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Principal bundles, connections, and curvature

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1 Introduction

This thesis is based on lectures from John Morgan in “Gauge Theory and Topology of Four-Manifolds” [FW91], a collection of lectures on the analytic, topological, and differential geometric aspects of gauge theory. This paper serves as a survey of the preliminary ideas and concepts that are essential to the understanding of gauge theory. This is an attempt to fill in details that are left out in the notes and expand on ideas with an undergraduate-level audience in mind.

This paper will cover concepts and major results regarding principal bundles, and connections, and briefly touch on curvatures. The first section will introduce the definition of principal bundles and certain properties with examples that are often seen in the context of gauge theory. The second section focuses on the 3 types of connection introduced in the lectures: geometric connection, connection as a one form, and covariant derivative. The majority of the section is spent on developing the relationship between geometric connections and their corresponding one-forms and understanding the space of connection one-forms. Lastly, we introduce the idea of curvatures on principal bundles with respect to the three types of connections.

In this thesis, we assume an understanding of some differential geometry such as manifolds, tangent spaces, and differential forms, and fiber bundles. Some books I used to understand some background in differential geometry include “Introduction to Smooth Manifolds” by John Lee [Lee00], “Manifolds and Differential Geometry” by Jeffrey Lee [Lee09], “Topics in differential geometry” by Peter Michor [Mic08].

2 Principal Bundles

2.1 Principal Bundles

Definition 2.1. *([FW91]Def. 2.2.1) A (right) principal G-bundle consists of a triple* (P, B, π) *where* $\pi : P \to B$ *is a map and a continuous, free right action* $P \times G \to P$ *with respect to which* π *is invariant and so that* π *induces a homomorphism between the quotient space of this action and B. Furthermore, there is an open covering* $\{U_\alpha\}$ *of B over which all the above data are isomorphic to the product data. That is to say for each* α *there exists a commutative diagram*

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\phi_\alpha} & U_\alpha \times G \\ \big\downarrow{\scriptstyle \pi} & & \big\downarrow{\scriptstyle p_1} \\ U_\alpha & \xrightarrow{=} & U_\alpha \end{array} \tag{1}$$

where ϕ_α *is a homomorphism that is equivariant with respect to the right G- actions and* p_1 *is the projection onto the first factor.*

Other sources may replace the condition of the induced homoeomorphism between the quotient space and B with the transitive property of the action. The action's transitivity leads to the "collapsing" of the fiber of (P, B, π); if we consider the quotient space $P/\sim_G$, where $p \sim p'$ if $p = p'g$ for some $g \in G$, since the action of G is transitive, we have that

$$P/\sim_G \cong B$$

as for all $p \in \pi^{-1}(b)$, we have that $p \cdot G = \{p \cdot g | \forall g \in G\} = \pi^{-1}(b)$.

For a trivial example, consider the topological product $\{1, -1\} \times S^1$ over the space S^1. Let $g = O(1)$, then this is a trivial principal bundle.

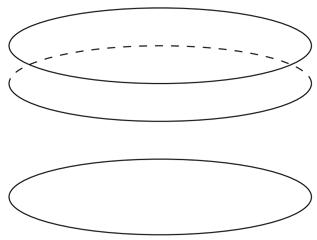


Figure 1: trivial $O(1)$ bundle over S^1

We can construct a nontrivial example by having also S^1 as the base space and the set $\{-1, 1\}$ as fiber over S^1 with one twist by gluing by -1.

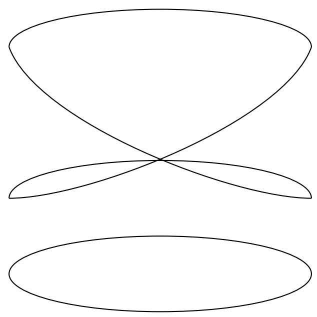


Figure 2: nontrivial $O(1)$ bundle over S^1

This space also admits a group action from $O(1)$.

For any manifold M, one can construct a natural principal bundle, the frame bundle, from the tangent bundle TM. Let $\pi : TM \to M$ be the projection map. Take $x \in M$ and note $\pi^{-1}(x) = T_x M \subset TM$ is a real n dimensional vector space. Let B_x be the set of all bases and $\beta_x = \{v_1, \cdots, v_n\}$ be an element of B_x denoting a set of basis vectors for $T_x M$, then construct a new principal bundle $\tilde{\pi} : E \to M$ where E is the new total space such that

$$E|_x = \tilde{\pi}^{-1}(x) = \{(x, \beta_x) | \text{ for all } \beta_x \in B_x\}$$

is the data over each point of M. Since $T_x M$ is a real vector space, there is a natural action from $GL(n, \mathbb{R})$ on B_x (linear functions map basis to basis). and we know this action to be free and transitive.

Another important notion of principal bundles is that they naturally determine their associated vector bundles.

Let $\pi : P \to B$ be a principal bundle with a G action. Let V be a vector space of which G has a natural action; if G was a linear lie group embedded into $GL(n, \mathbb{R})$, then V can be the vector space $\mathbb{R}^n$. Let G act on the left on the product

$$G \times (P \times V) \to P \times V$$

$$g \cdot (p, v) = (p \cdot g^{-1}, g \cdot v),$$

and then consider the quotient space of $P \times V$ by the action of G action:

$$P \times_G V := (P \times V)/G = (P \times V)/\sim$$

$$(p, v) \sim (pg^{-1}, gv), \ g \in G.$$

We denote the orbits by

$$[(p, v)] = \{G \cdot (p, v)\}$$

We also have a natural projection map

$$\pi : P \times_G V \to B$$

$$\pi([(p, v)]) = \pi(p)$$

The well-definedness of π is obvious as the action preserves the fiber; i.e., if $[(p_1, v_1)] = [(p_2, v_2)]$, then there is some $g \in G$ such that $p_2 = p_1 g$ and $v_2 = g v_1$, and then $\pi(p_2) = \pi(p_1 g) = \pi(p_1)$.

We can then claim that $\pi : P \times_G V \to B$ is a vector bundle. To show that such space has the appropriate trivialization we need the result of the following lemma:

Lemma 2.2. *Let M, $\tilde{M}$, and N be manifolds such that G acts on the right on M and $\tilde{M}$, and acts on the left on N. If we have an isomorphism*

$$\phi : M \longrightarrow \tilde{M}$$

that is equivariant with respect to the G action, then there is an isomorphism

$$\psi : M \times_G N \longrightarrow \tilde{M} \times_G N$$

$$[(m, n)] \mapsto [(\phi(m), n)]$$

Proof. We begin by checking the well definedness of ψ. Consider $[(m_1, n_1)] = [(m_2, n_2)] \in M \times_G N$, so $(m_1, n_1) = (m_2 g^{-1}, g n_2)$ for some $g \in G$. Then,

$$\psi([(m_1, n_1)]) = [(\phi(m_1), n_1)]$$

and

$$\psi([(m_2, n_2)]) = [(\phi(m_2), n_2)]$$

and given that ϕ is equivariant to the group action,

$$\phi(m_1) = \phi(m_2 g^{-1}) = \phi(m_2) g^{-1}$$

we have that $(\phi(m_1), n_1) = g(\phi(m_2), n_2)$, so $[(\phi(m_1), n_1)] = [(\phi(m_2), n_2)]$, meaning ψ is indeed well-defined.

To see the injectivity of ψ, let $\psi[(m_1, n_1)] = \psi[(m_2, n_2)]$, so

$$[(\phi(m_1), n_1)] = [(\phi(m_2), n_2)]$$

meaning

$$(\phi(m_1), n_1) = (\phi(m_2) g^{-1}, g n_2)$$

once again, by the equivariance of ϕ, we have

$$\phi(m_1) = \phi(m_2)g^{-1} = \phi(m_2 g^{-1})$$

and since ϕ is an isomorphism, we can claim that $m_1 = m_2 g^{-1}$ implying that $(m_1, n_1) = g(m_2, n_2)$, or equivalently $[(m_1, n_1)] = [(m_2, n_2)]$.

For surjectivity, take $[(m_2, n_2)] \in \tilde{M} \times_G N$, since ϕ is an isomorphism, there is some $m_1 \in M$ such that $\phi(m_1) = m_2$, so $\psi([(m_1, n_1)]) = [(m_2, n_2)]$.

We then have that ψ is an isomorphism. □

Proposition 2.3. *The associated bundle $\tilde{\pi} : P \times_G V \to B$ is a vector bundle. On top of that, the transition functions for the associated vector bundle is the same as the transition functions for P.*

Proof. First, notice that

$$\tilde{\pi}^{-1}(U_\alpha) = \{[(p, v)] \mid \pi(p) \in U_\alpha\}$$

$$= \pi^{-1}(U_\alpha) \times_G V \tag{2}$$

By the triviality of the principal bundle from (1), we have $\phi_\alpha : \pi^{-1}(U_\alpha) \xrightarrow{\sim} U_\alpha \times G$, and by the last lemma, we have

$$\psi_\alpha : \tilde{\pi}^{-1}(U_\alpha) \xrightarrow{\sim} (U_\alpha \times G) \times_G V \tag{3}$$

$$[(p, v)] \mapsto [(\phi_\alpha(p), v)]$$

Next, we want to show that there is an isomorphism

$$\tilde{\phi}_\alpha : U_\alpha \times V \xrightarrow{\sim} (U_\alpha \times G) \times_G V \tag{4}$$

$$(b, v) \mapsto [(b, e), v]$$

To prove injectivity, say $\tilde{\phi}_\alpha(b_1, v_1) = \tilde{\phi}_\alpha(b_2, v_2)$, so $[(b_1, e), v_1] = [(b_2, e), v_2]$. Since G acts trivially on B, $b_1 = b_2$. If for some $g \in G$, we have $eg^{-1} = e$, then $g = e$, so $e[(b_1, e), v_1] = [(b_2, e), v_2]$, meaning $v_1 = v_2$. So $\tilde{\phi}_\alpha$ is injective.

To show surjectivity, take $[(b, g), v] \in (U_\alpha \times G) \times_G V$. Since $[(b, g), v] = [(b, e), gv]$, we know that $\tilde{\phi}_\alpha(b, gv) = [(b, g), v]$, so this map is also surjective, which prove that $\tilde{\phi}_\alpha$ is an isomorphism.

Altogether (2), (3), and (4) show that the bundle is indeed a vector bundle where the compositions

$$\pi_a^{-1}(U_\alpha) \xrightarrow[\sim]{\psi_\alpha} (U_\alpha \times G) \times_G V \xrightarrow[\sim]{\phi_\alpha'^{-1}} U_\alpha \times V \tag{5}$$

$$[(p, v)] \mapsto [(\phi_\alpha(p), v)] = [(\pi(p), g), v] \mapsto [\pi(p), v]$$

are the trivializations $P \times_G V$.

Now let U_α and U_β be open sets with nontrivial intersection, then

$$\phi_\alpha \phi_\beta^{-1} : U_\alpha \bigcap U_\beta \times G \to U_\alpha \bigcap U_\beta \times G \tag{6}$$

$$(b, g) \mapsto (b, \Phi_{\alpha\beta}(b) g) \tag{7}$$

where $\Phi_{\alpha\beta} : U_\alpha \bigcap U_\beta \to G$ is the transition function for P over $U_\alpha \bigcap U_\beta$. Similarly, consider:

$$\phi'_\alpha \phi_\beta'^{-1} : (U_\alpha \bigcap U_\beta \times G) \times_G V \to (U_\alpha \bigcap U_\beta \times G) \times_G V$$

knowing that $(b, g) \mapsto (b, g(\Phi_{\alpha\beta}(b))$ from (7), we can say

$$[(b, g), v] \mapsto [(b, g\Phi_{\alpha\beta}(b)), v] \tag{8}$$

Rewriting (8) as corresponding elements in $(U_\alpha \bigcap U_\beta) \times V$ with respect to the trivialization in (5):

$$(b, gv) \mapsto (b, \Phi_{\alpha\beta}(b)gv)$$

So the transition function for $\pi_a : P \times_G V \to B$ over $U_\alpha \bigcap U_\beta$ is $\tilde{\Phi}_{\alpha\beta} : b \mapsto \Phi_{\alpha\beta}(b)$ $\square$

3 Connection

3.1 Geometric connection

The most geometrically intuitive way of understanding connections is by the notion of distributions, or horizontal bundles. Recall that a *distribution* $\mathcal{D}$ on a manifold M is a sub-vector bundle of the tangent bundle TM, i.e., $\mathcal{D} \subseteq TM$. In other words a distribution is a smoothly varying family of subspaces of the tangent spaces to M. If $f : M \to N$ is a C^∞ morphism of manifolds, then a distribution $\mathcal{D} \subseteq TM$ is called *horizontal* if the composition $\mathcal{D} \hookrightarrow TM \stackrel{Tf}{\to} f^*TN = TN \times_N M$ is an isomorphism.

Definition 3.1 (Geometric connection)**.** *([FW91]p.65) A geometric connection on a principal G-bundle $\pi : P \to B$ is a horizontal distribution $\mathcal{H}$ on P that is invariant under the action of G.*

We now explain what we mean by $\mathcal{H}$ being invariant under the actin of G. We are given the action

$$P \times G \to P.$$

Taking derivatives, we have

$$TP \times TG = TP \times (T_eG \times G) \longrightarrow TP$$

and restricting to $TP \times G = TP \times \{0\} \times G$ gives the action

$$TP \times G \to TP.$$

In other words, for each $g \in G$, we have an isomorphism $g : P \to P$, and taking derivatives we have an associated map $TP \to TP$. To say that $\mathcal{H}$ is invariant under the action of G we mean that the following diagram commutes:

$$\begin{array}{ccc} TP \times G & \longrightarrow & TP \\ \uparrow & & \uparrow \\ \mathcal{H} \times G & \longrightarrow & \mathcal{H} \end{array}$$

In other words for each $p \in P$ and each $g \in G$, we have

$$\mathcal{H}_p \cdot g = \mathcal{H}_{pg} \tag{9}$$

Another equivalent definition comes naturally after defining a vertical bundle.

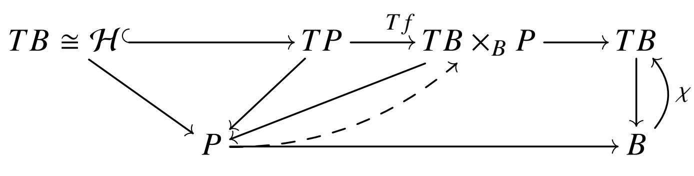
Definition 3.2 (vertical tangent bundle)**.** *([FW91]Lemma 2.7.1) Let $P \to B$ be a smooth fiber bundle. The vertical tangent bundle TP^V is the kernel of differential $T\pi : TP \to TB$.*

A connection $\mathcal{H}$ is an invarriant complement of TP^V in the sense that for each $p \in P$, we have

$$T_pP = T_pP^V \oplus \mathcal{H}_p \tag{10}$$

.

With the isomorphism between $\mathcal{H}$ and π^*TB, given a vector field on B, we get a vector field on P. Since vector fields can be integrated, this means that given a smooth path $\gamma : [0,1] \to B$, then restricting the principal bundle to γ, we obtain a vector field, and so given any lift of $\gamma(0)$, there is a unique path lifting γ to P passing through that point, and tangent to the vector field at each point.



Lemma 3.3 ([FW91, Def. 2.7.2])**.** *Suppose that $\mathcal{H}$ is a connection for $P \to B$. Let $\gamma : [0,1] \to B$ be a smooth path and $e \in \pi^{-1}(\gamma(0))$. Then there is a unique path $\tilde{\gamma} : [0,1] \to P$ such that $\tilde{\gamma}(0) = e$, $\pi \circ \tilde{\gamma}$, and $\tilde{\gamma}'(t)$ is contained in the horizontal space $\mathcal{H}_{\tilde{\gamma}(t)}$ for all $t \in [0,1]$.*

The existence of a unique lifting is a result from ODE.

Consider p_0 and $p_0 \cdot g$ in $\pi^{-1}(b_0)$ and a path γ from b_0 to b_1 in B, from the lemma we know that there is a unique path $\tilde{\gamma}$ from p_0 to a unique $p_1 \in \pi^{-1}(b_1)$. Put $\tilde{\gamma}_g = \tilde{\gamma} \cdot g$ where the group action is carried out pointwise. We know that $\tilde{\gamma}_g$ is still continuous since the group action is required to be continuous, and it is a path starting at $\tilde{\gamma}(0) \cdot g = p_0 \cdot g$, and since G actions preserve fibres, $\pi \circ \tilde{\gamma}_g = \gamma$, so $\tilde{\gamma}_g$ is in fact the unique lifted path starting at $p_0 \cdot g$.

With this lemma, one can see that a connection determines an isomorphism between fibers $\pi^{-1}(b_0) \to \pi^{-1}(b_1)$ for any $b_0, b_1 \in B$; let γ be a path between b_0 and b_1, fixing $p_0 \in \pi^{-1}(b_0)$, from the last lemma we know that there is a unique path $\tilde{\gamma}_0$ such that $\tilde{\gamma}(1) = p_1$ which decides a unique $p_1 \in \pi^{-1}(b_1)$ such that $\tilde{\gamma}(1) = p_1$. By letting G act on $\tilde{\gamma}$ we get an isomorphism from $\pi^{-1}(b_0)$ to $\pi^{-1}(b_1)$ where $\tilde{\gamma}(0)g \to \tilde{\gamma}(1)g$ for all $g \in G$.

3.2 Connection as a one form

An equivalent way of looking at connections is through a Lie algebra valued one form with specific properties, one of which involves a unique one form, the Maurer-Cartan form ω_{mc}:

Definition 3.4. *The Maurer-Cartan form $\omega_{mc} \in \Omega^1(G, \mathcal{G})$ is a unique one form invariant under left multiplication by G and whose value at the identity of G is the identity map from $TG_e \to \mathcal{G}$. This form is sometimes denoted $g^{-1}dg$.*

Lemma 3.5 ([FW91, Lemma 2.8.1]). *A connection on a smooth principal bundle $\pi : P \to B$ is equivariant to a differential one form $\omega \in \Omega^1(P, \mathcal{G})$ with the following properties:*

- *Under right multiplication by G the form ω transforms via the adjoint representation of G on $\mathcal{G}$;*

$$\omega_{p \cdot g}(\tau \cdot g) = g^{-1}\omega_p(\tau) \cdot g \tag{11}$$

- *For any $p \in P$, consider the embedding $R_p : G \to P$ given by $R_p(g) = p \cdot g$. Then the pullback $R_p^*(\omega) = \omega_{MC}$.*

Proof. Let us start with a one form with the given properties. Let $\mathcal{H}_p$ denote the kernel of the linear map $\omega_p : T_pP \to \mathcal{G}$. Consider the left exact sequence

$$0 \to \mathcal{H}_p \hookrightarrow T_pP \xrightarrow{\omega_p} \mathcal{G}$$

where the exactness comes from the definition of $\mathcal{H}_p$. From the local trivialization in (1)

$$\phi_\alpha : P_{U_\alpha} \cong U_\alpha \times G$$

taking the differential at p yields

$$T_pP \cong T_bB \times \mathcal{G} \tag{12}$$

Now, consider the differential of the map R_p at e, which gives $TR_p|_e : T_eG \to T_pP$, and notice that $\omega_p \circ TR_p|_e = R_p^*(\omega) = Id_{\mathcal{G}}$ since $R_p^*(\omega) = \omega_{MC}$, which implies that ω_p is surjective.

Since $T_pP \cong T_bB \times \mathcal{G}$, the form ω_p must map T_bB to 0 in $\mathcal{G}$, so $\mathcal{H}_p \cong T_bB$, meaning that $\mathcal{H}_p$ is "horizontal".

From the first condition in (11), we can infer that

$$\omega_{pg}(\tau g) = 0 \Longleftrightarrow g^{-1}\omega_p(\tau)g = 0 \Longleftrightarrow \omega_p(\tau) = 0$$

with the last implication given by the fact that if $g^{-1}\omega_p(\tau)g = 0$ then $\omega_p(\tau) = g^{-1} \cdot 0 \cdot g = 0$ and conversely if $\omega_p(\tau) = 0$ then of course $g^{-1}\omega_p(\tau)g = 0$. Therefore there is a one-to-one correspondence between $ker(\omega_{pg})$ and $ker(\omega_p)$ (specifically $ker(\omega_{pg}) = ker(\omega_p)g$), giving us the G invariance of a connection. We then conclude that $\mathcal{H}_p$ forms a connection.

Conversely, given a connection, define $\omega_p : T_pP \to \mathcal{G}$ to be

$$T_pP \xrightarrow{Pr_p} T_pP^V \xrightarrow{(TR_p)^{-1}} \mathcal{G} \tag{13}$$

where Pr_p is projection into the vertical tangent space T_pP^V with kernel $\mathcal{H}_p$.

We first check that such ω_p transforms under the adjoint representation as described in (11):

$$\omega_{p\cdot g}(\tau \cdot g) = g^{-1}\omega_p(\tau) \cdot g$$

Let $L_g : G \to G$ denote left multiplication on G, then based on the definition given, we have that $R_{pg}(g') = pgg'$, but notice also

$$R_p \circ L_g(g') = pgg'$$

for all $g' \in G$, so $R_{pg} = R_p \circ L_g$, which implies that $R_{pg}^{-1} = L_g^{-1} \circ R_p^{-1}$, and we conclude that

$$(TR_{pg})^{-1} = TL_g^{-1} \circ TR_p^{-1} \tag{14}$$

so we have

$$(TR_{pg})^{-1}(Pr)(\tau g) = g^{-1}(TR_p)^{-1}Pr(\tau g) \tag{15}$$

And we know that $\tau = v^{ver} + v^{hor}$ where $v^{ver} \in T_pP^v$ and $v^{hor} \in \mathcal{H}_p$ from (10), so $\tau g = v^{ver}g + v^{hor}g$. Since connections are G equivariant per (9) we have $v^{hor}g \in \mathcal{H}_{pg}$ and it is always true that $v^{ver}g \in T_{pg}P^v$, which implies that $Pr(\tau g) = Pr(\tau)g$. Combine that with (14) and (15):

$$\omega_{pg}(\tau g) = g^{-1}TR_p(Pr(\tau)g)$$

As we know that TR_p and group actions commute, we conclude that

$$\omega_{pg}(\tau g) = g^{-1}TR_pPr(\tau)g$$

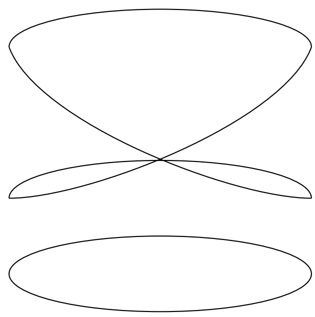
Next, we want to show that $R_p^*(\omega) = \omega_{MC}$. We show this by checking that $R_p^*(\omega)$ is invariant under left multiplication by G and has value at $e \in G$ is equaled to the identity map $I_{\mathcal{G}} : T_eG = \mathcal{G} \to \mathcal{G}$

Consider the sequence

$$\mathcal{G} \xrightarrow{TR_p} \mathcal{T}_pP \xrightarrow{Pr} T_pP^v \xrightarrow{(TR_p)^{-1}} \mathcal{G}$$

We know that $TR_p(\mathcal{G}) = T_pP^v$ since vertical tangent vectors are invariant to group actions, so $R_p^*(\omega) = TR_p \circ (TR_p)^{-1} = Id_{\mathcal{G}}$ □

Consider connections on example (2):



One can tell by looking at the figure that the horizontal distribution $\mathcal{H}$ is equal to the tangent bundle TP and since from Lemma 3.5 we know that the horizontal distribution is the kernel of the connection one-form, here we will have a trivial connection one-one. Next, we move on to discussing the space of connections and its structure.

Definition 3.6. *Let M be a manifold and V be some vector space. A k-form $\omega \in \Omega^k(M, V)$ is horizontal if for all $x \in M$ and $\{v_1, \cdots, v_k\} \in T_xM^v$ one has $\omega_x(v_1, \cdots, v_k) = 0$.*

Lemma 3.7. *Let $\pi : P \to B$ be a principal bundle with adjoint bundle* ad P. *Given $\eta \in \Omega^1(P, \mathcal{G})$, there is $\omega \in \Omega^1(B, \mathrm{ad}P)$ with $\pi^*(\omega) = \eta$ if and only if η is horizontal and transforms via adjoint representation of G under right multiplication.*

Proof. To be specific, $\pi^*(\omega) \in \Omega^1(P, \pi^*(\mathrm{ad}P))$ where $\pi^*(\mathrm{ad}P)$ is the pull back of ad$P$

therefore we need

$$Pr_p : \pi^*(\mathrm{ad}P)_p \to \mathcal{G}$$

to project onto the Lie algebra. So more precisely we have

$$\eta = Pr\pi^*(\omega) \in \Omega(P, \mathcal{G}) \tag{16}$$

Let us first show that elements in $\pi^*(\mathrm{ad}P)$ transform under the adjoint representation as well. Consider a locally trivial principal bundle $\pi : P \to B$, so $P = B \times G$. Then $\mathrm{ad}P = (B \times G) \times_G \mathcal{G} = B \times \mathcal{G}$. Consider

$$\begin{array}{ccc} \pi^*(\mathrm{ad}P) & \xrightarrow{\pi^*} & \mathrm{ad}P \\ {\scriptstyle \tilde{Pr}}\downarrow & & \downarrow{\scriptstyle Pr} \\ P & \xrightarrow{\pi} & B \end{array}$$

where $\pi^*(\mathrm{ad}P) = P \times_B \mathrm{ad}P = B \times G \times \mathcal{G}$, since $P \to B$ is assumed to be trivial.

Consider $\iota_p : \mathcal{G} \to \mathrm{ad}P$ where $\iota(v) = [p, v]$. Let us start with a one form $\eta \in \Omega^1(P, \mathcal{G})$ such that $\eta_{pg}(\tau g) = g^{-1}\eta_p(\tau)g$. For each $b \in B$ and $v \in T_bB$ pick some $p \in \pi^{-1}(b)$ and $\tau \in T\pi_p^{-1}(v)$. Then define $\omega \in \Omega^1(B, \mathrm{ad}P)$ pointwise:

$$\omega_b(v) = \iota_p(\eta_p(\tau)) \tag{17}$$

so

$$\pi^*(\omega)_p(\tau) = \omega_{\pi(p)}(T\pi_p(\tau)) = \omega_b(v) = \iota(\eta_p(\tau))$$

We then verify that this is independent of choices made in the first step. Let $\tilde{\tau} \in T\pi_p^{-1}(v)$; we want that

$$\omega_{\pi(p)}(T\pi_p(\tau)) = \omega_{\pi(p)}(T\pi_p(\tilde{\tau}))$$

Recall that $T_pP = T_{\pi(p)}B \oplus T_pP^V$, so $\tau = \tau^{hor} + \tau^{ver}$ and $\tilde{\tau} = \tilde{\tau}^{hor} + \tilde{\tau}^{ver}$ where τ^{hor} and $\tilde{\tau}^{hor} \in T_{\pi(p)}B$ and τ^{ver} and $\tilde{\tau}^{ver} \in T_pP^V$. But since $\tau^{hor} = T\pi_p(\tau) = T\pi_p(\tilde{\tau}) = \tilde{\tau}^{hor}$ and η vanishes on vertical tangent vectors, we have that $\omega_{\pi(p)}(T\pi_p(\tau)) = \omega_{\pi(p)}(T\pi_p(\tilde{\tau}))$.

Next, consider p and $\tilde{p} \in \pi^{-1}(b)$, given by the transitivity of the G action, there is some $g \in G$ such that $\tilde{p} = pg$. Since $\eta_{\tilde{p}}(\tau g) = g^{-1}\eta_p(\tau)g$, and

$$\iota_{\tilde{p}}(\eta_{\tilde{p}}(\tau g)) = [pg, g^{-1}\eta_p(\tau)g] = [p, \eta_p(\tau)] = \iota_p(\eta_p(\tau))$$

so the definition in (17) does not depend on p or τ.

Then, let $\eta \in \Omega^1(P, \mathcal{G})$ such that there is $\omega \in \Omega^1(B, \mathrm{ad}P)$ where $\eta = \pi^*(\omega)$. We want that $\eta_{pg}(\tau g) = g^{-1}\eta_p(\tau)g$, or $\pi^*\omega_{pg}(\tau g) = g^{-1}\pi^*\omega_p(\tau)g$. To be more precise, $\pi^*(\omega) \in \pi^*(\mathrm{ad}P)$, therefore we will need a projection map $i_p : \pi^*(\mathrm{ad}P)_p \longrightarrow \mathcal{G}$ so that $\eta = i\pi^*(\omega) \in \Omega(P, \mathcal{G})$.

Consider

$$\pi^*\omega_p(\tau) = (p, \omega_{\pi(p)}T\pi_p(\tau)) = (p, [p, v_p]) \in \pi^*(\mathrm{ad}P)$$

then

$$\pi^*\omega_{pg}(\tau g) = (pg, \omega_{\pi(pg)}T\pi_{pg}(\tau g)) = (pg, [pg, v_{pg}])$$

But notice that since $\pi \circ R_g = \pi$ as π is G invariant, by taking the derivative on both sides we have that $T\pi \circ TR_g = T\pi$, meaning $T\pi_{pg}(\tau g) = T\pi_{pg}TR_g|p(\tau) = T\pi_p(\tau)$.

This implies that $\omega_{\pi(pg)}(T\pi_{pg}(\tau g)) = \omega_{\pi(p)}(T\pi_p(\tau)$, so $[p, v_p] = [pg, v_{pg}]$. From the equivalence relation in ad$P$, this implies that $v_{pg} = g^{-1}v_pg$. Consequently,

$$\pi^*\omega_{pg}(\tau g) = (pg, [pg, g^{-1}v_pg])$$

Then

$$\eta_p(\tau) = Pr_p(p, [p, v_p]) = v_p$$

$$\eta_{pg}(\tau g) = Pr_{pg}(pg, [pg, g^{-1}v_pg]) = g^{-1}v_pg$$

so

$$\eta_{pg}(\tau g) = g^{-1}\eta_p(\tau)g$$

We conclude that there is a one-to-one correspondence between forms in $\Omega^1(B, \mathrm{ad}P)$ and horizontal forms that transform under the adjoint representation in $\Omega^1(P, \mathcal{G})$ □

Lemma 3.8 ([FW91, Lemma. 2.9.2]). *If P has a connection, then the space of all connections on P is an affine space; the underlying vector space is* $\Omega^1(B, \mathrm{ad}P)$*, the space of one-forms on B with values in the vector bundle* $\mathrm{ad}P$.

Proof. Consider two connection one forms ω_1 and ω_2, both of which satisfy the two conditions in lemma 3.8, then from the first condition we get that

$$(\omega_1 - \omega_2)_{pg}(\tau g) = g^{-1}((\omega_1 - \omega_2)_p(\tau))g$$

since ω_1 and ω_2 each transforms under the adjoint representation.

Recall that $\omega_{1p} \circ TR_p|_e = Id_{\mathcal{G}}$ and $\omega_{2p} \circ TR_p|_e = Id_{\mathcal{G}}$. Then for each $v \in T_pP^V$ we have that $v = TR_p|_e(u_v)$ for some $u_v \in \mathcal{G}$ and

$$\omega_{1p}(v) = \omega_{1p}(TR_p|_e(u_v)) = u_v$$

and similarly

$$\omega_{2p}(v) = \omega_{2p}(TR_p|_e(u_v)) = u_v$$

therefore $(\omega_{1p} - \omega_{2p})(v) = 0$ for all $v \in T_pP^V$

So we have that the difference of two connection forms transforms under the adjoint representation and is horizontal. So we can apply lemma 3.10 and conclude that connection one forms from an affine vector space over $\Omega^1(B, \mathrm{ad}P)$ □

One can always construct a global connection one form by the local triviality of principal bundles. The method is illustrated in the lemma below.

Lemma 3.9 ([FW91, Lemma. 2.9.23]). *Any smooth principal bundle* $P \to B$ *has a connection A. The space of all connections is an affine space whose underlying vector space is identified with* $\Omega^1(B, \mathrm{ad}P)$

Proof. Let $\{U_\alpha\}$ be an open cover of B over which P is trivial. Since

$$P|_{U_\alpha} \xrightarrow[\sim]{\phi_\alpha} U_\alpha \times G$$

which implies that

$$TP|_{U_\alpha} \xrightarrow[\sim]{T\phi_\alpha} TB|_{U\alpha} \times \mathcal{G}$$

Therefore $T\phi_\alpha^{-1}(TB|_{U\alpha})$ is a horizontal distribution, and by lemma 3.8, this gives rise to a local connection one form ω_α on U_α. Let $\{\lambda_\alpha\}$ be a partition of unity subordinate to the covering $\{U_\alpha\}$. Then form

$$\omega = \Sigma_\alpha \lambda_\alpha \omega_\alpha$$

which is a connection one form on all of P. □

3.3 Connection as covariant derivative

Suppose that ω is a connection one-form on a principal bundle $\pi : P \to B$, and suppose that $W \to B$ is a vector bundle associated to this principal bundle and a linear action of G on a vector space V. We can use the connection to differentiate sections of W, producing one-forms with values in W. This covariant differentiation is a linear operator denoted as $\nabla_A : \Omega^0(B, W) \to \Omega^1(B, W)$. This is defined on some section σ of $W \to B$ where

$$\sigma(b) = [p(b), v(b)] \tag{18}$$

with p being a section of $P \to B$ and v being a smooth function from B to V. Then $\nabla_A(\sigma)(b)$ evaluated on $\tau_b \in T_bB$ is equal to

$$[p(b), \omega_{p(b)}(Tp_b(\tau_b))(v(b)) + Tv_b(\tau_b)] \tag{19}$$

Notice that $Tp_b(\tau_b)$ is an element of $T_{p(b)}P$, and $\omega_b(Tp_b(\tau_b))$ is an element of $\mathcal{G}$, and since we are using some representation $\rho : G \to Aut(V)$, by taking the derivative $T\rho_e$:

$\mathcal{G} \to End(V)$, by which $\omega_b(Tp_b(\tau_b))$ acts on $v(b)$.

One can check that this expression does not depend on the choice of p or v for (18). Consider some $\tilde{p}$ and $\tilde{v}$ such that $[\tilde{p}(b), \tilde{v}(b)] = [p(b), v(b)]$. This implies that there is some $g \in G$ such that

$$\tilde{p}(b) = p(b)g \tag{20}$$

and

$$\tilde{v}(b) = g^{-1}v(b)g \tag{21}$$

(since $[p, v] = [pg, g^{-1}vg]$). Then, following the definition of the covariant derivative from (19)

$$\nabla_A(\sigma)(\tau_b) = [\tilde{p}(b), \omega_{\tilde{p}(b)}T\tilde{p}_b(\tau_b)(\tilde{v}(b)) + T\tilde{v}_b(\tau_b)]$$

To unpack the expression, first notice that since $\tilde{p} = pg$, so $T\tilde{p} = Tpg$, and because $\tilde{p}(b) = p(b)g$, we have that

$$\omega_{\tilde{p}(b)}T\tilde{p}_b(\tau_b) = \omega_{p(b)g}(Tp_b((\tau_b)g))$$

From the properties of a connection one form from (11), we have that

$$\omega_{p(b)g}(Tp_b((\tau_b)g)) = g^{-1}\omega_{p(b)}Tp_b(\tau_b)g$$

From (21), we have $\tilde{v}(b) = g^{-1}v(b)g$, so

$$\omega_{\tilde{p}(b)}T\tilde{p}_b(\tau_b)(\tilde{v}(b)) = g^{-1}\omega_{p(b)}Tp_b(\tau_b)gg^{-1}v(b)g = g^{-1}\omega_{p(b)}Tp_b(\tau_b)(v(b))g$$

Similarly, since $\tilde{v}(b) = g^{-1}v(b)g$, we have $T\tilde{v}(b) = g^{-1}Tv(b)g$, so

$$[\tilde{p}(b), \omega_{\tilde{p}(b)}T\tilde{p}_b(\tau_b)(\tilde{v}(b)) + T\tilde{v}_b(\tau_b)] = [p(b)g, g^{-1}\omega_{p(b)}Tp_b(\tau_b)(v(b))g + g^{-1}Tv(b)g]$$

by the construction of the associated vector bundle

$$[p(b), \omega_{p(b)} T p_b(\tau_b)(v(b)) + T v_b(\tau_b)] = [p(b)g, g^{-1}\omega_{p(b)} T p_b(\tau_b)(v(b))g + g^{-1} T v(b) g]$$

We conclude that the choice of p and v does not matter.

Another important property of the covariant derivative is that given any section $\sigma : B \to W$ and real-valued function f on B the covaraint derivative satisfies:

$$\nabla_A(f\sigma) = f\nabla_A(\sigma) + df \otimes \sigma$$

4 Curvature

4.1 Geometric Curvature

We begin with a geometric understanding included in Morgans's notes. Let $P \to B$ be a smooth principal G bundle and let ad$P$ be the vector bundle associated to P and the adjoint action of G on its Lie algebra $\mathcal{G}$. Suppose that A is a connection on P which we view as a horizontal distribution. Now fix some $b \in B$ and two linearly independent tangent vectors τ_1, τ_2 at B. Consider local coordinate system $(\chi_1, \cdots, x_k)$ centered at b such that $\partial/\partial\chi_1|_0 = \tau_1$ and $\partial/\partial\chi_2|_0 = \tau_2$. Then consider the rectangle $[0,\epsilon] \times [0,\epsilon]$ in the (χ_1, χ_2) subspace. We lift the four sides of this rectangle in a counterclockwise fashion beginning with the side on the χ_1- axis. Let $p \in P$ be the point to which the initial point lifts. There is no guarantee that the end of the last side will be equal to p, but it will lie within the same fiber and therefore can be expressed at $p \cdot g$ for some $g \in G$, and more specifically, this g depends on ϵ, as one can imagine the smaller ϵ the closer to e our g will be, therefore we denote it as $g(\epsilon)$. Then consider

$$K_A(\epsilon) = -\frac{log(g(\epsilon))}{\epsilon^2}$$

which is an element of the Lie algebra.

Lemma 4.1 ([FW91, Lemma. 3.1.1])**.** *The element in $\mathcal{G}$ given by*

$$K_A(p, \tau_1, \tau_2) = lim_{\epsilon \to 0} K_A(\epsilon)$$

depends only on p, τ_1, τ_2. Furthermore, the point

$$[p, K_A(e, \tau_1, \tau_2)] \in \text{ad}P$$

depends only on τ_1, τ_2 and is bilinear and skew-symmetric in these variables. It is given *by evaluating a two-form on B with values in* ad$P$*, denoted F_A, on (τ_1, τ_2). This two-form F_A is called the curvature of A.*

We can also define the two-form in terms of vector fields. Suppose χ_1, χ_2 are vector fields on B and let $\tilde{\chi}_1, \tilde{\chi}_2$ be lifted through $\mathcal{H}$ on P. Let $\tilde{\chi} = [\tilde{\chi}_1, \tilde{\chi}_2]$, we can use the horizontal subspaces to project this vector field to a vertical vector field $\tilde{\chi}^v$ on P.

Lemma 4.2 ([FW91, Lemma. 3.1.3])**.** *The vector field $\tilde{\chi}^v$ is a G invariant vector field on P. As such it is equivalent to a section $\sigma(\chi_1, \chi_2)$ of* ad$P$.

Proof. First, notice that both $\tilde{\chi}_1$ and $\tilde{\chi}_2$ are horizontal and therefore G equivariant, so therefore so is $\tilde{\chi}$. At each point p, we know that $\tilde{\chi}(p) = \chi_p^V + \chi_p^H$ and since $g\tilde{\chi}(p) = \tilde{\chi}(p \cdot g)$ so $\tilde{\chi}(p \cdot g) = \chi_p^V \cdot g + \chi_p^H \cdot g$ and since $\chi_p^H \cdot g \in H_{p \cdot g}$, we know that $\chi_p^V \cdot g = \chi_{p \cdot g}^V$. So we conclude that $\tilde{\chi}^v$ is G invariant.

Next, define

$$\sigma : B \to [p, (DR_p)^{-1}(\tilde{\chi}_p^v)]$$

which is a section of ad$P \to B$. Since we will be preselecting a $p \in \pi^{-1}(b)$, it is important to check that our final expression does not depend on p, so consider $p \cdot g \in \pi^{-1}(b)$ and

$$[pg, (TR_{pg})^{-1}(\tilde{\chi}_{pg}^v)] = [pg, g^{-1}TR_p(\tilde{\chi}^V) \cdot g]$$

This comes from the result of our former calculation: $TR_{pg}^{-1} = g^{-1} \circ TR_p^{-1}$ combined with the fact that $\tilde{\chi}^V$ is G invariant. Based on the construction of the adjoint bundle, we conclude that

$$[pg, g^{-1}TR_p(\tilde{\chi}^V) \cdot g] = [p, TR_p(\tilde{\chi}^V)]$$

Therefore the definition is well-defined. □

With the construction in mind, we can now define curvature in terms of this two-form.

Lemma 4.3 ([FW91, Lemma. 3.1.3]). *Keeping the notation from above, we have* $-\sigma(\chi_1, \chi_2) = F_A(\chi_1, \chi_2)$

Proof. The first thing to check is that $\sigma(\chi_1, \chi_2)$ is bilinear over the smooth functions acting on the vector fields. Consider

$$[\widetilde{f_1\chi_1}, \widetilde{f_2\chi_2}]$$

Since $\widetilde{f_1\chi_1} = f_1\tilde{\chi}_1$, we know that

$$\begin{aligned}
[f_1\widetilde{\chi_1}, f_2\widetilde{\chi_2}] &= f_1\tilde{\chi}_1(f_2)\tilde{\chi}_2 + f_2[f_1\tilde{\chi}_1, \tilde{\chi}_2] \\
&= f_1\tilde{\chi}_1(f_2)\tilde{\chi}_2 + f_2(\tilde{\chi}_2(f_1)\tilde{\chi}_1 + f_1[\tilde{\chi}_1, \tilde{\chi}_2]) \\
&= f_1\tilde{\chi}_1(f_2)\tilde{\chi}_2 + f_2\tilde{\chi}_2(f_1)\tilde{\chi}_1 + f_2 f_1[\tilde{\chi}_1, \tilde{\chi}_2]
\end{aligned}$$

Since $f_1\tilde{\chi}_1(f_2)$ and $f_2\tilde{\chi}_2(f_1)$ are just smooth functions, we know that $f_1\tilde{\chi}_1(f_2)\tilde{\chi}_2 + f_2\tilde{\chi}_2(f_1)\tilde{\chi}_1$ is a sum of two horizontal vector fields and therefore is horizontal as well. This tells us that

$$\sigma(\widetilde{f_1\chi_1}, \widetilde{f_2\chi_2}) = f_1 f_2(\tilde{\chi}_1, \tilde{\chi}_2)$$

and it depends only on χ_1 and χ_2 and is linear.

Then by evaluating the bracket at $(0, 0)$ we get the value for the curvature from lemma 2.13, thus the claim is proven. □

4.2 Curvature: in terms of connection one-form

We can also understand this two-form from the perspective of a connection one form.

Lemma 4.4 ([FW91, Lemma. 3.2.2]). *The two-form* $\pi^* F_A$ *on* P *with values in* $\mathcal{G}$ *is equal to*

$$d\omega_A + \omega_A \wedge \omega_A$$

This lemma can be proved by first noting that the defining qualities of $\pi^* F_A$ include giving the negative vertical projection when evaluated on two horizontal vector fields and vanishing if one of the vector fields is vertical. Then simply check that $d\omega_A + \omega_A \wedge \omega_A$ satisfies the same qualities, which mostly follows from the fact that the pullback of ω_A is the Maurer Cartan form.

4.3 Curvature: covariant derivative

We can also rephrase the two-form in terms of covariant derivatives. We can extend the covariant derivative

$$\nabla_A : \Omega^0(B, W) \to \Omega^1(B, W)$$

to

$$\nabla_A : \Omega^k(B, W) \to \Omega^{k+1}(B, W)$$

which will help with the following lemma.

Lemma 4.5 ([FW91, Lemma. 3.3.1])**.** *The linear operator* $\nabla_A \circ \nabla_A : \Omega^0(B, W) \to \Omega^2(B, W)$ *is given by multiplying by a two-form with values in* $End(E)$*. In fact,* $\nabla_A \circ \nabla_A(\sigma) = ad\rho(F_A)(\sigma)$

Since we have $\rho : G \to Aut(G)$, by taking the differential of this map we obtain a new representation $\rho : \mathcal{G} \to End(V)$ which extends to $End(E)$ where E is $P \times_G V$.

References

[FW91] Robert Friedman and John W.Morgan, *Gauge theory and the topology of four-manifolds*, American Mathematical Society and Institute for Advanced Study, 1991.

[Lee00] John Lee, *Introduction to smooth manifolds*, Springer, 2000.

[Lee09] Jeffrey Marc Lee, *Manifolds and differential geometry*, American Mathematical Society, 2009.

[Mic08] Peter W. Michor, *Topics differential geometry*, American Mathematical Society, 2008.