Qualitative Analysis of Some Nonlinear PDE systems

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Qualitative Analysis of Some Nonlinear PDE systems

Thesis directed by Prof. Congming Li

We mainly study an important and interesting class of nonlinear PDE systems, the Hardy-Littlewood-Sobolev (HLS) type system. In addition, we qualitatively study 3-wave resonance interaction (3WRI). HLS system plays crucial roles in geometric analysis, dynamics analysis of vacuum states, study of nonlinear Schrödinger equations, and many other research areas. 3WRI emerges from nonlinear optics, plasma physics, water wave etc.

Our goal is to develop some new idea and method to qualitatively analyze those systems. This involves many types of problems, e.g. existence and non-existence, asymptotic behavior near singularity or at infinity, stability etc.

Our study shows that nonlinear systems have brought many new challenges to us, where methods and tools in the past may be limited to some special cases or even not applicable. By developing new ideas we can provide insight into these problems and solve some of the challenges.

Dedication

To the memory of Shaoqing Li,

I wish I made you proud.

The same to Bizhen Chen.

To my parents and my family.

To my dear friends in Chengdu, in China,

in Boulder and in United States.

Thank you for being in my life.

Acknowledgements

Thank my PhD advisor, Professor Congming Li. It is a very special experience to work with Prof. Li. To describe it in short, I would say I feel a strong echo in watching *Whiplash* and reading *Starship Troopers* by Robert A. Heinlein. In the following I quote some of Prof. Li's words which I find interesting and also deep:

Don't say you can't, just do it. Don't trust any one, trust your own judgment. Think more on simple things. 努力不须勤奋，用心不用操心 *(Translated as: Try your best but not too hard, think but don't get overwhelmed). Better to keep your mouth shut than say something unclear or ambiguous.*

These are only a tiny tip of a iceberg and there are a lot more of his words that strike me like a barrel of ice water making me awake and alert. It may not be all original from Prof. Li and he probably learned somewhere, for example, *the best way of learning math and maybe everything is to explain to someone else.* Nevertheless, I learned so much from him during my PhD student life.

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Chapter 1

Introduction

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1.1 HLS type systems

The well known Hardy-Littlewood-Sobolev (HLS) inequality states (see e.g. [68]):

$$\int_{R^n}\int_{R^n}\frac{f(x)g(y)}{|x-y|^{n-\gamma}}dxdy \leq C(n,s,\gamma)||f||_r||g||_s \tag{1.1}$$

where $0<\gamma<n$, $1<s,r<\infty$, $\frac{1}{r}+\frac{1}{s}+\frac{n-\gamma}{n}=2$, $f\in L^r(\mathbb{R}^n)$ and $g\in L^s(\mathbb{R}^n)$.

Define an operator T such that $Tg(x):=\displaystyle\int_{R^n}\frac{g(y)}{|x-y|^{n-\gamma}}dy$, $\gamma\in(0,n)$, then the HLS inequality is sometimes also written as:

$$||Tg||_{\frac{ns}{n-s\gamma}} \leq C(n,s,\gamma)||g||_s, \text{ or } ||Tg||_p \leq C(n,s,\gamma)||g||_{\frac{np}{n+\gamma p}}, \tag{1.2}$$

where $\frac{n}{n-\gamma}<p<\infty$, and $1<s<n/\gamma$.

The best constant $C=C(n,s,\gamma)$ is the maximum of:

$$J(f,g)=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{f(x)g(y)}{|x-y|^{n-\gamma}}dxdy \tag{1.3}$$

with constraints $\|f\|_r=\|g\|_s=1$. This optimizing problem leads us to Euler-Lagrange equations on f and g,

$$\begin{cases} c_1 f^{r-1}(x)=\displaystyle\int_{\mathbb{R}^n}\frac{g(y)}{|x-y|^{n-\gamma}}dy, \\ c_2 g^{s-1}(x)=\displaystyle\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\gamma}}dy, \end{cases} \tag{1.4}$$

where c_1 and c_2 are constants. Let $u = c_1 f^{r-1}$, $v = c_2 g^{s-1}$, $p = \frac{1}{r-1}$, $q = \frac{1}{s-1}$, and c_2, we arrive at the following system of integral equations:

$$\begin{cases} u(x) = \displaystyle\int_{\mathbb{R}^n} \frac{v^q(y)}{|x-y|^{n-\gamma}} dy, \\ v(x) = \displaystyle\int_{\mathbb{R}^n} \frac{u^p(y)}{|x-y|^{n-\gamma}} dy, \end{cases} \tag{1.5}$$

with $u, v > 0,\ \ u \in L^{p+1},\ \ v \in L^{q+1}$, $0 < p < \infty,\ \ 0 < q < \infty,\ \ \frac{1}{p+1} + \frac{1}{q+1} = \frac{n-\gamma}{n}$.

For $pq > 1$, a solution of (1.5) is also a solution of:

$$\begin{cases} (-\Delta)^{\gamma/2} u = v^q, \quad u > 0, \ \text{in } \mathbb{R}^n, \\ (-\Delta)^{\gamma/2} v = u^p, \quad v > 0, \ \text{in } \mathbb{R}^n. \end{cases} \tag{1.6}$$

The fractional Laplacian can be defined in several ways, e.g. via Fourier transform (see [46]). For $0 < p, q < \infty$, we call (1.5) and (1.6) **Hardy-Littlewood-Sobolev (HLS) type systems** and short as **HLS systems**.

The Hardy-Littlewood-Sobolev type systems are related to abundant problems e.g. in geometric analysis, dynamics analysis of vacuum states, study of nonlinear Schrödinger equations. For instance, if $p = q = \frac{n+2}{n-2}$, and $u(x) = v(x)$,

$$-\Delta u = u^{(n+2)/(n-2)}, \quad u > 0, \ \text{in } \mathbb{R}^n, \tag{1.7}$$

is closely related to Yamabe problem. In the elegant paper [23], Gidas, Ni and Nirenberg classified all the solution to (1.7) by method of moving plane (MMP) as radial and decaying, and unique up to scaling and translation

$$u(x) = \left(\frac{\sqrt{n(n-2)}\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n-2}{2}}, \quad \lambda > 0,$$

with assumption that $u(x) = O(\frac{1}{|x|^{n-2}})$. R. Schoen pointed out that this result is equivalent to a geometric result due to Obata [48]: A Riemannian metric on S^n which is conformal to the standard metric and having constant scalar curvature, then up to a constant scalar factor, is the pullback of the standard metric under a conformal map of S^n to itself. Later, Caffarelli, Gidas and Spruck

[6] removed the growth assumption $u(x) = O(\frac{1}{|x|^{n-2}})$. Chen and Li [8] simplified their proof with Kelvin transform and MMP.

Now, consider a more general equation

$$- \Delta u = u^p, \;\; u \geq 0, \text{ in } \mathbb{R}^n, \tag{1.8}$$

For $1 \leq p < \frac{n+2}{n-2}$, Gidas and Spruck [24] proved that the above equation admits only zero solution. People are seeking an analogy of this result in system of equation (1.6) and call it Lane-Emden conjecture. The conjecture says the "subcritical" (explained below) system admits only zero solution, and it is still open for spatial dimension $n \geq 5$. Such Liouville type results are very useful in the study of potential singularities and *a priori* estimates via a blow up argument, cf. [34, 55, 61].

For $p > \frac{n+2}{n-2}$, it is still open that if the above equation admits non-radial solution (See [18]).

Due to their difference, $p = \frac{n+2}{n-2}$ is called **critical**, $1 \leq p < \frac{n+2}{n-2}$ **subcritical** and $p > \frac{n+2}{n-2}$ **supercritical**.

Then a more general example of HLS system is,

$$(-\Delta)^{\gamma/2} u = u^{(n+\gamma)/(n-\gamma)}, \;\; u > 0, \text{ in } \mathbb{R}^n, \tag{1.9}$$

which is critical in a similar sense. Chen, Li and Ou [11] showed that equation (1.9) is equivalent to:

$$u(x) = \int_{\mathbb{R}^n} \frac{u(y)^{\frac{n+\gamma}{n-\gamma}}}{|x-y|^{n-\gamma}} dy, \;\; u > 0 \text{ in } \mathbb{R}^n, \tag{1.10}$$

and classified all the solution to take the form

$$u(x) = C \left(\frac{\lambda}{\lambda^2 + |x - x_0|^2} \right)^{\frac{n-\gamma}{2}} \tag{1.11}$$

Therefore, in a similar fashion as the scalar case, we categorize the HLS type systems into three cases: **critical case** $\frac{1}{p+1} + \frac{1}{q+1} = \frac{n-\gamma}{n}$, **subcritical case** $\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-\gamma}{n}$, **supercritical case** $\frac{1}{p+1} + \frac{1}{q+1} < \frac{n-\gamma}{n}$. Our goal is to study the different properties in each case, where various analytic methods are applied.

The existence of solution for **critical/supercritical** HLS system (1.6) is established for integer γ via a shooting method with topological degree theory, cf. [36, 42]. Indeed, the existence can be obtained for much more general systems via this method. All the study of shooting method so far [63, 64, 69] requires a positivity condition on source term with which the solution has a nice monotone decaying property. However, this positivity condition is not necessary for degree shooting to be applicable. In Chapter 2, we present a theorem of existence of solution to systems that allow sign-changing source terms. Note that without the the positivity condition, many estimates fail, and thus we need to develop some new dynamic estimates to overcome these difficulties.

For the **subcritical** HLS type systems, in particular, $\gamma = 2$, i.e., the Lane-Emden system, we study the non-existence of solutions. The so-called Lane-Emden conjecture states that, for $\gamma = 2$, the subcritical HLS system (1.6), i.e. for $0 < p, q < \infty$, $\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}$,

$$\begin{cases} -\Delta u(x) = v^q(x), \;\; u \geq 0, \text{ in } \mathbb{R}^n, \\ -\Delta v(x) = u^p(x), \;\; v \geq 0, \text{ in } \mathbb{R}^n, \end{cases} \tag{1.12}$$

has $u = 0$ and $v = 0$ as the unique locally bounded solution.

The conjecture naturally generalize to the systems (1.5) or (1.6) in the subcritical cases with an additional condition, $pq > 1$. Notice that $pq > 1$ is a necessary condition for this conjecture to hold in high order HLS type systems. For example if $p = q = 1$ and $\gamma = 4$ we have solution $u = v = e^{w \cdot x}$ to (1.6) for $w \in \mathbb{R}^n$ with $|w| = 1$.

The conjecture is confirmed in the case of $n = 3, 4$, see [62, 55, 65]. For higher dimension, only some subregion of subcritical region is confirmed, and the conjecture is still open. In Chapter 3, we present a necessary and sufficient condition to the Lane-Emden conjecture, a condition that assumes the solution satisfies an energy estimate in certain form. We believe that this result may point out a possible path to approach the long standing and interesting conjecture, i.e., to prove the energy estimate mentioned above.

In Chapter 4, we consider discrete HLS inequality. Similar to continuous case, discrete HLS inequality corresponds to discrete HLS system. In the classic paper [40], Lieb studied the optimizer and the best constant of HLS inequality and obtained existence of both and symmetry property of the optimizer. In particular, he gave explicit best constant and optimizer (which is essentially unique) in the case of $p = q'$ or $p = 2$ or $q = 2$ via stereographic projection to recast equations on S^n. Here, we study the best constant and the optimizer of an extended discrete HLS system (setting $\gamma = 0$, $p = q = 2$ in discrete version of (1.1) and limiting the inequality on a finite domain), and we prove the existence and uniqueness of the optimizer, and give a sharp estimate of the best constant. Moreover, we obtain a symmetry and monotone decaying property of the optimizer.

1.2 Three-wave resonance interaction

Consider the 3-wave resonance interaction (3WRI) system,

$$\begin{cases} \partial_\tau A_1 + c_1 \cdot \nabla A_1 = i\gamma_1 \overline{A_2 A_3}, \\ \partial_\tau A_2 + c_2 \cdot \nabla A_2 = i\gamma_2 \overline{A_1 A_3}, \quad \text{in } \Omega, \\ \partial_\tau A_3 + c_3 \cdot \nabla A_3 = i\gamma_3 \overline{A_1 A_2}, \end{cases} \tag{1.13}$$

with periodic boundary condition, where Ω is a rectangle domain,

$$\Omega = \{x \in \mathbb{R}^n |\ |x_k| < a_k, k = 1, \cdots, n\}$$

and all $A_j's$ are are complex amplitude and periodic on Ω. $\gamma_j = \pm 1$, and c_j's are real non-zero constant vectors.

3WRI generates from various background, for example water waves, nonlinear optics and plasma physics. The system (5.1) holds when a resonance condition is satisfied,

$$k_1 \pm k_2 \pm k_3 = 0, \quad \omega_1 \pm \omega_2 \pm \omega_3 = 0,$$

where k_j's are wave vector (or wave number), ω_j's are wave frequency and A_j's are complex amplitude. k_j, ω_j and A_j characterize a linearization of the solution of a nonlinear problem by

$$u(x,t) = \sum_j A_j(x,t) \exp i(k_j \cdot x - \omega_j t).$$

The derivation of (5.1) is rather standard in e.g. nonlinear optics and can be found in e.g. Chap. 4 in [1] and [32].

3WRI with positive wave energy corresponds to the case that γ_j's do **not** have the same sign, which corresponds to

$$k_1 = k_2 + k_3, \quad \omega_1 = \omega_2 + \omega_3.$$

3WRI with negative wave energy corresponds to the case that γ_j's have the **same** sign, which corresponds to

$$k_1 + k_2 + k_3 = 0, \quad \omega_1 + \omega_2 + \omega_3 = 0.$$

The study of 3WRI has been focused on using inverse scattering transformation to construct exact solution and thus providing a numerical scheme for solving the system. The 1D-3WRI was approached by Zakharov and Manakov [71] (and independently by Kaup [29]), and the study 3D3WRI is developed along with the development of IST. We will discuss more detail in Chapter 5. However, IST only solves solution with fast decay at infinity and there is not so much study about 3WRI with periodic boundary condition; also, there is not much qualitative information that we can provide with the solution constructed by IST, e.g. can the solution to 3WRI develop a singularity in finite time? If yes, does all solution blow up in finite time?

In Chapter 5, we first present a regularity theorem that guarantees no singularity can develop in finite time for 3WRI with positive wave energy. Then for 3WRI with negative wave energy we classify all the blow-up solution for spatial uniform case. Last, we show a class of solution that blows up in finite time with "overlapping" initial value.

Chapter 2

Existence for critical and supercritical HLS system and more general system

The critical and supercritical HLS systems have been known to admit solution. Whereas the subcritical HLS systems in some important cases do not admit solution, which we call non-existence for subcritical HLS systems and will be discussed in next chapter. Here, we consider some general systems that include critical and supercritical HLS systems as special cases.

The method we use to obtain existence is called shooting method with topological degree theory, which is introduced independently by Liu-Guo-Zhang [42] and Li [37]. The original shooting method with degree theory applies to system with strictly positive source term f. By developing a new dynamic estimate, we have replaced the positivity condition imposed on the f with some mild conditions that allow sign-changing f.

In section 2.2, we prove the main result, an existence theorem 2.1. In section 2.3, we show the existence of solution to the some example systems. In fact, we show the nonexistence of solution to the Dirichlet problems corresponding to those systems. Then according to theorem 2.1, the original systems admit solution.

This chapter contains the work done in [14].

2.1 Introduction

Inspired by the study of the existence of solution to critical and supercritical HLS system,

$$\begin{cases} (-\triangle)^k u = v^p & \text{in } \mathbb{R}^n, \\ (-\triangle)^k v = u^q & \text{in } \mathbb{R}^n, \\ \quad u, v > 0, \end{cases} \tag{2.1}$$

with $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2k}{n}$ (take integer $k = \frac{\gamma}{2}$ in HLS system defined in chapter 1), we consider a more general system,

$$\begin{cases} -\Delta u_i = f_i(u) & \text{in } \mathbb{R}^n, \\ \quad u_i > 0 & \text{in } \mathbb{R}^n, \end{cases} \tag{2.2}$$

where $i = 1, 2, \cdots, L$. Denote $\mathbb{R}^L_+ = \{u \in \mathbb{R}^L | u_i > 0, \text{ for } i = 1, \cdots, L\}$, and throughout this chapter $f = (f_1, f_2, \cdots, f_L)$ is assumed continuous in $\overline{\mathbb{R}^L_+}$ and locally Lipschitz continuous in $\mathbb{R}^L_+$. Notice that (2.1) can be reduced to (2.2).

When $k = 1$, HLS system (2.1) is also called Lane-Emden system. An interesting phenomena about Lane-Emden system is that, there exists a dividing curve of parameters (p, q) introduced independently by Clément-De Figueiredo-Mitidieri [15] and Peletier-Van der Vorst [49, 70], namely,

$$\frac{1}{p+1} + \frac{1}{q+1} = 1 - \frac{2}{n}. \tag{2.3}$$

Below this curve, i.e. (p, q) satisfy $\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}$, people conjecture that the system admits no solution (this is still open for $n \geq 5$, and we will detail this in next chapter). On the other hand, it is known that on or above this curve the Lane-Emden system admits solutions.

The critical case of Lane-Emden system, i.e., (p, q) on the curve (2.3), is known to admit solution by concentration compactness. In [41] P.L. Lions has systematically developed the concept of concentration compactness, and people find great application of it in calculus of variations and nonlinear elliptic PDE, for example, to derive the existence of solution of the Lane-Emden system in critical case. However, the argument no longer applies for supercritical case, i.e. (p, q) satisfying $\frac{1}{p+1} + \frac{1}{q+1} < \frac{n-2}{n}$ (that is above the curve (2.3)). Instead, Serrin and Zou [64] used shooting method to get the existence of solutions of Lane-Emden system in the supercritical case.

A degree approach to shooting method is introduced by Liu, Guo and Zhang [42] and independently by C. Li in [37]. The method can be used to obtain existence of radial positive solution for a general system (2.2), which includes critical and supercritical cases of Lane-Emden systems (thus we can obtain existence of solution to both cases in a uniform way). The idea of the method is to relate the existence of solutions in whole (global) space to the non-existence to a corresponding (local) Dirichlet problem. Such idea generates from the earlier work of Berestycki, Lions and Peletier [4] who considered radial solution to a single equation, i.e. (2.2) with $L = 1$. Now, with system of equations, the degree theory will play a role in the shooting scheme. For more development of degree approach to shooting method, see [39, 69].

Besides some non-degeneracy and growth control requirements, the conditions places on f of system (2.2) in previous mentioned works by Li [39], Liu-Guo-Zhang [42], Serrin-Zou [64, 63], Villavert [69] all include that f needs to be **positive** (i.e. f_i is positive for all $i = 1, \cdots, L$). Although several important classes of systems such as critical and supercritical cases of (2.1) and etc. are covered by those works, there are cases of nonlinear Shrödinger type of systems that f does not need to be always positive. Here, we consider even more general cases where each f_i can change sign by replacing the condition f being positive by $\sum f_i \geq 0$ (see assumption (2.6)).

Since we are looking for positive radial solution $u(x) = u(|x|)$ of (2.2), the problem is equivalent to looking for global positive solution to the following ODE system,

$$\begin{cases} u_i''(r) + \dfrac{n-1}{r} u_i'(r) = -f_i(u(r)) \\ u_i'(0) = 0,\ u_i(0) = \alpha_i \text{ for } i = 1, 2, \cdots, L. \end{cases} \tag{2.4}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_L)$ is positive (i.e. each $\alpha_i > 0$) initial value for u.

By classical ODE theory, this initial value problem (2.4) has a unique solution $u(r, \alpha)$ for r in some maximum interval. Let $r_\alpha := \inf_{r\geq 0}\{r \in \mathbb{R} | u(r, \alpha) \in \partial\mathbb{R}^L_+\}$ (by $\partial\mathbb{R}^L_+$ we mean the boundary of $\mathbb{R}^L_+$, which sometimes we call "the wall"), so $r = r_\alpha$ is where $u(r, \alpha)$ touches the wall for the first time. There are two cases, case 1: $r_\alpha = \infty$, u never hits wall, then $u(r, \alpha)$ is a radial positive solution to (2.2), and we are done; case 2: $r_\alpha < \infty$, u hits wall in finite time, and we will show that the existence of solution of (2.2) will be **sufficiently determined by the non-existence**

of radial solution to its corresponding Dirichlet problem (2.5) **in the below,**

$$\begin{cases} -\Delta u_i = f_i(u) & \text{in } B_R, \\ u_i > 0 & \text{in } B_R, \\ u_i = 0 & \text{on } \partial B_R, \end{cases} \tag{2.5}$$

where $B_R := B_R(0)$ for any $R > 0$ and $i = 1, 2, \cdots, L$.

For the part of obtaining local non-existence, we follow Mitidieri's work in [44] by implementing Rellich-Pohožaev (see Pohožaev [53] and Rellich [60]) type of identities, and for completeness we will give proof for some examples in section 3. See also the pioneering work of Pohožaev [54] and Pucci and Serrin [56] about Pohožaev type of identities for general variational problem.

Here is our main result,

Theorem 2.1. *Given the nonexistence of radial solution to system* (2.5) *for all* $R > 0$*, the system* (2.2) *admits a radially symmetric solution of class* $C^{2,\alpha}(\mathbb{R}^n)$ *with* $0 < \alpha < 1$*, if* $f = (f_1(u), f_2(u), \cdots, f_L(u)) : \mathbb{R}^L \to \mathbb{R}^L$ *satisfies the following assumptions:*

(1) f *is continuous in* $\overline{\mathbb{R}^L_+}$ *and locally Lipschitz continuous in* $\mathbb{R}^L_+$*, and furthermore,*

$$\sum_{i=1}^{L} f_i(u) \geq 0 \ \text{in } \mathbb{R}^L_+; \tag{2.6}$$

(2) If $\alpha \in \partial\mathbb{R}^L_+$ *and* $\alpha \neq 0$*, i.e., for some permutation* $(i_1, \cdots, i_L)$*,* $\alpha_{i_1} = \cdots = \alpha_{i_m} = 0$*,* $\alpha_{i_{m+1}}, \cdots, \alpha_{i_L} > 0$ *where* m *is an integer in* $(0, L)$*, then* $\exists \delta_0 = \delta_0(\alpha) > 0$ *such that for* $\beta \in \mathbb{R}^L_+$ *and* $|\beta - \alpha| < \delta_0$*,*

$$\sum_{j=m+1}^{L} |f_{i_j}(\beta)| \leq C(\alpha) \sum_{j=1}^{m} f_{i_j}(\beta), \tag{2.7}$$

where C *is a non-negative constant that depends only on* α*.*

Remark 2.2. *(a) Notice that under assumption* (2.6)*, the components of* f *is allowed to be sign-changing. It is known that if* f_i*'s stay positive, there are many nice properties that we can use to obtain existence results. For example, Serrin and Zou's paper on Lane-Emden* *system [64] and on Hamiltonian type [63] and some recent work done by Li and Villavert [37, 39, 69] and Liu, Guo and Zhang's work in [42], all these works require f to be positive. If f_i changes sign, good properties are lost, which leads to estimates failing. Our work here is to derive dynamic estimate* (2.10) *and* (2.16) *under assumptions above, such that the degree theory approach to shooting method is applicable to show existence of solution with sign-changing f.*

(b) Assumption (2.7) *guarantees the continuity of target map (see definition 2.3) near "the wall". As we shall see in next section, the continuity of the target map plays a crucial role in obtaining existence of solution and the most analysis lies in proving such continuity.*

2.2 Proof of main result

In this section, we will first define a target map and prove its continuity. Then we apply degree theory to prove theorem 2.1.

2.2.1 Target map

For any real number $a > 0$, let $\Sigma_a = \{\alpha \in \overline{\mathbb{R}^L_+} | \sum_{i=1}^L \alpha_i = a\}$, and $B_a = \{\alpha \in \partial\mathbb{R}^L_+ | \sum_{i=1}^L \alpha_i \leq a\}$. Recall that for positive α (i.e. every $\alpha_i > 0$) we define $r_\alpha = \inf_{r\geq 0}\{r \in \mathbb{R} | u(r, \alpha) \in \partial\mathbb{R}^L_+\}$. As mentioned before, we can assume $r_\alpha < \infty$ since if $r_\alpha = \infty$ we get a solution to (2.2). Then we define a target map on a initial data of (2.4) as following,

Definition 2.3. *Let $u(r, \alpha)$ be a solution to* (2.4) *with initial value $\alpha \in \Sigma_a$, we define a map $\psi : \Sigma_a \to \partial\mathbb{R}^L_+$, such that*

$$\psi(\alpha) = \begin{cases} u(r_\alpha, \alpha) & \alpha \in \mathbb{R}^L_+, \\ \alpha & \alpha \in \partial\mathbb{R}^L_+. \end{cases} \tag{2.8}$$

Here we sketch shooting method with topological degree theory as follows. Fix any real number $a > 0$, and assume that for any initial value $\alpha \in \Sigma_a$ no global positive solution to (2.4) exists (i.e. $r_\alpha < \infty$), so we can define a target map. Hence, **step 1**, we show that, under some suitable assumptions on f, the target map ψ is continuous from Σ_a to B_a; **step 2**, by degree theory we show that ψ is onto, therefore $\exists \alpha_0 \in \Sigma_a$ such that $\psi(\alpha_0) = u(r_{\alpha_0}, \alpha_0) = 0$; **step 3**, note that by assumption $r_{\alpha_0} < \infty$, $u(r, \alpha_0)$ for $r \in [0, r_{\alpha_0}]$ is a solution to the Dirichlet problem (2.5) with $R = r_{\alpha_0}$, which makes a contradiction if we assume that system (2.5) admits no radial solution for any $R > 0$.

In what follows, we assume (2.4) admits no global positive solution, i.e. $r_\alpha < \infty$. We first show that under our assumptions (2.6) and (2.7) on f, the behavior of u is controlled in a good way such that ψ is continuous.

Lemma 2.4. *For any real number $a > 0$, let*

$$\Sigma_a = \{\alpha \in \overline{\mathbb{R}^L_+} | \sum_{i=1}^{L} \alpha_i = a\} \text{ and } B_a = \{\alpha \in \partial\mathbb{R}^L_+ | \sum_{i=1}^{L} \alpha_i \leq a\}.$$

The target map ψ defined in definition 2.3 is a continuous map from Σ_a to B_a if f satisfies assumptions (2.6) *and* (2.7)*.*

Proof. To see that ψ maps Σ_a to B_a, we need to notice that by assumption (2.6) $\Sigma_{i=1}^L f_i \geq 0$, so we solve from the ODE system (2.4) and get

$$\begin{aligned} \Sigma_{i=1}^L u_i(r, \alpha) &= \Sigma_{i=1}^L \alpha_i - \Sigma_{i=1}^L \int_0^r \int_0^s (\frac{\tau}{s})^{n-1} f_i(u(\tau)) d\tau ds \\ &\leq \Sigma_{i=1}^L \alpha_i, \end{aligned}$$

for $r \in [0, r_\alpha]$. Therefore, $\psi(\alpha) \in B_a$.

Next, we will show that ψ is continuous on Σ_a. Fix any $\overline{\alpha} \in \Sigma_a$, then $\overline{\alpha}$ lies on the boundary of $\mathbb{R}^L_+$ or in $\mathbb{R}^L_+$ ($\overline{\alpha} \neq 0$ since $a > 0$), and we will prove for these two cases.

Case 1. $\overline{\alpha} \in \partial\mathbb{R}^L_+$.

Suppose $\overline{\alpha}_{i_1} = \cdots = \overline{\alpha}_{i_m} = 0$, and $\overline{\alpha}_{i_{m+1}}, \cdots, \overline{\alpha}_{i_L} > 0$, for some integer m that $0 < m < L$.

By the second assumption (2.7), $\exists \delta_0 > 0$, such that for $\alpha \in \mathbb{R}^L_+$ satisfying $|\alpha - \overline{\alpha}| < \delta_0$ we have

$$\sum_{j=m+1}^{L} |f_{i_j}(\alpha)| \leq C(\overline{\alpha}) \sum_{j=1}^{m} f_{i_j}(\alpha). \tag{2.9}$$

Notice that we can choose $C = C(\overline{\alpha}) \geq 1$ in (2.9). First, we claim that $\sum_{1\leq j\leq m} f_{i_j}(\alpha) \geq 0$. Indeed, if $C = 0$, the term on the left of the inequality (2.9) must be zero, and due to the first assumption (2.6), $\sum_{1\leq j\leq m} f_{i_j}(\alpha) \geq 0$. If $C > 0$, then $\sum_{1\leq j\leq m} f_{i_j}(\alpha) \geq 0$ obviously. Since $\sum_{1\leq j\leq m} f_{i_j}(\alpha) \geq 0$ we choose $C \geq 1$.

For this δ_0 and C, we **claim** that:

Fix any $\delta < \frac{\delta_0}{2(3+L)C}$, and for $\alpha \in \Sigma_a$ such that $|\alpha - \overline{\alpha}| \leq \delta$, then for $r \in [0, r_\alpha]$

$$|u(r,\alpha) - \overline{\alpha}| \leq 2(3+L)C\delta < \delta_0. \tag{2.10}$$

As we will see in the following proof, (2.10) is a dynamic estimate, in the sense that if $|u(r,\alpha) - \overline{\alpha}| < \delta_0$ with $r \in [0, a_1) \subset [0, r_\alpha]$ for some $a_1 > 0$, then by (2.7) we have

$$\sum_{j=m+1}^{L} |f_{i_j}(u(r,\alpha)| \leq C(\overline{\alpha}) \sum_{1\leq j\leq m} f_{i_j}(u(r,\alpha)), \tag{2.11}$$

and this control enables us to push the range of r in $|u(r,\alpha) - \overline{\alpha}| < \delta_0$ further than a_1 and up to r_α.

Suppose the claim not true, then there exists $\alpha_0 \in \mathbb{R}^L_+$ satisfying $|\alpha_0 - \overline{\alpha}| \leq \delta$ and $a_1 \in (0, r_{\alpha_0})$ such that the equality of (2.10) holds at $r = a_1$ for the first time, i.e.

$$|u(r,\alpha_0) - \overline{\alpha}| \begin{cases} < 2(3+L)C\delta, & \text{if } r < a_1, \\ = 2(3+L)C\delta, & \text{if } r = a_1. \end{cases} \tag{2.12}$$

For $r \in (0, a_1)$ we have,

$$\begin{align} |u(r,\alpha_0) - \overline{\alpha}| &\leq |u(r,\alpha_0) - \alpha_0| + |\alpha_0 - \overline{\alpha}| \tag{2.13} \\ &\leq \sum_{j=1}^{m} |u_{i_j}(r,\alpha_0) - \alpha_{0i_j}| + \sum_{j=m+1}^{L} |u_{i_j}(r,\alpha_0) - \alpha_{0i_j}| + |\alpha_0 - \overline{\alpha}|. \tag{2.14} \end{align}$$

So, to estimate the second term of (2.14), we solve from (2.4) and get

$$\sum_{j=m+1}^{L} |u_{i_j}(r,\alpha_0) - \alpha_{0i_j}| = \sum_{j=m+1}^{L} |\int_0^r \int_0^s (\frac{\tau}{s})^{n-1} f_{i_j}(u(\tau)) d\tau ds|$$

Notice that since $C \geq 1$, $2(3+L)C\delta < \delta_0$, therefore assumption (2.7) can be applied on $u(r, \alpha_0)$ for $r \in (0, a_1)$,

$$\begin{aligned}
\sum_{j=m+1}^{L} |u_{i_j}(r,\alpha_0) - \alpha_{0i_j}| &\leq \sum_{j=m+1}^{L} \int_0^r \int_0^s |(\frac{\tau}{s})^{n-1} f_{i_j}(u(\tau))| d\tau ds \\
&\leq C \int_0^r \int_0^s (\frac{\tau}{s})^{n-1} \sum_{j=1}^{m} f_{i_j}(u(\tau)) d\tau ds \\
&= C \sum_{j=1}^{m} (\alpha_{0i_j} - u_{i_j}(r,\alpha_0)),
\end{aligned}$$

The first term of (2.14) can be estimated by

$$\begin{aligned}
\sum_{j=1}^{m} |u_{i_j}(r,\alpha_0) - \alpha_{0i_j}| &= \sum_{j=1}^{m} (\alpha_{0i_j} - u_{i_j}(r,\alpha_0))^+ + \sum_{j=1}^{m} (\alpha_{0i_j} - u_{i_j}(r,\alpha_0))^- \\
&\leq 2 \sum_{j=1}^{m} (\alpha_{0i_j} - u_{i_j}(r,\alpha_0))^+.
\end{aligned}$$

To see the inequality above, one needs to notice that due to (2.7), $\sum_{j=1}^{m} f_{i_j}(u) \geq 0$, so $\sum_{j=1}^{m} u_{i_j}(r, \alpha_0)$ is monotone decreasing on $[0, r_{\alpha_0}]$. So,

$$\sum_{j=1}^{m} (\alpha_{0i_j} - u_{i_j}(r,\alpha_0)) = \sum_{j=1}^{m} (\alpha_{0i_j} - u_{i_j}(r,\alpha_0))^+ - \sum_{j=1}^{m} (\alpha_{0i_j} - u_{i_j}(r,\alpha_0))^- \geq 0.$$

The last term of (2.14) is bounded by δ, and notice that $u_i(r, \alpha_0) > 0$, $i = 1, \cdots, L$ for $r \in (0, r_{\alpha_0})$, so we get

$$\begin{aligned}
|u(r,\alpha_0) - \overline{\alpha}| &\leq 2 \sum_{j=1}^{m} (\alpha_{0i_j} - u_{i_j}(r,\alpha_0))^+ + C \sum_{j=1}^{m} (\alpha_{0i_j} - u_{i_j}(r,\alpha_0)) + \delta \\
&\leq (2+C) \sum_{j=1}^{m} \alpha_{0i_j} + \delta \\
&\leq (2+C)L|\alpha_0 - \overline{\alpha}| + \delta \\
&\leq (3+C)L\delta,
\end{aligned}$$

where C is the same C in (2.7). Then we get a contradiction with (2.12) by taking $r = a_1$ in the above. Hence the claim is proved.

Notice that the claim and estimate (2.10) implies the continuity of ψ at $\overline{\alpha}$, therefore, we have proved case 1.

Case 2. $\overline{\alpha} \in \mathbb{R}^L_+$.

As above we assume $r_{\overline{\alpha}} < \infty$, and obviously $r_{\overline{\alpha}} > 0$. Let's assume at $r = r_{\overline{\alpha}}$, for some integer m that $0 < m \leq L$ ($m > 0$ because $\psi(\overline{\alpha}) = u(r_{\overline{\alpha}}, \overline{\alpha}) \in \partial\mathbb{R}^L_+$, i.e. u touches the wall at $r = r_{\overline{\alpha}}$), so suppose $u_{i_1}(r_{\overline{\alpha}}, \overline{\alpha}) = \cdots = u_{i_m}(r_{\overline{\alpha}}, \overline{\alpha}) = 0$, and $u_{i_{m+1}}(r_{\overline{\alpha}}, \overline{\alpha}), \cdots, u_{i_L}(r_{\overline{\alpha}}, \overline{\alpha}) > 0$. Let $\overline{\omega}(r) = \sum_{j=1}^m u_{i_j}(r, \overline{\alpha})$, and we claim that $\overline{\omega}'(r_{\overline{\alpha}}) < 0$.

By the continuity of $u(r, \overline{\alpha})$ with respect to r, $\exists \delta_1 > 0$ such that for $r \in (r_{\overline{\alpha}} - \delta_1, r_{\overline{\alpha}}]$,

$$|u(r, \overline{\alpha}) - \psi(\overline{\alpha})| < \delta_0. \tag{2.15}$$

If $m < L$, the assumption (2.7) is taking effect, and therefore $\sum_{j=1}^m f_{i_j} \geq 0$ for $r \in (r_{\overline{\alpha}} - \delta_1, r_{\overline{\alpha}}]$; if $m = L$ then by the assumption (2.6), we also have $\sum_{j=1}^m f_{i_j} \geq 0$ for $r \in (r_{\overline{\alpha}} - \delta_1, r_{\overline{\alpha}}]$. So, in $(r_{\overline{\alpha}} - \delta_1, r_{\overline{\alpha}}]$

$$-\frac{1}{r^{n-1}}(r^{n-1}\overline{\omega}'(r))' = \sum_{j=1}^m f_{i_j} \geq 0. \tag{2.16}$$

Also, since $\overline{\omega}(r) > 0$ for $r < r_{\overline{\alpha}}$ and $\overline{\omega}(r_{\overline{\alpha}}) = 0$, there must exist $r_0 \in (r_{\overline{\alpha}} - \delta_1, r_{\overline{\alpha}})$, such that $\overline{\omega}'(r_0) < 0$. So, for $r \in [r_0, r_{\overline{\alpha}}]$,

$$\overline{\omega}'(r) = (\frac{r_0}{r})^{n-1}\overline{\omega}'(r_0) - \int_{r_0}^r (\frac{\tau}{r})^{n-1} \sum_{j=1}^m f_{i_j}(u(\tau))d\tau < 0. \tag{2.17}$$

This proves the claim $\overline{\omega}'(r_{\overline{\alpha}}) < 0$. Then there exists $l_0 \in \{1, \cdots, m\}$ such that $u'_{i_{l_0}}(r_{\overline{\alpha}}, \overline{\alpha}) < 0$. Therefore, combining with the fact $u_{i_{l_0}}(r_{\overline{\alpha}}, \overline{\alpha}) = 0$, we see $u_{i_{l_0}}$ crosses the wall with a non-zero slope, i.e. there is a transversality at $r = r_{\overline{\alpha}}$, and by classical ODE stability theory (the continuous dependence on initial value) ψ is continuous at $\overline{\alpha}$. $\square$

2.2.2 Application of degree theory

Now, we recall some results in degree theory (in particular, the treatment modified by P.Lax, cf. [47]). Consider C^∞ oriented manifolds X_0, Y of dimension n (all manifolds are assumed to be paracompact) and an open subset $X \subseteq X_0$ with compact closure. For convenience, write $dy^1 \wedge \cdots \wedge dy^n = dy$. Then for a C^1 map $\phi : X \to Y$, the degree is defined as following:

Definition 2.5. *Let $\mu = f(y)dy$ be a C^∞ n-form with support contained in a coordinate patch Ω of y_0 and lying in $Y \setminus \{\phi(\partial X)\}$ such that $\int_Y \mu = 1$; set*

$$deg(\phi, X, y_0) = \int_X \mu \circ \phi. \tag{2.18}$$

Here are some properties of degree which we will refer to in our proof,

Proposition 2.6. *For y_1 close to y_0, $deg(\phi, X, y_0) = deg(\phi, X, y_1)$.*

It follows that the degree of a mapping is constant on any connected component C of $Y \setminus \{\phi(\partial X)\}$, and we can write degree as $deg(\phi, X, C)$.

Proposition 2.7. *If $y_0 \notin \phi(\overline{X})$, then $deg(\phi, X, y_0) = 0$.*

As a result, if $deg(\phi, X, y_0) \neq 0$, then $y_0 \in \phi(\overline{X})$.

An important property of degree is that, the notion can be extended to maps ϕ which are merely continuous, since we can approximate such ϕ by C^1 maps ϕ_n (see Property 1.5.3 in [47]). Also, degree is homotopy invariant (see Proposition 1.4.3 in [47]), which enables us to define degree for continuous map $\eta : \partial X \to \mathbb{R}^n \setminus y_0$ (see Property 1.5.4 in [47]). It leads to the following theorem (see Property 1.5.5 in [47]),

Theorem 2.8. *$deg(\eta, X, y_0)$ depends only on the homotopy class of $\eta : \partial X \to Y = \mathbb{R}^n$.*

The above theorem is also true if we change $\mathbb{R}^n$ to a hyperplane $T^n = \{\alpha \in \mathbb{R}^n | \Sigma_{i=1}^n \alpha_i = a\}$, i.e.,

Theorem 2.9. *$deg(\eta, X, y_0)$ depends only on the homotopy class of $\eta : \partial X \to Y = T^n$.*

Now we are prepared to prove the existence of solution to (2.2).

Proof of theorem 2.1. For any fixed real number $a > 0$, assume that (2.4) admits no global positive solution with any initial value $\alpha \in \Sigma_a$, so $r_\alpha < \infty$, and then we can define a target map ψ by (2.3).

Recall that $\Sigma_a = \{\alpha \in \overline{\mathbb{R}^L_+} | \sum_{i=1}^{L} \alpha_i = a\}$ and $B_a = \{\alpha \in \partial\mathbb{R}^L_+ | \sum_{i=1,\cdots,L} \alpha_i \leq a\}$, and by lemma 2.4 ψ is a continuous maps from $\Sigma_a \longrightarrow B_a$.

Let $\pi(\alpha) = \alpha + \frac{1}{L}(a - \sum\limits_{i=1,\cdots,L} \alpha_i)(1, \cdots, 1) : B_a \longrightarrow \Sigma_a$, then π is continuous with a continuous inverse $\pi^{-1}(\alpha) = \alpha - (\min\limits_{i=1,\cdots,L} \alpha_i)(1, \cdots, 1) : \Sigma_a \longrightarrow B_a$.

The map: $\phi = \pi \circ \psi : \Sigma_a \longrightarrow \Sigma_a$ is continuous and $\phi(\alpha) = \alpha$ on $\partial\Sigma_a$. Then let $\eta = id$ (the identity map) and for $n = L - 1$, $X = \Sigma_a \setminus \partial\Sigma_a$, $Y = T^n$ and then by theorem 2.9 we have $deg(\phi, X, \alpha) = deg(\eta, X, \alpha) = 1$ for any $\alpha \in X$. By property 2.7, ϕ is onto, which implies that ψ is also onto. this shows that there exists an $\alpha_0 \in \Sigma_a$ such that $\psi(\alpha_0) = 0$.

Since we assume that system (2.5) admits no solution, r_{α_0} corresponding to this α_0 cannot be finite, a contradiction. This completes the proof of Theorem 2.1. $\Box$

2.3 Examples

One of the simplest systems is that $f \equiv 0$ in (2.2), then $u \equiv C$ for some constant vector C is a solution. Let us point out that since f is not positive, this trivial system is not included in results of Li and Villavert [37, 39, 69], but it is included in our cases. In this section, we will show the existence of solution to some non-trivial systems. In the view of theorem 2.1, we only need to show their corresponding Dirichlet problems admit no radial solution, and verify its source term satisfy our assumptions in theorem 2.1.

The main tool of showing non-existence of solution (hence no radial solution) to Dirichlet problem is Rellich-Pohožaev type of identities. Mitidieri did the pioneering work in [44], and here we present a brief proof for completeness (see also Quittner-Souplet [58]). Here the Rellich-Pohožaev identities we need are the following,

Lemma 2.10. *For the following system,*

$$\begin{cases} -\Delta u = f(u,v) & \text{in } B, \\ -\Delta v = g(u,v) & \text{in } B, \\ u, v > 0 & \text{in } B, \\ u, v = 0 & \text{on } \partial B, \end{cases} \quad \text{in } \mathbb{R}^n, \tag{2.19}$$

we have

(i) for $\theta \in [0,1]$,

$$\int_B \nabla u \cdot \nabla v dx = -\int_B \theta v \Delta u + (1-\theta) u \Delta v dx, \tag{2.20}$$

(ii)

$$\int_B \Delta u (x \cdot \nabla v) + \Delta v (x \cdot \nabla u) - (n-2) \nabla u \cdot \nabla v dx = \int_{\partial B} (x \cdot \nu)(\frac{\partial u}{\partial \nu} \frac{\partial v}{\partial \nu}) d\sigma \geq 0, \tag{2.21}$$

where ν is the outward normal,

(iii)

$$\int_B \Delta u (x \cdot \nabla u) dx = \int_B \frac{n-2}{2} |\nabla u|^2 dx + \frac{1}{2} \int_{\partial B} x \cdot \nu |\nabla u|^2 d\sigma \geq 0. \tag{2.22}$$

Proof. Suppose there exists a positive solution (u, v) to (2.19).

(i) Since

$$-\int_B v \Delta u dx = \int_B \nabla u \cdot \nabla v dx = -\int_B u \Delta v dx,$$

for $\theta \in [0,1]$ we have

$$\int_B \nabla u \cdot \nabla v dx = -\int_B \theta v \Delta u + (1-\theta) u \Delta v dx.$$

(ii)

$$
\begin{aligned}
&\int_B \Delta u(x\cdot\nabla v)+\Delta v(x\cdot\nabla u)dx\\
&=\int_{\partial B}\frac{\partial u}{\partial\nu}(x\cdot\nabla v)+\frac{\partial v}{\partial\nu}(x\cdot\nabla u)d\sigma-\int_B(2\nabla u\cdot\nabla v+x_j\partial_i u\partial^2_{ij}v+x_j\partial_i v\partial^2_{ij}u)dx\\
&=\int_{\partial B}\frac{\partial u}{\partial\nu}(x\cdot\nabla v)+\frac{\partial v}{\partial\nu}(x\cdot\nabla u)d\sigma-\int_B(2\nabla u\cdot\nabla v+x\cdot\nabla(\nabla u\cdot\nabla v))dx\\
&=\int_{\partial B}\frac{\partial u}{\partial\nu}(x\cdot\nabla v)+\frac{\partial v}{\partial\nu}(x\cdot\nabla u)d\sigma-\int_B 2\nabla u\cdot\nabla v dx-\int_{\partial B}x\cdot\nu(\nabla u\cdot\nabla v))d\sigma+n\int_B\nabla u\cdot\nabla v dx.
\end{aligned}
$$

Notice the fact that $x=|x|\nu$, and $\nabla u=\frac{\partial u}{\partial\nu}\nu$ and $\nabla v=\frac{\partial v}{\partial\nu}\nu$ due to $u,v=0$ on ∂B, and after rearrangement we have the identity (2.21). Also, $\frac{\partial u}{\partial\nu}\le 0$ and $\frac{\partial v}{\partial\nu}\le 0$ on ∂B, so we have RHS of (2.21) ≥ 0.

(iii) If we let $u=v$, then (ii) gives

$$\int_B \Delta u(x\cdot\nabla u)dx=\int_B\frac{n-2}{2}|\nabla u|^2dx+\frac{1}{2}\int_{\partial B}x\cdot\nu|\nabla u|^2d\sigma\ge 0.$$

□

2.3.1 Sign-changing source terms

First we give a simple example that can be easily generated to some non-trivial sign-changing source terms systems. Consider the following system,

$$
\begin{cases}
-\Delta u=v^p-u^p, & \\
-\Delta v=u^p, & \text{in } \mathbb{R}^n,\\
u,v>0, &
\end{cases}
\tag{2.23}
$$

and its corresponding Dirichlet problem,

$$
\begin{cases}
-\Delta u=v^p-u^p & \text{in } B,\\
-\Delta v=u^p & \text{in } B,\\
u,v>0 & \text{in } B,\\
u,v=0 & \text{on } \partial B,
\end{cases}
\tag{2.24}
$$

where $B=B_R(0)\subset\mathbb{R}^n$ for any $R>0$. We have

Theorem 2.11. *If $p > \frac{n+2}{n-2}$, then (2.23) admits radial positive solution.*

Again the proof relies on the non-existence of solution to (2.24).

Lemma 2.12. *If $p > \frac{n+2}{n-2}$, then system (2.24) admits no solution for any $R > 0$.*

Proof. By Lemma 2.10, we merge the source terms into (2.21) (we replace $-u^p$ in the first source term by Δv, i.e. $-\Delta u = v^p + \Delta v$ and $-\Delta v = u^p$), and by (2.20) we have

$$
\begin{aligned}
&\text{LHS of (2.21)} = \int_B -(v^p + \Delta v)(x\cdot\nabla v) - u^p(x\cdot\nabla u) - (n-2)\nabla u\cdot\nabla v dx \\
&= \int_B -x\cdot\nabla\left(\frac{v^{p+1}}{p+1} + \frac{u^{p+1}}{p+1}\right) - \Delta v(x\cdot\nabla v) + (n-2)\left(\theta v\Delta u + (1-\theta)u\Delta v\right)dx \\
&= \int_B -x\cdot\nabla\left(\frac{v^{p+1}}{p+1} + \frac{u^{p+1}}{p+1}\right) - \Delta v(x\cdot\nabla v) + (n-2)\left(-\theta(v^{p+1} + v\Delta v) - (1-\theta)u^{p+1}\right)dx \\
&= \int_B \left\{\left(\frac{n}{p+1} - (n-2)\theta\right)v^{p+1} + \left(\frac{n}{p+1} - (n-2)(1-\theta)\right)u^{p+1}\right\}dx \\
&\quad - \int_B \Delta v(x\cdot\nabla v) + (n-2)\theta v\Delta v dx
\end{aligned}
$$

So, by (2.22), (2.21) becomes

$$
\begin{aligned}
&\int_B \left\{\left(\frac{n}{p+1} - (n-2)\theta\right)v^{p+1} + \left(\frac{n}{p+1} - (n-2)(1-\theta)\right)u^{p+1}\right\}dx \\
&= \int_B \Delta v(x\cdot\nabla v) + (n-2)\theta v\Delta v dx + \int_{\partial B}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\frac{\partial v}{\partial\nu}\right)d\sigma \\
&= \int_B \frac{n-2}{2}|\nabla v|^2 dx + \frac{1}{2}\int_{\partial B} x\cdot\nu|\nabla u|^2 d\sigma - \int_B \theta(n-2)|\nabla v|^2 dx + \int_{\partial B}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\frac{\partial v}{\partial\nu}\right)d\sigma \\
&= \int_B (1-2\theta)\frac{n-2}{2}|\nabla v|^2 dx + \frac{1}{2}\int_{\partial B} x\cdot\nu|\nabla u|^2 d\sigma + \int_{\partial B}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\frac{\partial v}{\partial\nu}\right)d\sigma.
\end{aligned}
$$

Take $\theta = \frac{1}{2}$, and we have

$$
\int_B \left(\frac{n}{p+1} - \frac{n-2}{2}\right)\left(v^{p+1} + u^{p+1}\right)dx = \frac{1}{2}\int_{\partial B} x\cdot\nu|\nabla u|^2 d\sigma + \int_{\partial B}(x\cdot\nu)\left(\frac{\partial u}{\partial\nu}\frac{\partial v}{\partial\nu}\right)d\sigma \geq 0. \quad (2.25)
$$

So, by assumption $p > \frac{n+2}{n-2}$ the LHS of the above identity < 0, a contradiction.

□

Proof of Theorem 2.11. Directly we see that (2.23) satisfies our first main assumption (2.6). To see (2.7) is satisfied, just notice that if $u = 0$ then $|f_2| = 0 \leq f_1 = v^p$, and if $v = 0$ then $|f_1| = u^p \leq f_2$. Then for a neighborhood of such (u, v) (i.e. $uv = 0$), (2.7) holds.

So, combined with Lemma 2.12, Theorem 2.11 follows from Theorem 2.1.

□

Remark 2.13. *We thank the referees for pointing out that, actually system* (2.23) *can be solved as follows. Suppose* $-\Delta w = w^p$*, let* $u = \lambda w$ *and* $v = \nu w$*, then we can find suitable* λ, ν *such that* (u, v) *is a solution.*

However, consider a similar system,

$$\begin{cases} -\Delta u = v^p + v^q - u^p, & \\ -\Delta v = u^p, & \text{in } \mathbb{R}^n, \\ u, v > 0, & \end{cases} \tag{2.26}$$

where $p \neq q$ *and* $p, q > \frac{n+2}{n-2}$*. This system cannot be solved trivially as above, but we can show the existence of solution to it by Theorem 2.1. We only need to show the nonexistence of solution to corresponding Dirichlet problem (to show source terms satisfying assumptions* (2.6) *and* (2.7) *is similar to the proof in Theorem 2.11). By Lemma 2.10, we merge* $-\Delta u = v^p + v^q + \Delta v$ *and* $-\Delta v = u^p$ *into* (2.21)*, and take* $\theta = \frac{1}{2}$ *then we have an identity similar to* (2.25)*,*

$$\begin{aligned} &\int_B \left(\frac{n}{p+1} - \frac{n-2}{2}\right)\left(v^{p+1} + u^{p+1}\right) + \left(\frac{n}{q+1} - \frac{n-2}{2}\right) v^{q+1} dx \\ = \quad & \frac{1}{2}\int_{\partial B} x \cdot \nu |\nabla u|^2 d\sigma + \int_{\partial B} (x \cdot \nu)\left(\frac{\partial u}{\partial \nu}\frac{\partial v}{\partial \nu}\right) d\sigma > 0. \end{aligned} \tag{2.27}$$

Then for $p, q > \frac{n+2}{n-2}$ *and* $p \neq q$*, the above equation cannot hold and therefore the nonexistence of solution to the Dirichlet problem follows.*

2.3.2 Conservative source terms

In this section, we consider a system that f has a potential function F, i.e. $f = \nabla F$, and $F(0) = 0$.

Type I. Consider the following system,s

$$\begin{cases} -\Delta u_i = \frac{\partial F}{\partial u_i}, & \\ & \text{in } \mathbb{R}^n, \\ u_i > 0, & \end{cases} \tag{2.28}$$

where $i = 1, \ldots, L$. The corresponding Dirichlet problem is

$$\begin{cases} -\Delta u_i = \frac{\partial F}{\partial u_i} & \text{in } B, \\ u_i > 0 & \text{in } B, \\ u_i = 0 & \text{on } \partial B, \end{cases} \tag{2.29}$$

where $i = 1, \ldots, L$, and $B = B_R(0) \subset \mathbb{R}^n$ for any $R > 0$.

In [56], Pucci and Serrin have showed that for a general variational problem,

$$\int_\Omega \mathcal{F}(x, u, Du)dx = 0,$$

there exists Pohožaev type of identity that can give a sufficient condition on the nonexistence of solution to Dirichlet problem on a bounded star-shaped domain. In this case, the system (2.28) corresponds to a vector-valued extremal of the variational problem with

$$\mathcal{F}(x, u, Du) = \frac{1}{2}\sum_{k=1}^{L} |p^k|^2 - F(u),$$

where $p^k = Du_k$. So, theorem 6 in Pucci-Serrin [56] leads to the following result.

Lemma 2.14. *For $u \in \mathbb{R}^L_+$, F is a C^1 function of u and satisfies $F(0) = 0$ and*

$$\frac{n-2}{2} u^k \frac{\partial F}{\partial u^k} - nF(u) > 0, \text{ for } u \neq 0, \tag{2.30}$$

then system (2.29) *admits no nontrivial solution.*

Proof. Suppose there exists a solution $u = (u_1, \cdots, u_L)$ to system (2.29). By Lemma 2.10, for $i = 1, \cdots, L$,

$$\int_B \Delta u_i (x \cdot \nabla u_i) dx - \int_B \frac{n-2}{2} |\nabla u_i|^2 dx = \frac{1}{2} \int_{\partial B} x \cdot \nu |\nabla u_i|^2 d\sigma \geq 0.$$

Sum the LHS of the above identities up and get

$$\begin{aligned} 0 &\leq \int_B -\frac{\partial F}{\partial u_i}(x \cdot \nabla u_i) + \frac{n-2}{2} u_i \Delta u_i dx \\ &= \int_B -x \cdot \nabla F(u(x)) - \frac{n-2}{2} u_i \frac{\partial F}{\partial u_i} dx \\ &= \int_B nF - \frac{n-2}{2} u_i \frac{\partial F}{\partial u_i} dx, \end{aligned}$$

which contradicts to (2.30). $\square$

Therefore, it follows from Theorem 2.1 and Lemma 2.14 that

Theorem 2.15. *For system* (2.28)*, if f satisfies the assumptions* (2.6) *and* (2.7)*, and additionally if $f = \nabla F$, where $F(0) = 0$, and F satisfies* (2.30) *for $u \neq 0$, then system* (2.28) *admits a radially symmetric solution of class $C^2(\mathbb{R}^n)$.*

Here is an example system of **Type I** (2.28): Let the potential function be

$$F(u,v) = -(u-v)^2 + v^p u + u^q v, \text{with } p,q > \frac{n+2}{n-2}, \tag{2.31}$$

and then

$$\begin{cases} -\Delta u = F_u = -2(u-v) + v^p + qu^{q-1}v, & \\ -\Delta v = F_v = 2(u-v) + pv^{p-1}u + u^q, & \text{in } \mathbb{R}^n. \\ u, v > 0, & \end{cases} \tag{2.32}$$

We can verify that F satisfies the condition of Theorem 2.15. $F(0,0) = 0$, and $F_u + F_v \geq 0$ ((2.6) is satisfied). Let $u = 0$ then $F_u = 2v + v^p \geq |F_v| = 2v$, similarly if $v = 0$, then $F_v = 2u + u^q \geq |F_u| = 2u$, so we can see that (2.7) is satisfied in a neighborhood. Last, direct computation shows that

$$\begin{aligned} \frac{n-2}{2}(uF_u + vF_v) - nF &= \frac{n-2}{2}(-2(u-v)^2 + (p+1)uv^p + (q+1)u^q v) - n(-(u-v)^2 + v^p u + u^q v) \\ &\geq 2(u-v)^2 + (\frac{n-2}{2}(p+1) - n)uv^p + (\frac{n-2}{2}(q+1) - n)vu^q \\ &> 0, \text{ for } (u,v) \neq (0,0), \end{aligned}$$

so (2.30) is satisfied. Also, notice that F_u and F_v are sign-changing functions, for example, let $v = 0$, then $F_u < 0$ and let $u = 0$ then $F_u > 0$.

Type II. In the following example, we give another class of systems with potential type of source terms.

Consider the following system, in $\mathbb{R}^n$,

$$\begin{cases} -\Delta u = f_1 = \frac{\partial F}{\partial v}, \\ -\Delta v = f_2 = \frac{\partial F}{\partial u}, \\ \quad u, v > 0, \end{cases} \tag{2.33}$$

and corresponding Dirichlet problem

$$\begin{cases} -\Delta u = \frac{\partial F}{\partial v}, & \text{in } B \\ -\Delta v = \frac{\partial F}{\partial u}, & \text{in } B \\ \quad u, v > 0, & \text{in } B \\ \quad u, v = 0, & \text{on } \partial B \end{cases} \tag{2.34}$$

where $B = B_R(0) \subset \mathbb{R}^n$ for any $R > 0$. Similarly we have the following local-nonexistence lemma obtained by Mitidieri in [44],

Lemma 2.16. *Let F be a C^1 function of u, v and satisfies $F(0) = 0$. In addition, if for $(u, v) \neq (0, 0)$, there exists $\theta \in [0, 1]$ such that*

$$(n-2)\left(\theta u \frac{\partial F}{\partial u} + (1-\theta) v \frac{\partial F}{\partial v}\right) - nF(u, v) > 0, \tag{2.35}$$

then system (2.34) admits no nontrivial solution.

Then it follows from Theorem 2.1 and Lemma 2.16 that

Theorem 2.17. *For system (2.33), if $f_1 = F_v$ and $f_2 = F_u$ satisfies the assumptions (2.6) and (2.7), and additionally if $F(0) = 0$, and F satisfies (2.35) for $(u, v) \neq (0, 0)$, then system (2.33) admits a radially symmetric solution of class $C^2(\mathbb{R}^n)$.*

Proof of Lemma 2.16. Suppose there exists a solution (u, v). By Lemma 2.10, we merge the source terms to (2.21) and get,

$$
\begin{aligned}
0 &\le \int_B \Delta u(x\cdot\nabla v) + \Delta v(x\cdot\nabla u) - (n-2)\nabla u\cdot\nabla v dx \\
&= \int_B -F_v(x\cdot\nabla v) - F_u(x\cdot\nabla u) + (n-2)(\theta v\Delta u + (1-\theta)u\Delta v)dx \\
&= \int_B -x\cdot\nabla F - (n-2)(\theta vF_v + (1-\theta)uF_u)dx \\
&= \int_B nF - (n-2)(\theta vF_v + (1-\theta)uF_u)dx,
\end{aligned}
$$

which contradicts to (2.35). $\square$

Here is an example system of **Type II** (2.33): Let the potential function be

$$F(u,v) = \frac{|u-v|^{r+1}}{r+1} + \frac{u^{q+1}}{q+1} + \frac{v^{p+1}}{p+1}, \text{with } p,q,r > \frac{n+2}{n-2}, \tag{2.36}$$

and

$$\begin{cases} -\Delta u = F_v = -|u-v|^{r-1}(u-v) + v^p, \\ -\Delta v = F_u = |u-v|^{r-1}(u-v) + u^q, \quad \text{in } \mathbb{R}^n. \\ \quad u, v > 0, \end{cases} \tag{2.37}$$

We verify that F satisfies conditions in Theorem 2.17. $F(0) = 0$, and $F_u + F_v \ge 0$ ((2.6) is satisfied). Let $u = 0$ then $F_v = v^r + v^p \ge |F_u| = v^r$, similarly if $v = 0$, then $F_u = u^r + u^p \ge |F_v| = u^r$, so we can see that (2.7) is satisfied in a neighborhood. Last, direct computation shows that by taking $\theta = \frac{1}{2}$,

$$
\begin{aligned}
\frac{n-2}{2}(uF_u + vF_v) - nF &= \frac{n-2}{2}(|u-v|^{r+1} + u^{q+1} + v^{p+1}) - n(\frac{|u-v|^{r+1}}{r+1} + \frac{u^{q+1}}{q+1} + \frac{v^{p+1}}{p+1}) \\
&= (\frac{n-2}{2} - \frac{n}{r+1})|u-v|^{r+1} + (\frac{n-2}{2} - \frac{n}{q+1})u^{q+1} + (\frac{n-2}{2} - \frac{n}{p+1})v^{p+1} \\
&> 0, \text{for } (u,v) \ne (0,0),
\end{aligned}
$$

so (2.35) is satisfied. Also, notice that F_u and F_v are sign-changing functions, for example, let $v = 0$, then $F_v < 0$ for $u \ne 0$, and let $u = 0$ then $F_v > 0$ for $v \ne 0$.

Chapter 3

Non-existence for a subcritical HLS system: Lane-Emden conjecture

In this chapter, we consider a special case, $\gamma = 2$, of subcritical HLS system, i.e. the Lane-Emden system (3.1). The so-called Lane-Emden conjecture states that the Lane-Emden system admits only zero as non-negative solution. Should the conjecture be true, it has great application in singular analysis and *a priori* estimates for a large class of nonlinear systems.

The full conjecture is still open. Here we present a necessary and sufficient condition to the Lane-Emden conjecture. This condition is an energy type of integral estimate on solutions to subcritical Lane-Emden system. To approach the long standing and interesting conjecture, we believe that one plausible path is to refocus on establishing this energy type estimate.

This chapter contains the work in [12].

3.1 Introduction

Consider the subcritical Lane-Emden system,

$$\begin{cases} -\Delta u = v^p, \\ -\Delta v = u^q, \end{cases} \quad \text{in } \mathbb{R}^n, \tag{3.1}$$

where $u, v \geq 0$, and

$$\frac{1}{p+1} + \frac{1}{q+1} > \frac{n-2}{n}, \tag{3.2}$$

For critical and supercritical cases, $\frac{1}{p+1} + \frac{1}{q+1} \leq \frac{n-2}{n}$, see chapter 2. People guess that the following statement holds and call it the Lane-Emden conjecture:

Conjecture. $u = v \equiv 0$ *is the unique nonnegative solution for system* (3.1).

In [43, 45] Mitidieri has proved the nonexistence for **radial** solution in subcritical case, which settles the radial case. The full Lane-Emden conjecture is still open and only partial results are known. Many researchers have made contribution in pushing the progress forward, and we shall briefly present some important recent developments of the Lane-Emden conjecture.

Denote the scaling exponents of system (3.1) by

$$\alpha = \frac{2(p+1)}{pq-1}, \quad \beta = \frac{2(q+1)}{pq-1}, \quad \text{for } pq > 1. \tag{3.3}$$

Then subcritical condition (3.2) is equivalent to

$$\alpha + \beta > n - 2, \quad \text{for } pq > 1. \tag{3.4}$$

For p, q in the following region

$$pq \leq 1, \text{ or } pq > 1 \text{ and } \max\{\alpha, \beta\} \geq n - 2, \tag{3.5}$$

(3.1) admits no positive entire **supersolution**, cf. Serrin and Zou [62]. This implies the conjecture for $n = 1, 2$. Also, the conjecture is true for

$$\min\{\alpha, \beta\} \geq \frac{n-2}{2}, \text{ with } (\alpha, \beta) \neq (\frac{n-2}{2}, \frac{n-2}{2}), \tag{3.6}$$

cf. Busca and Manásevich [5]. Note that (3.6) covers the case that both (p, q) are subcritical, i.e. $\max\{p, q\} \leq \frac{n+2}{n-2}$, with $(p, q) \neq (\frac{n+2}{n-2}, \frac{n+2}{n-2})$, which is treated earlier, cf. de Figueiredo and Felmer [19] and Reichel and Zou [59]. Also, Mitidieri [45] has proved that the system admits no radial positive solution. Chen and Li [10] have proved that any solution with finite energy must be radial, therefore combined with Mitidieri [45], no finite-energy non-trivial solution exists.

For $n = 3$, the conjecture is solved by two papers. First, Serrin and Zou [62] proved that there is no positive solution with polynomial growth at infinity. Then Poláčik, Quittner and Souplet [55] removed the growth condition. In fact, they proved that no bounded positive solution implies no positive solution. To sum up, the result of Poláčik et al. has two important consequences: one is that combining with Serrin and Zou's result, one can prove the conjecture for $n = 3$; the other is that proving the Lane-Emden conjecture is equivalent to proving nonexistence of bounded positive solution. Thus, we always assume that (u, v) are bounded in this chapter.

For $n = 4$, the conjecture is recently solved by Souplet [65]. In [62], Serrin and Zou used the integral estimates to derive the nonexistence results. Souplet further developed the approach of integral estimates and solved the conjecture for $n = 4$ along with the case $n = 3$. In higher dimensions, this approach provides a new subregion where the conjecture holds, but the problem of full range in high dimensional space still seems stubborn. Souplet has proved that if

$$\max\{\alpha, \beta\} > n - 3, \tag{3.7}$$

then (3.1) with (p, q) satisfying (3.2) has no positive solution. Notice that (3.7) covers (3.2) only when $n \leq 4$, and when $n \geq 5$ (3.7) covers a subregion of (3.2).

The approach developed by Souplet in [65] is also effective on non-existence of positive solution to Hardy-Hénon type equations and systems (cf. [20, 21, 51, 52]):

$$\begin{cases} -\Delta u = |x|^a v^p, \\ -\Delta v = |x|^b u^q, \end{cases} \quad \text{in } \mathbb{R}^n.$$

This approach can also be applied to more general elliptic systems, for further details, we refer to [66] and [57]. Moreover, a natural extension and application of this tool is the high order Lane-Emden system which was done by Arthur, Yan and Zhao [3].

In this chapter, we point out that the key to the Lane-Emden conjecture is obtaining a certain type of energy estimate. This estimate is in fact a necessary and sufficient condition to the conjecture. Connecting the estimate and the conjecture is a laborious work and needs to combine many types of estimates. We believe that with the result here people can refocus on proving the crucial estimate and thus solve the conjecture.

Theorem 3.1. *Let $n \geq 3$ and (u, v) be a non-negative bounded solution to* (3.1). *Assume there* *exists an* $s > 0$ *satisfying* $n - s\beta < 1$ *such that*

$$\int_{B_R} v^s \leq CR^{n-s\beta}, \tag{3.8}$$

then $u, v \equiv 0$ *provided* $0 < p, q < +\infty$ *and* $\frac{1}{p+1} + \frac{1}{q+1} > 1 - \frac{2}{n}$.

Remark 3.2. *(a) Energy estimate* (3.8) *is a necessary condition to the Lane-Emden conjecture. One just needs to notice that when* $u, v \equiv 0$, (3.8) *is obviously satisfied. The key to the proof of Theorem 3.1 is to show* (3.8) *is sufficient.*

(b) If $p \geq q$, *the assumption on* v *is weaker than the corresponding assumption on* u *due to a comparison principle between* u *and* v *(i.e. Lemma 3.8).*

In other words, if $p \geq q$, *and we assume for some* $r > 0$, *such that* $n - r\alpha < 1$,

$$\int_{B_R} u^r \leq CR^{n-r\alpha}. \tag{3.9}$$

Then (3.9) *implies* (3.8) *by Lemma 3.8.*

(c) By taking $s = p$ *Theorem 3.1 recovers the result in [65].*

(d) A technical issue is that the standard $W^{2,p}$*-estimate used in [65] is not enough to establish Theorem 3.1 (see the footnote of Proposition 3.14). To overcome this difficulty, a mixed type* $W^{2,p}$*-estimate is introduced in Lemma 3.5.*

Remark 3.3. *(a) It is worthy to point out an interesting role that the coefficient "1" of the nonlinear source term plays in the Lane-Emden system. Consider the following system*

$$\begin{cases} -\Delta u = c_1(x) v^p, \\ -\Delta v = c_2(x) u^q, \end{cases} \quad in \quad \mathbb{R}^n, \tag{3.10}$$

where $0 < a \leq c_1(x), c_2(x) \leq b < \infty$ *and* $x \cdot \nabla c_1(x), x \cdot \nabla c_2(x) \geq 0$ *for some positive constants* $a, b > 0$. *We can also have the following Rellich-Pohožaev type identity for some* *constants* d_1, d_2 *such that* $d_1 + d_2 = n - 2$*,*

$$\begin{aligned}&\int_{B_R} (\frac{nc_1}{p+1} - d_1 c_1 + \frac{x \cdot \nabla c_1(x)}{p+1}) v^{p+1} + (\frac{nc_2}{q+1} - d_2 c_2 + \frac{x \cdot \nabla c_2(x)}{q+1}) u^{q+1} dx \\ &= R^n \int_{\mathbb{S}^{n-1}} \frac{c_1(R) v^{p+1}(R)}{p+1} + \frac{c_2(R) u^{q+1}(R)}{q+1} d\sigma \\ &+ R^{n-1} \int_{\mathbb{S}^{n-1}} d_1 v' u + d_2 u' v d\sigma + R^n \int_{\mathbb{S}^{n-1}} (v'u' - R^{-2} \nabla_\theta u \cdot \nabla_\theta v) d\sigma.\end{aligned} \tag{3.11}$$

By the constrains on $c_1(x), c_2(x)$*, we can have the left terms (LT) in* (3.11) *as*

$$LT \geq \delta_0 \int_{B_R} v^{p+1} + u^{q+1} dx, \quad \textit{for some} \quad \delta_0 > 0. \tag{3.12}$$

The argument in [65] is also valid for this case, and we still can prove nonexistence for $n \leq 4$ *and for* $\max(\alpha, \beta) > n - 3, n \geq 5$.

On the other hand, for $c_1(x), c_2(x)$ *such that* $x \cdot \nabla c_1(x), x \cdot \nabla c_2(x) < 0$*, there exist non-zero solutions of* (3.10) *in some subcritical cases (see Lei and Li [35] for detail).*

(b) Theorem 3.1 is still true if we consider (3.10) *with* $0 < a \leq c_1(x), c_2(x) \leq b < \infty$ *and* $x \cdot \nabla c_1(x), x \cdot \nabla c_2(x) \geq 0$*. And the proof is very similar to the case* $c_1 = c_2 = 1$*. So here, we only prove for* $c_1 = c_2 = 1$*.*

The complete solution of the Lane-Emden conjecture may be a longstanding work. Hence, it will be interesting to consider the Lane-Emden conjecture under some conditions weaker than (3.8).

Open problem 1. Can we prove the Lane-Emden conjecture under the following pointwise asymptotic:

$$|v(x)| \leq C|x|^{-\gamma}, \quad \text{for some} \quad 0 < \gamma < \beta.$$

Open problem 2. Can we prove the Lane-Emden conjecture under the following integral asymptotic:

$$\int_{B_R} v^s \leq CR^\delta, \quad \text{for some} \quad s > 0, \quad 0 < \delta < 1.$$

Clearly, if problem 2 is solved, problem 1 directly follows by choosing sufficiently large s.

In Section 3.2, we provide a few preliminary results. Some simplified proofs are given for the completeness and convenience of readers. One of the difficulty in the proof of Theorem 3.1 was to control the embedding index, and we derived a varied form of $W^{2,p}$-estimate (see Lemma 3.16) to solve this problem. In Section 3.3, we give the proof of Theorem 3.1. Our proof by classifying the argument into two cases hopefully can deliver the idea and the structure of the proof to readers in a clearer way.

3.2 Preliminaries

Throughout this chapter, the standard Sobolev embedding on $\mathbb{S}^{n-1}$ is frequently used. Here we make some conventions about the notations. Let D denote the gradient with respect to standard metric on manifold. Let $n \geq 2$, $j \geq 1$ be integers and $1 \leq z_1 < \lambda \leq +\infty$, $z_2 \neq (n-1)/j$. For $u = u(\theta) \in W^{j,z_1}(\mathbb{S}^{n-1})$, we have

$$\|u\|_{L^{z_2}(\mathbb{S}^{n-1})} \leq C\left(\|D_\theta^j u\|_{L^{z_1}(\mathbb{S}^{n-1})} + \|u\|_{L^1(\mathbb{S}^{n-1})}\right), \tag{3.13}$$

where

$$\begin{cases} \frac{1}{z_2} = \frac{1}{z_1} - \frac{j}{n-1}, & \text{if } z_1 < (n-1)/j, \\ z_2 = \infty, & \text{if } z_1 > (n-1)/j, \end{cases}$$

and $C = C(j, z_1, n) > 0$. Although C may be different from line to line, we always denote the universal constant by C. For simplicity, in what follows, for a function $f(r,\theta)$, we define

$$\|f\|_p(r) = \|f(r,\cdot)\|_{L^p(\mathbb{S}^{n-1})}, \tag{3.14}$$

if no risk of confusion arises. Also let s, p, q be defined as in Theorem 3.1 and

$$l = s/p, \quad k = \frac{p+1}{p}, \quad m = \frac{q+1}{q}.$$

By Remark 3.2 (b) and Lemma 3.8, throughout the chapter, we always assume $p \geq q$. At last, we set

$$F(R) = \int_{B_R} u^{q+1} dx.$$

3.2.1 Basic Inequalities

Let us start with a basic yet important fact. Considering L^t-norm on B_{2R}, we can write

$$\|f\|_{L^t(B_{2R})}^t = \int_0^{2R} \|f(r)\|_{L^t(\mathbb{S}^{n-1})}^t r^{n-1} dr,$$

then by a standard measurement argument (cf. [62], [65]) one can prove that:

Lemma 3.4. *Let $f_i \in L^{p_i}_{loc}(\mathbb{R}^n)$, and $i = 1, \ldots, N$, then for any $R > 0$, there exists $\tilde{R} \in [R, 2R]$ such that*

$$\|f_i\|_{L^{p_i}(\mathbb{S}^{n-1})}(\tilde{R}) \leq (N+1) R^{-\frac{n}{p_i}} \|f_i\|_{L^{p_i}(B_{2R})}, \text{ for each } i = 1, \ldots, N.$$

The following lemma is a varied $W^{2,p}$-estimate which seems not to appear in any literature, so we give a simple proof.

Lemma 3.5. *Let $1 < \gamma < +\infty$ and $R > 0$. For $u \in W^{2,\gamma}(B_{2R})$, we have*

$$\|D^2 u\|_{L^\gamma(B_R)} \leq C \left(\|\Delta u\|_{L^\gamma(B_{2R})} + R^{\frac{n}{\gamma} - (n+2)} \|u\|_{L^1(B_{2R})} \right)$$

where $C = C(\gamma, n) > 0$.

Proof. First we deal with functions in $C^2(B_2) \cap C^0(\overline{B_2})$. By standard elliptic $W^{2,p}$-estimate, we have

$$\|D^2 u\|_{L^\gamma(B_1)} \leq C(\|\Delta u\|_{L^\gamma(B_{\frac{3}{2}})} + \|u\|_{L^\gamma(B_{\frac{3}{2}})}). \tag{3.15}$$

By Lemma 3.4, $\exists \tilde{R} \in [\frac{7}{4}, 2]$ such that on $B_{\tilde{R}}$, u can be written as $u = w_1 + w_2$, where respectively w_1 and w_2 are solutions to

$$\begin{cases} \Delta w_1 = \Delta u, & \text{in } B_{\tilde{R}}, \\ w_1 = 0, & \text{on } \partial B_{\tilde{R}}, \end{cases}$$

and

$$\begin{cases} \Delta w_2 = 0, & \text{in } B_{\tilde{R}}, \\ w_2 = u, & \text{on } \partial B_{\tilde{R}}, \end{cases}$$

and additionally,

$$\int_{\partial B_{\tilde{R}}} ud\sigma \leq C\|u\|_{L^1(B_2)}. \tag{3.16}$$

By standard $W^{2,p}$-estimate with homogeneous boundary condition, we have

$$\|w_1\|_{L^\gamma(B_{\frac{3}{2}})} \leq \|w_1\|_{W^{2,\gamma}(B_{\frac{3}{2}})} \leq C\|\Delta w_1\|_{L^\gamma(B_{\tilde{R}})}.$$

Since w_2 can be solved explicitly by Poisson formula on $B_{\tilde{R}}$, we see that by (3.16) for any $x \in B_{\frac{3}{2}} \subsetneq B_{\tilde{R}}$, $w_2(x)$ can be bounded pointwisely by

$$|w_2(x)| \leq C\int_{\partial B_{\tilde{R}}} |u| \leq C\|u\|_{L^1(B_2)}.$$

So,

$$\|w_2\|_{L^\gamma(B_{\frac{3}{2}})} \leq C\|u\|_{L^1(B_2)}.$$

Hence,

$$\begin{aligned}\|u\|_{L^\gamma(B_{\frac{3}{2}})} &\leq \|w_1\|_{L^\gamma(B_{\frac{3}{2}})} + \|w_2\|_{L^\gamma(B_{\frac{3}{2}})}\\ &\leq C(\|\Delta u\|_{L^\gamma(B_{\tilde{R}})} + \|u\|_{L^1(B_2)}).\end{aligned}$$

Therefore, (3.15) becomes

$$\|D^2u\|_{L^\gamma(B_1)} \leq C(\|\Delta u\|_{L^\gamma(B_2)} + \|u\|_{L^1(B_2)}).$$

Then the lemma follows from a dilation and approximation argument. □

Lemma 3.6 (Interpolation inequality on B_R)**.** *Let $1 \leq \gamma < +\infty$ and $R > 0$. For $u \in W^{2,\gamma}(B_R)$, we have*

$$\|D_x u\|_{L^1(B_R)} \leq C\left(R^{n(1-\frac{1}{\gamma})+1}\|D_x^2 u\|_{L^\gamma(B_R)} + R^{-1}\|u\|_{L^1(B_R)}\right),$$

where $C = C(\gamma, n) > 0$.

3.2.2 Pohožaev Identity, Comparison Principle and Energy Estimates

For system (3.1) we have a Rellich-Pohožaev identity, which is the starting point of the proof of Theorem 3.1,

Lemma 3.7. *Let $d_1, d_2 \geq 0$ and $d_1 + d_2 = n-2$, then*

$$\int_{B_R} (\frac{n}{p+1} - d_1)v^{p+1} + (\frac{n}{q+1} - d_2)u^{q+1}dx$$
$$= R^n \int_{\mathbb{S}^{n-1}} \frac{v^{p+1}(R)}{p+1} + \frac{u^{q+1}(R)}{q+1}d\sigma + R^{n-1}\int_{\mathbb{S}^{n-1}} d_1 v'u + d_2 u'v d\sigma + R^n \int_{\mathbb{S}^{n-1}} (v'u' - R^{-2}\nabla_\theta u \cdot \nabla_\theta v)d\sigma.$$

The following comparison principle is somewhat well known. An alternative proof can be found in [65].

Lemma 3.8 (Comparison Principle)**.** *Let $p \geq q > 0, pq > 1$ and (u, v) be a positive bounded solution of* (3.1)*. Then we have the following comparison principle,*

$$v^{p+1}(x) \leq \frac{p+1}{q+1}u^{q+1}(x),\ x \in \mathbb{R}^n.$$

Proof. Let $l = (\frac{p+1}{q+1})^{\frac{1}{p+1}}$, $\sigma = \frac{q+1}{p+1}$. So $l^{p+1}\sigma = 1$, and $\sigma \leq 1$. Denote

$$\omega = v - lu^\sigma.$$

We will show that $\omega \leq 0$.

$$\begin{aligned}
\Delta\omega &= \Delta v - l\nabla \cdot (\sigma u^{\sigma-1}\nabla u) \\
&= \Delta v - l\sigma(\sigma-1)|\nabla u|^2 - l\sigma u^{\sigma-1}\Delta u \\
&\geq -u^q + l\sigma u^{\sigma-1}v^p \\
&= u^{\sigma-1}((\frac{v}{l})^p - u^{q+1-\sigma}) \\
&= u^{\sigma-1}((\frac{v}{l})^p - u^{\sigma p}).
\end{aligned}$$

So, $\Delta\omega > 0$ if $w > 0$. Now, suppose $w > 0$ for some $x \in \mathbb{R}^n$, and there are two cases:

Case 1: $\exists x_0 \in \mathbb{R}^n$, such that $\omega(x_0) = \max_{\mathbb{R}^n} \omega(x) > 0$, and $\Delta\omega(x_0) \leq 0$. However, when $w > 0$, $\Delta\omega > 0$, a contradiction.

Case 2: There exists a sequence $\{x_m\}$ with $|x_m| \to +\infty$, such that $\lim_{m\to+\infty} \omega(x_m) = \max_{\mathbb{R}^n} \omega(x) > c_0 > 0$ for some constant c_0.

Let $\omega_R(x) = \phi(\frac{x}{R})\omega(x)$, where $\phi(x) \in C_0^\infty(B_1)$ is a cutoff function and $\phi(x) \equiv 1$ in $B_{\frac{1}{2}}$. Since $\omega_R(x) = 0$ on ∂B_R, there exists an $x_R \in B_R$ such that $\omega_R(x_R) = \max_{B_R} \omega_R(x)$ and $\lim_{R\to+\infty} \omega(x_R) = \max_{\mathbb{R}^n} \omega(x) > 0$. Also,

$$0 = \nabla\omega_R(x_R) = \phi(\frac{x_R}{R})\nabla\omega(x_R) + \frac{1}{R}\nabla\phi(\frac{x_R}{R})\omega(x_R).$$

As $\phi(\frac{x_R}{R}) \geq c_1 > 0$ for some constant c_1 (in fact, $\phi(\frac{x_R}{R}) \to 1$) and $\omega(x_R)$ is bounded since u, v are bounded in $\mathbb{R}^n$, we see that $\nabla\omega(x_R) \to 0$ as $R \to +\infty$. So,

$$\begin{aligned} 0 \geq \Delta\omega_R(x_R) &= \frac{1}{R^2}\Delta\phi(\frac{x_R}{R})\omega(x_R) + \frac{2}{R}\nabla\phi(\frac{x_R}{R}) \cdot \nabla\omega(x_R) + \phi(\frac{x_R}{R})\Delta\omega(x_R) \\ \Rightarrow 0 &\geq \Delta\omega(x_R) + O(\frac{1}{R^2}) \end{aligned}$$

Since $\omega(x_R) > c_0/2$ for sufficiently large R, $\Delta\omega(x_R) > c_2 > 0$ for some constant c_2, a contradiction.

□

Remark 3.9. *For general Lane-Emden type system* (3.10)*, we can choose*

$$w = v - Clu^\sigma, \quad \textit{where} \quad C^{p+1} = \sup_{x\in\mathbb{R}^n} \frac{c_2(x)}{c_1(x)}.$$

By the same arguments, we can also get the desired comparison principle.

Next we prove a group of energy estimates which are crucial to the entire argument. As Theorem 3.1 points out, better energy estimates are the key to the Lane-Emden conjecture. Unfortunately, efforts have been made so far only provide the following inequalities, which are first obtained by Serrin and Zou [62] (1996). Here we give a simpler proof than the original one for the convenience of readers.

Lemma 3.10. *Let $p, q > 0$ with $pq > 1$. For any positive solution (u, v) of* (3.1)

$$\int_{B_R} u \leq CR^{n-\alpha}, \ \textit{and} \int_{B_R} v \leq CR^{n-\beta}, \tag{3.17}$$

$$\int_{B_R} u^q \leq CR^{n-q\alpha}, \ \textit{and} \int_{B_R} v^p \leq CR^{n-p\beta}. \tag{3.18}$$

Proof. Without loss of generality, we can assume that $p \geq q$.

Let $\phi \in C^\infty(B_R(0))$ be the first eigenfunction of $-\Delta$ in B_R and λ be the eigenvalue. By definition and rescaling, it is easy to see that $\phi \mid_{\partial B_R} = 0$ and $\lambda \sim \frac{1}{R^2}$. By normalizing, one gets $\phi \geq c_0 > 0$ on $B_{R/2}$ for some constant c_0 independent of R, $\phi(0) = \|\phi\|_\infty = 1$. So, multiplying (3.1) by ϕ then integrating by parts on B_R we have,

$$\int_{B_R} \phi u^q = -\int_{B_R} \phi \Delta v = \int_{\partial B_R} v \frac{\partial \phi}{\partial n} d\sigma + \lambda \int_{B_R} \phi v.$$

By Hopf's Lemma we know that $\frac{\partial \phi}{\partial n} < 0$ on ∂B_R, so

$$\int_{B_R} \phi u^q \leq \lambda \int_{B_R} \phi v. \tag{3.19}$$

Similarly, we have

$$\int_{B_R} \phi v^p \leq \lambda \int_{B_R} \phi u. \tag{3.20}$$

Applying Lemma 3.8 to (3.19), we have

$$\frac{1}{R^2} \int_{B_R} \phi v \geq C \int_{B_R} \phi v^{\frac{q(p+1)}{q+1}}.$$

Notice that $\frac{q(p+1)}{q+1} > 1$ as $pq > 1$, so by Hölder inequality

$$\begin{aligned}\int_{B_R} \phi v^{\frac{q(p+1)}{q+1}} &\geq (\int_{B_R} \phi v)^{\frac{q(p+1)}{q+1}} (\int_{B_R} \phi)^{-(\frac{q(p+1)}{q+1}-1)} \\ &\geq C(\int_{B_R} \phi v)^{\frac{q(p+1)}{q+1}} R^{-n\frac{qp-1}{q+1}}.\end{aligned}$$

So,

$$\begin{aligned}&\frac{1}{R^2} \int_{B_R} \phi v \geq C(\int_{B_R} \phi v)^{\frac{q(p+1)}{q+1}} R^{-n\frac{qp-1}{q+1}} \\ \Rightarrow &\int_{B_R} \phi v \leq C R^{n-\beta}.\end{aligned}$$

Therefore, by (3.19)

$$\int_{B_R} \phi u^q \leq C R^{n-\beta-2} = C R^{n-q\alpha}.$$

Now, **Case 1:** If $q > 1$, then by Hölder inequality

$$\int_{B_R} \phi u \leq (\int_{B_R} \phi u^q)^{\frac{1}{q}} (\int_{B_R} \phi)^{\frac{1}{q'}} \leq CR^{\frac{n}{q}-\alpha} R^{\frac{n}{q'}} = CR^{n-\alpha}, \quad \frac{1}{q} + \frac{1}{q'} = 1.$$

Mean while, by (3.20)

$$\int_{B_R} \phi v^p \leq CR^{n-\alpha-2} = CR^{n-p\beta}.$$

This finishes the proof for Case 1.

Case 2: Assume that $q \leq 1$. To prove this trickier case, we begin with a lemma of energy-type estimate,

Lemma 3.11. *If $\Delta u \leq 0$, then for $\gamma \in (0,1)$, $\eta \in C_0^\infty(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \frac{4}{\gamma^2} |D(u^{\frac{\gamma}{2}})|^2 \eta^2 = \int \eta^2 |Du|^2 u^{\gamma-2} \leq C \int |D\eta|^2 u^\gamma. \tag{3.21}$$

Proof. Multiply $\eta^2 u^{\gamma-1}$ to $\Delta u \leq 0$ then integrate over the whole space. $\square$

We rewrite (3.21) as

$$\int_{B_R} |Du|^2 u^{\gamma-2} \leq \frac{C_\gamma}{R^2} \int_{B_{2R}} u^\gamma \tag{3.22}$$

where $C_\gamma \to +\infty$ as $\gamma \to 1$. From Poincaré's Inequality, we have

$$|f|_{\frac{na}{n-a},\Omega_R} \leq C(n,a,\Omega) \left(|Df|_{a,\Omega_R} + |R|^{\frac{n-a}{a}} |f_{\Omega_R}| \right), \tag{3.23}$$

where

$$f_{\Omega_R} = ⨍_{\Omega_R} f = \frac{1}{|\Omega_R|} \int_{\Omega_R} f, \quad \Omega_R = \{Rx | x \in \Omega\}.$$

Next we prove a variation of embedding inequality,

Lemma 3.12. *For any $l \geq 1$,*

$$|f^l|_{\frac{an}{n-a},\Omega_R} \leq C(n,a,\Omega) \left(|D(f^l)|_{a,\Omega_R} + |R|^{\frac{n-a}{a}} |f_{\Omega_R}|^l \right) \tag{3.24}$$

Proof. By (3.23),

$$\begin{aligned}|f^l|_{\frac{an}{n-a},\Omega_R} &\le C(n,a,\Omega)\left(|D(f^l)|_{a,\Omega_R}+|R|^{\frac{n-a}{a}}|(f^l)_{\Omega_R}|\right)\\ &\le C(n,a,\Omega)\left(|D(f^l)|_{a,\Omega_R}+|R|^{\frac{n-a}{a}-n}\int_{\Omega_R} f^l dx\right)\\ &\le C(n,a,\Omega)\left\{|D(f^l)|_{a,\Omega_R}+|R|^{\frac{n-a}{a}-n}(\int_{\Omega_R} f dx)^{\theta l}(\int_{\Omega_R} f^{l\frac{an}{n-a}}dx)^{(1-\theta)l\frac{n-a}{lan}}\right\},\quad \theta=\frac{1-\frac{n-a}{na}}{l-\frac{n-a}{na}}\\ &\le C(n,a,\Omega)|D(f^l)|_{a,\Omega_R}+\frac{1}{2}|f^l|_{\frac{an}{n-a},\Omega_R}+C(n,a,\Omega)|R|^{\frac{n-a}{a}}|f_{\Omega_R}|^l.\end{aligned}$$

In getting the last inequality, we have used the Young's inequality. So we get (3.24). □

Let $l\ge 1$, $\theta\le 2q<2$, $\gamma=l\theta<1$, $f=u^{\frac{\theta}{2}}$, $a=2$. Then

$$\begin{aligned}|f^l|_{\frac{2n}{n-2},B_R} &\le C\left(|Df^l|_{2,B_R}+R^{\frac{n-2}{2}}|f_{B_R}|^l\right)\\ &\le C\left(|D(u^{\frac{l\theta}{2}})|_{2,B_R}+R^{\frac{n-2}{2}}|(u^{\frac{\theta}{2}})_{B_R}|^l\right)\\ &\le \frac{C}{R}\left(\int_{B_{2R}} u^{l\theta}\right)^{\frac{1}{2}}+R^{\frac{n-2}{2}}\left(⨍_{B_R} u^{\frac{\theta}{2}}\right)^l.\end{aligned}\tag{3.25}$$

The last term on the right can be estimate by Hölder and the fact that $⨍_{B_R} u^q\le CR^{-q\alpha}$ since $\frac{\theta}{2}<q$. This yields that

$$\int_{B_R} u^{\frac{n}{n-2}\theta l}\le C\left(R^{-\frac{2n}{n-2}}\left(\int_{B_{2R}} u^{l\theta}\right)^{\frac{n}{n-2}}+R^{n-\frac{n}{n-2}l\theta\alpha}\right).\tag{3.26}$$

This means if $⨍_{B_R} u^{l\theta}\le CR^{-l\theta\alpha}$, we have $⨍_{B_R} u^{\frac{n}{n-2}l\theta}\le CR^{-\frac{n}{n-2}l\theta\alpha}$ provided $l\theta<1$. By $⨍_{B_R} u^q\le CR^{-q\alpha}$, one gets

$$⨍_{B_R} u^s\le C(s)R^{-s\alpha},\quad \text{for}\quad s<\frac{n}{n-2}\tag{3.27}$$

where $C(s)\to+\infty$ as $s\to\frac{n}{n-2}$.

By taking $s=1$, the above inequality immediately leads to

$$\int_{B_R} u\le CR^{n-\alpha}.$$

Since $pq>1$ and we assume that $p\ge q$, q must be greater than 1, then by Hölder and (3.20) we get

$$\int_{B_R} v^p\le CR^{n-p\beta}.$$

This finishes the proof of Lemma 3.10. □

3.2.3 Key Estimates on $\mathbb{S}^{n-1}$

Now that we have energy inequalities (3.18), in our assumption (3.8) we can always assume $s \geq p$. Since $l = \frac{s}{p}$, we have $l \geq 1$. The following estimates for quantities on sphere $\mathbb{S}^{n-1}$ are necessary to the proof.

Proposition 3.13. *For $R \geq 1$, there exists $\tilde{R} \in [R, 2R]$ such that for $l = \frac{s}{p} \geq 1$, $k = \frac{p+1}{p}$ and $m = \frac{q+1}{q}$, we have*

$$\|u\|_1(\tilde{R}) \leq CR^{-\alpha},\ \|v\|_1(\tilde{R}) \leq CR^{-\beta},$$
$$\|D_x^2 u\|_l(\tilde{R}) \leq CR^{-\frac{lp\beta}{l+\varepsilon}},\ \|D_x^2 v\|_{1+\varepsilon}(\tilde{R}) \leq CR^{-\frac{q\alpha}{1+\varepsilon}},$$
$$\|D_x u\|_1(\tilde{R}) \leq CR^{1-\frac{\alpha+2}{1+\varepsilon}},\ \|D_x v\|_1(\tilde{R}) \leq CR^{1-\frac{\beta+2}{1+\varepsilon}},$$
$$\|D_x^2 u\|_k(\tilde{R}) \leq C(R^{-n}F(2R))^{\frac{1}{k}},\ \|D_x^2 v\|_m(\tilde{R}) \leq C(R^{-n}F(2R))^{\frac{1}{m}}.$$

In view of Lemma 3.4, to prove Proposition 3.13, we shall give the corresponding estimates on B_{2R} first. We use the varied $W^{2,p}$-estimate (i.e. Lemma 3.5) to achieve this.

Proposition 3.14. *For $R \geq 1$, we have*

$$\begin{cases} \|u\|_{L^1(B_R)} & \leq CR^{n-\beta}, \\ \|v\|_{L^1(B_R)} & \leq CR^{n-\alpha}, \end{cases} \tag{3.28}$$

$$\begin{cases} \|D_x^2 u\|_{L^{l+\varepsilon}(B_R)}^{l+\varepsilon} \leq CR^{n-lp\beta}, \ \text{with } l = \frac{s}{p} \geq 1, \\ \|D_x^2 v\|_{L^{1+\varepsilon}(B_R)}^{1+\varepsilon} \leq CR^{n-q\alpha}, \end{cases} \tag{3.29}$$

$$\begin{cases} \|D_x u\|_{L^1(B_R)} & \leq CR^{n+1-\frac{\alpha+2}{1+\varepsilon}}, \\ \|D_x v\|_{L^1(B_R)} & \leq CR^{n+1-\frac{\beta+2}{1+\varepsilon}}, \end{cases} \tag{3.30}$$

and let $k = \frac{p+1}{p}$, $m = \frac{q+1}{q}$,

$$\begin{cases} \|D_x^2 u\|_{L^k(B_R)}^k \leq CF(2R), \\ \|D_x^2 v\|_{L^m(B_R)}^m \leq CF(2R). \end{cases} \tag{3.31}$$

Proof. Some frequently used facts include, $q\alpha = \beta + 2$, $p\beta = \alpha + 2$ and hence $n - kp\beta < 0$ (due to (3.4)) and therefore $l < k$ (since $n - lp\beta \geq 0$).

Estimates (3.28) directly follow from (3.17) in Lemma 3.10.

For the first estimate of (3.29), after applying Lemma 3.5, the mixed type $W^{2,p}$-estimate[1] , we get

$$\|D_x^2 u\|_{L^{l+\varepsilon}(B_R)}^{l+\varepsilon} \leq C \left(\|\Delta u\|_{L^{l+\varepsilon}(B_{2R})}^{l+\varepsilon} + R^{n-(l+\varepsilon)(n+2)} \|u\|_{L^1(B_{2R})}^{l+\varepsilon} \right).$$

Then we use the assumed estimate (3.8) and Lemma 3.10 to get

$$\begin{aligned}
\|D_x^2 u\|_{L^{l+\varepsilon}(B_R)}^{l+\varepsilon} &\leq C \left(\int_{B_{2R}} v^{p(l+\varepsilon)} dx + R^{n-(l+\varepsilon)(n+2)} R^{(l+\varepsilon)(n-\alpha)} \right) \\
&\leq C \left(R^{n-pl\beta} + R^{n-(l+\varepsilon)(2+\alpha)} \right) \\
&\leq C R^{n-pl\beta},
\end{aligned}$$

where the last inequality is due to $\alpha + 2 = p\beta$. For the second estimate of (3.29),

$$\begin{aligned}
\|D_x^2 v\|_{L^{1+\varepsilon}(B_R)}^{1+\varepsilon} &\leq C \left(\|\Delta v\|_{L^{1+\varepsilon}(B_{2R})}^{1+\varepsilon} + R^{n-(1+\varepsilon)(n+2)} \|v\|_{L^1(B_{2R})}^{1+\varepsilon} \right) \\
&\leq C \left(\int_{B_{2R}} u^{q(1+\varepsilon)} dx + R^{n-(1+\varepsilon)(n+2)} R^{(1+\varepsilon)(n-\beta)} \right) \\
&\leq C \left(R^{n-q\alpha} + R^{n-(1+\varepsilon)(\beta+2)} \right) \\
&\leq C R^{n-q\alpha}.
\end{aligned}$$

For the first estimate of (3.30), by Lemma 3.6,

$$\begin{aligned}
\|D_x u\|_{L^1(B_R)} &\leq C \left(R^{n(1-\frac{1}{1+\varepsilon})+1} \|D_x^2 u\|_{L^{1+\varepsilon}(B_R)} + R^{-1} \|u\|_{L^1(B_R)} \right) \\
&\leq C \left(R^{n(1-\frac{1}{1+\varepsilon})+1} R^{\frac{n-p\beta}{1+\varepsilon}} + R^{-1} R^{n-\alpha} \right) \\
&\leq C R^{n+1-\frac{\alpha+2}{1+\varepsilon}},
\end{aligned}$$

The second estimate in (3.30) can be obtained by a similar process. Last, the fact that $n-(p+1)\beta <$

[1] Notice that with the standard $W^{2,p}$-estimate, we end up with a term of $\|u\|_{l+\epsilon}$ which cannot be estimated by any energy bound.

0 gives

$$\begin{aligned}
\|D_x^2 u\|_{L^k(B_R)}^k &\le C\left(\int_{B_{2R}} |\Delta u|^k dx + R^{n-k(n+2)}(\int_{B_{2R}} |u| dx)^k\right)\\
&\le C\left(\int_{B_{2R}} v^{p+1} dx + R^{n-k(n+2)} R^{k(n-\alpha)}\right)\\
&\le C\left(F(2R) + R^{n-(p+1)\beta}\right)\\
&\le CF(2R),
\end{aligned}$$

and hence the first estimate in (3.31) follows, and similarly we get the second estimate. □

Proof of Proposition 3.13: By Lemma 3.4, $\exists \tilde{R} \in [R, 2R]$, Proposition 3.13 follows from Proposition 3.14 immediately. □

Lemma 3.15. *There exists $M > 0$ such that $\exists \{R_j\} \to +\infty$,*

$$F(4R_j) \le MF(R_j).$$

Proof. Suppose not, then for any $M > 0$ and any $\{R_j\} \to +\infty$, we have

$$F(4R_j) > MF(R_j).$$

Take $M > 5^n$ and $R_{j+1} = 4R_j$ with $R_0 > 1$. Therefore,

$$F(R_j) > M^j F(R_0),$$

which leads to a contradiction with $F(R_j) \le CR_j^n \le C(4^j R_0)^n$. □

3.3 Proof of Liouville Theorem

From now on, without loss of generality, we may assume $p \ge q$. By Lemma 3.8, $\|v\|_{L^{p+1}(B_R)}^{p+1} \le \|u\|_{L^{q+1}(B_R)}^{q+1}$. By the Rellich-Pohožaev type identity in Lemma 3.7, we can denote

$$F(R) := \int_{B_R} u^{q+1} \le CG_1(R) + CG_2(R), \tag{3.32}$$

where

$$G_1(R) = R^n \int_{\mathbb{S}^{n-1}} u^{q+1}(R) d\sigma, \tag{3.33}$$

$$G_2(R) = R^n \int_{\mathbb{S}^{n-1}} (|D_x u(R)| + R^{-1} u(R))(|D_x v(R)| + R^{-1} v(R)) d\sigma. \tag{3.34}$$

Heuristically, we are aiming for estimate as

$$G_i(R) \leq C R^{-a_i} F^{1-\delta_i}(4R), \quad i = 1, 2. \tag{3.35}$$

Then by Lemma 3.15 there exists a sequence $\{R_j\} \to +\infty$ such that

$$G_i(R_j) \leq C R^{-a_i} F^{1-\delta_i}(R_j), \quad i = 1, 2.$$

Suppose there are infinitely many R_j's such that $G_1(R_j) \geq G_2(R_j)$, then take that subsequence of $\{R_j\}$ and still denote as $\{R_j\}$. We do the same if there are infinitely many R_j's such that $G_1(R_j) \leq G_2(R_j)$. So, there are only two cases we shall deal with: there exists a sequence $\{R_j\} \to +\infty$ such that

Case 1: $G_1(R_j) \geq G_2(R_j)$. Then we prove $a_1 > 0$, $\delta_1 > 0$. So, $F^{\delta_1}(R_j) \leq C R_j^{-a_1} \to 0$,

Case 2: $G_1(R_j) \leq G_2(R_j)$. Then we prove $a_2 > 0$, $\delta_2 > 0$. So, $F^{\delta_2}(R_j) \leq C R_j^{-a_2} \to 0$.

Then we conclude that $F(R) \equiv 0$.

Surprisingly, for both cases $a_i \approx (\alpha + \beta + 2 - n)\delta_i$. Indeed, we have

Theorem 3.16. *For $F(R)$ defined as* (3.32) *and α, β defined as* (3.3)*, there exists a sequence $\{R_j\} \to +\infty$ such that*

$$F(R_j) \leq C R_j^{-(\alpha+\beta+2-n)+o(1)}.$$

Hence, Theorem 3.1 is a direct consequence of Theorem 3.16, and we only need to prove Theorem 3.16 for case 1 and 2.

3.3.1 Case 1: Estimate for $G_1(R)$

According to previous discussion in the introduction, we assume that

$$p \geq q > 0, \quad pq > 1, \quad \beta \leq \alpha < n - 2, \quad n \geq 3,$$

hence in particular

$$p > \frac{n}{n-2}. \tag{3.36}$$

Remark 3.17. *For systems* (3.10) *with double bounded coefficients,* (3.36) *is a necessary condition for existence of positive solution, see [35].*

In addition to our assumption that $n - s\beta < 1$, since we have energy inequalities (3.18), we can assume $s \geq p$. Also, if $n - s\beta < 0$, (3.8) implies $v \equiv 0$ and hence $u \equiv 0$. So, we assume $n - s\beta \geq 0$. Let $l = \frac{s}{p}$, then

$$l \geq 1, \text{ and } \frac{n-1}{p\beta} < l \leq \frac{n}{p\beta}. \tag{3.37}$$

It is worthy to point out that, what the proof of Lane-Emden conjecture really needs is a "breakthrough" on the energy estimate (3.18). s in (3.8) needs not be very large but enough to satisfy $n - s\beta < 1$. In other words, s can be very close to $\frac{n-1}{\beta}$, and it is sufficient to prove Theorem 3.1.

The strategies of attacking G_1 and G_2 are the same. Basically, first by Hölder inequality we split the quantities on sphere $\mathbb{S}^{n-1}$ into two parts. One has a lower (than original) index after embedding, and the other has a higher one. Then we estimate the latter part by $F(R)$, and thus we get a feedback estimate as (3.35).

Let

$$k = \frac{p+1}{p}.$$

Since $p\beta = \alpha + 2$, $n - (p+1)\beta = n - 2 - (\alpha + \beta) < 0$ by (3.4). Thus, $n - kp\beta < 0$ as $n - lp\beta \geq 0$, it follows that $l < k$.

Subcase 1.1 $\frac{1}{l} \geq \frac{2}{n-1} + \frac{1}{q+1}$.

Note that in this subcase, since $l \geq 1$, we must have $n \geq 4$ (i.e., $n \neq 3$). By (3.36) we see that $k = 1 + \frac{1}{p} < 1 + \frac{n-2}{n} = \frac{2}{n}(n-1) \leq \frac{n-1}{2}$. Take

$$\frac{1}{\mu} = \frac{1}{k} - \frac{2}{n-1}.$$

So, $W^{2,k}(\mathbb{S}^{n-1}) \hookrightarrow L^{\mu}(\mathbb{S}^{n-1})$.

Take

$$\frac{1}{\lambda} = \frac{1}{l} - \frac{2}{n-1} \geq \frac{1}{q+1}.$$

Then $W^{2,l+\varepsilon}(\mathbb{S}^{n-1}) \hookrightarrow L^{\lambda}(\mathbb{S}^{n-1})$.

Direct verification shows that $\frac{1}{\mu} = \frac{1}{k} - \frac{2}{n-1} \leq \frac{1}{q+1}$ which is due to (3.2), so we have

$$\frac{1}{\mu} \leq \frac{1}{q+1} \leq \frac{1}{\lambda}.$$

Then by Hölder inequality and Sobolev embedding (3.13), we have (with notation (3.14))

$$\|u\|_{q+1}(R) \leq \|u\|_{\lambda}^{\theta}\|u\|_{\mu}^{1-\theta}(R) \tag{3.38}$$

$$\leq C(R^2\|D_x^2u\|_{l+\varepsilon}(R) + \|u\|_1(R))^{\theta}(R^2\|D_x^2u\|_k(R) + \|u\|_1(R))^{1-\theta}, \tag{3.39}$$

where $\theta \in [0,1]$ and

$$\frac{1}{q+1} = \frac{\theta}{\lambda} + \frac{1-\theta}{\mu}. \tag{3.40}$$

Since l can be 1 (then $W^{2,p}$-estimate fails for $\|D_x^2u\|_1(R)$), we add an ε to l for later use of $W^{2,p}$-estimate. ε can be any real positive number and later will be chosen sufficiently small.

To get desired estimate, we have requirements in form of inequalities involving parameters, such as $\alpha, \beta, \varepsilon$ and etc. To verify those requirements very often we just verify strict inequalities with $\varepsilon = 0$ because such inequalities continuously depend on ε.

So, by (3.33) and (3.39)

$$G_1(R) \leq CR^n\left((R^2\|D_x^2u\|_{l+\varepsilon}(R) + \|u\|_1(R))^{\theta}(R^2\|D_x^2u\|_k(R) + \|u\|_1(R))^{1-\theta}\right)^{q+1}. \tag{3.41}$$

Then by Proposition 3.13, there exists $\tilde{R} \in [R, 2R]$ such that

$$\begin{aligned} G_1(\tilde{R}) &\leq CR^n \left((R^2 R^{-\frac{lp\beta}{l+\varepsilon}} + R^{-2-\alpha})^\theta (R^2 (R^{-n} F(4R))^{\frac{1}{k}} + R^{-\alpha})^{1-\theta} \right)^{q+1} \\ &\leq CR^n \left(R^{2-\frac{lp\beta\theta}{l+\varepsilon} - \frac{n(1-\theta)}{k}} F^{\frac{1-\theta}{k}}(4R) \right)^{q+1} \\ &\leq R^{-a_1} F^{1-\delta_1}(4R), \end{aligned}$$

where the last inequality is due to $R^{-\frac{n}{k}} > R^{-\alpha-2}$ and

$$a_1 = a_1^\varepsilon = (q+1)(\frac{lp\beta\theta}{l+\varepsilon} + \frac{np(1-\theta)}{p+1} - 2 - \frac{n}{1+q}), \tag{3.42}$$

$$1 - \delta_1 = \frac{(1-\theta)p(q+1)}{p+1}. \tag{3.43}$$

Since for sufficiently small ε, $a_1^\varepsilon > 0$ and $\delta_1 > 0$ are just a perturbation of

$$a_1^0 > 0, \text{ and } \delta_1 > 0, \tag{3.44}$$

we only need to prove (3.44) is true.

Since $lp = s$, $p\beta = \alpha + 2$ and $q\alpha = \beta + 2$,

$$\begin{aligned} a_1^0 &= p\beta\theta(q+1) + (1-\delta_1)n - 2(q+1) - n \\ &= (q+1)(p\beta\theta - 2) - \delta_1 n \\ &= (q+1)(p\beta(\theta-1) + p\beta - 2) - \delta_1 n \\ &= (q+1)(-\alpha(1-\delta_1) + \alpha) - \delta_1 n \\ &= \delta_1((q+1)\alpha - n) \\ &= (\alpha + \beta + 2 - n)\delta_1. \end{aligned}$$

So we just need to prove $\delta_1 > 0$. By (3.42) and (3.40) we have

$$\begin{aligned}
&(1-\theta)p(q+1) < p+1 \\
\Leftrightarrow &\frac{\frac{1}{l} - \frac{2}{n-1} - \frac{1}{1+q}}{\frac{1}{l} - \frac{1}{k}}(q+1) < k \\
\Leftrightarrow &(\frac{1}{l} - \frac{2}{n-1})(q+1) - 1 < \frac{k}{l} - 1 \\
\Leftrightarrow &\frac{1}{l}(q+1-1-\frac{1}{p}) < \frac{2}{n-1}(q+1) \\
\Leftrightarrow &\frac{pq-1}{s} < \frac{2(q+1)}{n-1} \\
\Leftrightarrow &n-1 < s\beta,
\end{aligned}$$

and the last inequality is included in our assumption. So, we have proved subcase 1.1.

Subcase 1.2 $\frac{1}{l} < \frac{2}{n-1} + \frac{1}{q+1}$.

As discussed in the beginning of subcase 1.1, $k < \frac{n-1}{2}$ if $n > 3$. Since $l < k$, $\frac{1}{l} > \frac{2}{n-1}$ for $n > 3$. When $n = 3$, since $l \geq 1$ by (3.37), $\frac{1}{l} \leq 1 = \frac{2}{n-1}$.

Therefore, for $n > 3$, take

$$\frac{1}{\lambda} = \frac{1}{l} - \frac{2}{n-1} < \frac{1}{q+1},$$

and for $n = 3$, take

$$\lambda = \infty,$$

so we have

$$W^{2,l+\varepsilon}(\mathbb{S}^{n-1}) \hookrightarrow L^{\lambda}(\mathbb{S}^{n-1}),\ n \geq 3.$$

So,

$$\|u\|_{q+1}(R) \leq C\|u\|_{\lambda}(R) \leq C(R^2\|D_x^2 u\|_{l+\varepsilon}(R) + \|u\|_1(R)).$$

Therefore, by Proposition 3.13 there exists $\tilde{R} \in [R, 2R]$ such that

$$G_1(\tilde{R}) \le CR^n(R^2\|D_x^2 u\|_{l+\varepsilon}(R) + \|u\|_1(R))^{q+1} \tag{3.45}$$
$$\le CR^n(R^2 R^{-\frac{lp\beta}{l+\varepsilon}} + R^{-\alpha})^{q+1} \tag{3.46}$$
$$\le CR^{n+(2-\frac{lp\beta}{l+\varepsilon})(q+1)}. \tag{3.47}$$

So,

$$\begin{aligned} F(\tilde{R}) &\le CR^{n+(2-\frac{lp\beta}{l+\varepsilon})(q+1)} \\ &\le CR^{n+(2-p\beta)(q+1)+\frac{\varepsilon p\beta}{l+\varepsilon}(q+1)} \\ &\le CR^{-(\alpha+\beta+2-n)+\frac{\varepsilon p\beta}{l+\varepsilon}(q+1)}. \end{aligned}$$

Since ε can be arbitrarily small,

$$F(\tilde{R}) \le CR^{-(\alpha+\beta+2-n)+o(1)}.$$

Thus, we have proved Case 1.

3.3.2 Case 2: Estimate for $G_2(R)$.

Let

$$m = \frac{q+1}{q}.$$

Subcase 2.1 $m < n-1$.

With $z, z' > 0$ and $\frac{1}{z} + \frac{1}{z'} = 1$, (3.34) becomes,

$$\begin{aligned} G_2(R) &\le CR^n \||D_x u| + R^{-1}u\|_z \||D_x v| + R^{-1}v\|_{z'}(R) \\ &\le CR^n(\|D_x u\|_z(R) + R^{-1}\|u\|_z(R))(\|D_x v\|_{z'}(R) + R^{-1}\|v\|_z(R)) \\ &\le CR^n(\|D_x u\|_z(R) + R^{-1}\|u\|_1(R))(\|D_x v\|_{z'}(R) + R^{-1}\|v\|_1(R)), \end{aligned} \tag{3.48}$$

where the last inequality is due to

$$\|u\|_z(R) \le C(R\|D_x u\|_z(R) + \|u\|_1(R)), \text{ and } \|v\|_{z'}(R) \le C(R\|D_x v\|_{z'}(R) + \|v\|_1(R)).$$

Assume there exists z (we shall check the existence later) such that by Sobolev Embedding (3.13),

$$\begin{aligned} \|D_x u\|_z(R) &\leq \|D_x u\|_{\rho_1}^{\tau_1}(R)\|D_x u\|_{\gamma_1}^{1-\tau_1}(R) \\ &\leq C(R\|D_x^2 u\|_{l+\varepsilon}(R) + \|D_x u\|_1(R))^{\tau_1}(R\|D_x^2 u\|_k(R) + \|D_x u\|_1(R))^{1-\tau_1}, \end{aligned} \tag{3.49}$$

$$\begin{aligned} \|D_x v\|_{z'}(R) &\leq \|D_x v\|_{\rho_2}^{\tau_2}(R)\|D_x v\|_{\gamma_2}^{1-\tau_2}(R) \\ &\leq C(R\|D_x^2 v\|_{1+\varepsilon}(R) + \|D_x v\|_1(R))^{\tau_2}(R\|D_x^2 v\|_m(R) + \|D_x v\|_1(R))^{1-\tau_2}, \end{aligned} \tag{3.50}$$

where $\tau_1, \tau_2 \in [0,1]$ and

$$\frac{1}{z} = \frac{\tau_1}{\rho_1} + \frac{1-\tau_1}{\gamma_1}, \tag{3.51}$$

$$\frac{1}{z'} = \frac{\tau_2}{\rho_2} + \frac{1-\tau_2}{\gamma_2}, \tag{3.52}$$

and since $l < k \leq m < n-1$, define

$$\frac{1}{\rho_1} = \frac{1}{l} - \frac{1}{n-1}, \quad \frac{1}{\gamma_1} = \frac{1}{k} - \frac{1}{n-1}, \tag{3.53}$$

$$\frac{1}{\rho_2} = 1 - \frac{1}{n-1}, \quad \frac{1}{\gamma_2} = \frac{1}{m} - \frac{1}{n-1}. \tag{3.54}$$

So, we have

$$W^{1,l+\varepsilon}(\mathbb{S}^{n-1}) \hookrightarrow L^{\rho_1}(\mathbb{S}^{n-1}),\ W^{1,k}(\mathbb{S}^{n-1}) \hookrightarrow L^{\gamma_1}(\mathbb{S}^{n-1}),$$

$$W^{1,1+\varepsilon}(\mathbb{S}^{n-1}) \hookrightarrow L^{\rho_2}(\mathbb{S}^{n-1}),\ W^{1,m}(\mathbb{S}^{n-1}) \hookrightarrow L^{\gamma_2}(\mathbb{S}^{n-1}).$$

To verify the existence of such z, by (3.51)-(3.54), we expect that

$$\max\left\{\frac{1}{k} - \frac{1}{n-1}, \frac{1}{n-1}\right\} \leq \frac{1}{z} \leq \min\left\{\frac{1}{l} - \frac{1}{n-1}, \frac{1}{q+1} + \frac{1}{n-1}\right\}. \tag{3.55}$$

Thus, we need to verify, (i) $\frac{1}{k} - \frac{1}{n-1} \leq \frac{1}{l} - \frac{1}{n-1}$, (ii) $\frac{1}{n-1} \leq \frac{1}{l} - \frac{1}{n-1}$, (iii) $\frac{1}{n-1} \leq \frac{1}{q+1} + \frac{1}{n-1}$, (iv) $\frac{1}{k} - \frac{1}{n-1} \leq \frac{1}{q+1} + \frac{1}{n-1}$.

Since $l < k$, (i) is true. (ii) holds for $n > 3$ as discussed at the beginning of subcase 1.2 $\frac{1}{l} > \frac{1}{k} > \frac{2}{n-1}$; for $n = 3$, take $s = p$ and then $l = 1$, so (ii) still holds. (iii) is obvious. (iv) is equivalent to $\frac{1}{p+1} + \frac{1}{q+1} \geq 1 - \frac{2}{n-1}$, which is guaranteed by (3.2).

So, we put (3.49) and (3.50) in (3.48) and get

$$
\begin{aligned}
G_2(R) \leq\ & CR^{n+2}(\|D_x^2 u\|_{l+\varepsilon}(R) + R^{-1}\|D_x u\|_1(R) + R^{-2}\|u\|_1(R))^{\tau_1} \\
& \times(\|D_x^2 u\|_k(R) + R^{-1}\|D_x u\|_1(R) + R^{-2}\|u\|_1(R))^{1-\tau_1} \\
& \times(\|D_x^2 v\|_{1+\varepsilon}(R) + R^{-1}\|D_x v\|_1(R) + R^{-2}\|v\|_1(R))^{\tau_2} \\
& \times(\|D_x^2 v\|_m(R) + R^{-1}\|D_x v\|_1(R) + R^{-2}\|v\|_1(R))^{1-\tau_2}.
\end{aligned} \tag{3.56}
$$

Then by Proposition 3.13, there exists $\tilde{R} \in [R, 2R]$ such that

$$
\begin{aligned}
G_2(\tilde{R}) &\leq CR^{n+2} R^{\frac{-p\beta\tau_1}{1+\varepsilon/l}} \left((R^{-n}F(4R))^{\frac{1}{k}} + R^{-\frac{\alpha+2}{1+\varepsilon}} + R^{-\alpha-2}\right)^{1-\tau_1} \\
&\quad \times R^{-\frac{q\alpha\tau_2}{1+\varepsilon/l}} \left((R^{-n}F(4R))^{\frac{1}{m}} + R^{-\frac{\beta+2}{1+\varepsilon}} + R^{-\beta-2}\right)^{1-\tau_2} \\
&\leq CR^{-a_2^\varepsilon} F^{1-\delta_2}(4R),
\end{aligned}
$$

where the last inequality is due to $R^{-\frac{n}{k}} > R^{-\alpha-2}$ and $R^{-\frac{n}{m}} > R^{-\beta-2}$. Meanwhile,

$$
a_2 = a_2^\varepsilon = -n - 2 + \frac{p\beta\tau_1}{1+\varepsilon/l} + \frac{q\alpha\tau_2}{1+\varepsilon/l} + n\frac{1-\tau_1}{k} + n\frac{1-\tau_2}{m}, \tag{3.57}
$$

$$
1 - \delta_2 = \frac{1-\tau_1}{k} + \frac{1-\tau_2}{m}. \tag{3.58}
$$

Similar to subcase 1.1, we only need to prove

$$
a_2^0 > 0, \ \ \delta_2 > 0.
$$

Surprisingly, similar to $a_1 \approx (\alpha+\beta+2-n)\delta_1$, we have $a_2 \approx (\alpha+\beta+2-n)\delta_2$ since we can prove $a_2^0 = (\alpha+\beta+2-n)\delta_2$. Indeed,

$$
\begin{aligned}
a_2^0 &= -n - 2 + p\beta(\tau_1 - 1) + p\beta + q\alpha(\tau_2 - 1) + q\alpha + n(1-\delta_2) \\
&= -n - 2 - p\beta k\frac{1-\tau_1}{k} - q\alpha m\frac{1-\tau_2}{m} + \alpha + \beta + 4 + n(1-\delta_2) \\
&= \alpha + \beta + 2 - n - (\alpha+\beta+2)(1-\delta_2) + n(1-\delta_2) \\
&= (\alpha+\beta+2-n)\delta_2,
\end{aligned}
$$

where the third equality above is due to $p\beta k = (p+1)\beta = (q+1)\alpha = q\alpha m$ and $(p+1)\beta = \alpha+\beta+2$. So, we only need to prove $\delta_2 > 0$ or equivalently by (3.51), (3.52) and (3.58),

$$
(m - \frac{k}{l})\frac{1}{z} + (\frac{k}{n-1} + (m-1)(k-1))\frac{1}{l} + \frac{m-2}{n-1} - (m-1) > 0, \tag{3.59}
$$

To achieve this, we take the upper bound of $\frac{1}{z}$ in (3.55) and see whether (3.59) holds.

Case 2.1.1 If $\frac{1}{l} - \frac{1}{n-1} \geq \frac{1}{q+1} + \frac{1}{n-1}$, then let $\frac{1}{z} = \frac{1}{q+1} + \frac{1}{n-1}$, and (3.59) becomes,

$$\begin{aligned}
&(\frac{1}{pq} - \frac{p+1}{p(q+1)})\frac{1}{l} + \frac{q+1}{q}(\frac{1}{n-1} + \frac{1}{q+1}) + \frac{1-q}{(n-1)q} - \frac{1}{q} > 0 \\
&\Leftrightarrow (\frac{1}{pq} - \frac{p+1}{p(q+1)})\frac{1}{l} + \frac{2}{q(n-1)} > 0 \\
&\Leftrightarrow -\frac{2}{\beta s} + \frac{2}{n-1} > 0 \\
&\Leftrightarrow s\beta > n-1.
\end{aligned}$$

Case 2.1.2 If $\frac{1}{l} - \frac{1}{n-1} < \frac{1}{q+1} + \frac{1}{n-1}$, then let $\frac{1}{z} = \frac{1}{l} - \frac{1}{n-1}$, and (3.59) becomes,

$$\begin{aligned}
&(m - \frac{k}{l})(\frac{1}{l} - \frac{1}{n-1}) + (\frac{k}{n-1} + (m-1)(k-1))\frac{1}{l} + \frac{m-2}{n-1} - (m-1) > 0 \\
&\Leftrightarrow -\frac{k}{l^2} + (m + \frac{k}{n-1} + \frac{k}{n-1} + (m-1)(k-1))\frac{1}{l} + \frac{m-2}{n-1} - (m-1) > 0 \\
&\Leftrightarrow -\frac{k}{l^2} + (\frac{p}{p+1} + \frac{1}{q} + \frac{2}{n-1})\frac{k}{l} > \frac{2}{n-1} + \frac{1}{q} \\
&\Leftrightarrow -\frac{k}{l^2} + (1 + k(\frac{2}{n-1} + \frac{1}{q}))\frac{1}{l} - (\frac{2}{n-1} + \frac{1}{q}) > 0 \\
&\Leftrightarrow (\frac{k}{l} - 1)(\frac{1}{l} - (\frac{2}{n-1} + \frac{1}{q})) < 0 \\
&\Leftrightarrow \frac{1}{k} < \frac{1}{l} < \frac{2}{n-1} + \frac{1}{q}.
\end{aligned}$$

Notice that $\frac{1}{l} < \frac{2}{n-1} + \frac{1}{q}$ holds under the assumption of case 2.1.2, and $\frac{1}{k} < \frac{1}{l}$ since $l < k$. In all, (3.59) always holds under our assumption $n - s\beta < 1$.

Subcase 2.2 $m \geq n-1$.

First, we have for any $\gamma \in [1, \infty)$,

$$W^{1,m}(\mathbb{S}^{n-1}) \hookrightarrow L^{\gamma}(\mathbb{S}^{n-1}).$$

Then we claim $\frac{1}{l} > \frac{1}{n-1}$. Suppose $\frac{1}{l} \leq \frac{1}{n-1}$, then $k > l \geq n-1$, hence $p \leq \frac{1}{n-2}$, which is not possible due to (3.36). Take $\frac{1}{z} = \frac{1}{l} - \frac{1}{n-1}$ then

$$W^{1,l+\varepsilon}(\mathbb{S}^{n-1}) \hookrightarrow L^{z}(\mathbb{S}^{n-1}).$$

Therefore, by Sobolev embedding and (3.48)

$$
\begin{aligned}
G_2(R) &\le CR^n(\|D_x u\|_z(R) + R^{-1}\|u\|_1(R))(\|D_x v\|_{z'}(R) + R^{-1}\|v\|_1(R)) \\
&\le CR^{n+2}(\|D_x^2 u\|_{l+\varepsilon} + R^{-1}\|D_x u\|_1 + R^{-2}\|u\|_1)(\|D_x^2 v\|_m + R^{-1}\|D_x v\|_1 + R^{-2}\|v\|_1).
\end{aligned}
$$

Similarly to previous work, there exists a $\tilde{R} \in [R, 2R]$ such that

$$
\begin{aligned}
G_2(\tilde{R}) &\le CR^{n+2}R^{\frac{-p\beta}{1+\varepsilon/l}}\left((R^{-n}F(4R))^{\frac{1}{m}} + R^{-\frac{\beta+2}{1+\varepsilon}} + R^{-\beta-2}\right) \\
&\le CR^{-a_2^{\varepsilon}}F^{1-\delta_2}(4R),
\end{aligned}
$$

where

$$
a_2 = a_2^{\varepsilon} = -n - 2 + \frac{p\beta}{1+\varepsilon/l} + \frac{n}{m}, \tag{3.60}
$$

$$
1 - \delta_2 = \frac{1}{m}. \tag{3.61}
$$

Direct verification shows that

$$
a_2^0 = (\alpha + \beta + 2 - n)\delta_2,
$$

and obviously $\delta_2 > 0$ so $\alpha_2^0 > 0$.

Thus, we have proved Case 2.

Chapter 4

Discrete HLS system

In this chapter, we consider discrete version of HLS system. Actually, there are two ways of considering discrete HLS system. One way is to discretise critical and supercritical HLS inequality and consider its optimization problem. As in continuous case, discrete HLS system is the Euler-Lagrange equation of the optimization problem of the discrete HLS inequality. People find that the discrete system shares some similar properties with continuous system, e.g. symmetry and decaying property of solution; however, whether the best constant corresponding to discrete HLS inequality (4.4) in critical case can be attained by its optimizer is still open [27].

Another way of considering discrete HLS system is to push the HLS inequality into an extreme case: $\mu = n$, where original HLS inequality fails. However, discrete HLS inequality can survive if we limit the inequality on a finite domain. Here we study such a finite form of HLS inequality with $\mu = n$ and $p = q$. We give estimate for the best constant with logarithm correction and study its corresponding optimizer. For the optimizer, we prove the uniqueness and a symmetry property. Also, by using a discrete version of maximum principle, we prove certain monotonicity of this optimizer.

This chapter contains the work in [13].

4.1 Introduction

Recall the Hardy-Littlewood-Sobolev (HLS) inequality [68],

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{|x-y|^{\mu}}\,dx\,dy \le C_{p,\mu,n}\|f\|_p\|g\|_q \tag{4.1}$$

for any $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ provided that

$$0 < \mu < n, 1 < p, q < \infty \text{ with } \frac{1}{p} + \frac{1}{q} + \frac{\mu}{n} = 2.$$

$C_{p,\mu,n}$ is the best constant for (4.1), and proved by Lieb [40] that, such $C_{p,\mu,n}$ and corresponding maximizing pair (f, g) exists. In particular, Lieb also gave the explicit f abd $C_{p,\mu,n}$ in the case $p = q$. The method Lieb used was to examine the Euler-Lagrange equation of the maximizing pair (f, g) with stereographic projection to exploit the symmetry of f. This idea is inherited in [38] and here to find the sharp estimate of best constant of a finite form of HLS in a critical case: $p = q = 2$, and hence $\mu = n$.

Following the idea that the maximizer of HLS satisfies corresponding E-L equations (see chapter 1), the study of the HLS inequality and weighted inequality later generalized by Stein and Weiss [67] is naturally related to the studies of various of integral equations. For recent results, see [8, 11, 9] and a brief summary can be found in [7]. These works have studied regularity and radial symmetry of solutions of such integral systems, and introduced a method of moving plane in an integral form which is proved to be a powerful tool. In [26], the result of integral system corresponding to HLS (4.1) is improved to all cases, i.e. the condition $p, q \geq 1$ is removed. Here we do not use the method of moving plane directly, but borrowing its idea, we use a maximum principle to deal with a discrete problem and prove the symmetry of the solution.

First, let's have a look at the discrete and 1-dimensional version of HLS inequality (4.1), the Hardy-Littlewood-Pólya (HLP) inequality [22]: if $a \in l^p(\mathbb{Z})$ and $b \in l^q(\mathbb{Z})$ and

$$0 < \mu < 1, 1 < p, q < \infty \text{ with } \frac{1}{p} + \frac{1}{q} + \mu = 2,$$

then

$$\sum_{r \neq s} \frac{a_r b_s}{|r - s|^\mu} \leq C \|a\|_p \|b\|_q \tag{4.2}$$

where $r, s \in \mathbb{Z}$ and the constant C depends on p and q only.

For this HLP inequality (4.2), let's consider the critical case: $p = q = 2$ and $\mu = 2 - \frac{1}{p} - \frac{1}{q} = 1$, for which the original HLP fails, but we can compromise and get a finite form of HLP. In [38], the inequality is extended to the critical case as: If $a, b \in l^p(\mathbb{Z})$, then

$$\sum_{r \neq s, 1 \leq r,s \leq N} \frac{a_r b_s}{|r-s|} \leq \lambda_N \|a\|_2 \|b\|_2. \tag{4.3}$$

where λ_N is the best constant for (4.3), and $\lambda_N = 2\ln N + O(1)$.

Remark 4.1. *One of the reasons that we consider discrete version of HLS instead of the original inequality is, when $\mu = 1$ the integrand on the left side of HLS* (4.1) *is not always integrable on a finite domain for L^p functions. So we can extend* (4.2) *to* (4.3)*, but we cannot extend HLS inequality* (4.1) *to the critical case unless we discretise it.*

Consider the discrete HLS, if $a, b \in l^p(\mathbb{Z}^n)$, and

$$0 < \mu < n, 1 < p, q < \infty \text{ with } \frac{1}{p} + \frac{1}{q} + \frac{\mu}{n} = 2,$$

then

$$\sum_{r \neq s} \frac{a_r b_s}{|r-s|^\mu} \leq C \|a\|_p \|b\|_q \tag{4.4}$$

where $r, s \in \mathbb{Z}^n$ and the constant C depends on p and q only. We can extend (4.4) to a finite form in the corresponding critical case: $p = q = 2$ and $\mu = n$, in the following way:

Theorem 4.2. *If $r, s \in \mathbb{Z}^n$ and $1 \leq r_i, s_i \leq N$ where r_i, s_i are integers and $1 \leq i \leq n$, then $a_r, b_s \in \mathbb{R}^L$, where $L = N^n$. let*

$$\lambda_N = \max_{\|a\|_2 = \|b\|_2 = 1} \sum_{r \neq s} \frac{a_r b_s}{|r-s|^n} \tag{4.5}$$

So, we have an extension of HLS inequality

$$\sum_{r \neq s} \frac{a_r b_s}{|r-s|^n} \leq \lambda_N \|a_r\|_2 \|b_s\|_2 \tag{4.6}$$

where the two statements below holds

(i) $|S^{n-1}| \ln N - o(\ln N) < \lambda_N < |S^{n-1}| \ln N + o(\ln N)$.

(ii) $\exists! \overline{a^N} = \overline{b^N}$ *and* $\|\overline{a^N}\|_2 = 1$ *such that the equality in* (4.6) *holds, and* $\overline{a^N} \in \mathbb{R}^L_+$ *where* $L = N^n$.

Let's call the triplet $(\overline{a^N} = \overline{b^N}, \lambda_N)$ the optimizer of (4.6) since it is unique, and there are some properties of the optimizer. First, as a consequence of the uniqueness, we have symmetry property of the optimizer in the following sense,

Theorem 4.3. *Let $(\overline{a^N}, \lambda_N)$ be the optimizer. $\Phi : S \to S$ is an isometric map, where $S = \{r \in \mathbb{Z}_+^n | 1 \leq r_i \leq N\}$. Then $\overline{a^N_{\Phi(r)}} = \overline{a^N_r}$.*

Second, the optimizer has certain monotone decaying property. For convenience of writing, let's change the range of r_i from $[1, N]$ to $[-N, N]$, which makes no essential change to the results above, and we have the monotone decaying property for this special case,

Theorem 4.4. *If $(\overline{a^N}, \lambda_N)$ is the optimizer and $r \in \mathbb{Z}^n$, $-N \leq r_i \leq N$ for $1 \leq i \leq n$, then $a \in \mathbb{R}_+^L$, where $L = (2N+1)^n$, and $\overline{a^N}$ has a monotone decaying property from its central element: For $1 \leq i \leq n$,*

$$\begin{cases} \overline{a^N_{(r_i, r')}} \leq \overline{a^N_{(r_i - 1, r')}}, 1 \leq r_i \leq N \\ \overline{a^N_{(r_i, r')}} \geq \overline{a^N_{(r_i - 1, r')}}, -N+1 \leq r_i \leq 0 \end{cases} \tag{4.7}$$

To prove Theorem (4.4), we use the following maximum principle,

Theorem 4.5 (Maximum Principle)**.** *Let $\mathbb{R}_+^L$ be the positive cone in $\mathbb{R}^L$, i.e., if $a \in \mathbb{R}_+^L$ then every element of a is positive. Suppose a linear equation:*

$$u = Au + f \tag{4.8}$$

where $A : \overline{\mathbb{R}_+^L} \to \overline{\mathbb{R}_+^L}$ with $\|A\|_2 < 1$, and $f \in \overline{\mathbb{R}_+^L}$, then $\exists! u$ satisfies (4.8) and $u \in \overline{\mathbb{R}_+^L}$. In other words, $(I - A)^{-1} \in \overline{\mathbb{R}_+^{L \times L}}$.

This Maximum Principle follows directly from standard contracting mapping iteration. It is a discrete version of maximum principle analogous to the usual versions in PDE. To see this, let's look at a typical maximum principle: let $\Omega \subset \mathbb{R}$ be an open bounded and connected domain with smooth boundary $\partial\Omega$. Let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of following equation,

$$\begin{cases} -\Delta u = f \geq 0 \text{ in } \Omega \\ \quad u = 0 \text{ on } \partial\Omega \end{cases} \tag{4.9}$$

Then by maximum principle $u \geq 0$ in Ω. Actually, by strong maximum principle, $u > 0$ or $u \equiv 0$ in Ω.

So, Theorem 4.5 is indeed saying that if $(I - A)u = f \in \overline{\mathbb{R}^L_+}$, then $u \in \overline{\mathbb{R}^L_+}$. Corresponding to strong maximum principle, in Theorem 4.5 if every entry of A is strictly positive, it is easy to see that $u \in \mathbb{R}^L_+$. For more general symmetric linear operators, there is also maximum principle, and one can check [33] for details.

4.2 Best Constant Estimate in High Dimension Space

In this section, we shall present proof of part (i) of Theorem 4.2.

Proof. **Step 1.** $\lambda_N \geq |S^{n-1}| \ln N - o(\ln N)$.

Let $a = b$, and

$$a_r = N^{-\frac{n}{2}}, 1 \leq r_i \leq N, 1 \leq i \leq n \tag{4.10}$$

So, $\|a\|_2 = 1$.

By the definition of λ_N, we have

$$\begin{aligned}
\lambda_N &\geq \sum_{r \neq s} \frac{a_r a_s}{|s-r|^n} = N^{-n} \sum_{r \neq s} \frac{1}{|s-r|^n} \\
&= N^{-n} \{ 2^n \sum_{x_n=1}^{N-1} \cdots \sum_{x_1=1}^{N-1} \frac{(N-x_1)\cdots(N-x_n)}{(x_1^2 + \cdots x_n^2)^{\frac{n}{2}}} - o(N^n \ln N) \} \\
&\geq (\frac{2}{N})^n \int_0^{\frac{\pi}{2}} \cdots \int_0^{\frac{\pi}{2}} \int_1^N \frac{(N - r\cos\phi_1)\cdots}{r^n} r^{n-1} dr d\phi_1 \cdots d\phi_n - o(\ln N) \\
&= (\frac{2}{N})^n |S^{n-1}| 2^{-n} N^n \ln N - o(\ln N) \\
&= |S^{n-1}| \ln N - o(\ln N)
\end{aligned}$$

Step 2. $\lambda_N \leq |S^{n-1}| \ln N + o(\ln N)$

Let $J(a,b) = \sum_{r \neq s} \frac{a_r b_s}{|r-s|^n}$. Hence, $\lambda_N = \max_{\|a\|_2 = \|b\|_2 = 1} J(a,b)$, i.e. we will maximize $J(a,b)$ under the constraints $\|a\|_2 = \|b\|_2 = 1$ (in fact, we use $\frac{1}{2}\|a\|_2^2 = \frac{1}{2}\|b\|_2^2 = \frac{1}{2}$). Therefore, we conduct Euler-Lagrange equations and by compactness: $\exists \|\overline{a^N}\|_2 = \|\overline{b^N}\|_2 = 1$ such that $\lambda_N = J(\overline{a^N}, \overline{b^N})$

and,

$$\begin{cases} \lambda_1 \overline{a_r^N} = \sum_{s \neq r} \dfrac{\overline{b_s^N}}{|s-r|^n} \\ \\ \lambda_2 \overline{b_s^N} = \sum_{r \neq s} \dfrac{\overline{a_r^N}}{|r-s|^n} \end{cases} \tag{4.11}$$

where $r, s \in \mathbb{Z}^n$ and $1 \leq r_i, s_i \leq N$.

For convenience, write (4.11) in matrix form,

$$\begin{cases} \lambda_1 \overline{a^N} = A\overline{b^N} \\ \\ \lambda_2 \overline{b^N} = A\overline{a^N} \end{cases} \tag{4.12}$$

Left multiply the first equation of (4.12) by a^T, the second equation by b^T, and by the fact that A is symmetric and $\|\overline{a^N}\|_2 = \|\overline{b^N}\|_2 = 1$, one sees that

$$\begin{aligned} \lambda_1 = \lambda_1 \|\overline{a^N}\|_2^2 &= \overline{a^N}^T A \overline{b^N} = J(\overline{a^N}, \overline{b^N}) \\ &= \overline{b^N}^T A^T \overline{a^N} = \lambda_2 \|\overline{b^N}\|_2^2 = \lambda_2 \end{aligned}$$

and since $\lambda_N = J(\overline{a^N}, \overline{b^N})$, we have $\lambda_1 = \lambda_2 = \lambda_N$.

Now, let $b_{s_0} = \max\{|\overline{a_r^N}|, |\overline{b_s^N}|\} > 0$, so, $\overline{b_{s_0}^N} \lambda_N = \sum_{r \neq s_0} \frac{\overline{a_r^N}}{|r-s_0|^n}$, which leads to

$$\begin{aligned} \lambda_N &= \sum_{r \neq s_0} \frac{\overline{a_r^N}}{b_{s_0}|r-s_0|^n} \leq \sum_{r \neq s_0} \frac{1}{|r-s_0|^n} \\ &\leq \sum_{r \neq (\frac{N}{2}, \cdots, \frac{N}{2}) = m_0} \frac{1}{|r-m_0|^n} \\ &\leq \int_\Sigma \int_{\frac{1}{2}}^{\frac{\sqrt{2}N}{2}} \frac{1}{r^n} r^{n-1} dr d\sigma \\ &\leq |S^{n-1}|(\ln(\sqrt{2}N)) = |S^{n-1}|(\ln N + \frac{1}{2}\ln 2) \\ &= |S^{n-1}| \ln N + o(\ln N) \end{aligned}$$

Part (ii) will be shown later in section 3. □

Lemma 4.6. *If (a, b, λ_N) satisfies $\|a\|_2 = \|b\|_2 = 1$ and makes the equality of (4.6) hold, then $a, b \in \overline{\mathbb{R}_+^L} \cup \overline{\mathbb{R}_-^L}$.*

Notice that if there is a sign change among the elements of a and b, (a, b) must not be an optimizer since $|\sum a_i b_i| < \sum |a_i||b_i|$. So the Lemma holds, and it means that we can assume the triplet $(\overline{a^N}, \overline{b^N}, \lambda_N)$ above to satisfy $\overline{a^N}, \overline{b^N} \in \overline{\mathbb{R}^L_+}$.

Now, let's introduce a notation,

Definition 4.7. (a, b, λ_N) *such that*

- $\|a\|_2 = \|b\|_2 = 1$
- $a, b \in \overline{\mathbb{R}^L_+}$
- *The equality of* (4.6) *holds*

is called an optimizer or solution of optimization of (4.6).

Obviously, $(\overline{a^N}, \overline{b^N}, \lambda_N)$ is an optimizer. Next, we are going to prove part (ii) of Theorem 4.2, i.e., the optimizer is unique in positive cone and $\overline{a^N} = \overline{b^N}$.

4.3 Uniqueness of The Optimizer

From previous discussion we see that, an optimizer of (4.6), $(\overline{a^N}, \overline{b^N}, \lambda_N)$, satisfies Euler-Lagrange equations(4.11). We are going to show the optimizer is unique in positive cone by showing the solution of the Euler-Lagrange equations in the positive cone $\mathbb{R}^L_+$ where $L = N^n$ is unique. Considering the following equations,

$$\begin{cases} \lambda_1 a_r = \displaystyle\sum_{s \neq r} \frac{b_s}{|s-r|^n} \\ \lambda_2 b_s = \displaystyle\sum_{r \neq s} \frac{a_r}{|r-s|^n} \end{cases} \tag{4.13}$$

where $\|a\|_2 = \|b\|_2 = 1$, $r = (r_i) \in \mathbb{Z}^n$, and $1 \leq r_i \leq N$, $1 \leq i \leq n$. $a, b \in \mathbb{R}^L$, where $L = N^n$. By Lemma 4.6, we only need to study solution of (4.13) in the positive cone $\overline{\mathbb{R}^L_+}$.

In the proof, we will use the following simple map,

Definition 4.8. *Let* $T : \overline{\mathbb{R}^L} \to \overline{\mathbb{R}^L_+}$ *such that* $(Ta)_i = |a_i|$ *for* $1 \leq i \leq L$.

Theorem 4.9. *If* $(a, b, \lambda_1, \lambda_2)$ *is a solution of* (4.13), *where* $a, b \in \overline{\mathbb{R}^L_+}$, *then* $\lambda_1 = \lambda_2 = \lambda_N$, *and* $a = b \in \mathbb{R}^L_+$ *is unique.*

Proof. **Step 1.** $\lambda_1 = \lambda_2$.

This is similar to step 2 of Theorem 4.2. So, let $\lambda = \lambda_1 = \lambda_2$.

Step 2. $a, b \in \mathbb{R}^L_+$.

Since

$$\begin{aligned} \lambda a_r &= \sum_{s \neq r} \frac{b_s}{|s-r|^n} \\ &= \frac{1}{\lambda} \sum_t \sum_{s \neq r,t} \left(\frac{1}{|r-s|^n} \frac{1}{|t-s|^n}\right) a_t \\ &= \frac{1}{\lambda} \sum_t C(r,t) a_t \end{aligned}$$

we have $\lambda^2 a = Ca$, where $C = A^T A$ and A is a symmetric matrix. So C is non-negative definite. Since $C(r,t) > 0$, $a \in \overline{\mathbb{R}^L_+}$ and $a \neq 0$ for $\|a\| = 1$, the last term above is strictly positive. Therefore, $a, b \in \mathbb{R}^L_+$.

Let $0 \leq \mu_1 \leq \mu_2 \cdots \leq \mu_L$ be the eigenvalues of C. Then $\exists \overline{\xi_L} \in \mathbb{R}^L$, s.t. $C\overline{\xi_L} = \mu_L \overline{\xi_L}$, and $\|\overline{\xi_L}\| = 1$, and $\overline{\xi_L} \notin \overline{\mathbb{R}^L_-}$. We can assume the last property because eigenvectors appear in pairs with opposite signs. Also, by theory of adjoint operators, $\mu_L = \sup_{\|\xi\|=1} < \xi, C\xi > = < \xi_L, C\xi_L >$.

Step 3. $\exists \overline{\xi_L} \in \mathbb{R}^L_+, \|\overline{\xi_L}\| = 1$, and $\mu_{L-1} < \mu_L$.

First, $\exists \overline{\xi_L} \in \overline{\mathbb{R}^L_+}$. If not, then $\overline{\xi_L} \notin \overline{\mathbb{R}^L_+} \cup \overline{\mathbb{R}^L_-}$.

Then we have

$$\mu_L = \overline{\xi_L}^T C \overline{\xi_L} \tag{4.14}$$

$$< (T\overline{\xi_L})^T C (T\overline{\xi_L}) \tag{4.15}$$

$$\leq \max_{\|\xi\|=1} \xi^T C \xi = \mu_L \tag{4.16}$$

where T is defined in Definition 4.8. A contradiction. So, $\exists \overline{\xi_L} \in \overline{\mathbb{R}^L_+}$, and since $C\overline{\xi_L} = \mu_L \overline{\xi_L}$, $\overline{\xi_L} \in \mathbb{R}^L_+$.

The argument above also shows that $\mu_{L-1} < \mu_L$. If not, $\mu_{L-1} = \mu_L$, then by a similar argument as above $\exists \xi_{L-1} \in \mathbb{R}_+^L$, s.t. $C\xi_{L-1} = \mu_L \xi_{L-1}$, and moreover $\xi_{L-1} \perp \xi_L$ which is impossible.

Step 4. $a = b = \overline{\xi_L}, \lambda = \lambda_N = \sqrt{\mu_L}$.

Considering $\lambda^2 a = Ca$,

(1) If $\lambda^2 \neq \mu_L$, then $a \perp \overline{\xi_L}$. Since $a \in \mathbb{R}_+^L$ by step 2, this is impossible. So, $\lambda^2 = \mu_L$.

(2) Since $Ca = \mu_L a, C\overline{\xi_L} = \mu_L \overline{\xi_L}$, and by the fact that $\mu_{L-1} < \mu_L$ and $\|a\| = \|\overline{\xi_L}\| = 1$, $a = \overline{\xi_L}$. Similarly, $b = \overline{\xi_L}$.

(3) If $(\overline{a^N}, \overline{b^N}, \lambda_N)$ is an optimizer of (4.6) in the positive cone, it is a solution of (4.13). So, $a = \overline{a^N} = b = \overline{b^N}$, $\lambda_N^2 = \lambda^2 = \mu_L$, and $\lambda, \lambda_N > 0$, so $\lambda = \lambda_N$.

□

Hence we can prove part (ii) of Theorem 4.2. The same as the 3rd argument of step 4 above, since an optimizer $(\overline{a^N}, \overline{b^N}, \lambda_N)$ is a solution of (4.13), part (ii) follows from Theorem 4.9.

Remark 4.10. *At the time of this writing, thanks to Professor Dongsheng Li of Jiaotong University in Xi'an, we find that uniqueness follows directly from Perron's Theorem [50]. So the proof above can be much simplified.*

Corollary 4.11. *λ is increasing as N increases.*

Proof. Let λ_N and A_N be a solution and coefficient matrix of (4.13). So,

$$\begin{aligned}
\lambda_N &= \max_{\|\xi\|=1} \xi^T A_N \xi = \overline{\xi_N}^T A_N \overline{\xi_N} \\
&= (\overline{\xi_N}, 0)^T A_{N+1} (\overline{\xi_N}, 0) \\
&< \max_{\|\xi\|=1} \xi^T A_{N+1} \xi = \lambda_{N+1}
\end{aligned}$$

where $(\overline{\xi_N}, 0)$ means $(\overline{\xi_N}, 0) \in \mathbb{R}^L$ and $L = (N+1)^n$, and arranging $\overline{\xi_N}$ to take the first N^n entries and stuffing the rest with zeros. Then calculate in blocks of matrices. □

4.4 Symmetry of The Optimizer

From section 2 we see the uniqueness of the optimizer of (4.6) in positive cone. So, from this point, we use $(a = b, \lambda)$ instead of $(\overline{a^N}, \overline{b^N}, \lambda_N)$ when referring the optimizer of (4.6) for simplicity, and next we prove Theorem 4.3.

Proof. From (4.13) we have

$$\lambda a_r = \sum_{s \neq r} \frac{a_s}{|s-r|^n}$$

then

$$\lambda a_{\Phi(r)} = \sum_{s \neq \Phi(r)} \frac{a_s}{|s-\Phi(r)|^n} = \sum_{t \neq r} \frac{a_{\Phi(t)}}{|\Phi(t)-\Phi(r)|^n} = \sum_{t \neq r} \frac{a_{\Phi(t)}}{|t-r|^n}$$

So, $\bar{a} = (a)_{\Phi(r)}$ is also a solution to (4.13). Then, by uniqueness of the solution, $\bar{a} = a$. So, $a_{\Phi(r)} = a_r$. $\square$

Example 4.1. *If a is an optimizer, then $a_{(r_i, r')} = a_{(N-r_i+1, r')}$ for $1 \leq i \leq N$.*

4.5 Monotone Property of The Optimizer

For convenience of writing, we change the range of r_i's from $1 \leq r_i \leq N$ to $-N \leq r_i \leq N$ which makes no change to the results above essentially. The following is the proof of Theorem 4.4.

Proof. We are only going to show inequalities (4.7) are true for $i = 1$ for simplicity. Consider $d^{(1)}_r = a_{(r_1-1, r')} - a_{(r_1, r')}$, where $1 \leq r_1 \leq N$ and $-N \leq r_i \leq N$, $2 \leq i \leq n$. So $d^{(1)} \in \mathbb{R}^{N(2N+1)^{(n-1)}}$.

Then by applying Theorem 4.3, we have

$$
\begin{aligned}
d_r^{(1)} &= \frac{1}{\lambda}\Big(\sum_{s\neq(r_1-1,r')}\frac{a_s}{|s-(r_1-1,r')|^n}-\sum_{s\neq(r_1,r')}\frac{a_s}{|s-(r_1,r')|^n}\Big)\\
&= \frac{1}{\lambda}\Big(\sum_{\substack{t=(t_1,t')\neq(r_1,r'),\\ -N+1\le t_1\le N+1}}\frac{a_{(t_1-1,t')}}{|t-(r_1,r')|^n}-\sum_{s\neq(r_1,r')}\frac{a_{(s_1,s')}}{|s-(r_1,r')|^n}\Big)\\
&= \frac{1}{\lambda}\Big(\sum_{\substack{t=(t_1,t')\neq(r_1,r'),\\ 1\le t_1\le N}}\frac{d_t^{(1)}}{|t-(r_1,r')|^n}+\sum_{\substack{t=(-t_1+1,t')\neq(r_1,r'),\\ 1\le t_1\le N}}\frac{-d_t^{(1)}}{|(-t_1+1,t')-(r_1,r')|^n}\\
&\quad+\underbrace{\sum_{t=(N+1,t')\neq(r_1,r')}\frac{a_{(N,t')}}{|t-(r_1,r')|^n}-\sum_{t=(-N,t')\neq(r_1,r')}\frac{a_{(-N,t')}}{|t-(r_1,r')|^n}\Big)}_{f(r)}\\
&= \frac{1}{\lambda}\Big(\sum_{\substack{(t_1,t')\neq r\\ 1\le t_1\le N}}\Big(\frac{1}{|(t_1,t')-r|^n}-\frac{1}{|(-t_1+1,t')-r|^n}\Big)d_t^{(1)}+\frac{-d_r^{(1)}}{|2r_1-1|^n}+f(r)\Big)
\end{aligned}
$$

Also by Theorem 4.3, $a_{(N,t')} = a_{(-N,t')}$, easily one sees that $f(r) \geq 0$.

So, for $1 \leq r_1 \leq N$

$$
\Big(\lambda+\frac{1}{|2r_1-1|^n}\Big)d_r^{(1)} = \sum_{\substack{(t_1,t')\neq r\\ 1\le t_1\le N}}\Big(\frac{1}{|(t_1,t')-r|^n}-\frac{1}{|(-t_1+1,t')-r|^n}\Big)d_t^{(1)}+f(r)
$$

Write the above equations in matrix form,

$$
d^{(1)} = Ad^{(1)} + F \tag{4.17}
$$

where $(F)_r = \frac{1}{(\lambda+\frac{1}{|2r_1-1|^n})}f(r)$, and

$$
A(r,t) = \begin{cases} \dfrac{1}{(\lambda+\frac{1}{|2r_1-1|^n})}\Big(\dfrac{1}{|(t_1,t')-r|^n}-\dfrac{1}{|(-t_1+1,t')-r|^n}\Big), & r\neq t\\[2ex] 0, & r=t \end{cases}
$$

It is easy to see that entries of A and F are non-negative. So, $A : \overline{\mathbb{R}_+^L} \to \overline{\mathbb{R}_+^L}$, where $L = N(2N+1)^{(n-1)}$, and $F \in \overline{\mathbb{R}_+^L}$. Therefore, provided $\|A\| < 1$, then by Theorem 4.5 (Maximum Principle) we get $d^{(1)} \in \overline{\mathbb{R}_+^L}$, hence (4.7) is proved. So, the only thing left to prove is $\|A\| < 1$.

Notice that if C, D are symmetric matrices such that $C, D : \mathbb{R}_+^L \to \mathbb{R}_+^L$, for some positive integer L, then $\|C\| \le \|C + D\|$, because

$$\|C\| = \max_{\|\xi\|=1} \xi^T C \xi = \bar{\xi}^T C \bar{\xi} \le \bar{\xi}^T (C + D) \bar{\xi} \le \max_{\|\xi\|=1} \xi^T (C + D) \xi = \|C + D\|$$

Let

$$C(r,t) = \begin{cases} \dfrac{1}{(\lambda + \frac{1}{|2r_1 - 1|^n})} \dfrac{1}{|(t_1, t') - r|^n}, & r \ne t \\ 0, & r = t \end{cases}$$

and

$$D(r,t) = \begin{cases} \dfrac{1}{(\lambda + \frac{1}{|2r_1 - 1|^n})} \dfrac{1}{|(-t_1 + 1, t') - r|^n}, & r \ne t \\ 0, & r = t \end{cases}$$

So,

$$\begin{aligned} \|A\| &\le \|A + D\| = \|C\| \\ &\le \frac{1}{\lambda + \delta(N)} \|A_N\| \end{aligned}$$

where A_N is the matrix of (4.13) of the case that $-N \le r_i \le N$ and $1 \le i \le n$, so $\|A_N\| = \lambda$. So, $\|A\| < 1$. $\square$

Chapter 5

Qualitative analysis of three-wave resonance interaction

In this chapter, we qualitatively analyze three-wave resonance interaction (3WRI) with periodic boundary condition. First, for 3WRI with positive wave energy, we present a regularity theorem for all spatial dimension. Second, for 3WRI with negative wave energy, we present a class of solution in general spatial dimension that will blow up in finite time. Moreover, a complete classification of spatial uniform solution is given for this particular system.

5.1 Introduction

Consider the 3-wave resonance interaction (3WRI) system,

$$\begin{cases} \partial_\tau A_1 + c_1 \cdot \nabla A_1 = i\gamma_1 \overline{A_2 A_3}, \\ \partial_\tau A_2 + c_2 \cdot \nabla A_2 = i\gamma_2 \overline{A_1 A_3}, \quad \text{in } \Omega, \\ \partial_\tau A_3 + c_3 \cdot \nabla A_3 = i\gamma_3 \overline{A_1 A_2}, \end{cases} \tag{5.1}$$

with periodic boundary condition, where Ω is a rectangle domain,

$$\Omega = \{x \in \mathbb{R}^n |\ |x_k| < a_k, k = 1, \cdots, n\}$$

and all $A_j's$ are are complex amplitude and periodic on Ω. $\gamma_j = \pm 1$, and c_j's are real non-zero constant vectors. The derivation of (5.1) is rather standard in e.g. nonlinear optics and can be found in e.g. Chap. 4 in [1] and [32].

Here we are interested in the qualitative property of solution to (5.1). Namely, we are interested in determining if a solution possesses either finite blow up or global existence. Depending on whether γ_j's have the same sign, the system (5.1) can be classified into two cases, and each case models different physical phenomena.

γ_j's **not having the same sign** corresponds to 3WRI with positive wave energy, which generates from various physical backgrounds e.g. nonlinear optics and water waves. In fact most 3WRI studies in the past focused on this case, and an important tool is inverse scattering transformation (IST), with which people construct solution and solve the system numerically for some special initial value, e.g. "separable" initial value [30], or more general initial value [31]. The 1D-3WRI was approached by Zakharov and Manakov [71] (and independently by Kaup [29]) with inverse scattering transformation, but the boundary condition they considered is that the solution decays sufficiently fast as $|x| \to \infty$. A detailed review of the 1-D problem can be found e.g. in [32]. For 3-D problem, Ablowitz and Haberman's work [2] leads Cornille [17] to reformulate the problem into integral equations and Kaup gave explicitly general inverse-scattering solution in a series of papers [30, 31]. See also some recent developments in [25].

However, it seems there has not been any regularity analysis of 3WRI in the case of γ_j's **not** having the same sign. Here, we prove that the system (5.1) cannot develop singularity in finite time given that the coefficients c_j, $j = 1, 2, 3$, are the same.

Theorem 5.1. *For all space dimension, if initial data is continuously differentiable in space, γ_j's do not have the same sign and c_j's $j = 1, 2, 3$ are the same, then the solution to system* (5.1) *exists globally in time.*

The case that γ_j's have **the same sign** corresponds to 3WRI with negative wave energy, and (5.1) becomes (5.2). One interesting phenomena about such 3WRI system is that the solution can blow up in finite time. Coppi et al. [16] first found such instability of 3WRI with negative wave energy in plasma physics, which they called "explosively unstable".

Heuristically, the transportation wave equation is non-dispersive, and the nonlinear term should enhance the amplitude in a superlinear way. One can compare this to a different system, the 3-D wave equation, where the positive feedback from the nonlinear term needs to race with the dispersive tendency (See a classic paper of F. John [28]). So, finite-time blow up should not be a surprise in system (5.1) when γ_j's have the same sign. However, note that complex amplitude means phase interaction and c_j's can be different, and both cause difficulty in analysis. This chapter is dedicated to determining if a solution will blow up in finite time given any initial value.

Besides finite-in-time blow-up solution, system (5.2) obviously admits global-in-time existing solution, e.g., two of A_j's are zero and the third is a constant. A natural question is whether (5.2) admits other non-trivial globally existing solutions. In this chapter, we show that the system admits globally existing and decaying solution. Moreover, we see that the spatial uniform solution of (5.2) can be completely classified in terms of blowup or not. The spatial uniform case of (5.2) reduces to an ODE system which has been well studied for decades. Here we state a classification result which may be known to other researchers through various analysis approach. Nevertheless, in Section 5.3.1 we give a analytic proof for convenience of the reader.

Theorem 5.2. *A necessary and sufficient condition of finite-in-time blow-up solution to* (5.4) *is that the initial condition satisfies one of the following,*

(1) Only one of $A_j(0)$'s is zero, i.e., $|A_3(0)| \geq |A_1(0)| > |A_2(0)| = 0$;

(2) $(\theta_1 + \theta_2 + \theta_3)(0) = \frac{3\pi}{2}$, one of $A_j(0)$'s is strictly less than the other two and none is zero, i.e., $|A_3(0)| \geq |A_1(0)| > |A_2(0)| > 0$;

(3) $(\theta_1 + \theta_2 + \theta_3)(0) \neq \frac{3\pi}{2}$ (implicitly none of $A_j(0)$'s is zero),

where the indexes $\{1, 2, 3\}$ allow any permutation and θ_j's are from $A_j = r_j e^{i\theta_j}$, $j = 1, 2, 3$.

Remark 5.3. *We think an interesting question is, whether all globally existing solution to* (5.2) *is spatially uniform, and this is so far open.*

We also describe a new class of solutions that blow up in finite time (Theorem 5.4),

Theorem 5.4. *Suppose $\theta_j(x, 0) = \theta_j(0)$, $j = 1, 2, 3$, and $(\theta_1+\theta_2+\theta_3)(0) = \frac{\pi}{2}$, and then the solution of* (5.2) *blows up in finite time.*

The mathematical technique used here is elementary, and the results are based on the method of characteristics. We believe that a natural step after this is to implement perturbation theory.

This chapter is organized this way. In section 5.2, we focus on 3WRI with positive wave energy and prove Theorem 5.1. In section 5.3, we study 3WRI with negative wave energy. In section 5.3.1, we focus on space-independent case and prove Theorem 5.2. In section 5.3.2, we consider the general case, and after briefly discussing the well-posedness of (5.1) we prove Theorem 5.4.

5.2 3WRI with positive wave energy

Consider system (5.1) with $\gamma_1 = 1$, $\gamma_2, \gamma_3 = -1$, and $c_1 = c_2 = c_3 = c$.

Proof of Theorem 5.1. Multiply (5.1) by $\overline{A_j}$ and take conjugate of the system then multiply by A_j, we have

$$\partial_\tau |A_1|^2 + c \cdot \nabla |A_1|^2 = i(A_1 A_2 A_3 + \overline{A_1 A_2 A_3}),$$

$$\partial_\tau |A_2|^2 + c \cdot \nabla |A_2|^2 = -i(A_1 A_2 A_3 + \overline{A_1 A_2 A_3}),$$

$$\partial_\tau |A_3|^2 + c \cdot \nabla |A_3|^2 = -i(A_1 A_2 A_3 + \overline{A_1 A_2 A_3}).$$

Hence, we have

$$\partial_\tau (|A_1|^2 + |A_2|^2) + c \cdot \nabla |A_1|^2 + c \cdot \nabla |A_2|^2 = 0,$$

$$\partial_\tau (|A_1|^2 + |A_3|^2) + c \cdot \nabla |A_1|^2 + c \cdot \nabla |A_3|^2 = 0.$$

These lead to

$$|A_1(x + c\tau, \tau)|^2 + |A_2(x + c\tau, \tau)|^2 = K_1(x),$$

$$|A_1(x + c\tau, \tau)|^2 + |A_3(x + c\tau, \tau)|^2 = K_2(x).$$

Since the initial data of A_j's are smooth, we know that $K_1(x)$ and $K_2(x)$ must be smooth and bounded. Hence, $|A_j|$, $j = 1, 2, 3$, must be bounded for all time. □

Remark 5.5. *The proof above is obviously valid for general domain and boundary condition.*

5.3 3WRI with negative wave energy

Now, if γ_j's have the same sign, (5.1) becomes

$$\begin{cases} \partial_\tau A_1 + c_1 \cdot \nabla A_1 = i\overline{A_2 A_3}, \\ \partial_\tau A_2 + c_2 \cdot \nabla A_2 = i\overline{A_1 A_3}, \quad \text{in } \Omega, \\ \partial_\tau A_3 + c_3 \cdot \nabla A_3 = i\overline{A_1 A_2}, \end{cases} \tag{5.2}$$

with periodic boundary condition. The constants of motion are,

$$K_1 = \int_\Omega |A_1|^2 - |A_2|^2 dx, \quad K_2 = \int_\Omega |A_1|^2 - |A_3|^2 dx, \tag{5.3}$$

where K_1, K_2 are constants.

5.3.1 Spatially uniform case

There are many classical studies (for reference, see section I. D. of [32]) of space-independent 3WRI which is an ODE system. Here we give a complete classification of space independent solutions of the negative wave energy case of 3WRI,

$$\begin{cases} \partial_\tau A_1 = i\overline{A_2 A_3}, \\ \partial_\tau A_2 = i\overline{A_1 A_3}, \\ \partial_\tau A_3 = i\overline{A_1 A_2}. \end{cases} \tag{5.4}$$

The system (5.4) admits three constants of the motion:

$$\begin{gathered} K_1 = |A_1|^2 - |A_2|^2, \quad K_2 = |A_1|^2 - |A_3|^2, \\ H = A_1 A_2 A_3 + \overline{A_1 A_2 A_3}, \end{gathered} \tag{5.5}$$

where K_1, K_2 and H are constants. The first two constants are also called Manley-Rowe relations.

By direct calculation we rewrite (5.4) as

$$\partial_\tau r_j^2 = 2 r_1 r_2 r_3 \sin(\theta_1 + \theta_2 + \theta_3), \text{ for } j = 1, 2, 3, \tag{5.6}$$

$$\begin{aligned} r_1 \partial_\tau \theta_1 &= r_2 r_3 \cos(\theta_1 + \theta_2 + \theta_3), \\ r_2 \partial_\tau \theta_2 &= r_1 r_3 \cos(\theta_1 + \theta_2 + \theta_3), \\ r_3 \partial_\tau \theta_3 &= r_1 r_2 \cos(\theta_1 + \theta_2 + \theta_3), \end{aligned} \tag{5.7}$$

and (5.5) becomes

$$K_1 = r_1^2 - r_2^2, \qquad K_2 = r_1^2 - r_3^2,$$
$$H = 2r_1 r_2 r_3 \cos(\theta_1 + \theta_2 + \theta_3). \tag{5.8}$$

Remark 5.6. (5.6)-(5.7) *holds only when none of r_j's is zero. This is because θ_j is not defined when $r_j = 0$. Actually, we should pay special attention to the situation when A_j's touch zero since* (5.4) *does not satisfy Lipchitz condition there. However, we can take advantage of* (5.5). *For instance A_1 touches zero at $\tau = \tau_0$ and $A_2(\tau_0), A_3(\tau_0) \neq 0$, then $\partial_\tau A_j(\tau_0) \neq 0$, i.e., $|A_1|$ will increase and hence $|A_2|, |A_3|$ will increase since K_1, K_2 are constant. By* (5.6), *we must have* $\sin(\theta_1 + \theta_2 + \theta_3) > 0$ *and this implies $(\theta_1 + \theta_2 + \theta_3)(\tau_0+) \in (0, \pi)$. Moreover, we know $H \equiv 0$ and hence $(\theta_1 + \theta_2 + \theta_3)(\tau) \equiv \frac{\pi}{2}$ for $\tau \in (\tau_0, \infty)$. With a little extra effort we can show that the solution must blow up in finite time.*

If two of A_j's, say A_1, A_2, are zero at $\tau = \tau_0$, the solution of (5.4) *has to be an equilibrium, see the proof of Theorem 5.8.*

Proposition 5.7. *If $(\theta_1 + \theta_2 + \theta_3)(0) \neq \frac{\pi}{2}$ or $\frac{3\pi}{2}$, then none of A_j's will touch zero.*

Proof. Since $H \neq 0$, $A_j \neq 0$. □

Theorem 5.8. *A necessary and sufficient condition for the solution of* (5.4) *to exist globally in time is that, the initial data satisfyå either of the following,*

(i) $|A_3(0)| \geq |A_1(0)| = |A_2(0)| = 0$;

(ii) $|A_3(0)| \geq |A_1(0)| = |A_2(0)| > 0$ *and* $(\theta_1 + \theta_2 + \theta_3)(0) = \frac{3\pi}{2}$;

where the indexes $\{1, 2, 3\}$ *allow any permutation.*

Proof. *We first prove the sufficiency.*

For (ii) by (5.8) $H, K_1 \equiv 0$ and $K_2 \leq 0$. So, $r_1 = r_2$.

Notice that our choice of initial value ensures that all r_j, $j = 1, 2, 3$ decay until one of them touches zero. Let's define τ_0 to be the first time that one of r_j's touches zero. If $\tau_0 = \infty$ then we have a global decaying solution. So, we assume $\tau_0 < \infty$ and to sum up:

(1) r_j, $j = 1, 2, 3$ decays in $[0, \tau_0]$;

(2) $(\theta_1 + \theta_2 + \theta_3)(\tau) = \frac{3\pi}{2}$ in $[0, \tau_0)$;

Let $d_0 := |A_3(0)| - |A_1(0)| \geq 0$. **Case 1:** If $d_0 = 0$, we can solve the system (5.6) on interval $[0, \tau_0)$ by

$$r_j = \frac{r_j(0)}{1 + 2r_j(0)\tau}. \tag{5.9}$$

Thus $r_j(\tau_0) \neq 0$, which contradicts with the assumption on r_0. So, (5.9) can extend to $[0, \infty)$.

Case 2: If $d_0 > 0$, by (5.6) and the facts $r_1 = r_2$, $r_3(\tau) \leq r_3(0)$ on $[0, \tau_0)$ we have

$$\partial_\tau r_1 \geq -r_3(0) r_1.$$

Hence by Grönwall's inequality,

$$r_1 = r_2 \geq r_1(0) e^{-r_3(0)\tau}. \tag{5.10}$$

Similar to case 1, $r_j(\tau_0) \neq 0$ contradicts the assumption on r_0, so (5.10) can extend to $[0, \infty)$.

For (i), the initial data obviously admits an equilibrium solution, and we claim that this is the only solution (this is not obvious since we lose Lipchitz condition of (5.4) if some A_j's are zero). Suppose a non-constant solution $A_1(\tau), A_2(\tau), A_3(\tau)$ with initial condition $|A_3(0)| \geq |A_1(0)| = |A_2(0)| = 0$ (so $H \equiv 0$) and $|A_3| \geq |A_1| = |A_2| > 0$ at $\tau = \tau_0$ (since $K_1 \equiv 0$, $r_1 \equiv r_2$). Then if we change the time direction of (5.4) by letting $\tau = \tau_0 - \eta$ and $B_j(\eta) = A_j(\tau_0 - \eta)$ (and corresponding new r_j and θ_j), $j = 1, 2, 3$, then we get a new system,

$$\begin{cases} \partial_\eta B_1 = -i\overline{B_2 B_3}, \\ \partial_\eta B_2 = -i\overline{B_1 B_3}, \\ \partial_\eta B_3 = -i\overline{B_1 B_2}, \end{cases} \tag{5.11}$$

with initial value $|B_3(0)| \geq |B_2(0)| = |B_1(0)| > 0$ at $\eta = 0$. From $H \equiv 0$ (H, K_1, K_2 do not change) and $r_j, j = 1, 2, 3$, decaying for some interval, we see that $\theta_1 + \theta_2 + \theta_3 = \frac{\pi}{2}$ for that interval. So, similar to the proof for (ii), it has a unique solution that decays to a limit $(0, 0, A_3(\tau = 0))$ as

$\eta \to \infty$ and never touches zero in finite time. This contradicts with $(B_1(\tau_0), B_2(\tau_0), B_3(\tau_0)) = (A_1(0), A_2(0), A_3(0)) = (0, 0, A_3(0))$.

Now we turn to the necessity. If the initial data does not satisfy either condition in Theorem 5.8, we have the following cases,

(1) Only one of $A_j(0)$'s is zero, i.e., $|A_3(0)| \geq |A_1(0)| > |A_2(0)| = 0$;

(2) $(\theta_1 + \theta_2 + \theta_3)(0) = \frac{3\pi}{2}$, one of $A_j(0)$'s is strictly less than the other two and none is zero, i.e., $|A_3(0)| \geq |A_1(0)| > |A_2(0)| > 0$;

(3) $(\theta_1 + \theta_2 + \theta_3)(0) \neq \frac{3\pi}{2}$ (implicitly none of $A_j(0)$'s is zero).

Again, the above indexes $\{1, 2, 3\}$ allow any permutation. We are going to show that solutions to the above initial conditions will blow up in finite time.

For **case 1**, by Remark 5.6, we know $(\theta_1 + \theta_2 + \theta_3)(\tau) \equiv \frac{\pi}{2}$ for $\tau \in (0, \infty)$, and by (5.6) and (5.8) we have,

$$\partial_\tau r_1^2 = 2r_1\sqrt{r_1^2 - K_1}\sqrt{r_1^2 - K_2},$$

where $K_1, K_2 < 0$. Hence it is easy to see r_1 blows up in finite time.

For **case 2**, by (5.6) we know that A_2 will touch zero first in finite time, call this time as τ_0. Then $\partial_\tau A_2(\tau_0) \neq 0$ and by Remark 5.6 we know $(\theta_1 + \theta_2 + \theta_3)(\tau) = \frac{\pi}{2}$ for $\tau > \tau_0$. This implies a finite-in-time blow up.

For **case 3**, let $\Theta = \theta_1 + \theta_2 + \theta_3$. If $\Theta(0) \in (-\frac{\pi}{2}, \frac{\pi}{2}]$ (we change $\frac{3\pi}{2}$ to $-\frac{\pi}{2}$ for convenience), **step 1.** we show that Θ increases and approaches $\frac{\pi}{2}$. By (5.7) we have

$$\partial_\tau \Theta = (\frac{r_2 r_3}{r_1} + \frac{r_1 r_3}{r_2} + \frac{r_1 r_2}{r_3}) \cos\Theta.$$

By 5.8 we have $|r_1 r_2 r_3| \geq \frac{|H|}{2}$. Suppose $r_j > \eta_1$, $j = 1, 2, 3$ for some $\eta_1 > 0$, then there exists some constant $\eta_0 > 0$ such that $(\frac{r_2 r_3}{r_1} + \frac{r_1 r_3}{r_2} + \frac{r_1 r_2}{r_3}) > \eta_0$. Now without loss of generality suppose $r_1 \to 0$, then $r_2 r_3 \to \infty$, hence $\frac{r_2 r_3}{r_1} \to \infty$. In both cases, we see that $\cos\Theta$ is positive since $\Theta \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, and hence Θ will approach $\frac{\pi}{2}$.

Step 2. There exists $\tau_1 > 0$ such that $\Theta > \frac{\pi}{4}$ for $\tau > \tau_1$. Then by (5.6) for $\tau > \tau_1$

$$\begin{aligned}\partial_\tau r_1 &\geq C r_2 r_3 \\ &\geq C\sqrt{r_1^2 - K_1}\sqrt{r_1^2 - K_2},\end{aligned}$$

where the second inequality is due to (5.8). So a finite blow up in r_1 can be easily derived.

Similarly we have a finite blow up for $\Theta(0) \in (\frac{\pi}{2}, \frac{3\pi}{2})$. $\square$

Hence Theorem 5.2 directly follows from Theorem 5.8.

5.3.2 General case

First we briefly discuss the well-posedness of (5.1). The local existence and uniqueness is guaranteed by method of characteristics. Also, the spatially periodic boundary condition is directly satisfied if initial data are periodic, since all c_j's are constant vectors.

Now we consider (5.2), and by direct calculation we rewrite the system as

$$\partial_\tau r_j^2 + c_j \cdot \nabla r_j^2 = 2r_1 r_2 r_3 \sin(\theta_1 + \theta_2 + \theta_3), \text{ for } j = 1, 2, 3, \tag{5.12}$$

$$\begin{aligned} r_1(\partial_\tau \theta_1 + c_1 \cdot \nabla\theta_1) &= r_2 r_3 \cos(\theta_1 + \theta_2 + \theta_3), \\ r_2(\partial_\tau \theta_2 + c_2 \cdot \nabla\theta_2) &= r_1 r_3 \cos(\theta_1 + \theta_2 + \theta_3), \\ r_3(\partial_\tau \theta_3 + c_3 \cdot \nabla\theta_3) &= r_1 r_2 \cos(\theta_1 + \theta_2 + \theta_3), \end{aligned} \tag{5.13}$$

To outline the proof, if initially A_j is lined up in the same direction at each point, and the summation of θ_j's equal $\frac{\pi}{2}$, then all θ_j's preserve at all time. Hence (5.13) is moved out of the equations. Then we are left to analyze (5.12).

Proof of Theorem 5.4. **Step 1.** We show that $(\theta_1 + \theta_2 + \theta_3)(x, \tau) \equiv \frac{\pi}{2}$.

Note that implicitly, the initial condition guarantees that $A_j(x, 0)$'s are nowhere zero and bounded. So, the short time existence and uniqueness of solution to (5.12) is provided by method of characteristics. Hence, $\theta_j(x, \tau) \equiv \theta_j(0)$ and $(\theta_1 + \theta_2 + \theta_3)(x, \tau) \equiv \frac{\pi}{2}$.

Step 2. (5.12) becomes

$$\begin{cases} \partial_\tau r_1 + c_1 \cdot \nabla r_1 = r_2 r_3, \\ \partial_\tau r_2 + c_2 \cdot \nabla r_2 = r_1 r_3, \\ \partial_\tau r_3 + c_3 \cdot \nabla r_3 = r_1 r_2, \end{cases} \quad \text{in } \Omega.$$

By method of characteristics, the corresponding characteristic equation for r_1 is

$$\begin{cases} \frac{dr_1}{d\tau} = r_2 r_3, \\ \frac{dx}{d\tau} = c_1, \end{cases} \tag{5.14}$$

with initial data $x(0) = \xi$ and $r_1(\xi, 0) = \phi_1(\xi)$. Since $r_2, r_3 \geq 0$, for fixed ξ, $r_1(\xi, \tau)$ is monotone increasing as τ increases. We say r_1 is "monotone increasing" along characteristics. Similarly, r_2, r_3 are monotone increasing along their characteristics $x = c_2\tau + \xi$ and $x = c_3\tau + \xi$.

Since $r_j(0)$'s are strictly positive in Ω, the characteristic equations also imply that $r_j(x, \tau) > 0$ on $\overline{\Omega}$ for all $\tau > 0$. This eliminates the possible zero at the boundary of Ω. Hence after any $\tau = \tau_0 > 0$, r_j's must have a positive infimum in $\overline{\Omega}$. Suppose

$$f(\tau) = \min_{x\in\Omega} \{r_1(x, \tau), r_2(x, \tau), r_3(x, \tau)\},$$

then $f(\tau) > 0$ for $\tau > \tau_0 > 0$. We **claim** that $f(\tau)$ is Lipschitz, and hence from (5.14) we see

$$\frac{df}{d\tau} \geq f^2,$$

hold in weak sense. This implies that f blows up in finite time, since for any positive test function $v \in C_0^1(\mathbb{R})$ and if g is the solution of

$$\begin{cases} \dfrac{dg}{d\tau} = g^2, \\ g(\tau_0) = C_0, \text{ and } 0 < C_0 < f(\tau_0), \end{cases}$$

then $w = f - g$ satisfies

$$-\int_{\tau_0}^{T} wv' ds \geq \int_{\tau_0}^{T} \beta(s) w(s) v(s) ds,$$

where $\beta(\tau)$ is non-negative (since f, g are positive) and suppose $[\tau_0, T]$ is the support of v. By choosing suitable v we see that

$$w(T) \geq C \int_{\tau_0}^{T} \beta(s) w(s) ds,$$

where $v(s)$ on the right is absolved in $\beta(s)$. Hence by Grönwall's inequality $w \geq 0$. Then g blowing up in finite time implies that f blows up in finite time.

We finish the proof by proving the claim. Let $\bar{r}_1(\tau) = \min_{x \in \Omega} r_1(x, \tau) = r_1(x_*, \tau)$,. Note that r_1 is monotone increasing along characteristics, then $\bar{r}_1(\tau)$ must be monotone increasing. Hence,

$$0 \leq \frac{\bar{r}_1(\tau + \Delta\tau) - \bar{r}_1(\tau)}{\Delta\tau} \leq \frac{r_1(x_*, \tau + \Delta\tau) - r_1(x_*, \tau)}{\Delta\tau} \leq |r_1|_{C^1}.$$

So $\bar{r}_1(\tau)$ is Lipschitz. Similarly $\bar{r}_2, \bar{r}_3$ are Lipschitz. Finally, $f = \min\{\bar{r}_1, \bar{r}_2, \bar{r}_3\}$ is Lipschitz. $\square$

Remark 5.9. *Although θ_j's are constant, r_j's vary in space. So, for the future perturbation work, one only needs to perturb θ_j's.*

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