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THE EDGE ADMITTANCE OF A WIDE MICROSTRIP PATCH SEEN BY AN OBLIQUELY INCIDENT WAVE

by

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Abstract

The reflection coefficient of a TEM wave obliquely incident at the truncated upper edge of a slab loaded parallel-plate waveguide is computed from a knowledge of both the tangential electric field and normal magnetic field on the top of the slab beyond the truncated edge. To approximate these aperture fields, we use the static edge fields, which results in the correct value of the reflection coefficient to first order when the substrate is electrically thin. This development is the cornerstone of some promising techniques in the analysis of microstrip patch antennas and transmission lines.

1. Introduction

Analysis of microstrip patch antennas can be carried out by several techniques. A "numerically exact" approach can be used in which an integral equation for the surface electric current on the patch is formulated and solved using, e.g., a moment method [1]. This method can be cumbersome, since the kernel of the integral equation is a Sommerfeld integral which itself requires a fair amount of numerical analysis to compute. Moreover, the unknown current must be found over the entire two-dimensional surface of the patch, while the only peculiar behavior it actually exhibits is near the edge where its singularities appear.

Other approximate methods are used which provide insight into the physics of the problem. The cavity model [2] is widely used, in which the fields are reckoned mainly to be under the patch and obey a two-dimensional equation. Opinions differ, however, on what conditions should be enforced on these fields at the edges of the patch. Different ways of accounting for the fringing fields at the edge can result in resonant frequency shifts of the same order as the bandwidth, making design a difficult procedure [3]. Once accurate fringe fields at the edges are available, however, the radiation field of the patch can be accurately computed using the equivalent aperture or magnetic-current method [4,5].

This report, following the approach of [6], will address the problem of reflection of an incident TEM wave at the edge of a semi-infinite patch conductor. The general case of oblique incidence will be treated and the simpler case of normal incidence will be derived from the more general approach.

This problem has an exact formal solution by the Wiener-Hopf method [7,8], but extracting simple but accurate approximations for the case of an electrically thin substrate is a formidable mathematical task [3]. Also, the extension of the Wiener-Hopf method to deal with the finite patch of arbitrary shape does not seem to be possible.

We will formulate the problem as an integral equation for the electric and magnetic fields in the "aperture" which extends beyond the edge of the semi-infinite plane conductor along the top of the substrate to infinity. We find that an accurate expression for the reflection coefficient for the thin substrate limit can be obtained by using known expressions for the static electric and magnetic fields near the edge. The idea is similar to one used by Leppington and Levine [9] to find the edge correction for the capacitance of a circular disk capacitor. This method has been shown to be generalizable to a patch of fairly arbitrary shape, and offers considerable efficiency insofar as it deals with an integral equation along the edge of the patch and not on its surface. This work will be reported in a forthcoming report. A time dependence $e^{i\omega t}$ is used throughout.

2. Magnetic Field Aperture Integral Relation

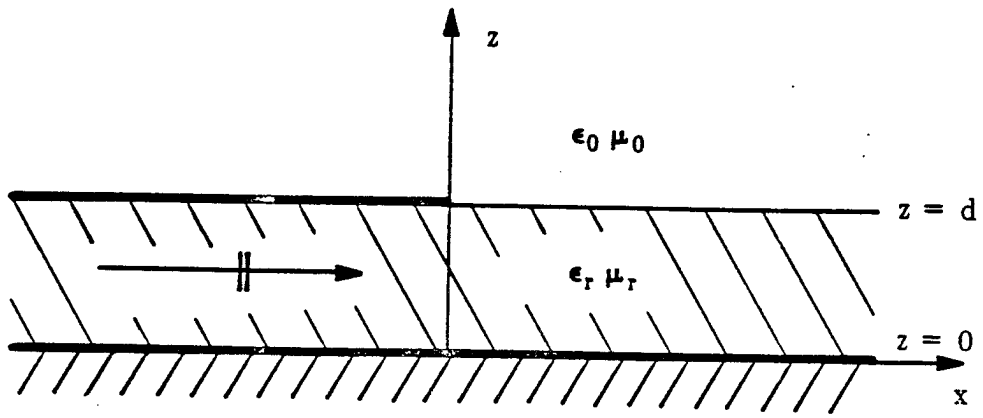
Consider the geometry of Figure 1. A TEM wave in the parallel plate region $0 < z < d$ is obliquely incident at the edge of the upper conductor, which occupies $z = d$ from $-\infty < x < 0$. We consider the incident wave

$$E_z^i = e^{-ik_0 n(x\cos\phi - y\sin\phi)} \quad (0 < z < d) \tag{1}$$

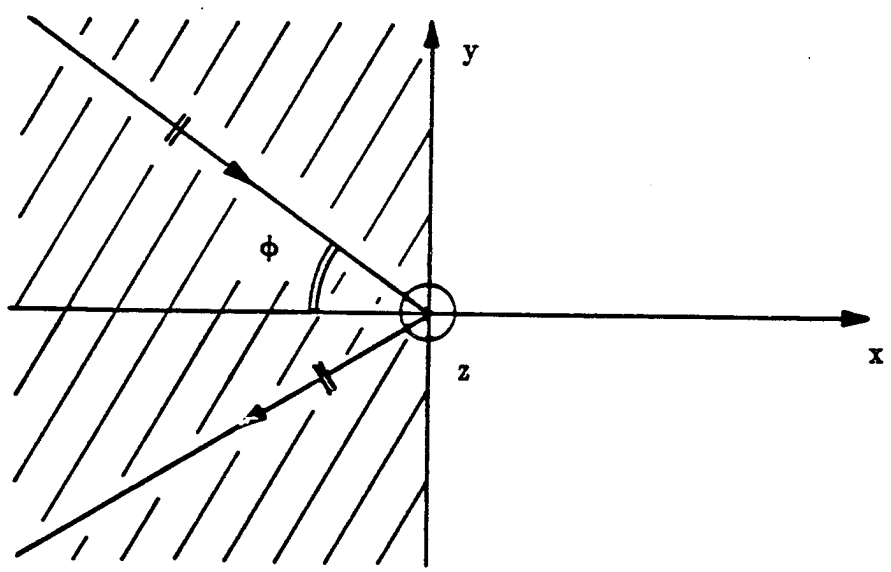
where $k_0 = \omega\sqrt{\epsilon_0\mu_0}$, $n = \sqrt{\epsilon_r\mu_r}$, ϵ_r and μ_r are the relative permittivity and permeability of the substrate, and ϕ is the angle between the direction of the incident wave and the normal to the edge of the upper plate. Define

$$\alpha = n\sin\phi \tag{2}$$

The complete incident field is then given by



Side view



Top view

Figure 1. TEM wave incident at the edge of a semi-infinite patch. (a) side view; (b) top view.

$$E_z^i = e^{-ik_0(\sqrt{n^2-\alpha^2}x - \alpha y)}$$

$$\bar{H}_t^i = \frac{-1}{\mu_r \zeta_0}(\alpha \bar{a}_x + \sqrt{n^2-\alpha^2}\,\bar{a}_y)\, e^{-ik_0(\sqrt{n^2-\alpha^2}x - \alpha y)} \tag{3}$$

where $\zeta_0 = \sqrt{\mu_0/\epsilon_0}$. We choose $\mathrm{Re}(\sqrt{n^2-\alpha^2}) \geq 0$ for definiteness. All field quantities behave as $e^{ik_0\alpha y}$ and this factor will be omitted throughout. Define the following Fourier transform pair:

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(\lambda)\, e^{-ik_0\lambda x}\, d\lambda$$

$$\tilde{f}(\lambda) = \frac{k_0}{2\pi} \int_{-\infty}^{\infty} f(x)\, e^{ik_0\lambda x}\, dx \tag{4}$$

and the convolution product

$$\int_{-\infty}^{\infty} \tilde{f}(\lambda)\tilde{g}(\lambda)\, e^{-ik_0\lambda x}\, d\lambda = \frac{k_0}{2\pi} \int_{-\infty}^{\infty} f(x')g(x-x')dx' \tag{5}$$

We can express all the components of the tangential scattered fields in terms of $\tilde{E}_z$ and $\tilde{H}_z$ as follows:

$$\tilde{\bar{E}}_t^s = \frac{1}{\alpha^2+\lambda^2}\left[\zeta_0\mu_r(\alpha\bar{a}_x + \lambda\bar{a}_y)\,\tilde{H}_z^s - \frac{i}{k_0}(\lambda\bar{a}_x - \alpha\bar{a}_y)\frac{\partial\tilde{E}_z^s}{\partial z}\right]$$

$$\tilde{\bar{H}}_t^s = \frac{1}{\alpha^2+\lambda^2}\left[-\frac{\epsilon_r}{\zeta_0}(\alpha\bar{a}_x + \lambda\bar{a}_y)\,\tilde{E}_z^s - \frac{i}{k_0}(\lambda\bar{a}_x - \alpha\bar{a}_y)\frac{\partial\tilde{H}_z^s}{\partial z}\right] \tag{6}$$

The solution of the wave equation in the transform domain yields at once:

$$\tilde{E}_z^s = E_1 e^{-k_0 u_0 z}$$

$$(z > d) \tag{7}$$

$$\tilde{H}_z^s = H_1 e^{-k_0 u_0 z}$$

(where $u_0 = (\lambda^2 + \alpha^2 - 1)^{1/2}$ and we require $\mathrm{Re}(u_0) \geq 0$ because of the physical constraints of the problem), and

$$\tilde{E}_z^s = E_{2s} \sinh(k_0 u_n z) + E_{2c} \cosh(k_0 u_n z)$$

$$(0 > z > d) \tag{8}$$

$$\tilde{H}_z^s = H_{2s} \sinh(k_0 u_n z) + H_{2c} \cosh(k_0 u_n z)$$

where $u_n = (\lambda^2 + \alpha^2 - n^2)^{1/2}$ and $\mathrm{Re}(u_n) \geq 0$.

The metallic boundary condition on the ground plane requires $\partial\tilde{E}_z^s/\partial z$ and $\tilde{H}_z^s$ to vanish at $z = 0$, yielding from (8)

$$\begin{aligned} E_{2s} &= 0 \\ H_{2c} &= 0 \end{aligned} \tag{9}$$

Also the continuity of $\tilde{\bar{E}}_z^s$ at the interface $z = d$ enforces, directly from (6), continuity of $\mu_r \tilde{H}_z^s$ and $\partial\tilde{E}_z^s/\partial z$

$$\begin{aligned} H_1 &= \mu_r H_{2s} e^{k_0 u_0 d} \sinh(k_0 u_n d) \\ u_0 E_1 &= -u_n E_{2c} e^{k_0 u_0 d} \sinh(k_0 u_n d) \end{aligned} \tag{10}$$

The integral relation will emerge from the continuity of the tangential component of the magnetic field outside the patch in the aperture plane $z = d$. Forcing H_y to be continuous

$$\int_{-\infty}^{\infty} \frac{e^{-ik_0\lambda x}}{\alpha^2+\lambda^2} e^{-k_0u_0d} \left(- \frac{\lambda}{\zeta_0} E_1 - i\alpha u_0 H_1 \right) d\lambda$$

$$= \int_{-\infty}^{\infty} \frac{e^{-ik_0\lambda x}}{\alpha^2+\lambda^2} \cosh(k_0u_nd) \left(- \frac{\lambda\epsilon_r}{\zeta_0} E_{2c} + i\alpha u_n H_{2s} \right) d\lambda$$

$$- \frac{\sqrt{n_2-\alpha^2}}{\mu_r\zeta_0} e^{-ik_0\sqrt{n^2-\alpha^2}\,x} \qquad (x > 0) \tag{11}$$

Combining (10) with (11) we can write the integral relation in the form

$$\int_{-\infty}^{\infty} \frac{e^{-ik_0\lambda x}}{\alpha^2+\lambda^2} \left[\frac{\lambda E_1}{\zeta_0} \left(1 + \frac{\epsilon_r u_0}{u_n} \coth(k_0u_nd) \right) + i\alpha H_1 \left(u_0 + \frac{u_n}{\mu_r} \coth(k_0u_nd) \right) \right] d\lambda$$

$$= \frac{\sqrt{n^2-\alpha^2}}{\mu_r\zeta_0} e^{-ik_0\sqrt{n^2-\alpha^2}\,x} e^{k_0u_0d} \qquad (x > 0) \tag{12}$$

3. Integral Equation in the Space Domain

We can obtain the x-component of the scattered electric field directly from (6):

$$\tilde{E}_x^s = \frac{\alpha\zeta_0}{\alpha^2+\lambda^2} \tilde{H}_z^s - \frac{i\lambda}{k_0(\alpha^2+\lambda^2)} \frac{\partial \tilde{E}_z^s}{\partial z} \qquad (z = d^+) \tag{13}$$

Replacing $\tilde{E}_z^s$ and $\tilde{H}_z^s$ by their expressions (7), we have

$$E_1 = \left[(\alpha^2 + \lambda^2) \tilde{E}_x^s - \alpha\zeta_0 \tilde{H}_z^s \right] \frac{e^{k_0u_0d}}{i\lambda u_0}$$

$$H_1 = \tilde{H}_z^s e^{k_0u_0d} \tag{14}$$

We now introduce (14) into (12) in order to get an integral relation for $\tilde{E}_x^s$ and $\tilde{H}_z^s$ only

$$\int_{-\infty}^{\infty} \frac{e^{-ik_0\lambda x}}{\alpha^2+\lambda^2} \left[\frac{1}{i\zeta_0u_0} ((\alpha^2 + \lambda^2) \tilde{E}_x^s - \alpha\zeta_0 \tilde{H}_z^s) \left(1 + \frac{\epsilon_r u_0}{u_n} \coth(k_0u_nd) \right) \right.$$

$$+ i\alpha \tilde{H}_z^s \left(u_0 + \frac{u_n}{\mu_r} \coth(k_0 u_n d) \right) \Bigg] = \frac{\sqrt{n^2-\alpha^2}}{\mu_r \zeta_0} e^{-ik_0\sqrt{n^2-\alpha^2}x} \qquad (z = d^+) \tag{15}$$

Or, grouping the terms in (15),

$$\frac{1}{i\zeta_0} \int_{-\infty}^{\infty} e^{-ik_0\lambda x} \tilde{E}_x^s \left[\frac{1}{u_0} + \frac{\epsilon_r}{u_n} \coth(k_0 u_n d) \right] d\lambda$$

$$+ i\alpha \int_{-\infty}^{\infty} e^{-ik_0\lambda x} \tilde{H}_z^s \left[\frac{1}{u_0} + \frac{1}{\mu_r u_n} \coth(k_0 u_n d) \right] d\lambda$$

$$= \frac{\sqrt{n^2-\alpha^2}}{\mu_r \zeta_0} e^{-ik_0\sqrt{n^2-\alpha^2}x} \qquad (z = d^+) \tag{16}$$

Recalling the convolution theorem (5), (16) becomes in the space domain

$$\frac{k_0}{2\pi i \zeta_0} \int_{-\infty}^{\infty} E_x^s(x') [G_1(x-x') + n^2 G_2(x-x')] dx'$$

$$+ \frac{i\alpha k_0}{2\pi} \int_{-\infty}^{\infty} H_z^s(x') [G_1(x-x') + G_2(x-x')] dx'$$

$$= \frac{\sqrt{n^2-\alpha^2}}{\mu_r \zeta_0} e^{-ik_0\sqrt{n^2-\alpha^2}x} \qquad (x > 0;\ z = d^+) \tag{17}$$

where the kernels G_1 and G_2 are defined by

$$G_1(x) = \int_{-\infty}^{\infty} e^{-ik_0\lambda x} \frac{d\lambda}{u_0} \tag{18}$$

$$G_2(x) = \frac{1}{\mu_r} \int_{-\infty}^{\infty} e^{-ik_0\lambda x} \coth(k_0 u_n d) \frac{d\lambda}{u_n} \tag{19}$$

and are evaluated in Appendix A:

$$G_1(x) = 2K_0(k_0\sqrt{\alpha^2-1}\,|x|) \tag{20}$$

$$G_2(x) = \frac{-i\pi}{k_0\mu_r d}\left[\frac{e^{-ik_0\sqrt{n^2-\alpha^2}\,|x|}}{\sqrt{n^2-\alpha^2}} + 2\sum_{m=1}^{\infty}\frac{e^{-ik_0\left(n^2-\alpha^2-\frac{m^2\pi^2}{k_0^2d^2}\right)^{1/2}|x|}}{\left(n^2-\alpha^2-\frac{m^2\pi^2}{k_0^2d^2}\right)^{1/2}}\right] \tag{21}$$

4. Reflection Coefficient

All the field quantities are normalized to the incident electric field whose amplitude is unity

$$E_z^i = e^{-ik_0\sqrt{n^2-\alpha^2}\,x} \tag{22}$$

$$\tilde{E}_z^s|_{z=d^+} = E_1\, e^{-k_0u_0d}\,; \qquad \tilde{E}_z^s|_{z=d^-} = E_{2c}\cosh(k_0u_nd) \tag{23}$$

The electric charge densities in the z=d plane are given as follows:

$$\text{incident charge density} \quad \rho^i = -\,\epsilon_0\epsilon_r\, e^{-ik_0\sqrt{n^2-\alpha^2}\,x} \tag{24}$$

$$\text{scattered charge density} \quad \tilde{\rho}^s = \epsilon_0\left[E_1\, e^{-k_0u_0d} - E_{2c}\,\epsilon_r\cosh(k_0u_nd)\right] \tag{25}$$

and therefore the total charge density in the space domain is

$$\rho(x) = -\,\epsilon_0\epsilon_r\, e^{-ik_0\sqrt{n^2-\alpha^2}\,x} + \epsilon_0\int_{-\infty}^{\infty} e^{-ik_0\lambda x}\,E_1 e^{-k_0u_0d}\,d\lambda$$
$$-\,\epsilon_0\epsilon_r\int_{-\infty}^{\infty} e^{-ik_0\lambda x}\,E_{2c}\cosh(k_0u_nd)d\lambda \tag{26}$$

Introducing (10) and (14), we have

$$\rho(x) = -\,\epsilon_0\epsilon_r e^{-ik_0\sqrt{n^2-\alpha^2}\,x}$$
$$-i\epsilon_0\int_{-\infty}^{\infty}\frac{e^{-ik_0\lambda x}}{\lambda}\left[(\alpha^2+\lambda^2)\tilde{E}_x^s - \alpha\zeta_0\tilde{H}_z^s\right]\left[\frac{1}{u_0} + \frac{\epsilon_r}{u_n}\coth(k_0u_nd)\right]d\lambda \tag{27}$$

Using the convolution product (5) and taking the derivative with respect to x,

$$\frac{d\rho(x)}{dx} = ik_0\sqrt{n^2-\alpha^2}\,\epsilon_0\epsilon_r e^{-ik_0\sqrt{n^2-\alpha^2}\,x} + \frac{\epsilon_0 k_0}{2\pi i}\int_{-\infty}^{\infty} E_x^s(x')G_3(x-x')dx'$$

$$- \frac{\epsilon_0 k_0 \alpha \zeta_0}{2\pi i}\int_{-\infty}^{\infty} H_z^s(x')G_4(x-x')dx' \tag{28}$$

where

$$G_3(x) = -ik_0\int_{-\infty}^{\infty} e^{-ik_0\lambda x}(\alpha^2+\lambda^2)\left(\frac{1}{u_0} + \epsilon_r\frac{\coth(k_0u_nd)}{u_n}\right)d\lambda \tag{29}$$

$$G_4(x) = -ik_0\int_{-\infty}^{\infty} e^{-ik_0\lambda x}\left[\frac{1}{u_0} + \epsilon_r\frac{\coth(k_0u_nd)}{u_n}\right]d\lambda \tag{30}$$

Pulling out the operator $\frac{1}{k_0^2}\left[-\frac{d^2}{dx^2} + k_0^2\alpha^2\right]$

$$G_3(x) = \frac{i}{k_0}\left[\frac{d^2}{dx^2} - k_0^2\alpha^2\right]\int_{-\infty}^{\infty} e^{-ik_0\lambda x}\left(\frac{1}{u_0} + \frac{\epsilon_r}{u_n}\coth(k_0u_nd)\right)d\lambda$$

or

$$G_3(x) = \frac{i}{k_0}\left[\frac{d^2}{dx^2} - k_0^2\alpha^2\right][G_1(x) + n^2G_2(x)] \tag{31}$$

Similarly,

$$G_4(x) = -ik_0[G_1(x) + n^2G_2(x)] \tag{32}$$

Thus we can write

$$\frac{d\rho(x)}{dx} = \frac{d\rho^i(x)}{dx} + \frac{\epsilon_0}{2\pi}\int_{-\infty}^{\infty} E_x^s(x')\left[\frac{d^2}{dx^2} - k_0^2\alpha^2\right]\left[G_1(x-x') + n^2G_2(x-x')\right]dx'$$

$$+ \frac{\epsilon_0 k_0^2 \alpha \zeta_0}{2\pi} \int_{-\infty}^{\infty} H_z^s(x') \left[G_1(x-x') + n^2 G_2(x-x') \right] dx'$$

and identify the derivative of the reflected charge density due to the dominant TEM mode from the corresponding part of G_2

$$\frac{d\rho^{\text{refl.}}(x)}{dx} = \frac{-i\epsilon_0 n^2}{2\mu_r k_0 d} \int_{-\infty}^{\infty} E_x^s(x') \left[\frac{d^2}{dx^2} - k_0^2 \alpha^2 \right]$$

$$\left[\frac{e^{-ik_0\sqrt{n^2-\alpha^2}\,|x-x'|}}{\sqrt{n^2-\alpha^2}} \right] dx' - \frac{i\epsilon_0 k_0 \alpha \zeta_0 n^2}{2\mu_r d} \int_{-\infty}^{\infty} H_z^s(x') \left[\frac{e^{-ik_0\sqrt{n^2-\alpha^2}\,|x-x'|}}{\sqrt{n^2-\alpha^2}} \right] dx' \qquad (33)$$

Since the magnetic charges and current are only present for $x' > 0$ and we observe the reflection coefficient for $x < 0$,

$$\frac{d\rho^{\text{refl.}}(x)}{dx} = \frac{i\epsilon_0 k_0 n^2}{2\mu_r d\sqrt{n^2-\alpha^2}} e^{ik_0\sqrt{n^2-\alpha^2}\,x}$$

$$\left(n^2 \int_0^{\infty} E_x^s(x') e^{-ik_0\sqrt{n^2-\alpha^2}\,x'} dx' - \alpha\zeta_0 \int_0^{\infty} H_z^s(x') e^{-ik_0\sqrt{n^2-\alpha^2}\,x'} dx' \right) \qquad (34)$$

But, assuming the limiting absorption principle:

$$\rho^{\text{refl.}}(x) = \int_{-\infty}^{x} \left(\frac{d\rho^{\text{refl.}}(x)}{dx} \right) dx$$

and we conclude that

$$\rho^{\text{refl.}}(x) = \frac{\epsilon_0 \epsilon_r e^{ik_0\sqrt{n^2-\alpha^2}\,x}}{2d(n^2-\alpha^2)}$$

$$\left(n^2 \int_0^{\infty} E_x^s(x') e^{-ik_0\sqrt{n^2-\alpha^2}\,x'} dx' - \alpha\zeta_0 \int_0^{\infty} H_z^s(x') e^{-ik_0\sqrt{n^2-\alpha^2}\,x'} dx' \right) \qquad (35)$$

Since $\rho^i(x) = -\epsilon_0 \epsilon_r e^{-ik_0\sqrt{n^2-\alpha^2}\,x}$, we can readily get the reflection coefficient

$$\Gamma = \frac{\rho^{\text{refl.}}(0)}{\rho^{i}(0)} = \frac{-1}{2d(n^2-\alpha^2)}$$

$$\left[n^2 \int_0^\infty E_x^s(x') e^{-ik_0\sqrt{n^2-\alpha^2}\,x'}\, dx' - \alpha\zeta_0 \int_0^\infty H_z^s(x') e^{-ik_0\sqrt{n^2-\alpha^2}\,x'}\, dx' \right] \tag{36}$$

5. Evaluation of the Reflection Coefficient

For the same reason as described in [6], the expression (36) for Γ is not likely to give the desired accuracy when we approximate the integrand. We need to compute $(1 - \Gamma)$ and this is done by combining (36) with the integral equation (17) and evaluating the equation for $x = 0^+$.

$$\Gamma = \frac{-i\mu_r k_0}{2\pi\sqrt{n^2-\alpha^2}} \left[n^2 \int_0^\infty E_x^s(x') G_2^{TEM}(x') dx' - \alpha\zeta_0 \int_0^\infty H_z^s(x') G_2^{TEM}(x') dx' \right]$$

where

$$G_2^{TEM}(x) = \frac{-\pi i}{k_0\mu_r d\sqrt{n^2-\alpha^2}} e^{-ik_0\sqrt{n^2-\alpha^2}\,|x|} \tag{37}$$

$$1-\Gamma = \frac{i\mu_r k_0}{\pi\sqrt{n^2-\alpha^2}} \int_0^\infty (\alpha\zeta_0 H_z^s(x') - E_x^s(x'))\, K_0(k_0\sqrt{\alpha^2-1}\,x')\, dx'$$

$$+ \frac{1}{d\sqrt{n^2-\alpha^2}} \int_0^\infty (\alpha\zeta_0 H_z^s(x') - n^2 E_x^s(x')) \sum_{m=1}^{\infty} \frac{e^{-ik_0\left(n^2-\alpha^2-\frac{m^2\pi^2}{k_0^2 d^2}\right)^{1/2} x'}}{\left(n^2-\alpha^2-\frac{m^2\pi^2}{k_0^2 d^2}\right)^{1/2}}\, dx' \tag{38}$$

Since $\sqrt{1-\alpha^2} = -i\sqrt{\alpha^2-1}$, we can rewrite

$$K_0(k_0\sqrt{\alpha^2-1}\,x') = \frac{-i\pi}{2} H_0^{(2)}(k_0\sqrt{1-\alpha^2}\,x') \quad .$$

and approximate the Hankel function of the second kind for small argument:

$$H_0^{(2)}(k_0\sqrt{1-\alpha^2}x') = 1 - \frac{2i}{\pi}\ln\left[\frac{k_0\sqrt{1-\alpha^2}x'}{2}\right] - \frac{2i\gamma}{\pi} + 0\left(k_0^2(1-\alpha^2)x'^2\ln(k_0\sqrt{1-\alpha^2}x')\right)$$

where γ is Euler's constant 0.57721 ... . This is equivalent to assuming that the dominant contribution is due to the fields close to the edge. We also take the quasi-static approximation of the summation:

$$\sum_{m=1}^{\infty} \frac{e^{-ik_0\left(n^2-\alpha^2-\frac{m^2\pi^2}{k_0^2d^2}\right)^{1/2}x'}}{\left(n^2-\alpha^2-\frac{m^2\pi^2}{k_0^2d^2}\right)^{1/2}} = i\sum_{m=1}^{\infty} \frac{e^{-k_0\left(\frac{m^2\pi^2}{k_0^2d^2}+\alpha^2-n^2\right)^{1/2}x'}}{\left(\frac{m^2\pi^2}{k_0^2d^2}+\alpha^2-n^2\right)^{1/2}}$$

$$\approx i\sum_{m=1}^{\infty} \frac{e^{-\frac{m\pi x'}{d}}}{\frac{m\pi}{k_0 d}} = \frac{-ik_0 d}{\pi}\ln\left(1-e^{-\frac{\pi x'}{d}}\right)$$

Equation (38) then becomes

$$1-\Gamma = \frac{\mu_r k_0}{\sqrt{n^2-\alpha^2}}\int_0^{\infty}\left[\zeta_0\alpha H_z^s(x') - E_x^s(x')\right]$$

$$\left[\frac{1}{2} - \frac{i}{\pi}\ln\left(\frac{k_0\sqrt{1-\alpha^2}x'}{2}\right) - \frac{i\gamma}{\pi}\right]dx' - \frac{ik_0}{\pi\sqrt{n^2-\alpha^2}}\int_0^{\infty}\left[\alpha\zeta_0 H_z^s(x') - n^2E_x^s(x')\right]$$

$$\ln\left(1-e^{-\frac{\pi x'}{d}}\right)dx' \tag{39}$$

6. Approximate Evaluation of Γ

The strategy here will be to make a reasonable approximation of the fields in order to evaluate Γ. The expression (39) is for now useless, the exact distribution of the fields being unknown. If the substrate is electrically thin, $kd \ll 1$, it is reasonable to suppose that near the edge ($kx \ll 1$), $E_x(x)$ and $H_z(x)$ should be proportional to the corresponding static field distributions for the same geometry.

Let us denote the static electric field by $E_{ox}(x)$ and assume a unit voltage between the top plate and the ground plane

$$\int_0^\infty E_{ox}(x')dx' = 1 \tag{40}$$

At a point $x < 0$ a few d behind the edge underneath the upper plate, we should still be in the quasistatic region near the edge and yet all the higher cutoff modes ($m \geq 1$) have attenuated to negligible levels, leaving a TEM field of

$$E_z^s(x,z) = 1 + \Gamma \quad (x < 0;\ |x| >> d;\ k|x| << 1)$$

and a voltage from the upper to lower plate of $-(1 + \Gamma)d$. Therefore, the quasistatic approximation to $E(x)$ should be

$$E_x^s(x) \approx -(1 + \Gamma)dE_{ox}(x) \tag{41}$$

For the magnetic field, we proceed in a similar way. We define a static field $H_{0z}(x)$ subject to the normalization

$$\int_0^\infty H_{0z}(x')dx' = 1 \tag{42}$$

In the static limit, the electric and magnetic problems are decoupled. This is not the case for the dynamic problem, where Faraday's law imposes in the plane $z = d^+$

$$\frac{\partial E_y^s(x)}{\partial x} - \frac{\partial E_x^s(x)}{\partial y} = -i\omega\mu_0 H_z^s$$

Integrating with respect to x and using both edge and radiation conditions

$$\int_0^\infty H_z^s(x')dx' = \frac{\alpha}{\zeta_0}\int_0^\infty E_x^s(x')dx' \tag{43}$$

Directly from (41), we now have

$$H_z^s(x,z = d^+) \approx \frac{-\alpha d}{\zeta_0}(1 + \Gamma)H_{0z}(x,z = d^+) \qquad (x > 0) \tag{44}$$

Introducing (41) and (44) into (39), we get

$$\begin{aligned}\frac{1-\Gamma}{1+\Gamma} = {} & \frac{i\mu_r k_0 d}{\pi\sqrt{n^2-\alpha^2}}(1-\alpha^2)\left(\frac{\pi}{2i} + \ln\left(\frac{2}{k_0 d\sqrt{1-\alpha^2}}\right) - \gamma\right) \\ & - \frac{ik_0 d\mu_r}{\pi\sqrt{n^2-\alpha^2}}\int_0^\infty E_{0x}(x')\left[\ln\left(\frac{x'}{d}\right) + \epsilon_r \ln\left(1 - e^{-\frac{\pi x'}{d}}\right)\right]dx' \\ & + \frac{ik_0 d\mu_r\alpha^2}{\pi\sqrt{n^2-\alpha^2}}\int_0^\infty H_{0z}(x')\left[\ln\left(\frac{x'}{d}\right) + \frac{1}{\mu_r}\ln\left(1 - e^{-\frac{\pi x'}{d}}\right)\right]dx'\end{aligned} \tag{45}$$

The integral involving the electric field is solved in [6]

$$\int_0^\infty E_{0x}(x')\left[\ln\left(\frac{x'}{d}\right) + \epsilon_r \ln\left(1 - e^{-\pi x'/d}\right)\right]dx' = \ln 2 + 2\epsilon_r Q_0(-\delta_\epsilon) - \epsilon_r \ln(2\pi) - 1 \tag{46}$$

where

$$Q_0(-\delta_\epsilon) = \sum_{m=1}^{\infty}(-\delta_\epsilon)^m \ln m \quad ; \quad \delta_\epsilon = \frac{\epsilon_r - 1}{\epsilon_r + 1}$$

Using duality

$$\int_0^\infty H_{0z}(x')\left[\ln\left(\frac{x'}{d}\right) + \frac{1}{\mu_r}\ln\left(1 - e^{-\pi x'/d}\right)\right]dx' = \ln 2 + \frac{2}{\mu_r}Q_0(\delta_\mu) - \frac{1}{\mu_r}\ln(2\pi) - 1 \tag{47}$$

where $\delta_\mu = \dfrac{\mu_r - 1}{\mu_r + 1}$. Thus finally

$$\frac{1-\Gamma}{1+\Gamma} = \frac{ik_0 d}{\pi\sqrt{n^2-\alpha^2}}\left\{\mu_r(1-\alpha^2)\left[1-\frac{i\pi}{2}-\gamma-\ln(k_0 d\sqrt{1-\alpha^2})\right]\right.$$

$$\left.-n^2(2Q_0(-\delta_\epsilon)-\ln 2\pi)+\alpha^2(2Q_0(\delta_\mu)-\ln 2\pi)\right\} \tag{48}$$

This result for the edge admittance agrees to order of kd with [3], as shown in Appendix B.

7. The Particular Case of Normal Incidence

The case of a wave perpendicularly incident at the edge is easily treated by setting $\alpha = 0$, i.e., no y-variation along the edge. We consider an incident wave

$$E_z^i = e^{-ik_0 nx} \quad ; \quad H_y^i = \frac{-e^{-ik_0 nx}}{\zeta} \qquad (0 < z < d) \tag{49}$$

where $\zeta = \zeta_0\sqrt{\mu_r/\epsilon_r}$.

In this particular situation, the edge admittance (48) reduces to [6]:

$$\frac{1-\Gamma}{1+\Gamma} = k_0 d\zeta\left\{\frac{1}{2}-\frac{i}{\pi}\left[\ln(k_0 d)+\gamma-1+2n^2 Q_0(-\delta_\epsilon)-n^2\ln 2\pi\right]\right\} \tag{50}$$

8. Conclusion

We have shown that a rigorous result for the edge reflection can be duplicated up to terms of order (kd) using the knowledge of certain integrals of the static fields near the edge. In the general case of oblique incidence, both the electric and magnetic fields need to be considered.

In the case of an arbitrarily shaped patch, we can generalize the relations (41) and (44), defining an equivalent edge voltage and current as follows:

$$E_x^s(x,\ell) \simeq V(\ell)E_{0x}(x)$$

$$H_z^s(x,\ell) \simeq I(\ell)H_{0z}(x)$$

where ℓ is the coordinate along the edge. The y-dependence along the edge is now replaced by an ℓ-dependence. In particular, (43) is replaced by

$$I(\ell)\int_0^\infty H_{0z}(x')dx' = \frac{-i}{\omega\mu_0}\frac{\partial V(\ell)}{\partial \ell}\int_0^\infty E_{0x}(x)dx$$

Or with the normalizations used above, (40) and (42),

$$I(\ell) = \frac{-i}{\omega\mu_0}\frac{\partial V(\ell)}{\partial \ell}$$

The approach used here generalizes the situation of normal incidence treated in [6] and provides a building block for the analysis of arbitrarily shaped two-dimensional microstrip structures.

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Appendix A

In this appendix, we will evaluate the two kernels,

$$G_1(x) = \int_{-\infty}^{\infty} e^{-ik_0\lambda x} \frac{d\lambda}{u_0}$$

$$G_2(x) = \frac{1}{\mu_r} \int_{-\infty}^{\infty} e^{-ik_0\lambda x} \coth(k_0 u_n d) \frac{d\lambda}{u_n}$$

First to calculate G_1, we make a change of variable $\omega = k_0\lambda$,

$$G_1 = \int_{-\infty}^{\infty} \frac{e^{-ik_0\lambda x}}{\sqrt{\alpha^2 + \lambda^2 - 1}} d\lambda = \int_{-\infty}^{\infty} \frac{e^{-i\omega x}}{\left(\alpha^2 + \frac{\omega^2}{k_0^2} - 1\right)^{1/2}} \frac{d\omega}{k_0}$$

$$= 2\int_0^{\infty} \frac{\cos\omega x}{\sqrt{\omega^2 + k_0^2(\alpha^2 - 1)}} d\omega \tag{A1}$$

Perform another change of variable $\omega = tk_0\sqrt{\alpha^2 - 1}$

$$G_1 = 2\int_0^{\infty} \frac{\cos(tk_0\sqrt{\alpha^2-1}\ x)}{\sqrt{t^2 + 1}} dt \tag{A2}$$

We now recall the following integral representation of the modified Bessel function of the second kind [10]:

$$K_0(x) = \int_0^{\infty} \frac{\cos(xt)}{\sqrt{t^2 + 1}} dt \quad (x > 0) \tag{A3}$$

Thus,

$$G_1(x) = 2K_0(k_0\sqrt{\alpha^2-1}\ |x|) \tag{A4}$$

In order to evaluate $G_2(x)$ we proceed as follows

$$G_2(x) = \frac{1}{\mu_r} \int_{-\infty}^{\infty} e^{-ik_0\lambda x} \frac{\coth(\sqrt{\alpha^2+\lambda^2-n^2}\ k_0 d)}{\sqrt{\lambda^2 + \alpha^2 - n^2}} d\lambda \tag{A5}$$

Recall the following series representation of coth(x)

$$\coth(x) = \frac{1}{x} + 2x \sum_{m=1}^{\infty} \frac{1}{x^2+m^2\pi^2}$$

yielding

$$\coth(\sqrt{\alpha^2+\lambda^2-n^2}\ k_0 d) = \frac{1}{\sqrt{\alpha^2+\lambda^2-n^2}\ k_0 d} + 2k_0 d\sqrt{\alpha^2+\lambda^2-n^2}$$

$$\sum_{m=1}^{\infty} \frac{1}{k_0^2 d^2(\alpha^2+\lambda^2-n^2) + m^2\pi^2} \tag{A6}$$

Equation (A5) then becomes

$$G_2(x) = \frac{1}{k_0\mu_r d} \int_{-\infty}^{\infty} e^{-ik_0\lambda x} \frac{d\lambda}{\lambda^2 - (n^2-\alpha^2)}$$

$$+ \frac{2}{k_0\mu_r d} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-ik_0\lambda x} d\lambda}{\lambda^2 - \left(n^2-\alpha^2-\frac{m^2\pi^2}{k_0^2 d^2}\right)} \tag{A7}$$

To be able to carry an integration in the complex plane, we need to precisely know the location of the poles in the λ-plane.

$$\lambda^2 - (n^2-\alpha^2) = (\lambda - \xi)(\lambda + \xi) \tag{A8}$$

where $\xi = \sqrt{n^2-\alpha^2}$.

We choose the branch of ξ such that $\mathrm{Re}(\xi) > 0$ and be consistent from here on. Directly from the principle of limiting absorption, $\mathrm{Im}(k_0 n) < 0$ and $\mathrm{Im}(n) < 0$. Consider the two following cases:

(i) If $\mathrm{Re}(n) > |\alpha|$, then $\mathrm{Re}(\xi) > 0$ and $\mathrm{Im}(\xi) < 0$.

(ii) If $\mathrm{Re}(n) < |\alpha|$, then we have in the α-plane the situation described in Figure A1, and

$$\sqrt{n - \alpha} = -i\sqrt{\alpha - n} \tag{A9}$$

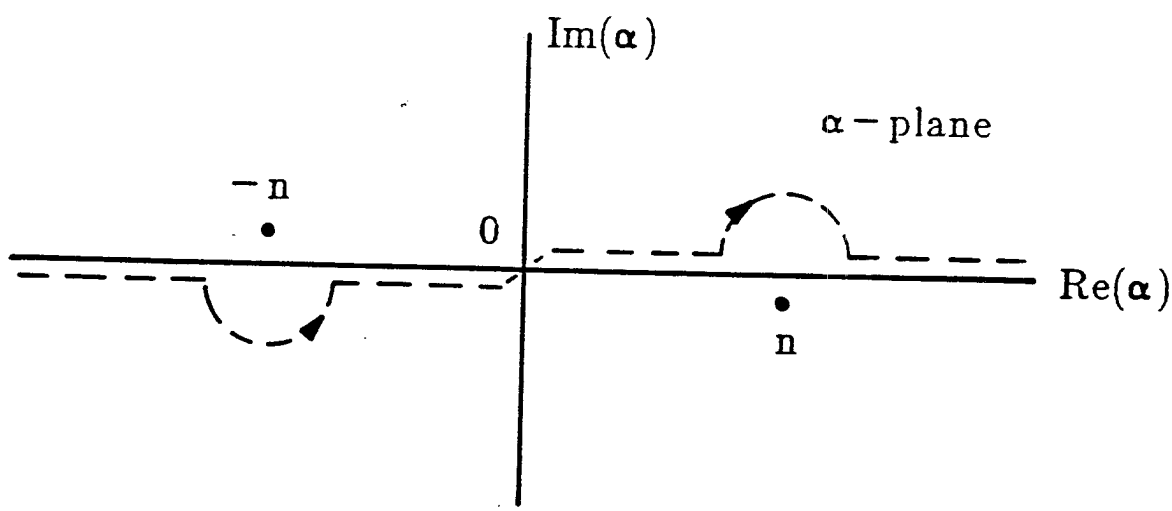

Figure A1: Poles in the α-plane.

Altogether we have in the λ-plane the migration of the poles given in Figure A2 and

$$\sqrt{n^2 - \alpha^2} = -i\sqrt{\alpha^2 - n^2} \tag{A10}$$

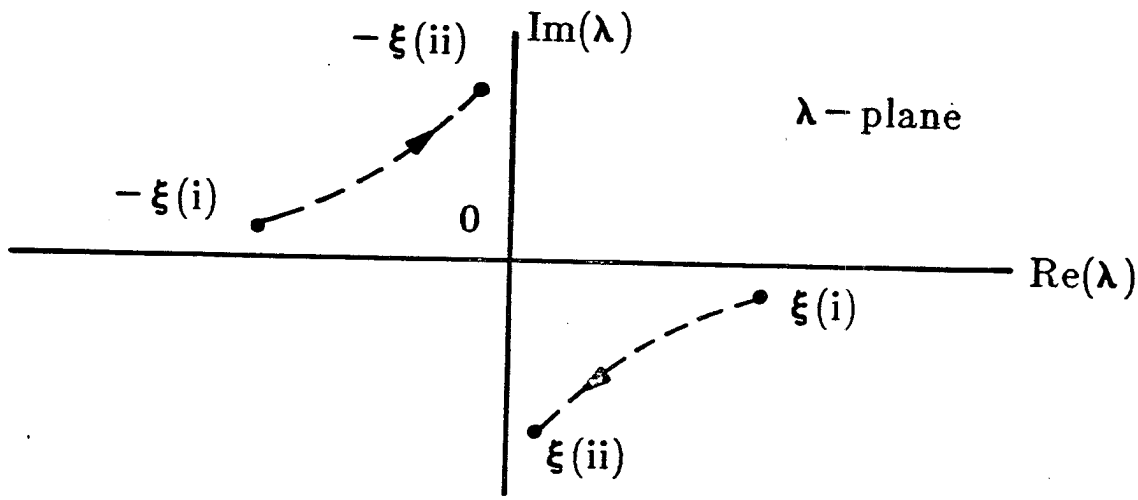

Figure A2: Poles in the λ-plane.

Let

$$I_1 = \frac{1}{k_0\mu_r d}\int_{-\infty}^{\infty} e^{-ik_0\lambda x}\,\frac{d\lambda}{(\lambda - \sqrt{n^2-\alpha^2})(\lambda + \sqrt{n^2-\alpha^2})}$$

and integrate I_1 along the integration contour represented in Figure A3, closing the contour in the lower half-plane ($x > 0$).

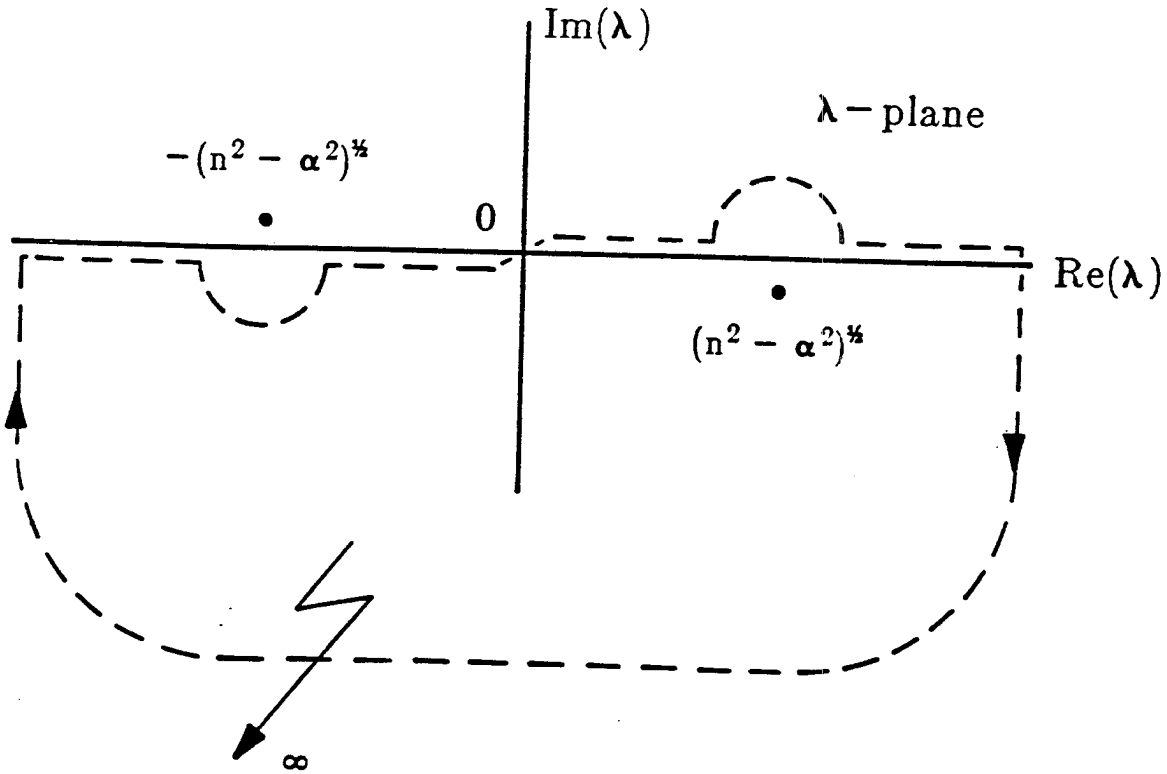

Figure A3: Integration contour in the λ-plane.

The residue theorem yields at once

$$I_1 = -\frac{\pi i}{k_0 \mu_r d \sqrt{n^2-\alpha^2}} e^{-ik_0\sqrt{n^2-\alpha^2}\,|x|} \tag{A11}$$

Similarly, the second term in the right-hand side of (A7) becomes

$$I_2 = \frac{2}{k_0 \mu_r d} \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-ik_0\lambda x}\, d\lambda}{\left(\lambda - \left(n^2 - \alpha^2 - \frac{m^2\pi^2}{k_0^2 d^2}\right)^{1/2}\right)\left(\lambda + \left(n^2 - \alpha^2 - \frac{m^2\pi^2}{k_0^2 d^2}\right)^{1/2}\right)}$$

$$= \frac{-2\pi i}{k_0 \mu_r d} \sum_{m=1}^{\infty} \frac{e^{-ik_0\left(n^2 - \alpha^2 - \frac{m^2\pi^2}{k_0^2 d^2}\right)^{1/2} |x|}}{\left(n^2 - \alpha^2 - \frac{m^2\pi^2}{k_0^2 d^2}\right)^{1/2}} \tag{A12}$$

Finally,

$$G_2(x) = \frac{-\pi i}{k_0 \mu_r d} \frac{e^{-ik_0\sqrt{n^2-\alpha^2}\,|x|}}{\sqrt{n^2 - \alpha^2}} - \frac{2\pi i}{k_0 \mu_r d} \sum_{m=1}^{\infty} \frac{e^{-ik_0\left(n^2-\alpha^2-\frac{m^2\pi^2}{k_0^2 d^2}\right)^{1/2} |x|}}{\left(n^2 - \alpha^2 - \frac{m^2\pi^2}{k_0^2 d^2}\right)^{1/2}} \tag{A13}$$

Appendix B

In this appendix, the result for the edge admittance (48) is shown to be the same to order (kd) as the one given in [3]. If we recall the equations (12), (19) and (20) of [3], the reflection coefficient Γ for the TEM wave is given as

$$\Gamma = e^{i\chi(\alpha)} \tag{B1}$$

with

$$\chi(\alpha) \approx \frac{2k_0 d}{\pi\sqrt{n^2-\alpha^2}} \left\{(1-\alpha^2)\,\mu_r\left[\ln\left(\sqrt{\alpha^2-1}\,k_0 d\right) + \gamma - 1\right] + n^2[2Q_0(-\delta_\epsilon) - \ln 2\pi] - \alpha^2\,[2Q_0(\delta_\mu) - \ln 2\pi]\right\} \tag{B2}$$

Furthermore, for small kd, we can approximate

$$\frac{1-\Gamma}{1+\Gamma} \approx \frac{-i\chi}{2} \tag{B4}$$

Also, the branch of the square root has been chosen as

$$\sqrt{1-\alpha^2} = -i\sqrt{\alpha^2-1}$$

Thus,

$$\ln\left(\sqrt{\alpha^2-1}\,k_0 d\right) = \frac{i\pi}{2} + \ln\left(\sqrt{1-\alpha^2}\,k_0 d\right) \tag{B5}$$

With the help of (B2) and (B5), (B4) can be written as

$$\frac{1-\Gamma}{1+\Gamma} \approx \frac{ik_0 d}{\pi\sqrt{n^2-\alpha^2}} \left\{ \mu_r(1-\alpha^2) \left[1 - \frac{i\pi}{2} - \gamma - \ln(k_0 d\sqrt{1-\alpha^2}) \right] \right.$$

$$\left. - n^2(2Q_0(-\delta_\epsilon) - \ln 2\pi) + \alpha^2(2Q_0(\delta_\mu) - \ln 2\pi) \right\} \tag{B6}$$

which checks exactly with (48).