

## Branching diffusion with particle interactions

János Engländer\*      Liang Zhang†

### Abstract

A  $d$ -dimensional branching diffusion,  $Z$ , is investigated, where the linear attraction or repulsion between particles is competing with an Ornstein-Uhlenbeck drift, with parameter  $b$  (we take  $b > 0$  for inward O-U and  $b < 0$  for outward O-U). This work has been motivated by [4], where a similar model was studied, but without the drift component.

We show that the large time behavior of the system depends on the interaction and the drift in a nontrivial way. Our method provides, *inter alia*, the SLLN for the non-interactive branching (inward) O-U process.

First, regardless of attraction ( $\gamma > 0$ ) or repulsion ( $\gamma < 0$ ), a.s., as  $t \rightarrow \infty$ , the center of mass of  $Z_t$  converges to the origin when  $b > 0$ , while escapes to infinity exponentially fast (rate  $|b|$ ) when  $b < 0$ .

We then analyze  $Z$  as viewed from the center of mass, and finally, for the system as a whole, we provide a number of results and conjectures regarding the long term behavior of the system; some of these are scaling limits, while some others concern local extinction.

**Keywords:** spatial branching process; branching Ornstein-Uhlenbeck process; branching diffusion; interaction; center of mass; most recent common ancestor;  $\Delta$ -transform, Strong Law of Large Numbers, local extinction.

**AMS MSC 2010:** Primary 60J60, Secondary 60J80.

Submitted to EJP on December 26, 2015, final version accepted on October 31, 2016.

## 1 Introduction: Branching motion with drift and self-interaction

### 1.1 Model

We consider a branching diffusion in  $\mathbb{R}^d$ , where the motion component is an Ornstein-Uhlenbeck (O-U) process, and dyadic branching occurs in each time unit. (Dyadic branching means precisely two offspring.) In addition, we introduce interaction between particles, namely either *attraction* or *repulsion*.

---

\*Department of Mathematics, University of Colorado, Boulder, CO-80309-0395, USA  
 E-mail: janos.englander@colorado.edu

†Oracle Corporation. 10075 Westmoor Dr., Nr. 200, Westminster, CO-80021, USA  
 E-mail: Liang.Zhang-2@Colorado.EDU

Let  $Z$  denote the process and  $Z_t^i$  the  $i^{\text{th}}$  particle<sup>1</sup> in time  $[m, m + 1)$ , where  $m = 0, 1, 2, \dots$ . As branching is unit time, in the time interval  $[m, m + 1)$  there are  $2^m$  particles in total. Without interaction from other particles,  $Z_t^i$  is an Ornstein-Uhlenbeck process with drift parameter  $b \in \mathbb{R}$ , corresponding to the elliptic operator

$$\frac{1}{2}\Delta - bx \cdot \nabla$$

on  $\mathbb{R}^d$ . (Note the negative sign of the drift. It is somewhat unusual, but it fits our setup better, because of the sign of the interaction parameter  $\gamma$ , introduced below.) If  $b > 0$ , then we have an ‘inward’ O-U process; if  $b < 0$ , then we have an ‘outward’ O-U process. If  $b = 0$ , then it is a Brownian motion.

As far as the aforementioned interaction is concerned, let us fix the *interaction parameter*  $\gamma \neq 0$ . We assume that the  $i^{\text{th}}$  particle  $Z_t^i$ , on the time interval  $[m, m + 1)$ , ‘feels’ a drift caused by attraction/repulsion of all other particles as

$$\frac{1}{2^m} \sum_{j=1}^{2^m} \gamma \cdot (Z_t^j - \cdot) dt,$$

and so  $Z_t^i$  satisfies the following stochastic differential equation:

$$dZ_t^i = dW_t^{i,m} - bZ_t^i dt + \frac{1}{2^m} \sum_{j=1}^{2^m} \gamma \cdot (Z_t^j - Z_t^i) dt.$$

If  $\gamma > 0$ , then this means that particles attract each other, whereas if  $\gamma < 0$ , then this means that they repel each other.

In the stochastic differential equation above, the  $\{W_t^{i,m}\}_{1 \leq i \leq 2^m}$  are independent Brownian motions on  $[m, m + 1)$ . In other words, the infinitesimal generator of the  $i$ th particle in the time interval is

$$\frac{1}{2}\Delta + \left( \frac{1}{2^m} \sum_{j=1}^{2^m} \gamma \cdot (Z_t^j - x) - bx \right) \cdot \nabla.$$

**Notation.** Throughout the paper, the symbol  $\xrightarrow{w}$  (or just  $\Rightarrow$ ) will denote *weak convergence* of finite measures; the symbol  $\xrightarrow{v}$  will denote *vague convergence*. By a *bounded rational rectangle* we will mean a set  $B \subset \mathbb{R}^d$  of the form  $B = I_1 \times I_2 \times \dots \times I_d$ , where  $I_i$  is a bounded interval with rational endpoints for each  $1 \leq i \leq d$ . The family of all bounded rational rectangles will be denoted by  $\mathcal{R}$ . The symbol  $X \oplus Y$  will denote the independent sum of the random variables  $X$  and  $Y$ . As usual,  $\mathcal{N}(\mu, \sigma^2)$  will denote the normal distribution with mean  $\mu$  and variance  $\sigma^2$ ;  $\text{Leb}$  will denote  $d$ -dimensional Lebesgue measure. Finally, for  $z \in \mathbb{R}$ ,  $[z]$  will denote the largest integer which is less than or equal to  $z$ .

The following criterion will be used in the paper; we omit the standard proof, which follows from the Portmanteau Theorem and the well known condition in Theorem 2.2 in [3].

**Lemma 1.1** (Weak convergence). *Let  $\mu_1, \mu_2, \dots$  and  $\mu$  be probability measures on  $\mathbb{R}^d$  and  $\mu \ll \text{Leb}$ . Then  $\mu_n \Rightarrow \mu$  if and only if  $\lim_{n \rightarrow \infty} \mu_n(B) = \mu(B)$  for all  $B \in \mathcal{R}$ .*

## 1.2 Motivation

This paper has been motivated by [4], where a similar model was studied. There the motion was Brownian motion ( $b = 0$ ), and it has been shown that the center of the system

<sup>1</sup>We can use an arbitrary labeling, as long as it is independent of the spatial motion.

is a Brownian motion, being slowed down such that it tends to a ‘terminal position’  $N$  almost surely, and  $N$  is a  $d$ -dimensional, normally distributed random variable, with mean zero. If  $P^x$  denotes the probability conditioned on  $N = x$ ,  $x \in \mathbb{R}^d$ , then the following theorem was demonstrated for  $\gamma > 0$  (attraction):

$$2^{-n} Z_n(dy) \Rightarrow \left(\frac{\gamma}{\pi}\right)^{d/2} \exp(-\gamma|y-x|^2) dy, P^x - a.s.,$$

as  $n \rightarrow \infty$ , for almost all  $x \in \mathbb{R}^d$ , where  $Z(dy)$  denotes the discrete measure-valued process corresponding to the interacting branching particle system. For  $\gamma < 0$ , a conjecture was stated.

A similar model for superdiffusions has been introduced and studied by H. Gill recently [6] and results analogous to those in [4], were obtained. The toolsets used in the two papers are very different though. Gill’s paper utilizes the so-called *historical calculus* of E. Perkins.

It should be mentioned that, although our original motivation was to analyze the compound effect of the drift and the interaction, it turns out that our method yields an elementary proof for the Strong Law of Large Numbers for the case of a *non-interactive* branching (inward) Ornstein-Uhlenbeck process as well. See Example 7.5.

Finally, for classical results on limit theorems for branching particle systems (without interaction), see the fundamental monograph [1], and the more recent article [5].

### 1.3 Existence and uniqueness

In this section, we show the existence and uniqueness for this system (process). Actually, it is easy to see<sup>2</sup> that we only need to show that, given the initial value in time interval  $[m, m + 1)$ , the system exists and is unique.

Now, in the time interval  $[m, m + 1)$ , we can look at the  $2^m$  interacting particles (diffusions) as a single  $2^m d$ -dimensional Brownian motion with a drift  $\mathbf{d} : \mathbb{R}^{2^m d} \rightarrow \mathbb{R}^{2^m d}$  defined as

$$\mathbf{d}(x_1, x_2, \dots, x_{d-1}, x_d, \dots, x_{2^m d}) := \gamma(\beta_1, \beta_2, \dots, \beta_{d-1}, \beta_d, \dots, \beta_{2^m d})^T,$$

where  $\beta_k = 2^{-m} \gamma(x_{\bar{k}} + x_{d+\bar{k}} + \dots + x_{(2^m-1)d+\bar{k}}) - (\gamma + b)x_k$ . Here  $\bar{k} - k$  is a multiple of  $d$  and  $0 < \bar{k} \leq d$ . As the drift  $\mathbf{d}$  is Lipschitz, the existence and uniqueness of our system follows from the uniqueness/existence theorem for stochastic differential equations in several dimensions.

### 1.4 Summary of our results

In this paper, we will provide the following results/conjectures.

Case 1:  $b > 0, b + \gamma > 0$ : As  $n \rightarrow \infty$ ,

$$2^{-n} Z_n(dy) \xrightarrow{w} \left(\frac{\gamma + b}{\pi}\right)^{d/2} \exp(-(\gamma + b)|y|^2) dy, a.s.$$

Case 2:  $b > 0, b + \gamma = 0$ : Then

$$(2\pi)^{d/2} n^{d/2} 2^{-n} Z_n(dy) \xrightarrow{v} \text{Leb}, a.s.$$

Case 3:  $b > 0, b + \gamma < 0$ : We will conjecture that if  $\frac{\log 2}{d} \leq |b + \gamma|$ , then  $Z$  suffers local extinction, while if  $\frac{\log 2}{d} > |b + \gamma|$ , then

$$2^{-n} e^{d|b+\gamma|n} Z_n(dy) \xrightarrow{v} \text{Leb}, a.s.$$

<sup>2</sup>Otherwise use ‘concatenating’ for the processes.

Case 4:  $b < 0$  (outward O-U process): In the attractive ( $\gamma > 0$ ) case, we will show that  $Z_n(dy) \xrightarrow{v} 0$  a.s. (local extinction), irrespective of the relationship between  $\gamma$  and  $|b|$ ; the repulsive case is still open.

## 2 The center of mass

**Definition 2.1** (COM). For  $t \in [m, m + 1)$ , there are  $2^m$  particles, denoted by  $\{Z_t^i\}, i = 1, 2, \dots, 2^m$ , moving in space. Hence, letting  $m := \lfloor t \rfloor$ , we define the center of mass (COM) as

$$\bar{Z}_t := \frac{1}{2^m} \sum_1^{2^m} Z_t^i.$$

In this section we are going to show that as  $t \rightarrow \infty$ :

- (1) if  $b > 0$ , then the center of mass converges to the origin, no matter if attraction or repulsion holds;
- (2) if  $b < 0$ , then it will tend to infinity with ‘speed’  $e^{-bt}$ .

The significance of this result is that the attraction/repulsion for  $Z_t^i$  is given by

$$\frac{1}{2^m} \sum_{j=1}^{2^m} \gamma(Z_t^j - Z_t^i) dt = \gamma(\bar{Z}_t - Z_t^i) dt,$$

where  $\bar{Z}_t$  is as above. Hence, one can replace the interaction between particles by the interaction with the center of mass. Therefore, as a first step, we will study the large time behavior of the center of mass  $\bar{Z}_t$ .

Before stating our first result, we note that in this section, we will be interested in a.s. and  $L^2$  convergence of the COM. Since it is easy to see that these limits can be verified coordinate-wise, we assume  $d = 1$  for this section. (The reader should keep in mind that the results work for any  $d \geq 1$ .)

Our main results here will concern the behavior of the COM in the attractive/repulsive case. But we need some preliminary lemmas first. Below we give two results regarding a general, one-dimensional stochastic differential equation

$$\begin{cases} dX_t = \beta(t)dW_t - bX_t dt, \\ X_0 = 0 \text{ a.s.}, \end{cases} \tag{2.1}$$

where we assume that  $b > 0$  and that  $\beta(\cdot) > 0$  is locally Lipschitz. Here  $\beta(t)$  can be considered a time change of the Brownian part. We assume that  $\beta(t)$  converges to 0 as  $t \rightarrow \infty$ , that is that the Brownian motion is slowing down completely. We then want to determine the limiting distribution.

**Lemma 2.2** (Convergence to zero). *Let  $X$  be the solution of (2.1).*

- (a) *If  $\lim_{t \rightarrow \infty} \beta(t) = 0$ , then  $\lim_{t \rightarrow \infty} X_t = 0$  in  $L^2$ .*
- (b) *Assume in addition, that  $\beta(t)$  is decreasing in  $t$ , and that*

$$\sum_{m=1}^{\infty} m\beta^2(m) < \infty.$$

*Then  $\lim_{t \rightarrow \infty} X_t = 0$  a.s.*

*Proof.* (a) Assume, that  $X_t = X(t)$  is of the form  $X(t) = X_1(t)X_2(t)$ , with  $X_1(0) = 0$  and  $X_1, X_2$  being of finite variation. Keeping the product rule for  $dX(t)$  in mind, set

$$X_2(t)dX_1(t) = \beta(t)dW_t$$

and

$$X_1(t) dX_2(t) = -bX(t) dt = -bX_1(t)X_2(t) dt,$$

that is,  $dX_2(t) = -bX_2(t) dt$ .

We obtain  $X_2(t) = Ce^{-bt}$ ,  $C \neq 0$ , and thus  $dX_1(t) = C^{-1}\beta(t)e^{bt}dW_t$ . Then  $X(t) := e^{-bt} \int_0^t \beta(s)e^{bs} dW_s$  satisfies (2.1); by uniqueness, it is in fact *the* solution to the equation.

Since  $X_t$  is centered Gaussian, the claim is tantamount to

$$\text{Var}(X_t) = e^{-2bt} \int_0^t \beta^2(s)e^{2bs} ds \rightarrow 0$$

(use Itô-isometry). Let  $\lim_{t \rightarrow \infty} \int_0^t \beta^2(s)e^{2bs} ds = \infty$  (otherwise the statement is trivial), and use L'Hospital's rule:

$$\lim_{t \rightarrow \infty} \text{Var}(X_t) = \lim_{t \rightarrow \infty} \frac{\int_0^t \beta^2(s)e^{2bs} ds}{e^{2bt}} = \lim_{t \rightarrow \infty} \frac{\beta^2(t)}{2b} = 0.$$

(b) We need to show that for any  $\epsilon > 0$ , we have

$$P(\sup\{|X_t| : m \leq t < m + 1\} > \epsilon, \text{ i.o.}) = 0.$$

Let

$$A_m := \{\sup\{|X_t| : m \leq t < m + 1\} > \epsilon\}.$$

Then, by the Borel-Cantelli lemma, it is sufficient to show that

$$\sum_{m=1}^{\infty} P(A_m) < \infty. \tag{2.2}$$

Denote  $Y_t := e^{bt}X_t = \int_0^t \beta(s)e^{bs} dW_s$ , and note that  $Y_t$  is a martingale as it is an Itô integral. Thus  $|Y_t|$  is a submartingale.

We have  $P(A_m) < P(\sup\{|Y_t| : m \leq t < m + 1\} > \epsilon e^{bm})$ . By Doob's inequality,

$$P(\sup\{|Y_t| : m \leq t < m + 1\} > \epsilon e^{bm}) < \frac{E(|Y_{m+1}|^2)}{\epsilon^2 e^{2bm}}.$$

As  $E(Y_t) = 0$ , we have

$$E(|Y_t|^2) = \text{Var}(Y_t) = \int_0^t \beta^2(s)e^{2bs} ds,$$

and thus

$$\frac{E(|Y_{m+1}|^2)}{\epsilon^2 e^{2bm}} = \frac{\int_0^{m+1} \beta^2(s)e^{2bs} ds}{\epsilon^2 e^{2bm}}.$$

It remains to prove that

$$\sum_{m=1}^{\infty} e^{-2bm} \int_0^{m+1} \beta^2(s)e^{2bs} ds < \infty.$$

For  $e^{-2bm} \int_0^{m+1} \beta^2(s)e^{2bs} ds$ , we break up the expression into two parts:

$$\begin{aligned} & e^{-2bm} \int_0^{m+1} \beta^2(s)e^{2bs} ds \\ &= e^{-2bm} \int_0^{(m+1)/2} \beta^2(s)e^{2bs} ds + e^{-2bm} \int_{(m+1)/2}^{m+1} \beta^2(s)e^{2bs} ds =: I_1^m + I_2^m. \end{aligned}$$

We show now that both  $\sum_{m=1}^{\infty} I_1^m$  and  $\sum_{m=1}^{\infty} I_2^m$  are finite.

$I_1^m$  summable:

$$I_1^m = e^{-b(m-1)} \int_0^{(m+1)/2} \beta^2(s)e^{2bs-bm-b} ds \leq e^{-(m-1)} \int_0^{(m+1)/2} \beta^2(s) ds.$$

As  $\beta$  decreases to 0, there is a constant  $C$  such that  $\beta(s) < C$  for all  $s \geq 0$ . Then  $\int_0^{(m+1)/2} \beta^2(s) ds < C^2(m+1)/2$ . For large  $m$ ,  $m-1 > (m+1)/2$ , and so

$$I_1^m \leq C^2 e^{-b(m-1)} (m-1),$$

yielding that  $\sum_{m=1}^{\infty} I_1^m \leq C^2 \sum_{m=0}^{\infty} m e^{-bm} = C^2 (e^b + e^{-b} - 2)^{-1} < \infty$ .

$I_2^m$  summable:

$$\begin{aligned} I_2^m &= e^{-2bm} \int_{(m+1)/2}^{m+1} \beta^2(s)e^{2bs} ds \leq e^{2b} \int_{(m+1)/2}^{m+1} \beta^2(s) ds \\ &\leq e^{2b} (m+1)/2 \cdot \beta^2((m+1)/2). \end{aligned}$$

Note that  $(m+1)/2 \leq 2\lfloor(m+1)/2\rfloor$  for  $m \geq 1$ , and, since  $\beta$  is a decreasing function, one has

$$\begin{aligned} \sum_{m=1}^{\infty} I_2^m &\leq \sum_{m=1}^{\infty} e^{2b} (m+1)/2 \cdot \beta^2((m+1)/2) \\ &\leq 2e^{2b} \sum_{m=1}^{\infty} \lfloor(m+1)/2\rfloor \beta^2(\lfloor(m+1)/2\rfloor) \leq 4e^{2b} \sum_{n=1}^{\infty} n \beta^2(n). \end{aligned}$$

Since, by assumption,  $\sum_{m=1}^{\infty} m \beta^2(m) < \infty$ , we have  $\sum_{m=1}^{\infty} I_2^m < \infty$ .

Since  $I_1^m$  and  $I_2^m$  are summable, the summability condition (2.2) indeed holds.  $\square$

Next, we describe the evolution of the COM using a stochastic differential equation.

**Lemma 2.3** (SDE for COM). *On  $[0, \infty)$ , the process  $\bar{Z}$  satisfies the stochastic differential equation  $d\bar{Z}_t = \beta(t)dW_t - b\bar{Z}_t dt$  with  $\beta(t) := 2^{-m/2}$ , for  $t \in [m, m+1)$ , where  $W$  is a standard Brownian motion. Consequently,*

$$Z_t = e^{-bt} \int_0^t \beta(s)e^{bs} dW_s. \tag{2.3}$$

*Proof.* Consider the time interval  $[m, m + 1)$ , and recall the definition of the center of mass:  $\bar{Z}_t = 2^{-m} \sum_{i=1}^{2^m} Z_t^{i,m}$ . For each  $i$ , the particle  $Z_t^{i,m}$ 's motion satisfies the stochastic differential equation

$$dZ_t^{i,m} = dW_t^{i,m} + \left( \gamma 2^{-m} \sum_{j=1}^{2^m} (Z_t^{j,m} - Z_t^{i,m}) - b Z_t^{i,m} \right) dt,$$

where  $\gamma$  is the interaction coefficient, and  $b$  is the drift part of the Brownian motion. In our case, we consider  $b > 0$ .

Taking averages on both sides, the center of mass  $\bar{Z}_t$  will thus satisfy

$$d\bar{Z}_t = 2^{-m} \sum_{i=1}^{2^m} dW_t^{i,m} - b \bar{Z}_t dt.$$

As the Brownian components of different particles are independent, Brownian scaling yields that

$$2^{-m} \sum_{i=1}^{2^m} dW_t^{i,m} = 2^{-m/2} d\widetilde{W}_t,$$

where  $\widetilde{W}_t$  is standard Brownian motion in the time interval  $[m, m + 1)$ . We thus have

$$d\bar{Z}_t = 2^{-m/2} d\widetilde{W}_t - b \bar{Z}_t dt.$$

Hence, in the time interval  $[m, m + 1)$ , we have an Ornstein-Uhlenbeck process, while on  $[0, \infty)$ , the process  $\bar{Z}$  satisfies the general stochastic differential equation in the statement.  $\square$

After these preparations, we now turn to the attractive case.

**Theorem 2.4** (COM; Attraction). *If  $b > 0$ , then  $\lim_{t \rightarrow \infty} \bar{Z}_t = 0$  a.s.*

*Proof.* Recall the stochastic differential equation satisfied by COM. In order to prove the theorem, it is sufficient to check that the conditions of Lemma 2.2 are satisfied.

Clearly,  $\beta$  is decreasing and locally Lipschitz, as  $\beta(t) = 2^{-m/2}$  for  $t \in [m, m + 1)$ , and furthermore,

$$\sum_{m=1}^{\infty} m \beta^2(m) = \sum_{m=1}^{\infty} m 2^{-m} < \infty.$$

Thus,  $\lim_{t \rightarrow \infty} \bar{Z}_t = 0$  a.s.  $\square$

For the repulsive case ( $b < 0$ ), we have the following theorem.

**Theorem 2.5** (Exponential escape of the COM for repulsion). *For  $b < 0$ ,  $\lim_{t \rightarrow \infty} e^{bt} \bar{Z}_t = \mathcal{N}$  a.s., where  $\mathcal{N}$  is a normal variable with mean zero and*

$$\text{Var}(\mathcal{N}) = \frac{1 - e^{2b}}{|b|(2 - e^{2b})} \cdot I_d.$$

(Here  $I_d$  is the identity matrix.)

*Proof.* By the independence of the coordinate processes, it is enough to consider  $d = 1$ . In order to show the existence of the limit and to identify it, we are going to utilize the Dambis-Dubins-Schwarz Theorem. We will use the shorthand  $\bar{X}_t := e^{bt}Z_t$ .

More precisely, we are going to show that there exists a Brownian motion  $B$  on the same probability space where  $Z$  is defined, such that  $\bar{X}_t = B_{s(t)}$ ,  $P$ -a.s. Here  $t \mapsto s(t)$  is a deterministic time-change of  $t$ , mapping  $[0, \infty)$  to a finite interval, satisfying that  $\lim_{t \rightarrow \infty} s(t) = T$ , where

$$T = T(b) := \frac{1 - e^{2b}}{|b|(2 - e^{2b})}. \tag{2.4}$$

Let  $\langle U \rangle$  denote the increasing process for a process  $U$ . It then follows that

$$\lim_{t \rightarrow \infty} \bar{X}_t = \lim_{t \rightarrow \infty} B_{\langle \bar{X} \rangle_t} = B_{\lim_{t \rightarrow \infty} \langle \bar{X} \rangle_t} = B_T.$$

To achieve all these, recall first that by (2.3),  $\bar{X}_t = \int_0^t \beta(s)e^{bs} dW_s$ , and thus, it is a continuous martingale. Therefore by the Dambis-Dubins-Schwarz Theorem (see e.g. Theorem V.1.6 in [8]),  $\bar{X}_t$  is a time-changed Brownian motion:

$$\bar{X}_t = B_{\langle \bar{X} \rangle_t}, \text{ a.s.}$$

Since the increasing process is deterministic in this case, we have that

$$s(t) := \langle \bar{X} \rangle_t = \text{Var}(\bar{X}_t) = \int_0^t \beta^2(s)e^{2bs} ds,$$

where  $\beta(s) := 2^{-m/2}$  for  $s \in [m, m + 1)$ . Thus,  $\bar{X}_t = B_{s(t)}$ , almost surely, and furthermore,

$$\lim_{t \rightarrow \infty} s(t) = \sum_{m=0}^{\infty} \int_m^{m+1} 2^{-m} e^{2bs} ds = \sum_{m=0}^{\infty} 2^{-m} \cdot \frac{e^{2b(m+1)} - e^{2bm}}{2b}.$$

To evaluate the infinite sum, one can use Abel’s (summation by part) formula, which leads to:

$$\lim_{t \rightarrow \infty} s(t) = \sum_{m=0}^{\infty} 2^{-m} \cdot \frac{e^{2b(m+1)} - e^{2bm}}{2b} = \frac{e^{2b} - 1}{b(2 - e^{2b})} = T,$$

completing the proof. □

We note that without the Dambis-Dubins-Schwarz Theorem, much more elementary, standard arguments still prove the existence of the almost sure limit, but only along certain ‘discrete time skeletons.’

**Remark 2.6** (Exponential speed of COM). As  $\lim_{t \rightarrow \infty} e^{bt}\bar{Z}_t$  exists a.s. and  $e^{|b|t} \rightarrow \infty$ , the point  $\bar{Z}_t$  will tend to infinity, almost surely, with ‘speed’  $e^{|b|t}$  in the sense that  $\bar{Z}_t \approx e^{|b|t} \cdot \mathcal{N}$ . Furthermore, even in higher dimensions, it is clear by symmetry considerations that the angular component of  $\bar{Z}_t$  will be uniformly distributed.

Finally, it is easy to see that  $\lim_{b \rightarrow 0} T(b) = 2$ , in accordance with the already studied drift-less case. ◇

### 3 The system as viewed from the center (‘relative system’)

Having described the motion of the center of mass  $\bar{Z}_t$ , in order to study the whole system, we need to investigate the ‘relative system’, that is the system as viewed from  $\bar{Z}_t$ .



**Definition 3.1** (Relative system). Denote  $Y_t^i := Z_t^i - \bar{Z}_t$ . The particle system  $\{Y_t^i\}_{i=1}^{2^{[t]}}$  will be called the *relative system*, or *the system*, as viewed from the center of mass.

We focus on the behavior of the relative system in this section. We will use the shorthand  $\sigma_m^2 := 1 - 2^{-m}$ .

First, we want to determine the stochastic differential equation for  $Y_t^i$ . It can be obtained by direct computation, as follows. Fixing the time interval  $[m, m + 1)$ , recall that for each  $i$ , the particle  $Z_t^{i,m}$ 's motion satisfies

$$dZ_t^{i,m} = dW_t^{i,m} + \left( \gamma 2^{-m} \sum_{j=1}^{2^m} (Z_t^{j,m} - Z_t^{i,m}) - b Z_t^{i,m} \right) dt,$$

while  $\bar{Z}_t$  satisfies that

$$d\bar{Z}_t = 2^{-m} \sum_{i=1}^{2^m} dW_t^{i,m} - b \bar{Z}_t dt.$$

Subtracting the second equation from the first, one has

$$\begin{aligned} dY_t^i &= d(Z_t^i - \bar{Z}_t) = \\ &\sigma_m^2 dW_t^{i,m} + \sum_{j \neq i} -2^{-m} dW_t^{j,m} + \left( \gamma \bar{Z}_t - \gamma Z_t^{i,m} - b Z_t^{i,m} + b \bar{Z}_t \right) dt = \\ &\sigma_m^2 dW_t^{i,m} + \sum_{j \neq i} -2^{-m} dW_t^{j,m} - (\gamma + b) Y_t^i dt. \end{aligned}$$

As  $\{W_t^{i,m}\}$  are independent standard Brownian motions, a short computation shows that  $\sigma_m^2 W_t^{i,m} \oplus_{j \neq i} -2^{-m} W_t^{j,m}$  is a Brownian motion with variance  $\sigma_m^2 t$  at time  $t > 0$ . Hence,

$$dY_t^i = \sigma_m d\widetilde{W}_t^i - (\gamma + b) Y_t^i dt,$$

where  $\widetilde{W}_t^i$  is a driving standard Brownian motion for  $Y_t^i$ , such that

$$\sigma_m \widetilde{W}_t^i = \sigma_m^2 W_t^{i,m} \oplus_{j \neq i} -2^{-m} W_t^{j,m}.$$

When  $t \rightarrow \infty$ , (i.e.,  $m \rightarrow \infty$ ), the process  $Y^i$  will asymptotically satisfy the equation

$$dY_t^i = d\widetilde{W}_t^i - (\gamma + b) Y_t^i dt, \tag{3.1}$$

yielding that, for large times, the motion of  $Y_t^i$  is very close to the one governed by (3.1), namely, an

- (i) inward O-U process, if  $\gamma + b > 0$ ;
- (ii) outward O-U process, if  $\gamma + b < 0$ ;
- (iii) Brownian motion, if  $\gamma + b = 0$ .

As a next step, we need to study the correlation between the particles of  $\{Y_t^i\}$  for  $t > 0$ . As  $\sum_i \widetilde{W}_t^i = 0$ , they are obviously not independent.

First we determine the ‘degree of freedom’ of  $\{\widetilde{W}_t^i\}$ . Similarly to [4], one can show that the degree of freedom of  $\{\widetilde{W}_t^i\}$  is  $2^m - 1$ . To explain what this means, fix  $m \geq 1$  and for  $t \in [m, m + 1)$  let  $Y_t := (Y_t^1, \dots, Y_t^{2^m})^T$ , where  $()^T$  denotes transposed. (This is a vector of length  $2^m$  where each component itself is a  $d$ -dimensional vector; one can actually view it as a  $2^m \times d$  matrix too.) We then have

$$dY_t = \sigma_m d\widetilde{W}_t^{(m)} - \gamma Y_t dt,$$

where

$$\widetilde{W}^{(m)} = \left( \widetilde{W}^{m,1}, \dots, \widetilde{W}^{m,2^m} \right)^T$$

and

$$\widetilde{W}_\tau^{m,i} = \sigma_m^{-1} \left( W_\tau^{m,i} - 2^{-m} \bigoplus_{j=1}^{2^m} W_\tau^{m,j} \right), \quad i = 1, 2, \dots, 2^m$$

are mean zero, correlated Brownian motions.

Just like in subsection 1.3, here we can also consider  $Y$  as a single  $2^m d$ -dimensional diffusion. Each of its components is an Ornstein-Uhlenbeck process with asymptotically unit diffusion coefficient.

By independence, it is enough to consider the  $d = 1$  case.

Let us first describe the distribution of  $\widetilde{W}_t^{(m)}$  for  $t \geq 0$  fixed. Recall that  $\{W_s^{m,i}, s \geq 0; i = 1, 2, \dots, 2^m\}$  are independent Brownian motions. By definition,  $\widetilde{W}_t^{(m)}$  is a  $2^m$ -dimensional multivariate normal:

$$\begin{aligned} \widetilde{W}_t^{(m)} &= \sigma_m^{-1} \cdot \begin{pmatrix} 1 - 2^{-m} & -2^{-m} & \dots & -2^{-m} \\ -2^{-m} & 1 - 2^{-m} & \dots & -2^{-m} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & \dots & \cdot \\ -2^{-m} & -2^{-m} & \dots & 1 - 2^{-m} \end{pmatrix} W_t^{(m)} \\ &=: \sigma_m^{-1} \mathbf{A}^{(m)} W_t^{(m)}, \end{aligned}$$

where  $W_t^{(m)} = (W_t^{m,1}, \dots, W_t^{m,2^m})^T$ , yielding

$$dY_t = \mathbf{A}^{(m)} dW_t^{(m)} - \gamma Y_t dt.$$

Since we are viewing the system from the center of mass,  $\widetilde{W}_t^{(m)}$  is a *singular* multivariate normal and thus  $Y$  is a degenerate diffusion. The ‘true’ dimension of  $\widetilde{W}_t^{(m)}$  is  $\text{rank}(\mathbf{A}^{(m)})$ . Then the same argument as in [4], yields that  $\text{rank}(\mathbf{A}^{(m)}) = 2^m - 1$ , and the above comment about the degrees of freedom should be understood in this sense.

Moreover, the driving Brownian motions  $\{\widetilde{W}_t^i\}$  will be exactly the same as in [4], and thus they have asymptotically vanishing correlation (see Remark 12 in [4]).

The relative system thus coincides with the drift-less one in [4], if  $\gamma$  is replaced by  $\gamma + b$ .

#### 4 A useful transformation: the process $Z^\Delta$

We first make an important observation, making the last sentence of the previous section more general: we notice that  $\gamma$  and  $b$  are ‘interchangeable’ in the following sense.

**Lemma 4.1** (Interchangeable coefficients). Suppose that we have two branching particle systems, and  $Y$  and  $\mathcal{Y}$  represent the relative systems for them. Denote by  $b_1, \gamma_1$  and  $b_2, \gamma_2$  the corresponding coefficients of  $Y$  and  $\mathcal{Y}$ . (Here  $\gamma_i = 0$  is allowed.) Assume that  $b_1 + \gamma_1 = b_2 + \gamma_2$ . Then the laws of  $Y$  and  $\mathcal{Y}$  are the same.

*Proof.* Fix  $m \geq 0$  and  $i = 1, 2, \dots, 2^m$ . Then  $Y^i$  and  $\mathcal{Y}^i$  satisfy the same stochastic differential equation

$$dY_t^i = (1 - 2^{-m})^{1/2} d\widetilde{W}_t^i - \gamma Y_t^i dt$$

in the interval  $t \in [m, m + 1)$ , where  $\gamma = b_1 + \gamma_1 = b_2 + \gamma_2$ . Thus, the fact that single particles have the same law in the two systems, follows by induction, along with the

existence and uniqueness of the solution for stochastic differential equations. (We know that for  $m = 0$  they start with the same initial value.)

The fact that even the *joint* distributions of the two particle systems agree, follows the same way as we proved existence and uniqueness for the model in subsection 1.3, except that now the independent driving Brownian motions must be replaced by  $\sigma_m^{-1} \mathbf{A}^{(m)} W_t^{(m)}$  in  $[m, m + 1)$  (recall (3.2)). Since the piecewise Lipschitz-ness of the coefficients is preserved, the existence and uniqueness theorem is still in force.  $\square$

We now define a transformation which plays a crucial role in this paper.

**Definition 4.2** ( $\Delta$ -transformation). Consider  $Z$  with  $\gamma$  and  $b$  given and let  $\mathcal{Z} = Z^\Delta$  be another system with parameters

$$\begin{aligned} \gamma^\Delta &:= \gamma - \Delta; \\ b^\Delta &:= b + \Delta. \end{aligned}$$

Since  $b + \gamma = b^\Delta + \gamma^\Delta$ , we know by Lemma 4.1 that the corresponding relative systems  $Y$  and  $\mathcal{Y}$  have the same law.

Consider  $Z$  and  $\mathcal{Z}$  on the same probability space. Then

$$Z_t(B) = Y_t(B - \bar{Z}_t) \stackrel{d}{=} \mathcal{Y}_t(B - \bar{Z}_t) = \mathcal{Z}_t(B - (\bar{Z}_t - \bar{Z}_t)), \tag{4.1}$$

for  $B \subset \mathbb{R}^d$  Borel and  $t \geq 0$ . In fact,

$$Z \text{ has the same law as the process } t \mapsto \mathcal{Z}_t(\cdot - D_t), \tag{4.2}$$

where  $D_t := \bar{Z}_t - \bar{Z}_t$ . The behavior of  $D_t$  for large times depends on the signs of  $b$  and  $b + \gamma$ . E.g. if they are both positive, then  $D_t$  tends to the origin almost surely.

In particular, with appropriate transformations we can ‘knock out’ either the interaction or the O-U drift:

(a) **Representation with non-interactive system:** pick  $\Delta := \gamma$ . That is, let the non-interactive process  $\mathcal{Z}$  correspond to  $\gamma^\Delta = 0$  and  $b^\Delta := b + \gamma$ . Then (4.2) gives a remarkable link between the interactive and the non-interactive models. This connection is reminiscent of the one found in [6] (see Remark 3.2 there).

(b) **Eliminating the O-U drift:** pick  $\Delta := -b$ . That is, the motion component of  $Z^\Delta$  is just a Brownian motion, similarly to [4]. Then (4.2) gives a link between our model and the drift-less one studied in [4].

## 5 Outline of the strategy of the rest of the proofs

In light of the previous section, we could choose to base the analysis of the relative system on the corresponding result in [4] when  $b + \gamma > 0$  (by eliminating the drift – see part (b) in the previous section), or on the results on the global system in Example 11 of [5] when  $b + \gamma < 0$  (by eliminating the interaction – see part (a) in the previous section).

In the second case, we should handle the problem that the setting is different in [5] in that the branching is not unit time but rather exponential.

On top of that, the method of the proof in both [4] and [5] requires the introduction of two non-trivial auxiliary functions, related to the model.

Instead of choosing one of the paths alluded to above, we decided to give a completely elementary proof in the next section for the Strong Law for the relative system in our case, when  $b + \gamma > 0$ . It does not use the complicated machinery of [4] or [5], and it is done in unit time. The proof only uses some calculations involving the most recent common ancestors of particles and some covariance estimates.

In particular, it gives a new, elementary proof for the Strong Law for the global system, for the case of a non-interactive branching (inward) Ornstein-Uhlenbeck process. See Example 7.5.

## 6 Proof of SLLN for the relative system when $b + \gamma > 0$

### 6.1 General comments

Recall that  $\sigma_m^2 := 1 - 2^{-m}$ , and that for the relative system  $Y$ ,

$$dY_t^i = \sigma_m d\widetilde{W}_t^i - (b + \gamma)Y_t^i dt$$

for  $i = 1, 2, 3, \dots, 2^m$  in the interval  $t \in [m, m + 1)$ . Recall also that

$$\sigma_m d\widetilde{W}_t^i = \sigma_m^2 dW_t^{i,m} \bigoplus_{j \neq i} -2^{-m} dW_t^{j,m}.$$

Assuming  $b + \gamma > 0$ , our goal is to find  $\lim_{m \rightarrow \infty} \frac{1}{2^m} \sum_{i=1}^{2^m} Y_t^i(B)$  for a generic Borel set  $B \subset \mathbb{R}^d$ . (Here, we consider  $Y_t$  as a random measure, and that is why we may write  $Y_t^i(B) = 1_B(Y_t^i)$ ,  $i = 1, 2, 3, \dots, 2^m$ .) For  $d = 1$ , this will be achieved in Theorem 6.11 and subsequently, it will be upgraded to higher dimensions; before these, we will prove several preparatory results.

**Remark 6.1** (Integer vs. continuous times). When taking the limit, we will just consider integer times. This is somewhat weaker than considering continuous times, however, since the model is about unit time branching, we did not have sufficient motivation to go into the technical details as to how one upgrades the limit along integer times to a limit along continuous times. (There are existing techniques though, going back to the work of Asmussen and Hering, see [1, 5].)  $\diamond$

As mentioned above, for simplicity we will first treat  $d = 1$ , and then show that the main result we got also works for higher dimensions.

Next, let us sketch our *strategy* of the investigation:

- (1) Find the correlation between the particle positions  $Y_m^i, 1 \leq i \leq 2^m; m \geq 1$ .
- (2) Use the correlations in (1) to control the correlation between  $Y_m^i(B)$ , where  $B \subset \mathbb{R}^d$ .
- (3) Establish the Strong Law of Large Numbers for  $Y$ , that is, find a measurable function  $f \geq 0$  such that for  $B \subset \mathbb{R}^d$  Borel,

$$\lim_{m \rightarrow \infty} \frac{1}{2^m} \sum_{i=1}^{2^m} Y_m^i(B) = \int_B f(x) dx \text{ a.s.}$$

**Assumption 6.2** (No drift). In this section, when studying the relative system, we will assume that  $b = 0$  and  $\gamma > 0$ . We can do this without the loss of generality, since given  $\gamma$  and  $b$ , one may apply a  $\Delta$ -transformation with  $\Delta := -b$ , that is, eliminate the O-U drift. Then  $b^\Delta = 0$  and  $\gamma^\Delta = b + \gamma$ , and  $Y$  and  $\mathcal{Y}$  have the same law by Lemma 4.1.

### 6.2 Crucial estimates

Now let us focus on the distribution of the random variables  $Y_t^i, 1 \leq i \leq 2^{\lfloor t \rfloor}, t \geq 0$ . Since  $dY_t^i = \sigma_m d\widetilde{W}_t^i - \gamma Y_t^i dt$ , and since for the general differential equation

$$dY_t = a dW_t - rY_t dt$$

( $a, r > 0$ ), with initial value  $Y_0$ , the solution is

$$Y_t = e^{-rt} \left[ \int_0^t e^{rs} d(aW_s) + Y_0 \right],$$

it follows, by conditioning on  $Y_m^i$ , that

$$Y_{m+1}^i = e^{-\gamma} \left[ \int_m^{m+1} e^{\gamma(s-m)} d(\sigma_m \widetilde{W}_s^i) + Y_m^i \right] \text{ a.s.}$$

We know that  $\{\widetilde{W}_s^i\}_{0 \leq s < 1}$  is a Brownian motion on the time interval  $[m, m + 1]$  and that it is independent of  $Y_m^i$ . Also, by symmetry, the distributions of  $Y_t^i$  and  $Y_t^j$  are the same, so we will just write  $Y_t^i$ .

We thus have  $Y_{m+1}^i = e^{-\gamma}(Y_m^i \oplus X_m^i)$ , where

$$X_m^i = \int_m^{m+1} e^{\gamma(s-m)} d(\sigma_m \widetilde{W}_s^i) \tag{6.1}$$

is a normal variable with distribution  $\mathcal{N}(0, \sigma_m^2(2\gamma)^{-1}(e^{2\gamma} - 1))$ , and  $Y_0^i = 0$ . Note that  $Y_m^i$  and  $X_m^i$  (or more generally,  $X_m$  and  $Y_m$ ) are independent, as  $X_m$  is defined by a stochastic integral on  $[m, m + 1]$ .

**Proposition 6.3** (Variance; single particle). *For any integer  $m \geq 0$ ,*

$$\text{Var}(Y_{m+1}^i) = \sigma^2(Y_{m+1}^i) = \frac{e^{2\gamma} - 1}{2\gamma} e^{-2m\gamma} \left( \frac{1 - e^{2m\gamma}}{1 - e^{2\gamma}} - \frac{1 - (\frac{1}{2}e^{2\gamma})^m}{2 - e^{2\gamma}} \right).$$

*Proof.* Since  $Y_{m+1} = e^{-\gamma}(Y_m \oplus X_m)$  and  $Y_0 = 0$ , we can compute the variance of  $Y_{m+1}$  using recursion, due to the independence of  $Y_m$  and  $X_m$ . First, we have

$$\begin{aligned} \sigma^2(Y_{m+1}) &= e^{-2\gamma} (\sigma^2(Y_m) + \sigma^2(X_m)) \\ &= e^{-2\gamma} \left( \sigma^2(Y_m) + \frac{1}{2\gamma} \left( 1 - \frac{1}{2^m} \right) (e^{2\gamma} - 1) \right), \end{aligned}$$

and  $\sigma^2(Y_0) = 0$ .

Thus, for convenience, we denote  $a_m := \sigma^2(Y_m)$  and

$$b_m := \sigma^2(X_m) = \frac{1}{2\gamma} \left( 1 - \frac{1}{2^m} \right) (e^{2\gamma} - 1).$$

Since  $a_{m+1} = e^{-2\gamma}(a_m + b_m)$ , we have the following recursion:

$$\begin{aligned} a_{m+1} &= e^{-2\gamma}(e^{-2\gamma}(a_{m-1} + b_{m-1}) + b_m) \\ &= (e^{-2\gamma})^2 a_{m-1} + (e^{-2\gamma})^2 b_{m-1} + e^{-2\gamma} b_m \\ &= (e^{-2\gamma})^3 a_{m-2} + (e^{-2\gamma})^3 b_{m-2} + (e^{-2\gamma})^2 b_{m-1} + e^{-2\gamma} b_m \\ &= (e^{-2\gamma})^{m+1} a_0 + (e^{-2\gamma})^{m+1} b_0 \\ &\quad + \dots + (e^{-2\gamma})^3 b_{m-2} + (e^{-2\gamma})^2 b_{m-1} + e^{-2\gamma} b_m. \end{aligned}$$

Using that  $a_0 = b_0 = 0$ , summation leads to

$$a_{m+1} = \frac{e^{2\gamma} - 1}{2\gamma} e^{-2m\gamma} \left[ \frac{1 - e^{2m\gamma}}{1 - e^{2\gamma}} - \frac{1 - (\frac{1}{2}e^{2\gamma})^m}{2 - e^{2\gamma}} \right], \quad m \geq 1.$$

Using the definition of  $a_{m+1}$ , the proof is complete. □

We now need to analyze the covariance between  $Y_m^i$  and  $Y_m^j$  for  $1 \leq i, j \leq 2^m$  and  $m \geq 1$ . Here the notion of the *most recent common ancestor* (MRCA) becomes important.

**Definition 6.4** (MRCA). Consider  $m \geq 0$  and the particles  $Y_m^i, Y_m^j, 1 \leq i \neq j \leq 2^m$ . A common ancestor is a particle considered at some integer time interval  $[a - 1, a)$  with  $0 < a < m$ , which is an ancestor to both  $Y_m^i$  and  $Y_m^j$ . The MRCA of the particles is a common ancestor such that  $a$  is maximal. This  $a$  is then called the *splitting time* of the particles.

Note that in this definition, we consider particles living in different time intervals  $[m, m + 1)$  as different particles.

As different pairs of particles  $(Y_m^i, Y_m^j)$  with different MRCA's may have different covariances, we need to take into account the MRCA of  $(Y_m^i, Y_m^j)$ . Hence, with splitting time  $a > 0$ , we will write  $(Y_{m,a}^i, Y_{m,a}^j)$  for  $(Y_m^i, Y_m^j)$ .

**Theorem 6.5** (Covariance bound). For  $i, j, m \in \mathbb{N}$  with  $m \geq 1$  and  $1 \leq i \neq j \leq 2^m$ ,

$$\text{Cov}(Y_{m,a}^j, Y_{m,a}^i) < Cm(e^{-2\gamma(m-a)} + 2^{-m}),$$

where  $C$  is a constant which only depends on  $\gamma$ .

*Proof.* Since  $Y_{m+1}^i = e^{-\gamma}(Y_m^i \oplus X_m^i)$  for any 'relative particle,' it follows by recursion that

$$Y_m^i = e^{-\gamma(m-a)}Y_a^i + \sum_{n=a}^{m-1} e^{-\gamma(m-n)}X_n^i.$$

Here  $X_n^i$  is the process as in (6.1) on the unit time interval  $[n, n + 1]$ . From the preceding formula we can easily compute  $\text{Cov}(Y_m^i, Y_m^j)$ , once we know the corresponding  $\text{Cov}(X_n^i, X_n^j)$  for  $n = a, a + 1, \dots, m - 1$ . For the  $i$ th particle ( $i$  can be replaced by  $j$ , of course),

$$X_n^i = \int_n^{n+1} e^{\gamma(s-n)} d(\sigma_n \widetilde{W}_s^i),$$

where  $\sigma_n \widetilde{W}_s^i = -\frac{1}{2^n} \bigoplus_{k \neq i} W_s^k \oplus \sigma_n^2 W_s^i$  and  $\{W_s^k, k = 1, 2, 3, \dots, 2^n\}$ , as well as  $\widetilde{W}^i$  are standard independent Brownian motions on the unit time interval  $[n, n + 1]$ .

Using that the  $\{W_s^k, k = 1, 2, \dots, 2^n\}$  are independent,  $\text{Cov}(X_n^i, X_n^j)$  can now be computed. Indeed, the stochastic integrals  $X_n^i, X_n^j$  are just linear combinations of the stochastic integrals  $A_k := \int_n^{n+1} e^{\gamma(s-n)} dW_s^k, k = 1, 2, 3, \dots$ , that is,

$$\begin{aligned} X_n^i &= -\frac{1}{2^n} \sum_{k \neq i} A_k + \sigma_n A_i; \\ X_n^j &= -\frac{1}{2^n} \sum_{k \neq j} A_k + \sigma_n A_j. \end{aligned}$$

Note that, by the Itô-isometry,

$$\text{Var}(A_k) = \frac{e^{2\gamma} - 1}{2\gamma}.$$

Therefore, using the independence of the  $A_k$ 's,

$$\begin{aligned} &\text{Cov}(X_n^i, X_n^j) \\ &= \text{Cov} \left( -\frac{1}{2^n} \sum_{k \neq i} A_k + \left(1 - \frac{1}{2^n}\right) A_i, -\frac{1}{2^n} \sum_{k \neq j} A_k + \left(1 - \frac{1}{2^n}\right) A_j \right) \\ &= \frac{1}{2^{2n}} \sum_{k \neq i, j} \text{Var}(A_k) - \frac{1}{2^n} \left(1 - \frac{1}{2^n}\right) \text{Var}(A_i) - \frac{1}{2^n} \left(1 - \frac{1}{2^n}\right) \text{Var}(A_j) \end{aligned}$$

$$\begin{aligned}
 &= \left( \frac{1}{2^{2n}} (2^n - 2) - \frac{2}{2^n} \left( 1 - \frac{1}{2^n} \right) \right) \frac{e^{2\gamma} - 1}{2\gamma} \\
 &= -\frac{1}{2^n} \frac{e^{2\gamma} - 1}{2\gamma}.
 \end{aligned} \tag{6.2}$$

Since  $Y_m^i = e^{-\gamma(m-a)} Y_a^i + \sum_{n=a}^{m-1} e^{-\gamma(m-n)} X_n^i$ , and the  $\{X_k^i\}$  are independent for different  $k$ 's and finally, since  $Y_a^i = Y_a^j$ , it follows that<sup>3</sup>

$$\text{Cov}(Y_m^i, Y_m^j) = e^{-2\gamma(m-a)} \text{Var}(Y_a^i) + \sum_{k=a}^{m-1} e^{-2\gamma(m-k)} \text{Cov}(X_k^i, X_k^j). \tag{6.3}$$

According to Proposition 6.3,

$$\text{Var}(Y_a^i) = \frac{e^{2\gamma} - 1}{2\gamma} e^{-2\gamma(a-1)} \left[ \frac{1 - e^{2\gamma(a-1)}}{1 - e^{2\gamma}} - \frac{1 - (\frac{1}{2}e^{2\gamma})^{a-1}}{2 - e^{2\gamma}} \right].$$

Using this along with (6.2) and (6.3), one obtains that

$$\begin{aligned}
 \text{Cov}(Y_m^i, Y_m^j) &= \frac{e^{2\gamma} - 1}{2\gamma} \left( e^{-2\gamma(m-1)} \left[ \frac{1 - e^{2\gamma(a-1)}}{1 - e^{2\gamma}} - \frac{1 - (\frac{1}{2}e^{2\gamma})^{a-1}}{2 - e^{2\gamma}} \right] \right. \\
 &\quad \left. + \sum_{k=a}^{m-1} e^{-2\gamma(m-k)} \left( -\frac{1}{2^k} \right) \right).
 \end{aligned}$$

It then follows that

$$\text{Cov}(Y_m^i, Y_m^j) < C_0(I_1 + I_2),$$

where

$$\begin{aligned}
 I_1 &= e^{-2\gamma(m-1)} \left[ \frac{1 - e^{2\gamma(a-1)}}{1 - e^{2\gamma}} - \frac{1 - (\frac{1}{2}e^{2\gamma})^{a-1}}{2 - e^{2\gamma}} \right], \\
 I_2 &= \sum_{k=a}^{m-1} e^{-2\gamma(m-k)} \left( -\frac{1}{2^k} \right),
 \end{aligned}$$

and  $C_0 = C_0(\gamma) := \frac{e^{2\gamma}-1}{2\gamma}$ . As  $\gamma > 0$ , one has  $|I_1| < C_1 e^{-2\gamma(m-a)}$ , where  $C_1$  is a constant which only depends on  $\gamma$ . Moreover,  $I_2 = e^{-2m\gamma} \sum_{k=a}^{m-1} \left( -(\frac{e^{2\gamma}}{2})^k \right)$ . Then, by an easy computation,

$$|I_2| < C_2 \left( m e^{-2m\gamma} + \frac{1}{2^m} \right),$$

where  $C_2$  is also a constant which only depends on  $\gamma$ .

Hence,  $\text{Cov}(Y_m^i, Y_m^j)$  is bounded from above by

$$C_0 \left( C_1 e^{-2\gamma(m-a)} + C_2 \left( m e^{-2m\gamma} + \frac{1}{2^m} \right) \right) < C m (e^{-2\gamma(m-a)} + 2^{-m}),$$

where  $C$  is a constant that only depends on  $\gamma$ . □

<sup>3</sup>In this formula,  $i > 2^k$  may occur. Then  $X_k^i$  stands for the normal variable associated with the ancestor of  $i$  at time  $k$ . The definition is similar to that of  $X_n^i$ .

Recall that our goal is to prove the existence of  $\lim_{m \rightarrow \infty} 2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i)$  for Borel sets  $B \subset \mathbb{R}^d$  and identify them. To achieve this, we will use the Borel-Cantelli Lemma in conjunction with the Chebyshev inequality, and so we need to estimate  $\text{Var} \left( 2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i) \right)$ . Clearly,

$$\text{Var} \left( \sum_{i=1}^{2^m} 1_B(Y_m^i) \right) = \sum_{i=1}^{2^m} \text{Var}(1_B(Y_m^i)) + \sum_{\substack{i=1 \\ i \neq j}}^{2^m} \sum_{j=1}^{2^m} \text{Cov}(1_B(Y_m^i), 1_B(Y_m^j)),$$

and

$$\text{Var}(1_B(Y_m^i)) = P(Y_m^i \in B) - P^2(Y_m^i \in B) \leq 1/4.$$

We need to compute  $\text{Cov}(1_B(Y_m^i), 1_B(Y_m^j))$  for  $i \neq j$ , and in order to do that, we need to analyze the splitting time (still denoted by  $a$ ) for the particles  $Y_m^i$  and  $Y_m^j$ . The following lemma on joint normal distribution will be useful.

**Lemma 6.6** (Covariance for indicators). *Let  $(X, Y)$  be a joint normal vector such that its marginals  $X$  and  $Y$  are standard normal, and denote  $\rho := \text{Cov}(X, Y)$ . Then there exists an absolute constant  $C > 0$  such that*

$$\text{Cov}(1_B(X), 1_B(Y)) \leq C|\rho|$$

holds for all Borel sets  $B \subset \mathbb{R}$  and all  $|\rho| < 1/2$ .

*Proof.* Plugging in the joint and marginal densities

$$\frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{x^2+y^2+2\rho xy}{2(1-\rho^2)}} \text{ and } \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

one obtains

$$\begin{aligned} \psi(\rho, B) &:= \text{Cov}(1_B(X), 1_B(Y)) \\ &= P(X \in B, Y \in B) - P(X \in B) \cdot P(Y \in B) \\ &= \frac{1}{2\pi} \iint_{B \times B} \frac{1}{\sqrt{1-\rho^2}} e^{-\frac{x^2+y^2+2\rho xy}{2(1-\rho^2)}} - e^{-\frac{x^2-y^2}{2}} \, dx dy \\ &=: \frac{1}{2\pi} \iint_{B \times B} f(x, y, \rho) \, dx dy. \end{aligned}$$

Clearly,  $\psi(0, B) = 0$ . Since  $f \in C^\infty(\mathbb{R} \times \mathbb{R} \times (-1/2, 1/2))$ ,

$$\psi'(\rho, B) = \frac{1}{2\pi} \iint_{B \times B} \frac{\partial f(x, y, \rho)}{\partial \rho} \, dx dy,$$

where

$$\frac{\partial f(x, y, \rho)}{\partial \rho} = e^{-\frac{x^2+y^2+2\rho xy}{2(1-\rho^2)}} \cdot \left( \frac{\rho}{(1-\rho^2)^{\frac{3}{2}}} - \frac{1}{\sqrt{1-\rho^2}} \frac{k(x, y, \rho)}{4(1-\rho^2)^2} \right),$$

with  $k(x, y, \rho) := 2xy2(1-\rho^2) + 4\rho(x^2 + y^2 + 2\rho xy)$ . For  $|\rho| \leq \frac{1}{2}$ , we have  $\frac{x^2+y^2}{2} \leq x^2 + y^2 + 2\rho xy$ , i.e.  $x^2 + y^2 + 4\rho xy \geq 0$ , and thus,

$$\exp \left\{ -\frac{x^2 + y^2 + 2\rho xy}{2(1-\rho^2)} \right\} \leq \exp \left\{ -\frac{x^2 + y^2}{4} \right\}.$$



Also,  $k(x, y, \rho) \leq 2(x^2 + y^2) + 2(2(x^2 + y^2)) \leq 6(x^2 + y^2)$ . Hence, with some  $C > 0$  constant,

$$\left| \frac{\partial f(x, y, \rho)}{\partial \rho} \right| \leq C e^{-\frac{x^2+y^2}{4}} (1 + x^2 + y^2), \quad \forall |\rho| \leq 1/2.$$

Consequently,

$$\begin{aligned} |\psi'(\rho, B)| &\leq \frac{1}{2\pi} \iint_{B \times B} C e^{-\frac{x^2+y^2}{4}} (1 + x^2 + y^2) \, dx dy \\ &\leq \frac{1}{2\pi} \iint_{\mathbb{R} \times \mathbb{R}} C e^{-\frac{x^2+y^2}{4}} (1 + x^2 + y^2) \, dx dy \\ &\leq C \int_0^\infty e^{-r^2/4} r(r^2 + 1) dr =: C' < \infty, \end{aligned}$$

yielding

$$|\psi(\rho, B)| = \left| \int_0^\rho \psi'(s, B) ds \right| \leq \int_0^{|\rho|} |\psi'(s, B)| ds \leq C' |\rho|.$$

□

**Remark 6.7.** More generally, let both  $X$  and  $Y$  be  $\mathcal{N}(0, \sigma^2)$ -distributed. If  $(X, Y)$  is joint normal with  $\text{Cov}(X, Y) = \rho$ , then we can scale  $X$  and  $Y$  to use Lemma 6.6. Indeed,  $\frac{X}{\sigma}$  and  $\frac{Y}{\sigma}$  are then standard normal variables, and  $\text{Cov}(\frac{X}{\sigma}, \frac{Y}{\sigma}) = \frac{\rho}{\sigma^2}$ . From Lemma 6.6, if  $\frac{|\rho|}{\sigma^2} < 1/2$  ( $2|\rho| < \sigma^2$ ), then

$$\text{Cov}(1_B(X), 1_B(Y)) = \text{Cov}\left(1_{\frac{B}{\sigma}}\left(\frac{X}{\sigma}\right), 1_{\frac{B}{\sigma}}\left(\frac{Y}{\sigma}\right)\right) \leq \frac{C|\rho|}{\sigma^2}. \tag{6.1}$$

◇

Returning to the question of covariances with splitting time  $a$ , note that  $\{Y_m^i\}$  are linear combinations of a number of underlying Brownian motions; hence  $(Y_m^i, Y_m^j)$  will be joint normal. From Theorem 6.5, we know that  $\text{Cov}(Y_m^i, Y_m^j) \leq C(\gamma)m(e^{-2\gamma(m-a)} + 2^{-m})$ . From (6.1), we then have the following corollary.

**Corollary 6.8.** *With some constant  $C > 0$  (that depends only on  $\gamma$ ),*

$$\text{Cov}(1_B(Y_m^i), 1_B(Y_m^j)) \leq \frac{Cm(e^{-2\gamma(m-a)} + 2^{-m})}{\sigma^2(Y_m^i)},$$

*provided  $2m(e^{-2\gamma(m-a)} + 2^{-m}) < \sigma^2(Y_m^i)$ . (Of course,  $\sigma^2(Y_m^i) = \sigma^2(Y_m^j)$ .)*

**Remark 6.9.** From Proposition 6.3, it follows that  $\lim_{m \rightarrow \infty} \sigma^2(Y_m^i) = \frac{1}{2\gamma}$ . Therefore, if we write  $a(m)$  in place of  $a$ , and  $\lim_{m \rightarrow \infty} (m - a(m)) = \infty$ , then the condition in Corollary 6.8 will always be true for large enough  $m$ 's. ◇

### 6.3 Strong Law for the relative system

First we need a technical (but very simple) lemma that shows that it will be sufficient to handle the one dimensional case.

**Lemma 6.10** (Control by coordinates). *Consider an open rectangle  $B$  in  $\mathbb{R}^d$ , that is,  $B = B_1 \times B_2 \times B_3 \cdots \times B_d$ , where  $B_i$  is an open interval in  $\mathbb{R}$  for  $i = 1, 2, \dots, d$ . Let  $X = (X_1, X_2, \dots, X_d)$  and  $Y = (Y_1, Y_2, \dots, Y_d)$  be two random vectors in  $\mathbb{R}^d$  satisfying that the pairs  $(X_1, Y_1), (X_2, Y_2), \dots, (X_d, Y_d)$  are independent. Then*

$$|\text{Cov}(1_B(X), 1_B(Y))| \leq \sum_{i=1}^d |\text{Cov}(1_{B_i}(X_i), 1_{B_i}(Y_i))|.$$

*Proof.* One has

$$\begin{aligned} & \text{Cov}(1_B(X), 1_B(Y)) \\ &= P(X \in B, Y \in B) - P(X \in B)P(Y \in B) \\ &= P(X_1 \in B_1, \dots, X_d \in B_d, Y_1 \in B_1, \dots, Y_d \in B_d) \\ &\quad - P(X_1 \in B_1, \dots, X_d \in B_d)P(Y_1 \in B_1, \dots, Y_d \in B_d). \end{aligned}$$

Using the assumption,

$$\begin{aligned} & P(X_1 \in B_1, \dots, X_d \in B_d) \\ &\quad = P(X_1 \in B_1)P(X_2 \in B_2) \cdots P(X_d \in B_d); \\ & P(Y_1 \in B_1, \dots, Y_d \in B_d) \\ &\quad = P(Y_1 \in B_1)P(Y_2 \in B_2) \cdots P(Y_d \in B_d), \end{aligned}$$

and

$$\begin{aligned} & P(X_1 \in B_1, \dots, X_d \in B_d, Y_1 \in B_1, \dots, Y_d \in B_d) \\ &\quad = P(X_1 \in B_1, Y_1 \in B_1)P(X_2 \in B_2, Y_2 \in B_2) \cdots P(X_d \in B_d, Y_d \in B_d). \end{aligned}$$

Using the shorthands  $a_i := P(X_i \in B_i, Y_i \in B_i)$ ,  $b_i := P(X_i \in B_i)P(Y_i \in B_i)$ , one has  $\text{Cov}(1_{B_i}(X_i), 1_{B_i}(Y_i)) = a_i - b_i$ , and from the computation above,

$$\text{Cov}(1_B(X), 1_B(Y)) = a_1 a_2 \cdots a_d - b_1 b_2 \cdots b_d.$$

Therefore the statement becomes

$$|a_1 a_2 \cdots a_d - b_1 b_2 \cdots b_d| \leq |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_d - b_d|.$$

Use that  $0 \leq a_i \leq 1$  and  $0 \leq b_i \leq 1$  and induction on  $d$  as follows. The statement is true for  $d = 1$ , and if it is true for some  $d \geq 1$ , then

$$\begin{aligned} & |a_1 a_2 \cdots a_d a_{d+1} - b_1 b_2 \cdots b_d b_{d+1}| \\ &= |(a_1 a_2 \cdots a_d - \cdots - b_1 b_2 \cdots b_d) a_{d+1} + (a_{d+1} - b_{d+1}) b_1 b_2 \cdots b_d| \\ &\leq |(a_1 a_2 \cdots a_d - \cdots - b_1 b_2 \cdots b_d) a_{d+1}| + |(a_{d+1} - b_{d+1}) b_1 b_2 \cdots b_d| \\ &\leq |(a_1 a_2 \cdots a_d - \cdots - b_1 b_2 \cdots b_d)| + |a_{d+1} - b_{d+1}| \\ &\leq |a_1 - b_1| + |a_2 - b_2| + \cdots + |a_d - b_d| + |a_{d+1} - b_{d+1}|, \end{aligned}$$

and so it is also true for  $d + 1$ . □

Now we are ready to prove the strong law.

**Theorem 6.11 (SLLN).** Assume that  $\gamma + b > 0$  and let  $B \subset \mathbb{R}^d$  be a Borel set. Then, almost surely,

$$\lim_{m \rightarrow \infty} \left( 2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i) - P(Y_m^1 \in B) \right) = 0.$$

Note: Here, of course  $\lim_{m \rightarrow \infty} P(Y_m^1 \in B)$  exists. That is, similarly to [4],  $\lim_{m \rightarrow \infty} 2^{-m} Y_m(dy) = f(y) dy$  a.s. in the weak topology, where  $f$  is the density for  $\mathcal{N}\left(0, \frac{1}{2^\gamma} I_d\right)$  in  $\mathbb{R}^d$ . (Since the labeling is independent of the spatial positions,  $Y_m^1$  is just a generic particle.)

*Proof.* First, because of Assumption 6.2 and Lemma 6.10, we may and will assume that  $d = 1$ ,  $b = 0$  and  $\gamma > 0$ .

Now, we need to show that for any  $\epsilon > 0$ , a.s., only finitely many of the events

$$A_m := \left\{ \left| 2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i) - E(1_B(Y_m^1)) \right| > \epsilon \right\}$$

will occur. By the Borel-Cantelli Lemma and Chebyshev's inequality, it is enough to show that

$$\sum_1^\infty P(A_m) \leq \epsilon^{-2} \sum_1^\infty \text{Var} \left( 2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i) \right) < \infty.$$

(Note that  $E \left( 2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i) \right) = E(1_B(Y_m^1))$ .) Thus, it remains to show that

$$\sum_{m=1}^\infty \text{Var} \left( 2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i) \right) < \infty. \tag{6.2}$$

To this end, note that

$$\begin{aligned} \text{Var} \left( 2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i) \right) &= \frac{1}{2^{2m}} \left( \sum_{i=1}^{2^m} \text{Var}(1_B(Y_m^i)) \right. \\ &\quad \left. + \sum_{i=1}^{2^m} \sum_{j=1, j \neq i}^{2^m} \text{Cov}(1_B(Y_m^i), 1_B(Y_m^j)) \right). \end{aligned} \tag{6.3}$$

Since  $\text{Var}(1_B(Y_m^i)) \leq 1/2 - 1/4 = 1/4$ , we have

$$\text{Var} \left( 2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i) \right) \leq \frac{1}{2^{2m}} \left( \frac{2^m}{4} + 2^m \sum_{j \neq i} \text{Cov}(1_B(Y_m^i), 1_B(Y_m^j)) \right),$$

for a fixed  $i$ , since the roles of  $i$  and  $j$  are symmetric.

Hence, one needs to analyze  $\sum_{j \neq i} \text{Cov}(1_B(Y_m^i), 1_B(Y_m^j))$  for a fixed  $i$ . We know that  $\text{Cov}(1_B(Y_m^i), 1_B(Y_m^j))$  depends on the time  $a$  when the MRCA of these particles splits. We thus need to distinguish between 'close relatives' and other pairs.

Notice that there are  $2^k$  particles which have the MRCA at time  $m - k$  with  $Y_m^i$ . From Theorem 6.5, we know that

$$\text{Cov}(Y_m^j, Y_m^i) < Cm(e^{-2\gamma(m-a)} + 2^{-m}),$$

if the MRCA of  $i$  and  $j$  is  $a$ . If  $a = a(m) = m/2$ , (or just  $m - a$  tends to  $\infty$ ), then the righthand side converges to zero as  $m \rightarrow \infty$ . As already seen,  $\lim_{m \rightarrow \infty} \text{Var}(Y_m^i) = \frac{1}{2\gamma}$ , and so we may apply Lemma 6.6 and the remark following it for  $a = a(m) \leq \frac{m}{2}$  and large  $m$ . That is, we could choose a large  $N$  such that for all  $m > N$ , the condition

$$\frac{\text{Cov}(Y_m^i, Y_m^j)}{\text{Var}(Y_m^i)} \leq \frac{1}{2}, \tag{6.4}$$

is satisfied.

Now, the important point is that *the majority of particle-pairs have  $a \leq m/2$ , that is, they are not 'close' relatives.* Indeed, the number of pairs with  $a > m/2$  (close relatives) is  $2^m(1 + 2 + 4 + \dots + 2^{\lfloor \frac{m-1}{2} \rfloor})$ , which is bounded from above by  $2^{m+\frac{m}{2}+1}$ .

As we know, for any pair, we have  $\text{Cov}(Y_m^i, Y_m^j) \leq \text{Var}(Y_m^i) \leq \frac{1}{4}$ . Thus, for all of those pairs with  $a > m/2$ , the total covariance will be controlled by  $2^{m+\frac{m}{2}+1} \cdot \frac{1}{4} = 2^{m+\frac{m}{2}-1}$ .

Moreover, as discussed above, for the pairs with  $a \leq m/2$ , we may apply Lemma 6.6 and the remark following it, yielding

$$\begin{aligned} \sum_{j \neq i} \text{Cov}(1_B(Y_m^i), 1_B(Y_m^j)) &\leq \sum_{k=\lfloor \frac{m+1}{2} \rfloor}^{m-1} 2^k C m (e^{-2\gamma k} + 2^{-m}) \\ &\leq C m \sum_{k=1}^{m-1} \left( \left( \frac{2}{e^{2\gamma}} \right)^k + 2^{k-m} \right), \end{aligned}$$

where  $C$  only depends on  $N, \gamma, B$ . Consequently,

$$\begin{aligned} \text{Var} \left( 2^{-m} \sum_{i=1}^{2^m} 1_B(Y_m^i) \right) &\leq \frac{1}{4 \times 2^m} + \frac{2^{\frac{m}{2}-1}}{2^m} + \frac{C m}{2^m} \left( \frac{1 - \left(\frac{2}{e^{2\gamma}}\right)^m}{1 - \frac{2}{e^{2\gamma}}} + (1 - 2^{-m}) \right) \\ &\leq \frac{C_1 m}{2^m} + \frac{C_2 m}{e^{2\gamma m}} + \frac{C_3 m^2}{2^m} + \frac{1}{2^{\frac{m}{2}+1}} \\ &\leq \frac{C_0 m}{e^{2\gamma m}} + \frac{C_0 m^2}{2^m} + \frac{1}{2^{\frac{m}{2}+1}}. \end{aligned}$$

Here  $C_0$  is a constant which only depends on  $N, \gamma, B$ . Given that  $\gamma > 0$  and the three terms on the righthand side are all summable in  $m$ , (6.2) holds and the proof is complete.  $\square$

## 7 The distribution of the particle system

Now we have collected enough information to describe the large time behavior the system as a whole.

### 7.1 Preparation

Below we describe the system's behavior as it depends on the parameters  $\gamma, b$ . The statements about the large time behavior of the branching particle system will follow from the behavior of the center of mass (Theorem 2.4 and Theorem 2.5), that of the relative system (Theorem 6.11), and finally, from the following proposition.

**Proposition 7.1** (Independence). *The tail  $\sigma$ -algebra  $\mathcal{T}$  of  $\bar{Z}$  is independent of the relative system  $Y$ .*

*Proof.* The proof of this proposition is exactly the same as the corresponding proof of Lemma 14 in [4].  $\square$

Recall that  $Z(dy)$  denotes the discrete measure-valued process corresponding to the interacting branching system. The following notion will be important.

**Definition 7.2** (Local extinction). We say that  $Z$  suffers *local extinction*, if

$$Z_n(dy) \xrightarrow{v} 0, \text{ as } n \rightarrow \infty, \text{ a.s.} \tag{7.1}$$

Since  $Z$  is a discrete particle system, (7.1) is tantamount to the property that for all  $B \subset \mathbb{R}^d$  ball, there exists an almost surely finite random time  $T_B$  such that

$$P(Z_n(B) = 0, \forall n \geq T_B) = 1.$$

**7.2 Inward drift in the motion component**

We now turn to the first results about the behavior of  $Z$ , starting with the case of an inward drift ( $b > 0$ ). In fact, we distinguish between three further sub-cases.

**Case 1:**  $b > 0, b + \gamma > 0$ .

As we have demonstrated, the center of mass converges to zero as  $t \rightarrow \infty$ , and the relative system will become an inward O-U process with parameter  $\gamma + b$ . Putting Theorem 6.11 together with the a.s. converge of the COM (Theorem 2.4), and finally, with Proposition 7.1, we arrive at the following conclusion.

**Theorem 7.3** (Inward drift, weak repulsion or attraction). *Assume that  $b > 0$ , and  $b + \gamma > 0$ . Then*

$$2^{-n} Z_n(dy) \Rightarrow \left(\frac{\gamma + b}{\pi}\right)^{d/2} \exp(-(\gamma + b)|y|^2) dy,$$

almost surely.

**Remark 7.4.** Since  $\mathcal{R}$  is a countable family, the weak limit in the previous theorem is actually equivalent to the statement that for all  $B \in \mathcal{R}$ ,

$$\lim_{n \rightarrow \infty} 2^{-n} Z_n(B) = \int_B \left(\frac{\gamma + b}{\pi}\right)^{d/2} \exp(-(\gamma + b)|y|^2) dy, \text{ a.s.}$$

(See Lemma 1.1.) ◇

**Example 7.5** (Non-interactive branching O-U process). Consider the case  $\gamma = 0, b > 0$ , that is, the case of a non-interacting branching (inward) O-U process with parameter  $b$ . The proof goes through in this case as well, and we obtain that for all  $B \in \mathcal{R}$ ,

$$\lim_{n \rightarrow \infty} 2^{-n} Z_n(B) = \int_B \left(\frac{b}{\pi}\right)^{d/2} \exp(-b|y|^2) dy, \text{ a.s.,}$$

complementing the exponential-clock results in [5].

**Case 2:**  $b > 0, b + \gamma = 0$ .

**Theorem 7.6** (Inward drift, balanced by repulsion). *In this case,*

$$\lim_{n \rightarrow \infty} C n^{d/2} 2^{-n} Z_n(dy) = \text{Leb} \text{ a.s.,} \tag{7.2}$$

where  $C := (2\pi)^{d/2}$ .

*Proof.* The proof is based on an argument which ‘switches off’ the interaction. In order to accomplish this, we are going to utilize the lemma on interchangeability (Lemma 4.1). Namely, we match the relative system with that of another system without interaction.

This other system is the one with  $b = \gamma = 0$  (branching Brownian motion without interaction). As far as the behavior of this second system is concerned, it is well known (see [9, 2]), that (7.2) holds.

Even though in [9, 2], the decomposition into COM and a relative system was not considered, we do that now. Doing so is useful, because by Lemma 4.1, the relative system is the same for the two processes, even though the behavior of the COM is not: for the original system it converges to the origin almost surely (Theorem 2.4), and for the non-interacting BBM it has an almost sure (Gaussian) limit (see [4]).

Now use the fact that Leb is translation invariant. In both systems, one has

$$Z_t(B) = Y_t(B + \bar{Z}_t),$$

hence

$$Cn^{d/2}2^{-n}Z_n(B) = Cn^{d/2}2^{-n}Y_n(B + \bar{Z}_n).$$

By conditioning on the almost sure limit of  $\bar{Z}_n$ , and using Proposition 7.1, the translation invariance of the Lebesgue measure yields (7.2) for the relative system in the  $b = \gamma = 0$  case.

But then, by Lemma 4.1, the same holds for the relative system in the original model. Since the COM converges to the origin almost surely for the original model, using Proposition 7.1, we conclude that the scaling limit (7.2) also holds for the original system.  $\square$

**Case 3:**  $b > 0, b + \gamma < 0$ .

As the relative system behaves asymptotically like an outward O-U process, we have a conjecture similar to the one in [4]. In our case, however, the center of mass tends to 0 as  $t \rightarrow \infty$ . Thus, the conjecture will take the following form:

**Conjecture 7.7** (Inward drift, strong repulsion). *The following dichotomy holds for the long term behavior of the process:*

1. If  $\frac{\log 2}{d} \leq |b + \gamma|$ , then  $Z$  suffers local extinction.
2. If  $\frac{\log 2}{d} > |b + \gamma|$ , then almost surely,

$$2^{-n}e^{d|b+\gamma|n}Z_n(dy) \xrightarrow{v} \text{Leb.}$$

**Remark 7.8.** The intuitive explanation of the dichotomy in the conjecture is as follows. Even though the motion has a strong inward component (forcing the center of mass to tend to the origin, according to Theorem 2.4), this is offset by the even stronger repulsion term.

This combined effect is then competing with the mass creation (the ‘rate’ of mass creation in this case can be taken  $\log 2$ ): if mass creation is stronger, then the Law of Large Numbers is still in force; otherwise the mass creation is no longer able to compensate the fact that particles are ‘being pushed away.’  $\diamond$

### 7.3 Outward drift in the motion component

This case is more difficult than the case of the inward drift. The result below is quite natural once the decomposition of the process (COM plus relative system) is established, however, it may be somewhat surprising if one is just given the definition of the model with the pairwise particle interactions.

**Case 4:**  $b < 0$  (Outward drift)

In this case, according to Theorem 2.5, the center of mass converges to infinity a.s. as  $t \rightarrow \infty$ , and so the question is, intuitively, whether this effect will be compensated by the large number of particles.

The next result says that for outward drift and attraction, the system *always* exhibits local extinction, irrespective of the relationship between the drift size  $|b|$  and the attraction parameter  $\gamma$ .

**Theorem 7.9** (Outward drift with attraction). *For  $b < 0$  and  $\gamma > 0$ ,  $Z$  suffers local extinction:*

$$Z_n(dy) \xrightarrow{v} 0 \text{ a.s.}$$

**Remark 7.10.** As far as the relative system’s behavior is concerned, that is of course given by Theorem 6.11.  $\diamond$

*Proof.* The proof is based on comparing the speed of the COM with that of the relative system. Let  $M_t^{\text{rel}}$  denote the radius of the smallest ball containing the support of  $Y_t$  for  $t \geq 0$ . It is clear that the relation

$$\lim_{t \rightarrow \infty} \frac{M_t^{\text{rel}}}{|\bar{Z}_t|} = 0 \text{ a.s.}, \tag{7.3}$$

proven below, implies local extinction; we will verify it by distinguishing between three sub-cases.

- (a) When  $b = \gamma = 0$ , the non-interactive branching Brownian motion will be denoted by  $\mathcal{Z}$  ( $b = \gamma = 0$ ). Let  $Y^{\text{non-int}}$  be the relative system for  $\mathcal{Z}$  and let  $M_t^{\text{non-int}}$  be the radius of the smallest ball containing the support of  $Y^{\text{non-int}}$ .

Using the Borel-Cantelli lemma, and then bounding the union by the sum, for  $c > \sqrt{2 \log 2}$  we obtain

$$P(\mathcal{Z}_n(\mathbb{R}^d \setminus B(0, cn)) > 0 \text{ i.o.}) = 0.$$

Indeed, this follows from the fact that

$$\sum_n \sum_{i=1}^{2^n} P(\mathcal{Z}_n^i(\mathbb{R}^d \setminus B(0, cn)) > 0) < \infty,$$

which, in turn, follows from the estimate<sup>4</sup>

$$P(\mathcal{Z}_n^i(\mathbb{R}^d \setminus B(0, cn)) > 0) = \exp(-c^2 n/2 + o(n)),$$

where  $\{\mathcal{Z}_n^i; 1 \leq i \leq 2^n\}$  are the particles forming  $\mathcal{Z}_n$ .

(The asymptotic ‘speed of the support’ of  $\mathcal{Z}$  is actually exactly  $\sqrt{2 \log 2}$ , but we don’t need this fact here. Cf. [7].)

We now claim that  $b + \gamma = 0$  implies

$$M_t^{\text{rel}} = \mathcal{O}(t), t \rightarrow \infty, \text{ a.s.} \tag{7.4}$$

Indeed, recall that by  $\Delta$ -transform invariance,  $Y \stackrel{d}{=} Y^{\text{non-int}}$ . But  $M_t^{\text{non-int}} = \mathcal{O}(t)$ , as  $t \rightarrow \infty$  a.s., because it is well known that

$$\max\{|z| \mid z \in \text{supp}(\mathcal{Z}_t)\} = \mathcal{O}(t) \text{ a.s.},$$

and because of the existence of  $\lim_{t \rightarrow \infty} \bar{Z}_t$  a.s.

Turning back to  $Z$ , if  $b + \gamma = 0$ , then (7.4) and the fact that  $\bar{Z}_t$  escapes to infinity exponentially fast (according to our Theorem 2.5) yields (7.3).

- (b) If  $b + \gamma > 0$ , then by  $\Delta$ -transform invariance, for the relative system, we may assume that it is non-interacting and the particles are performing inward Ornstein-Uhlenbeck motions with parameter  $b + \gamma > 0$ . The proof is then the same as in (a), given that  $P(\mathcal{Z}_n^i(\mathbb{R}^d \setminus B(0, cn)) > 0)$  is even smaller now.
- (c) Finally, for the case when  $b + \gamma < 0$  and  $\gamma > 0$ , we still have (7.3). To see this, recall first that the logarithmic escape rate of  $\bar{Z}_t$  is  $-bt$ . On the other hand, a calculation similar<sup>5</sup> to the one in Example 11 in [5] reveals that the logarithmic rate of spread of the relative system is  $-(b + \gamma)t$ , which yields (7.3). (Since the probability that at least one particle is present in a set  $B$  is trivially dominated by the expected particle number in that set, that is,  $P(\mathcal{Z}_n(B) > 0) \leq E(\mathcal{Z}_n(B))$ , the calculation reduces to computing certain expectations. In our case it is even easier than in [5], as the total population size is deterministic.)

<sup>4</sup>Use Brownian scaling and the Gaussian upper tail estimate.

<sup>5</sup>There the clock is exponential, whereas here it is unit time.

Hence, in all three cases, local extinction occurs.  $\square$

We conclude with posing an open problem:

**Open problem** (Outward drift with repulsion). Describe the large-time behavior of the relative/global system for  $b, \gamma < 0$ . (As far as the relative system's behavior is concerned, the conjecture is the same as in Conjecture 7.7.)

## 8 The behavior of $\bar{Z}$ for a drift $b(\cdot)$ bounded between positive constants

So far we have been working under the assumption that the drift is linear:  $b(x) = bx$ . Assume now instead, that the drift satisfies that

$$0 < a < b(x) < b.$$

For simplicity, we still start with  $d = 1$ . Given that

$$dZ_t^i = dW_t^i + b(Z_t^i) dt + \frac{1}{2^m} \sum_{j=1}^{2^m} \gamma(Z_t^j - Z_t^i) dt,$$

the motion of the center of mass  $\bar{Z}_t$  satisfies the equation

$$d\bar{Z}_t = \frac{1}{2^m} \sum_{i=1}^{2^m} dW_t^i + \frac{1}{2^m} \sum_{i=1}^{2^m} b(Z_t^i) dt.$$

As  $a < b(x) < b$ , we know that  $a < \frac{1}{2^m} \sum_{i=1}^{2^m} b(Z_t^i) < b$ . We thus have

$$\frac{1}{2^m} \sum_{i=1}^{2^m} dW_t^i + a dt < d\bar{Z}_t < \frac{1}{2^m} \sum_{i=1}^{2^m} dW_t^i + b dt.$$

Integration on both sides yields

$$\frac{1}{2^m} \sum_{i=1}^{2^m} W_t^i + at < \bar{Z}_t < \frac{1}{2^m} \sum_{i=1}^{2^m} W_t^i + bt.$$

Since (as we have discussed previously)  $\frac{1}{2^m} \sum_{i=1}^{2^m} W_t^i$  is a Brownian motion being slowed down, and  $b > a > 0$ , the center  $\bar{Z}_t$  will tend to  $+\infty$  with an 'essentially constant speed' a.s. (For  $a < b(x) < b < 0$ , it tends to  $-\infty$  with essentially constant speed a.s.) More precisely, we have the following result.

**Theorem 8.1** (Positive, constant-like drift). *Assume that the drift satisfies  $0 < a < b(x) < b$ . Then, almost surely,*

$$0 < a < \liminf_{t \rightarrow \infty} t^{-1} \bar{Z}_t \leq \limsup_{t \rightarrow \infty} t^{-1} \bar{Z}_t < b.$$

Finally, a similar result holds for  $d > 1$ , when replacing  $\bar{Z}_t$  with  $|\bar{Z}_t|$ , as one can consider the statement coordinate-wise.

## References

- [1] Asmussen, S. and Hering, H.: Branching processes. Progress in Probability and Statistics, **3**. Birkhäuser, Boston, 1983. x+461 pp. MR-0701538



- [2] Biggins, J. D.: Uniform convergence of martingales in the branching random walk. *Ann. Probab.* **20**, 137–151. MR-1143415
- [3] Billingsley, P.: Convergence of probability measures. Second edition. Wiley Series in Probability and Statistics, *John Wiley and Sons*, New York, 1999. x+277. MR-1700749
- [4] Engländer, J.: The center of mass for spatial branching processes and an application for self-interaction. *Electron. J. Probab.* **15** (2010), no. 63, 1938–1970. MR-2738344
- [5] Engländer, J., Harris, S. C and Kyprianou, A. E.: Strong law of large numbers for branching diffusions. *Ann. Inst. Henri Poincaré Probab. Stat.* **46** (2010), no. 1, 279–298. MR-2641779
- [6] Gill, H.: A super Ornstein-Uhlenbeck process interacting with its center of mass. *Ann. Probab.* **41** (2013), no. 2, 989–1029. MR-3077532
- [7] Kyprianou, A. E.: Asymptotic radial speed of the support of supercritical branching Brownian motion and super-Brownian motion in  $\mathbb{R}^d$ . *Markov Process. Related Fields* **11** (2005), no. 1, 145–156. MR-2099406
- [8] Revuz, D. and Yor, M.: Continuous martingales and Brownian motion. Third edition. Grundlehren der Mathematischen Wissenschaften, **293**. *Springer*, Berlin, 1999. xiv+602 pp. MR-1725357
- [9] Watanabe, S.: Limit theorem for a class of branching processes. 1967 Markov Processes and Potential Theory (Proc. Sympos. Math. Res. Center, Madison, Wis., 1967) pp. 205–232, *Wiley*, New York. MR-0237007

**Acknowledgments.** We are indebted to the referees and to the Associate Editor for their close reading of the manuscript and for their valuable suggestions.

---

# Electronic Journal of Probability

## Electronic Communications in Probability

---

### Advantages of publishing in EJP-ECP

- Very high standards
- Free for authors, free for readers
- Quick publication (no backlog)
- Secure publication (LOCKSS<sup>1</sup>)
- Easy interface (EJMS<sup>2</sup>)

### Economical model of EJP-ECP

- Non profit, sponsored by IMS<sup>3</sup>, BS<sup>4</sup>, ProjectEuclid<sup>5</sup>
- Purely electronic

### Help keep the journal free and vigorous

- Donate to the IMS open access fund<sup>6</sup> (click here to donate!)
- Submit your best articles to EJP-ECP
- Choose EJP-ECP over for-profit journals

---

<sup>1</sup>LOCKSS: Lots of Copies Keep Stuff Safe <http://www.lockss.org/>

<sup>2</sup>EJMS: Electronic Journal Management System <http://www.vtex.lt/en/ejms.html>

<sup>3</sup>IMS: Institute of Mathematical Statistics <http://www.imstat.org/>

<sup>4</sup>BS: Bernoulli Society <http://www.bernoulli-society.org/>

<sup>5</sup>Project Euclid: <https://projecteuclid.org/>

<sup>6</sup>IMS Open Access Fund: <http://www.imstat.org/publications/open.htm>