathcal{E}^\infty(X)_O = \mathcal{C}^\infty(O)/\mathcal{J}^\infty(X,O)$, and $\mathcal{J}_O^\infty = \mathcal{J}^\infty(X,O)$, for any $O \subset \mathbb{R}^n$ that is open with $X \subset O$. Consider the restriction map $\rho_i : \mathcal{E}^\infty(X)_{O_i} \to \mathcal{E}^\infty(X)_P$ induced from the restriction of smooth functions on O_i to P for $i = 1, 2$.

Lemma 3.5.1 The map ρ_i is an isomorphism.

Proof: Since the sheaf of smooth functions is a soft sheaf, the map ρ_i is surjective. Let $f + \mathcal{J}_{O_i}^\infty \in \mathcal{E}^\infty(O_i)$ if $\rho_i\left(f + \mathcal{J}_{O_i}^\infty\right) = 0 + \mathcal{J}_P^\infty$ then $f|_P \in \mathcal{J}^\infty(X,P)$. This means that f vanishes in a neighborhood of X, and hence all of its partials vanish on X. Therefore $f + \mathcal{J}_{O_i}^\infty = 0 + \mathcal{J}_{O_i}^\infty$, and ρ_i is injective. $\square$

Therefore the following statement is justified:

Corollary 3.5.1 The $\mathbb{R}$-algebra of Whitney functions

$$\mathcal{E}^\infty(X) = \mathcal{C}^\infty(O)/\mathcal{J}^\infty(X,O)$$

is independent of the choice of open set O containing X.

Because of this it is frequently useful to make a wise choice of O. Usually that will mean that O is chosen so that X is a homotopy equivalent to O.

3.6 Whitney Functions on a Smooth Cone Space

The purpose of this section is to describe the local behavior of the sheave of Whitney functions on a smooth cone space. The set up is rather technical, but the result is what should be expected.

Theorem 3.6.1 Let $(X, \mathcal{S}, n)$ be a smooth cone space. For every $x \in X$, let $S \in \mathcal{S}$ be the strata containg x. Then there is an open set $U \subset X$, a positive integer d, a stratified cone space L of depth d such that

$$\mathcal{E}_X^\infty|_U \cong \mathcal{E}_{(S\cap U)\times CL}^\infty.$$

Here L and CL can be chosen so that L is a closed subset of the unit sphere S^l of dimension l in $\mathbb{R}^{l+1}$ and CL has vertex at $(U \times S) \times 0$, with every point in $(U \cap S) \times CL$ described by (s, tx) where $0 \leq t < 1$, $x \in L$, and $s \in U \cap S$.

Proof: Let $U_x \subset X$ be a connected open set in X containing x, that is the domain of a cone chart $\psi_x : \phi(U_x) \to C_d \subset \mathbb{R}^m$ of depth d. Since $x \in S$ then $\phi(x) \in \phi(S)$. Let $W_x \subset \mathbb{R}^n$ be an open set with $\phi(U_x) = W_x \cap \phi(X)$ and $V_x \subset \mathbb{R}^m$ open, with C_d closed in V_x. There is a smooth diffeomorphism $\Psi_x : W_x \to V_x$ such that $\Psi_x|_{\phi(U_x)} = \psi_x$. Now $\phi(x)$ is a point of C_d so there exists a connected open neighborhood A_x of x in C_d, a cone space L_x of depth $d-1$, and a stratified diffeomorphism $k_{\phi(x)} : A_x \to \phi(S) \cap A_x \times CL_x$. The map k_x is the restriction of a smooth diffeomorphism $K_x : B_x \to U^{l+1}$ from an open set B_x in $\mathbb{R}^m$ to an open set U^{k+1} in $\mathbb{R}^m \times \mathbb{R}^{k+1}$ containing the $\phi(S) \cap A_x \times CL_x$. Thus the open set $\psi_x^{-1}(A_x) \subset \phi(X)$ is stratified diffeomorphic to a space of the form $\phi(S) \cap A_x \times CL_x \subset \mathbb{R}^m \times \mathbb{R}^{l+1}$ via the map $k_{\phi(x)}\psi_x$, which is the restriction of a smooth diffeomorphism $G = K_x\Psi_x$ whose domain is restricted to $\psi_x^{-1}(A_x)$. The map G induces a map smooth functions $G^* : \mathcal{C}^\infty\left(U^{l+1}\right) \to \mathcal{C}^\infty(V_x)$which is an isomorphism since G is a diffeomorphism. By the chain rule, and the fact that G^* is an isomorphism, G^* maps the ideal of flat functions on $\phi(S) \cap A_x \times CL_x$ exactally onto the ideal of flat functions on $\psi_x^{-1}(A_x)$, and similarly with its inverse. Hence there is an isomorphism $G^* : \mathcal{E}^\infty(\phi(S) \cap A_x \times CL_x) \to \mathcal{E}^\infty\left(\psi_x^{-1}(A_x)\right)$. $\square$

3.7 Whitney functions on a Semi-Analytic Set

As in the above section this section examines the sheaf of Whitney functions locally. The results are not as nice as the previous section, but still quite useful. This is due to Hironaka's desingularization theorem. Let $X \subset \mathbb{R}^n$ be a semi-analytic set, let $x \in X$ and $U \subset \mathbb{R}^n$ open. Call $A = X \cap U$. By Hironaka there exists a open set $V \subset \mathbb{R}^n$ and a proper, surjective, real-analytic map $\pi : V \to U$ such that $B = \pi^{-1}(A)$ is locally a union on quadrants, and π is an isomorphism from$V \setminus B$ to $U \setminus A$, with analytic inverse $s : U \setminus A \to V \setminus B$. Choose coordinates on U and V so that $x = 0$ in U and $0 \in B$. The coordinates on U will be represented by x_i's and the coordinates on V will be represented by y_j's.

The goal of this section will be to show that the ideal $\mathcal{J}^{\infty}(A,U) \subset \mathcal{C}^{\infty}(U)$, is isomorphic to the ideal $\mathcal{J}^{\infty}(B,V) \subset \mathcal{C}^{\infty}(V)$. To do this one defines the map $S : \mathcal{J}^{\infty}(B,V) \to \mathcal{J}^{\infty}(A,U)$, for each $g \in \mathcal{J}^{\infty}(B,V)$, by the formula

$$Sg(x) = \begin{cases} g(s(z)) & z \in U \setminus A \\ 0 & z \in A \end{cases}$$

The difficulty is to prove that this is a well defined map. What needs to be checked is that $g(s(z))$ vanishes to all orders as z approaches A. Since s is well defined for all $z \in U \setminus A$, all partial derivatives to all orders exist, and they will be a product of partials of g with partials of s. The partials of g vanish to all orders as z approaches A. Hence one needs to show that the partials of S do not blow up exponentially as z approaches A. Thus one wants to show that for each $\alpha \in \mathbb{N}^n$ there are constants $k_\alpha, r_\alpha > 0$ such that

$$|D^{\alpha} s_i(x)| \leq \frac{k_\alpha}{d(x,A)^{r_\alpha}},$$

for $1 \leq i \leq n$.

Lemma 3.7.1 The map S is well defined.

Proof Because both U and V are smooth analytic manifolds, then the matrix of partial derivatives $J\pi$, and Js of each has non-zero determinate(where defined) and compose to be the identity. That is to say that for any $x \in U \setminus A$,

$$J\pi(s(x))\, Js(x) = I_n.$$

By Cramers rule this means that one can write the partial derivative of the i-th coordinate function s_i of s with respect to the j-th variable x_j of U as

$$\frac{ds_i}{dx_j} = \frac{\|C_{ij}(J\pi(s(x)))\|}{\|J\pi(s(x))\|}.$$

Where the vertical bars $\|\bullet\|$ stand for the determinate, and $C_{ij}(\bullet)$ means the ij-th cofactor matrix of the matrix $\bullet$.

Let $K \subset \mathbb{R}^n$ be a compact subset contained in U. Since π is proper $L = \pi_{-1}(K)$ is also compact. Thus on L each partial derivative of π is bounded so the numerator can be bounded above by some constant M, since it is a sum of products of partial derivatives of π. Furthermore $\|J\pi\|$ is an analytic function on V. Thus the Lojasiawicz's inequality,[13]:Theorem 4.1, says that there exists $C, r > 0$ such that for any $y \in L$ the following is true:

$$|\|J\pi(y)\|| \geq Cd(y, B)^r .$$

Thus it follows, for any $0 \leq i, j \leq n$, that there are a constants $c, r > 0$ such that for any $x \in K$,

$$|\frac{ds_i}{dx_j}(x)| \leq \frac{c}{d(s(x), B)^r}.$$

One now needs to show that there exists constants k and s such that for any $x \in K$

$$d(s(x), B) \geq kd(x, A)^s .$$

This can be done in such a way that $s = 1$ and

$$k = n \sup_{z \in K, 1 \leq i, j \leq n} |\frac{d\pi_i}{dy_j}(z)|.$$

Let $b \in B$, $y \in V$, and γ a rectifiable curve in V connecting y to b in V, then $\pi(\gamma)$ is a curve in U connecting $x = \pi(y) \in U$ to $a = \pi(b) \in A$. Lemma 6.11 of Bierstone and Millman[15] says, not only is $\pi(\gamma)$ rectifiable but its length can be bounded as follows

$$|\pi(\gamma)| \leq n|\gamma| \sup_{z \in \gamma, 1 \leq i, j \leq n} |\frac{d\pi_i}{dy_j}(z)|.$$

According to Hironaka[9], every sub-analytic set, and hence every semi-analytic set, can be given the structure of a Whitney (B) stratified space. Hence, by the work of Mather in control theory[14], it posses a system of control data. This implies that there exists an open set W, containing A, that is homotopy equivalent to A. Furthermore this implies that one can choose a compact set $Z \subset U$ in such a way that $A \subset Z \subset W$, and that for every point in $z \in Z$ there is a curve γ that satisfies the condition that $|\gamma| = d(z, A)$. This comes from the fact that each stratum in A has a tubular neighborhood, and these neighborhoods are compatible with one another.

In the above situation it is sufficient or one to work with $K = Z$. This is because one is only interested in the behavior of the function s near A.

The partials of π are defined and continuous on L, so there is a constant M that only depends on π and L such that, for any γ in L,

$$\sup_{z \in \gamma, 1 \leq i,j \leq n} |\frac{d\pi_i}{dy_j}(z)| \leq M.$$

Combining this with the previous statement, and that the length of a curve between two points is always at least as large as the distance between the two points, one sees that

$$d(x, a) \leq |\pi(\gamma)| \leq nd(y, b) M,$$

Where γ is chosen to be the curve in L connecting y to b with $|\gamma| = d(y, b)$.

Now note that since $a = \pi(b) \in A$, then

$$d(x, A) \leq d(x, a).$$

This yields

$$d(x, A) \leq nMd(y, b).$$

Furthermore this is true for all points $b \in B$, so if one takes the infemum over all $b \in B$, and writes, $y = s(x)$, then one achieves the desired result

$$C' d(x, A) \leq d(s(x), B),$$

Where the constant C' only depends on π and K.

By induction, and basic calculus, one can, for any $\alpha \in \mathbb{N}^n$, for any $x \in U \setminus A$, that the function $D^\alpha s_i(x)$ is a quotient $P(x)/Q(x)$ where $P(x)$ is a finite sum of finite products of partial derivatives of π, all of order less than α, evaluated at $s(x)$, and $Q(x) = \|J\pi(s(x))\|^{m_\alpha}$ for some integer $m_\alpha \geq 1$. Thus for the same reasons as above one can write, for each $x \in K$, that

$$|D^\alpha s_i(x)| \leq \frac{c_\alpha}{d(x,A)^{m_\alpha}},$$

for $1 \leq i \leq n$.

This means that the partials of $g(s(z))$, for $z \in K \setminus A$ vanish to all orders. This is because the partials of g vanish faster than $d(x,A)^k$ for all $k \in \mathbb{N}$.

□

Corollary 3.7.1 There is an isomorphism of non-unital $\mathbb{R}$-algebras.

$$\mathcal{J}^\infty(A,U) \cong \mathcal{J}^\infty(B,V).$$

Proof: One immedeatly see that the following identities are true

$$S\pi^* = 1_{\mathcal{J}^\infty(A,U)}, \quad \text{and} \quad \pi^* S = 1_{\mathcal{J}^\infty(B,V)}.$$

Both the map π^* and the map S are easily seen to be algebra morphisms. □

Thus there is the commutative diagram which shows to what extent one can understand the Whitney functions local behavior on a semi-algebraic set.

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{J}^\infty(A,U) \arrow[r] \arrow[d, "\pi^*"', "\cong"] & \mathcal{C}^\infty(U) \arrow[r] \arrow[d, "\pi^*"'] & \mathcal{E}^\infty(A) \arrow[r] \arrow[d, "\pi^*"'] & 0 \\
0 \arrow[r] & \mathcal{J}^\infty(B,V) \arrow[r] & \mathcal{C}^\infty(V) \arrow[r] & \mathcal{E}^\infty(B) \arrow[r] & 0
\end{tikzcd}$$

This result will be needed when this same idea is extended to differential forms.

Chapter 4

The Whitney-deRham Complex

4.1 The deRham Complex

The deRham complex will be used extensively in what follows. It will only be necessary to work with the deRham complex on an open set in $\mathbb{R}^n$, hence for completion a quick and easy definition is given here. For the proper definition on should consult Lee[10], Bott and Tu[25], or Madsen and Tornehave[24].

Let $x_1, x_2, \ldots, x_n$ be coordinates on $\mathbb{R}^n$.

Definition 4.1.1 Define Ω^* to be the exterior algebra on an n-dimensional $\mathbb{R}$-vector space. To be more precise Ω^* is the graded $\mathbb{R}$-algebra generated by the symbols $dx_1, dx_2, \ldots, dx_n$ each of which has degree $+1$, with the relations $(dx_i)^2 = 0$ and $dx_i dx_j = -dx_j dx_i$ when $i \neq j$. As an $\mathbb{R}$-vector space the k-th piece Ω^k of Ω^* has a basis given by all $dx_{i_1} dx_{i_2} \ldots dx_{i_k}$ with $i_1 < i_2 < \cdots < i_k$.

Let $O \subset \mathbb{R}^n$ be an open set.

Definition 4.1.2 The algebra of smooth differential forms on O is defined to be

$$\Omega^*_{\mathrm{dR}}(O) = \mathcal{C}^\infty(O) \otimes_{\mathbb{R}} \Omega^*.$$

Therefore if $\omega \in \Omega^k_{\mathrm{dR}}(O)$ then ω can be uniquely written as a sum

$$\sum_{i_1 < i_2 < \ldots < i_k} f_{i_1 i_2 \cdots i_k} dx_{i_1} dx_{i_2} \cdots dx_{i_k},$$

with $f_{i_1 i_2 \cdots i_k} \in \mathcal{C}^\infty(O)$. When no confusion will occur this sum will frequently be abbreviated as $\sum f_I dx_I$. The algebra $\Omega^*_{\mathrm{dR}}(O)$ is naturally graded with grading induced from Ω^*.

Definition 4.1.3 Define the differential $d : \Omega^k_{\text{dR}}(O) \to \Omega^{k+1}_{\text{dR}}(O)$ recursively by first defining d on $\Omega^0_{\text{dR}}(O) = \mathcal{C}^\infty(O)$. For any $f \in \mathcal{C}^\infty(O)$

$$d(f) = \sum_{i=1}^{k} \frac{df}{dx_i} dx_i.$$

Now if a $f dx_{i_1} \cdots dx_{i_k}$ is a simple tensor in $\Omega^k_{\text{dR}}(O)$ define d as

$$d(f dx_{i_1} \cdots dx_{i_k}) = df dx_{i_1} \cdots dx_{i_k}$$

and extend by linearity to all k-forms.

If $\omega \in \Omega^k_{\text{dR}}(O)$ and $\tau \in \Omega^l_{\text{dR}}(O)$ then their product $\omega \wedge \tau \in \Omega^{k+l}_{\text{dR}}(O)$. By virtue of the fact that d acts by taking partial derivatives $d(fg) = d(f)g + fd(g)$ for all $f, g \in \mathcal{C}^\infty(O)$. Hence, since the dx_i's anti-commute

$$d(\omega \wedge \tau) = d(\omega) \wedge \tau + (-1)^k \omega \wedge d(\tau).$$

Similarly it is clear that

$$\omega \wedge \tau = (-1)^{kl} \tau \wedge \omega.$$

Definition 4.1.4 Therefore, $(\Omega^*_{\text{dR}}(O), d, \wedge)$ forms a CDGA (commutative differential graded algebra). The cohomology $\mathrm{H}^*_{dR}(O)$ of this algebra is called the deRham cohomology of O, and is a commutative graded algebra with the product induced from $\wedge$.

Definition 4.1.5 For each $k \geq 0$, define the presheaf Ω^k_O by the assignment

$$U \mapsto \mathcal{C}^\infty(U) \otimes_{\mathbb{R}} \Omega^k,$$

for each open $U \subset O$, with restriction maps $\rho_{VU} \otimes 1_{\Omega^k}$.

Each sheaf Ω^k_O is a $\mathcal{C}^\infty_O$-module. Hence since $\mathcal{C}^\infty_O$ is a fine sheaf, so is Ω^k_O. Notice also that the differential $d : \Omega^k_{\text{dR}}(U) \to \Omega^{k+1}_{\text{dR}}(U)$ induces a differential $d : \Omega^k_O \to \Omega^{k+1}_O$. Furthermore there is an injective map $\eta : \mathbb{R}_X \to \mathcal{C}^\infty_O$ of sheaves, induced from the map on sections, $\eta' : \mathbb{R} \to \mathcal{C}^\infty(U)$, that sends the real number r to the constant function r on U.

Lemma 4.1.1 (Poincaré's Lemma) For each point $x \in U$ there is a basis of open, contractible neighborhoods at x such that for any V in this basis the chain complex

$$0 \longrightarrow \mathbb{R}_V \longrightarrow \Gamma\left(V, \Omega_O^0\right) \longrightarrow \Gamma\left(V, \Omega_O^1\right) \longrightarrow \dots$$

is exact.

These facts translate into deRham's theorem

Theorem 4.1.1 (deRham's Theorem) The complex Ω_O^* comprises a fine resolution the locally constant sheaf $\mathbb{R}_O$. Hence after applying the global sections functor and taking cohomology one gets

$$\mathrm{H}_{dR}^*\left(O\right) \cong \mathrm{H}^*\left(O, \mathbb{R}\right).$$

4.2 The Whitney-deRham Complex

Let $X \subset O$ be a relatively closed subset of $\mathbb{R}^n$. Recall from above the ideal of smooth functions on O that are flat on X is denoted $\mathcal{J}^\infty\left(X, O\right)$ and consists of all smooth functions on O whose partials to all orders vanish when restricted to X, and the $\mathbb{R}$-algebra of Whitney functions, $\mathcal{E}^\infty\left(X\right)$, on X is is the quotient $\mathcal{C}^\infty\left(O\right)/\mathcal{J}^\infty\left(X, O\right)$. Let $J : \mathcal{C}^\infty\left(O\right) \to \mathcal{E}^\infty\left(X\right)$ be the quotient map.

Definition 4.2.1 Define the Whitney-deRham complex, to be

$$\Omega_{\mathrm{W}}^*\left(X\right) = \mathcal{E}^\infty\left(X\right) \otimes_{\mathbb{R}} \Omega^*,$$

With the differential $d_W = d + \mathcal{J}^\infty$. This is a CDGA with product is $\wedge + \mathcal{J}^\infty$. The cohomology, $\mathrm{H}_W^*\left(X\right)$, of this complex is called the Whitney-deRHam cohomology. With the wedge product $\mathrm{H}_W^*\left(X\right)$ is a commutative graded algebra.

Note that the quotient map J induces a map

$$J^* = J \otimes 1_{\Omega^*} : \Omega_{\mathrm{dR}}^*\left(O\right) \to \Omega_{\mathrm{W}}^*\left(X\right).$$

Since J is a surjective algebra homomorphism so then is J^*. In fact, J^* is a morphism of CDGA's.

Definition 4.2.2 The differential graded ideal

$$\Omega_r^*(X,O) = \mathcal{J}^\infty(X,O)\,\Omega_{\mathrm{dR}}^*(O) = \ker J^*,$$

Is called the complex if relative Whitney differential forms, and its cohomology, $\mathrm{H}_r^*(X,O)$, is called the relative Whintey-deRham cohomology.

One now has a short exact sequcence of chain complexes,

$$0 \longrightarrow \Omega_r^*(X,O) \longrightarrow \Omega_{\mathrm{dR}}^*(O) \xrightarrow{J^*} \Omega_{\mathrm{W}}^*(X) \longrightarrow 0.$$

Similar to the previous section one has, for each $k \geq 0$, the fine sheaf $\Omega_{W,X}^k$ of Whitney differential forms on X whose sectional space over the open set $U \subset X$ is exactly $\Omega_{\mathrm{W}}^k(U)$. One also has the map, $J^*\eta' : \mathbb{R} \to \mathcal{E}_X^\infty$, which remains injective.

One may wonder if the complex $\Omega_{W,X}^*$ also satisfies Poincaré's Lemma, thereby making $\Omega_{W,X}^*$ a fine resolution of the locally constant sheaf. Below it will be shown that this is in fact true for all smooth cone spaces with a global singular chart, and for all semi-analytic subsets of Euclidean space. In particular it is true for smooth manifolds, manifolds with isolated singularities, real-(semi)analytic sets, and real-(semi)algebraic sets. Thus it is also true for all complex analytic and algebraic subsets of some $\mathbb{C}^n$.

4.3 The Tangent Bundle

As above consider a locally closed subset $X \subset \mathbb{R}^n$ contained in an open set $O \subset \mathbb{R}^n$, and let $x_1, x_2, \ldots, x_n$ be coordinates on $\mathbb{R}^n$.

Definition 4.3.1 Define the $\mathbb{R}$-vector space T to be the $\mathbb{R}$-vector space generated by the basis $\frac{d}{dx_1}, \frac{d}{dx_2}, \ldots \frac{d}{dx_n}$. The space of vector fields on O is defined to be $\mathcal{X}O = \mathcal{C}^\infty(O) \otimes_{\mathbb{R}} T$. Likewise one can define the Whitney vector fields on X to be $\mathcal{X}_W X = \mathcal{E}^\infty(X) \otimes_{\mathbb{R}} T$. For each $k \geq 0$ define the pairing

$$\langle , \rangle : T \otimes_{\mathbb{R}} \Omega^k \to \Omega^{k-1}$$

by defining

$$\left\langle \frac{d}{dx_j}, dx_{i_1} \cdots dx_{i_k} \right\rangle = \begin{cases} 0 & \forall l, j \neq i_l \\ dx_{i_1} \cdots \widehat{dx_{i_l}} \cdots dx_{i_k} & \exists l, j = i_l \end{cases}$$

Extend this by linearity to the entire space. The hat on a specific term means that term has been removed from the product and replaced with a 1.

This induces the following contraction operators

$$\langle , \rangle : \mathcal{X}O \otimes_{\mathbb{R}} \Omega^k_{\mathrm{dR}}(O) \to \Omega^{k-1}_{\mathrm{dR}}(O),$$

$$\langle , \rangle_W : \mathcal{X}_W O \otimes_{\mathbb{R}} \Omega^k_{\mathrm{W}}(X) \to \Omega^{k-1}_{\mathrm{W}}(X).$$

Both will be written using the $\langle , \rangle$ notation.

4.4 Poincaré's Lemma for smooth manifolds

Theorem 4.4.1 If X is a locally closed smooth submanifold of $\mathbb{R}^n$ for some n, and O is an open set in $\mathbb{R}^n$ with X closed in O, then the homomorphism of commutative differential graded algebras

$$J^* : \Omega^*_{\mathrm{dR}}(O) \to \Omega^*_{\mathrm{W}}(X)$$

is a quasi-isomorphism.

Proof:

By the tubular neighborhood theorem it can be assumed that X is a deformation retract of O by a smooth homotopy $h : O \times I \to O$ relative to X. Denote $h_t(x) = h(x,t)$ for a fixed t. Then $h_0 = 1_X$, $h_1(x) \in X$ for all $x \in O$, and $h_t(x) = x$ for all $x \in X$ and all $t \in I$.

Define the operator $K^* : \Omega^*_{\mathrm{dR}}(O \times I) \to \Omega^{*-1}_{\mathrm{dR}}(O)$ by the formula

$$K^k(\omega) = \int_0^1 \left\langle \frac{d}{dt}, \omega \right\rangle dt$$

for all $\omega \in \Omega^k_{\mathrm{dR}}(O \times I)$ and all $k \geq 0$. The composition of this operator with h^*, satisfies the following formula

$$K^{k+1} h^{k+1} d(\omega) + dK^k h^k(\omega) = h^k_1(\omega) - h^k_0(\omega).$$

It is clear that taking partial derivatives in the O directions commutes with integrating in the t direction. Thus if $\omega \in \Omega_r^k(X, O)$ then $K^k h^k \omega \in \Omega_r^{k-1}(X, O)$. Furthermore, since $h_1(x) \in X$, for all $x \in O$, then $h_1^*(\omega) = 0$. Thus if $d\omega = 0$, the the above formula says that ω is a boundary in $\Omega_r^k(X, O)$. Therefore the chain complex $\Omega_r^*(X, O)$ is acyclic, in the sense that $\mathrm{H}_r^k(X, O) = 0$ for all $k \geq 0$. This chain complex is exactly the kernel of the map J^*, which is surjective by definition. Therefore the map J^* is a quasi-isomorphism.□

4.5 Poincaré's Lemma for smooth cone spaces

Let $(X, \mathcal{S}, n)$ be a smooth cone space. Consider the complex of sheaves $\Omega_{W,X}^*$ on X. To prove a Whitney-deRham type theorem, as in the previous section, one need to prove the Poincaré lemma for this sheaf of complexes.

Theorem 4.5.1 For any $x \in X$, there exists a basis of contractible open neighborhoods of x in X such that for each element $U \subset X$ of this basis, the complex $\Omega_{\mathrm{W}}^*(U)$ is acyclic.

Proof: Since the sheaf of Whitney functions on X is locally isomorphic to the sheaf of Whitney functions on $(S \cap U) \times CL$ for some open connected set U in X and cone space L of depth d, it is sufficient to prove the theorem for $X = Y \times CL \subset \mathbb{R}^m \times B$. Here Y is a locally closed smooth submanifold of $\mathbb{R}^m$ for some m, L is embedded in the unit sphere S^l in the closed unit ball $B = B^{l+1}$, and every point in CL is of the form tx with $0 \leq t < 1$ and $x \in L$.

Let T be a tubular neighborhood of Y in $\mathbb{R}^m$, then X is closed in $T \times B$. One has that $\Omega_{\mathrm{W}}^*(X) \cong \Omega_{\mathrm{dR}}^*(T \times B) / \Omega_r^*(X, T \times B)$. Now one has the radial homotopy $h : T \times B \to T \times B$ given by $h(y, tx, s) = (y, stx)$. This contracts $T \times B$ to $T \times \{0\}$ relative to X.

By applying the operator K^* as defined in the previous section, one achieves the same results. That $K^* h^* \omega \in \Omega_r^{*-1}(X, T \times B)$ whenever $\omega \in \Omega_r^*(X, T \times B)$. This means that the map

$$h^0 : \Omega_{\mathrm{W}}^*(X) \to \Omega_{\mathrm{W}}^*(Y \times \{0\})$$

is well defined and is a homotopy equivalence.

This gives the desired result since the complexes $\Omega^*_{\mathrm{W}}(Y \times \{0\})$ and $\Omega^*_{\mathrm{dR}}(Y)$ are homotopy equivalent, as per the previous section.

□

As stated above the sheaf of Whitney functions $\mathcal{E}^\infty(X)$ is a fine sheaf on X and thus each $\Omega^k_{\mathrm{W}}(X)$ is a fine sheaf of modules over $\mathcal{E}^\infty(X)$. Thus the sheaf of complexes $\Omega^*_{W,X}$ is a fine resolution of the locally constant sheaf $\mathbb{R}_X$, on X. Therefore the chain complex $\Omega^*_{\mathrm{W}}(X)$ is quasi-isomorphic to the singular cochain complex $C^*(X)$ as chain complexes.

Furthermore the above proof actually shows that when U is an open sets that deformation retracts onto X then the map $J : \Omega^*_U \to \Omega^*_{W,X}$ is locally a quaisi-isomorphism of differential graded algebras, thus it is globally a quasi-isomorphism of differential graded algebras.

Hence the following result:

Theorem 4.5.2 Let $(X, \mathcal{S}, n)$ be a smooth cone space with X a closed subset of an open set O in $\mathbb{R}^n$ such that X is a deformation retract of O. The quotient map

$$J^* : \Omega^*_{\mathrm{dR}}(O) \to \Omega^*_{\mathrm{W}}(X)$$

is a quasi-isomorphism of differential graded algebras.

4.6 Poincaré's Lemma for Semi-Analytic Sets

4.6.1 Local Union of Quadrants

The study of a local union of quadrants is similar to studying locally complete intersections, or local normal crossings in algebraic geometry. One first proves theorems about this type of space, and then extends them to more general types of spaces using some sort of resolution of singularities. This is exactly what is done here.

Definition 4.6.1 A closed subspace $X \subset \mathbb{R}^n$ is call a local union of quadrants, if for every point $x \in X$ there is an open neighborhood U of x in $\mathbb{R}^n$, an open neighborhood V of the origin in $\mathbb{R}^n$, and a smooth diffeomorphism $\phi : U \to V$, such that if $x_1, x_2, \ldots x_n$ are coordinates on $\mathbb{R}^n$ centered at the origin then $\phi(U)$ is defined by a collection of equations $\{x_i = 0\}_{i\in\mathcal{I}}$ and $\{\pm x_j > 0\}_{j\in\mathcal{J}}$, where $\mathcal{I}, \mathcal{J} \subset \{1, 2, \cdots, n\}$. For clarity purposes it should also be specified that $\phi(x) = 0$ where 0 is the origin in $\mathbb{R}^n$. Is said that $\phi(U)$ is the union of quadrants in $\mathbb{R}^n$.

For this type of space it is easy to prove that the Whitney-deRham cohomology of X is isomorphic to the singular cohomology of X. The only thing that needs to be done is to prove Poincaré's lemma.

Lemma 4.6.1 Let X be a local union of quadrants, then the sheaf of chain complexes $\Omega^*_{W,X}$ is exact.

Proof: This proof is identical to the proof of 4.5.1$\square$

Theorem 4.6.1 Let X be a local union of quadrants, then there is an isomorphism

$$\mathrm{H}^*(X, \mathbb{R}) \cong H^*\left(\Gamma\left(X, \Omega^*_{W,X}\right)\right) = \mathrm{H}^*_W(X)$$

Corollary 4.6.1 Let $X \subset \mathbb{C}^n$ be an algebraic set that is a local normal crossing, then $\mathrm{H}^*_W(X) \cong \mathrm{H}^*(X, \mathbb{R})$.

If X is a local union of quadrants, then X is semi-analytic, and hence, by 2.8.1 there is an open set U in $\mathbb{R}^n$ such that X is a deformation retract of U. Then there is the following exact sequence of chain complexes

$$0 \longrightarrow \mathcal{J}^\infty(X, U)\,\Omega^*_{\mathrm{dR}}(U) \xrightarrow{I} \Omega^*_{\mathrm{dR}}(U) \xrightarrow{J^*} \Omega^*_{\mathrm{W}}(X) \longrightarrow 0\cdot$$

The above theorem proves more than just the fact that $\mathrm{H}^*_W(X) \cong \mathrm{H}^*(X, \mathbb{R})$. It actually shows that locally J^* is a quasi-isomorphism of CDGA's. Thus one may conclude that J^* in the above diagram is a quasi-isomorphism of CDGA's. This is restated in the following corollary.

Corollary 4.6.2 When $X \subset \mathbb{R}^n$ is locally a union of quadrants, and $U \subset \mathbb{R}^n$ is an open subset such that X is a deformation retract of U in $\mathbb{R}^n$ then the quotient map

$$J : \Omega^*_{\mathrm{dR}}(U) \to \Omega^*_{\mathrm{W}}(X)$$

is a quasi-isomorphism of CDGA's. In particular the complex

$$\Omega_r^*(X,U) = \mathcal{J}^\infty(X,U)\,\Omega_{\mathrm{dR}}^*(U)$$

is acyclic.

4.6.2 Semi-analytic sets

Recall from section 3.7 that the main fact about semi-analytic sets used is the embedded desingularization theorem by Hironaka. Let $X \subset \mathbb{R}^n$ is a semi-analytic set, $x \in X$, and $U \subset \mathbb{R}^n$ open. Let $A = X \cap U$. There exists an open set $V \subset \mathbb{R}^n$, and a proper surjective analytic map $\pi : V \to U$, such that $B = \pi^{-1}(A)$ is a local union of quadrants, and $\pi| : V \setminus B \to U \setminus A$ is an isomorphism, with analytic inverse $s : U \setminus A \to V \setminus B$. Note that because both A and B are semi-analytic sets one can choose, by 2.8.1, both U and V in such a way that A is a deformation retract of U, and B is a deformation retract of V. Furthermore U can be chosen so that A is contractible to x, in A.

This data yields the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_r^*(A,U) & \xrightarrow{I} & \Omega_{\mathrm{dR}}^*(U) & \xrightarrow{J_{AU}} & \Omega_{\mathrm{W}}^*(A) & \longrightarrow & 0 \\
 & & {\scriptstyle\pi^*}\downarrow & & {\scriptstyle\pi^*}\downarrow & & {\scriptstyle\pi^*}\downarrow & & \\
0 & \longrightarrow & \Omega_r^*(B,V) & \xrightarrow{I} & \Omega_{\mathrm{dR}}^*(V) & \xrightarrow{J_{BV}} & \Omega_{\mathrm{W}}^*(B) & \longrightarrow & 0.
\end{array}$$

Because of the fact that B is locally a union of quadrants, the previous section implies that J_{BV} is a quasi-isomorphism of commutative differential graded algebras.

Lemma 4.6.2

$$\mathrm{H}_r^*(B,V) = 0.$$

□

Using the map $s : U \setminus A \to V \setminus B$, the inverse to $\pi|$, define the map

$$S : \Omega_r^*(B,V) \to \Omega_r^*(A,U)$$

by the following formula

$$S\omega(z) = \begin{cases} \omega(s(z)) & \text{if } z \in S \setminus A \\ 0 & \text{if } z \in A \end{cases} \qquad \text{for } \omega \in \Omega_r^*(B, V)$$

This map is well defined since it is just the extension of the map S defined in section 3.7, from the 0-th graded piece to all of $\Omega_r^*(B, V)$. Furthermore it is a morphism of commutative differential graded algebras.

It should be noted that this is one place where it is necessary to work with forms that vanish to all orders on a subset. If ω vanished only to finite orders then $S\omega$ would not have smooth coefficients.

Proposition 4.6.1 The map

$$S : \Omega_r^*(B, V) \to \Omega_r^*(A, U)$$

is an isomorphism of commutative differential graded algebras. Therefore

$$\mathrm{H}_r^*(A, U) = 0.$$

Proof: This is true because the map S from section 3.7 is an isomorphism, and it extends to well defined map S as above, because the complexes of relative differential forms are free on the ideal of flat functions.

Furthermore, it is easy to see that for all $\omega \in \Omega_r^*(B, V)$ and $\tau \in \Omega_r^*(A, U)$ the following equalities hold:

$$\pi^* S\tau = \tau \text{ and } S\pi^*\omega = \omega.$$

Thus the desired result.$\square$

From this proposition it follows that

Theorem 4.6.2 The map

$$J_{AU} : \Omega_{\mathrm{dR}}^*(U) \to \Omega_{\mathrm{W}}^*(A)$$

is a quasi-isomorphism of commutative differential graded algebras.

Proof: This follows since the kernel of J_{AU} is acyclic, and J_{AU} is surjective.$\square$

Theorem 4.6.3 Let $X \subset \mathbb{R}^n$ be a semi-analytic set. Then the Whitney-deRham cohomology on X,

$$\mathrm{H}^*_W(X) \cong \mathrm{H}^*(X, \mathbb{R}),$$

a graded algebras.

Proof: By 2.8.1 there exists, $W \subset \mathbb{R}^n$, an open set containing the semi-analytic set X such that X is a deformation retract of W. Then the above work shows that the quotient map

$$J_{WX} : \Omega^*_{\mathrm{dR}}(W) \to \Omega^*_{\mathrm{W}}(X)$$

is locally a quasi-isomorphism. Thus it is globally a quasi-isomorphism. Since it is induced from a morphism of sheaves. Furthermore since W is a smooth manifold the deRham cohomology of W is isomorphic to the singular cohomology of W. Since W is homotopy equivalent to X then the singular cohomology of W is isomorphic to the singular cohomology of X.$\square$

The following statement summarizes the above work.

Corollary 4.6.3 Let $X \subset \mathbb{R}^n$ be a semi-analytic set, let $W \subset \mathbb{R}^n$ be an open set containing the semi-analytic set X such that X is a deformation retract of W. The quotients map

$$J_{WX} : \Omega^*_{\mathrm{dR}}(W) \to \Omega^*_{\mathrm{W}}(X)$$

is a quasi-isomorphism of commutative graded differential algebras.$\square$

Chapter 5

Real Homotopy Theory

This chapter discusses an application of the above work in the area of real homotopy theory. The first several sections introduce the basic terminology needed to understand the results, and the last section states the main results. Most of the main results have already been proved in the above work, and only need to be interpreted correctly. If one is interested in reading more about real and rational homotopy theory there are several nice resources: Félix, Halperin, and Thomas's book[23], Griffiths, and Morgans' book[16], a very nice paper by Kathyren Hess[8], and many more. The following introduction will introduce only what is needed for this paper and is in no way comprehensive.

5.1 Introduction

Rational homotopy theory can be thought of as homotopy theory modulo torsion. One can pass from the homotopy groups $\pi_* (X)$ of a space X, to the rational homotopy groups $\pi_*^{\mathbb{Q}} (X)$ of X simply by tensoring

$$\pi_*^{\mathbb{Q}} (X) = \pi_* (X) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Of course the problem with this is that in general $\pi_1 (X)$ is not an abelian group. This can be resolved in certain cases [22]. In order to avoid complications it will be assumed from now on that X is simply connected, unless stated otherwise.

Let $\mathbb{K}$ be any field containing $\mathbb{Q}$.

Definition 5.1.1 The $\mathbb{K}$ homotopy type of X is the isomorphism class of graded $\mathbb{K}$ vector space represented by

$$\pi_*(X) \otimes_{\mathbb{Z}} \mathbb{K}.$$

Throughout this paper $\mathbb{K}$ will only be either $\mathbb{Q}$ or $\mathbb{R}$. The symbol $\mathbb{K}$ will be used when the result holds for any field containing $\mathbb{Q}$.

In order to study the real homotopy type of X one must proceed in a rather round about way, via commutative differential graded algebras, Sullivan algebras, and minimal Sullivan algebras.

5.2 Minimal Sullivan Algebras and CDGA's

Definition 5.2.1 A commutative differential graded algebra (CDGA) A over a field $\mathbb{K}$ is graded $\mathbb{K}$-vector space $A = \{A^n\}_{n \in \mathbb{Z}}$, along with a graded multiplication

$$\mu : A^m \otimes_{\mathbb{K}} A^n \to A^{m+n} : a \otimes b \mapsto ab,$$

which is commutative in the sense that if $a \in A^m$ and $b \in A^n$ then

$$ab = (-1)^{mn} ba.$$

Furthermore A is endowed with a differential, which is a $\mathbb{K}$-linear map

$$d : A^n \to A^{n+1} : a \mapsto da,$$

and satisfies the following property: For $a \in A^m$ and $b \in A$

$$d(ab) = (da)b + (-1)^m a(db).$$

That is to say that d is a degree $+1$, $\mathbb{K}$-derivation of A into itself.

There are many examples of CDGA's that arise naturally. The ones that will be of the most interest in this paper are listed below.

Example 5.2.1 Let X be a topological space, the singular cohomology ring $H^*(X;\mathbb{K})$ with coefficients in $\mathbb{K}$ is a CDGA, where the product is given by the cup product, and the differential is zero.

Example 5.2.2 Let M be a smooth manifold, the de Rham complex $\Omega^*_{\mathrm{dR}}(X)$ on X with the wedge product and the exterior derivative is a CDGA over $\mathbb{R}$

Example 5.2.3 Given a locally closed subset A of $\mathbb{R}^n$ the Whitney-de Rham $\Omega^*_{\mathrm{W}}(X)$ complex is a CDGA over $\mathbb{R}$. This is because it is a quotient of the de Rham complex on an open neighborhood of X by a differential graded ideal.

A commutative graded algebra (CGA) is a CDGA with zero differential, and a CDGA can be made into an CGA by forgetting the differential.

Example 5.2.4 Let $V = \{V_k\}_{k\in\mathbb{N}}$ be a graded $\mathbb{K}$-vector space. There is a natural way to build a CGA out of this data. Let

$$TV = \bigoplus_{n\geq 0} V^{\otimes n}$$

be the tensor algebra, with multiplication defined by concatenation of tensors. Define

$$SV = TV/J,$$

Where J is the differential graded ideal of TV generated by all elements of the form

$$a \otimes b - (-1)^{mn}\, b \otimes a \text{ where } a \in V_m, b \in V_n, \forall m, n.$$

Then SV forms a CGA with the multiplication induced from TV.

It should be noted that if V is a differential graded Lie algebra over $\mathbb{K}$, then $S(V^*)$, where V^* is the dual vector space of V, can be endowed with a differential that is induced by the dual of the Lie bracket in V, thereby making it a CDGA. One should consult [21], or [22] for more details.

The above example is needed to define (minimal) Sullivan algebras, which are fundamental in the study of rational homotopy theory.

Definition 5.2.2 A Sullivan CDGA, or simply a Sullivan Algebra, is an CDGA of the form SV for some graded vector space $V = \{V_k\}_{k\in\mathbb{Z}}$ over $\mathbb{K}$ with $V_k = 0$ for all $k < 0$. Furthermore it is required that V have a basis $\{v_\alpha\}_{\alpha\in J}$ where J is a partially ordered set such that for any $\beta \in J$

$$dv_\beta \in S(V_{<\beta}),$$

where $V_{<\beta}$ is the graded vector space spanned by $\{v_\alpha\}_{\alpha<\beta}$.

A Sullivan algebra is called minimal if the following condition is satisfied:

$$d(V) \subset \left(S^+V\right)\left(S^+V\right)$$

Where S^+V consists of all the elements with nonzero tensor degree in SV.

This minimality condition is simply requiring that the image of d be in the image of the multiplication map when it is restricted to elements of strictly positive degree.

Definition 5.2.3 A CDGA $A = \{A^n\}_{n\in\mathbb{Z}}$ is called connected if $A^n = 0$ for all $n < 0$ and $A^0 = \mathbb{K}$. Similarly A is called simply connected, or 1-connected, if $A^1 = 0$.

Definition 5.2.4 Given a CDGA A, a minimal model for A is a minimal Sullivan algebra SV along with a morphism of CDGAs $m_A : SV \to A$ that induces an isomorphism on cohomology groups.

With the above definitions it is now possible to explain why minimal Sullivan algebras are so important.

Proposition 5.2.1 ([23]§13) To any connected, simply-connected CDGA A there exists a minimal model SV_A that is unique in the following sense: If B is any connected, simply-connected CDGA that is homotopy equivalent to A then the minimal model SV_B associated to B is isomorphic to SV_A, by a unique isomorphism

Since homotopy of CDGA's will not be used often in this paper one should consult section 13 of [23] for the precise definition of what it means for two CDGA's to be homotopy equivalent.

The above proposition says that referring to the minimal model of a CDGA is justified without qualification.

5.3 Piecewise Linear Differential Forms

This section describes a canonical way of associating to a topological space a CDGA, $A_{PL}(X)$, called the CDGA of piecewise linear differential forms on X. A good reference for this is [23]§10.

Before defining the CDGA $A_{PL}(X)$ there are several prerequisites that need to be discussed. Namely one first needs to define a simplicial CDGA A_{PL} which should be thought of as the algebra of polynomial differential forms on the standard simplices $|\Delta_\bullet|$ as subsets of $\mathbb{R}^{\bullet+1}$. Then the algebra $A_{PL}(X)$ is the collection of simplicial morphisms from the set of singular simplices in X to A_{PL}. This has the effect of assigning to each simplex in X a polynomial differential form that is compatible with the face and degeneracy maps.

Definition 5.3.1 The category of ordinal numbers Δ is defined to have objects being ordered sets $\Delta_n = \{0 < 1 < 2 < \cdots < n\}$ for each $n \in \mathbb{N}$. Morphisms in this category are order preserving functions.

Definition 5.3.2 Let $\mathcal{C}$ be any category. A simplicial object in C is a functor

$$K : \Delta^o \to \mathcal{C}$$

Where Δ^o is the opposite category of Δ.

Proposition 5.3.1 (Proposition 8.1.3 [27]) To specify a simplicial object in $\mathcal{C}$ it is sufficient to specify an objects K_n and morphisms $\partial_i : K_{n+1} \to K_n$ for $0 \leq i \leq n+1$, and $s_j : K_n \to K_{n+1}$ for $0 \leq j \leq n$ for each $n \in \mathbb{N}$. These maps are required to satisfy the following simplicial relations

$$\begin{aligned}
\partial_i \partial_j &= \partial_{j-1} \partial_i && , i < j \\
s_i s_j &= s_{j+1} s_i && , i \leq j \\
\partial_i s_j &= s_{j-1} \partial_i && , j < i \,\cdot \\
\partial_i s_j &= id && , i = j, j+1 \\
\partial_i s_j &= s_j \partial_{i-1} && , i > j+1
\end{aligned}$$

The maps ∂_i are called face maps, and the maps s_j are called degeneracy maps.

The algebra A_{PL} is a simplicial CDGA. This means that one must specify a CDGA for each $n \in \mathbb{N}$ along with the above maps.

Definition 5.3.3 Fix a natural number n. Define the graded $\mathbb{R}$-vector space $V = \{V_j\}_{j\geq 0}$ with V_0 having basis $\{t_i\}_{i=0}^{n}$, V_1 having basis $\{dt_i\}_{i=0}^{n}$, and $V_j = 0$ for $j \geq 2$. Let

$$A_{PL,n} = SV/I,$$

where I is the differential graded ideal generated by the elements $t_0 + t_1 + \cdots + t_n - 1$ and $dt_0 + dt_1 + \cdots + dt_n$. Define the differential d by requiring that

$$d\,(t_i) = dt_i, \text{ and } d\,(dt_i) = 0.$$

The above construction gives a CDGA for each $n \in \mathbb{N}$. The maps ∂_i and s_j are defined in terms of the basis t_i and dt_i. This can be done as long as the ideal I is in the kernel of the maps.

Definition 5.3.4 Define the CDGA morphisms $\partial_i : A_{PL,n+1} \to A_{PL,n}$ and $s_j : A_{PL,n} \to A_{PL,n+1}$ for $0 \leq i \leq n+1$, and $0 \leq j \leq n$, by defining them on generators as follows

$$\partial_i t_k = \begin{cases} t_k & ,k < i \\ 0 & ,k = i \\ t_{k-1} & ,k > i \end{cases} \text{, and } s_j t_k = \begin{cases} t_k & ,k < j \\ t_k + t_{k+1} & k = j \\ t_{k+1} & k > j \end{cases}.$$

It is easy to see that the above definition is well defined and that ∂_i and s_j do in fact CDGA morphisms. Furthermore one can check that these maps satisfy the simplicial identities. Therefore the above construction defines a simplicial CDGA A_{PL}.

There is a natural simplicial set associated to each topological space. The set of singular simplicies.

Definition 5.3.5 Let X be a topological space. The set of singular simplicies in X is defined for each $n \in \mathbb{N}$ by

$$S_n\,(X) = \{\sigma : |\Delta_n| \to X, \quad \sigma \text{ is continuous }\}.$$

Where $|\Delta_n|$ is the standard simplex in $\mathbb{R}^{n+1}$. The face and degeneracy maps are defined by including $|\Delta_n|$ into $|\Delta_{n+1}|$ as the i-th face, and by collapsing the j-th face of $|\Delta_{n+1}|$ respectively these operations induce maps on $S_n\,(X)$ by precomposing σ with the respective inclusion or quotient, thus defining a simplicial set.

The CDGA of polynomial differential forms on X can now be defined.

Definition 5.3.6 Let X be a topological space then the CDGA of polynomial differential forms on X is defined to be the set of all simplicial homomorphism between $S_\bullet(X)$ and $A_{PL,\bullet}$. That is to say it is the set of morphisms in the category of simplicial sets

$$A_{PL}(X) = \hom_{\Delta_\bullet}(S_\bullet(X), A_{PL,\bullet})$$

Proposition 5.3.2 ([23]§10 – §12) The assignment $X \mapsto A_{PL}(X)$ defines a functor from the category $\mathcal{TOP}_{CW}$, of topological spaces that are homotopy equivalent to a CW-complex, to the category $\mathcal{CDGA}_{\mathbb{R}}$ of commutative differential graded algebras over $\mathbb{R}$. Furthermore this functor is invariant on homotopy classes of topological spaces in the following sense: If X is homotopy equivalent to Y then the CDGAs $A_{PL}(X)$ and $A_{PL}(Y)$ are homotopy equivalent, and hence have isomorphic minimal Sullivan algebras.

Definition 5.3.7 The minimal model for X is a the minimal Sullivan algebra SV_X along with the CDGA morphism $\phi_X : SV_X \to A_{PL}(X)$.

Notice that both SV_S and ϕ_X are uniquely defined up to isomorphism. The connection between minimal models and rational homotopy theory shows up via the graded $\mathbb{R}$-vector space V_X on which the minimal model of X is defined.

Proposition 5.3.3 ([23]§15) The real homotopy groups of a simply connected space of finite type are determined by V_X. In fact

$$V_{X,*} \cong \hom_{\mathbb{R}}(\pi_*(X), \mathbb{R}).$$

It should be noted that the above proposition holds when $\mathbb{R}$ is replaces by $\mathbb{Q}$, but since we are only interested in the real homotopy type of X this is sufficient.

5.4 Main Results

The reason the above statement is so useful is because one may compute SV_X from any CDGA that is quasi-isomorphic to the CDGA $A_{PL}(X)$. This is where the work in this paper comes in. In the 4.6.3 it is shown that $\Omega^*_{\mathrm{W}}(X)$ calculates the singular cohomology of X when X is semi-analytic or a smooth cone space with global singular chart. However what has been shown is that $\Omega^*_{\mathrm{W}}(X)$ is in fact quasi-isomorphic as CDGAs to the deRham complex $\Omega^*_{\mathrm{dR}}(U)$ on an open neighborhood U of X in the ambient space $\mathbb{R}^n$, such that X is a deformation retract of U. The existence of such a U is guaranteed by Corollary 2.8.1.

Theorem 5.4.1 Let X be a semi-analytic set in $\mathbb{R}^n$, or a smooth cone space with a global singular chart into $\mathbb{R}^n$, then the Whitney-deRham complex determines the real homotopy type of X, and conversely the real homotopy groups of X determine the Whitney-deRham complex up to homotopy equivalence of CDGAs.

Proof: As stated in 4.6.3, there is a quasi-isomorphism of CDGA's between $\Omega^*_{\mathrm{dR}}(U)$ and $\Omega^*_{\mathrm{W}}(X)$. Furthermore [23]§11 shows that the complex $\Omega^*_{\mathrm{dR}}(U)$ is quasi-isomorphic as CDGAs to $A_{PL}(U)$. Since U is homotopy equivalent to X then $A_{PL}(U)$ is quasi-isomorphic to $A_{PL}(X)$. Hence there is a chain of quasi-isomorphism of CDGAs between $\Omega^*_{\mathrm{W}}(X)$ and $A_{PL}(X)$. Therefore they have isomorphic minimal models, and thus the desired result.$\square$

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