Lattices of Supercharacter Theories

by

Jonathan P. Lamar

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M.A., University of Colorado, 2015

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Nathaniel Thiem

Richard M. Green

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Lattices of Supercharacter Theories

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The set of supercharacter theories of a finite group forms a lattice under a natural partial order. An active area of research in the study of supercharacter theories is the classification of this lattice for various families of groups. One other active area of research is the formation of Hopf structures from compatible supercharacter theories over indexed families of groups. This thesis therefore has two goals. First, we will classify the supercharacter theory lattice of the dihedral groups D_{2n} in terms of their cyclic subgroups of rotations, as well as for some semidirect products of the form $\mathbb{Z}_n \rtimes \mathbb{Z}_p$. Second, we will construct a pair of combinatorial Hopf algebras from natural supercharacter theories on the alternating and finite special linear groups and relate them using the theory of combinatorial Hopf algebras, as developed by Aguiar, Bergeron, and Sottile in 2006.

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Contents

Figures

Figure

Chapter 1

Introduction

A supercharacter theory of a finite group is a combinatorial tool for studying the character theory of that group. It is defined by a pair of partitions: one of the irreducible characters and one of the conjugacy classes. These partitions are subject to conditions (made precise in Chapter 2) that ensure that one can think of the supercharacter theory as an approximation of the usual character theory. Since the character theory of a group may evade classification, this may be a useful guiding principle.

The history of supercharacter theory began with the work of André (see [And95], [And02], and [AN06]) and Yan (see [Yan01]), who studied certain characters (called "basic characters") of the groups $U_n(q)$ of $n \times n$ unipotent upper triangular matrices over finite fields $\mathbb{F}_q$. These characters can easily be indexed and have the property that each irreducible character appears as a constituent of precisely one basic character. Moreover, the trivial character is basic, and products of basic characters decompose into sums of basic characters. Hence, the basic characters form a computationally friendly approximation to the usual character theory. Although the classification of the irreducible characters of $U_n(q)$ is equivalent to the enumeration of pairs of $n \times n$ invertible matrices up to simultaneous similarity, and as such is a "wild" problem (see [Hig60], [VLA03], [Pol66], [Ser00]), this approximation has yielded partial solutions to previously intractible problems. For example, in [ACDS04], the authors used this supercharacter theory to provide bounds on the rate of convergence to equilibrium of a well-known random walk on $U_n(q)$.

In [DI08], Diaconis and Isaacs formally defined the notion of a supercharacter theory for an arbitrary finite group and studied a particular supercharacter theory for a family of finite groups known as algebra groups. These and related supercharacter theories were studied in subsequent papers (for example, see [DT09], [MT09], [TV09], [Thi10], [AAB+12], [ABT13], and [And15]) where priority was placed on using supercharacter theory to ease computation of character values and building combinatorial Hopf-theoretic structures for nested families of groups with given supercharacter theories in a manner analogous to [Mac98]. In [Hen12], Hendrickson established a one-to-one correspondence between supercharacter theories of a group and central Schur rings over that group, connecting supercharacter theories with this earlier work (see [Tam70]). In [BBF+12], Brumbaugh et al. connect Gauss and Ramanujan sums to supercharacter theories of cyclic groups.

As we will see below, the set $\mathrm{SCT}(G)$ of all supercharacter theories of a fixed group G forms a lattice under a natural partial order. The first author to directly attempt to classify the lattice of all supercharacter theories of a fixed group was Hendrickson in [Hen08]. In this paper, he classified the supercharacter theories of cyclic groups of prime order, while in [Hen12], he classified the supercharacter theories of arbitrary cyclic groups using the previous work of Leung and Man (see [LM96] and [LM98]) on Schur rings. In [BHH14], the authors studied the combinatorial properties of the lattice of supercharacter theories of a cyclic group. In a recent preprint ([Lan17]), Lang has classified the supercharacter theories of $\mathbb{Z}_p \times \mathbb{Z}_2 \times \mathbb{Z}_2$ for odd primes p.

In recent years, some attempts have been made to classify the supercharacter theory lattices of families of nonabelian groups. Lang first made some progress for semidirect products of abelian groups in [Lan14]. Aliniaeifard has described a sublattice of supercharacter theories for any group G, each element corresponding to a collection of normal subgroups of G (see [Ali15] and [Ali16] for a complete description), and more recently, Aliniaeifard and Burkett in [AB17] constructed a PSH algebra using actions of a Galois group on the irreducible characters of finite general linear groups. Most recently, in [Wyn17], Wynn has classified supercharacter theories of Frobenius groups and has made some general conclusions about supercharacter theories of Camina pairs. Moreover, that thesis also contains a classification of the supercharacter theories of dihedral groups, a major result of this thesis. We acknowledge Wynn's result, however the classification provided in this thesis was proven independently and without knowledge of his work.

1.1 Organization

There are two goals in this thesis: to provide classifications of the supercharacter theory lattices for some families of nontrivial semidirect products of cyclic groups, and to discuss Hopf-algebraic constructions arising from supercharacter theories of nested families of groups.

Chapters 2 and 3 are mostly preliminary. Chapter 2 provides the background material required for the rest of this thesis, in particular the definitions and basic results in the study of supercharacter theories, finite posets and lattices, and Hopf algebras. In Chapter 3, we discuss the lattice $\mathrm{SCT}(G)$ of all supercharacter theories of a finite group G and prove some of the basic results concerning the combinatorics of this lattice and its sublattices as they relate to the algebra of G. We will also discuss an algorithm for computing $\mathrm{SCT}(G)$ given knowledge of its character table, and the use of that algorithm in the classification of all groups with only two supercharacter theories. We also discuss some results and conjectures regarding the lattices of supercharacter theories of alternating groups.

The main result of Chapter 4 is an explicit classification of the supercharacter theories of the dihedral groups D_{2n} of order $2n$ using their cyclic subgroups of rotations. We do this first by classifying the sublattice of characteristic supercharacter theories (defined in Chapter 3) by embedding a canonical sublattice of $\mathrm{SCT}(\mathbb{Z}_n)$ into $\mathrm{SCT}(D_{2n})$, and obtaining the remainder of the characteristic supercharacter theories through the use of a coarsening map and a refining map which glue and split parts, respectively. We obtain the non-characteristic supercharacter theories by generalizing the splitting map. The final section of Chapter 4 is devoted to generalizations of this strategy to semidirect products of the form $\mathbb{Z}_n \rtimes \mathbb{Z}_p$. We provide a classification of $\mathrm{SCT}(\mathbb{Z}_n \rtimes \mathbb{Z}_p)$ for a subfamily of these semidirect products and discuss some partial results and conjectures for a more general case.

In Chapter 5, we will construct a pair of Hopf algebras using compatible families of supercharacter theories of the alternating and finite special linear groups which are induced by the natural actions of the symmetric and finite general linear groups, respectively. In each construction (one being a rough q-analogue of the other), we obtain supercharacters indexed by equivalence classes of familiar combinatorial objects and with compatible super-induction and super-restriction functors. Finally, we relate these Hopf algebras explicitly using the theory of combinatorial Hopf algebras as introduced by Aguiar, Bergeron, and Sottile in [ABS06].

Chapter 2

Preliminaries

In this chapter, we will develop background material on character theory of finite groups, supercharacter theories and finite posets and lattices.

2.1 The Artin–Wedderburn theorem

Let k be an algebraically closed field and let A be a k-vector space. Suppose A has the additional structure of a unital ring A such that for all $a, b \in A$ and all $z \in k$, the condition

$$z(a \cdot b) = (za) \cdot b = a \cdot (zb)$$

is met. Then A is called a k**-algebra**. We say A is **finite-dimensional** if it is finite-dimensional as a vector space. For the remainder of this chapter, A will always refer to a finite-dimensional k-algebra unless otherwise specified. While many of these results hold in greater generality, this thesis is only concerned with this case.

Let A be a k-algebra and let M be a finite-dimensional k-vector space equipped with an action of A on M, i.e., for each $a \in A$, there is a k-vector space automorphism $\rho_a : M \to M$ subject to the following compatibility conditions for all $a, b \in A$, $m \in M$, and $z \in k$:

$$(\rho_a + \rho_b)(m) = \rho_a(m) + \rho_b(m);$$

$$\rho_{za}(m) = z\rho_a(m);$$

$$\rho_{a \cdot b}(m) = \rho_a(\rho_b(m)).$$

Such an object M is called an A**-module**. We generally write $a \cdot m$ in place of $\rho_a(m)$. If M and N are two A-modules, then an A**-module homomorphism** is a k-vector space transformation $\varphi : M \to N$ with the additional property that for all $a \in A$ and $m \in M$,

$$\varphi(a \cdot m) = a \cdot \varphi(m).$$

Let $\mathrm{Hom}_A(M, N)$ denote the set of all A-module homomorphisms from M to N. Note that this is not simply the set of all k-vector space transformations. That latter set is denoted $\mathrm{Hom}_k(M, N)$; we always have containment $\mathrm{Hom}_A(M, N) \subseteq \mathrm{Hom}_k(M, N)$.

We will call an A-module M **irreducible** or **simple** if it has no proper nontrivial submodules. We will call an algebra A **semisimple** if every (left or right) A-module M decomposes as a direct sum of simple modules.

Lemma 2.1 (Schur's lemma)**.** *Let A be a k-algebra. If V and W are irreducible A-modules, then any nonzero element in $\mathrm{Hom}_A(V, W)$ has an inverse in $\mathrm{Hom}_A(W, V)$. In particular if k is algebraically closed, then $\mathrm{Hom}_A(V, V) = k \cdot \mathrm{id}_V$ is the set of all scalar multiplications on V.*

Theorem 2.2 (Artin–Wedderburn)**.** *Let A be a finite-dimensional k-algebra, where k is an algebraically closed field. Then A is semisimple if and only if A is isomorphic to a direct product of matrix rings*

$$A \cong M_{n_1}(k) \times \cdots \times M_{n_s}(k)$$

for a unique set of integers $n_1, \ldots, n_s$.

If A acts on itself by left (respectively right) multiplication, the resulting module is called the left (respectively right) **regular** A-module.

An element $a \in A$ with the property $a \cdot a = a$ is called **idempotent**. If a is an idempotent that lies in the center $Z(A)$, then a is called a **central idempotent**. Finally, a **primitive idempotent** $a \in A$ is an idempotent with the property that if a can be written as a sum of idempotents a_1 and a_2, i.e., $a = a_1 + a_2$, then one of a_1 or a_2 is equal to a and the other is equal to the zero element $0 \in A$.

Corollary 2.3. *[CR90, Theorem 3.22] Let A be a finite-dimensional k-algebra. Then A is semisimple if and only if A can be decomposed as a direct sum of ideals*

$$A = A_1 \oplus \cdots \oplus A_s,$$

with each A_i of the form $A \cdot e_i$, where e_i is a primitive central idempotent. Moreover, each A_i is a ring with identity e_i and A is isomorphic as a ring to the direct product of the rings A_i.

The following corollary of the Artin–Wedderburn Theorem is most useful to this chapter.

Corollary 2.4. *[Isa76, Corollary 1.17] Let A be a finite-dimensional semisimple algebra over an algebraically closed field and suppose $A = M_{n_1}(k) \times \cdots \times M_{n_s}(k)$ is a direct product of s matrix rings, as in Theorem 2.2. Then*

(1) A has exactly s isomorphism classes of simple modules, and representatives $M_1, \dots, M_s$ may be chosen so that $\dim(M_i) = n_i$ for all i.

(2) $s = \dim(Z(A))$;

(3) $n_1^2 + \cdots + n_s^2 = \dim(A)$;

(4) the left regular A-module A decomposes as a direct sum

$$A \cong \bigoplus_{i=1}^{s} M_i^{\oplus \dim(M_i)}.$$

2.2 Character theory of finite groups

Now we turn our attention to group algebras. If k is a field and G is a finite group, the group algebra kG is defined to be the set of formal sums of the form

$$\sum_{g \in G} a_g g,$$

where each a_g is an element of k, and with multiplication defined by the formula

$$\left(\sum_{g \in G} a_g g\right) \cdot \left(\sum_{h \in G} b_h h\right) = \sum_{g \in G} a_g b_h g h = \sum_{g \in G} \left(\sum_{h \in G} a_h b_{h^{-1}g}\right) g.$$

There is another product, known as the **Hadamard product**, which is defined by the formula

$$\left(\sum_{g\in G} a_g g\right) \cdot \left(\sum_{h\in G} b_h h\right) = \sum_{g,h\in G} a_g b_g g.$$

Theorem 2.5 (Maschke)**.** *Let G be a finite group, k be a field whose characteristic does not divide the order of G, and let V be a kG-module with kG-submodule U. Then there exists a kG-submodule W of V such that $V = U \oplus W$.*

Corollary 2.6. *If G is a finite group, the group algebra $\mathbb{C}G$ is semisimple.*

For the remainder of this thesis, G will always be a finite group. A (complex) **representation** of G is a group homomorphism of the form $\rho : G \to \mathrm{GL}(V)$, where V is a vector space over $\mathbb{C}$. While one can study representations and characters over other fields, this thesis is only concerned with the complex setting. Hence, when we speak of a representation, we will always be referring to a complex representation unless otherwise specified. Two representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(V)$ of G are said to be **equivalent** if there is a vector space isomorphism $\varphi : V \to W$ such that $\varphi \circ \rho(g) = \tau(g) \circ \varphi$ for all $g \in G$.

Let G be a finite group and let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. Then V is a $\mathbb{C}G$-module with action defined by

$$\left(\sum_{g\in G} a_g g\right) \cdot v = \sum_{g\in G} a_g \rho(g)(v)$$

for all $v \in V$ and $\sum_g a_g g \in \mathbb{C}G$. Conversely, if M is a $\mathbb{C}G$-module, then we can define a representation $\rho : G \to \mathrm{GL}(M)$ by the rule

$$\rho(g)(m) = g \cdot m.$$

It is easy to show that this defines a representation, and that these two identifications are inverse to each other. Thus, equivalence classes of representations of G are in one-to-one correspondence with isomoprhism classes of $\mathbb{C}G$-modules.

A representation $\rho : G \to \mathrm{GL}(V)$ is called **irreducible** if the corresponding $\mathbb{C}G$-module V is irreducible. Two representations $\rho : G \to \mathrm{GL}(V)$ and $\sigma : G \to \mathrm{GL}(W)$ are called **isomorphic** if their corresponding $\mathbb{C}G$-modules V and W are isomorphic, or equivalently, if there exists a vector space isomorphism $\varphi : V \to W$ such that $\varphi^{-1}\sigma(g)\varphi = \rho(g)$ for all $g \in G$.

Let $\rho : G \to \mathrm{GL}(V)$ be a representation and let $\mathrm{tr} : \mathrm{GL}(V) \to \mathbb{C}$ denote the familiar trace function. The composition $\mathrm{tr} \circ \rho : G \to \mathbb{C}$, is called the **character afforded by** ρ, and is often denoted χ_ρ or χ_V.

Proposition 2.7. *[Isa76, Corollary 2.9] Two $\mathbb{C}$-representations ρ and σ of G are isomorphic if and only if the corresponding characters χ_ρ and χ_σ are equal.*

Thus, each character corresponds to an isomorphism class of representations. In this way, the character theory of a group captures the essential information of the representation theory of that group.

A character χ of G is called **irreducible** if it is afforded by an irreducible representation. Let $\mathrm{Irr}(G)$ denote the set of all irreducible characters of G and let $\mathrm{Cl}(G)$ denote the set of conjugacy classes of G. A complex-valued function $f : G \to \mathbb{C}$ is called a **class function** if $f(hgh^{-1}) = f(g)$ for all $g, h \in G$. The $\mathbb{C}$-vector space of all class functions of G is denoted $\mathrm{cf}(G)$. For each conjugacy class $K \in \mathrm{Cl}(G)$, define the **conjugacy class identifier** to be the function $\delta_K : G \to \mathbb{C}$ given by

$$\delta_K(g) = \begin{cases} 1 & : \quad g \in K \\ 0 & : \quad g \notin K \end{cases}.$$

Evidently, the conjugacy class identifier functions defined above form a natural basis for $\mathrm{cf}(G)$. Hence, $\dim(\mathrm{cf}(G)) = \#\mathrm{Cl}(G)$.

Proposition 2.8. *Let χ be a character of G. Then χ is a class function.*

Proof. Let $g, h \in G$ and let ρ be a representation that affords χ. Then

$$\begin{aligned}\chi(h^{-1}gh) &= \mathrm{tr}(\rho(h^{-1}gh))\\ &= \mathrm{tr}(\rho(h^{1}g)\rho(h))\\ &= \mathrm{tr}(\rho(h)\rho(h^{-1}g))\\ &= \mathrm{tr}(\rho(g))\\ &= \chi(g).\end{aligned}$$

Therefore, χ is constant on the conjugacy classes of G. ∎

We can define two products on the space of class functions. The **pointwise product** of two class functions f and g is the function $f \cdot g$ given by

$$(f \cdot g)(x) = f(x)g(x).$$

The **convolution product** of two class functions f and g is the function $f * g$ given by

$$(f * g)(x) = \frac{1}{|G|} \sum_{y \in G} f(y)g(y^{-1}x).$$

We may define an inner product on $\mathrm{cf}(G)$ by the following formula:

$$\langle f, g \rangle = \frac{1}{|G|} \sum_{x \in G} f(x)\overline{g(x)},$$

where the bar over $g(x)$ denotes complex conjugation. The conjugacy class identifier functions are clearly orthogonal with respect to this inner product.

Proposition 2.9. *[CR90, Proposition 9.24] If χ and ψ are characters of G afforded by the $\mathbb{C}G$-modules V and W respectively, then $\langle \chi, \psi \rangle = \dim(\mathrm{Hom}_{\mathbb{C}G}(V, W))$.*

Corollary 2.10. *The irreducible characters are linearly independent and orthonormal with respect to the above inner product.*

Proof. Orthonormality follows from Lemma 2.1. Suppose

$$\sum_{\chi \in \mathrm{Irr}(G)} c_\chi \chi = 0$$

for some constants $c_\chi \in \mathbb{C}$. Then for any $\psi \in \mathrm{Irr}(G)$, we have

$$c_\psi = \left\langle \sum_{\chi \in \mathrm{Irr}(G)} c_\chi \chi, \psi \right\rangle = 0,$$

hence the irreducible characters are linearly independent. ■

Proposition 2.11. *The irreducible characters form a basis of* $\mathrm{cf}(G)$*.*

Proof. This follows from Corollaries 2.10 and 2.4. ■

Let n be the number of conjugacy classes of G, let $g_1, g_2, \ldots, g_n$ be representatives for the conjugacy classes, and label the irreducible characters $\chi_1, \chi_2, \ldots, \chi_n$. The **character table** of a finite group is the $n \times n$ matrix whose (i, j)-entry is $\chi_i(g_j)$.

Theorem 2.12 (Orthogonality Relations)**.** *[Isa76, Corollary 2.14ff] For all $h \in G$, we have*

$$\frac{1}{|G|} \sum_{g \in G} \chi_i(gh)\chi_j(h^{-1}) = \delta_{i,j} \frac{\chi_i(h)}{\chi_i(e)}, \tag{2.1}$$

where $\delta_{i,j}$ is the Kronecker delta and e denotes the identity of G. For all $g, h \in G$, we have

$$\frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\overline{\chi(h)} = \delta_{g,h} \cdot |C_G(g)|. \tag{2.2}$$

It is easy to see that the conjugacy class identifiers form a basis of primitive central idempotents with respect to the pointwise product on $\mathrm{cf}(G)$. Thus, $\mathrm{cf}(G)$ is a commutative semisimple $\mathbb{C}$-algebra with respect to this product. One of the consequences of the above theorem is that the scaled irreducible characters $\{\chi(e)\chi \,:\, \chi \in \mathrm{Irr}(G)\}$ form a basis of primitive central idempotents with respect to the convolution product on $\mathrm{cf}(G)$, and therefore $\mathrm{cf}(G)$ is a commutative[1] semisimple $\mathbb{C}$-algebra with respect to this product as well.

Note there is a one-to-one correspondence between complex-valued functions on G and elements of $\mathbb{C}G$: if $f : G \to \mathbb{C}$ is a function, then we can associate f with the element $\sum_{g \in G} f(g)g \in$

[1] While it isn't immediate that the convolution product is commutative here, but it follows because the functions are class functions.

$\mathbb{C}G$. Under this identification, the convolution product of functions corresponds to the usual product of elements of $\mathbb{C}G$, the pointwise product corresponds to the Hadamard product of elements of $\mathbb{C}G$, and $\mathrm{cf}(G)$ corresponds to $Z(\mathbb{C}G)$.

There are several canonical ways of constructing new characters from old, all of which will appear later. Let G be a finite group, let H be a subgroup of G, and let χ be a character of H. The **induced character**, $\mathrm{Ind}_H^G(\chi)$,[2] is defined by the following formula: for $g \in G$,

$$\mathrm{Ind}_H^G(\chi)(g) = \sum_{t \in T} \chi^0(t^{-1}gt) = \frac{1}{|H|} \sum_{x \in G} \chi^0(x^{-1}gx), \tag{2.3}$$

where T is a set of left coset representatives for H in G and χ^0 is the function that agrees with χ on H and vanishes on $G \smallsetminus H$. An equivalent definition of the induced character is the following: if χ is the character of H afforded by the $\mathbb{C}H$-module V, then $\mathrm{Ind}_H^G(\chi)$ is the character of G afforded by the $\mathbb{C}G$-module $\mathbb{C}G \otimes_{\mathbb{C}H} V$.

Let G be a finite group, let H be a subgroup of G, and let χ be a character of G. The **restricted character** $\mathrm{Res}_H^G(\chi)$ is the usual restriction of χ to H.[3] That $\mathrm{Res}_H^G(\chi)$ is a class function follows from the observation that every conjugacy class of H is a subset of a conjugacy class of G. That $\mathrm{Res}_H^G(\chi)$ is a character of H is a consequence of the following result (and the fact that characters of a group are precisely the class functions that are nonnegative integer linear combinations of irreducible characters).

Proposition 2.13 (Frobenius Reciprocity). *Let G be a finite group, let H be a subgroup of G, and let χ and ψ be characters of G and H, respectively. Then*

$$\langle \chi, \mathrm{Ind}_H^G(\psi) \rangle_G = \langle \mathrm{Res}_H^G(\chi), \psi \rangle_H.$$

Let G and H be finite groups, let $\pi : G \to H$ be a surjective homomorphism, and let ψ be a character of H. The **inflated character** $\mathrm{Inf}_H^G(\psi)$ is the character defined by the composition $\psi \circ \pi$. We will usually see inflation in the context of normal subgroups and quotients. If G is a finite

[2] We will sometimes write $\mathrm{Ind}(\chi)$ if there is no chance of confusion, especially in Chapter 5. Some authors will also write χ^G if H is clear from context.

[3] or $\mathrm{Res}(\chi)$ if there is no chance of confusion

group with normal subgroup K, and π is the canonical projection $G \to G/K$, then the inflated character $\mathrm{Inf}_{G/K}^{G}(\psi)$ has the formula

$$\mathrm{Inf}_{G/K}^{G}(\psi)(g) = \psi(gK).$$

If, on the other hand, χ is a character of G, then we may form the **deflated character**, $\mathrm{Def}_{G/K}^{G}(\chi)$, by averaging over cosets; i.e., $\mathrm{Def}_{G/K}^{G}(\chi)$ is defined by the formula

$$\mathrm{Def}_{G/K}^{G}(\chi)(gK) = \frac{1}{|K|}\sum_{k\in K}\chi(gk).$$

It is a straightforward calculation to show that if $f \in \mathrm{cf}(G)$, then $\mathrm{Def}_{G/K}^{G}(f) \in \mathrm{cf}(G/K)$. By the following result, these operations are adjoint in a manner similar to that of Proposition 2.13. This result (together with the fact that characters are the class functions that are nonnegative integer linear combinations of irreducible characters) also imples that $\mathrm{Def}_{G/K}^{G}(\chi)$ is a character of G/K.

Proposition 2.14 (Reciprocity for Inflation and Deflation)**.** *Let G be a finite group with normal subgroup K, and let χ and ψ be characters of G and G/K, respectively. Then*

$$\langle \chi, \mathrm{Inf}_{G/K}^{G}(\psi)\rangle_G = \langle \mathrm{Def}_{G/K}^{G}(\chi), \psi\rangle_{G/K}.$$

2.3 Supercharacter theories of finite groups

Let G be a finite group. Let X be a subset of $\mathrm{Irr}(G)$. The **Wedderburn sum** associated to X is the character

$$\sigma_X = \sum_{\chi\in X}\chi(e)\chi.$$

A **supercharacter theory** of G is an ordered pair $S = (\mathcal{K}, \mathcal{X})$, where $\mathcal{K}$ is a partition of G into unions of conjugacy classes, $\mathcal{X}$ is a partition of $\mathrm{Irr}(G)$, and such that the following conditions are met:

(1) $|\mathcal{K}| = |\mathcal{X}|$;

(2) $\{e\} \in \mathcal{K}$;

(3) for each $X \in \mathcal{X}$, the Wedderburn sum σ_X is constant on the parts of $\mathcal{K}$.

If $S = (\mathcal{K}, \mathcal{X})$ is a supercharacter theory, then we call the parts of $\mathcal{K}$ the **superclasses** of S, or S-**superclasses**, and we call the the functions σ_X the **supercharacters of** S, or S-**supercharacters**. By [DI08, Lemma 2.1], any class function that is constant on the parts of $\mathcal{K}$ is a linear combination of S-supercharacters. Let $|S|$ denote the number of superclasses (equivalently the number of supercharacters) of S. Let $\mathrm{SCT}(G)$ denote the set of all supercharacter theories of G.

Remark. (1) By [DI08, Theorem 2.2(c)], the partitions $\mathcal{K}$ and $\mathcal{X}$ uniquely determine each other; in particular, $\mathcal{K}$ is the unique coarsest partition of G on whose parts the Wedderburn sums are constant.

(2) We will very often need to refer to the superclass and supercharacter partitions of several supercharacter theories at a time. Rather than name every partition in question, we will use the shorthand $\mathcal{K}(S)$ and $\mathcal{X}(S)$ to denote the superclass and supercharacter partitions, respectively, of the supercharacter theory S.

For any group G, let $\mathbf{1}_G$ denote the trivial character of G. We can define two extreme supercharacter theories, which are (using the notation of [Hen08])

$$m(G) = \big(\mathrm{Cl}(G), \big\{\{\chi\} : \chi \in \mathrm{Irr}(G)\big\}\big),$$

and

$$M(G) = \big(\big\{\{e\}, G \smallsetminus \{e\}\big\}, \big\{\{\mathbf{1}_G\}, \mathrm{Irr}(G) \smallsetminus \{\mathbf{1}_G\}\big\}\big).$$

These supercharacter theories are distinct for all groups G of order greater than 2. In [BLLW17], it is shown that the only groups for which these are all of the supercharacter theories are the cyclic group of order 3, the symmetric group S_3, and the finite symplectic group $\mathrm{Sp}(6,2)$ (see Chapter 3 for a summary of that result). Note that the supercharacters of $m(G)$ are simply the scaled irreducible characters $\chi(e)\chi$ for $\chi \in \mathrm{Irr}(G)$. This is the reason that supercharacter theories can be thought of as generalizations (coarsenings?) of the character theory of G.

For an algebraic interpretation of $\mathrm{SCT}(G)$, consider the following. To every supercharacter theory $S = (\mathcal{K}, \mathcal{X})$, there is an associated algebra $\mathrm{scf}_S(G)$ of **superclass functions**, i.e., complex-valued functions on G that are constant on the parts of $\mathcal{K}$. One basis for $\mathrm{scf}_S(G)$ is the set of **superclass identifier functions**, which are the functions $\{\delta_K : K \in \mathcal{K}(S)\}$ given by the formula

$$\delta_K(g) = \begin{cases} 1 & : \quad g \in K \\ 0 & : \quad g \notin K \end{cases}$$

for $K \in \mathcal{K}(S)$. These are primitive central idempotents with respect to the pointwise product. Since the supercharacters are linearly independent and $|\mathcal{X}| = |\mathcal{K}|$, it follows that the supercharacters also form a basis for $\mathrm{scf}_S(G)$. It follows by [DI08, Theorem 2.2(b)] that (suitably scaled) supercharacters are idempotents with respect to the convolution product. Thus $\mathrm{scf}_S(G)$ forms a commutative semisimple subalgebra of $\mathrm{cf}(G)$ with respect to both product structures.

Let S be a supercharacter theory of G. Let $\mathcal{K} = \{K_1, \ldots, K_n\}$ and $\mathcal{X} = \{X_1, \ldots, X_n\}$ be an ordering on the superclasses and supercharacters, respectively. Let $g_1, \ldots, g_n$ be representatives for the superclasses of S. The **supercharacter table** associated to S is the $n \times n$ matrix whose (i, j)-entry is $\sigma_{X_i}(g_j)$. Then we may generalize the first orthogonality relation (2.1) in the following manner.

Proposition 2.15. *For all $h \in G$, we have*

$$\frac{1}{|G|} \sum_{g \in G} \sigma_{X_i}(gh)\sigma_{X_i}(h^{-1}) = \delta_{i,j}\sigma_{X_i}(h). \tag{2.4}$$

In a direct analogy to the previous section, it is easy to see that the superclass identifiers form a basis of primitive central idempotents with respect to the pointwise product on $\mathrm{scf}_S(G)$. Thus, $\mathrm{scf}_S(G)$ is a commutative semisimple $\mathbb{C}$-algebra with respect to this product. The main consequence of (2.4) is that the Wedderburn sums for S form a basis of primitive central idempotents with respect to the convolution product on $\mathrm{scf}_S(G)$, and therefore $\mathrm{scf}_S(G)$ is a commutative semisimple $\mathbb{C}$-algebra with respect to this product as well. Therefore, one can define a supercharacter theory as a subalgebra of $\mathrm{cf}(G)$ (equivalently, $\mathbb{Z}(\mathbb{C}G)$) that is closed with respect to both products defined on this algebra (cf. Section 1 of [And14]). Such objects are related to Schur rings (see [Wie64], [LM96], [LM98]). Briefly, we call a subring S of $\mathbb{C}G$ a **Schur ring** over G if there exists a partition $\{D_1, \ldots, D_t\}$ of G such that

(1) S is spanned by the sums $\sum_{x \in D_i} x$ for $1 \leq i \leq t$, and

(2) if we define $D_i^{-1} = \{x^{-1} : x \in D_i\}$ for all $1 \leq i \leq t$, then for all i, there exists some $1 \leq j \leq t$ such that $D_i^{(-1)} = D_j$.

A **central Schur ring** is a Schur ring over G that lies in $Z(\mathbb{C}G)$. Then (see [Hen12, Proposition 2.4]), supercharacter theories are in bijection with central Schur rings over G.

2.3.1 New supercharacter theories from old

Let G be a finite group with normal subgroup N and let S be a supercharacter theory of N. If the conjugation action of G on N fixes each superclass (equivalently, this action fixes each supercharacter), then we call S a G**-invariant** supercharacter theory, and we write $\mathrm{InvSCT}_G(N)$ for the set of all G-invariant supercharacter theories of N.[4] There is a unique minimal element of $\mathrm{InvSCT}_G(N)$, denoted $m_G(N)$, whose superclass and supercharacter partitions are the orbits of the actions of G on N and $\mathrm{Irr}(N)$, respectively. Hence, the G-invariant supercharacter theories are simply those whose superclass partitions consist of unions of G-conjugacy classes and whose supercharacters are (up to scalar multiplication) sums of restrictions of irreducible characters of G.

2.3.1.1 Restrictions and deflations

If $S = (\mathcal{K}, \mathcal{X})$ is a supercharacter theory of a finite group G and N is a subgroup, we call N S**-normal** if it is a union of S-superclasses (or equivalently, if $\mathrm{Irr}(G/N)$ is a union of parts of $\mathcal{X}$). In this setting, we can introduce two new supercharacter theories, which are defined in [Hen08], as follows. The first is the **restricted supercharacter theory** $S_N \in \mathrm{SCT}(N)$, whose superclasses

[4] This subset (later we will see that it is a sublattice) of $\mathrm{SCT}(N)$ was defined by Hendrickson in [Hen08], where it is denoted $\mathrm{SCT}_G(N)$.

are merely those that lie in N:

$$S_N = \Big(\big\{K \in \mathcal{K} \,:\, K \subseteq N\big\}, \big\{X_N \,:\, X \in \mathcal{X},\, X \not\subseteq \mathrm{Irr}(G/N)\big\} \cup \big\{\{\mathbf{1}_G\}\big\}\Big), \tag{2.5}$$

where X_N denotes the set of irreducible constituents of $\mathrm{Res}_N^G(\sigma_X)$. The second is the **deflated supercharacter theory** $S^{G/N} \in \mathrm{SCT}(G/N)$, whose supercharacter blocks are merely those which lie in $\mathrm{Irr}(G/N)$:

$$\begin{aligned} S^{G/N} = \Big(&\big\{\{N\}\big\} \cup \big\{\{gN \,:\, g \in K\} \,:\, K \in \mathcal{K},\, K \not\subseteq N\big\}, \\ &\big\{X \in \mathcal{X} \,:\, X \subseteq \mathrm{Irr}(G/N)\big\}\Big). \end{aligned} \tag{2.6}$$

2.3.1.2 Products

In this section, we will discuss three different constructions for products of two supercharacter theories: the direct product, the $*$-product, and the Δ-product. While only the $*$-product will be used in the classification of $\mathrm{SCT}(D_{2n})$, the others are necessary to state the classification of the supercharacter theories of cyclic groups.

The first and simplest product construction is the **direct product**. If H and K are two finite groups, $S = (\mathcal{K}, \mathcal{X}) \in \mathrm{SCT}(H)$ and $T = (\mathcal{L}, \mathcal{Y}) \in \mathrm{SCT}(K)$, then the direct product of S and T, denoted $S \times T$, is the supercharacter theory of $H \times K$ with superclass partition

$$\mathcal{K} \times \mathcal{L} = \{K \times L \,:\, K \in \mathcal{K},\, L \in \mathcal{L}\}$$

and supercharacter partition

$$\mathcal{X} \times \mathcal{Y} = \{X \times Y \,:\, X \in \mathcal{X},\, Y \in \mathcal{Y}\}.$$

It is an easy calculation that $S \times T$ is a valid supercharacter theory of $H \times K$.

Next, let G be a finite group with normal subgroup N. Then G acts on N via automorphisms and the partition of N into G-orbits is the superclass partition of the minimal G-invariant supercharacter theory $m_G(N)$ of N. The superclasses of $m_G(N)$ are simply the G-conjugacy classes that are subsets of N. Thus, G-invariant supercharacter theories of N are those whose superclasses are unions of G-conjugacy classes. Consequently, the most naïve putative method of combining a G-invariant supercharacter theory S of N with a supercharacter theory T of G/N could be to pull back the nonidentity superclasses of T and combine them with the superclasses of S. It is perhaps surprising that this method does in fact produce a supercharacter theory of G, which Hendrickson calls the **$*$-product** of S and T, and denotes $S *_N T$.

Proposition 2.16. *[Hen08, Theorem 3.5] Let $S = (\mathcal{K}, \mathcal{X})$ be a G-invariant supercharacter theory of N and $T = (\mathcal{L}, \mathcal{Y})$ be a supercharacter theory of G/N. Then there is a supercharacter theory of G given by $S *_N T = (\mathcal{M}, \mathcal{Z})$, whose superclass partition is*

$$\mathcal{M} = \mathcal{K} \cup \left\{ \bigcup_{gN \in L} gN \,:\, L \in \mathcal{L} \smallsetminus \{\{eN\}\} \right\} \tag{2.7}$$

and whose supercharacter partition is

$$\mathcal{Z} = \left\{ X^G \,:\, X \in \mathcal{X} \smallsetminus \{\mathbf{1}_N\} \right\} \cup \mathcal{Y}, \tag{2.8}$$

where $X^G = \{\mathrm{Irr}(G|\chi) : \chi \in X\}$ and the elements of $\mathrm{Irr}(G/N)$ are identified as elements of $\mathrm{Irr}(G)$ through inflation.

Let G be a finite group and let N be a normal subgroup. We say a supercharacter theory of G **factors over** N if it can be written as a $*$-product of a G-invariant supercharacter theory of N with a supercharacter theory of G/N. The unique maximal supercharacter theory of G that factors over N is $M(N) *_N M(G/N)$ and we denote this supercharacter theory by $MM_N(G)$. The unique minimal supercharacter theory of G that factors over N is $m_G(N) *_N m(G/N)$ and we denote this supercharacter theory by $mm_N(G)$. Finally, there is a construction in [Hen08] known as the Δ-product, which we will summarize here.

Proposition 2.17. *[Hen08, Theorem 4.1] Let G be a finite group with normal subgroups $N \lhd M \lhd G$, let $S = (\mathcal{K}, \mathcal{X}) \in \mathrm{InvSCT}_G(M)$ and let $T = (\mathcal{L}, \mathcal{Y}) \in \mathrm{SCT}(G/N)$. Suppose*

(a) N is S-normal,

(b) M/N is T-normal, and

(c) $S^{M/N} = T_{M/N}$.

Then we can form a supercharacter theory of G *called the* Δ*-product of* S *and* T*, denoted* $S \,\Delta\, T = (\mathcal{M}, \mathcal{Z})$*, whose superclass partition is*

$$\mathcal{M} = \mathcal{K} \cup \Big\{ \bigcup_{gN \in L} gN \,:\, L \in \mathcal{L},\, L \not\subseteq M/N \Big\}, \tag{2.9}$$

and whose supercharacter partition is

$$\mathcal{Z} = \mathcal{Y} \cup \big\{ X^G \,:\, X \in \mathcal{X},\, X \not\subseteq \operatorname{Irr}(M/N) \big\}, \tag{2.10}$$

where X^G *denotes the set of constituents of* $\operatorname{Ind}_M^G(\sigma_X)$*.*

One can check that if $M = N$, then the Δ-product reduces to the $*$-product.

2.4 Finite posets and lattices

We will now summarize some basic facts about finite posets and lattices. For a thorough reference, see [Sta02, Chapter 3].

2.4.1 Posets

A **partially ordered set**, or **poset**, is an ordered pair $(P, \leq)$, where P is a set and $\leq$ is a partial order on P, i.e., a relation on P satisfying the following properties:

(1) (reflexivity) for all $s \in P$, $s \leq s$;

(2) (antisymmetry) for all $s, t \in P$, if $s \leq t$ and $t \leq s$, then $s = t$;

(3) (transitivity) for all $s, t, u \in P$, if $s \leq t$ and $t \leq u$, then $s \leq u$.

We will often write $x < y$ if x and y are elements of a poset P with $x \leq y$ and $x \neq y$. If the partial order is clear form context, we will generally refer to a poset $(P, \leq)$ by its underlying set P. In this thesis, we will primarily be concerned with finite posets (i.e., posets whose underlying sets are finite).

Two elements x and y of a poset P are called **comparable** if $x \leq y$ or $y \leq x$. Otherwise, we say x and y are **incomparable**. Let P be a poset and let S be a subset of P. The **upper ideal** generated by S is the subposet $\{x \in P : x \geq s \text{ for some } s \in S\}$. The **lower ideal** generated by S is the subposet $\{x \in P : x \leq s \text{ for some } s \in S\}$.

Let $(S, \leq_S)$ and $(T, \leq_T)$ be posets. A function $f : S \to T$ is **order-preserving** if $s \leq_S t$ implies $f(s) \leq_T f(t)$. Two posets S and T are said to be **isomorphic** if there exist order-preserving functions $f : S \to T$ and $g : T \to S$ such that $f \circ g = \mathrm{id}_T$ and $g \circ f = \mathrm{id}_S$.

If $(P, \leq)$ is a poset and Q is a subset of the underlying set P, we can give Q the structure of a poset by restricting the partial order $\leq$ to Q. The subset Q is then called a **subposet** of P, and the inclusion function $i : Q \to P$ is an order-preserving injection.

If $(P, \leq)$ is a poset and $x, y \in P$, we say y **covers** x if $x < y$ and for all $z \in P$, $x \leq z \leq y$ implies $z \in \{x, y\}$. The relation $x < y$ is called a **covering relation**. Given the data of a poset P and its covering relations, we may form a directed graph (V, E), known as the **Hasse diagram** of P, as follows. The vertices are the elements of the underlying set of P, and there is an edge from x to y if y covers x.

Example 2.18. Let $P = \{x \in \mathbb{Z} : x \mid 12\}$ be the poset of divisors of 12, with partial order given by divisibility, i.e., we say $x \leq y$ if y is divisible[5] by x. The Hasse diagram of P is shown in Figure 2.1.

Let S and T be posets. We can give the cartesian product $S \times T$ the structure of a poset as follows: we define $(s_1, t_1) \leq (s_2, t_2)$ if $s_1 \leq s_2$ in S and $t_1 \leq t_2$ in T.

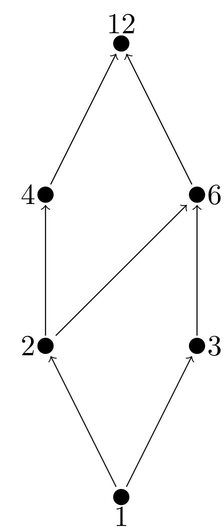
Let S be a poset and let $s, t \in S$. We define the interval $[s, t]$ to be the set of all $u \in S$ such that $s \leq u$ and $u \leq t$. The intervals $(s, t]$, $[s, t)$, and (s, t) are defined analogously.

2.4.2 Lattices

Let S be a finite poset and let T be a subset of S. An element $u \in S$ is called an **upper bound** for T if $t \leq u$ for all $t \in T$. If s is an upper bound for T and $s \leq u$ for all upper bounds

[5] In [Sta02, Chapter 3], the poset of divisors of n is denoted D_n.

Figure 2.1: The Hasse diagram of divisors of 12



u of T, then s is called the **supremum** (or least upper bound) of T and is denoted $s = \sup(T)$. Evidently, each T has at most one supremum.

Similarly, an element $l \in S$ is called a **lower bound** for T if $l \leq t$ for all $l \in T$. If a point s exists which is a lower bound for T, and $l \leq s$ for all lower bounds u of T (i.e., s is the greatest lower bound), then s is called the **infimum** of T and is denoted $s = \inf(T)$. Evidently, for each set T, $\inf(T)$ is unique, if it exists.

Now, let $s, t \in S$. We define the **meet** of s and t, denoted $s \wedge t$, to be the infimum of the set $\{s, t\}$, if it exists. We define the **join** of s and t, denoted $s \vee t$, to be the supremum of the set $\{s, t\}$, if it exists. A poset L is called a **lattice** if $s \wedge t$ and $s \vee t$ exist for all $s, t \in L$. Note that every finite lattice L has a unique top (respectively bottom) element, which is obtained by taking the supremum (respectively infimum) of the full lattice L.

Let L be a finite lattice and let 0 and 1 be the bottom and top elements of L, respectively. An element $x \in L$ that covers 0 is called an **atom** of L and an element $x \in L$ that is covered by 1 is called a **co-atom**. For any $z \in L \setminus \{0, 1\}$, there exists an atom x and a co-atom y (neither of which are necessarily unique) such that $x \leq z \leq y$.

A poset P with the property that $s \wedge t$ (respectively $s \vee t$) exists for all $s, t \in P$ is called a **meet semilattice** (respectively **join semilattice**).

Lemma 2.19. *[Sta02, Proposition 3.3.1] Let P be a finite meet (respectively join) semilattice with a top (respectively bottom) element. Then P is a lattice.*

Proposition 2.20. *[Sta02] Let L be a lattice. Then the following properties hold for all $s, t \in L$:*

(1) The operations $\vee$ and $\wedge$ are associative, commutative, and idempotent;

(2) (absorption laws) $s \wedge (s \vee t) = s = s \vee (s \wedge t)$;

(3) $s \wedge t = s \Leftrightarrow s \vee t = t \Leftrightarrow s \leq t$.

Let L and M be lattices. A **lattice homomorphism**, or **lattice map**, from L to M is an order-preserving function $f : L \to M$ with the additional properties that $f(x \wedge y) = f(x) \wedge f(y)$

and $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L$.

Proposition 2.21. *If L and M are finite lattices, then the product $L \times M$ is a lattice with respect to the product partial order. The meet and join operations are defined by*

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2) \text{ and } (x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2).$$

Example 2.22 (Refinement partial order on partitions)**.** Let S be a finite set and let $\mathcal{P}$ and $\mathcal{Q}$ be partitions of S. We say $\mathcal{P}$ is a **refinement** of $\mathcal{Q}$ (or equivalently, $\mathcal{Q}$ is a **coarsening** of $\mathcal{P}$) and write $\mathcal{P} \leq \mathcal{Q}$ if each part of $\mathcal{P}$ is a subset of a part of $\mathcal{Q}$, or equivalently, if each part of $\mathcal{Q}$ is a union of parts of $\mathcal{P}$. The set of all partitions of S forms a lattice with respect to this relation. If S contains n elements, then this poset is commonly referred to as Π_n (see [Sta02, Example 3.1.1(d)]).

If $\mathcal{P}$ and $\mathcal{Q}$ are partitions of S, then their meet and join can be described explicitly. Conveniently, the parts of $\mathcal{P} \wedge \mathcal{Q}$ are simply the intersections $P \cap Q$ for $P \in \mathcal{P}$ and $Q \in \mathcal{Q}$, excluding those intersections that are empty. The parts of $\mathcal{P} \vee \mathcal{Q}$ can be described as follows. If $x \in S$, then there is a unique part P of $\mathcal{P}$ containing x. Let $R_0 = P$, for $k > 0$, let

$$R_{2k+1} = \bigcup_{\substack{Q \in \mathcal{Q} \\ Q \cap R_{2k} \neq \emptyset}} Q,$$

and for $k > 1$, let

$$R_{2k} = \bigcup_{\substack{P \in \mathcal{P} \\ P \cap R_{2k-1} \neq \emptyset}} P.$$

Then $R_0, R_1, \ldots$ is an increasing sequence of subsets of S and therefore stabilizes eventually. This limit is the part of $\mathcal{P} \vee \mathcal{Q}$ containing x.

2.5 Hopf algebras and the ring of symmetric functions

In this section, we will provide all of the necessary background to discuss graded Hopf algebras and particularly the Hopf algebra of symmetric functions, which is used extensively in Chapter 5. For a thorough reference on Hopf algebras, see for example [DNR00]. For references on the Hopf algebra of symmetric functions, see [Mac98] or [Zel06].

2.5.1 Hopf algebras

Recall the definition of a k-algebra at the beginning of Section 2.1. If we define a multiplication map $m : A \otimes_k A \to A$ by $a \otimes b \mapsto ab$, a unit map $u : k \to A$ by $x \mapsto x \cdot 1_A$, and an identity map $\mathrm{id}_A : A \to A$ by $a \mapsto a$, then that definition is equivalent to the condition that m and u satisfy commutativity of the following diagrams:[6]

$$
\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "m \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes m"'] & A \otimes A \arrow[d, "m"] \\
A \otimes A \arrow[r, "m"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
 & A \otimes A \arrow[dd, "m"] & \\
k \otimes A \arrow[ur, "u \otimes \mathrm{id}"] \arrow[dr] & & A \otimes k. \arrow[ul, "\mathrm{id} \otimes u"'] \arrow[dl] \\
 & A &
\end{tikzcd}
\tag{2.11}
$$

We can dualize (2.11) to obtain the following definition. A k**-coalgebra** is a k-vector space C with maps $\Delta : C \to C \otimes C$ and $\varepsilon : C \to k$, called the **co-multiplication** and **co-unit**, respectively, that satisfy commutativity of the following diagrams.

$$
\begin{tikzcd}
C \otimes C \otimes C & C \otimes C \arrow[l, "\Delta \otimes \mathrm{id}"'] \\
C \otimes C \arrow[u, "\mathrm{id} \otimes \Delta"] & C \arrow[l, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & C \otimes C \arrow[dl, "\varepsilon \otimes \mathrm{id}"'] \arrow[dr, "\mathrm{id} \otimes \varepsilon"] & \\
k \otimes C & & C \otimes k \\
 & C \arrow[ul] \arrow[ur] \arrow[uu, "\Delta"'] &
\end{tikzcd}
\tag{2.12}
$$

A k**-bialgebra** is a k-vector space B that is both an algebra and coalgebra, and such that the co-multiplication and co-unit are algebra morphisms.

A **Hopf algebra over** k is a k-bialgebra H with a map $S : H \to H$, called the **antipode**, which satisfies the following commutative diagram:

$$
\begin{tikzcd}
H \otimes H \arrow[d, "S \otimes \mathrm{id}"'] & H \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "u \circ \varepsilon"] & H \otimes H \arrow[d, "\mathrm{id} \otimes S"] \\
H \otimes H \arrow[r, "m"] & H & H \otimes H. \arrow[l, "m"']
\end{tikzcd}
\tag{2.13}
$$

If the field is understood, then we simply refer to H as a Hopf algebra.

A **graded bialgebra over** k is a bialgebra B over k such that B has a decomposition as a direct sum of k-subspaces

$$B = \bigoplus_{n \geq 0} B_n,$$

[6] Unless otherwise specified, all tensor products in this thesis are taken over the ground field.

and such that the maps m, u, Δ, ε, and S all respect this decomposition, i.e., $m(B_i \otimes B_j) \subseteq B_{i+j}$ for all $i, j \geq 0$, $u(k) \subseteq B_0$, $\Delta(B_n) \subseteq \bigoplus_{i+j=n} B_i \otimes B_j$ for all $n \geq 0$, and $\varepsilon(B_n) = 0$ for all $n > 0$. Moreover, we say B is **connected** if $B_0 \cong k$.

When defining a Hopf algebra, we often make no mention of its antipode. The following result makes this possible.

Proposition 2.23. *[GR15, Proposition 1.36] Let B be a graded bialgebra. If B is also connected, then B is a Hopf algebra and the antipode $S : B \to B$ is given by the formula*

$$S(h) = -h - \sum_{i=1}^{n-1} S(h_{1,j})h_{2,n-j}, \tag{2.14}$$

where the elements $h_{i,j}$ come from the equation

$$\Delta(h) = h \otimes 1 + 1 \otimes h + \sum_{j=1}^{n-1} h_{1,j} \otimes h_{2,n-j}.$$

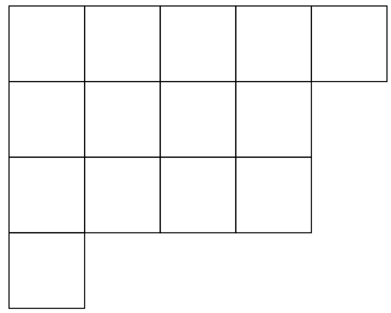
An element h of a graded Hopf algebra H is called **primitive** if $\Delta(h) = 1 \otimes h + h \otimes 1$. By (2.14), we have $S(h) = -h$ for any primitive element h.

2.5.2 The ring of symmetric functions

Let n be a nonnegative integer. An **integer partition** of size n is a tuple of nonnegative integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, whose entries are weakly decreasing and sum to n. We often identify two integer partitions if they differ only by a string of zeros. Let $\mathscr{P}$ denote the set of all integer partitions. If $\lambda = (\lambda_1, \ldots, \lambda_k)$, where $\lambda_i > 0$ for all i, then we write $\ell(\lambda) = k$ and refer to $\ell(\lambda)$ as the **length** of λ. For $\lambda \in \mathscr{P}$, let $|\lambda| = \lambda_1 + \cdots + \lambda_{\ell(\lambda)}$ denote the **size** of λ. We will often discuss an integer partition λ in terms of its **Ferrers diagram**: the diagram of left-justified rows of boxes that consists of λ_1 boxes in the top row, λ_2 boxes in the second row, and so on, as in Figure 2.2.

For each $n \geq 0$, let $\mathbb{C}[x_1, \ldots, x_n]$ be the $\mathbb{C}$-vector space of polynomials in n indeterminate variables with complex coefficients, and let S_n act on this space by permuting the variables. Let Sym_n denote the vector space $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$ of polynomials that are fixed under this action. Let Sym_n^k denote the subspace of Sym_n of homogeneous polynomials of degree k. Then Sym_n is graded

Figure 2.2: A typical Ferrers diagram, this one of the partition $\lambda = (5, 4, 4, 1)$



by homogeneous degree:

$$\mathrm{Sym}_n = \bigoplus_{k \geq 0} \mathrm{Sym}_n^k.$$

For all $m \geq n$, there is a surjective map $\rho_{m,n} : \mathrm{Sym}_m \to \mathrm{Sym}_n$ which is given by setting the variables $x_{n+1}, \ldots, x_m$ equal to zero. Moreover, this map respects the above grading, so we may write $\rho_{m,n} = \bigoplus_{k \geq 0} \rho_{m,n}^k$, where $\rho_{m,m}^k : \mathrm{Sym}_m^k \to \mathrm{Sym}_n^k$. Now, let Sym^k be the inverse limit of the subspaces Sym_n^k along the maps $\rho_{m,n}^k$ and for each n, let $\rho_n^k : \mathrm{Sym}^k \to \mathrm{Sym}_n^k$ be the map obtained from the universal property of inverse limits. Finally, define the **ring of symmetric functions** to be the direct sum

$$\mathrm{Sym} = \bigoplus_{k \geq 0} \mathrm{Sym}^k.$$

Then Sym is the inverse limit of the Sym_n in the category of graded rings, hence an element of Sym is uniquely determined by its images under the maps $\rho_n = \bigoplus_k \rho_n^k$ for $n \geq 0$.

The elements of Sym can be realized as polynomials of bounded degree in countably many indeterminates $x_1, x_2, \ldots$ that are fixed under all permutations of the variables by elements of any symmetric group. In order to discuss these elements efficiently, we will need the following notation. If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers, let

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

For any integer partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ of n, define the **monomial symmetric function**

$$m_\lambda = \sum_\sigma \sigma \cdot x^\lambda,$$

where the sum is over all elements of the filtered union of all symmetric groups, and the action is by permutation of the variables. Then the monomial symmetric functions form an algebraically independent basis for Sym. We will use the monomial symmetric functions to define other bases. For each integer $r \geq 0$, let

$$e_r = m_{(1^r)} = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}.$$

These elements are commonly known as the **elementary symmetric functions**. Let

$$h_r = \sum_{|\lambda|=n} m_\lambda = \sum_{|\alpha|=n} x^\alpha,$$

where for any tuple of integers $\alpha = (\alpha_1, \ldots, \alpha_k)$, we denote $|\alpha| = \alpha_1 + \cdots + \alpha_k$. The h_r are known as the **complete homogeneous symmetric functions**. Next, let

$$p_r = m_{(r)} = \sum_{i \geq 1} x_i^r.$$

These are known as the **power-sum symmetric functions**. The power-sum symmetric functions form a basis for the primitive elements of Sym.

If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is any integer partition, we will write p_λ for the product $p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}$. We will use similar notation e_λ and h_λ for the corresponding products of these basis elements.

The final canonical basis of Sym is defined as follows. For any m-tuple $\alpha = (\alpha_1, \ldots, \alpha_m)$, let

$$a_\alpha = a_\alpha(x_1, \ldots, x_m) = \sum_{\sigma \in S_m} \operatorname{sgn}(\sigma) \sigma \cdot x^\alpha.$$

Let $m \geq n$. For any integer partition $\lambda = (\lambda_1, \ldots, \lambda_m)$ of size n (written as an m-tuple, so that some entries may be zero), let $\lambda + \delta = (\lambda_1 + m - 1, \lambda_2 + m - 2, \ldots, \lambda_m)$. Then $a_{\lambda+\delta}$ is divisible by a_λ (see [Mac98, Chapter I.3]), so we can define the **Schur polynomial**

$$s_\lambda(x_1, \ldots, x_m) = \frac{a_{\lambda+\delta}}{a_\lambda}.$$

Since both $a_{\lambda+\delta}$ and a_λ are anti-symmetric, it follows that $s_\lambda(x_1, \ldots, x_m)$ is a homogeneous symmetric polynomial of degree n. Moreover, we have $\rho_{m_1,m_2}(s_\lambda(x_1, \ldots, x_{m_1})) = s_\lambda(x_1, \ldots, x_{m_2})$ for any $m_1 \geq m_2$. The **Schur function** s_λ is defined to be the unique element of Sym whose image under ρ_m is $s_\lambda(x_1, \ldots, x_m)$ for all $m \geq n$.

If λ and μ are partitions of n, a **Young tableau** of shape λ and weight μ is a filling of the Ferrers diagram of λ with integers between 1 and n such that for each i, the number of is in the filling is μ_i. A Young tableau is called **semistandard** or **column-strict** if the numbers in each column are strictly increasing and the numbers in each row are weakly increasing. A Young tableau is called **standard** if the numbers in each row and column are strictly increasing. For two partitions λ and μ, let $K_{\lambda,\mu}$ be the number of semistandard Young tableaux of shape λ and weight μ. Then the transition matrix from the monomial basis to the Schur function basis is

$$s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu.$$

The entries $K_{\lambda,\mu}$ of the transition matrix are known as **Kostka numbers**.

The structure coefficients of multiplication with respect to the Schur function basis are of particular importance. We write

$$s_\lambda \cdot s_\mu = \sum_\nu c^\nu_{\lambda,\mu} s_\nu.$$

The coefficients $c^\nu_{\lambda,\mu}$ are the famous **Littlewood–Richardson coefficients**. Their combinatorics are well-studied: see [Ful97, Chapter 5] for a thorough treatment. If $\mu = (1^k)$, then the Littlewood–Richardson coefficients specialize to a simple rule, known as the **Pieri formula**: $c^\nu_{\lambda,(1^r)} = 1$ if ν is obtained from λ by adding r boxes, no two in the same row, and otherwise $c^\nu_{\lambda,(1^r)} = 0$.

Chapter 3

Lattices of supercharacter theories

Let G be a finite group and let $\mathrm{SCT}(G)$ denote the set of all supercharacter theories of G. In this chapter, we will define a partial order on $\mathrm{SCT}(G)$ which gives this set the structure of a lattice. We will then discuss order-preserving actions on $\mathrm{SCT}(G)$ and some interesting subposets and sublattices obtained by taking fixed points under various actions. Much of this work is inspired by Hendrickson's study of the structure of the supercharacter theories of cyclic groups in [Hen08] and subsequent work, and we will cite his results accordingly. In Section 3.4, we will discuss some smaller results and conjectures, including an algorithm for computing $\mathrm{SCT}(G)$ given the character table of G.

3.1 Definitions and main results

We will use the refinement partial order defined in Example 2.22 to give $\mathrm{SCT}(G)$ the structure of a lattice. To that end, the following result is due to [Hen12].[1]

Proposition 3.1. *[Hen12, Corollary 3.4] Let $S = (\mathcal{K}, \mathcal{X})$ and $T = (\mathcal{L}, \mathcal{Y})$ be supercharacter theories of a finite group G. Then with respect to the refinement partial orders on G and $\mathrm{Irr}(G)$, $\mathcal{K} \leq \mathcal{L}$ if and only if $\mathcal{X} \leq \mathcal{Y}$.*

Proof. Suppose $\mathcal{K} \leq \mathcal{L}$. Then for all $Y \in \mathcal{Y}$, the T-supercharacter σ_Y is constant on the parts of

[1] Since we are developing these results in a different order than Hendrickson does, we will provide alternate proofs of some results.

$\mathcal{K}$. Thus, we have

$$\sigma_Y = \sum_{X \in \mathcal{X}} c_X \sigma_X$$

for some complex constants c_X. In fact, by examining coefficients of irreducible characters, one sees that $c_X \in \{0, 1\}$ for all X, whence Y is a union of parts of $\mathcal{X}$. Thus, we have $\mathcal{X} \leq \mathcal{Y}$.

Conversely, suppose $\mathcal{X} \leq \mathcal{Y}$. For any subset A of G, let δ_A denote the indicator function of that set. Then because the T-supercharacters form a basis for $\mathrm{scf}_T(G)$, it follows that for any $L \in \mathcal{L}$, there exist constants c_Y such that

$$\begin{aligned}
\delta_L &= \sum_{Y \in \mathcal{Y}} c_Y \sigma_Y \\
&= \sum_{Y \in \mathcal{Y}} \sum_{X \subseteq Y} c_Y \sigma_X \\
&= \sum_{Y \in \mathcal{Y}} \sum_{X \subseteq Y} \sum_{K \in \mathcal{K}} c_Y \sigma_X(K) \delta_K.
\end{aligned}$$

Thus, δ_L is constant on the parts of $\mathcal{K}$, which implies that L is a union of parts of $\mathcal{K}$. Therefore, it follows that $\mathcal{K} \leq \mathcal{L}$. ∎

Using the above proposition, we may define a partial order on $\mathrm{SCT}(G)$ as follows: if $S = (\mathcal{K}, \mathcal{X})$ and $T = (\mathcal{L}, \mathcal{Y})$ are supercharacter theories of G, we say $S \leq T$ if $\mathcal{K}$ is a refinement of $\mathcal{L}$, or equivalently if $\mathcal{X}$ is a refinement of $\mathcal{Y}$. By examining superclass identifier functions, we see that for $S, T \in \mathrm{SCT}(G)$, we have $S \leq T$ if and only if $\mathrm{scf}_T(G) \subseteq \mathrm{scf}_S(G)$. Note that $S \leq T$ certainly implies $|T| \leq |S|$, but the converse need not be true. However, if $S \leq T$ and $|T| = |S| - 1$, then $S < T$ is a covering relation in $\mathrm{SCT}(G)$.

Proposition 3.2. *[Hen12, Proposition 3.3] If $S = (\mathcal{K}, \mathcal{X})$ and $T = (\mathcal{L}, \mathcal{Y})$ are supercharacter theories of a group G, then their lattice-theoretic join $S \vee T$ in $\mathrm{SCT}(G)$ exists, and moreover, it is given by*

$$S \vee T = (\mathcal{K} \vee \mathcal{L}, \mathcal{X} \vee \mathcal{Y}).$$

Since SCT(G) is finite, it follows by Lemma 2.19 that SCT(G) is in fact a lattice with the meet of two supercharacter theories S and T defined by

$$S \wedge T = \bigvee_{U \leq S,T} U.$$

Unfortunately, the meet of two supercharacter theories need not be their mutual refinement. In fact there are meets of the form

$$(\mathcal{K}, \mathcal{X}) \wedge (\mathcal{L}, \mathcal{Z}) = (\mathcal{M}, \mathcal{Z}),$$

where neither $\mathcal{M}$ nor $\mathcal{Z}$ is equal to the appropriate mutual refinement, or where one of $\mathcal{M}$ or $\mathcal{Z}$ is equal to the appropriate mutual refinement and the other is not, as demonstrated in the following example. However, the superclasses and supercharacters of $(\mathcal{K}, \mathcal{X}) \wedge (\mathcal{L}, \mathcal{Y})$ are refinements of $\mathcal{K} \wedge \mathcal{L}$ and $\mathcal{X} \wedge \mathcal{Y}$, respectively.

Example 3.3. By direct calculation, we can find examples of all four possibilities for the meet of two supercharacter theories.

(1) Let G be any group and let $S = (\mathcal{K}, \mathcal{X})$ be any supercharacter theory. If $m(G) = (\mathcal{L}, \mathcal{Y})$, then $S \wedge m(G) = (\mathcal{K} \wedge \mathcal{L}, \mathcal{X} \wedge \mathcal{Y})$.

(2) Let $G = \langle x \rangle$ be a cyclic group of order 6 and let its character group be written $\langle \xi \rangle$, where $\xi^j(x^i) = e^{\frac{2\pi ij}{6}}$. Let $S = (\mathcal{K}, \mathcal{X})$ and $T = (\mathcal{L}, \mathcal{Y})$ be the supercharacter theories with the following partitions.

$$\mathcal{K} = \big\{\{e\}, \{x, x^3, x^5\}, \{x^2, x^4\}\big\},$$

$$\mathcal{L} = \big\{\{e\}, \{x, x^5\}, \{x^2, x^4\}, \{x^3\}\big\},$$

$$\mathcal{X} = \big\{\{\xi^0\}, \{\xi, \xi^4\}, \{\xi^2, \xi^5\}, \{\xi^3\}\big\},$$

$$\mathcal{Y} = \big\{\{\xi^0\}, \{\xi, \xi^5\}, \{\xi^2, \xi^4\}, \{\xi^3\}\big\}.$$

Then one can check directly that these form supercharacter theories of G. Their meet is therefore $m(G)$, since $\mathcal{X} \wedge \mathcal{Y}$ contains only singleton sets. However, the meet of $\mathcal{K}$

and $\mathcal{L}$ contains nonsingleton parts, hence $\mathcal{K} \wedge \mathcal{L}$ is not singleton. Therefore if we write $S \wedge T = (\mathcal{M}, \mathcal{Z})$, then $\mathcal{X} \wedge \mathcal{Y} = \mathcal{Z}$ but $\mathcal{K} \wedge \mathcal{L} \neq \mathcal{M}$.

(3) Again, let $G = \langle x \rangle$ be a cyclic group of order 6 and let its character group be written $\langle \xi \rangle$ as above. Let $S' = (\mathcal{K}', \mathcal{X}')$ and $T' = (\mathcal{L}', \mathcal{Y}')$ be the images of S and T through the isomorphism between G and its character group given by $x \mapsto \xi$. Then S' and T' have the following partitions:

$$\mathcal{K}' = \big\{\{e\}, \{x, x^4\}, \{x^2, x^5\}, \{x^3\}\big\},$$

$$\mathcal{L}' = \big\{\{e\}, \{x, x^5\}, \{x^2, x^4\}, \{x^3\}\big\},$$

$$\mathcal{X}' = \big\{\{\xi^0\}, \{\xi, \xi^3, \xi^5\}, \{\xi^2, \xi^4\}\big\},$$

$$\mathcal{Y}' = \big\{\{\xi^0\}, \{\xi, \xi^2\}, \{\xi^2, \xi^4\}, \{\xi^3\}\big\}.$$

Then we have $S' \wedge T' = m(G)$, since $\mathcal{K}' \wedge \mathcal{L}'$ contains only singleton sets. However, $\mathcal{X}' \wedge \mathcal{Y}'$ is not singleton. Therefore if we write $S' \wedge T' = (\mathcal{M}', \mathcal{Z}')$, then $\mathcal{K}' \wedge \mathcal{L}' = \mathcal{M}'$ but $\mathcal{X}' \wedge \mathcal{Y}' \neq \mathcal{Z}'$.

(4) Let $G = \langle x \rangle$ be a cyclic group of order 8 and let its character group be written $\langle \xi \rangle$, where $\xi^j(x^i) = e^{\frac{2\pi i j}{8}}$. Let $S = (\mathcal{K}, \mathcal{X})$ and $T = (\mathcal{L}, \mathcal{Y})$ be the supercharacter theories with the following partitions:

$$\mathcal{K} = \big\{\{e\}, \{x, x^3\}, \{x^2, x^6\}, \{x^4\}, \{x^5, x^7\}\big\},$$

$$\mathcal{L} = \big\{\{e\}, \{x, x^3, x^5, x^7\}, \{x^2\}, \{x^4\}, \{x^6\}\big\},$$

$$\mathcal{X} = \big\{\{\xi^0\}, \{\xi, \xi^7\}, \{\xi^2, \xi^6\}, \{\xi^3, \xi^5\}, \{\xi^4\}\big\},$$

$$\mathcal{Y} = \big\{\{\xi^0\}, \{\xi, \xi^5\}, \{\xi^2, \xi^6\}, \{\xi^3, \xi^7\}, \{\xi^4\}\big\}.$$

Then $S \wedge T = m(G)$, but both $\mathcal{K} \wedge \mathcal{L}$ and $\mathcal{X} \wedge \mathcal{Y}$ have nonsingleton parts. Therefore if we write $S \wedge T = (\mathcal{M}, \mathcal{Z})$, then $\mathcal{K} \wedge \mathcal{L} \neq \mathcal{M}$ but $\mathcal{X} \wedge \mathcal{Y} \neq \mathcal{Z}'$.

In Chapter 4, we will classify $\mathrm{SCT}(D_{2n})$ for all n, as well as $\mathrm{SCT}(\mathbb{Z}_n \rtimes \mathbb{Z}_p)$ for some values of n and p. One of the main strategies for classifying $\mathrm{SCT}(G)$ that we use in that chapter is to analyze the supercharacter theories that factor as $*$-products over a given normal subgroup $N \trianglelefteq G$ and use the set of those as a core sublattice from which the remainder of $\mathrm{SCT}(G)$ is derived. By the following lemma, the sublattice of $*$-products over N has a nice decomposition in terms of supercharacter theories of N and of G/N.

Lemma 3.4. *[BHH14, Lemma 2.2] Let G be a group and let N be a normal subgroup of G. Then the map*

$$f : \mathrm{InvSCT}_G(N) \times \mathrm{SCT}(G/N) \to \mathrm{SCT}(G);\ (S,T) \mapsto S *_N T$$

is an injection of lattices whose image is the interval $[mm_N(G), MM_N(G)]$.

We can generalize this result to Δ-products. Recall the superclass and supercharacter partitions of $S \,\Delta\, T$, where $S = (\mathcal{K}, \mathcal{X}) \in \mathrm{SCT}_G(M)$ and $T = (\mathcal{L}, \mathcal{Y}) \in \mathrm{SCT}(G/N)$: they are $(\mathcal{M}, \mathcal{Z})$, where

$$\mathcal{M} = \mathcal{K}(S \,\Delta\, T) = \mathcal{K} \cup \left\{\widetilde{L} : L \in \mathcal{L}, L \not\subseteq M/N\right\} \tag{3.1}$$

and

$$\mathcal{Z} = \mathcal{X}(S \,\Delta\, T) = \mathcal{Y} \cup \left\{X^G : X \in \mathcal{X}, X \not\subseteq \mathrm{Irr}(M/N)\right\}. \tag{3.2}$$

Lemma 3.5. *Let G be a finite group with normal subgroups $1 < H \leq K$, and let*

$$\mathcal{A} = \left\{S \in \mathrm{SCT}_G(K) : S \leq MM_H(K)\right\}$$

be the lower ideal of $\mathrm{SCT}_G(K)$ consisting of the supercharacter theories for which H is supernormal. Then the Δ-product provides a lattice embedding of $\mathcal{A} \times \mathrm{SCT}(G/H)$ into $\mathrm{SCT}(G)$.

Proof. The following statements are immediate consequences of (3.1) and (3.2): Δ is injective, $S_1 \,\Delta\, T_1 \leq S_2 \,\Delta\, T_2$ if and only if $S_1 \leq S_2$ and $T_1 \leq T_2$, and Δ preserves joins (recall that $(S_1, T_1) \vee (S_2, T_2) = (S_1 \vee S_2, T_1 \vee T_2)$). So, we just need to show that Δ preserves the meet operation of $\mathcal{A} \times \mathrm{SCT}(G/H)$, which is given by $(S_1, T_1) \wedge (S_2, T_2) = (S_1 \wedge S_2, T_1 \wedge T_2)$.

Let $(S_1, T_1), (S_2, T_2) \in \mathcal{A} \times \mathrm{SCT}(G/H)$, let $S_3 = S_1 \wedge S_2$, and let $T_3 = T_1 \wedge T_2$. Then we have $S_3 \in \mathcal{A}$, so it follows that $S_3 \wedge T_3 \in \mathrm{SCT}(G)$. Write $U = (S_1 \,\Delta\, T_1) \wedge (S_2 \,\Delta\, T_2)$. By definition, we have $U \leq S_1 \Delta T_1, S_2 \Delta T_2$, so it follows that $\mathcal{K}(U)$ refines $\{K, G \setminus K\}$, hence K is supernormal with respect to U. By the monotonicity of Δ, we have $S_3 \Delta T_3 \leq U$, hence every U-superclass outside of K is a union of H-cosets. By [Hen08, Proposition 4.3], this is true if and only if U is a Δ-product over H and K, and therefore U is equal to $S_3 \Delta T_3$. This completes the proof. ∎

3.2 Compatible actions on G and $\mathrm{Irr}(G)$

Let A be a group that acts on both G and on $\mathrm{Irr}(G)$. We say these actions are **compatible** if the identity

$$(a \cdot \chi)(g) = \chi(a^{-1} \cdot g) \tag{3.3}$$

holds for every $\chi \in \mathrm{Irr}(G)$, $g \in G$, and $a \in A$.

Lemma 3.6. *Let G be a group and let A be a group that has compatible actions on G and on $\mathrm{Irr}(G)$. Then the action of A on G must permute the conjugacy classes of G.*

Proof. Recall that g and h lie in the same conjugacy class if and only if $\chi(g) = \chi(h)$ for all $\chi \in \mathrm{Irr}(G)$. Thus if A has compatible actions on G and $\mathrm{Irr}(G)$, then for any $a \in A$ and any g and h in the same conjugacy class of G, we have for all $\chi \in \mathrm{Irr}(G)$ that

$$\chi(a \cdot g) = (a^{-1} \cdot \chi)(g) = (a^{-1} \cdot \chi)(h) = \chi(a \cdot h).$$

Therefore, the action of A on G must permute the conjugacy classes of G. ∎

Lemma 3.7. *Let G be a group and let A be a group that has compatible actions on G an on $\mathrm{Irr}(G)$. Then the identity e is a fixed point of the action of A on G and the trivial character $\mathbf{1}$ is a fixed point of the action of A on $\mathrm{Irr}(G)$.*

Proof. Let χ be the regular character of G. and let α be an element of A. Then we have

$$\alpha^{-1} \cdot \chi = \sum_{\psi \in \mathrm{Irr}(G)} \psi(e) \alpha^{-1} \cdot \psi.$$

Thus $\alpha^{-1} \cdot \chi$ is a character with positive degree, and so we have $\chi(\alpha \cdot e) = (\alpha^{-1} \cdot \chi)(e) > 0$. Since χ is identically zero on $G \setminus \{e\}$, it follows that $\alpha \cdot e = e$.

Now, let $g \in G$ and let $\alpha \in A$. Then $\alpha \cdot \mathbf{1}(g) = \mathbf{1}(\alpha^{-1} \cdot g) = 1$, and therefore $\alpha \cdot \mathbf{1} = \mathbf{1}$. ∎

Let A be a subgroup of $\mathrm{Aut}(G)$ and consider the natural action of A on G. For $\alpha \in A$ and $\chi \in \mathrm{Irr}(G)$, let $\alpha \cdot \chi$ be the class function defined by $\chi \circ \alpha^{-1}$. Then $\alpha \cdot \chi$ is an irreducible character of G, and this defines an action of A on $\mathrm{Irr}(G)$. Moreover, we have

$$(\alpha \cdot \chi)(g) = \chi(\alpha^{-1}(g)) = \chi(\alpha^{-1} \cdot g)$$

for all $\alpha \in A$, $\chi \in \mathrm{Irr}(G)$, and $g \in G$. Thus, these actions are compatible, and subgroups of this form will be primary examples of groups with compatible actions on G.

Let $n = |G|$ and let $\mathrm{Gal}(G)$ denote the Galois group of the cyclotomic extension $\mathbb{Q}[\zeta_n]$, where $\zeta_n = e^{2\pi i/n}$. Any subgroup A of $\mathrm{Gal}(G)$ acts on irreducible characters by post-composition, i.e., if $\tau \in \mathrm{Gal}(G)$ and $\chi \in \mathrm{Irr}(G)$, then $\tau \cdot \chi = \tau \circ \chi$. We can define an action of A on G as follows. For each $\tau \in A$, there is a unique integer $m_\tau < n$ that is relatively prime to n and such that $\tau(\zeta_n) = \zeta_n^{m_\tau}$. For any $\tau \in A$ and $g \in G$, define[2] $\tau \cdot g = g^{m_{(\tau^{-1})}}$. Notice that this is not an action via automorphisms in general.

Lemma 3.8. *Let A be a subgroup of* $\mathrm{Gal}(G)$. *Then A acts compatibly on G and on* $\mathrm{Irr}(G)$ *with the actions defined above.*

Proof. Let $g \in G$, let $\chi \in \mathrm{Irr}(G)$, and let $\tau \in A$. Let ρ be an irreducible representation affording χ, let n be the order of g, and let $k = \chi(1)$. Then by [Isa76, Lemma 2.15], $\rho(g)$ is similar the diagonal matrix $\mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_k)$, where each ε_i is an n-th root of unity. Thus, $\rho(g^{m_\tau}) = \rho(g)^{m_\tau}$ is

[2] Diaconis and Isaacs notably use a different action of A on G: they define $\tau \cdot g = g^{m_\tau}$. While this fails to satisfy (3.3) with the action of A on $\mathrm{Irr}(G)$, it does produce the same orbits. We will see below that this is sufficient for their purpose.

similar to the diagonal matrix $\mathrm{diag}(\varepsilon_1^{m_\tau}, \ldots, \varepsilon_k^{m_\tau})$. Thus, we have

$$\begin{aligned}(\tau \cdot \chi)(g) &= \tau \cdot \left(\sum_{i=1}^{k} \varepsilon_i\right) \\ &= \left(\sum_{i=1}^{k} \varepsilon_i^{m_\tau}\right) \\ &= \mathrm{tr}(\rho(g^{m_\tau})) \\ &= \chi(\tau^{-1} \cdot g).\end{aligned}$$

Therefore, these actions are compatible. ∎

The following construction is first discussed in [DI08], and studied extensively in [Kel14, Chapter 6].

Proposition 3.9. *[Kel14, Theorem 5.23] Let A be a group with compatible actions on G and on* $\mathrm{Irr}(G)$*. Let $\mathcal{K}$ be the set of A-orbits in G and let $\mathcal{X}$ be the set of A-orbits in* $\mathrm{Irr}(G)$*. Then $(\mathcal{K}, \mathcal{X})$ forms a supercharacter theory of G.*

We will denote the above supercharacter theory $m_A(G)$. A supercharacter theory which is of the form $m_A(G)$ for some subgroup A of $\mathrm{Aut}(G)$ is called **automorphic**. A supercharacter theory of the form $m_A(G)$ for some subgroup of $\mathrm{Gal}(G)$ is called **Galois**. We will write $\mathrm{AutSCT}(G)$ and $\mathrm{GalSCT}(G)$ to denote the sets of automorphic and Galois supercharacter theories, respectively. The maximal Galois supercharacter theory of G is precisely the finest supercharacter theory whose supercharacters table is rational. Rational supercharacter theories are studied extensively in [Kel14]. Any supercharacter theory with rational entries is invariant under the action of $\mathrm{Gal}(G)$ in the sense that each supercharacter is fixed under this action. We will discuss invariant supercharacter theories in more detail below.

Lemma 3.10. *Suppose A has compatible actions on G and on* $\mathrm{Irr}(G)$ *and let B be a subgroup of A. Then the restrictions of these actions to B are compatible and we have $m_B(G) \leq m_A(G)$.*

Proof. Since $B \subseteq A$, the restrictions to B of the actions of A are compatible. Moreover, each

B-orbit on G is contained in an A-orbit, and hence we have $\mathcal{K}(m_B(G)) \leq \mathcal{K}(m_A(G))$. Therefore, it follows that $m_B(G) \leq m_A(G)$. ∎

Lemma 3.11. *Let G be a finite group and let H be a group with compatible actions on G and on* $\mathrm{Irr}(G)$*. Then for any two subgroups A and B of H, we have $m_A(G) \vee m_B(G) = m_{\langle A,B\rangle}(G)$.*

Proof. By [DI08, Theorem 2.2(c)], it suffices to show that $\mathcal{K}(m_A(G) \vee m_B(G)) = \mathcal{K}(m_A(G)) \vee \mathcal{K}(m_B(G))$. By Lemma 3.10, we have $\mathcal{K}(m_A(G)) \vee \mathcal{K}(m_B(G)) \leq \mathcal{K}(m_{\langle A,B\rangle}(G))$, so we just need to show that $\mathcal{K}(m_{\langle A,B\rangle}(G)) \leq \mathcal{K}(m_A(G)) \vee \mathcal{K}(m_B(G))$. We can write any element $c \in \langle A, B\rangle$ as a product of the form $c = c_1c_2 \cdots c_n$, where each c_i lies either in A or B. Now let $x \in G$ and consider the orbit of x under the action of $\langle A, B\rangle$. Let c be as above, let $x_0 = x$, and for each i, let $x_i = x_{i-1}^{c_i}$. Then for all i, x_{i-1} and x_i lie in either the same block of $\mathcal{K}(m_A(G))$ or of $\mathcal{K}(m_B(G))$, and therefore lie in the same block of $\mathcal{K}(m_A(G)) \vee \mathcal{K}(m_B(G))$. Hence, x and x^c lie in the same block of $\mathcal{K}(m_A(G)) \vee \mathcal{K}(m_B(G))$. Since this holds for all $x \in G$ and $c \in \langle A, B\rangle$, it follows that $\mathcal{K}(m_{\langle A,B\rangle}(G)) \leq \mathcal{K}(m_A(G)) \vee \mathcal{K}(m_B(G))$, and therefore that $m_A(G) \vee m_B(G) = m_{\langle A,B\rangle}(G)$. ∎

Corollary 3.12. *For any finite group G, the subposets* $\mathrm{AutSCT}(G)$ *and* $\mathrm{GalSCT}(G)$ *are both join-closed.*

Proof. Let S and T be elements of $\mathrm{AutSCT}(G)$. Then there are subgroups A and B of $\mathrm{AutSCT}(G)$ such that $S = m_A(G)$ and $T = m_B(G)$. By Lemma 3.11, it follows that $S \vee T = m_{\langle A,B\rangle}(G)$. But $\langle A, B\rangle$ is a subgroup of $\mathrm{Aut}(G)$, therefore we have that $S \vee T \in \mathrm{AutSCT}(G)$. The proof that $\mathrm{GalSCT}(G)$ is join-closed is identical. ∎

A natural question to ask is whether either of these subposets is meet-closed. It is currently not known whether either is meet-closed. Let A and B be subgroups of $\mathrm{Gal}(G)$, and let E and F be the field extensions of $\mathbb{Q}$ corresponding to A and B, respectively. Then $E = \mathbb{Q}[\zeta_d]$ and $F = \mathbb{Q}[\zeta_e]$ for divisors e and d of $|G|$. Then, with the notation of [AB17], it follows that $\mathcal{X}(m_A(G)) = \mathrm{Irr}_d(G)$ and $\mathcal{X}(m_B(G)) = \mathrm{Irr}_e(G)$. Now, $m_A(G) \wedge m_B(G)$ corresponds to the Schur ring $\langle \mathrm{scf}_{m_A(G)}(G), \mathrm{scf}_{m_B(G)}(G)\rangle$, which is spanned by $\mathrm{Irr}_d(G) \cup \mathrm{Irr}_e(G)$, and $m_{A\cap B}(G)$ corresponds to the Schur ring spanned by $\mathrm{Irr}_{\gcd(d,e)}(G)$. Thus $\mathrm{GalSCT}(G)$ is meet-closed if and only if every element of $\mathrm{Irr}_{\gcd(d,e)}(G)$ lies in the span of $\mathrm{Irr}_d(G) \cup \mathrm{Irr}_e(G)$.

Conjecture 3.13. *The subposet* $\mathrm{GalSCT}(G)$ *is meet-closed.*

If the conjecture holds, then by Lemmas 3.10 and 3.11, we obtain a lattice embedding of the lattice of subgroups of $\mathrm{Gal}(G)$ into $\mathrm{SCT}(G)$.

Example 3.14. Unfortunately, meet-closure fails for $\mathrm{AutSCT}(G)$. Let $\langle x \rangle$ and $\langle y \rangle$ be cyclic groups of orders 8 and 2, respectively and let G be their semidirect product, with action of $\langle y \rangle$ on $\langle x \rangle$ given by $y^{-1}xy = x^5$. Then we have calculated using Algorithm 3.30 (which appears in Section 3.4 later in this chapter) that if we define

$$\mathcal{K}_1 = \big\{\{e\}, \{x^4\}, \{y, x^4y\}, \{x^2, x^6\}, \{x^2y, x^6y\}, \{x, x^3, x^5, x^7\}, \{xy, x^3y, x^5y, x^7y\}\big\}$$

and

$$\mathcal{K}_2 = \big\{\{e\}, \{x^4\}, \{y, x^4y\}, \{x^2, x^6\}, \{x^2y, x^6y\}, \{x, x^5, xy, x^5y\}, \{x^3, x^7, x^3y, x^7y\}\big\},$$

then $\mathcal{K}_1$ and $\mathcal{K}_2$ are the superclass partitions of automorphic supercharacter theories of G whose meet has the superclass partition $\mathcal{K}_1 \wedge \mathcal{K}_2$ (although $\mathcal{X} \wedge \mathcal{Z}$ is not the supercharacter partition of this meet), and this superclass partition is not the set of orbits of any subgroup of $\mathrm{Aut}(G)$.

3.3 Supercharacter theories fixed by an action on $\mathrm{SCT}(G)$

If A has compatible actions on G and on $\mathrm{Irr}(G)$, then these actions together yield an action of A on $\mathrm{SCT}(G)$ in the following manner. If $S = (\mathcal{K}, \mathcal{X})$ is a supercharacter theory and $\alpha \in A$, we define

$$\alpha \cdot \mathcal{K} = \big\{\{\alpha \cdot g \,:\, g \in K\} \,:\, K \in \mathcal{K}\big\} \tag{3.4}$$

and

$$\alpha \cdot \mathcal{X} = \big\{\{\alpha \cdot \chi \,:\, \chi \in X\} \,:\, X \in \mathcal{X}\big\}. \tag{3.5}$$

Lemma 3.15. *With respect to the actions defined above, the ordered pair* $(\alpha \cdot \mathcal{K}, \alpha \cdot \mathcal{X})$ *is a supercharacter theory of* G.

Proof. First, note that

$$\begin{aligned}|\alpha\cdot\mathcal{K}| &= |\mathcal{K}| \\ &= |\mathcal{X}| \\ &= |\alpha\cdot\mathcal{X}|.\end{aligned}$$

Next, note that by Lemma 3.7 that $\alpha\cdot e = e$, therefore $\{e\}\in\alpha\cdot\mathcal{K}$. Now, let $X\in\alpha\cdot\mathcal{X}$ and let $K\in\alpha\cdot\mathcal{K}$. Then X is of the form $\{\alpha\cdot\chi : \chi\in X'\}$ for some $X'\in\mathcal{X}$ and K is of the form $\{\alpha\cdot x : x\in K'\}$ for some $K'\in\mathcal{K}$. Thus, if g and h lie in K, then $\alpha^{-1}\cdot g$ and $\alpha^{-1}\cdot h$ lie in K', hence we have

$$\begin{aligned}\sigma_X(g) &= \sum_{\chi\in X'}(\alpha\cdot\chi)(g) \\ &= \sum_{\chi\in X'}\chi(e)\chi(\alpha^{-1}\cdot g) \\ &= \sum_{\chi\in X'}\chi(e)\chi(\alpha^{-1}\cdot h) \\ &= \sigma_X(h).\end{aligned}$$

Thus, the Wedderburn sums of parts of $\alpha\cdot\mathcal{X}$ are constant on the parts of $\alpha\cdot\mathcal{K}$. Therefore, we have proven directly that $(\alpha\cdot\mathcal{K},\alpha\cdot\mathcal{X})$ is a supercharacter theory of G. ∎

Let $\alpha\cdot S$ denote the supercharacter theory in the previous lemma. This supercharacter theory defines a group action of A on $\mathrm{SCT}(G)$, which is order-preserving. By [DI08, Theorem 2.2(f)], it follows that $\mathrm{Gal}(G)$ acts trivially on $\mathrm{SCT}(G)$. However, the action of $\mathrm{Aut}(G)$ on $\mathrm{SCT}(G)$ is more interesting and, moreover, yields a canonical sublattice of $\mathrm{SCT}(G)$, which we study in this section.

Call a supercharacter theory S **characteristic** if $\alpha\cdot S = S$ for all $\alpha\in\mathrm{Aut}(G)$, and write $\mathrm{CharSCT}(G)$ for the subposet of all characteristic supercharacter theories of G. More generally, if A is any group with compatible actions on G and on $\mathrm{Irr}(G)$, we will call a supercharacter theory S **A-characteristic** if $\alpha\cdot S = S$ for all $\alpha\in A$.

It should be clear that the notion of being characteristic here shares a property of the usual notion of being characteristic of a subgroup: If a supercharacter theory is defined by its uniqueness in the possession of a property invariant with respect to the action of $\mathrm{Aut}(G)$, then it must be characteristic. In much the same way, knowledge of the characteristic supercharacter theories of a group G may aid in the classification of the full lattice $\mathrm{SCT}(G)$. We will see an explicit example of this in Chapter 4. It is our belief that the general strategy of first characterizing $\mathrm{CharSCT}(G)$ could be applied to other families of groups.

It is natural to ask what the relationship is between the subposets $\mathrm{GalSCT}(G)$, $\mathrm{AutSCT}(G)$, and $\mathrm{CharSCT}(G)$. While there is not much to say in general, we list some small results here.

Proposition 3.16. *Let $m_A(G)$ be the automorphic supercharacter theory induced by $A \subseteq \mathrm{Aut}(G)$. If $A \trianglelefteq \mathrm{Aut}(G)$, then $m_A(G)$ is characteristic.*

Proof. Write $m_A(G) = (\mathcal{K}, \mathcal{X})$. We glue characters by the rule $\chi \sim \psi$ if $\psi = \sigma \cdot \chi$ for some $\sigma \in A$. Now let $\chi \in \mathrm{Irr}(G)$ and let X be the part of $\mathcal{X}$ containing χ. Let $\tau \in \mathrm{Aut}(G)$ and consider $\tau \cdot X$. It suffices to show that this set lies in $\mathcal{X}$, i.e., that $\tau \cdot X$ is the part of $\mathcal{X}$ that contains $\tau \cdot \chi$. Let $\psi \in X$ and let Y be the part of $\mathcal{X}$ containing $\tau \cdot \chi$. Then $\psi = \sigma \cdot \chi$ for some $\sigma \in A$. Hence

$$\begin{aligned}\tau \cdot \psi &= (\tau\sigma) \cdot \chi \\ &= (\tau\sigma\tau^{-1}) \cdot (\tau \cdot \chi) \\ &\sim \tau \cdot \chi,\end{aligned}$$

so that $\tau \cdot X \subseteq Y$. But this is true for all $\chi \in \mathrm{Irr}(G)$ and $\tau \in \mathrm{Aut}(G)$, so the reverse inclusion is implied. Therefore, $m_A(G)$ is characteristic. ∎

Proposition 3.17. *Let G be a finite group. Then we have* $\mathrm{GalSCT}(G) \subseteq \mathrm{CharSCT}(G)$.

Proof. Let $m_A(G) \in \mathrm{GalSCT}(G)$ be the Galois supercharacter theory induced by the subgroup $A \subseteq \mathrm{Gal}(G)$. For $\chi, \psi \in \mathrm{Irr}(G)$, write $\chi \sim \psi$ if χ and ψ lie in the same block of $\mathcal{X}(m_A(G))$, or equivalently, if $\psi = \sigma \cdot \chi$ for some $\sigma \in A$. It suffices to show for any $\alpha \in \mathrm{Aut}(G)$ that $\alpha \cdot \chi \sim \alpha \cdot \psi$ if and only if $\chi \sim \psi$. Since function composition is associative, it follows that

$$\alpha \cdot (\sigma \cdot \chi) = \sigma \circ \chi \circ \alpha^{-1} = \sigma \cdot (\alpha \cdot \chi^{\alpha})$$

for any $\sigma \in A$ and $\alpha \in \mathrm{Aut}(G)$. Thus, $\psi = \sigma \cdot \chi$ if and only if $\alpha \cdot \psi = \sigma \cdot (\alpha \cdot \chi)$, which proves our claim. Hence $m_A(G) \in \mathrm{CharSCT}(G)$. ∎

The following result is important, as it proves that $\mathrm{CharSCT}(G)$ is a sublattice of $\mathrm{SCT}(G)$ and not merely a subposet.

Proposition 3.18. *Let G be a finite group and let S and T be characteristic supercharacter theories of G. Then $S \wedge T$ and $S \vee T$ are characteristic.*

Proof. Let $M \in \mathcal{K}(S \wedge T)$. Then there exist $K \in \mathcal{K}(S)$ and $L \in \mathcal{K}(T)$ such that $M \subseteq K \cap L$. Let $\sigma \in \mathrm{Aut}(G)$. Then $\sigma \cdot (K \cap L) = (\sigma \cdot K) \cap (\sigma \cdot L)$, hence $\sigma \cdot M \subseteq (\sigma \cdot K) \cap (\sigma \cdot L)$. Thus, $\sigma \cdot (S \wedge T) \leq S, T$, and therefore $\sigma \cdot (S \wedge T) \leq S \wedge T$. But $|S \wedge T| = |\sigma \cdot (S \wedge T)|$, which implies that $\sigma \cdot (S \wedge T) = S \wedge T$, and therefore $S \wedge T$ is characteristic.

Now let $M \in \mathcal{K}(S \vee T)$. Then there exist subsets $\{K_i : i \in I\} \subseteq \mathcal{K}(S)$ and $\{L_j : j \in J\} \subseteq \mathcal{K}(T)$ such that

$$M = \bigcup_{i \in I} K_i = \bigcup_{j \in J} L_j.$$

Hence for any $\sigma \in \mathrm{Aut}(G)$, we have

$$\sigma \cdot M = \bigcup_{i} (\sigma \cdot K_i) = \bigcup_{j} (\sigma \cdot L_j),$$

and hence $\sigma \cdot M$ is a union of parts of $\mathcal{K}(S)$ and of $\mathcal{K}(T)$. Consequently, $\sigma \cdot (S \vee T) \geq S, T$, and we conclude as above that $\sigma \cdot (S \vee T) = S \vee T$. ∎

Another important result is that the Δ-product (and therefore also the $*$-product) respects the property of being characteristic.

Lemma 3.19. *Let $N \leq M$ be characteristic subgroups of G, let $S = (\mathcal{K}, \mathcal{X}) \in \mathrm{SCT}_G(M)$ be such that N is supernormal, and let $T = (\mathcal{L}, \mathcal{Y}) \in \mathrm{SCT}(G/N)$. Then $S \,\Delta\, T$ is characteristic if and only if S is A-characteristic and T is B-characteristic, where $A \subseteq \mathrm{Aut}(M)$ and $B \subseteq \mathrm{Aut}(G/N)$ are the images of $\mathrm{Aut}(G)$ under the canonical maps $\mathrm{Res}^G_M : \mathrm{Aut}(G) \to \mathrm{Aut}(M)$ and $\mathrm{Def}^G_{G/N} : \mathrm{Aut}(G) \to \mathrm{Aut}(G/N)$, respectively.*

Proof. Suppose S is A-characteristic and T is B-characteristic. Let $\alpha \in \mathrm{Aut}(G)$ and consider the action of α on $S \mathbin{\Delta} T$. Since N is characteristic, α permutes the superclasses of S via the action of $\mathrm{Res}^G_M(\alpha)$. Let L be a superclass of T and let $\widetilde{L} = \bigcup_{gN \in L} gN$ be its preimage under the canonical map $G \to G/N$. Similarly, let L' be the image of L under $\mathrm{Def}^G_{G/N}(\alpha)$ and let $\widetilde{L}'$ be its preimage. We are done if we can show that $\alpha(L) = L'$. But this is verified directly:

$$\alpha(\widetilde{L}) = \alpha\left(\bigcup_{x \in L} xN\right) = \bigcup_{x \in L} \alpha(xN) = \bigcup_{x \in L} \alpha(x)N = \widetilde{L}'.$$

Thus, $\mathrm{Aut}(G)$ permutes the superclasses of $S \mathbin{\Delta} T$, so this supercharacter theory is characteristic.

Conversely, suppose $S \mathbin{\Delta} T$ is characteristic. Then its superclasses are permuted by $\mathrm{Aut}(G)$. Since M is a characteristic subgroup, the superclasses contained in $\mathcal{K}$ are permuted amongst themselves, and the superclasses that are inherited from $\mathcal{L}$ are permuted amongst themselves. This immediately implies that S is A-characteristic. As before, let $\alpha \in \mathrm{Aut}(G)$ and consider the action of $\mathrm{Def}^G_{G/N}(\alpha)$ on $\mathcal{L}$. Let $L \in \mathcal{L}$ and let $\widetilde{L}$ be its preimage under the canonical map $G \to G/N$. Then by the preceeding remarks, $\alpha(\widetilde{L}) = \widetilde{L}'$ for some $L' \in \mathcal{L}$. If $L = \{g_1, \ldots, g_a\}$ and $L' = \{h_1, \ldots, h_b\}$, then we have $\bigcup_{i=1}^{a} \alpha(g_i)N = \bigcup_{j=1}^{b} h_jN$. That these are disjoint unions of distinct cosets implies that $\mathrm{Def}^G_{G/N}(\alpha)(L) = L'$. Thus, T is B-characteristic. ∎

In the definition of the $*$-product in Chapter 2, we defined the notion of a G-invariant supercharacter theory of N, where N is a normal subgroup of G. More generally, we can let A be any group with compatible actions on G and $\mathrm{Irr}(G)$, and define an **A-invariant supercharacter theory of** G to be any supercharacter theory S of G such that the actions of A on $\mathcal{K}(S)$ and $\mathcal{X}(S)$ (given in (3.4) and (3.5)) are trivial. Let $\mathrm{InvSCT}_A(G)$ denote the set of all A-invariant supercharacter theories of G.

Lemma 3.20. *Let G be a group, let A be a group which acts compatibly on G and on $\mathrm{Irr}(G)$, and let $S \in \mathrm{SCT}(G)$. Then S is A-invariant if and only if its superclasses and supercharacters are unions of A-orbits in G and $\mathrm{Irr}(G)$, respectively.*

Proof. Suppose S is A-invariant. Then for all $K \in \mathcal{K}(S)$ and $\alpha \in A$, we have $\alpha \cdot K = K$. Thus for each $g \in K$, the A-orbit of g is a subset of K, and therefore K is a union of A-orbits. By a similar argument, the parts of $\mathcal{X}$ are unions of A-orbits.

Conversely, suppose the parts of $\mathcal{K}(S)$ and $\mathcal{X}(S)$ are unions of A-orbits on G and on $\mathrm{Irr}(G)$, respectively. Then for each $K \in \mathcal{K}(S)$, $g \in K$, and $\alpha \in A$, we have $\alpha \cdot g \in K$, hence $\alpha \cdot K = K$. Thus, A acts trivially on $\mathcal{K}(S)$. By a similar argument, A acts trivially on $\mathcal{X}(S)$, and therefore S is A-invariant. ∎

Hence, $m_A(G)$ is the minimal A-invariant supercharacter theory of G and $\mathrm{InvSCT}_A(G)$ is equal to the interval $[m_A(G), M(G)]$. As an interval, it is clear that $\mathrm{InvSCT}_A(G)$ is a sublattice of $\mathrm{SCT}(G)$.

Example 3.21. (1) As explained in Chapter 2, if G acts on its normal subgroup N by conjugation, then $\mathrm{InvSCT}_G(N)$ is the set of supercharacter theories of N whose superclasses are unions of G-conjugacy classes.

(2) If $A = \mathrm{Gal}(G)$, then A-invariant supercharacter theories are precisely those whose supercharacter tables contain only rational values.

(3) If $A = \mathrm{Aut}(G)$, then A-invariant supercharacter theories are those whose superclass and supercharacter partitions are unions of $\mathrm{Aut}(G)$-orbits. This is not to be confused with the sublattice of characteristic supercharacter theories (the fixed points of the action of $\mathrm{Aut}(G)$ on $\mathrm{SCT}(G)$), whose superclass and supercharacter partitions may be permuted nontrivially.

3.3.1 Cyclic groups

Let $\mathbb{Z}_n$ denote the cyclic group of order n. The supercharacter theories of cyclic groups are classified in [Hen08] and [Hen12], using work of Leung and Man on Schur rings (see [LM96] and [LM98]). We digress to restate some of Hendrickson's results on cyclic groups of prime order to obtain an isomoprhism between $\mathrm{SCT}(\mathbb{Z}_p)$ and the lattice of divisors of $p-1$.

Lemma 3.22. *[Hen08, Lemma 6.9] Let G be a cyclic group of prime order. Then every supercharacter theory of G is automorphic.*

If G is cyclic, the action of $\mathrm{Aut}(G)$ on G is permutation isomorphic to the action of $\mathrm{Gal}(G)$ on $\mathrm{Irr}(G)$. Since $\mathrm{Gal}(G)$ acts trivially on $\mathrm{SCT}(G)$ (see, e.g., [DI08, Theorem 2.2]), this proves the following result.

Lemma 3.23. *[Hen08, Lemma 6.2] If G is a cyclic group, then $\mathrm{Aut}(G)$ acts trivially on $\mathrm{SCT}(G)$.*

Furthermore, if G is cyclic of prime order, the action of $\mathrm{Aut}(G)$ on G is transitive on the nonidentity elements of G, which proves the following lemma.

Lemma 3.24. *[Hen08, Lemma 6.4] Let G be a cyclic group of prime order and let $(\mathcal{K}, \mathcal{X})$ be a supercharacter theory. Then every nontrivial superclass contains the same number of elements.*

Tying all of these results together, we can explicitly describe an isomorphism between $\mathrm{SCT}(\mathbb{Z}_p)$ and the lattice of divisors of $p-1$.

Proposition 3.25. *Let $G = \langle x \rangle$ be a cyclic group of prime order $|G| = p$. Then for each divisor d of $p-1$, there exists a unique supercharacter theory of G whose nontrivial superclasses all have size d. In fact, these are all of the supercharacter theories of G.*

Proof. Since G has prime order, its automorphism group is cyclic of order $p-1$, say $\mathrm{Aut}(G) = \langle \sigma \rangle$, and note that $\mathrm{Aut}(G)$ is in fact faithful on G. List the nontrivial elements of G as $x_0, x_1, \ldots, x_{p-1}$, where $\sigma \cdot x_i = x_{i+1}$ for all i (i is taken modulo $p-1$). Let d be a divisor of $p-1$, and let $e = (p-1)/d$. Then one checks that the supercharacter theory induced by $\langle \sigma^e \rangle$ has a nontrivial superclass of size d; indeed one such class is

$$K_i = \{x_0, x_e, \ldots, x_{(d-1)e}\}.$$

By Lemma 3.24, it follows that all other nontrivial superclasses have size d.

Now suppose $S = (\mathcal{K}, \mathcal{X})$ and $T = (\mathcal{L}, \mathcal{Y})$ are two supercharacter theories with superclasses of size d. By Lemma 3.22, these are both induced by subgroups $\langle \alpha \rangle$ and $\langle \beta \rangle$ of $\mathrm{Aut}(G)$, respectively.

Since both subgroups are faithful and transitive on the nontrivial superclasses, they both have the same order, whence they are equal. Now, let $K = \{x_0, \ldots, x_d\}$ and $L = \{y_0, \ldots, y_d\}$ be nontrivial superclasses of S and T, respectively. We may, without loss of generality, assume $\alpha \cdot x_i = x_{i+1}$ and $\alpha \cdot y_i = y_{i+1}$. Then as $\mathrm{Aut}(G)$ is transitive on $G \setminus \{1\}$, there is some $\gamma \in \mathrm{Aut}(G)$ such that $\gamma \cdot x_0 = y_0$. Then acting by α yields that $\gamma \cdot K = L$. But $\gamma \cdot S = S$, so $L \in \mathcal{K}$. But this is true for all $L \in \mathcal{L}$, therefore $T = S$. ∎

Corollary 3.26. *Let G be a cyclic group of prime order and let $S = (\mathcal{K}, \mathcal{X})$ and $T = (\mathcal{L}, \mathcal{Y})$ be supercharacter theories. Then we have*

$$S \wedge T = (\mathcal{K} \wedge \mathcal{L}, \mathcal{X} \wedge \mathcal{Y}).$$

The proof of Proposition 3.25 relies heavily on the fact that the orbits of any subgroup of $\mathrm{Aut}(G)$ coincide with the orbits of some subgroup of $\mathrm{Gal}(G)$. The following example shows that this is not true in general.

Example 3.27. Let $G = M_{11}$ be the Mathieu group of order 7920. Then the automorphism group and character table of G are both known. Since the outer automorphism group of G is trivial, it follows that $m_A(G) = m(G)$ for any subgroup A of $\mathrm{Aut}(G)$. However, the character table of G contains entries that are not rational. Thus, $m_{\mathrm{Gal}(G)}(G)$ is a strict coarsening of $m(G)$.

3.4 Miscellany

In the final section of this chapter, we will discuss some small results and directions for future work.

3.4.1 Computing $\mathrm{SCT}(G)$

In [Hen08, Appendix A], Hendrickson developed an algorithm for computing the supercharacter theory lattice of a finite group G, given that group's character table. That algorithm builds superclass partitions part-by-part by checking that the new parts are well-behaved with respect to the others. Here, we develop a "dualized" algorithm which builds supercharacter partitions part-by-part using similar ideas, albeit with Wedderburn sums rather than superclasses. Much of what follows appears in [BLLW17], and although Algorithm 3.30 appears first in this thesis, it is a direct adaptation of [Hen08, Algorithm A.9].

A **partial partition** of $\operatorname{Irr}(G)$ is a set $\mathcal{X}$ of pairwise disjoint subsets of $\operatorname{Irr}(G)$. Let $\mathcal{X}$ be a partial partition of $\operatorname{Irr}(G)$ and let $\mathcal{A}$ be the subalgebra of $\operatorname{cf}(G)$ generated by $\{\sigma_X : X \in \mathcal{X}\}$. Define an equivalence relation on $\operatorname{Irr}(G)$ such that $\chi \sim \psi$ if and only if the coefficients of $\chi(e)\chi$ and $\psi(e)\psi$ in a are the same (i.e., if $\frac{1}{\chi(e)}\langle a, \chi\rangle = \frac{1}{\psi(e)}\langle a, \psi\rangle$) for all $a \in \mathcal{A}$. Then the **support** of $\mathcal{X}$, denoted $\mathcal{S}(\mathcal{X})$, is the partition of $\operatorname{Irr}(G)$ into equivalence classes with respect to this relation. The algebra $\mathcal{A}$ lies in the span of $\{\sigma_Y : Y \in \mathcal{S}(\mathcal{X})\}$, so each part of $\mathcal{X}$ is a union of parts of $\mathcal{S}(\mathcal{X})$. If $\mathcal{X} \subseteq \mathcal{S}(\mathcal{X})$, then we say $\mathcal{X}$ is **admissible**.

Lemma 3.28. *Let $\mathcal{X}$ be a partial partition of $\operatorname{Irr}(G)$. Suppose there exists a supercharacter theory $S = (\mathcal{Y}, \mathcal{L})$ such that $\mathcal{X} \subseteq \mathcal{Y}$. Then $\mathcal{Y}$ is a refinement of $\mathcal{S}(\mathcal{X})$ and $\mathcal{X}$ is admissible.*

Proof. Let $\mathcal{A}$ be the subalgebra of $\operatorname{cf}(G)$ generated by $\{\sigma_X : X \in \mathcal{X}\}$. Then we have

$$\mathcal{A} = \langle \sigma_X : X \in \mathcal{X}\rangle \subseteq \langle \sigma_Y : Y \in \mathcal{Y}\rangle = \operatorname{Span}\big\{\sigma_Y : Y \in \mathcal{Y}\big\},$$

where the last equality follows from [And14, Lemma 2.1]. Let $Y \in \mathcal{Y}$ be a supercharacter and let $\chi, \psi \in Y$; then for all $a \in \mathcal{A}$, the coefficients of $\chi(1)\chi$ and $\psi(1)\psi$ are the same because $a \in \operatorname{Span}\{\sigma_Z : Z \in \mathcal{Y}\}$. Then χ and ψ lie in the same part of $\mathcal{S}(\mathcal{X})$ by definition, so it follows that $\mathcal{Y} \leq \mathcal{S}(\mathcal{X})$.

Now for each part X of $\mathcal{X}$, it follows that because X is a union of parts of $\mathcal{F}(\mathcal{X})$, we can choose a part $Y \in \mathcal{S}(\mathcal{X})$ such that $Y \subseteq X$. Then since $\mathcal{Y} \leq \mathcal{S}(\mathcal{X})$, the set Y must be a union of parts of $\mathcal{Y}$, and since the only part of $\mathcal{Y}$ overlapping X is X itself, we have $X \subseteq Y$. Hence we have $X = Y \in \mathcal{S}(\mathcal{X})$. We conclude that $\mathcal{X} \subseteq \mathcal{S}(\mathcal{X})$, and the proof is complete. ∎

By Lemma 3.28, the support of a partial partition $\mathcal{X}$ is coarser than every supercharacter theory that contains the parts of $\mathcal{X}$ as supercharacters. By applying this observation to a single set of characters $X \subseteq \mathrm{Irr}(G)$, we derive a necessary condition for the existence of a supercharacter theory in which X is a supercharacter. Let G be a group and let X be a subset of $\mathrm{Irr}(G)$. We say X is **good** if $X \in \mathcal{F}(\{X\})$; otherwise we say X is **bad**.

Corollary 3.29. *Let G be a group with exactly n irreducible characters.*

(1) Let $S \in \mathrm{SCT}(G)$ and let $X \in \mathcal{X}(S)$. Then X is good.

(2) Let X be a subset of $\mathrm{Irr}(G)$. Then X is bad if and only if there exist characters $\chi, \psi \in X$ and an integer $j \in \{2, \ldots, n\}$ such that the coefficients of $\chi(1)\chi$ and $\psi(1)\psi$ in σ_X^j differ.

Proof. Let $S = (\mathcal{Y}, \mathcal{L})$ and let X be a supercharacter. Then $\{X\} \subseteq \mathcal{Y}$, so by Lemma 3.28, we know $\{X\} \subseteq \mathcal{S}(\{X\})$. Then by definition X is good, proving part (a).

Let X be a subset of $\mathrm{Irr}(G)$, and suppose X is bad. Now, X is a union of parts of $\mathcal{S}(\{X\})$, but X is not itself a part of $\mathcal{S}(\{X\})$, because it is bad. Therefore there exist elements $\chi, \psi \in X$ that lie in different parts of $\mathcal{S}(\{X\})$, so there exists some element $a \in \langle \sigma_X \rangle$ such that the coefficients of $\chi(1)\chi$ and $\psi(1)\psi$ in a are different. Now because $\langle \sigma_X \rangle \subseteq \mathrm{cf}(G)$ is at most n dimensional, it follows that $\langle \sigma_X \rangle$ is spanned by $\{\sigma_X, \sigma_X^2, \ldots, \sigma_X^n\}$. Then if the coefficient of $\chi(1)\chi$ in σ_X^j were equal to that of $\psi(1)\psi$ for all $j \in \{1, \ldots, n\}$, it would follow that the coefficients of $\chi(1)\chi$ and $\psi(1)\psi$ would be identical in the element a of $\langle \sigma_X \rangle$, a contradiction. Hence there exists some $j \in \{1, \ldots, n\}$ such that $\chi(1)\chi$ and $\psi(1)\psi$ have different coefficients in σ_X^j, and j is certainly not 1.

On the other hand, if there exist different elements $\chi, \psi \in X$ and an integer j such that $\chi(1)\chi$ and $\psi(1)\psi$ have different coefficients in $\sigma_X^j \in \langle \sigma_X \rangle$, then χ and ψ lie in different parts of $\mathcal{S}(\{X\})$, so X is not a part of $\mathcal{S}(\{X\})$. This completes the proof of part (b). ∎

Algorithm 3.30 (SCTFinder)**.** Given a group G with exactly n irreducible characters and an admissible partial partition $\mathcal{X}$ of $\mathrm{Irr}(G)$, this algorithm returns the supercharacter partition of every supercharacter theory of G that contains the parts of $\mathcal{X}$ as supercharacters.

1. Label the irreducible characters of G as $\mathrm{Irr}(G) = \{\chi_1, \ldots, \chi_n\}$.

2. Let k be the smallest integer such that χ_k is not contained in any part of $\mathcal{X}$.

3. Let S be the part of $\mathcal{S}(\mathcal{X})$ containing χ_k.

4. For each subset Y of S containing χ_k, do:

   a*. Let $\mathcal{Y} = \mathcal{X} \cup \{Y\}$.

   b*. If $\mathcal{Y}$ is admissible, then do:

      i. If $\mathcal{Y}$ is a full partition of $\mathrm{Irr}(G)$, record it as a supercharacter theory.

      ii. Otherwise, call this algorithm with $\mathcal{Y}$ in place of $\mathcal{X}$.

   c*. Otherwise, continue.

By [DI08, Theorem 2.2], a supercharacter partition $\mathcal{X}$ uniquely determines its supercharacter theory; indeed the superclass partition is the unique coarsest partition of G on whose parts the Wedderburn sums of $\mathcal{X}$ are constant, and this is easily computed using the character table. Thus, Algorithm 3.30 returns all supercharacter theories whose supercharacter partitions contain $\mathcal{X}$, and consequently, calling this algorithm on $\{\{\mathbf{1}_G\}\}$ will return all supercharacter theories. The choice of a character χ_k to lie in the new part Y is arbitrary, but choosing the index to be minimal eliminates repetition. Thus, supercharacter theories are only found once. The author has implemented Algorithm 3.30 using the Sage computer algebra system [The17]; this may be found at `github.com:jonathanlamar/sct_finder`.

Turning our attention to the group $G = \mathrm{Sp}(6,2)$, it will follow from Corollary 3.29 that this group has only the two trivial supercharacter theories if we show that the only good subsets of $\mathrm{Irr}(G) \setminus \{\mathbf{1}_G\}$ are the whole set and its singleton subsets. By that same corollary, we need only to check powers of Wedderburn sums against their underlying sets' supports. The following algorithm does exactly this.

Algorithm 3.31. Given any group G and any subset $X = \{\chi_i : i \in I\}$ of $\mathrm{Irr}(G)$, this algorithm determines if X is good. Suppose G has precisely n conjugacy classes with representatives $g_1, \ldots, g_n$ and irreducible characters $\chi_1, \ldots, \chi_n$, and let T be the character table, represented as an $n \times n$ matrix whose i, j entry is $\chi_i(g_j)$.

(1) Form the Wedderburn sum σ_X corresponding to X. Then with respect to the conjugacy class identifier basis, σ_X takes the form

$$\sigma_X = \sum_{j=1}^{n} \sigma_X(g_j)\delta_{[g_j]},$$

where $\delta_{[g_j]}$ is the indicator function of the conjugacy class containing g_i.

(2) For each $k \in \{2, \dots, n\}$, do the following.

(a) Consider the kth tensor power of σ_X; with respect to the above basis, we have

$$\sigma_X^k = \sum_{j=1}^{n} \sigma_X(g_j)^k \delta_{[g_j]}.$$

With respect to the irreducible character basis, write

$$\sigma_X^k = \sum_{i=1}^{n} c_{i,k}\chi_i(1)\chi_i.$$

(b) Set $m = \min(I)$. If there exists $i \in I$ for which $c_{i,k} \neq c_{m,k}$, return bad.

(c) Otherwise, continue.

(3) Return good.

We can then record the output of this function for each subset of $\mathrm{Irr}(\mathrm{Sp}(6,2))$. This method will not find nontrivial supercharacter theories, however it can negatively answer the question of whether any exist. In order to reduce runtime, we record the output only on subsets of $\mathrm{Irr}(\mathrm{Sp}(6,2))$ which we did not already know to be good, i.e., proper nonsingleton subsets of $\mathrm{Irr}(\mathrm{Sp}(6,2)) \setminus \{\mathbf{1}_{\mathrm{Sp}(6,2)}\}$. Since Algorithm 3.31 returns bad for each of these sets, it follows that $\mathrm{Sp}(6,2)$ has exactly two supercharacter theories.

Combining the above result with [BLLW17, Corollary 3] and [BLLW17, Theorem 4], we obtain a list of all finite groups with only trivial supercharacter theories.

Theorem 3.32. *[BLLW17] The groups $\mathbb{Z}_3$, S_3, and $\mathrm{Sp}(6,2)$ each have only two supercharacter theories. These are the only finite groups with only two supercharacter theories.*

3.4.2 Alternating groups

Fix $n \geq 7$ and consider the alternating group A_n. The automorphism group of A_n is isomorphic to S_n and its action on A_n is equivalent to conjugation by this group. The nontrivial orbits of the action of S_n on $\mathrm{Cl}(A_n)$ are therefore easy to describe: these are the **split** classes, i.e., those pairs of conjugacy classes in A_n whose union form a single conjugacy class in S_n. It is well-known (see [FH04, Section 5.1]) that an S_n-conjugacy class which lies in A_n splits if and only if its elements have a cycle type consisting of distinct odd integers. The behavior of the irreducible characters of S_n under restriction to A_n is classified by the following result.

Proposition 3.33. *[FH04, Proposition 5.1] Let χ^λ be an irreducible character of S_n, and let $\psi^\lambda = \mathrm{Res}^{S_n}_{A_n}(\chi^\lambda)$. Then exactly one of the following holds:*

(1) χ^λ is not equal to $\chi^{\lambda'}$, ψ^λ is irreducible and equal to its conjugate, and $\mathrm{Ind}^{S_n}_{A_n}(\psi^\lambda) = \chi^\lambda + \chi^{\lambda'}$;

(2) χ^λ is equal to $\chi^{\lambda'}$, $\psi^\lambda = \psi^\lambda_1 + \psi^\lambda_2$, where ψ^λ_1 and ψ^λ_2 are irreducible and conjugate but not equal, and $\mathrm{Ind}^{S_n}_{A_n}(\psi^\lambda_1) = \mathrm{Ind}^{S_n}_{A_n}(\psi^\lambda_2) = \chi^\lambda$.

Each irreducible character of A_n arises uniquely in this way.

Lemma 3.34. *Let $\lambda \vdash n$ be any partition whose corresponding S_n-conjugacy class lies in A_n and splits there. Then the partition $\mathcal{K}$ of $\mathrm{Cl}(A_n)$ whose parts are all singletons, save for the union of the two A_n-conjugacy classes of elements of cycle type λ, determines a supercharacter theory of A_n.*

Proof. Let μ be the symmetric partition of n whose diagonal hook lengths are the parts of λ: $h_\mu(i,i) = \lambda_i$ for all i. Then the symmetric group character χ^μ splits in A_n, i.e., $\mathrm{Res}^{S_n}_{A_n}(\chi^\mu) = \chi^\mu_1 + \chi^\mu_2$. Let $\mathcal{X}$ be the following partition of $\mathrm{Irr}(A_n)$:

$$\mathcal{X} = \{\psi^\mu_1, \psi^\mu_2\} \cup \big(\mathrm{Irr}(A_n) \smallsetminus \{\psi^\mu_1, \psi^\mu_2\}\big).$$

We claim that $(\mathcal{K}, \mathcal{X})$ form the desired supercharacter theory. Clearly the trivial character is a part of $\mathcal{X}$, since (1^n) is not symmetric. Moreover, $\mathcal{K}$ and $\mathcal{X}$ have the same number of parts, so it suffices to show that the Wedderburn sums σ_X for $X \in \mathcal{X}$ are constant on the only nontrivial superclass. But this follows immediately from [FH04, Proposition 5.3]. ∎

Let $S(\lambda)$ denote the supercharacter theory obtained by gluing the two split conjugacy classes of cycle type λ. Then $S(\lambda)$ is clearly an atom of $\mathrm{SCT}(A_n)$, since it has only one superclass which is not a conjugacy class.

Conjecture 3.35. *For all $n \geq 7$, every proper supercharacter theory of A_n is a join of atoms of the form $S(\lambda)$.*

Since the join of all of these atoms is the minimal S_n-invariant supercharacter theory $m_{S_n}(A_n)$, the above conjecture holds if and only if $m_{S_n}(A_n)$ is the unique coatom of $\mathrm{SCT}(A_n)$.

Conjecture 3.36. *For all $n \geq 7$, $m_{S_n}(A_n)$ is a coatom of* $\mathrm{SCT}(A_n)$.

This is a useful waypoint to classifying $\mathrm{SCT}(A_n)$, because any supercharacter theory S that lies outside of the join-closure of atoms can be joined with $m_{A_n}(S_n)$, and this join will be $M(A_n)$. This implies that S is relatively coarse. In fact, it must clump everything except possibly some halves of split characters. Thus S is likely to be $M(A_n)$.

Chapter 4

Supercharacter theories of dihedral groups

Let D_{2n} be the dihedral group of order $2n$. In this chapter, we will provide an explicit classification of both $\mathrm{SCT}(D_{2n})$ and $\mathrm{CharSCT}(D_{2n})$ using the subgroup of rotations. Wynn independently obtained similar results in [Wyn17], however the work that comprises the present chapter was complete and a preprint (see [Lam16]) was made available before the publication of that thesis. Moreover, we believe this method of classification would be useful in any attempt at enumerating the supercharacter theories of D_{2n}. In Section 4.4, we will prove some partial results and state some conjectures regarding the classification of $\mathrm{SCT}(\mathbb{Z}_n \rtimes \mathbb{Z}_p)$, where $n \geq 1$ and p is a prime divisor of $|\mathrm{Aut}(\mathbb{Z}_n)|$.

In [Wyn17], Wynn used the structures of Camina pairs—and specifically of Frobenius groups—in his classification. We pause to discuss this work.

A **Frobenius group** is a finite group G with a proper and nontrivial subgroup H, called the **Frobenius complement**, with the property that $H \cap H^g = 1$ for all $g \in G \smallsetminus H$. Every Frobenius group has a unique normal subgroup K, called the **Frobenius kernel**, with the property that for all $k \in K \smallsetminus \{e\}$, the centralizer in H of k is trivial. A Frobenius group is an instance of a more general object, known as a Camina pair. A **Camina pair** is an ordered pair (G, N), where G is a finite group and N is a normal subgroup of G with the property that for all $g \in G \smallsetminus N$, $gN \subseteq \mathrm{Cl}(g)$. An equivalent condition is the following: for all $\chi \in \mathrm{Irr}(G|N)$, χ vanishes on $G \smallsetminus N$.

The first classification in [Wyn17] that overlaps with this chapter is [Wyn17, Theorem 1.2]. In informal terms, this theorem states that if G is a Frobenius group with Frobenius kernel K, then any supercharacter theory $S \in \mathrm{SCT}(G)$ either factors as a $*$-product over K, or else S factors as a Δ-product over normal subgroups which sit in relation to K and which have a particular structure (see Section 4.2.1 for a precise statement). Since every odd dihedral group (i.e., D_{2n}, where n is odd) is a Frobenius group with Frobenius kernel $K = \langle r \rangle$, this theorem provides a complete classification of the supercharacter theories of odd dihedral groups. In Section 4.2.1, we discuss the equivalence between this result and ours.

The classification of $\mathrm{SCT}(D_{2n})$ proven by Wynn is the following theorem, which expands the above result to arbitrary n, but which is less structured.

Theorem 4.1. *[Wyn17, Theorem 6.2] Let S be a supercharacter theory of D_{2n} other than $M(D_{2n})$. Then S is either a $*$-product or a Δ-product.*

Our classification will have a more combinatorial flavor. We will classify the sublattice $\mathrm{CharSCT}(D_{2n})$ by first defining $\mathcal{P}$ to be the $*$-products over the subgroup of all rotations (star products over a subgroup of index two have a particularly trivial formula), and obtaining the remainder of $\mathrm{CharSCT}(D_{2n})$ through two order-preserving functions φ and ψ, whose domains are subposets $\mathcal{Q}$ and $\mathcal{R}$ of $\mathcal{P}$, respectively, and whose ranges are both subposets of $\mathrm{CharSCT}(D_{2n})$. Then $\mathrm{CharSCT}(D_{2n})$ is simply the disjoint union $\mathcal{P} \sqcup \varphi(\mathcal{Q}) \sqcup \psi(\mathcal{R})$. Moreover, φ and ψ may both be defined in terms of gluing two superclasses (in the case of φ) and splitting a superclass in two (in the case of ψ). See Figure 4.1 for a visual depiction of this process, and Theorem 4.7 for a precise statement of the classification.

We will then classify the non-characteristic supercharacter theories of D_{2n} by first generalizing ψ into two order-preserving functions ψ_0 and ψ_1, both defined on the same domain $\mathcal{S}$ which contains $\mathcal{R}$ as a subposet. Then $\mathrm{SCT}(D_{2n})$ is the union $\mathrm{CharSCT}(D_{2n}) \cup \psi_0(\mathcal{S}) \cup \psi_1(\mathcal{S})$. The maps ψ_0 and ψ_1 may both be defined in terms of splitting a specific superclass into two smaller superclasses in two different ways. Moreover, it will follow that $\psi_0(S) = \psi_1(S) = \psi(S)$ for all $S \in \mathcal{R}$ and that the nontrivial $\mathrm{Aut}(D_{2n})$-orbits on $\mathrm{SCT}(D_{2n})$ are precisely the sets $\{\psi_0(S), \psi_1(S)\}$ for $S \in \mathcal{S} \setminus \mathcal{R}$. See Theorem 4.8 for a precise statement of this result.

4.1 Definitions and main results

Write

$$D_{2n} = \langle r, s \,:\, r^n = s^2 = e,\, srs = r^{-1}\rangle = \langle r\rangle \rtimes \langle s\rangle.$$

The character tables of dihedral groups can be computed easily. For odd n, they take the form

class rep.	e	$r^k;\ 1 \le k \le \frac{n-1}{2}$	s
class size	1	2	n
$\mathbf{1}_{D_{2n}}$	1	1	1
λ	1	1	-1
$\chi_m,\ 1 \le m \le \frac{n-1}{2}$	2	$2\cos\left(\frac{2\pi km}{n}\right)$	0

(4.1)

while for even n, they take the form

class rep.	e	$r^k;\ 1 \le k \le \frac{n-2}{2}$	$r^{\frac{n}{2}}$	s	rs
class size	1	2	1	$\frac{n}{2}$	$\frac{n}{2}$
$\mathbf{1}_{D_{2n}}$	1	1	1	1	1
λ	1	1	1	-1	-1
μ_0	1	$(-1)^k$	$(-1)^{\frac{n}{2}}$	1	-1
μ_1	1	$(-1)^k$	$(-1)^{\frac{n}{2}}$	-1	1
$\chi_m,\ 1 \le m \le \frac{n-2}{2}$	2	$2\cos\left(\frac{2\pi km}{n}\right)$	$2\cos(\pi m)$	0	0

(4.2)

Remark. We will define $\chi_{\frac{n}{2}} := \mu_0 + \mu_1$ if n is even, and define $\chi_{\frac{n}{2}} \in X$ if $\mu_0, \mu_1 \in X$. This allows us to define supercharacter theories of D_{2n} in terms of the indices of the characters χ_m regardless of the parity of n.

Let $\mathcal{P}$ be the image of $\mathrm{InvSCT}_{D_{2n}}(\langle r\rangle)$ under the $*$-product with the unique supercharacter theory $m(D_{2n}/\langle r\rangle)$ of $D_{2n}/\langle r\rangle$ and note that by Lemmas 3.4 and 3.19, $\mathcal{P}$ is a sublattice of $\mathrm{CharSCT}(D_{2n})$ isomorphic to $\mathrm{InvSCT}_{D_{2n}}(\langle r\rangle)$. Moreover, if $T = (\mathcal{K}, \mathcal{X})$ is an element of

$\mathrm{InvSCT}_{D_{2n}}(\langle r\rangle)$, then we can describe its image $T *_{\langle r\rangle} m(D_{2n}/\langle r\rangle) = (\mathcal{L}, \mathcal{Y})$ easily: the superclass partition is $\mathcal{L} = \mathcal{K} \cup \{s\langle r\rangle\}$ and the supercharacters are $\mathbf{1}_{D_{2n}}$ and λ along with $\mathrm{Ind}_{\langle r\rangle}^{D_{2n}}(\sigma_X)$ for X in $\mathcal{X} \setminus \{\mathbf{1}_{\langle r\rangle}\}$. One can also check directly that the superclass partition of $mm_{\langle r\rangle}(D_{2n})$ is

$$\{s\langle r\rangle\} \cup \left\{\{r^k, r^{n-k}\} \,:\, 0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \right\},$$

(so $mm_{\langle r\rangle}(D_{2n}) = m(D_{2n})$ if n is odd). Finally, $MM_{\langle r\rangle}(D_{2n})$ has the superclass partition

$$\big\{\{e\}, s\langle r\rangle, \langle r\rangle \setminus \{e\}\big\}.$$

Thus, a supercharacter theory of D_{2n} factors over $\langle r\rangle$ if and only if it contains $s\langle r\rangle$ as a superclass.

Our aim is to use these factorizable supercharacter theories to produce all others through gluing and splitting maps which will be defined below. In order to define the domains of these maps, we will need the following set of factorizable supercharacter theories.

We will need the following terminology and lemmas. Say that a supercharacter theory of D_{2n} **glues reflections** if $s\langle r\rangle$ is a subset of a superclass of that supercharacter theory. Evidently if n is odd, then every supercharacter theory of D_{2n} glues reflections. If n is even, then note that because μ_0 and μ_1 are the only irreducible characters of D_{2n} whose values differ on $s\langle r^2\rangle$ and $sr\langle r^2\rangle$ and $\mu_0 + \mu_1 \equiv 0$ on $s\langle r\rangle$, it follows that a supercharacter theory $S = (\mathcal{K}, \mathcal{X})$ glues reflections if and only if μ_0 and μ_1 lie in the same block of $\mathcal{X}$.

Lemma 4.2. *Let $G = N \rtimes H$ be a semidirect product of two finite groups N and H and let $S = (\mathcal{K}, \mathcal{X})$ be a supercharacter theory of G. Let K be a part of $\mathcal{K}$ that contains all of $G \setminus N$. Then $G \setminus K$ is a S-supernormal subgroup of G contained in N.*

Proof. We will use the following notation: for any set C of elements of G, let $\underline{C} = \sum_{g\in C} g$ denote the sum in $\mathbb{C}G$ of the elements of C. Since $K \neq \{e\}$, it follows that $G \setminus K$ contains $\{e\}$. Let $g \in G \setminus K$. Then $g^{-1} \in G \setminus K^{-1}$, where $K^{-1} = \{k^{-1} : k \in K\}$. By [DI08, Theorem 2.2], we have $K^{-1} \in \mathcal{K}$. But $G \setminus N$ is closed under inversion, whence $K \cap K^{-1} \neq \emptyset$, and hence $K = K^{-1}$. Therefore, $G \setminus K$ is closed under inversion. Let $g_1, g_2 \in G \setminus K$ and let $L_1, L_2 \in \mathcal{K}$ be the respective superclasses containing these elements. Then $L_1, L_2 \subseteq N$, so $\underline{L_1} \cdot \underline{L_2}$ is a sum of elements of N, and in particular, we have $g_1 \cdot g_2 \in G \setminus K$. Therefore, $G \setminus K$ is a subgroup of N. The rest of the claim follows from the observation that $G \setminus K$ is a union of S-superclasses. ∎

We can specialize the above result to D_{2n} as follows.

Corollary 4.3. *Let $S = (\mathcal{K}, \mathcal{X})$ be a supercharacter theory of D_{2n} that glues reflections and let $K \in \mathcal{K}$ be the part containing $s\langle r\rangle$. Then $D_{2n} \setminus K$ is a subgroup of $\langle r\rangle$.*

The following lemma allows us to quickly detect which elements of $\mathrm{SCT}(D_{2n})$ are fixed under the action of $\mathrm{Aut}(D_{2n})$.

Lemma 4.4. *A supercharacter theory S of D_{2n} (for any n) is characteristic if and only if either $s\langle r\rangle$ is a union of S-superclasses or S glues reflections.*

Proof. Let S be a characteristic supercharacter theory. For $i = 0, 1$, let K_i be the superclass containing $sr^i\langle r^2\rangle$. If τ is the automorphism defined by $r \mapsto r$ and $s \mapsto sr$, then τ transposes $s\langle r^2\rangle$ and $sr\langle r^2\rangle$, hence τ transposes K_0 and K_1. Because τ fixes all rotations, it follows that either $K_0 = K_1$, or $K_i = sr^i\langle r^2\rangle$ for $i = 0, 1$.

Conversely, write $S = (\mathcal{K}, \mathcal{X})$ and suppose $s\langle r^2\rangle$ and $sr\langle r^2\rangle$ are parts of $\mathcal{K}$. Then n is necessarily even, since these are separate conjugacy classes. Now, μ_0 and μ_1 lie in different parts of $\mathcal{X}$, say X_0 and X_1, respectively. Let τ be as before, and note that because τ permutes $s\langle r^2\rangle$ and $sr\langle r^2\rangle$ and fixes all rotations, it follows that $\tau \cdot \mathcal{K} = \mathcal{K}$ and therefore $\tau \cdot \mathcal{X} = \mathcal{X}$. Thus, since τ transposes μ_0 and μ_1 and fixes all other characters, it follows that $X_i = \{\mu_i\}$ for $i = 0, 1$. Taking the join $S \vee mm_{\langle r\rangle}(D_{2n})$ glues X_0 to X_1 and $s\langle r^2\rangle$ to $sr\langle r^2\rangle$ and preserves all other superclasses and supercharacters. Thus, $S \vee mm_{\langle r\rangle}(D_{2n})$ factors over $\langle r\rangle$. By Lemma 3.19, it follows that $S \vee mm_{\langle r\rangle}(D_{2n})$ is characteristic. Since $\mathrm{Aut}(D_{2n})$ fixes $X_0 \cup X_1$ and $s\langle r\rangle$, it follows that S is characteristic.

Next, write $S = (\mathcal{K}, \mathcal{X})$ and suppose $s\langle r\rangle$ is a subset of a superclass. Then if n is even, it follows that μ_0 and μ_1 lie in the same part of $\mathcal{X}$. Write $s\langle r\rangle \cup A$ for the part of $\mathcal{K}$ containing $s\langle r\rangle$ and write $\{\lambda\} \cup B$ for the part of $\mathcal{X}$ containing λ. We claim that A is fixed under the action of

$\mathrm{Aut}(D_{2n})$. Let $r^k \in A$ and let j be coprime to n. Then for any part X of $\mathcal{X}$ that does not contain λ or $\mathbf{1}_{D_{2n}}$ as constituents, say $X = \{\chi_\ell : \ell \in I\}$ (where I may contain $n/2$), we have

$$\sigma_X(r^k) = 2\big(\zeta_n^{\frac{n}{2}}\big)^k + \sum_{\ell \in I \smallsetminus \{\frac{n}{2}\}} 2(\zeta_n^{k\ell} + \overline{\zeta_n}^{k\ell}) = 0$$

if $n/2 \in I$, and otherwise

$$\sigma_X(r^k) = \sum_{\ell \in I} 2(\zeta_n^{k\ell} + \overline{\zeta_n}^{k\ell}) = 0.$$

In each of the above equations, we have a polynomial $f(x) \in \mathbb{Z}[x]$ that is satisfied by ζ_n. Any such polynomial $f(x)$ is divisible by the nth cyclotomic polynomial, and is therefore satisfied by ζ_n^j for any j coprime to n. Hence we may replace ζ_n with ζ_n^j in these equations, which yields $\sigma_X(r^{kj}) = 0$. Thus, every supercharacter of S that does not contain λ or $\mathbf{1}_{D_{2n}}$ as constituents agrees on r^k and r^{kj}. Since $\rho_{D_{2n}}$ and $\mathbf{1}_{D_{2n}}$ agree on these elements and

$$\sigma_{\{\lambda\} \cup B} = \rho_{D_{2n}} - \mathbf{1}_{D_{2n}} - \sum_{\substack{X \in \mathcal{X} \\ \chi(1) > 1 \forall \chi \in X}} \sigma_X,$$

it follows that every supercharacter of S agrees on r^k and r^{kj}, and so these elements lie in the same part of $\mathcal{K}$. This implies that A is fixed under the action of $\mathrm{Aut}(D_{2n})$. Next, write $MM_{\langle r \rangle}(D_{2n}) = (\mathcal{L}, \mathcal{Y})$, where

$$\mathcal{L} = \big\{\{e\}, s\langle r \rangle, \langle r \rangle \smallsetminus \{e\}\big\}$$

and

$$\mathcal{Y} = \big\{\{\mathbf{1}_{D_{2n}}\}, \{\lambda\}, \mathrm{Irr}(D_{2n}) \smallsetminus \{\mathbf{1}_{D_{2n}}, \lambda\}\big\}.$$

Then

$$\mathcal{K} \wedge \mathcal{L} = \big\{s\langle r \rangle, A\big\} \cup \big(\mathcal{K} \smallsetminus \big\{s\langle r \rangle \cup A\big\}\big)$$

and

$$\mathcal{X} \wedge \mathcal{Y} = \big\{\{\lambda\}, B\big\} \cup \big(\mathcal{X} \smallsetminus \big\{\{\lambda\} \cup B\big\}\big),$$

and it is not hard to show that these partitions form a supercharacter theory, namely $S \wedge MM_{\langle r \rangle}(D_{2n})$.[1]

This supercharacter theory factors over $\langle r \rangle$, hence it is characteristic by Lemma 3.19. Since we

[1] One shows that $(\mathcal{K} \wedge \mathcal{L}, \mathcal{X} \wedge \mathcal{Y})$ is a supercharacter theory by showing that the characters λ and σ_B are constant on the parts of $\mathcal{K} \wedge \mathcal{L}$.

have already shown that A is fixed under the action of $\mathrm{Aut}(D_{2n})$, and $s\langle r\rangle$ is also fixed, it follows that $\mathrm{Aut}(D_{2n})$ permutes the remaining superclases amongst themselves. But these are precisely the superclasses of $\mathcal{K}$ not equal to $s\langle r\rangle \cup A$. Therefore, we have shown that S is characteristic. ∎

Corollary 4.5. *Every supercharacter theory of an odd dihedral group is characteristic.*

For each divisor d of n, let $S_d = \big(m_{D_{2n}}(\langle r^d\rangle) *_{\langle r^d\rangle} M(\langle r\rangle/\langle r^d\rangle)\big) *_{\langle r\rangle} M(D_{2n}/\langle r\rangle)$. We will need the superclass and supercharacter partitions of S_d in the classification, so we will state and prove them here as a proposition.

Proposition 4.6. *With S_d defined as above, we have*

$$\mathcal{K}(S_d) = \big\{\{e\}, s\langle r\rangle, \{r^k \,:\, d \textit{ does not divide } k\}\big\} \cup \big\{\{r^k, r^{-k}\} \,:\, d \textit{ divides } k\big\}$$

and

$$\mathcal{X}(S_d) = \Big\{\{\mathbf{1}_{D_{2n}}\}, \{\lambda\}, \Big\{\chi_k \,:\, \frac{n}{d} \textit{ divides } k\Big\}\Big\} \cup \bigcup_{\ell=1}^{\lfloor \frac{n}{2d} \rfloor} \Big\{\Big\{\chi_k \,:\, k \equiv \pm\ell \bmod \frac{n}{d}\Big\}\Big\}.$$

Proof. Let

$$S = m_{D_{2n}}\big(\langle r^d\rangle\big) *_{\langle r^d\rangle} M\big(\langle r\rangle/\langle r^d\rangle\big) \in \mathrm{SCT}(\langle r\rangle),$$

so that

$$S_d = S *_{\langle r\rangle} m\big(D_{2n}/\langle r\rangle\big).$$

It is not hard to calculate directly that

$$\mathcal{K}(S) = \big\{\{e\}\big\} \cup \big\{\{r^k, r^{-k}\} \,:\, d \mid k\big\} \cup \big\{\{r^k \,:\, d \nmid k\}\big\},$$

and hence

$$\mathcal{K}(S_d) = \big\{\{e\}, s\langle r\rangle\big\} \cup \big\{\{r^k, r^{-k}\} \,:\, d \mid k\big\} \cup \big\{\{r^k \,:\, d \nmid k\}\big\}.$$

To calculate the supercharacter partition of S_d, we first write

$$\mathcal{X}(S) = \big\{\{\mathbf{1}_{\langle r\rangle}\}, \mathrm{Irr}(\langle r\rangle/\langle r^d\rangle) \setminus \{\mathbf{1}_{\langle r\rangle}\}\big\} \cup \big\{Y^{\langle r\rangle} \,:\, Y \in \mathcal{X}(m_{D_{2n}}(\langle r^d\rangle)) \setminus \{\{\mathbf{1}_{\langle r^d\rangle}\}\}\big\}.$$

Consider the parts $Y^{\langle r\rangle}$. Each is of the form $Y = \{\xi_k, \overline{\xi_k}\}$ for some $1 \le k < n/d$, where ξ_k is the character defined by $r^d \mapsto \zeta_{\frac{n}{d}}^k$, where for any j. Then

$$\begin{aligned}
\mathrm{Ind}_{\langle r^d\rangle}^{\langle r\rangle}(\xi_k + \overline{\xi_k})(r^j) &= \sum_{i=0}^{d-1}(\xi_k + \overline{\xi_k})^0(r^{-i}r^jr^i) \\
&= d\cdot(\xi_k + \overline{\xi_k})^0(r^j) \\
&= \begin{cases} d\cdot(\zeta_n^{kj} + \overline{\zeta_n^{kj}}) & : \quad d \text{ divdes } j \\ 0 & : \quad d \text{ does not divide } j \end{cases}.
\end{aligned}$$

Suppose d divides j and note that $\zeta_n^{kj} = \zeta_n^{\ell j}$ if $\ell \equiv k \bmod n/d$. Moreover there are precisely d such integers $\ell \in \{1, \ldots, n-1\}$, namely $k, k+n/d, \ldots, k+(d-1)n/d$. Thus,

$$\begin{aligned}
\mathrm{Ind}_{\langle r^d\rangle}^{\langle r\rangle}(\xi_k + \overline{\xi_k})(r^j) &= \sum_{\substack{1\le \ell<n \\ \ell\equiv k \bmod \frac{n}{d}}} (\zeta_n^{\ell j} + \overline{\zeta_n^{\ell j}}) \\
&= \sum_{\substack{1\le \ell\le \frac{n}{2} \\ \ell\equiv \pm k \bmod \frac{n}{d}}} \eta_\ell(r^j),
\end{aligned}$$

where η_ℓ is the character defined by $r \mapsto \zeta_n^\ell$. Now if d does not divide j, then

$$\begin{aligned}
\sum_{\substack{1\le \ell<n \\ \ell\equiv k \bmod \frac{n}{d}}} (\zeta_n^{\ell j} + \overline{\zeta_n^{\ell j}}) &= \sum_{\substack{1\le \ell\le \frac{n}{2} \\ \ell\equiv \pm k \bmod \frac{n}{d}}} 2\cos\left(\frac{2\pi \ell j}{n}\right) \\
&= \sum_{a=0}^{d-1} 2\cos\left(\frac{2\pi j}{n}\left(\frac{n}{d}a + k\right)\right) \\
&= \sum_{a=0}^{d-1} 2\Bigg[\cos\left(\frac{2\pi ja}{d}\right)\cos\left(\frac{2\pi jk}{n}\right) \\
&\qquad - \sin\left(\frac{2\pi ja}{d}\right)\sin\left(\frac{2\pi jk}{n}\right)\Bigg] \\
&= \cos\left(\frac{2\pi jk}{n}\right)\left[\sum_{a=0}^{d-1} 2\cos\left(\frac{2\pi ja}{d}\right)\right] \\
&\qquad - \sin\left(\frac{2\pi jk}{n}\right)\left[\sum_{a=0}^{d-1} 2\sin\left(\frac{2\pi ja}{d}\right)\right].
\end{aligned} \tag{4.3}$$

But since d does not divide j, we have

$$\sum_{a=0}^{d-1} 2\cos\left(\frac{2\pi ja}{d}\right) = \sum_{a=0}^{d-1} 2\sin\left(\frac{2\pi ja}{d}\right) = 0,$$

hence (4.3) is equal to zero. Thus, we have proven that

$$\mathrm{Ind}_{\langle r^d\rangle}^{\langle r\rangle}(\sigma_Y) = \sum_{\substack{1\le \ell\le \frac{n}{2} \\ \ell\equiv \pm k \bmod \frac{n}{d}}} \eta_\ell,$$

and so we have

$$\mathcal{X}(S) = \Big\{\{\mathbf{1}_{\langle r\rangle}\}, \Big\{\eta_\ell \,:\, \frac{n}{d} \mid \ell\Big\}\Big\} \cup \bigcup_{k=1}^{\lfloor \frac{n}{2d}\rfloor} \Big\{\Big\{\eta_\ell \,:\, \ell \equiv \pm k \bmod \frac{n}{d}\Big\}\Big\}.$$

Finally, we have

$$\mathcal{X}(S_d) = \big\{\{\mathbf{1}_{D_{2n}}\}, \mathrm{Irr}(D_{2n}/\langle r\rangle) \smallsetminus \{\{\mathbf{1}_{D_{2n}}\}\}\big\} \cup \big\{Z^{D_{2n}} \,:\, Z \in \mathcal{X}(S) \smallsetminus \{\{\mathbf{1}_{\langle r\rangle}\}\}\big\}.$$

Consider the parts $Z^{D_{2n}}$. Evidently $\mathrm{Ind}_{\langle r\rangle}^{D_{2n}}(\eta_k) = \chi_k$, so

$$\begin{aligned}\mathcal{X}(S_d) = \big\{&\{\mathbf{1}_{D_{2n}}\}, \mathrm{Irr}(D_{2n}/\langle r\rangle) \smallsetminus \{\{\mathbf{1}_{D_{2n}}\}\}\big\} \\ &\cup \bigcup_{k=1}^{\lfloor \frac{n}{2d}\rfloor} \Big\{\Big\{\chi_\ell \,:\, \ell \equiv \pm k \bmod \frac{n}{d};\ 1 \le \ell \le \frac{n}{2}\Big\}\Big\}.\end{aligned}$$

■

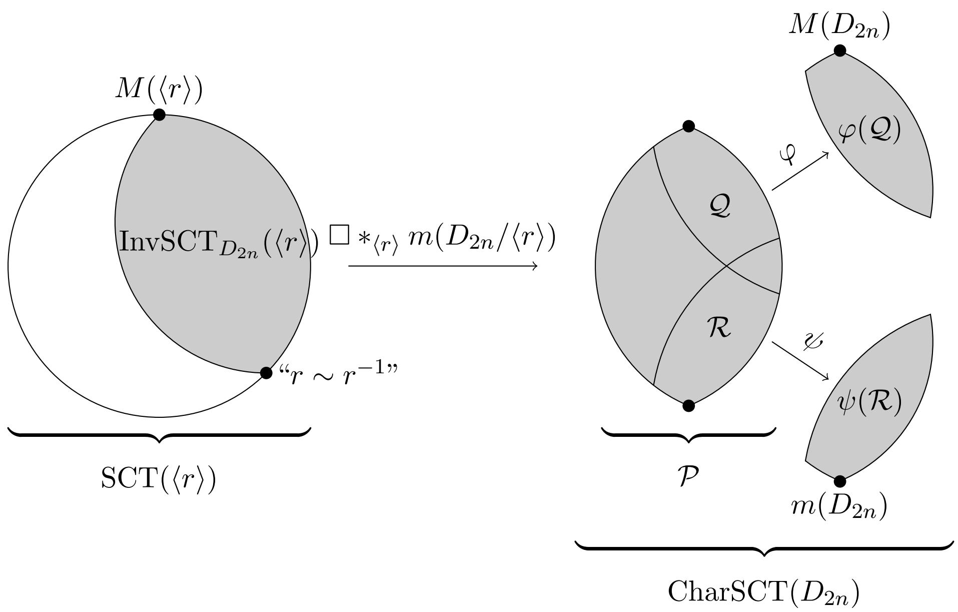
Let $\mathcal{Q}$ be the upper ideal of $\mathcal{P}$ generated by the supercharacter theories S_p for all prime divisors p of n, and if n is even, let $\mathcal{R}$ be the subposet of $\mathcal{P}$ consisting of those supercharacter theories for which $\chi_{\frac{n}{2}}$ is a supercharacter (if n is odd, let $\mathcal{R} = \emptyset$). For any $S \in \mathcal{Q}$, we may produce a new supercharacter theory $\varphi(S) \in \mathrm{CharSCT}(D_{2n})$ by gluing the parts containing r and s. Moreover if n is even, then for any $T \in \mathcal{R}$, we may produce a new supercharacter theory $\psi(T) \in \mathrm{CharSCT}(D_{2n})$ by splitting the two conjugacy classes of reflections into distinct superclasses. The classification of $\mathrm{CharSCT}(D_{2n})$ is given by Theorem 4.7.

Theorem 4.7. *The characteristic supercharacter theories of D_{2n} may be expressed as a disjoint union of the form*

$$\mathrm{CharSCT}(D_{2n}) = \varphi(\mathcal{Q}) \sqcup \mathcal{P} \sqcup \psi(\mathcal{R}).$$

In other words, every characteristic supercharacter theory of D_{2n} is either an $*$-product over $\langle r\rangle$, or it is the image of an $*$-product over $\langle r\rangle$ under one of the maps φ or ψ.

Figure 4.1: The process of obtaining the characteristic supercharacter theories of D_{2n} for arbitrary n.



Theorem 4.7 may be summed up by Figure 4.1.

Let $\mathcal{S}$ be the subposet of supercharacter theories of $\mathrm{CharSCT}(D_{2n})$ which glue reflections and whose superclass partitions satisfy the condition that if K is a superclass containing only powers of r, then these powers are all of the same parity (we will say these supercharacter theories **respect parity**). An equivalent (although less intuitive) definition of $\mathcal{S}$ is as follows. By Corollary 4.3, $\mathcal{S}$ is the set of supercharacter theories of the form $S = T *_{\langle r^d \rangle} M(D_{2n}/\langle r^d \rangle)$, where d divides n, $T \in \mathrm{InvSCT}_{D_{2n}}(\langle r^d \rangle)$, and such that $\mathrm{Res}^{D_{2n}}_{\langle r^d \rangle}(\mu_0)$ is a T-superclass function.

For $i = 0, 1$, define order-preserving injective functions $\psi_i : \mathrm{CharSCT}(D_{2n}) \to \mathrm{SCT}(D_{2n})$ as follows. For $S = (\mathcal{K}, \mathcal{X}) \in \mathcal{R}$, let $\psi_i(S) = (\mathcal{L}_i, \mathcal{Y}_i)$ be the following supercharacter theory. If $s\langle r \rangle \cup A \cup B$ is the S-superclass containing the reflections, where A contains only even powers of r and B only odd powers of r (and one or both may be empty), then ψ_i refines $\mathcal{K}$ by distinguishing parity, i.e.,

$$\mathcal{L}_0 = \big\{ s\langle r^2 \rangle \cup A, sr\langle r^2 \rangle \cup B \big\} \cup \big(\mathcal{K} \setminus \{ s\langle r \rangle \cup A \cup B \} \big)$$

and

$$\mathcal{L}_1 = \big\{ s\langle r^2 \rangle \cup B, sr\langle r^2 \rangle \cup A \big\} \cup \big(\mathcal{K} \setminus \{ s\langle r \rangle \cup A \cup B \} \big).$$

The supercharacter partition is defined by removing μ_i from the part X of $\mathcal{X}$ that contains μ_0 and μ_1, so that

$$\mathcal{Y}_0 = \big\{ \{\mu_0\}, X \setminus \{\mu_0\} \big\} \cup \big(\mathcal{X} \setminus \{X\} \big)$$

and

$$\mathcal{Y}_1 = \big\{ \{\mu_1\}, X \setminus \{\mu_1\} \big\} \cup \big(\mathcal{X} \setminus \{X\} \big).$$

Note that if A and B are both empty, then $\{\mu_0, \mu_1\} \in \mathcal{X}$ and $\psi_0 = \psi_1 = \psi$, where ψ is the map defined in Theorem 4.7, and in this case, the supercharacter theory lies in $\mathcal{R}$. In fact, $\mathcal{R}$ is the subposet of $\mathcal{S}$ consisting of the supercharacter theories that respect parity and for which $\chi_{\frac{n}{2}}$ is a superclass function. We claim that every noncharacteristic supercharacter theory is of the form $\psi_i(S)$ for some $i = 0, 1$ and some $S \in \mathcal{R}$ for which $s\langle r \rangle$ is not a superclass.

Theorem 4.8. *Let $S = (\mathcal{K}, \mathcal{X})$ be a supercharacter theory of D_{2n} that is not characteristic, and let $\tau \in \mathrm{Aut}(D_{2n})$ be the automorphism that sends s to sr and fixes all rotations. Then we have $S \vee S^\tau \in \mathcal{S}$ and $S = \psi_i(S \vee S^\tau)$ for some $i = 0, 1$.*

4.2 Proof of Theorem 4.7

Theorem 4.7 will follow from a series of technical lemmas.

Lemma 4.9. *Every characteristic supercharacter theory of D_{2n} either glues reflections, or is covered by a unique factorizable supercharacter theory that glues reflections.*

Proof. Let S be characteristic. If S does not glue reflections, then it follows by Lemma 4.4 that n is even and $s\langle r^2\rangle$ and $sr\langle r^2\rangle$ are superclasses. Since $s\langle r^2\rangle$ and $sr\langle r^2\rangle$ lie in different superclasses, it follows that μ_0 and μ_1 lie in different parts of the supercharacter partition of S, say X_0 and X_1 respectively. Let τ be the automorphism of D_{2n} defined by $r \mapsto r$ and $s \mapsto sr$. Then τ transposes μ_0 and μ_1, but it fixes all other irreducible characters. Hence, $X_0 = \{\mu_0\}$ and $X_1 = \{\mu_1\}$. Taking the join $S \vee mm_{\langle r\rangle}(D_{2n})$, we see that $S \vee mm_{\langle r\rangle}(D_{2n})$ factors over $\langle r\rangle$ and that $|S \vee mm_{\langle r\rangle}(D_{2n})| = |S| - 1$, which implies that $S \vee mm_{\langle r\rangle}(D_{2n})$ covers S. This supercharacter theory also contains $\{\mu_0, \mu_1\}$ in its supercharacter partition. That this is the unique factorizable supercharacter theory covering S follows from the observation that any factorizable supercharacter theory must glue reflections, and that any supercharacter theory which glues reflections must glue μ_0 and μ_1. ∎

Remark. One can check that any factorizable supercharacter theory containing $\{\mu_0, \mu_1\}$ may be refined in a manner reverse of Lemma 4.9. Thus, these factorizable covers are precisely the set of supercharacter theories whose supercharacter partitions each contain $\{\mu_0, \mu_1\}$ as a part, and this refinement is precisely the map ψ described in Theorem 4.7.

Lemma 4.9 provides a characterization of the elements of $\mathrm{CharSCT}(D_{2n})$ which contain $s\langle r^2\rangle$ and $sr\langle r^2\rangle$ as superclasses. All that remains to prove Theorem 4.7 is to classify the elements of $\mathrm{CharSCT}(D_{2n})$ for which $s\langle r\rangle$ is properly contained in a superclass. This is done using the map φ

defined in Theorem 4.7 which glues superclasses. In order to show that φ yields the remainder of CharSCT(D_{2n}), we will need the following sets of factorizable supercharacter theories.

For each divisor d of n, let $mM_{\langle r^d\rangle}(D_{2n}) = m_{D_{2n}}(\langle r^d\rangle) *_{\langle r^d\rangle} M(D_{2n}/\langle r^d\rangle)$. The superclass and supercharacter partitions of $mM_{\langle r^d\rangle}(D_{2n})$ are

$$\begin{aligned}\mathcal{K}(mM_{\langle r^d\rangle}(D_{2n})) &= \big\{\{e\}, s\langle r\rangle \cup \{r^k : d \text{ does not divide } k\}\big\} \cup \big\{\{r^k, r^{-k}\} : d \text{ divides } k\big\},\\ \mathcal{X}(mM_{\langle r^d\rangle}(D_{2n})) &= \Big\{\{\mathbf{1}_{D_{2n}}\}, \{\lambda\}\Big\} \cup \Big\{\chi_k : \frac{n}{d} \text{ divides } k\Big\} \cup \bigcup_{k=1}^{\lfloor \frac{n}{2d}\rfloor} \Big\{\Big\{\chi_\ell : k \equiv \pm\ell \bmod \frac{n}{d}\Big\}\Big\}\end{aligned} \tag{4.4}$$

respectively. We will also use the maximal $*$-products $MM_{\langle r^d\rangle}(D_{2n})$ in the next two results. The superclass and supercharacter partitions of $MM_{\langle r^d\rangle}(D_{2n})$ are

$$\begin{aligned}\mathcal{K}(MM_{\langle r^d\rangle}(D_{2n})) &= \big\{\{e\}, s\langle r\rangle \cup \{r^k : d \text{ does not divide } k\}, \{r^k : d \text{ divides } k\}\big\},\\ \mathcal{X}(MM_{\langle r^d\rangle}(D_{2n})) &= \Big\{\{\mathbf{1}_{D_{2n}}\}, \{\lambda\} \cup \Big\{\chi_k : \frac{n}{d} \text{ divides } k\Big\}, \Big\{\chi_k : \frac{n}{d} \text{ does not divide } k\Big\}\Big\}\end{aligned} \tag{4.5}$$

respectively.

Remark. If $d = 1$, then $mM_{\langle r^d\rangle}(D_{2n})$ and $MM_{\langle r^d\rangle}(D_{2n})$ reduce to the minimal and maximal $*$-products $mm_{\langle r\rangle}(D_{2n})$ and $MM_{\langle r\rangle}(D_{2n})$, respectively. If $d = n$, then $mM_{\langle r^d\rangle}(D_{2n})$ and $MM_{\langle r^d\rangle}(D_{2n})$ coincide and are equal to $M(D_{2n})$.

We claim that the two sets of supercharacter theories

$$\{mM_{\langle r^d\rangle}(D_{2n}) : d \mid n\}$$

and

$$\{MM_{\langle r^d\rangle}(D_{2n}) : d \mid n\}$$

are the minimal and maximal nonfactorizable supercharacter theories which glue reflections, respectively. This is the content of the following two lemmas.

Lemma 4.10. *Suppose $S \in \mathrm{SCT}(D_{2n})$ is any supercharacter that glues reflections and that does not factor over $\langle r\rangle$. Then S is a refinement of at least one of the supercharacter theories $MM_{\langle r^d\rangle}(D_{2n})$ for some divisor d of n.*

Proof. Write $S = (\mathcal{K}, \mathcal{X})$. Since S does not factor over $\langle r \rangle$, the part K of $\mathcal{K}$ containing $s\langle r \rangle$ also contains some rotation r^k, and the part X of $\mathcal{X}$ containing λ also contains some nonlinear character χ_ℓ (which may be $\mu_0 + \mu_1$). By Lemma 4.4, S is characteristic. Since $\mathrm{Aut}(D_{2n})$ fixes λ, it follows that X is a nontrivial union of orbits under this action.

Write $X = \{\lambda\} \cup \{\chi_\ell : \ell \in J\}$. It will suffice to find some divisor d of n so that X is of the form $\{\lambda\} \cup \{\chi_\ell : n/d \mid \ell\}$. Let $m = \gcd(J)$. We claim that m is a divisor of n. To this end, let $\ell \in J$ and let $(\ell, n) = a$. Then $\ell = a \cdot b$ for some b with $(b, n) = 1$. But this implies that χ_a and χ_ℓ are related by some automorphism of D_{2n}, which by the preceding paragraph implies that $a \in J$. Hence m divides a, whence m divides n. Next, let $d = n/m$ and let $X_d = \{\lambda\} \cup \{\chi_\ell : n/d \mid \ell\}$ be as in (4.5). We claim that $X = X_d$. Indeed, $X \subseteq X_d$ by the definition of d, so we just need to show that $X_d \smallsetminus X = \emptyset$.

If $r^k \in K$, then $\sigma_X(r^k) = \sigma_X(s) = -1$. But, if $n/2 \in J$, then we have

$$\begin{aligned}
\sigma_X(r^k) &= \lambda(r^k) + \chi_{\frac{n}{2}} + \sum_{\substack{\ell \in J \\ \ell \neq \frac{n}{2}}} 2\chi_\ell(r^k) \\
&= 1 + 2\big(\zeta_n^{\frac{n}{2}}\big)^k + \sum_{\substack{\ell \in J \\ \ell \neq \frac{n}{2}}} 2\big(\zeta_n^{k\ell} + \zeta_n^{-k\ell}\big) \\
&= -1,
\end{aligned}$$

while if $n/2 \notin J$, then we have

$$\begin{aligned}
\sigma_X(r^k) &= \lambda(r^k) + \sum_{\ell \in J} 2\chi_\ell(r^k) \\
&= 1 + \sum_{\ell \in J} 2\big(\zeta_n^{k\ell} + \zeta_n^{-k\ell}\big) \\
&= -1.
\end{aligned}$$

In either case, we have a set of powers of ζ_n^k whose sum is -1. But this is only possible if k is not divisible by d, hence K contains only rotations of this form. Thus, for all $g \in K$, we have $\sigma_{X_d}(g) = \sigma_X(g) = -1$, and hence $\sigma_{X_d \smallsetminus X}(g) = 0$. On the other hand, if k is divisible by d (including $k = 0$), then we have $\sigma_{X_d \smallsetminus X}(r^k) = 4|X_d \smallsetminus X|$. Hence, we have

$$\sigma_{X_d \smallsetminus X} = C \cdot \rho_{D_{2n}/\langle r^d \rangle}$$

for some constant C, where $\rho_{D_{2n}/\langle r^d \rangle}$ is the regular character of $D_{2n}/\langle r^d \rangle$. But this implies that

$$\sigma_{X_d \smallsetminus X} = C \cdot \sum_{\substack{\chi \in \mathrm{Irr}(D_{2n}) \\ \langle r^d \rangle \subseteq \mathrm{Ker}(\chi)}} \chi(1)\chi,$$

which implies that $\langle \lambda, \sigma_{X_d \smallsetminus X} \rangle = C$, and hence that $C = 0$. But this implies that $\sigma_{X_d \smallsetminus X} \equiv 0$, and therefore $X = X_d$. Since $MM_{\langle r^d \rangle}(D_{2n})$ has a three part supercharacter partition of the form

$$\big\{\{\mathbf{1}_{D_{2n}}\}, X_d, \mathrm{Irr}(D_{2n}) \smallsetminus (\{\mathbf{1}_{D_{2n}}\} \cup X_d)\big\},$$

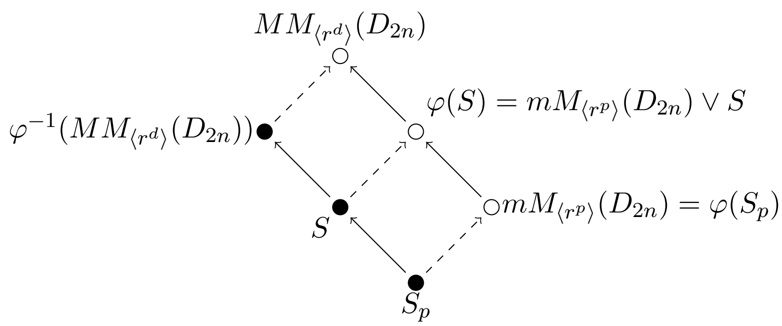
and we have proven that X_d is a part of $\mathcal{X}$, it follows that $\mathcal{X}$ is a refinement of the supercharacter partition of $MM_{\langle r^d \rangle}(D_{2n})$. ∎

Lemma 4.11. *Suppose* $S \in \mathrm{CharSCT}(D_{2n})$ *glues reflections and does not factor as a* $*$*-product over* $\langle r \rangle$*. Then* S *is a coarsening of at least one of the supercharacter theories* $mM_{\langle r^d \rangle}(D_{2n})$ *for some prime divisor* d *of* n*.*

Proof. Let K be the superclass of S that contains $s\langle r \rangle$, and note that by Corollary 4.3, $D_{2n} \smallsetminus \{K\}$ is a subgroup of $\langle r \rangle$, which is necessarily of the form $\langle r^d \rangle$ for some divisor d of n. Hence K is of the form $K = s\langle r \rangle \cup \{r^k : d \nmid k\}$. Since S does not factor over $\langle r \rangle$, we can assume that $d \neq n$. Let p be any prime divisor of n/d. Let $0 \le k < n$. If p does not divide k, then certainly n/d does not divide k. Hence, the set $K_p = s\langle r \rangle \cup \{r^k : p \nmid k\}$ is a subset of K. Since this is the only part of $\mathcal{K}(mM_{\langle r^p \rangle}(D_{2n}))$ that is not a conjugacy class (see (4.4)), it easily follows that $mM_{\langle r^p \rangle}(D_{2n}) \le S$. ∎

Since the $mM_{\langle r^d \rangle}(D_{2n})$ and $MM_{\langle r^d \rangle}(D_{2n})$ play an important role in the classification of $\mathrm{CharSCT}(D_{2n})$, it is worth illustrating how they are related. If $S \in \mathcal{Q}$, then there is some prime divisor p of n such that $S_p \le S$. Moreover, we have $mM_{\langle r^p \rangle}(D_{2n}) \le \varphi(S)$ and $\varphi(S) = S \vee mM_{\langle r^p \rangle}(D_{2n})$. We also have that $mM_{\langle r^p \rangle}(D_{2n}) \le MM_{\langle r^d \rangle}(D_{2n})$ for some divisor d of n, and that d is divisible by p, and that by (4.4) and (4.5), $mM_{\langle r^p \rangle}(D_{2n}) \le MM_{\langle r^d \rangle}(D_{2n})$ if and only if p divides d. This is shown in Figure 4.2.

Figure 4.2: The relationship between the minimal and maximal characteristic supercharacter theories of D_{2n} that do not factor as $*$-products over $\langle r\rangle$. Black nodes represent supercharacter theories which factor as $*$-products over $\langle r\rangle$, while hollow nodes represent their images under φ. The covering relations induced by φ are drawn as dashed arrows.



Proof of Theorem 4.7. By Lemma 4.4, a supercharacter theory $S = (\mathcal{K}, \mathcal{X})$ of D_{2n} is characteristic if and only if either $s\langle r\rangle$ is either a union of superclasses, or a subset of a superclass. If $s\langle r\rangle$ is a union of superclasses, then either $s\langle r\rangle \in \mathcal{K}$ (in which case $S \in \mathcal{P}$) or $s\langle r^2\rangle, sr\langle r^2\rangle \in \mathcal{K}$. In this latter case, it follows by Lemma 4.9 that $S = \psi(T)$ for some $T \in \mathcal{R}$. If $s\langle r\rangle$ is a proper subset of a superclass, then by Lemma 4.10 and Lemma 4.11, we have $mM_{\langle r^p\rangle}(D_{2n}) \leq S \leq MM_{\langle r^d\rangle}(D_{2n})$ for some divisors p and d of n, with p prime. Let $C_d = (M(\langle r^d\rangle) *_{\langle r^d\rangle} M(\langle r\rangle/\langle r^d\rangle)) *_{\langle r\rangle} M(D_{2n}/\langle r\rangle)$, so that

$$\mathcal{K}(C_d) = \big\{\{e\}, s\langle r\rangle, \{r^k \,:\, d \nmid k\}, \{r^k \,:\, d \mid k\}\big\}$$

and

$$\mathcal{X}(C_d) = \Big\{\{\mathbf{1}_{D_{2n}}\}, \{\lambda\}, \Big\{\chi_k \,:\, \frac{n}{d} \mid k\Big\}, \Big\{\chi_k \,:\, \frac{n}{d} \nmid k\Big\}\Big\}.$$

Then we have that $\varphi(C_d) = MM_{\langle r^d\rangle}(D_{2n})$ and $\varphi(S_p) = mM_{\langle r^p\rangle}(D_{2n})$. To complete the proof, it is enough to show that $C_d \wedge S \in \mathcal{Q}$ and that $\varphi(C_d \wedge S) = S$. One can check directly that $\mathcal{K}(C_d \wedge S) = \mathcal{K}(C_d) \wedge \mathcal{K}(S)$ and[2] $\mathcal{X}(C_d \wedge S) = \mathcal{X}(C_d) \wedge \mathcal{X}(S)$, from which both claims follow. ∎

Example 4.12. Let us pause and consider two small examples. By Lemma 4.4, D_{2n} has non-characteristic supercharacter theories if and only if n is even. We can compute the sublattices of characteristic supercharacter theories of D_{12} and D_{30} directly, using Hendrickson's classification and Theorem 4.7. Beginning with D_{12}, we consider the lattice of D_{12}-invariant supercharacter theories of $\langle r\rangle$. They are A, B, C, and D, where

$$\mathcal{K}(A) = \big\{\{e\}, \{r, r^5\}, \{r^2, r^4\}, \{r^3\}\big\},$$

$$\mathcal{K}(B) = \big\{\{e\}, \{r, r^2, r^4, r^5\}, \{r^3\}\big\},$$

$$\mathcal{K}(C) = \big\{\{e\}, \{r, r^3, r^5\}, \{r^2, r^4\}\big\}, \quad \text{and}$$

$$\mathcal{K}(D) = \big\{\{e\}, \{r, r^2, r^3, r^4, r^5\}\big\}.$$

Let $A' = A *_{\langle r\rangle} M(D_{12}/\langle r\rangle)$ and similarly define B', C', and D'. Then $\mathcal{R} = \{A', C'\}$ and $\mathcal{Q} = \{B', C', D'\}$. In particular, $C' = S_2$ and $B' = S_3$, so that $\varphi(C') = mM_{\langle r^2\rangle}(D_{2n})$ and $\varphi(B') =$

[2] If $\{\lambda\} \cup Y$ is the part of $\mathcal{X}(S)$ containing λ, then $\mathcal{X}(S) \wedge \mathcal{X}(C_d)$ has parts $\{\lambda\}$, Y, and all other parts are parts of $\mathcal{X}(S)$. It follows that these characters are constant on the parts of $\mathcal{K}(S) \wedge \mathcal{K}(C_d)$.

$mM_{\langle r^3\rangle}(D_{2n})$. Thus, the sublattice of characteristic supercharacter theories of D_{12} takes the form shown in Figure 4.3. Now consider D_{30}. The lattice of D_{30}-invariant supercharacter theories of $\langle r\rangle$ takes the form shown in Figure 4.4, where

$$\begin{aligned}
\mathcal{K}(A) &= \{\{e\},\{r,r^{14}\},\{r^2,r^{13}\},\{r^3,r^{12}\},\{r^4,r^{11}\},\{r^5,r^{10}\},\{r^6,r^9\},\{r^7,r^8\}\},\\
\mathcal{K}(B) &= \{\{e\},\{r,r^4,r^{11},r^{14}\},\{r^2,r^7,r^8,r^{13}\},\{r^3,r^{12}\},\{r^5,r^{10}\},\{r^6,r^9\}\},\\
\mathcal{K}(C) &= \{\{e\},\{r,r^2,r^4,r^5,r^7,r^8,r^{10},r^{11},r^{13},r^{14}\},\{r^3,r^{12}\},\{r^6,r^9\}\},\\
\mathcal{K}(D) &= \{\{e\},\{r,r^2,r^4,r^7,r^8,r^{11},r^{13},r^{14}\},\{r^3,r^6,r^9,r^{12}\},\{r^5,r^{10}\}\},\\
\mathcal{K}(E) &= \{\{e\},\{r,r^4,r^6,r^9,r^{11},r^{14}\},\{r^2,r^3,r^7,r^8,r^{12},r^{13}\},\{r^5,r^{10}\}\},\\
\mathcal{K}(F) &= \{\{e\},\{r,r^2,r^4,r^5,r^7,r^8,r^{10},r^{11},r^{13},r^{14}\},\{r^3,r^6,r^9,r^{12}\}\},\\
\mathcal{K}(G) &= \{\{e\},\{r,r^2,r^3,r^4,r^6,r^7,r^8,r^9,r^{11},r^{12},r^{13},r^{14}\},\{r^5,r^{10}\}\}, \quad \text{and}\\
\mathcal{K}(H) &= \{\{e\},\{r,r^2,r^3,r^4,r^5,r^6,r^7,r^8,r^9,r^{10},r^{11},r^{11},r^{12},r^{13},r^{14}\}\}.
\end{aligned}$$

Let $A' = A *_{\langle r\rangle} M(D_{30}/\langle r\rangle)$ and similarly define B', C', D', E', F', G', H'. Then $\mathcal{R}$ is empty by definition and $\mathcal{Q} = \{C', F', G', H'\}$. In particular, $C' = S_3$ and $G' = S_5$, so that $\varphi(C') = mM_{\langle r^3\rangle}(D_{2n})$ and $\varphi(G') = mM_{\langle r^5\rangle}(D_{2n})$. Thus, the lattice of characteristic supercharacter theories of D_{30} takes the form shown in Figure 4.5 We will see later that while this is the full lattice of supercharacter theories of D_{30}, there exist non-characteristic supercharacter theories of D_{12}.

Example 4.13. If p is an odd prime, then $\mathrm{SCT}(D_{2p})$ has a particularly simple structure. Let us first recall the structure of $\mathrm{SCT}(\langle r\rangle)$, as classified in Proposition 3.25. For each divisor d of $p-1$, there is a unique supercharacter theory whose nontrivial superclasses all have size d, and these superclasses are the orbits of the action of the unique subgroup of $\mathrm{Aut}(\langle r\rangle)$ of size d. The superclass partitions of the supercharacter theories of D_{2p} can therefore be obtained by taking the superclass partitions of the supercharacter theories of $\langle r\rangle$ whose nontrivial parts all have an even number of elements, and including $s\langle r\rangle$ as an additional superclass. Thus, $\mathrm{SCT}(D_{2p})$ is isomorphic to the lattice of even divisors of $p-1$, with an additional top element.

Figure 4.3: The characteristic supercharacter theories of D_{12}. Black nodes represent supercharacter theories which factor as $*$-products over $\langle r\rangle$, while hollow nodes represent their images under φ and hatched nodes represent their images under ψ. The covering relations induced by φ are drawn as dashed arrows, while those induced by ψ are drawn as dotted arrows.

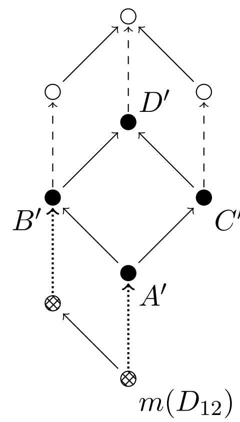


Figure 4.4: The D_{30}-invariant supercharacter theories of $\langle r\rangle$.

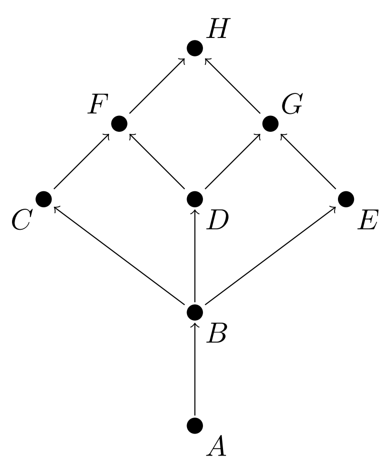
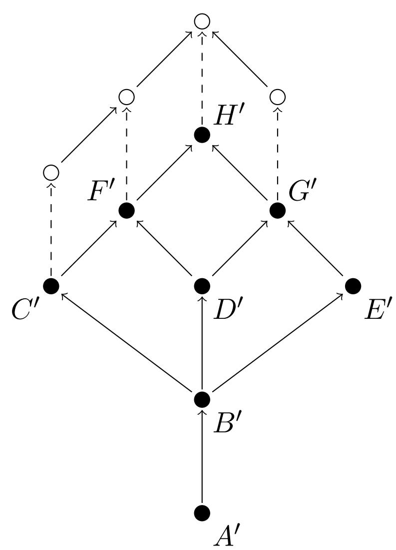


Figure 4.5: The supercharacter theory lattice of D_{30}. Hollow nodes represent the image of φ. The covering relations induced by φ are drawn as dashed arrows.



4.2.1 An alternate version

We can restate Theorem 4.7 in a slightly different way.

For each divisor d of n, let $\mathcal{P}_d = [mM_{\langle r^d\rangle}(D_{2n}), MM_{\langle r^d\rangle}(D_{2n})]$ be the interval of supercharacter theories lying between $mM_{\langle r^d\rangle}(D_{2n})$ and $MM_{\langle r^d\rangle}(D_{2n})$. If $d = 1$, then $mM_{\langle r^d\rangle}(D_{2n})$ reduces to $mm_{\langle r\rangle}(D_{2n})$, hence $\mathcal{P}_1 = \mathcal{P}$. If $d = n$, then $mM_{\langle r^d\rangle}(D_{2n})$ and $MM_{\langle r^d\rangle}(D_{2n})$ coincide and are equal to $M(D_{2n})$. Let $\mathcal{R}$ and $\psi : \mathcal{R} \to \mathrm{CharSCT}(D_{2n})$ be defined as in Theorem 4.7.

Theorem 4.14. *If n is odd, then we can express the characteristic supercharacter theories of D_{2n} as a disjoint union of the form*

$$\mathrm{CharSCT}(D_{2n}) = \bigsqcup_{d|n} \mathcal{P}_d.$$

If n is even, then we may express $\mathrm{CharSCT}(D_{2n})$ *as a disjoint union of the form*

$$\mathrm{CharSCT}(D_{2n}) = \left(\bigsqcup_{d|n} \mathcal{P}_n\right) \sqcup \psi(\mathcal{R}).$$

This result is a consequence of Theorem 4.7 and the following lemma.

Lemma 4.15. *Suppose $S = (\mathcal{K}, \mathcal{X})$ is any supercharacter theory of D_{2n} that glues reflections. Then there exists a divisor d of n such that $mM_{\langle r^d\rangle}(D_{2n}) \leq S \leq MM_{\langle r^d\rangle}(D_{2n})$.*

Proof. First note that if S factors over $\langle r\rangle$, then $S \in \mathcal{P}_1$ and we are done. Assume S does not factor over $\langle r\rangle$. Let K be the superclass of S which contains $s\langle r\rangle$, and note that by Corollary 4.3, $D_{2n} \smallsetminus \{K\}$ is a subgroup of $\langle r\rangle$, which is necessarily of the form $\langle r^d\rangle$ for some divisor d of n. Thus, $K = s\langle r\rangle \cup \{r^k : d \nmid k\}$, and so $\mathcal{K}$ shares this superclass with the superclass partition of $mM_{\langle r^d\rangle}(D_{2n})$. Since this is the only $mM_{\langle r^d\rangle}(D_{2n})$-superclass that is not a conjugacy class, it easily follows that $mM_{\langle r^d\rangle}(D_{2n}) \leq S$.

Since $\langle r^d\rangle$ is S-supernormal, we may consider the deflated supercharacter theory $S^{D_{2n}/\langle r^d\rangle}$, as in (2.6). By inspecting the superclass partition of $S^{D_{2n}/\langle r^d\rangle}$, we observe that $S^{D_{2n}/\langle r^d\rangle} = M(D_{2n}/\langle r^d\rangle)$, which implies that the number of blocks of $\mathcal{X}$ contained in $\mathrm{Irr}(D_{2n}/\langle r^d\rangle)$ is two. Since one of these is the trivial character $\mathbf{1}_{D_{2n}}$, it follows that the other must be $\{\lambda\} \cup \{\chi_\ell : n/d \mid \ell\}$.

Thus, $\mathcal{X}$ contains two of the three blocks of $\mathcal{X}_d$. This easily implies that $\mathcal{X} \leq \mathcal{X}_d$, and therefore $S \leq MM_{\langle r^d \rangle}(D_{2n})$. ∎

Proof of Theorem 4.14. By Lemma 4.4, a supercharacter theory $S = (\mathcal{K}, \mathcal{X})$ of D_{2n} is characteristic if and only if either S glues reflections, or $s\langle r \rangle$ is a union of superclasses. If S glues reflections, then Lemma 4.15 implies that $S \in \mathcal{P}_d$ for some divisor d of n. If S does not glue reflections, then Lemma 4.9 implies that $S = \psi(T)$ for some $T \in \mathcal{Q}$. It is routine to check that the subposets $\mathcal{P}_d$ and $\psi(\mathcal{Q})$ are pairwise disjoint. ∎

Recall that if n is odd, then every supercharacter theory of D_{2n} is characteristic and D_{2n} is a Frobenius group with kernel $K = \langle r \rangle$. The precise statement of Wynn's Frobenius group classification is as follows.

Theorem 4.16. *[Wyn17, Theorem 1.2] Let G be a Frobenius group with Frobenius kernel K and let $S \in \mathrm{SCT}(G)$. Then either S factors as a $*$-product over K, or else there exist normal subgroups L and N with $L < K < N$ so that $S = S_N \ \Delta \ S^{G/L}$. Moreover, $S_N = T *_L M(N/L)$, where $T \in \mathrm{InvSCT}_G(L)$ and $S^{G/L} = M(N/L) *_{N/L} U$ for some $U \in \mathrm{SCT}(G/L/N/L)$.*

Let S be a supercharacter theory of D_{2n}, where n is odd, and suppose S does not factor over $\langle r \rangle$. Then by the above theorem, S factors as a Δ-product over normal subgroups L and N with $L < K < N$ and such that $S_N = T *_L M(N/L)$ for some $T \in \mathrm{InvSCT}_{D_{2n}}(L)$. Since L is a subgroup of $\langle r \rangle$, it is of the form $\langle r^d \rangle$ for some divisor d of n. Moreover, since $\langle r \rangle$ is maximal, it follows that $N = D_{2n}$, and therefore $S \in \mathcal{P}_d$. Therefore Theorem 4.14 offers the same classification as Theorem 4.16 in the case G is an odd dihedral group.

4.3 Proof of Theorem 4.8

Recall that if $S = (\mathcal{K}, \mathcal{X}) \in \mathcal{S}$, then $\psi_i = (\mathcal{L}_i, \mathcal{Y}_i)$, where

$$\mathcal{L}_i = \left\{ sr^i\langle r^2 \rangle \cup A, sr^{1-i}\langle r^2 \rangle \cup B \right\} \cup \left(\mathcal{K} \smallsetminus \{ s\langle r \rangle \cup A \cup B \} \right)$$

and

$$\mathcal{Y}_i = \big\{\{\mu_i\}, X \smallsetminus \{\mu_i\}\big\} \cup \big(\mathcal{X} \smallsetminus \{X\}\big).$$

Lemma 4.17. *If $S = (\mathcal{K}, \mathcal{X}) \in \mathcal{S}$, then $\psi_i(S) = (\mathcal{L}_i, \mathcal{Y}_i)$ forms a supercharacter theory for $i = 0, 1$.*

Proof. Assume $i = 0$ (the proof for $i = 1$ is identical) and let $(\mathcal{K}, \mathcal{X})$ and $(\mathcal{L}_i, \mathcal{Y}_i)$ be labeled as above. Immediately, we have that if $Y \in \mathcal{X} \smallsetminus \{X\}$, Then σ_Y is constant on the parts of $\mathcal{L}_i$. By assumption, the parts of $\mathcal{K} \smallsetminus \{s\langle r\rangle \cup A \cup B\}$ respect parity. Thus because $\mu_0(r^k) = \mu_0(r^k s) = (-1)^k$, it follows that μ_0 is constant on the parts of $\mathcal{L}_i$. Finally, because the regular character $\rho_{D_{2n}}$ is constant on the parts of $\mathcal{L}_i$ and

$$\sigma_{X \smallsetminus \{\mu_0\}} = \rho_{D_{2n}} - \mu_0 - \sum_{Y \in \mathcal{X} \smallsetminus \{X\}} \sigma_Y,$$

it follows that $\sigma_{X \smallsetminus \{\mu_0\}}$ is constant on the parts of $\mathcal{L}_i$. Therefore, $(\mathcal{L}_i, \mathcal{Y}_i)$ is a supercharacter theory. ∎

Before we prove Theorem 4.8, we pause to recall the rules for multiplying irreducible characters of D_{2n}, all of which are consequences of the character tables (4.1) and (4.2). First, we have

$$\mu_i^2 = \mathbf{1}_{D_{2n}}$$

for $i = 0, 1$,

$$\mu_0 \cdot \mu_1 = \lambda,$$

and

$$\mu_i \cdot \lambda = \mu_{1-i}$$

for $i = 0, 1$. Next, let $0 \leq i \leq 1$, $1 \leq m < \frac{n}{2}$, $0 \leq k < n$, and consider

$$\begin{aligned}
(\mu_i \cdot \chi_m)(r^k) &= 2(-1)^k \cos\Big(\frac{2\pi k m}{n}\Big) \\
&= 2\cos\Big(\frac{2\pi k \frac{n}{2}}{n}\Big) \cos\Big(\frac{2\pi k m}{n}\Big) + 2\sin\Big(\frac{2\pi \frac{n}{2}}{n}\Big) \sin\Big(\frac{2\pi k m}{n}\Big) \\
&= 2\cos\Big(\frac{2\pi k (\frac{n}{2} - m)}{n}\Big) \\
&= \chi_{\frac{n}{2} - m}(r^k).
\end{aligned}$$

Since $\mu_i \cdot \chi_m$ vanishes on $s\langle r\rangle$, it follows that

$$\mu_i \cdot \chi_m = \chi_{\frac{n}{2}-m}.$$

Finally, let $1 \leq \ell, m < \frac{n}{2}$, $0 \leq k < n$, and consider

$$\begin{aligned}(\chi_\ell \cdot \chi_m)(r^k) &= 4\cos\Big(\frac{2\pi k\ell}{n}\Big)\cos\Big(\frac{2\pi km}{n}\Big) \\ &= 2\Big(\cos\Big(\frac{2\pi k(\ell+m)}{n}\Big) + \cos\Big(\frac{2\pi k|\ell-m|}{n}\Big)\Big).\end{aligned} \tag{4.6}$$

Now, we can generalize the definition of χ_a to all integers a if we define $\chi_a(r^k) = 2\cos(\frac{2\pi ka}{n})$ and $\chi_a(sr^k) = 0$. In this case, $\chi_a = \chi_{-a}$ if a is negative, $\chi_0 = \lambda + \mathbf{1}_{D_{2n}}$, $\chi_{\frac{n}{2}} = \mu_0 + \mu_1$ as before, and $\chi_a = \chi_{\frac{n}{2}-a}$ for $a > \frac{n}{2}$. Thus, (4.6) becomes

$$\chi_{\ell+m}(r^k) + \chi_{|\ell-m|}(r^k).$$

Finally, since $\chi_\ell \cdot \chi_m$ vanishes on $s\langle r\rangle$, we have

$$\chi_\ell \cdot \chi_m = \chi_{\ell+m} + \chi_{|\ell-m|}.$$

Proof of Theorem 4.8. By Lemma 4.4, we have

$$\mathcal{K} = \big\{s\langle r^2\rangle \cup A, s\langle r^2\rangle \cup B\big\} \cup \big\{K_j \,:\, j = 1, \ldots, |S|-2\big\},$$

where at least one of A or B is nonempty. Because S does not glue reflections,

$$\mathcal{X} = \big\{\{\mu_0\} \cup Y, \{\mu_1\} \cup Z\big\} \cup \big\{X_j \,:\, j = 1, \ldots, |S|-2\big\},$$

and because S is not characteristic, at least one of Y or Z is nonempty. We may write $S \vee S^\tau = (\mathcal{K} \vee \mathcal{K}^\tau, \mathcal{X} \vee \mathcal{X}^\tau)$, where

$$\mathcal{K} \vee \mathcal{K}^\tau = \big\{s\langle r\rangle \cup A \cup B\big\} \cup \big\{K_j \,:\, j = 1, \ldots, |S|-2\big\}$$

and

$$\mathcal{X} \vee \mathcal{X}^\tau = \big\{\{\mu_0, \mu_1\} \cup Y \cup Z\big\} \cup \big\{X_j \,:\, j = 1, \ldots, |S|-2\big\}.$$

Thus, $S \vee S^\tau$ glues reflections, so this supercharacter theory is characteristic. Our goal is to show that one of Y or Z is empty; if this is true, then μ_i will be an S-superclass function for some

$i = 0, 1$. This will imply that $S \vee S^\tau$ respects parity and that $S = \psi_i(S \vee S^\tau)$, thus completing our proof.

By Corollary 4.3 and Lemma 4.15, we know that $\{K_j : j = 1, \ldots |S| - 2\}$ forms the superclass partition for the restricted supercharacter theory $S_{\langle r^d \rangle}$ for some divisor d of n, and that one of the parts of $\mathcal{X} \vee \mathcal{X}^\tau$ is equal to $\mathrm{Irr}(D_{2n}/\langle r^d \rangle) \smallsetminus \{\mathbf{1}_{D_{2n}}\}$. We consider two cases, and in each case show that (without loss of generality) Z is empty.

Case 1: Suppose this part is $\{\mu_0, \mu_1\} \cup Y \cup Z$. Then $\langle r^d \rangle \subseteq \ker(\mu_0)$, so d is even. Also, $\lambda \in Y \cup Z$, so without loss of generality we may assume $\lambda \in Y$. We claim that Z is empty. Assume not, write $Y = \{\lambda\} \cup Y'$ (where Y' may be empty), and consider

$$\begin{aligned}\sigma_{\{\mu_0, \lambda\} \cup Y'} \cdot \sigma_{\{\mu_1\} \cup Z} &= (\mu_0 + \lambda + \sigma_{Y'}) \cdot (\mu_1 + \sigma_Z) \\ &= \mu_0 \cdot \mu_1 + \mu_0 \cdot \sigma_Z + \lambda \cdot \mu_1 + \lambda \cdot \sigma_Z + \sigma_{Y'} \cdot \mu_1 + \sigma_{Y'} \cdot \sigma_Z,\end{aligned}$$

which after expanding becomes

$$\lambda + \sum_{\chi_m \in Z} 2\chi_{\frac{n}{2} - m} + \mu_0 + \sigma_Z + \sum_{\chi_\ell \in Y'} 2\chi_{\frac{n}{2} - \ell} + \sum_{\substack{\chi_\ell \in Y' \\ \chi_m \in Z}} 4(\chi_{\ell + m} + \chi_{|\ell - m|}), \tag{4.7}$$

Note that

$$Y \cup Z = \{\lambda\} \cup \Big\{\chi_m \ : \ 1 \le m < \frac{n}{2}, \ \frac{n}{d} \text{ divides } m\Big\}.$$

Moreover, Y and Z are disjoint sets of irreducible characters of the form $\chi_{\ell \cdot \frac{n}{d}}$, so there are disjoint subsets $I, J \subseteq \{\frac{n}{d}, \ldots, \frac{(d-2)n}{2d}\}$ such that $Y' = \{\chi_\ell : \ell \in I\}$, $Z = \{\chi_m : m \in J\}$, and $I \cup J = \{\frac{n}{d}, \ldots, \frac{(d-2)n}{2d}\}$. So because $\frac{n}{d} \mid \frac{n}{2}$, it follows that $\frac{n}{2} - k \in I \cup J$ for all $k \in I \cup J$. Thus, (4.7) becomes

$$\sigma_{\{\mu_0\} \cup Y} + 2\sigma_Z + \sum_{\substack{\ell \in I \\ m \in J}} 4(\chi_{\ell + m} + \chi_{|\ell - m|}). \tag{4.8}$$

Now, (4.8) is a linear combination of S-supercharacters by [DI08, Theorem 2.2]. Thus since we have assumed $Z \neq \emptyset$ and σ_Z appears in this equation with nonzero coefficient, it follows that μ_1 must also appear with nonzero coefficient. This can only happen if the index of one of the summands of the rightmost sum in (4.8) is $n/2$. Hence, there exist $\ell \in I$ and $m \in J$ with $\ell + m = n/2$. Hence we have $4\chi_{\frac{n}{2}}$ appearing in the rightmost sum of (4.8). But $4\chi_{\frac{n}{2}} = 4\mu_0 + 4\mu_1$, so this implies that the coefficient of μ_0, and therefore $\sigma_{\{\mu_0\}\cup Y}$, in (4.8) is at least 5. Thus, the coefficient of λ in this equation is at least 5, and therefore there exist $\ell \in I$ and $m \in J$ with $|\ell - m| = 0$. But this is impossible because I and J are disjoint, so we have derived a contradiction. Therefore, Z is empty.

Case 2: Now assume that one of the X_i is equal to $\mathrm{Irr}(D_{2n}/\langle r^d\rangle) \setminus \{\mathbf{1}_{D_{2n}}\}$; without loss, assume it is X_1, so that

$$X_1 = \{\lambda\} \cup \left\{\chi_m \,:\, \frac{n}{d} \text{ divides } m\right\}.$$

Then d is odd and, with notation as in (2.8), $\{\mu_0, \mu_1\} \cup Y \cup Z$ is of the form $W_0^{D_{2n}}$, where W_0 is some part of the supercharacter partition $\mathcal{W}$ of $S_{\langle r^d\rangle}$. Thus, by Frobenius reciprocity, $\mathrm{Res}^{D_{2n}}_{\langle r^d\rangle}(\sigma_{\{\mu_0,\mu_1\}\cup Y\cup Z})$ is equal to σ_{W_0} up to scalar multiplication by a positive integer C. Because $\mathcal{K}$ contains the parts $\{K_i : i = 1, \ldots, |S| - 2\}$, which are the superclasses of $S_{\langle r^d\rangle}$, it follows that $\sigma_{\{\mu_0\}\cup Y}$ and $\sigma_{\{\mu_1\}\cup Z}$ are constant on these parts. Thus, we have nonnegative integers c_W, d_W for $W \in \mathcal{W}$ such that

$$\mathrm{Res}^{D_{2n}}_{\langle r^d\rangle}(\sigma_{\{\mu_0\}\cup Y}) = \sum_{W\in\mathcal{W}} c_W \sigma_W$$

and

$$\mathrm{Res}^{D_{2n}}_{\langle r^d\rangle}(\sigma_{\{\mu_1\}\cup Z}) = \sum_{W\in\mathcal{W}} d_W \sigma_W.$$

But

$$\begin{aligned}\sum_{W\in\mathcal{W}} (c_W + d_W)\sigma_W &= \mathrm{Res}^{D_{2n}}_{\langle r^d\rangle}(\sigma_{\{\mu_0\}\cup Y}) + \mathrm{Res}^{D_{2n}}_{\langle r^d\rangle}(\sigma_{\{\mu_1\}\cup Z}) \\ &= \mathrm{Res}^{D_{2n}}_{\langle r^d\rangle}(\sigma_{\{\mu_0,\mu_1\}\cup Y\cup Z}) \\ &= C \cdot \sigma_{W_0}.\end{aligned}$$

Hence, it follows that $c_W = d_W = 0$ unless $W = W_0$, so the sets of irreducible constituents of the restrictions of $\sigma_{\{\mu_0\}\cup Y}$ and $\sigma_{\{\mu_1\}\cup Z}$ are both equal to W_0.

If ξ_m denotes the character of $\langle r^d \rangle$ given by $r^d \mapsto \zeta_{\frac{n}{d}}^m$, then we have

$$\begin{aligned}
\mathrm{Res}_{\langle r^d \rangle}^{D_{2n}}(\sigma_{\{\mu_0\}\cup Y}) &= \mathrm{Res}_{\langle r^d \rangle}^{D_{2n}}(\mu_0) + \sum_{\chi_\ell \in Y} 2\mathrm{Res}_{\langle r^d \rangle}^{D_{2n}}(\chi_m) \\
&= \xi_{\frac{n}{2d}} + \sum_{\chi_m \in Y} 2(\xi_m + \overline{\xi_m}) \\
&= c_{W_0} \sum_{\xi_m \in W_0} \xi_m.
\end{aligned}$$

Thus, Y can only contain χ_m if $m \equiv \frac{n}{2d} \bmod \frac{n}{d}$. Similarly, Z can only contain characters of this form, and any character of this form must lie in $Y \cup Z$, so that

$$Y \cup Z = \left\{ \chi_m \,:\, m \equiv \frac{n}{2d} \bmod \frac{n}{d} \right\}.$$

As with the previous case, we claim that one of Y or Z is empty. Suppose both Y and Z are nonempty, write $Y = \{\chi_\ell : \ell \in I\}$ and $Z = \{\chi_m : m \in J\}$, where I and J are disjoint and $I \cup J = \{(2k-1) \cdot \frac{n}{2d} : 1 \leq k \leq \frac{d-1}{2}\}$. Consider

$$\begin{aligned}
\sigma_{\{\mu_0\}\cup Y} \cdot \sigma_{\{\mu_1\}\cup Z} &= (\mu_0 + \sigma_Y) \cdot (\mu_1 + \sigma_Z) \\
&= \mu_0 \cdot \mu_1 + \mu_0 \cdot \sigma_Z + \sigma_Y \cdot \mu_1 + \sigma_Y \cdot \sigma_Z,
\end{aligned}$$

which, after expanding becomes

$$\lambda + \sum_{m \equiv \frac{n}{2d} \bmod \frac{n}{d}} 2\chi_{\frac{n}{2}-m} + \sum_{\substack{\ell \in I \\ m \in J}} 4(\chi_{\ell+m} + \chi_{|\ell-m|}). \tag{4.9}$$

Consider the first sum in (4.9). If $m \equiv \frac{n}{2d} \bmod \frac{n}{d}$, then $\frac{n}{2} - m \equiv 0 \bmod \frac{n}{d}$. Thus, (4.9) becomes

$$\sigma_{X_1} + \sum_{\substack{\ell \in I \\ m \in J}} 4(\chi_{\ell+m} + \chi_{|\ell-m|}). \tag{4.10}$$

Let $\ell \in I$ and $m \in J$. Then $\ell + m$ and $|\ell - m|$ are both nonzero integers less than n and $\ell + m, |\ell - m| \equiv 0 \bmod \frac{n}{d}$. Thus because I and J are nonempty, the rightmost sum in (4.10) is nonempty, and every summand belongs to X_1. Hence, the coefficient of σ_{X_1} in (4.10) is at least 3. Thus, the coefficient of λ in this equation is at least 3, and therefore there exist $\ell \in I$ and $m \in J$ with $|\ell - m| = 0$. But because I and J are disjoint, ℓ and m are always distinct. Therefore we have derived a contradiction, so one of Y or Z (without loss assume Z) is empty.

In both cases, we have shown (without losss of generality) that Z is empty, so we may write

$$\mathcal{X} = \big\{\{\mu_0\} \cup Y, \{\mu_1\}\big\} \cup \big\{X_j \,:\, j = 1, \ldots, |S| - 2\big\}.$$

Thus because μ_1 is an S-supercharacter, it follows that any two rotations r^k and r^ℓ lie in the same superclass only if $k \equiv \ell \bmod 2$, and that

$$A = \big\{r^k \,:\, d \not| k,\, 2 \not| k\big\}$$

and

$$B = \big\{r^k \,:\, d \not| k,\, 2 \mid k\big\}.$$

Therefore $S = \psi_1(S \vee S^\tau)$, and the proof is complete. ∎

Unfortunately, the classification provided by Theorem 4.8 does not immediately imply that the non-characteristic supercharacter theories of D_{2n} are $*$- or Δ-products. While this is admittedly a downside, we believe the covering relations highlighted in this classification are more than enough to reconstruct the lattice $\mathrm{SCT}(D_{2n})$ to a satisfactory degree of detail.

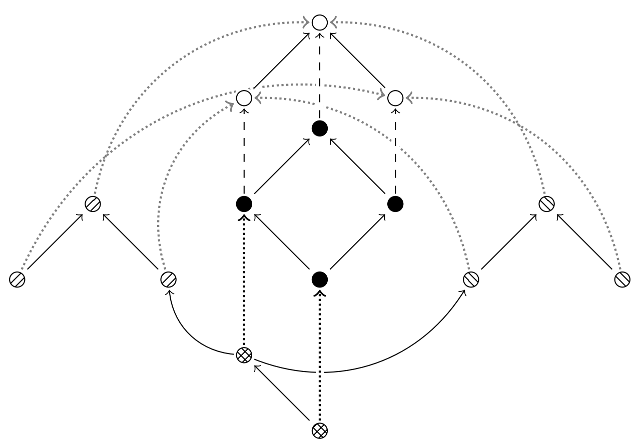
Example 4.18. Now that we have finished the classification, let us revisit Example 4.12. Using Theorem 4.8, we can compute the remainder of $\mathrm{SCT}(D_{12})$ and see how it fits with the sublattice that we computed earlier. With notation as in that example, one can compute that $\mathcal{S} = \{A', C', \varphi(B'), \varphi(C'), \varphi(D')\}$, so that there are six noncharacteristic supercharacter theories. Thus, $\mathrm{SCT}(D_{12})$ takes the form shown in Figure 4.6.

4.4 More general semidirect products

The remainder of this chapter is devoted to generalizations of the dihedral group classification to semidirect products of the form $\mathbb{Z}_n \rtimes \mathbb{Z}_p$, where p is a prime number. Let $\langle x \rangle$ be a cyclic group of order $n \geq 1$, let $\langle y \rangle$ be a cyclic group of prime order p, where p divides the order of $\mathrm{Aut}(\langle x \rangle)$. Let $\langle y \rangle$ act on $\langle x \rangle$ by extending the rule $y \cdot x = x^a$, where a is some positive integer with $1 < a < n$, $(a, n) = 1$, and $a^p \equiv 1 \bmod n$. Let us denote the semidirect product $G = \langle x \rangle \rtimes \langle y \rangle$ by

$$G = SD(n, p, a) = \langle x, y \,:\, x^n = y^p = e,\, yxy^{-1} = x^a \rangle. \tag{4.11}$$

Figure 4.6: The full lattice of supercharacter theories of D_{12}. Compare with Figure 4.3.



Let $\xi_0, \xi_1, \ldots, \xi_{n-1}$ be the irreducible characters of $\langle x \rangle$, where $\xi_i(x^j) = \zeta_n^{ij}$. Similarly, let $\psi_0, \ldots, \psi_p$ be the irreducible characters of $\langle y \rangle$, where $\psi_i(y^j) = \zeta_p^{ij}$. Since $G/\langle x \rangle \cong \langle y \rangle$, we may view the ψ_i as characters of G via the rule $\psi_i(x^j y^k) = \zeta_p^{ik}$.

Now, G acts on $\langle x \rangle$ and on $\mathrm{Irr}(\langle x \rangle)$ by the following rules:

$$(x^i y^j) x^k (x^i y^j)^{-1} = y^j \cdot x^k = x^{ka^j}$$

and

$$\begin{aligned}\big(\xi_\ell \cdot (x^i y^j)\big)(x^k) &= \xi_\ell\big((x^i y^j) x^k (x^i y^j)^{-1}\big) \\ &= \xi_\ell(x^{ka^j}) \\ &= \zeta_n^{\ell a^j k} \\ &= \xi_{\ell a^j}(x^k).\end{aligned}$$

Thus, the G-orbit of a character ξ_ℓ is $\{\xi_\ell, \xi_{\ell a}, \ldots, \xi_{\ell a^{p-1}}\}$, which has size 1 or p, since $\mathrm{Stab}_G(\xi_\ell) = \langle x \rangle \rtimes \mathrm{Stab}_{\langle y \rangle}(\xi_\ell)$ and $\langle y \rangle$ is a group of prime order. The orbits of size 1 coincide with integers ℓ such that $\ell \equiv \ell a \bmod n$. If we define $c = \gcd(n, a-1)$, then it is easy to see that $\ell \equiv \ell a \bmod n$ if and only if n/c divides ℓ. The number c is also significant because the commutator subgroup of G is $\langle x^c \rangle$ whenever p does not divide n.

For any irreducible character ξ_ℓ of $\langle x \rangle$, let $\widetilde{\xi_\ell}$ denote the function

$$\widetilde{\xi_\ell}(x^i y^j) = \xi_\ell(x^i).$$

This is not a character of G in general. If ξ_ℓ has an orbit of size 1 (equivalently, if n/c divides ℓ), then evidently $\mathrm{Stab}_G(\xi_\ell) = G$, and Mackey's method of little groups (see [CR90, Section 11]) implies that the functions $\widetilde{\xi_\ell}\psi_j$ for $0 \le j < p$ are irreducible characters. The orbits of size p can be indexed by integers ℓ such that n/c does not divide ℓ. For $\ell \in I$, Mackey's method also tells us that $\mathrm{Stab}_G(\xi_\ell) = \langle x \rangle$, and hence $\xi_\ell^G = \mathrm{Ind}_{\langle x \rangle}^G(\xi_\ell)$ is irreducible.

Thus,

$$\mathrm{Irr}(G) = \big\{\mathrm{Ind}_{\langle x \rangle}^G(\xi_\ell) \,:\, \ell \in I\big\} \cup \big\{\widetilde{\xi_{\ell \cdot \frac{n}{c}}}\psi_m \,:\, 0 \le \ell < c,\, 0 \le m < p\big\}, \tag{4.12}$$

where I is an index set (to be determined) for the orbits in $\mathrm{Irr}(\langle x\rangle)$ of size p.

A reasonable goal for expanding the classification of $\mathrm{SCT}(D_{2n})$ is to classify the supercharacter theories of $\mathbb{Z}_n \rtimes \mathbb{Z}_p$, at least in some tractable cases. If p does not divide n and $c = 1$, then G is a Frobenius group with Frobenius kernel $\langle x\rangle$. Thus Theorem 4.16 may be applied to classify $\mathrm{SCT}(G)$. In Section 4.4.1, we will provide a classification in the style of Theorem 4.14 which also implies that every supercharacter theory is characteristic.

We have observed empirically that the structure of $\mathrm{SCT}(SD(n, p, a))$ is well-behaved as long as p does not divide n and c is prime. If either of those conditions fail, then the lattice becomes much more complicated. In Section 4.4.2, we will describe examples highlighting this qualitative behavior. We will also summarize partial results and conjectures towards a classification of $\mathrm{SCT}(G)$ in this case.

4.4.1 p does not divide n and $c = 1$

4.4.1.1 More general results

Consider the normal subgroups of $G = SD(n, p, a)$, where p does not divide n, momentarily suspending the assumption that $c = 1$.

Lemma 4.19. *Let $G = SD(n, p, a)$, where p does not divide n. A normal subgroup N of G either takes the form $N = \langle x^d\rangle$ for some divisor d of n, or it takes the form $\langle x^d, y\rangle$ for some divisor d of c.*

Proof. Let N be a normal subgroup of G. If N is contained in $\langle x\rangle$, then N is of the form $\langle x^d\rangle$ for some divisor d of n. Suppose N is not contained in $\langle x\rangle$. Then N contains an element of the form $x^i y^j$ for some $j \neq 0$. Since

$$(x^i y^j)^k = (x^i)^{1+a^j+\cdots+(a^j)^{k-1}} y^{jk},$$

it follows that p divides the order of $x^i y^j$. Thus, N contains a subgroup of order p. Since N is normal, it contains all subgroups of G of order p. Hence, $y \in N$. Now $N \cap \langle x\rangle = \langle x^d\rangle$ for some divisor d of n, and one can check that x^d and y generate N. Therefore, $N = \langle x^d, y\rangle$. Now, conjugating y by x yields $x^{-1}yx = x^{a-1}y \in N$, which implies that d divides $a-1$. ∎

Lemma 4.20. *Let $G = SD(n,p,a)$, where p does not divide n and let $c = \gcd(n, a-1)$. Then G is a Frobenius group if and only if $c = 1$. In this case, the Frobenius kernel is $\langle x\rangle$ and the Frobenius complement is $\langle y\rangle$.*

Proof. If $c = 1$, then one can check that G is a Frobenius group with Frobenius kernel $\langle x\rangle$ and Frobenius complement $\langle y\rangle$. Conversely, suppose G is a Frobenius group, i.e., that there exists a proper nontrivial subgroup H such that $H \cap H^g = 1$ for all $g \in G \smallsetminus H$. Since H is not normal, it follows by the above lemma that H must contain y. Write $H = \langle x^d, y\rangle$. Then for all $g \in G \smallsetminus H$, $\langle x^d\rangle$ is a subgroup of H^g, which implies that $d = n$, and therefore $H = \langle y\rangle$. Now let $g \in G \smallsetminus \langle y\rangle$ and write $g = x^i y^j$. Then because $\langle y\rangle^g \cap \langle y\rangle = 1$, it follows that y^g, which is equal to $x^{i(1-a^{j+1})}y$, does not lie in $\langle y\rangle$. Equivalently, n/c does not divide i. But since this is true for all $1 \leq i < n$, it follows that $c = 1$. ∎

The following corollary holds independent of the assumption that p not divide n.

Corollary 4.21. *Let $G = SD(n,p,a)$ and let S be a supercharacter theory whose superclass partition has a part K containing all of $G \smallsetminus \langle x^d\rangle$ for some divisor d of n. Then $G \smallsetminus K$ is a normal subgroup of G.*

Proof. This is an immediate consequence of Lemma 4.2. ∎

It will also be helpful to classify the conjugacy classes of G.

Lemma 4.22. *For any group $G = SD(n,p,a)$, where p does not divide n, and any $0 \leq i < n$, $1 \leq j < p$, we have*

$$\mathrm{Cl}(x^i y^j) = \langle x^c\rangle x^i y^j.$$

Proof. Let $x^i y^j$ be given and conjugate by x. This yields

$$x x^i y^j x^{-1} = x^{i-(a^j-1)} y^j \in \langle x^c\rangle x^i y^j.$$

Conjugating by y yields

$$yx^iy^jy^{-1} = x^{ia}y^j.$$

Since c divides $a-1$, it follows that $ia \equiv i \bmod c$, so $x^{ia}y^j \in \langle x^c\rangle x^iy^j$. Thus, $\mathrm{Cl}(x^iy^j) \subseteq \langle x^c\rangle x^iy^j$ for all $0 \le i < n$ and $1 \le j < p$.

For the reverse containment, consider the order of x^iy^j; this is either p or $p \cdot c$. In the former case, x^iy^j generates a Sylow p-subgroup of G. By Sylow theory, $\langle x^iy^j\rangle$ is conjugate to $\langle y\rangle$. Since $N_G(\langle y\rangle) = \langle x^{n/c}, y\rangle$ is an abelian group (in fact cyclic, since $(p,c) = 1$), we have $\#\mathrm{Orb}_{N_G(\langle x^iy^j\rangle)}(x^iy^j) = 1$. Thus,

$$\#\mathrm{Cl}_G(x^iy^j) = \#\mathrm{Syl}_p(G) \cdot \#\mathrm{Orb}_{N_G(\langle x^iy^j\rangle)}(x^iy^j) = |G : N_G(\langle y\rangle)| = \frac{n}{c}.$$

Hence $\mathrm{Cl}(x^iy^j)$ and $\langle x^c\rangle x^iy^j$ are equal in this case.

If the order of x^iy^j is $p \cdot c$, then $\langle x^iy^j\rangle = N_G(P)$ for some $P \in \mathrm{Syl}_p(G)$. Thus because normalizers of Sylow subgroups are self-normalizing, we have $C_G(x^iy^j) = N_G(\langle x^iy^j\rangle) = \langle x^iy^j\rangle$. Consequently, $\#\mathrm{Cl}(x^iy^j) = n/c$, and therefore $\mathrm{Cl}(x^iy^j) = \langle x^c\rangle x^iy^j$. ∎

Thus $\langle x\rangle y^j$ is a union of c conjugacy classes, for any $1 \le j < p$. This lemma also implies that if S is any supercharacter theory for which $\langle x\rangle$ and $\langle x^c\rangle$ are supernormal, then S factors as a Δ-product over these subgroups.

Lemma 4.23. *For all $0 \le i < n$ and $0 < j < p$, the $\mathrm{Aut}(G)$-orbit of x^iy^j is a subset of the coset $\langle x\rangle y^j$.*

Proof. First, assume $i = 0$ and $j = 1$. Let $\alpha \in \mathrm{Aut}(G)$ and write $\alpha(y) = x^ky^\ell$, where $0 \le k < n$ and $1 \le \ell < p$. Then

$$\begin{aligned}\alpha(x) &= \alpha\big(yx^{a^{p-1}}y^{-1}\big)\\ &= x^ky^\ell\big(\alpha(x)\big)^{a^{p-1}}y^{-\ell}x^{-k}.\end{aligned}$$

Because α restricts to an automorphism of $\langle x\rangle$, we may write $\alpha(x) = x^m$, where $(m, n) = 1$. Hence,

$$\begin{aligned} x^m &= x^k y^\ell \left(x^m\right)^{a^{p-1}} y^{-\ell} x^{-k} \\ &= \left(x^{a^{p+\ell-1}}\right)^m. \end{aligned}$$

Thus, we have $a^{p+\ell-1} \equiv 1 \bmod n$ and hence $a^{\ell-1} \equiv 1 \bmod n$. Assume $\ell > 1$. Then n/c divides $1 + a + \cdots + a^{\ell-2}$. Since $\ell < p$, it follows that $\ell - 2 < p - 1$. Thus because n/c also divides $1 + a + \cdots + a^{p-1}$, it follows that n/c divides $1 + a + \cdots + a^{p-\ell}$. Thus, n/c divides $1 + a + \cdots + a^{\min(\ell-2, p-\ell)}$, and we can repeat the same argument inductively, arriving at a contradiction. Thus, $\ell = 1$, and therefore $\mathrm{Orb}_{\mathrm{Aut}(G)}(y) \subseteq \langle x\rangle y$. By a similar argument, we have $\mathrm{Orb}_{\mathrm{Aut}(G)}(y^j) \subseteq \langle x\rangle y^j$ for all $j \neq 0$. Finally, we can conclude that for any automorphism α and any element of the form $x^i y^j$ with $j \neq 0$, we have

$$\alpha(x^i y^j) = \alpha(x^i)\alpha(y^j) \in \alpha(x^i)\langle x\rangle y^j = \langle x\rangle y^j,$$

which completes the proof. ∎

4.4.1.2 The classification for $c = 1$

Now, let $G = SD(n, p, a)$ where p does not divide n and $c = \gcd(n, a-1) = 1$.[3] By Lemma 4.20, we can apply Theorem 4.16 to classify $\mathrm{SCT}(G)$. However, the goal of this section is to provide a classification which might generalize to prime c. We also wish to provide a more combinatorial classification which says something about the behavior of the lattice under the action of $\mathrm{Aut}(G)$.

In this case, every element of $\langle y\rangle$ has order p, and by Lemma 4.22, all conjugacy classes of elements of $\langle y\rangle$ are cosets of $\langle x\rangle$. For each divisor d of n, let $\mathcal{P}_d = [mM_{\langle x^d\rangle}(G), MM_{\langle x^d\rangle}]$, where, as in previous sections, $mM_{\langle x^d\rangle}(G) = m_G(\langle x^d\rangle) *_{\langle x^d\rangle} M(G/\langle x^d\rangle)$. Let $\mathcal{P}$ denote the subposet of all supercharacter theories that factor over $\langle x\rangle$. The classification of $\mathrm{SCT}(G)$ is the following theorem.

[3] Actually, the assumption that p does not divide n is superfluous. Indeed if $c = 1$, then $N_G(\langle y\rangle) = \langle y\rangle$, which implies that the number of Sylow p-subgroups of G is n. Thus by the Sylow theorems, p divides $n-1$, and consequently p does not divide n.

Theorem 4.24. *Let $G = SD(n,p,a)$ and suppose $\gcd(n,a-1) = 1$. Then we can express the supercharacter theories of D_{2n} as a disjoint union of the form*

$$\mathrm{SCT}(G) = \mathcal{P} \sqcup \bigsqcup_{\substack{d|n \\ d>1}} \mathcal{P}_d.$$

The proof of Theorem 4.24 is below. Turning our attention to the character table, we have $\ell \equiv \ell a \bmod n$ if and only if n divides ℓ, so there are only p linear characters, all inflated from $\langle y \rangle$. By Lemma 4.23, these linear characters are all fixed by $\mathrm{Aut}(G)$. By these remarks, the conjugacy classes and irreducible characters of G may be arranged in the form

$$\mathrm{Cl}(G) = \left\{C_1, \dots, C_p, D_1, \dots, D_{\frac{n-1}{p}}\right\}$$

and

$$\mathrm{Irr}(G) = \left\{\psi_1, \dots, \psi_p, \xi_1^G, \dots, \xi_{\frac{n-1}{p}}^G\right\},$$

where:

(1) $C_1 = \{e\}$;

(2) for all $1 \leq i \leq p$, C_i is the coset $y^i\langle x \rangle$; and

(3) for all $1 \leq j \leq (n-1)/p$, D_j is the conjugacy class of x^j.

Note that by examination of the character table, one can easily see that $mm_{\langle x \rangle}(G) = m(G)$.[4] With respect to the above notation, $MM_{\langle x \rangle}(G)$ is of the form $(\mathcal{K}, \mathcal{X})$, where

$$\mathcal{K} = \left\{C_1, C_2 \cup \cdots \cup C_p, D_1 \cup \cdots D_{(n-1)/p}\right\} \tag{4.13}$$

and

$$\mathcal{X} = \left\{\{\psi_1\}, \{\psi_2, \dots, \psi_p\}, \{\xi_1, \dots, \xi_{(n-1)/p}\right\}. \tag{4.14}$$

We will need these observations for the proof of Theorem 4.24.

[4] This is a property of Camina pairs in general, i.e., for any group G and any normal subgroup N, (G,N) is a Camina pair if and only if $m(G) = mm_N(G)$.

Proof of Theorem 4.24. Let S be a supercharacter theory and suppose S does not factor over $\langle x\rangle$. Write $S = (\mathcal{K}, \mathcal{X})$. Since $S \geq mm_{\langle x\rangle}(G)$, it follows that $S \not\leq MM_{\langle x\rangle}(G)$. Thus, there exists an S-superclass K that intersects $\langle x\rangle$ and $G \setminus \langle x\rangle$ and an S-supercharacter of the form $X \cup Y \in \mathcal{X}$ such that $X = \{\xi_i : i \in I\}$ and $Y = \{\psi_j : j \in J\}$ are both nonempty. Then we have

$$\begin{aligned}\sigma_{X\cup Y}(x^k) &= \sigma_Y(x^k) + \sigma_X(x^k)\\ &= |Y| + \sum_{i\in I}\sum_{r=0}^{p-1} \zeta_n^{kja^r}\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\sigma_{X\cup Y}(x^\ell y^m) &= \sigma_X(x^\ell y^m) + \sigma_Y(x^\ell y^m)\\ &= \sum_{i\in I} \psi_i(y^m)\\ &= \sum_{i\in I} \zeta_p^{mi}.\end{aligned} \tag{4.16}$$

Since x^k and $x^\ell y^m$ lie in the same superclass, these quantities are equal. Since (4.15) lies in $\mathbb{Q}[\zeta_n]$ and (4.16) lies in $\mathbb{Q}[\zeta_p]$, it follows that the shared value of these equations lies in $\mathbb{Q}[\zeta_n] \cap \mathbb{Q}[\zeta_p]$. But since p does not divide n, this intersection is $\mathbb{Q}$, and thefore (4.16) is integral. But p is prime, so it follows that $I = \{1, \ldots, p-1\}$.

Since all of the linear characters lie in $X \cup Y$, it follows that no supercharacter of S can distinguish between any two elements of $G \setminus \langle x\rangle$. Therefore, all of these elements lie in the same superclass, which is necessarily K. By Corollary 4.21, it follows that $G \setminus K$ is a subgroup of $\langle x\rangle$, which we denote $\langle x^d\rangle$. Finally, by examining $\mathcal{K}$, we can see that S lies in $\mathcal{P}_d$ as desired.

Finally, we just need to verify that the subposets $\mathcal{P}_d$ are disjoint. This is easy if we examine the superclass partitions of $mM_{\langle x^d\rangle}(G)$ and $MM_{\langle x^d\rangle}(G)$. The finer of these two supercharacter theories only has one nontrivial part, and that part is shared with the coarser of the two. ■

By Lemma 3.4, together with the observation that $\langle x^d\rangle$ is a characteristic subgroup of G for every divisor d of n, we arrive at the following corollary.

Corollary 4.25. *Let $G = SD(n,p,a)$, where p does not divide n let $c = \gcd(n, a-1)$. If $c = 1$, then every supercharacter theory of G is characteristic.*

Example 4.26. By Theorem 4.24, the lattice $\mathrm{SCT}(G)$ has a particularly simple structure when $G = SD(n,p,a)$, where n and p are distinct primes: every supercharacter theory other than $M(G)$ factors as a star product over $\langle x\rangle$. Thus by Lemma 3.4, we have a lattice isomorphism

$$\mathrm{SCT}(G) \cong \{M(G)\} \sqcup (\mathrm{InvSCT}_G(\langle x\rangle) \times \mathrm{SCT}(\langle y\rangle)).$$

Since $\langle x\rangle$ and $\langle y\rangle$ are both cyclic groups of prime order, their supercharacter theories are classified by Proposition 3.25: $\mathrm{InvSCT}_G(\langle x\rangle)$ is the subposet of $\mathrm{SCT}(\langle x\rangle)$ which consists of supercharacter theories whose nontrivial superclasses have sizes divisible by p, which is isomorphic to the lattice of divisors of $n-1$ which are divisible by p. Similarly, $\mathrm{SCT}(\langle y\rangle)$ is isomorphic to the lattice of divisors of $p-1$.

4.4.2 Conjectures and partial results

Let $G = SD(n,p,a)$, where p does not divide n and where $c = \gcd(n, a-1)$ is an arbitrary prime number. The remainder of this section will be devoted to some partial results and conjectures concerning the structure of $\mathrm{SCT}(G)$. Our first conjecture is based on example computations of $\mathrm{SCT}(SD(n,p,a))$ for different values of n, p, and a. We believe this conjecture is true for all $SD(n,p,a)$, without regard to the primality of c.

Conjecture 4.27. *Let $G = SD(n,p,a)$, where p does not divide n. Then every supercharacter theory of G is characteristic.*

Turning our attention to the character table, one can compute that G has conjugacy classes $\{x^{in/c}\}$ for $0 \le i < c$, $C_i = \{x^{ia^k} : 0 \le k < p\}$ for i in some index set I, which has size $(n-c)/p$, and $D_{i,j} = \langle x^c\rangle x^i y^j$ for $0 \le i < c$ and $0 \le j < p$. The character table of G takes the form

class rep.	e	$\{x^{i\frac{n}{c}}\}$	$D_{i,j}$	C_i
class size	1	1	$\frac{n}{c}$	p
$\widetilde{\xi}_{\ell\frac{n}{c}}\psi_m$	1	$\zeta_c^{i\ell\frac{n}{c}}$	$\zeta_c^{i\ell}\zeta_p^{jm}$	$\zeta_c^{i\ell}$
ξ_ℓ^G	p	$p\zeta_c^{i\ell}$	0	$\sum_{k=0}^{p-1}\zeta_n^{i\ell a^k}$

(4.17)

The following result rules out any approach to classifying SCT(G) which relies on the structure of Camina pairs. Notably, we cannot use [Wyn17, Theorem 3.1].

Proposition 4.28. *There is no normal subgroup N of G for which (G, N) is a Camina pair.*

Proof. Let N be a proper nontrivial normal subgroup of G. By Lemma 4.19, either $N = \langle x^c, y\rangle$ or N is equal to $\langle x^d\rangle$ for some proper nontrivial divisor d of n. In the former case, we can check that $|Nx| = \frac{n}{c} \cdot p$ and $|\mathrm{Cl}(x)| = p$, so because $c < n$, it follows that $|Nx| > |\mathrm{Cl}(x)|$, hence (G, N) is not a Camina pair. In the latter case, we know by Lemma 4.22 that $\mathrm{Cl}(g) = \langle x^c\rangle g$ for all $g \in G \smallsetminus \langle x\rangle$. Hence if (G, N) is a Camina pair, then c divides d. Hence $x^{\frac{n}{c}} \notin N$, but $\mathrm{Cl}(x^{\frac{n}{c}}) = \{x^{\frac{n}{c}}\}$, which contradicts the condition that $Nx^{\frac{n}{c}}$ be a subset of $\mathrm{Cl}(x^{\frac{n}{c}})$. Therefore, there are no normal subgroups N for which (G, N) is a Camina pair. ∎

The following example shows that any potential classification will not be as simple as Theorem 4.16.

Example 4.29. Let $G = SD(n, p, a)$ and let $c = \gcd(n, a - 1)$. Suppose c is prime and let

$$\mathcal{K} = \left\{\{e\}, \langle x^{\frac{n}{c}}\rangle \smallsetminus \{e\}, \langle x^c, y\rangle \smallsetminus \{e\}, \bigcup_{i=1}^{c-1} x^i\langle x^c, y\rangle\right\}$$

and

$$\mathcal{X} = \big\{\{\mathbf{1}_G\}, X_1, X_2, X_3\big\},$$

where

$$X_1 = \{\widetilde{\xi}_{\ell\frac{n}{c}} : \ell = 1, \ldots, c - 1\},$$

$$X_2 = \{\widetilde{\xi_0}\psi_m : m = 1, \ldots, p-1\},$$

and

$$X_3 = \mathrm{Irr}(G) \smallsetminus (\{\mathbf{1}_G\} \cup X_1 \cup X_2).$$

Then $S = (\mathcal{K}, \mathcal{X})$ is a supercharacter theory of G. The only subgroups of G which are supernormal with respect to S are $\langle x^c, y\rangle$ and $\langle x^{\frac{n}{c}}\rangle$, and it follows by considering the sizes of the superclasses that S cannot factor as a $*$- or Δ-product over either or both of these subgroups.

4.4.2.1 Towards a classification

Let $G = SD(n, p, a)$, where p does not divide n and $c = \gcd(n, a-1)$ is a prime number. For a normal subgroup N of G, write $mM_N(G)$ to denote the $*$-product $m_G(N) *_N M(G/N)$, and let $\mathcal{P}_d$ denote the interval $[mM_{\langle x^d\rangle}(G), MM_{\langle x^d\rangle}(G)]$, as in Section 4.4.1.2. We will see that as in those theorems, the subposets $\mathcal{P}_d$ capture everything too coarse to factor over $\langle x\rangle$.

Through example computations, we have observed a strong enough pattern to expand the above conjecture and state with some confidence that $\mathrm{SCT}(SD(n, p, a))$ is a disjoint union of the following form:

$$\mathcal{P} \sqcup \bigsqcup_{\substack{d|n \\ d>1}} \mathcal{P}_d \sqcup \mathcal{R} \sqcup \mathcal{S}, \tag{4.18}$$

where:

(1) $\mathcal{P}$ is the subposet of supercharacter theories of G that factor as $*$-products over $\langle x\rangle$;

(2) $\mathcal{R}$ is the subposet of supercharacter theories of G that are refinements of elements of $\mathcal{P}$; and

(3) $\mathcal{S}$ is a (apparently structured) subposet of "sporadic" supercharacter theories.

We can establish the relationship between $\mathcal{P}$ and the subposets $\mathcal{P}_d$ right away.

Proposition 4.30. *Let S be a supercharacter theory that does not factor over $\langle x\rangle$ and that is a coarsening of some supercharacter theory T which factors over $\langle x\rangle$. Then S lies in the subposet $\mathcal{P}_d$ for some nontrivial divisor d of n.*

Proof. If $S = M(G)$, then we may set $d = n$. Otherwise, assume $S < M(G)$. Since $S \geq mm_{\langle x\rangle}(G)$, it follows that $S \not\leq MM_{\langle x\rangle}(G)$. Thus there is a superclass $K \in \mathcal{K}(S)$ which intersects both $\langle x\rangle$ and $G \setminus \langle x\rangle$ nontrivially. There is also a part of $\mathcal{X}(S)$ of the form $X \cup Y$, where $X = \{\xi_\ell^G : \ell \in L\}$ and $Y = \{\psi_m : m \in M\}$ for nonempty subsets $L \subseteq \{1, \ldots, n-1\}$ and $M \subseteq \{1, \ldots, p-1\}$.

If $x^i y^j$ and x^k are elements of $K \setminus \langle x\rangle$ and $K \cap \langle x\rangle$, respectively, then

$$\sigma_{X\cup Y}(x^k) = \sigma_X(x^k) + \sigma_Y(x^k) = \sum_{\ell\in L}\sum_{r=0}^{p-1} \zeta_n^{k\ell a^r} + \#Y$$

and

$$\sigma_{X\cup Y}(x^i y^j) = \sigma_X(x^i y^j) + \sigma_Y(x^i y^j) = \sum_{m\in M} \zeta_p^{mj}.$$

Since the right hand sides of the above two equations are equal, their value lies in $\mathbb{Q}[\zeta_p] \cap \mathbb{Q}[\zeta_q] = \mathbb{Q}$, and is therefore equal to some integer s. In particular, it follows that the polynomial $\sum_{i\in I} x^i - s$ is divisible by $1 + x + \cdots + x^{p-1}$, whence $M = \{1, \ldots, p-1\}$ and $s = -1$. Since $S \geq mm_{\langle x\rangle}(G)$, the characters $\widetilde{\xi}_{\ell\frac{n}{c}}\psi_{m_1}$ and $\widetilde{\xi}_{\ell\frac{n}{c}}\psi_{m_2}$ lie in the same part of $\mathcal{X}(S)$ for all m_1 and m_2. In particular, no S-supercharacter can distinguish between any two elements of $G \setminus \langle x\rangle$, and therefore $G \setminus \langle x\rangle$ is a subset of a superclass, which is necessarily K. By Corollary 4.21, it follows that $G \setminus K$ is a subgroup of $\langle x\rangle$, which we denote by $\langle x^d\rangle$.

Finally, we consider $K = G \setminus \langle x^d\rangle$. This is the preimage of the nontrivial part of $\mathcal{K}(M(G/\langle x^d\rangle))$ under the canonical projection map. Hence,

$$\mathcal{K}(mM_{\langle x^d\rangle}(G)) = \big\{K\big\} \cup \big\{\mathrm{Cl}(g) : g \in G \setminus K\big\}$$

and

$$\mathcal{K}(MM_{\langle x^d\rangle}(G)) = \big\{\{e\}, K, G \setminus (K \cup \{e\})\big\}.$$

Therefore, $S \in \mathcal{P}_d$. ∎

There is little that we can say about the structure of $\mathcal{R}$ beyond some qualitative observations. Based on a limited number of examples which were computed using the SCTFinder algorithm (Algorithm 3.30), it appears that each element of this subposet is covered by a unique element of $\mathcal{P}$ and this $*$-product is recovered by gluing together the cosets of $\langle x\rangle$.

It also appears that $\mathcal{R}$ is a product of two smaller lattices: one which appears related to $\mathrm{SCT}(\mathbb{Z}_{n/c})$ and one which appears related to $\mathrm{SCT}(\mathbb{Z}_p)$. This can be found by considering the sublattice of $\mathcal{R}$ of supercharacter theories S for which $\xi_{\ell\frac{n}{c}}\psi_m$ is an S-supercharacter for all $\ell = 0,\dots,c-1$ and $m = 0,\dots,p-1$. Such a supercharacter theory S is "fully split" in the sense that if T is the minimal $*$-product coarser than S, then $\mathcal{X}(T)$ contains $\{\xi_{\ell\frac{n}{c}}\psi_m : m = 0,\dots,p-1\}$ as a part, for each $\ell = 0,\dots,c-1$. While these observations are merely conjecture, we can describe one notable covering relation in the following proposition.

Proposition 4.31. *The minimal factorable supercharacter theory $mm_{\langle x\rangle}(G)$ is an atom of* $\mathrm{SCT}(G)$.

Proof. This supercharacter theory has superclass and supercharacter partitions

$$\begin{aligned}\mathcal{K}(mm_{\langle x\rangle}(G)) &= \big\{\{e\}\big\} \cup \big\{\langle x\rangle y^j : j = 1,\dots,p-1\big\} \cup \big\{C_i : i \in I\big\} \text{ and}\\ \mathcal{X}(mm_{\langle x\rangle}(G)) &= \big\{\{\widetilde{\xi}_0\psi_m\} : m = 0,\dots,p-1\big\}\\ &\qquad\qquad \cup \big\{\{\widetilde{\xi}_{\ell\frac{n}{c}}\psi_m : m = 0,\dots,p-1\} : \ell = 1,\dots,c-1\big\}\\ &\qquad\qquad \cup \big\{\xi_\ell^G : \ell \in I\big\},\end{aligned}$$

respectively.

Let $S \le mm_{\langle x\rangle}(G)$ and suppose one of the parts of $\mathcal{K}(S)$ contains both $\langle x^c\rangle x^i y^j$ and $\langle x^c\rangle x^k y^j$ for some distinct i,k with $0 \le i,k < c$. We claim that this implies that $S = mm_{\langle x\rangle}(G)$. Note that because $\mathcal{K}(S)$ is strictly coarser than $\mathrm{Cl}(G)$, it follows that $\mathcal{X}(S)$ contains some non-singleton part X, which by the relation $S \le mm_{\langle x\rangle}(G)$ is necessarily a subset of the set $X_\ell := \{\widetilde{\xi}_{\ell\frac{n}{c}}\psi_m : m = 0,\dots,p-1\}$ for some $1 \le \ell < c$. Let J be an index set for the elements of X, i.e., $X = \{\widetilde{\xi}_{\ell\frac{n}{c}}\psi_m : m \in J\}$. Then

$$\sigma_X(x^i y^j) = \sum_{m\in J} \zeta_c^{i\ell}\zeta_p^{jm} = \sum_{m\in J} \zeta_c^{k\ell}\zeta_p^{jm} = \sigma_X(x^k y^j).$$

Hence, $(\zeta_c^{i\ell} - \zeta_c^{k\ell}) \cdot \sum_{m\in J} \zeta_p^{jm} = 0$. Since $i \neq k$, this equation implies that $\sum_{m\in J} \zeta_p^{jm} = 0$, which implies that $J = \{0, \dots, p-1\}$, and therefore $X = X_\ell$. In particular, we are done if $c = 2$, since in this case $\mathcal{X}(mm_{\langle x\rangle}(G))$ has only one non-singleton part, which is X_1.

Now assume $c > 2$, let $1 \leq \ell_1, \ell_2 < c$ be distinct, and suppose $X_{\ell_1} \in \mathcal{X}(S)$ but $X_{\ell_2} \notin \mathcal{X}(S)$. Then by the preceeding paragraph, $\{\widetilde{\xi}_{\ell_2 \frac{n}{c}} \psi_m\}$ is a part of $\mathcal{X}(S)$ for each $m = 0, \dots, p-1$. Thus,

$$(\widetilde{\xi}_{\ell_2 \frac{n}{c}} \psi_1)(x^i y^j) = \zeta_c^{i\ell_2} \zeta_p^j = \zeta_c^{k\ell_2} \zeta_p^j,$$

hence $i\ell_2 \equiv k\ell_2 \bmod c$. But since c is prime, this implies $i \equiv k \bmod c$. But this is a contradiction, since these were assumed to be distinct. Therefore, $S = mm_{\langle x\rangle}(G)$. ∎

We can fully classify $\mathcal{R}$ if we add the assumption that $c = 2$. Suppose $c = 2$ and let $\mathcal{G}$ be the subposet of $\mathcal{P}$ consisting of those supercharacter theories S for which $\mathcal{X}(S)$ contains the set $X_1 = \{\widetilde{\xi}_{\frac{n}{2}} \psi_m : m = 0, \dots, p-1\}$ as a part.

If $S \in \mathcal{G}$ factors as $(\mathcal{L}, \mathcal{Y}) *_{\langle x\rangle} (\mathcal{M}, \mathcal{Z})$ and thus corresponds to the data

$$\mathcal{K}(S) = \mathcal{L} \cup \left\{ \bigcup_{y^j \in M} \langle x\rangle y^j \ : \ M \in \mathcal{M} \smallsetminus \{\{e\}\} \right\}$$

and

$$\mathcal{X}(S) = \mathcal{Z} \cup \mathcal{Y}',$$

where

$$\mathcal{Y}' = \left\{ \mathrm{Irr}(G|\sigma_Y) \ : \ Y \in \mathcal{Y} \smallsetminus \{\{\mathbf{1}_{\langle y\rangle}\}\} \right\},$$

then $X_1 \in \mathcal{Y}'$. We can refine S to an element of $\mathcal{R}$, which we denote $\rho(S)$ and such that which corresponds to the pair of partitions

$$\mathcal{K}(\rho(S)) = \mathcal{L} \cup \left\{ \bigcup_{y^j \in M} \langle x^2\rangle y^j \ : \ M \in \mathcal{M} \smallsetminus \{\{e\}\} \right\}$$
$$\cup \left\{ \bigcup_{y^j \in M} \langle x^2\rangle x y^j \ : \ M \in \mathcal{M} \smallsetminus \{\{e\}\} \right\}$$

and

$$\mathcal{X}(\rho(S)) = \mathcal{Z} \cup \left\{ \widetilde{\xi}_{\frac{n}{2}} \sigma_Z : Z \in \mathcal{Z} \smallsetminus \{\{\mathbf{1}_{\langle y\rangle}\}\} \right\} \cup \mathcal{Y}' \smallsetminus \{X_1\}.$$

It is routine to check that these partitions define a supercharacter theory, and that $\rho(S) < S$.

Lemma 4.32. *Let $G = SD(n, p, a)$, where p is prime and $c = \gcd(n, a-1)$. If $c = 2$ and ρ and $\mathcal{G}$ are defined as above, then S covers $\rho(S)$ for all $S \in \mathcal{G}$.*

Proof. Suppose $\rho(S) \leq T < S$. Then $X_1 \notin \mathcal{X}(T)$. Then $X_1 \notin \mathcal{X}(T)$, so $\mathcal{X}(T)$ must contain a part which is a proper subset of X_1, say $X = \{\widetilde{\xi}_{\frac{n}{2}}\psi_m : m \in J\}$. Since $J \neq \{0, \ldots, p-1\}$, it follows that $\sum_{m\in J} \zeta_p^m \neq 0$. Thus,

$$\sigma_X(y^j) - \sigma(xy^j) = 2\sum_{m\in J}(\zeta^j)^m \neq 0,$$

Therefore, no part of $\mathcal{K}(T)$ contains both $\langle x^2\rangle y^j$ and $\langle x^2\rangle xy^j$ for any j. Hence $\widetilde{\xi}_{\frac{n}{2}}$ is a T-supercharacter, so $\widetilde{\xi}_{\frac{n}{2}}\sigma_Z$ is a T-supercharacter for any $Z \in \mathcal{Z}$. Therefore $T = \rho(S)$. ∎

Proposition 4.33. *Let $G = SD(n, p, a)$, where p is prime and $c = \gcd(n, a-1)$. If $c = 2$ and ρ and $\mathcal{G}$ are defined as above, then for all $S \in \mathcal{R}$, $S = \rho(T)$, where $T \in \mathcal{G}$ is obtained from S by coarsening $\mathcal{K}(S)$ according to the rule $y^j \sim xy^j$ for all $0 < j < p$ and coarsening $\mathcal{X}(S)$ according to the rule $\widetilde{\xi}_{\frac{n}{2}}\psi_{m_1} \sim \widetilde{\xi}_{\frac{n}{2}}\psi_{m_2}$ for all $0 \leq m_1, m_2 < p$.*

Proof. Since $S \in \mathcal{R}$, it follows that $S \leq MM_{\langle x\rangle}(G)$, hence $\langle x\rangle$ is S-normal and we can therefore consider $S_{\langle x\rangle} = (\mathcal{L}, \mathcal{Y})$ and $S^{G/\langle x\rangle} = (\mathcal{M}, \mathcal{Z})$. By [Hen08, Lemma 3.9], it follows that $T = S \vee mm_{\langle x\rangle}(G) = S_{\langle x\rangle} *_{\langle x\rangle} S^{G/\langle x\rangle}$. Thus, we just need to show that $\rho(T) = S$. Now, $mm_{\langle x\rangle}(G)$ has superclass and supercharacter partitions given by

$$\mathcal{K}(mm_{\langle x\rangle}(G)) = \big\{\{e\}\big\} \cup \big\{\langle x\rangle y^j : j = 1, \ldots, p-1\big\} \cup \big\{C_i : i \in I\big\} \text{ and}$$
$$\mathcal{X}(mm_{\langle x\rangle}(G)) = \big\{\{\widetilde{\xi}_0\psi_m\} : m = 0, \ldots, p-1\big\} \cup \big\{X_1\big\} \cup \big\{\xi_\ell^G : \ell \in I\big\},$$

respectively. Thus, the only blocks of $\mathcal{X}(S)$ which may differ from those of $\mathcal{X}(\rho(T))$ are the blocks which are subsets of X_1. Thus, it suffices to show that these parts are necessarily of the form $\{\widetilde{\xi}_{\frac{n}{2}}\psi_m : \psi_m \in Z\}$ for $Z \in \mathcal{Z} \setminus \{\{\mathbf{1}_{\langle y\rangle}\}\}$. Let $X \in \mathcal{X}(S)$ be a subset of X_1 and note that $\sigma_{X_1} = \widetilde{\xi}_{\frac{n}{2}}$ is an S-supercharacter. Then $\sigma_{X_1}\sigma_X$ is a linear combination of parts of $\mathcal{Z}$. By a similar argument, $\sigma_{X_1}\sigma_Z$ is a linear combination of parts $X \in \mathcal{X}(S)$ which are subsets of X_1. Therefore, $S = \rho(T)$. ∎

Corollary 4.34. *Let $G = SD(n, p, a)$, where p is prime and $c = \gcd(n, a-1)$. If $c = 2$, then $\mathcal{R}$ is* *order-isomorphic to* $\mathcal{A} \times \mathrm{SCT}(G/\langle x \rangle)$*, where* $\mathcal{A}$ *is the interval of supercharacter theories of* $\langle x \rangle$ *for which* $\xi_{\frac{n}{2}}$ *is a supercharacter.*

Much of the structure of the subposet $\mathcal{S}$ of "sporadic" supercharacter theories has so far evaded our classification. Based on observations, this subposet contains supercharacter theories which factor over $\langle x^c, y \rangle$, as well as the supercharacter theory defined in Example 4.29. For c prime, it appears that $|\mathcal{S}|$ is small compared to $|\mathrm{SCT}(G)|$ and the posets $\mathcal{S}$ and $\mathcal{P}$ are mostly incomparable.

Chapter 5

Hopf subalgebras and restriction supercharacter theories

In [AB17], the authors construct a combinatorial Hopf algebra in the sense of [ABS06] by gluing the irreducible characters of finite general linear groups by the action of a group of Galois automorphisms. In this chapter, we will construct a similar combinatorial Hopf algebra using the minimal $\mathrm{GL}_n(\mathbb{F}_q)$-invariant supercharacter theory of $\mathrm{SL}_n(\mathbb{F}_q)$ as the nth graded component, for each n. Combinatorially, this results in a Hopf subalgebra of the Hopf algebra of characters of the finite general linear groups. As a motivating example, we will first perform an analogous construction with the symmetric and alternating groups. These two constructions are explicitly related by the universality of Sym in the category of combinatorial Hopf algebras.

5.1 Hopf algebra over the alternating groups

For each $n \geq 0$, the action of S_n on A_n by conjugation produces a supercharacter theory of A_n whose superclasses are the S_n-conjugacy classes that lie in A_n and whose supercharacters are the irreducible constituents of the restrictions of irreducible characters of S_n. Combinatorially, this is the supercharacter theory obtained by clumping the pairs of "split" cycle-types, i.e., those pairs of A_n-conjugacy classes whose elements possess the same cycle-type.[1]

For $n \geq 0$, let $\mathrm{scf}(A_n)$ denote the algebra of complex-valued superclass functions on A_n with respect to these supercharacter theories (where $A_0 = S_0 = \{1\}$ by convention, so that $\mathrm{scf}(A_0) =$

[1] It is a standard exercise to prove that the S_n-conjugacy class of elements of cycle-type λ is a union of two distinct A_n-conjugacy classes if and only if the parts of λ are all distinct and odd.

$\mathrm{cf}(S_0) \cong \mathbb{C}$). Define

$$\mathbf{scf}(A) = \bigoplus_{n\geq 0} \mathrm{scf}(A_n).$$

Similarly, for all n, define $\mathrm{cf}(S_n)$ to be the algebra of complex-valued class functions on S_n and let $\mathbf{cf}(S) = \bigoplus_{n\geq 0} \mathrm{cf}(S_n)$. This is the familiar Hopf algebra that models the representation theory of symmetric groups, and which is isomorphic to the Hopf algebra of symmetric functions, denoted Sym (see [Mac98]). Our first goal is to define a Hopf algebra structure on $\mathbf{scf}(A)$ that realizes it as isomorphic to a Hopf subalgebra of $\mathbf{cf}(S)$ and examine its image in Sym, which we will denote by Alt.

5.1.1 The maps

Let us first examine the restriction map $\mathrm{Res} : \mathbf{cf}(S) \to \mathbf{scf}(A)$ which is given on graded components by the maps $\mathrm{Res}^{S_n}_{A_n}$. Let $\mathscr{P}$ be the set of all integer partitions. For any integer partition λ of size $|\lambda| = n$, let $\delta_\lambda \in \mathrm{cf}(S_n)$ be the identifier function for the conjugacy class of elements of S_n of cycle-type λ. The functions δ_λ for all $\lambda \in \mathscr{P}$ (respectively, their restrictions to the alternating groups) form a natural basis for $\mathbf{cf}(S)$ (respectively, $\mathbf{scf}(A)$). Evidently, the kernel of the restriction map is spanned by the identifiers of conjugacy classes which do not lie in alternating groups.

We can distinguish which partitions index the superclasses of A_n as follows. Let $\mathrm{sgn} : S_n \to \{-1, 1\}$ be the sign homomorphism whose kernel is A_n and (at the risk of abusing notation) for any integer partition $\lambda \in \mathscr{P}$, define $\mathrm{sgn}(\lambda)$ to be the value of $\mathrm{sgn}(w)$, where w is any element of $S_{|\lambda|}$ of cycle-type λ. In other words, $\mathrm{sgn}(\lambda) = 1$ if elements of cycle-type λ lie in A_n and $\mathrm{sgn}(\lambda) = -1$ otherwise. For a more combinatorial classification of the cycle-types which lie in A_n, we can formulate sgn as

$$\mathrm{sgn}(\lambda) = (-1)^{|\lambda|-\ell(\lambda)},$$

which is a consequence of [Mac98, Example I.7.1].

Let ι_A be the map

$$\begin{aligned} \iota_A : \mathbf{scf}(A) &\to \mathbf{cf}(S) \\ \operatorname{Res}^{S_n}_{A_n}(\delta_\lambda) &\mapsto \delta_\lambda = \operatorname{Res}^{S_n}_{A_n}(\delta_\lambda)^0 \end{aligned} \tag{5.1}$$

which takes a superclass function $f \in \mathrm{scf}(A_n)$ and maps it to f^0, where $f^0 : S_n \to \mathbb{C}$ denotes the class function

$$f^0(x) = \begin{cases} f(x) & : \quad x \in A_n \\ 0 & : \quad \text{otherwise} \end{cases}. \tag{5.2}$$

Recall that if $\lambda \in \mathscr{P}$ is an integer partition, then its **transpose** partition, denoted λ',[2] is obtained by reflecting the Ferrers diagram of λ about the main diagonal. Equivalently, $\lambda' = (\lambda'_1, \lambda'_2, \ldots)$, where for each i, $\lambda'_i = \#\{j : \lambda_j \geq i\}$. It is known that $\chi^{\lambda'} = \chi^\lambda \cdot \mathrm{sgn}$. This implies that[3]

$$\operatorname{Res}(\chi^{\lambda'}) = \operatorname{Res}(\chi^\lambda) \tag{5.3}$$

and (see, e.g., [FH04, Chapter 5.1])

$$\operatorname{Res}(\chi^\lambda)^0 = \frac{1}{2}(\chi^\lambda + \chi^{\lambda'}). \tag{5.4}$$

Thus, $\iota_A : \mathbf{scf}(A) \to \operatorname{Im}(\iota_A)$ and $\operatorname{Res}|_{\operatorname{Im}(\iota_A)} : \operatorname{Im}(\iota_A) \to \mathbf{scf}(A)$ are inverse isomorphisms and we may therefore define the Hopf maps to exploit this identification of $\mathbf{scf}(A)$ with points of $\mathbf{cf}(S)$ fixed under the symmetry of transposition.

Define the unit $u : \mathbb{C} \to \mathbf{scf}(A)$ to be the map $c \mapsto c \cdot \mathbf{1}_{A_0}$, where $\mathbf{1}_{A_0}$ is the trivial character of $A_0 = \{1\}$.

Define the co-unit $\varepsilon : \mathbf{scf}(A) \to \mathbb{C}$ as follows. If $f \in \mathbf{scf}(A)$ decomposes as $f = \sum_{n \geq 0} f_n$, where $f_n \in \mathrm{scf}(A_n)$ for all $n \geq 0$, then we may let

$$\varepsilon(f) = \langle f_0, \mathbf{1}_{A_0} \rangle.$$

[2] This is the only instance of terminology which differs from [Mac98]: that book refers to λ' as the **conjugate** partition. We have chosen a different word for obvious reasons.

[3] Throughout this chapter, we will write Ind and Res with no decoration, as it always occurs componentwise between S_n and A_n.

Define a product on $\mathbf{scf}(A)$ by linearly extending the following operation to arbitrary elements of $\mathbf{scf}(A)$. If $f \in \mathrm{scf}(A_n)$ and $g \in \mathrm{scf}(A_m)$, then we set

$$m_A(f,g) = \mathrm{Res}^{S_{n+m}}_{A_{n+m}}\big(\mathrm{Ind}^{S_{n+m}}_{S_n \times S_m}(f^0 \times g^0)\big). \tag{5.5}$$

If m_S denotes the product on $\mathbf{cf}(S)$, then we have an equivalent definition:

$$m_A = \mathrm{Res} \circ m_S \circ \iota_A^{\otimes 2}. \tag{5.6}$$

Define the comultiplication to be the linear extension of the following map: for all $f \in \mathrm{scf}(A_n)$, let

$$\Delta_A(f) = \sum_{i+j=n} \mathrm{Res}^{A_n}_{A_i \times A_j}(f). \tag{5.7}$$

If Δ_S denotes the coproduct on $\mathbf{cf}(S)$, then we can similarly form an equivalent definition of Δ_A:

$$\Delta_A = \mathrm{Res}^{\otimes 2} \circ \Delta_S \circ \iota_A. \tag{5.8}$$

5.1.1.1 Compatibility conditions

Proposition 5.1. *With the maps defined above,* $\mathbf{scf}(A)$ *forms a bialgebra isomorphic to a bi-subalgebra of* $\mathbf{cf}(S)$.

Proof. Recall that ι_A and $\mathrm{Res}|_{\mathrm{Im}(\iota_A)}$ are inverses. Hence, applying ι_A to (5.6), we obtain

$$\iota_A \circ m_A = m_S \circ \iota_A^{\otimes 2}.$$

Similarly, applying $\iota_A^{\otimes 2}$ to (5.8) yields

$$\iota_A^{\otimes 2} \circ \Delta_A = \Delta_S \circ \iota_A.$$

Thus if τ_S is the linear map on $\mathbf{cf}(S)^{\otimes 2}$ which exchanges the order of factors in simple tensors and τ_A is the analogous map on $\mathbf{scf}(A)^{\otimes 2}$, then we have

$$\begin{aligned}
\iota_A^{\otimes 2} \circ \Delta_A \circ m_A &= \Delta_S \circ m_S \circ \iota_A^{\otimes 2} \\
&= m_S^{\otimes 2} \circ (\mathrm{id}_S \otimes \tau_S \otimes \mathrm{id}_S) \circ \Delta_S^{\otimes 2} \circ \iota_A^{\otimes 2} \\
&= \iota_A^{\otimes 2} \circ m_A \circ (\mathrm{id}_A \otimes \tau_A \otimes \mathrm{id}_A) \circ \Delta_A^{\otimes 2}.
\end{aligned}$$

Therefore, applying $\mathrm{Res}^{\otimes 2}$ to this equation implies that Δ_A and m_A are compatible. The remaining compatibility conditions follow by similar methods. ■

As in [Mac98, Chapter I.7], let $\Psi : \bigcup_{n\geq 0} S_n \to \mathrm{Sym}$ be defined by $\Psi(w) = p_{\rho(w)}$, and define the map $\mathrm{ch}_S : \mathbf{cf}(S) \to \mathrm{Sym}$ by sending a class function $f \in \mathrm{cf}(S_n)$ to

$$\mathrm{ch}_S(f) = \langle f, \Psi\rangle_{S_n} = \sum_{|\lambda|=n} z_\lambda^{-1} f_\lambda p_\lambda,$$

where f_λ denotes the value of f on elements of cycle type λ and

$$z_\lambda = \prod_i i^{m_i(\lambda)} \lambda_i!,$$

where for all i, $m_i = m_i(\lambda)$ is the multiplicity of i in λ. Note that z_λ is the size of the centralizer in S_n of an element of cycle-type λ. It is well-known (again, see [Mac98, Chapter I.7]) that ch_S is a Hopf algebra isomorphism between $\mathbf{cf}(S)$ and Sym. Let ch_A denote the map $\mathrm{ch}_S \circ \iota_A$, so that for $f \in \mathrm{scf}(A_n)$,

$$\mathrm{ch}_A(f) = \langle f^0, \Psi\rangle_{S_n} = \sum_{\substack{|\lambda|=n \\ \mathrm{sgn}(\lambda)=1}} z_\lambda^{-1} f_\lambda p_\lambda.$$

Let Alt denote the image of $\mathbf{scf}(A)$ under this map. We summarize these relationships with a commutative square of bialgebras in (5.9).

$$\begin{tikzcd} \mathbf{cf}(S) \arrow[r, "\mathrm{ch}_S"] & \mathrm{Sym} \\ \mathbf{scf}(A) \arrow[u, "\iota_A"] \arrow[r, "\mathrm{ch}_A"'] & \mathrm{Alt} \arrow[u, hook] \end{tikzcd} \tag{5.9}$$

5.1.2 Structure coefficients

The Hopf isomorphism between $\mathbf{scf}(A)$ and Alt allows us to work entirely in the ring of symmetric functions. Two natural bases for $\mathbf{scf}(A)$ are the superclass identifier functions

$$\{\mathrm{Res}(\delta_\lambda) : \lambda \in \mathscr{P},\ \mathrm{sgn}(\lambda) = 1\}$$

and the supercharacters

$$\{\mathrm{Res}(\chi^\lambda) : \lambda \in I\},$$

where $I \subset \mathscr{P}$ is a complete set of representatives for $\mathscr{P}$ modulo the equivalence relation $\lambda \sim \lambda'$. We can compute the images of these bases under ch_A directly. Let $\lambda \in \mathscr{P}$ be an integer partition of n such that $\mathrm{sgn}(\lambda) = 1$. Since $\mathrm{Res}_{A_n}^{S_n}(\delta_\lambda)^0 = \delta_\lambda$, we have

$$\mathrm{ch}_A(\mathrm{Res}_{A_n}^{S_n}(\delta_\lambda)) = \mathrm{ch}_S(\delta_\lambda) = z_\lambda^{-1} p_\lambda. \tag{5.10}$$

Similarly, it follows by (5.4) that

$$\mathrm{ch}_A(\mathrm{Res}_{A_n}^{S_n}(\chi^\lambda)) = \mathrm{ch}_S\Big(\frac{1}{2}(\chi^\lambda + \chi^{\lambda'})\Big) = \frac{1}{2}(s_\lambda + s_{\lambda'}). \tag{5.11}$$

Thus, the analogous two bases for Alt are the scaled power-sums

$$\{z_\lambda^{-1} p_\lambda : \lambda \in \mathscr{P}, \mathrm{sgn}(\lambda) = 1\}$$

and the averaged Schur functions

$$\Big\{\frac{1}{2}(s_\lambda + s_{\lambda'}) : \lambda \in I\Big\}.$$

Recall the formula for the antipode of a graded and connected Hopf algebra, which was given by (2.14) in Proposition 2.23: for an element h of a graded and connected Hopf algebra H, we have

$$S(h) = -h - \sum_{i=1}^{n-1} S(h_{1,j}) h_{2,n-j},$$

where the elements $h_{i,j}$ come from the equation

$$\Delta(h) = h \otimes 1 + 1 \otimes h + \sum_{j=1}^{n-1} h_{1,j} \otimes h_{2,n-j}.$$

Recall also that the power-sum symmetric functions p_1 for $n \geq 1$ are primitive elements of Sym. Thus, the above equations imply that $S_{\mathrm{Sym}}(p_n) = -p_n$ for all n. Therefore, for any integer partition λ of size n and length $\ell(\lambda)$, we have

$$S_{\mathrm{Sym}}(p_\lambda) = (-1)^{\ell(\lambda)} p_\lambda = (-1)^n \omega(p_\lambda),$$

where ω is the involution defined by the formula $\omega(s_\lambda) = s_{\lambda'}$; see [Mac98, Chapter I, (2.13) and (3.8)]. Therefore, S is equal to $(-1)^n\omega$ on the nth graded component of Sym. Combining equations (5.9), (5.3), and (5.11), it follows that Alt (respectively $\mathbf{scf}(A)$) is closed with respect to S_{Sym} (respectively $S_{\mathbf{scf}(A)} \circ \iota_A$). By defining antipodes S_{Alt} and $S_{\mathbf{scf}(A)}$, it follows that (5.9) is a commutative square of Hopf algebras.

The structure coefficients for the scaled power-sums are well-known, but we will derive them for completeness. By definition, $p_\lambda = \prod_i p_{\lambda_i}$. Hence,

$$(z_\lambda^{-1} p_\lambda) \cdot (z_\mu^{-1} p_\mu) = \frac{z_{\lambda\cup\mu}}{z_\lambda z_\mu} z_{\lambda\cup\mu}^{-1} p_{\lambda\cup\mu}. \tag{5.12}$$

The co-product of a power-sum is computed using the above equation and the facts that Δ is an algebra morphism and that the power-sums p_i are primitive elements. Let λ be an integer partition of n and write $\lambda = (1^{m_1} 2^{m_2} \cdots n^{m_n})$. Then, $p_\lambda = \prod_i p_i^{m_i}$, so that

$$\begin{aligned}
\Delta(z_\lambda^{-1} p_\lambda) &= z_\lambda^{-1} \prod_i \Delta(p_i)^{m_i} \\
&= z_\lambda^{-1} \prod_i (p_i \otimes 1 + 1 \otimes p_i)^{m_i} \\
&= z_\lambda^{-1} \sum_{\lambda=\mu\cup\nu} \binom{m_i}{m_i(\mu)} p_\mu \otimes p_\nu \\
&= \sum_{\lambda=\mu\cup\nu} \frac{1}{z_\lambda} \frac{m_i!}{m_i(\mu)! m_i(\nu)!} p_\mu \otimes p_\nu.
\end{aligned}$$

Therefore,

$$\Delta(z_\lambda^{-1} p_\lambda) = \sum_{\lambda=\mu\cup\nu} (z_\mu^{-1} p_\mu) \otimes (z_\nu^{-1} p_\nu). \tag{5.13}$$

Note that the Littlewood–Richardson coefficients satisfy the following identity for all partitions α, β, γ:

$$c_{\alpha,\beta}^{\gamma} = c_{\alpha',\beta'}^{\gamma'}.$$

Thus, we have the structure coefficients of multiplication in Alt with respect to the supercharacter basis:

$$\begin{aligned}\frac{1}{2}(s_\lambda + s_{\lambda'}) \cdot \frac{1}{2}(s_\mu + s_{\mu'}) &= \frac{1}{4}(s_\lambda s_\mu + s_\lambda s_{\mu'} + s_{\lambda'} s_\mu + s_{\lambda'} s_{\mu'}) \\ &= \frac{1}{4}\sum_{\nu} \left(c^{\nu}_{\lambda,\mu} + c^{\nu}_{\lambda,\mu'} + c^{\nu}_{\lambda',\mu} + c^{\nu}_{\lambda',\mu'}\right) s^{\nu} \\ &= \frac{1}{2}\sum_{\nu\in I} (c^{\nu}_{\lambda,\mu} + c^{\nu}_{\lambda,\mu'} + c^{\nu}_{\lambda',\mu} + c^{\nu}_{\lambda',\mu'}) \cdot \frac{1}{2}(s_\nu + s_{\nu'}).\end{aligned} \tag{5.14}$$

We can also compute the structure coefficients of comultiplication in Alt with respect to this basis. In order to make the computation easier to follow, let $\mathbb{Z}/2\mathbb{Z} = \langle \tau \rangle$ act on $\mathscr{P}$ by transposition. Then,

$$\begin{aligned}\Delta\Big(\frac{1}{2}(s_\nu + s_{\nu'})\Big) &= \frac{1}{2}\sum_{\lambda,\mu} (c^{\nu}_{\lambda,\mu} + c^{\nu'}_{\lambda,\mu}) s_\lambda \otimes s_\mu. \\ &= \frac{1}{2}\sum_{\lambda,\mu\in I}\ \sum_{i,j,k\in\{0,1\}} c^{\tau^k\nu}_{\tau^i\lambda,\tau^j\mu} s_{\tau^i\lambda} \otimes s_{\tau^j\mu}.\end{aligned}$$

Now double-count the Littlewood–Richardson coefficients by introducing another index variable ℓ:

$$\begin{aligned}&= \frac{1}{4}\sum_{\lambda,\mu\in I}\ \sum_{i,j,k,\ell\in\{0,1\}} c^{\tau^k\nu}_{\tau^i\lambda,\tau^{j+\ell}\mu} s_{\tau^i\lambda} \otimes s_{\tau^j\mu} \\ &= \frac{1}{4}\sum_{\lambda,\mu\in I}\sum_{i}\ \sum_{j,k,\ell\in\{0,1\}} c^{\tau^{k-i}\nu}_{\lambda,\tau^{j+\ell-i}\mu} s_{\tau^i\lambda} \otimes s_{\tau^j\mu}.\end{aligned}$$

By re-indexing the variables ℓ and k, we can rewrite the above expression as follows.

$$\frac{1}{4}\sum_{\lambda,\mu\in I}\sum_{j}\ \sum_{k,\ell\in\{0,1\}} c^{\tau^k\nu}_{\lambda,\tau^{j+\ell}\mu}\Big(\sum_i s_{\tau^i\lambda}\Big) \otimes s_{\tau^j\mu}.$$

Now, we can similarly re-index ℓ to absorb j and rearrange the expression into the following form.

$$\frac{1}{4}\sum_{\lambda,\mu\in I}\ \sum_{k,\ell\in\{0,1\}} c^{\nu}_{\tau^{-k}\lambda,\tau^{\ell-k}\mu}\Big(\sum_i s_{\tau^i\lambda}\Big) \otimes \Big(\sum_j s_{\tau^j\mu}\Big).$$

Therefore,

$$\begin{aligned}&\Delta\Big(\frac{1}{2}(s_\nu + s_{\nu'})\Big) = \\ &\sum_{\lambda,\mu\in I} (c^{\nu}_{\lambda,\mu} + c^{\nu}_{\lambda,\mu'} + c^{\nu}_{\lambda',\mu} + c^{\nu}_{\lambda',\mu'})\Big(\frac{1}{2}(s_\lambda + s_{\lambda'})\Big) \otimes \Big(\frac{1}{2}(s_\mu + s_{\mu'})\Big).\end{aligned} \tag{5.15}$$

5.1.3 Co-commutativity and combinatorial Hopf algebras

A **combinatorial Hopf algebra** (CHA) is a pair (H, ζ), where H is a graded and connected Hopf algebra and $\zeta : H \to \mathbb{C}$ is an algebra morphism (usually called a **character**). Aguiar, Bergeron, and Sottile [ABS06] first defined CHAs and developed their theory. In that paper, they showed that $(\mathrm{Sym}, \zeta_{\mathrm{Sym}})$ is the terminal object in the category of co-commutative CHAs, where ζ_{Sym} is defined on the monomial basis by the formula

$$\zeta_{\mathrm{Sym}}(m_\lambda) = \begin{cases} 1 & : \quad \lambda = (n) \text{ or } \lambda = () \\ 0 & : \qquad \text{otherwise} \end{cases}. \tag{5.16}$$

hence applying (5.16), we obtain an equivalent definition

$$\zeta_{\mathrm{Sym}}(s_\lambda) = \begin{cases} 1 & : \quad \lambda = (n) \text{ or } \lambda = () \\ 0 & : \qquad \text{otherwise} \end{cases}. \tag{5.17}$$

As a Hopf subalgebra of Sym, it follows that Alt is co-commutative. Furthermore, it is graded and connected by construction. Thus for any algebra morphism $\zeta_{\mathrm{Alt}} : \mathrm{Alt} \to \mathbb{C}$, we obtain a unique Hopf algebra morphism $\psi_{\zeta_{\mathrm{Alt}}} : \mathrm{Alt} \to \mathrm{Sym}$ such that $\zeta_{\mathrm{Alt}} = \zeta_{\mathrm{Sym}} \circ \psi_{\zeta_{\mathrm{Alt}}}$. This result is restated below.

Theorem 5.2. *[ABS06, Theorem 4.3] For any co-commutative CHA (H, ζ_H), there is a unique Hopf algebra morphism $\psi_H : H \to \mathrm{Sym}$ with the property that $\zeta_H = \zeta_{\mathrm{Sym}} \circ \psi_H$. Moreover, this map is given by the formula*

$$\psi_H(h) = \sum_{\substack{\mu \in \mathscr{P} \\ |\mu| = n}} \zeta_\mu(h) m_\mu, \tag{5.18}$$

where for all $\mu = (\mu_1, \ldots, \mu_k)$, ζ_μ is the composition

$$H \xrightarrow{\Delta^{(k-1)}} H^{\otimes k} \twoheadrightarrow H_{\mu_1} \otimes \cdots \otimes H_{\mu_k} \xrightarrow{\zeta_H^{\otimes k}} \mathbb{C},$$

and where the unlabeled map is the tensor product of the canonical projections onto the graded components H_{μ_i}.

The universal property of Sym in this category ensures that the above map, for $H = \mathrm{Alt}$, reduces to the inclusion $\mathrm{Alt} \hookrightarrow \mathrm{Sym}$ if ψ_{Alt} is taken to be the restriction $\mathrm{Res}(\psi_{\mathrm{Sym}})$.

5.2 Hopf algebra over the special linear groups

There is a deep connection between the symmetric and general linear groups. One views S_n as the general linear group over the (nonexistent) field with one element, or equivalently, one views $\mathrm{GL}_n(q)$ as the natural q-analogue of S_n. Under these analogies, the alternating groups—being the derived subgroups of the symmetric groups—correspond to the special linear groups. We will show that there is a similar Hopf algebra structure for the family of supercharacter theories obtained by restricting the character theories of the general linear groups to their special linear counterparts.

Fix a prime power q and let $n \geq 0$. Let $G_n = \mathrm{GL}_n(q)$ and let $H_n = \mathrm{SL}_n(q)$, where we define $H_0 = G_0 = \{1\}$. The action of G_n on H_n by conjugation produces a supercharacter theory of H_n whose superclasses are the G_n-conjugacy classes in H_n and whose supercharacters are the irreducible constituents of the restrictions of irreducible characters of G_n to H_n. Let $\mathrm{cf}(G_n)$ denote the algebra of complex-valued class functions on G_n, and let $\mathrm{scf}(H_n)$ denote the algebra of complex-valued superclass functions on H_n, so that $\mathrm{scf}(H_0) = \mathrm{cf}(G_0) \cong \mathbb{C}$. Let $\mathbf{cf}(G) = \bigoplus_n \mathrm{cf}(G_n)$ and let $\mathbf{scf}(H) = \bigoplus_n \mathrm{scf}(H_n)$. We wish to describe the superclasses of H_n as combinatorial objects modulo some notion of symmetry in a manner similar to the previous section (i.e., integer partitions modulo transposition). This will be possible using the combinatorics of [Mac98, Chapter IV] and the notion of parallelism defined by Lehrer in [Leh73].

5.2.1 Combinatorial background

The following is a brief summary of the early sections of [Mac98, Chapter IV]. Let $M = \overline{\mathbb{F}}_q^{\times}$ be the algebraic closure of the finite field $\mathbb{F}_q$ and let $F : M \to M$ be the Frobenius endormorphism which sends an element $x \in M$ to x^q. Let Φ be the set of F-orbits in M. Then the conjugacy classes of G_n are indexed by functions $\mu : \Phi \to \mathscr{P}$ subject to the condition

$$\|\mu\| = \sum_{f \in \Phi} \deg(f)|\mu(f)| = n.$$

(Here we use $\deg(f)$ to denote the size of the orbit f. If we view elements of Φ as irreducible polynomials as outlined by Macdonald, then $\deg(f)$ is simply the degree of the polynomial corresponding to f.) Let

$$\mathcal{I} = \{\mu : \Phi \to \mathscr{P} \,:\, \|\mu\| < \infty\} \tag{5.19}$$

denote the set of all functions $\mu : \Phi \to \mathscr{P}$ satisfying the previous equation for some n.

We can describe the irreducible characters of G_n in a similar manner. For all n, let $M_n = \mathbb{F}_{q^n}^{\times} = M^{F^n}$, and let $L_n = \mathrm{Hom}(M_n, \mathbb{C}^{\times})$. Let $L = \varinjlim L_n$, where embeddings $L_m \to L_n$ are given for m dividing n by precomposition with the surjective norm maps $N_{n,m} : M_n \to M_m$. The Frobenius endomorphism F acts on each L_n by precomposition. These actions are compatible with the maps $L_m \to L_n$, and this induces an action of F on L with the property that $L_n = L^{F^n}$ for all n. Let Θ be the set of F-orbits in L. The irreducible characters of G_n are indexed by functions $\lambda : \Theta \to \mathscr{P}$ subject to the condition

$$\|\lambda\| = \sum_{\varphi \in \Theta} \deg(\varphi)|\lambda(\varphi)| = n.$$

Let

$$\mathcal{J} = \{\lambda : \Theta \to \mathscr{P} \,:\, \|\lambda\| < \infty\} \tag{5.20}$$

denote the set of all functions $\lambda : \Theta \to \mathscr{P}$ satisfying the previous equation for some n.

Let $X_{i,f}$ ($i \geq 1,\ f \in \Phi$) be independent variables over $\mathbb{C}$ and for any symmetric function $u \in \mathrm{Sym}$, let $u(f)$ denote the symmetric function $u(f) = u(X_f) = u(X_{1,f}, X_{2,f}, \ldots)$. For all $f \in \Phi$, let $\mathrm{Sym}(X_f)$ denote the Hopf algebra of symmetric functions in these variables. Similarly, let $Y_{i,\varphi}$ ($i \geq 1$, $\varphi \in \Theta$) be independent variables over $\mathbb{C}$ and for any symmetric function $u \in \mathrm{Sym}$, let $u(\varphi)$ denote the symmetric function $u(\varphi) = u(Y_{1,\varphi}, Y_{2,\varphi}, \ldots)$. For all $\varphi \in \Theta$, let $\mathrm{Sym}(Y_\varphi)$ denote the Hopf algebra of symmetric functions in these variables.

These constructions give us two representations of cf(GL) as tensor products of symmetric functions. There are isomorphisms

$$\begin{array}{ccccc}
\bigotimes_{f\in\Phi} \mathrm{Sym}(X_f) & \longleftarrow & \bigoplus_{n\geq 0} \mathrm{cf}(G_n) & \longrightarrow & \bigotimes_{\varphi\in\Theta} \mathrm{Sym}(Y_\varphi) \\
\tilde{P}_\mu & \longleftarrow\!\!\shortmid & \delta_\mu & & \\
 & & \chi^\lambda & \longmapsto & S_\lambda,
\end{array} \tag{5.21}$$

where $S_\lambda = \prod_{\varphi\in\Theta} s_{\lambda(\varphi)}(Y_\varphi)$ and $\tilde{P}_\mu$ is the scaled Hall–Littlewood symmetric function defined in [Mac98, Chapter IV.4].

5.2.1.1 Restriction of characters

Now let us recall some basic notions from [Leh73]. In this paper, Lehrer defines an **n-simplex** to be an F-orbit of size n in L. Let φ be an orbit, and list its elements $x_0, x_1, \ldots, x_{n-1}$ in such a way that $x_i = F^i(x_0)$. In particular, we have $F^n(x_i) = x_i$ for all i, and hence $\varphi \subseteq L^{F^n}$. By construction, we have $L^{F^n} = L_n$, where L_n is viewed as a subobject of L via its canonical embedding. Thus, since $L_n \cong \widehat{M}_n$, we may view n-simplexes as F-orbits of size n in $\widehat{M}_n$.

Two n-simplexes φ and ψ are said to be **adjacent** if there is some $\alpha \in L_1 \subseteq L_n$[4] such that if $\varphi = \{x_0, \ldots, x_{n-1}\}$, then $\psi = \{\alpha x_0, \ldots, \alpha x_{n-1}\}$. Whenever φ and ψ are related this way, we write $\psi = \alpha \cdot \varphi$. For each n, let Θ_n be the set of n-simplexes, so that $\Theta = \bigcup_n \Theta_n$. A **translation** is a map $\tau : \Theta_n \to \Theta_n$ such that there exists $\alpha \in L_1 \subseteq L_n$ with $\tau(\varphi) = \alpha \cdot \varphi$. Finally, the **parallel translation** of Θ induced by $\alpha \in L_1$ is the map $\tau_\alpha : \Theta \to \Theta$ defined by $\tau_\alpha(\varphi) = \alpha \cdot \varphi$. This defines a group action of L_1 on Θ, whence the maps τ_α have inverses, which are given by $\tau_\alpha^{-1} = \tau_{\alpha^{-1}}$.

If λ and μ are related by a parallel translation, call λ and μ **parallel** and write $\lambda \parallel \mu$. For each λ, let $[\lambda]$ denote the parallel class of λ. Note that $L_1 = \widehat{M_1} = \mathrm{Hom}(\mathbb{F}_q^\times, \mathbb{C}^\times) \cong \mathbb{F}_q^\times$, hence L_1 is cyclic. These definitions were used by Lehrer to prove the following classification.

Theorem 5.3. *[Leh73, Theorem 7, Theorem 8] Two irreducible characters χ^λ and χ^μ of G_n have the same restriction to H_n if and only if λ and μ are parallel. Moreover, the irreducible characters of H_n are indexed by pairs $([\lambda], i)$, where $[\lambda]$ is a parallel class and $1 \leq i \leq (q-1)/|[\lambda]|$.*

5.2.2 The maps

Throughout, we will write $\mathrm{Indf}_{G_i\times G_j}^{G_{i+j}}$ to denote parabolic induction from $G_i \times G_j$ to G_n. Similarly, we will denote its adjoint by $\mathrm{Resf}_{G_i\times G_j}^{G_{i+j}}$. The formula for the latter operation is as

[4] We write $L_1 \subseteq L_n$ as an abuse of notation. In reality, L_1 embeds into L_n by precomposition with the norm map $N_{n,1} : M_n \to M_1$.

follows. If u is a class function on G_{i+j} and $g_0 \in G_i \times G_j \subseteq G_{i+j}$, then

$$\mathrm{Resf}_{G_i \times G_j}^{G_{i+j}}(u)(g_0) = \frac{1}{|U_{i,j}|} \sum_{x \in U_{i,j}} u(g_0 x),$$

where $U_{i,j}$ is the unipotent radical of the parabolic subgroup $P_{i,j}$ of G_{i+j} containing the Levi subgroup $G_i \times G_j$.

Let $\mathrm{scf}(\mathrm{SL})$ denote the space $\bigoplus_{n \geq 0} \mathrm{scf}(H_n)$. We wish to define operations on $\mathrm{scf}(\mathrm{SL})$ which realize it as a graded and connected Hopf algebra.

Consider the restriction map $\mathrm{Res} : \mathrm{cf}(\mathrm{GL}) \to \mathrm{scf}(\mathrm{SL})$ which is given on the graded components by the maps $\mathrm{Res}_{H_n}^{G_n}$. For any function $\mu \in \mathcal{I}$, we may define δ_μ to be the identifier function of the conjugacy class indexed by μ. These identifier functions (respectively, their restrictions to the special linear groups) form a natural basis for $\mathrm{cf}(\mathrm{GL})$ (respectively, $\mathrm{scf}(\mathrm{SL})$). Evidently, the kernel of the restriction map is spanned by the identifiers of conjugacy classes which do not lie in special linear groups.

We can distinguish which functions $\mu \in \mathcal{I}$ index the superclasses of H_n as follows. Let $\det : G_n \to \mathbb{F}_q$ denote the determinant homomorphism whose kernel is H_n and (at the risk of abusing notation) for any function $\mu \in \mathcal{I}$ satisfying $\|\mu\| = n$, define $\det(\mu)$ to be the value of $\det(g)$ for any element $g \in G_n$ which lies in the conjugacy class indexed by μ. Then $\det(\mu) = 1$ if and only if μ indexes a partition which lies in H_n. For a more combinatorial classification of the conjugacy classes which lie in H_n, we can formulate det as follows.

Proposition 5.4. *[Mac98, Chapter IV.2, Example 2] For all $\mu \in \mathcal{I}$, we have*

$$\det(\mu) = \prod_{x \in M} x^{|\mu(x)|}.$$

Let ι_{SL} be the map

$$\begin{aligned} \iota_{\mathrm{SL}} : \mathrm{scf}(\mathrm{SL}) &\to \mathrm{cf}(\mathrm{GL}) \\ \mathrm{Res}_{H_n}^{G_n}(\delta_\mu) &\mapsto \delta_\mu = \mathrm{Res}_{H_n}^{G_n}(\delta_\mu)^0 \end{aligned} \tag{5.22}$$

which takes a superclass function $f \in \mathrm{scf}(H_n)$ and maps it to f^0, where $f^0 : G_n \to \mathbb{C}$ is defined as in (5.2).

The following result is analogous to (5.3).

Lemma 5.5. *[Leh73, 5.31 and Theorem 7] Two irreducible characters χ^λ and χ^μ of G_n have the same restriction to H_n if and only if $\mu = \lambda \circ \tau_\alpha$ for some $\alpha \in L_1$.*

By Clifford's Theorem (see [Isa76, Theorem 6.2] for a character-theoretic formulation), we obtain a formula analogous to (5.4). For any irreducible character χ^λ of G_n, we have

$$(\mathrm{Res}^{G_n}_{H_n}\chi^\lambda)^0 = \frac{1}{q-1}\mathrm{Ind}^{G_n}_{H_n}\mathrm{Res}^{G_n}_{H_n}\chi^\lambda = \frac{1}{q-1}\sum_{\tilde{\lambda}\|\lambda}\chi^{\tilde{\lambda}}. \tag{5.23}$$

Therefore, we have

$$\iota_{\mathrm{SL}} \circ \mathrm{Res}|_{\mathrm{Im}(\iota_{\mathrm{SL}})} = \mathrm{id}_{\mathrm{Im}(\iota_{\mathrm{SL}})}.$$

However, for any irreducible character χ^λ, we have

$$(\mathrm{Res} \circ \iota_{\mathrm{SL}})(\mathrm{Res}\chi^\lambda) = \frac{|[\lambda]|}{q-1}\mathrm{Res}\chi^\lambda.$$

Thus, ι_{SL} is a left-inverse for Res, but unlike the previous section, these maps are not full inverses. However, $\mathrm{Res} \circ \iota_{\mathrm{SL}}$ is clearly invertible, being a diagonal transformation of scf(SL).

Define the unit $u : \mathbb{C} \to \mathrm{scf(SL)}$ to be the map $c \mapsto c \cdot \mathbf{1}_{H_0}$, where $\mathbf{1}_{H_0}$ is the trivial character of $H_0 = \{1\}$.

Define the co-unit $\varepsilon : \mathrm{scf(SL)} \to \mathbb{C}$ as follows. If $f \in \mathrm{scf(SL)}$ decomposes as $f = \sum_{n\geq 0} f_n$, where $f_n \in \mathrm{scf}(H_n)$ for all n, then we may let

$$\varepsilon(f) = \langle f_0, \mathbf{1}_{H_0}\rangle.$$

Define a product on scf(SL) by linearly extending the following operation to arbitrary elements of scf(SL). If $f \in \mathrm{scf}(H_n)$ and $g \in \mathrm{scf}(H_m)$, then we set

$$m_{\mathrm{SL}}(f, g) = \mathrm{Res}^{G_{n+m}}_{H_{n+m}}\big(\mathrm{Indf}^{G_{n+m}}_{G_n\times G_m}(f^0 \times g^0)\big), \tag{5.24}$$

where $f^0 : G_n \to \mathbb{C}$ is the extension of f to G_n by zero, as in (5.2), and similarly for $g^0 : G_m \to \mathbb{C}$. If m_{GL} denotes the product on cf(GL), then

$$m_{\mathrm{SL}} = \mathrm{Res} \circ m_{\mathrm{GL}} \circ \iota_{\mathrm{SL}}^{\otimes 2}. \tag{5.25}$$

Define the coproduct by linearly extending the following rule: if $f \in \mathrm{scf}(H_n)$, then

$$\Delta_{\mathrm{SL}}(f) = \sum_{i+j=n} \mathrm{Res}^{G_i \times G_j}_{H_i \times H_j}\big(\mathrm{Resf}^{G_n}_{G_i \times G_j}(f^0)\big). \tag{5.26}$$

If Δ_{GL} denotes the coproduct on $\mathrm{cf}(\mathrm{GL})$, then we can similarly form an equivalent definition of Δ_{SL}:

$$\Delta_{\mathrm{SL}} = \mathrm{Res}^{\otimes 2} \circ \Delta_{\mathrm{GL}} \circ \iota_{\mathrm{SL}}. \tag{5.27}$$

5.2.3 Compatibility conditions

Proposition 5.6. *With the maps defined above,* $\mathrm{scf}(\mathrm{SL})$ *forms a bialgebra isomorphic to a bi-subalgebra of* $\mathrm{cf}(\mathrm{GL})$.

Proof. Recall that ι_{SL} is a left-inverse of $\mathrm{Res}|_{\mathrm{Im}(\iota_{\mathrm{SL}})}$ and that $\mathrm{Res} \circ \iota_{\mathrm{SL}}$ is invertible. Let ψ be the inverse of that map. Hence, applying ι_{SL} to (5.25), we obtain

$$\iota_{\mathrm{SL}} \circ m_{\mathrm{SL}} = m_{\mathrm{GL}} \circ \iota_{\mathrm{SL}}^{\otimes 2}.$$

Similarly, applying $\iota_{\mathrm{SL}}^{\otimes 2}$ to (5.27) yields

$$\iota_{\mathrm{SL}}^{\otimes 2} \circ \Delta_{\mathrm{SL}} = \Delta_{\mathrm{GL}} \circ \iota_{\mathrm{SL}}.$$

Thus if τ_{GL} is the linear map on $\mathrm{cf}(\mathrm{GL})^{\otimes 2}$ which exchanges the order of factors in simple tensors and τ_{SL} is the analogous map on $\mathrm{scf}(\mathrm{SL})^{\otimes 2}$, then we have

$$\begin{aligned}
\iota_{\mathrm{SL}}^{\otimes 2} \circ \Delta_{\mathrm{SL}} \circ m_{\mathrm{SL}} &= \Delta_{\mathrm{GL}} \circ m_{\mathrm{GL}} \circ \iota_{\mathrm{SL}}^{\otimes 2} \\
&= m_{\mathrm{GL}}^{\otimes 2} \circ (\mathrm{id}_{\mathrm{GL}} \otimes \tau_{\mathrm{GL}} \otimes \mathrm{id}_{\mathrm{GL}}) \circ \Delta_{\mathrm{GL}}^{\otimes 2} \circ \iota_{\mathrm{SL}}^{\otimes 2} \\
&= \iota_{\mathrm{SL}}^{\otimes 2} \circ m_{\mathrm{SL}} \circ (\mathrm{id}_{\mathrm{SL}} \otimes \tau_{\mathrm{SL}} \otimes \mathrm{id}_{\mathrm{SL}}) \circ \Delta_{\mathrm{SL}}^{\otimes 2}.
\end{aligned}$$

Therefore, applying $\psi^{\otimes 2} \circ \mathrm{Res}^{\otimes 2}$ to this equation implies that Δ_{SL} and m_{SL} are compatible. The remaining compatibility conditions follow by similar methods. ∎

Now, there is an isomorphism

$$\bigotimes_{f \in \Phi} \mathrm{Sym}(X_f) \longrightarrow \bigotimes_{\varphi \in \Theta} \mathrm{Sym}(Y_\varphi)$$

(cf. [SZ84]) which allows us to identify these two Hopf algebras as one and the same; call this Hopf algebra $\mathrm{Sym}_{\mathrm{GL}}$.

Recall (cf. [Mac98, Chapter IV.4]) that there is an isomorphism

$$\mathrm{ch}_{\mathrm{GL}} : \bigoplus_{n\geq 0} \mathrm{cf}(G_n) \longrightarrow \mathrm{Sym}_{\mathrm{GL}}$$

given on conjugacy class identifiers by $\delta_\mu \mapsto \tilde{P}_\mu$. Let $\mathrm{ch}_{\mathrm{SL}} = \mathrm{ch}_{\mathrm{GL}} \circ \iota_{\mathrm{SL}}$, and let $\mathrm{Sym}_{\mathrm{SL}}$ denote the image of scf(SL) under this map. We summarize these relationships with a commutative square of bialgebras in (5.28).

$$\begin{tikzcd} \mathrm{cf}(\mathrm{GL}) \arrow[r, "\mathrm{ch}_{\mathrm{GL}}"] & \mathrm{Sym}_{\mathrm{GL}} \\ \mathrm{scf}(\mathrm{SL}) \arrow[u, "\iota_{\mathrm{SL}}"] \arrow[r, "\mathrm{ch}_{\mathrm{SL}}"'] & \mathrm{Sym}_{\mathrm{SL}} \arrow[u, hook] \end{tikzcd} \tag{5.28}$$

5.2.4 Structure coefficients

As before, the Hopf isomorphism between scf(SL) and $\mathrm{Sym}_{\mathrm{SL}}$ allows us to work entirely with symmetric functions. Again, as before, the first natural question to ask is "What is a good basis for $\mathrm{Sym}_{\mathrm{SL}}$?" Two natural bases for scf(SL) are the superclass identifier functions

$$\{\mathrm{Res}(\delta_\mu) : \mu \in \mathcal{I},\ \det(\mu) = 1\}$$

and the supercharacters

$$\{\mathrm{Res}(\chi^\lambda) : \lambda \in J\},$$

where J is a complete set of representatives for the parallel classes of functions $\lambda \in \mathcal{J}$. We can compute the images of these bases under $\mathrm{ch}_{\mathrm{SL}}$ directly. By definition, we have

$$\mathrm{ch}_{\mathrm{SL}}(\delta_\mu) = \widetilde{P}_\mu. \tag{5.29}$$

for any $\mu \in \mathcal{I}$. Similarly, it follows by (5.23) that

$$\mathrm{ch}_{\mathrm{SL}}(\mathrm{Res}(\chi^\lambda)) = \frac{1}{q-1} \sum_{\tilde{\lambda} \| \lambda} S_{\tilde{\lambda}} \tag{5.30}$$

for any $\lambda \in \mathcal{J}$. Thus, the analogous two bases for $\mathrm{Sym}_{\mathrm{SL}}$ are the Hall–Littlewood functions

$$\{\widetilde{P}_\mu : \mu \in \mathcal{I},\, \det(\mu) = 1\}$$

and the averaged Schur basis

$$\left\{ \frac{1}{q-1} \sum_{\tilde{\lambda} \| \lambda} S_\lambda : \lambda \in J \right\}.$$

Recall that $S_\lambda = \prod_{\varphi \in \mathcal{J}} s_{\lambda(\varphi)}(Y_\varphi)$ and that $\mathrm{Sym}_{\mathrm{GL}} = \bigotimes_{\varphi \in \mathcal{J}} \mathrm{Sym}(Y_\varphi)$. Thus the antipode of $\mathrm{Sym}_{\mathrm{GL}}$ acts componentwise as the antipode of Sym, and therefore[5]

$$S_{\mathrm{Sym}_{\mathrm{GL}}}(S_\lambda) = (-1)^{\sum_{\varphi \in \mathcal{J}} |\lambda(\varphi)|} S_{\lambda'},$$

where λ' is the componentwise transpose of λ, i.e., $\lambda'(\varphi) = \lambda(\varphi)'$ for all $\varphi \in \mathcal{J}$.

Lemma 5.7. *Let $\lambda \in \Theta$. Then we have $[\lambda'] = \{\mu' : \mu \in [\lambda]\}$.*

Proof. Let τ_α be a parallel translation. Then we have

$$\tau_\alpha(\lambda') = \alpha \circ \lambda' = (\alpha \circ \lambda)' = (\tau_\alpha(\lambda))'.$$

The result follows from this equation. ∎

By Lemma 5.7, we have that $\lambda' \parallel \mu'$ if and only if $\lambda \parallel \mu$, and therefore (5.28) is a commutative square of Hopf algebras.

We can calculate the structure coefficients for the product and coproduct in $\mathrm{Sym}_{\mathrm{SL}}$ with respect to the averaged Schur basis, using the analogous calculations of (5.14) and (5.15) in the previous section. For any $\lambda, \mu, \nu \in \mathcal{J}$, let

$$C^\nu_{\lambda,\mu} = \prod_{\varphi \in \Theta} c^{\nu(\varphi)}_{\lambda(\varphi),\mu(\varphi)},$$

so that in $\mathrm{cf}(G_n)$ (as a consequence of the isomorphism $\mathrm{cf}(G_n) \cong \bigotimes_\varphi \mathrm{Sym}(Y_\varphi)$), we have structure coefficients

$$m_{\mathrm{GL}}(\chi^\lambda, \chi^\mu) = \sum_\nu C^\nu_{\lambda,\mu} \chi^\nu$$

[5] Capital S is used canonically in the literature for both Schur functions and antipodes of Hopf algebras. The author would like to apologize for the excess of Ss in this equation and reassure the reader that it is an isolated incident.

and

$$\Delta_{\mathrm{GL}}(\chi^{\nu}) = \sum_{\lambda,\mu} C^{\nu}_{\lambda,\mu} \chi^{\lambda} \otimes \chi^{\mu}.$$

These coefficients, much like their classical counterparts, are well-behaved with respect to parallelism.

Lemma 5.8. *For any partition-valued functions $\lambda, \mu, \nu \in \mathcal{J}$ and any parallel translation τ_γ, we have*

$$C^{\tau_\gamma \nu}_{\tau_\gamma \lambda, \tau_\gamma \mu} = C^{\nu}_{\lambda,\mu}.$$

Proof. By direct calculation, we have

$$\begin{aligned} C^{\tau_\gamma \nu}_{\tau_\gamma \lambda, \tau_\gamma \mu} &= \prod_{\varphi \in \Theta} c^{\tau_\gamma \nu(\varphi)}_{\tau_\gamma \lambda(\varphi), \tau_\gamma \mu(\varphi)} \\ &= \prod_{\varphi \in \Theta} c^{\nu(\tau_\gamma(\varphi))}_{\lambda(\tau_\gamma(\varphi)), \mu(\tau_\gamma(\varphi))}. \end{aligned}$$

By rearranging the factors in this product, we have

$$\begin{aligned} &= \prod_{\varphi \in \Theta} c^{\nu(\varphi)}_{\lambda(\varphi), \mu(\varphi)} \\ &= C^{\nu}_{\lambda,\mu}, \end{aligned}$$

which completes the proof. ■

Recall that parallel classes are orbits under the action of the cyclic group $L_1 = \langle \alpha \rangle$, which has order $q-1$. Let $\tau = \tau_\alpha$ be the parallel translation corresponding to α, so that for any index λ, we have

$$[\lambda] = \{\tau^i \lambda : i = 0, \ldots, q-2\}$$

(note this set may not have size $q-1$).

Proposition 5.9. *The structure coefficients for multiplication in* $\mathrm{Sym}_{\mathrm{SL}}$ *with respect to the averaged Schur basis are*

$$\left(\frac{1}{q-1} \sum_{\tilde{\lambda} \| \lambda} S_{\tilde{\lambda}} \right) \cdot \left(\frac{1}{q-1} \sum_{\tilde{\mu} \| \mu} S_{\tilde{\mu}} \right) = \frac{1}{q-1} \sum_{\nu \in J} \left(\sum_{\substack{\tilde{\lambda} \| \lambda \\ \tilde{\mu} \| \mu}} C^{\nu}_{\tilde{\lambda}, \tilde{\mu}} \right) \frac{1}{q-1} \sum_{\tilde{\nu} \| \nu} S_{\tilde{\nu}}.$$

for all $\lambda, \mu \in J$.

Proof. Let $\lambda, \mu \in J$. We calculate

$$\begin{aligned}
m_{\mathrm{GL}}\big((\mathrm{Res}\chi^{\lambda})^0, (\mathrm{Res}\chi^{\mu})^0\big) &= \frac{1}{(q-1)^2} m_{\mathrm{GL}}\big(\mathrm{IndRes}\chi^{\lambda}, \mathrm{IndRes}\chi^{\mu}\big) \\
&= \frac{1}{(q-1)^2} \sum_{\substack{\nu \in \mathcal{J} \\ \tilde{\lambda} \| \lambda \\ \tilde{\mu} \| \mu}} C^{\nu}_{\tilde{\lambda}, \tilde{\mu}} \chi^{\nu} \\
&= \frac{1}{(q-1)^2} \sum_{\nu \in J} \sum_{\substack{\tilde{\nu} \| \nu \\ \tilde{\lambda} \| \lambda \\ \tilde{\mu} \| \mu}} C^{\tilde{\nu}}_{\tilde{\lambda}, \tilde{\mu}} \chi^{\tilde{\nu}} \\
&= \frac{1}{(q-1)^2} \sum_{\nu \in J} \sum_{i,j,k} C^{\tau^k \nu}_{\tau^i \lambda, \tau^j \mu} \chi^{\tau^k \nu},
\end{aligned}$$

where i, j, and k index the parallel classes of λ, μ, and ν, respectively (and hence may not cover the range $0, 1, \ldots, q-2$). Exploiting symmetry, this equals

$$\frac{1}{(q-1)^2} \sum_{\nu \in J} \sum_{k} \left(\sum_{i,j} C^{\nu}_{\tau^{i-k}\lambda, \tau^{j-k}\mu} \right) \chi^{\tau^k \nu}.$$

By reindexing the i and j to absorb k, it follows that the innermost sum is independent of k, hence this is equal to

$$\begin{aligned}
&\frac{1}{q-1} \sum_{\nu \in J} \left(\sum_{i,j} C^{\nu}_{\tau^i \lambda, \tau^j \mu} \right) \frac{1}{q-1} \sum_{\tilde{\nu} \| \nu} \chi^{\tilde{\nu}} \\
&\quad = \frac{1}{q-1} \sum_{\nu \in J} \left(\sum_{i,j} C^{\nu}_{\tau^i \lambda, \tau^j \mu} \right) (\mathrm{Res}\chi^{\nu})^0,
\end{aligned}$$

and the calculation is complete. ∎

Proposition 5.10. *The structure coefficients for comultiplication in* $\mathrm{Sym}_{\mathrm{SL}}$ *with respect to the averaged Schur basis are*

$$\Delta\left(\frac{1}{q-1} \sum_{\tilde{\nu} \| \nu} S_{\tilde{\nu}} \right) = \sum_{\lambda, \mu \in J} \left(\sum_{\substack{\tilde{\nu} \| \nu \\ \ell = 0, \ldots, q-2}} C^{\tilde{\nu}}_{\lambda, \tau^{\ell}\mu} \right) \left(\frac{1}{q-1} \sum_{\tilde{\lambda} \| \lambda} S_{\tilde{\lambda}} \right) \otimes \left(\frac{1}{q-1} \sum_{\tilde{\mu} \| \mu} S_{\tilde{\mu}} \right).$$

for all $\nu \in J$.

Proof. Let $\nu \in J$. Then we have

$$\begin{aligned}
\Delta_{\mathrm{GL}}\big(\mathrm{Res}(\chi^\nu)^0\big) &= \Delta_{\mathrm{GL}}\left(\frac{1}{q-1}\sum_{\tilde{\nu}\|\nu}\chi^{\tilde{\nu}}\right)\\
&= \frac{1}{q-1}\sum_{\tilde{\nu}\|\nu}\sum_{\lambda,\mu} C^{\tilde{\nu}}_{\lambda,\mu}\chi^\lambda\otimes\chi^\mu.\\
&= \frac{1}{q-1}\sum_{\lambda,\mu\in J}\sum_{i,j,k} C^{\tau^k\nu}_{\tau^i\lambda,\tau^j\mu}\chi^{\tau^i\lambda}\otimes\chi^{\tau^j\mu}.
\end{aligned}$$

Now, overcount the Littlewood–Richardson coefficient products by introducing another index variable ℓ:

$$\begin{aligned}
&= \frac{1}{(q-1)^2}\sum_{\lambda,\mu\in J}\sum_{\substack{i,j,k\\ \ell=0,\dots,q-2}} C^{\tau^k\nu}_{\tau^i\lambda,\tau^{j+\ell}\mu}\chi^{\tau^i\lambda}\otimes\chi^{\tau^j\mu}\\
&= \frac{1}{(q-1)^2}\sum_{\lambda,\mu\in J}\sum_{i}\left(\sum_{\substack{j,k\\ \ell=0,\dots,q-2}} C^{\tau^{k-i}\nu}_{\lambda,\tau^{j+\ell-i}\mu}\right)\chi^{\tau^i\lambda}\otimes\chi^{\tau^j\mu}.
\end{aligned}$$

By re-indexing the variables ℓ and k to absorb i, it follows that the innermost sum is independent of i, hence we can rewrite the above expression as follows.

$$= \frac{1}{(q-1)^2}\sum_{\lambda,\mu\in J}\sum_{j}\left(\sum_{\substack{k\\ \ell=0,\dots,q-2}} C^{\tau^k\nu}_{\lambda,\tau^{j+\ell}\mu}\right)\left(\sum_i\chi^{\tau^i\lambda}\right)\otimes\chi^{\tau^j\mu}.$$

Now, we can similarly re-index ℓ to absorb j and rearrange the expression into the following form.

$$\begin{aligned}
&= \frac{1}{(q-1)^2}\sum_{\lambda,\mu\in J}\sum_{\substack{k\\ \ell=0,\dots,q-2}} C^{\tau^k\nu}_{\lambda,\tau^\ell\mu}\left(\sum_i\chi^{\tau^i\lambda}\right)\otimes\left(\sum_j\chi^{\tau^j\mu}\right)\\
&= \sum_{\lambda,\mu\in J}\left(\sum_{\substack{k\\ \ell=0,\dots,q-2}} C^{\tau^k\nu}_{\lambda,\tau^{j+\ell}\mu}\right)\mathrm{Res}(\chi^\lambda)^0\otimes\mathrm{Res}(\chi^\mu)^0.
\end{aligned}$$

∎

5.2.4.1 PSH-algebras

We pause now to discuss these constructions in the context of Zelevinsky's theory of PSH-algebras (see [Zel06] for a thorough reference). A Hopf algebra is called **positive** if it is spanned by a distinguished basis of homogeneous elements known as **irreducible** elements, and in such a way that all of the Hopf maps take **positive** elements (i.e., nonnegative $\mathbb{Z}$-linear combinations of irreducible elements) to positive elements. A **PSH-algebra** is defined to be a Hopf algebra which is connected, positive, and self-adjoint. The following structure theorem classifies PSH-algebras.

Theorem 5.11. *[Zel06, Chapter I.2ff] Let H be a PSH-algebra. If H has only one primitive irreducible elemtent, say x, then $H = \mathbb{C}[x]$, and therefore H is isomorphic to* Sym *as a Hopf algebra. Otherwise, H decomposes as a tensor product*

$$H = \bigotimes_{\alpha \in A} H_\alpha,$$

where A is the set of irreducible primitive elements in H and each H_α is a PSH-algebra with only one irreducible primitive element, namely α.

Both Alt and $\mathrm{Sym}_{\mathrm{SL}}$ are positive and connected: the averaged Schur functions $\{\frac{1}{2}(s_\lambda + s_{\lambda'} : \lambda \in I\}$ serve as a basis of irreducible elements for Alt, while the averaged Schur functions $\{\frac{1}{q-1}\sum_{\tilde{\lambda}\|\lambda} S_\lambda : \lambda \in J\}$ do so for $\mathrm{Sym}_{\mathrm{SL}}$. The next result proves that Alt is not self-adjoint.

By (5.14) and (5.15), it follows that Alt is not self-adjoint, and therefore not a PSH-algebra. In particular,

$$\langle xy, z\rangle = \frac{1}{2}\langle x \otimes y, z\rangle$$

holds for all $x, y, z \in$ Alt. Since Alt is a proper subalgebra of Sym, it is perhaps unsurprising that Alt fails to be a PSH-algebra itself.

It is not immediate whether $\mathrm{Sym}_{\mathrm{SL}}$ is self-adjoint; compare Propositions 5.9 and 5.10, which give expressions for the structure coefficients of multiplication and comultiplication in $\mathrm{Sym}_{\mathrm{SL}}$. It is our belief that in a manner analogous to Alt, $\mathrm{Sym}_{\mathrm{SL}}$ at best fails to be self-adjoint by a constant multiple of $1/(q-1)$.

5.2.5 Co-commutativity and combinatorial Hopf algebras

Recall the algebra morphism $\zeta_{\mathrm{Sym}} : \mathrm{Sym} \to \mathbb{C}$, which is defined in (5.16). By the co-commutativity of Sym, it follows that for all $\lambda, \mu, \nu \in \mathcal{J}$,

$$C^{\nu}_{\mu,\lambda} = C^{\nu}_{\lambda,\mu}.$$

Therefore, $\mathrm{Sym}_{\mathrm{GL}}$, and consequently also $\mathrm{Sym}_{\mathrm{SL}}$, is co-commutative. Furthermore, these Hopf algebras are graded and connected by construction. Thus for any algebra morphism $\zeta_{\mathrm{GL}} : \mathrm{Sym}_{\mathrm{GL}} \to \mathbb{C}$ (respectively $\zeta_{\mathrm{GL}} : \mathrm{Sym}_{\mathrm{GL}} \to \mathbb{C}$), we obtain a unique Hopf algebra morphism $\psi_{\zeta_{\mathrm{GL}}} : \mathrm{Sym}_{\mathrm{GL}} \to \mathrm{Sym}$ such that $\zeta_{\mathrm{GL}} = \zeta_{\mathrm{Sym}} \circ \psi_{\zeta_{\mathrm{GL}}}$ (respectively a Hopf algebra morphism $\psi_{\zeta_{\mathrm{SL}}} : \mathrm{Sym}_{\mathrm{SL}} \to \mathrm{Sym}$ such that $\zeta_{\mathrm{SL}} = \zeta_{\mathrm{Sym}} \circ \psi_{\zeta_{\mathrm{SL}}}$).

Let $\zeta_{\mathrm{GL}} : \mathrm{cf}(\mathrm{GL}) \to \mathbb{C}$ be defined by linearly extending the following operation to arbitrary elements of $\mathrm{cf}(\mathrm{GL})$. If $f \in \mathrm{cf}(G_n)$, then we set

$$\zeta_{\mathrm{GL}}(f) = \langle f, \mathbf{1}_{G_n} \rangle_{G_n}. \tag{5.31}$$

Then ζ_{GL} is a linear map. For $f \in \mathrm{cf}(G_n)$ and $g \in \mathrm{cf}(G_m)$, we have

$$\begin{aligned}
\zeta_{\mathrm{GL}}(m_{\mathrm{GL}}(f,g)) &= \langle \mathrm{Indf}^{G_{n+m}}_{G_n \times G_m}(f \times g), \mathbf{1}_{G_{n+m}} \rangle_{G_{n+m}} \\
&= \langle f \times g, \mathrm{Resf}^{G_{n+m}}_{G_n \times G_m}(\mathbf{1}_{G_{n+m}}) \rangle_{G_n \times G_m} \\
&= \langle f \times g, \mathbf{1}_{G_n} \times \mathbf{1}_{G_m} \rangle_{G_n \times G_m} \\
&= \langle f, \mathbf{1}_{G_n} \rangle_{G_n} \langle g, \mathbf{1}_{G_m} \rangle_{G_m} \\
&= \zeta_{\mathrm{GL}}(f) \zeta_{\mathrm{GL}}(g).
\end{aligned}$$

Therefore, ζ_{GL} is an algebra map.

For each n, let $\alpha_n \in \mathcal{J}$ be the element of $\mathcal{J}$ such that $\chi^{\alpha_n} = \mathbf{1}_{G_n}$. Then α_n is given by the formula

$$\alpha_n(\varphi) = \begin{cases} (1^n) & : \quad \varphi = \{\mathbf{1}_{M_1}\} \\ () & : \quad \text{otherwise} \end{cases}.$$

Lemma 5.12. *Let $\lambda \in \mathcal{J}$ have size $\|\lambda\| = n$ and let $\mu = (\mu_1, \ldots, \mu_k) \in \mathscr{P}$ have size $|\mu| = n$. Then*

$$\begin{aligned} C^{\lambda}_{\alpha_{\mu_1},\ldots,\alpha_{\mu_k}} &= c^{\lambda^{(1)}}_{(1^{\mu_1}),\ldots,(1^{\mu_k})} \\ &= K_{(\lambda^{(1)})',\mu}, \end{aligned}$$

where $\lambda^{(1)}$ is shorthand for $\lambda(\{\mathbf{1}_{M_1}\})$.

Proof. The first identity follows from the application of the formula for α_{μ_i} to the equation

$$C^{\lambda}_{\alpha_{m_1},\ldots,\alpha_{\mu_k}} = \prod_{\varphi \in \Theta} c^{\lambda(\varphi)}_{\alpha_{\mu_1}(\varphi),\ldots,\alpha_{\mu_k}(\varphi)}.$$

The second identity can be more generally stated as

$$c^{\lambda}_{(1^{\mu_1}),\ldots,(1^{\mu_k})} = K_{\lambda',\mu}.$$

The right hand side of the above equation is, by definition, equal to the number of semi-standard (i.e., column-strict) Young tableaux of shape λ' and weight μ. This is equal to the number of row-strict Young tableaux of shape λ and weight μ. By co-commutativity, the left hand side of the above equation expands as

$$c^{\lambda}_{(1^{\mu_1}),\ldots,(1^{\mu_k})} = \sum_{\nu_1,\ldots,\nu_{k-1}} c^{\lambda}_{\nu_{k-1},(1^{\mu_k})} c^{\nu_{k-1}}_{\nu_{k-2},(1^{\mu_{k-1}})} \cdots c^{\nu_1}_{(1^{\mu_1}),(1^{\mu_2})}.$$

By the Pieri formula, each Littlewood–Richardson coefficient $c^{\nu_i}_{\nu_{i-1},(1^{\mu_i})}$ (where $\nu_0 = (1^{\mu_1})$ and $\nu_k = \lambda$) is equal to the number of ways of building the partition ν_i from ν_{i-1} by adding μ_i boxes, no two in the same row. Hence, their product is equal to the number of ways of assembling λ by first adding μ_2 boxes to (1^{μ_1}), then μ_3 boxes, and so on, at each step adding no more than one box in each row. By keeping track of the step at which boxes are added, we observe that this product is equal to the number of row-strict Young tableaux of shape λ and weight μ. Therefore, the identity is proven. ∎

Then we can define the analogous character $\zeta_{\mathrm{Sym}_{\mathrm{GL}}} : \mathrm{Sym}_{\mathrm{GL}} \to \mathbb{C}$ as follows. For all $\lambda \in \mathcal{J}$ with $\|\lambda\| = n$,

$$\zeta_{\mathrm{Sym}_{\mathrm{GL}}}(S_\lambda) = \begin{cases} 1 & : \quad \lambda = \alpha_n \\ 0 & : \quad \text{otherwise} \end{cases}. \tag{5.32}$$

In [ABS06], the authors provide a constructive proof that Sym is the terminal co-commutative CHA (see Theorem 5.2 above for their formula). Let $\mathrm{Sym}(Y_{(1)})$ be the subalgebra of $\mathrm{Sym}_{\mathrm{GL}}$ spanned by the S_λ for those $\lambda : \Theta \to \mathscr{P}$ which are identically equal to the zero partition except at the orbit $\{\mathbf{1}_{M_1}\}$ (cf. (5.21)).

Proposition 5.13. *The map* $\psi_{\mathrm{Sym}_{\mathrm{GL}}} : \mathrm{Sym}_{\mathrm{GL}} \to \mathrm{Sym}$ *induced by the character* $\zeta_{\mathrm{Sym}_{\mathrm{GL}}} : \mathrm{Sym}_{\mathrm{GL}} \to \mathbb{C}$ *defined in* (5.32) *is given by the formula*

$$\psi_{\mathrm{Sym}_{\mathrm{GL}}}(S_\lambda) = \begin{cases} s_{(\lambda^{(1)})'} & : \quad S_\lambda \in \mathrm{Sym}(Y_{(1)}) \\ 0 & : \quad \textit{otherwise} \end{cases}. \tag{5.33}$$

Proof. Let $\lambda \in \mathscr{J}$ and write $\|\lambda\| = n$. With $\zeta_{\mathrm{Sym}_{\mathrm{GL}}}$ defined as in (5.32), it follows that for any integer partition $\mu = (\mu_1, \ldots, \mu_k) \in \mathscr{P}$ of size $|\mu| = n$,

$$\begin{aligned} \zeta_\mu(S_\lambda) &= \zeta^{\otimes k}_{\mathrm{Sym}_{\mathrm{GL}}}\Big(\big(\Delta^{(k-1)}\chi^\lambda\big)_{\mu_1,\ldots,\mu_k} \Big) \\ &= \zeta^{\otimes k}_{\mathrm{Sym}_{\mathrm{GL}}}\Big(\sum_{\substack{\nu_1,\ldots,\nu_k \in \mathscr{J} \\ \|\nu_i\| = \mu_i}} C^\lambda_{\nu_1,\ldots,\nu_k} S_{\nu_1} \otimes \cdots \otimes S_{\nu_k} \Big) \\ &= C^\lambda_{\alpha_{\mu_1},\ldots,\alpha_{\mu_k}} \\ &= K_{(\lambda^{(1)})',\mu}. \end{aligned}$$

Therefore, by Lemma 5.12,

$$\psi_{\mathrm{Sym}_{\mathrm{GL}}}(S_\lambda) = \sum_{\substack{\mu \in \mathscr{P} \\ |\mu| = n}} \zeta_\mu(S_\lambda) m_\mu = \sum_{\substack{\mu \in \mathscr{P} \\ |\mu| = n}} K_{(\lambda^{(1)})',\mu} m_\mu.$$

A property of Kostka numbers is that for any pair of partitions ν and μ, the Kostka number $K_{\nu,\mu}$ is zero if $|\nu| \neq |\mu|$ (see [Mac98, Chapter I, (6.5)]. Thus, the above equation is nonzero only if $|\lambda^{(1)}| = n$. Since we know a priori that $\|\lambda\| = n$, it follows that $|\lambda^{(1)}| = n$ if and only if $S_\lambda \in \mathrm{Sym}(Y_{(1)})$. Hence the above equation is zero for $S_\lambda \notin \mathrm{Sym}(Y_{(1)})$. Now suppose $\|\lambda\| = n$ and $|\lambda^{(1)}| = n$. Then using the known change of basis from the Schur functions to the monomial functions, we have

$$\psi_{\mathrm{Sym}_{\mathrm{GL}}}(S_\lambda) = \sum_{\substack{\mu \in \mathscr{P} \\ |\mu| = n}} K_{(\lambda^{(1)})',\mu} m_\mu = s_{(\lambda^{(1)})'},$$

and this completes the proof. ∎

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List of Symbols

$S \times T$	direct product of S and T, page 17
$S *_N T$	the $*$-product of S and T, page 18
$MM_N(G)$	$M(N) *_N M(G/N)$, page 18
$mm_N(G)$	$m_G(N) *_N m(G/N)$, page 18
$S \Delta T$	the Δ-product of S and T, page 18
$\mathscr{P}$	the set of all integer partitions, page 25
Sym	the Hopf algebra of symmetric functions, page 26
$c^{\nu}_{\lambda,\mu}$	Littlewood–Richardson coefficient, page 28

Chapter 3

$\leq$	refinement partial order of supercharacter theories, page 30
$S \vee T$	join of supercharacter theories, page 30
$S \wedge T$	meet of supercharacter theories, page 31
ζ_n	$e^{2\pi i/n}$, page 35
$m_A(G)$	minimal A-invariant supercharacter theory of G, page 36
$\mathrm{AutSCT}(G)$	automorphic supercharacter theories, page 36
$\mathrm{GalSCT}(G)$	Galois supercharacter theories, page 36
$\mathrm{CharSCT}(G)$	the set of characteristic supercharacter theories, page 39
$\mathrm{InvSCT}_A(G)$	A-invariant supercharacter theories of G, page 42
$\mathbb{Z}_n$	cyclic group of order n, page 43

Chapter 4

$\langle r \rangle$	subgroup of rotations of D_{2n}, page 54
$\langle s \rangle$	subgroup of reflections of D_{2n}, page 54
λ	linear character of D_{2n} whose kernel is $\langle r \rangle$, page 54
χ_ℓ	nonlinear irreducible character of D_{2n}, page 54
μ_i	linear character of D_{2n} whose kernel is $\langle r^2 \rangle$, page 54
$\chi_{\frac{n}{2}}$	$\mu_0 + \mu_1$, page 54
$\mathcal{P}$	sublattice of SCT(D_{2n}) of $*$-products over $\langle r \rangle$, page 54
S_d	$\big(m_{D_{2n}}(\langle r^d \rangle) *_{\langle r^d \rangle} M(\langle r \rangle / \langle r^d \rangle)\big) *_{\langle r \rangle} M(D_{2n}/\langle r \rangle)$, page 58
$\mathcal{Q}$	the upper ideal of $\mathcal{P}$ generated by S_p for prime p, page 60
$\mathcal{R}$	the subposet of $\mathcal{P}$ of supercharacter theories with $\chi_{\frac{n}{2}}$ as a supercharacter, page 60
φ	gluing map on SCT(D_{2n}), page 60
ψ	splitting map on CharSCT(D_{2n}), page 60
$\mathcal{S}$	subposet of supercharacter theories of D_{2n} that glue reflections and respect parity, page 62
$mM_{\langle r^d \rangle}(D_{2n})$	$m_{D_{2n}}(\langle r^d \rangle) *_{\langle r^d \rangle} M(D_{2n}/\langle r^d \rangle)$, page 64
$\mathcal{P}_d$	$[mM_{\langle r^d \rangle}(D_{2n}), MM_{\langle r^d \rangle}(D_{2n})]$, page 72
$SD(n,p,a)$	$G = \langle x \rangle \rtimes \langle y \rangle = \mathbb{Z}_n \rtimes \mathbb{Z}_p$ with action given by $x \mapsto x^a$, page 80
ξ_ℓ	the irreducible characters of $\langle x \rangle$, page 80
ψ_m	the irreducible characters of $\langle y \rangle$, page 80

c	$\gcd(n, a-1)$, page 81
$mM_N(G)$	$m_G(N) *_N M(G/N)$, page 90

Chapter 5

$\mathbf{scf}(A)$	the Hopf algebra of alternating group superclass functions, page 97
$\mathbf{cf}(S)$	the Hopf algebra of symmetric group class functions, page 97
Alt	the alternating group symmetric functions, page 97
$\mathrm{Res} : \mathbf{cf}(S) \to \mathbf{scf}(A)$	the restriction Hopf map, page 97
$\delta_\lambda \in \mathrm{cf}(S_n)$	the identifier function for elements of cycle-type λ, page 97
sgn	the sign homomorphism $S_n \to \mathbb{Z}_2$, page 97
ι_A	inclusion $\mathbf{scf}(A) \to \mathbf{cf}(S)$, page 98
λ'	transpose of λ, page 98
$\mathrm{ch}_S : \mathbf{cf}(S) \to \mathrm{Sym}$	characteristic map, page 100
z_λ	$\prod_i i^{m_i(\lambda)} \lambda_i!$, page 100
$m_i(\lambda)$	multiplicity of i in λ, page 100
ch_A	characteristic map, page 100
$\mathbf{cf}(G)$	the Hopf algebra of $\mathrm{GL}_n(\mathbb{F}_q)$-class functions, page 105
$\mathbf{scf}(H)$	the Hopf algebra of $\mathrm{SL}_n(\mathbb{F}_q)$ superclass functions, page 105
M	$\overline{\mathbb{F}}_q^\times$, page 105
$F : M \to M$	Frobenius endomorphism, page 105
Φ	F-orbits in M, page 105

M_n	$\mathbb{F}_{q^n}^\times = M^{F^n}$, page 106
L_n	$\mathrm{Hom}(M_n, \mathbb{C}^\times)$, page 106
L	$\varinjlim L_n$, page 106
$N_{n,m} : M_n \to M_m$	norm map, page 106
Θ	the set of F-orbits in L, page 106
$[\lambda]$	the parallel class of λ, page 107
ι_{SL}	the inclusion $\mathrm{scf}_{\mathrm{SL}} \to \mathrm{cf}_{\mathrm{GL}}$, page 108
$\mathrm{Sym}_{\mathrm{GL}}$	the Hopf algebra of GL_n symmetric functions, page 111
$\mathrm{ch}_{\mathrm{GL}} : \mathrm{cf}(\mathrm{GL}) \to \mathrm{Sym}_{\mathrm{GL}}$	characteristic map, page 111
$\mathrm{Sym}_{\mathrm{SL}}$	the Hopf algebra of SL_n symmetric functions, page 111
$C^{\nu}_{\lambda,\mu}$	$\prod_{\varphi \in \Theta} c^{\nu(\varphi)}_{\lambda(\varphi),\mu(\varphi)}$, page 112