On Momentum in Nonlinear Schrödinger Equations

by

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On Momentum in Nonlinear Schrödinger Equations

Thesis directed by Prof. Magdalena Czubak

We study the focusing nonlinear Schrödinger equation,

$$\begin{cases} iu_t + \Delta u = -|u|^{p-1}u, \\ u(x,0) = u_0(x), \end{cases} \tag{NLS}$$

as well as the focusing electromagnetic nonlinear Schrödinger equation

$$\begin{cases} iD_t u + \Delta_A u = -|u|^{p-1}u, \\ u(x,0) = u_0(x), \end{cases} \tag{emNLS}$$

with

$$0 < \frac{N}{2} - \frac{2}{p-1} < 1,$$

and for (emNLS) with certain potentials which decay at infinity. We obtain dichotomy results for (emNLS) for initial data below the ground state, and new scattering and blow up results above the ground state for (NLS). New methods include tools for studying the gradient of the modulus, $\nabla|u|$, of the complex valued solutions of (NLS) and (emNLS), nonlinear bounds for $H^1(\mathbb{R}^N)$ functions lying below the ground state, and the analysis of function spaces with nonradial symmetry.

Dedication

To Don and Rita.

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Chapter 1

Introduction

We study the focusing nonlinear Schrödinger equation,

$$\begin{cases} iu_t + \Delta u = -|u|^{p-1}u, \\ u(x,0) = u_0(x), \end{cases} \tag{NLS}$$

as well as the focusing electromagnetic nonlinear Schrödinger equation

$$\begin{cases} iD_t u + \Delta_A u = -|u|^{p-1}u, \\ u(x,0) = u_0(x). \end{cases} \tag{emNLS}$$

In both cases we consider an unknown $u(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$, and initial data $u_0 : \mathbb{R}^N \to \mathbb{C}$. In (emNLS), we replace the standard derivatives in (NLS) with covariant derivatives. A covariant partial derivative D_α is defined by

$$D_\alpha = \partial_\alpha + iA_\alpha, \qquad \alpha \in \{0, \ldots, N\}$$

for some real valued function $A_\alpha : \mathbb{R}^N \to \mathbb{R}$, using the standard notation

$$D_t = \partial_0 + iA_0.$$

Solutions of (NLS) satisfy a scaling property. For a fixed dimension N, and power p, if $u(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ solves (NLS), then for any $\lambda > 0$,

$$u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)$$

also solves (NLS). This rescaling has the property that for fixed t_0,

$$\|u_\lambda(x,t_0)\|_{\dot{H}^s(\mathbb{R}^N)} = \lambda^{s-\left(\frac{N}{2}-\frac{2}{p-1}\right)} \left\|u(x,\lambda^2 t_0)\right\|_{\dot{H}^s(\mathbb{R}^N)}, \tag{1.0.1}$$

where $\|u\|_{\dot{H}^s(\mathbb{R}^N)}$, can be defined using the Fourier transform by

$$\|u\|_{\dot{H}^s(\mathbb{R}^N)} = \||\xi|^s \widehat{u}(\xi)\|_{L^2_\xi(\mathbb{R}^N)}.$$

We define

$$s_c = \frac{N}{2} - \frac{2}{p-1},$$

so that scaling preserves the $\dot{H}^{s_c}(\mathbb{R}^N)$ norm. In this case we say that (NLS) is $\dot{H}^{s_c}(\mathbb{R}^N)$ critical. We will always choose the dimension, N and power p, such that $0 < s_c < 1$.

Observe that for a solution u defined on a time interval $(0,T)$, rescaling by $0 < \lambda < 1$ produces a solution defined on a larger time interval $(0, \frac{1}{\lambda^2}T)$. Furthermore, with N and p chosen so that $0 < s_c < 1$, for $0 < \lambda < 1$ and fixed time t_0, using (1.0.1),

$$\|u_\lambda(x,t_0)\|_{L^2(\mathbb{R}^N)} > \left\|u(x,\lambda^2 t_0)\right\|_{L^2(\mathbb{R}^N)} \qquad \text{and} \qquad \|u_\lambda(x,t_0)\|_{\dot{H}^1(\mathbb{R}^N)} < \left\|u(x,\lambda^2 t_0)\right\|_{\dot{H}^1(\mathbb{R}^N)}.$$

Because of this, for $0 < s_c < 1$, (NLS) is mass (or $L^2(\mathbb{R}^N)$) supercritical, and energy (or $\dot{H}^1(\mathbb{R}^N)$) subcritical. This is often called the intercritical regime.

The equation (emNLS) also has a scaling property, though it is more delicate than that of (NLS). If $u(x,t) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ solves (emNLS), then for any $\lambda > 0$,

$$u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)$$

solves a different (emNLS) equation with the rescaled potentials

$$\begin{cases} A_0'(x) = \lambda^2 A_0(\lambda x), \\ A_j'(x) = \lambda A_j(\lambda x) \qquad j \in \{1, \dots, N\}. \end{cases}$$

In the study of (emNLS), some properties which arise from scaling in (NLS) also hold for (emNLS) with fixed potentials, and so we place the same scaling assumption on dimension N and power p

for (emNLS) as we do for (NLS). That is, we consider N and p so that $0 < s_c < 1$ with s_c still defined by $s_c = \frac{N}{2} - \frac{2}{p-1}$.

For (NLS), we will consider initial data in the Sobolev space $H^1(\mathbb{R}^N)$, and for (emNLS) we will consider initial data in analogous electromagnetic spaces, defined carefully in Chapter 2.

Solutions of (NLS) have conserved quantities including

$$\text{mass:} \quad \|u(t)\|^2_{L^2(\mathbb{R}^N)} = \|u_0\|^2_{L^2(\mathbb{R}^N)},$$

$$\text{energy:} \quad E[u](t) \coloneqq \frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} = E[u_0], \quad \text{and}$$

$$\text{angular momentum:} \quad \mathbf{M}[u](t) \coloneqq \int \mathcal{P}[u(x,t)] \wedge \mathbf{x}\, dx = \mathbf{M}[u_0],$$

where

$$\mathcal{P}[u] = 2\,\mathrm{Im}\,[\overline{u}\,\nabla u],$$

and the operator $\wedge$ is a generalization of the cross product, to be described in Chapter 4.

Solutions of (emNLS) have conserved quantities including

$$\text{mass:} \quad \|u(t)\|^2_{L^2(\mathbb{R}^N)} \quad \text{and}$$

$$\text{energy:} \quad E_A[u](t) \coloneqq \frac{1}{2}\|\nabla_A u(t)\|^2_{L^2(\mathbb{R}^N)} - \frac{1}{p+1}\|u(t)\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} = E[u_0].$$

The equation (NLS) has a standing wave solution u_Q, so that at each time, u_Q produces equality in the Gagliardo-Nirenberg inequality, which is

$$\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \leq C_{GN}\,\|\nabla u\|^{\frac{N(p-1)}{2}}_{L^2(\mathbb{R}^N)}\,\|u\|^{2-\frac{(N-2)(p-1)}{2}}_{L^2(\mathbb{R}^N)},$$

for $u \in H^1(\mathbb{R}^N)$. We will call u_Q the ground state solution, and it will be significant for our general study of existence and blow up.

The quantities

$$\text{mass energy:} \quad E[u]\,\|u\|^{2\frac{1-s_c}{s_c}}_{L^2(\mathbb{R}^N)}, \quad \text{and}$$

$$\text{mass kinetic energy:} \quad \|\nabla u\|^2_{L^2(\mathbb{R}^N)}\,\|u\|^{2\frac{1-s_c}{s_c}}_{L^2(\mathbb{R}^N)},$$

are both invariant under the scaling for (NLS), and play a role in determining the global dynamics of a given solution. For (emNLS), mass energy and mass kinetic energy are defined in the same manner, using the covariant derivatives associated with the equation.

1.1 Brief context and history of the nonlinear Schrödinger equation

The nonlinear Schrödinger equation, (NLS), takes its name from the linear Schrödinger equation, which was introduced in 1926 to describe a wave function associated with a particle in the context of quantum physics [6].

It has been shown rigorously that in certain cases, solutions to (NLS) can exhibit norm concentration [44] and even blow up [29]. Blow up in this equation is a type of singularity formation.

The (NLS) equation arises in various physical phenomena. One influential context to the early study of global dynamics and blow up is that of nonlinear optics. In certain media, high intensity laser beams can cause changes in the medium which will have a focusing effect on the laser. This effect is often called self-focusing or self-trapping, and can be modeled by (NLS) [11].

An attractive feature of the context of lasers is that the mathematical phenomenon of blow up can be observed as a physically significant phenomenon. In particular, focusing from mathematical blow up can cause damage to a laser [11]. A given medium will have an electric field breakdown strength, past which the medium could be permanently damaged. It has been noted that self-focusing can lead to increased local energy density, which can surpass the electric field breakdown strength of a medium, causing damage to the medium (see [38] as well as the summary and review section of the same journal).

Individual chapters contain further history and context relevant to particular results in this thesis.

1.2 Chapter summaries and overview of results

This thesis consists of two papers completed during the author's graduate student career, as well as results which have not yet been released outside of this thesis.

Below we summarize each chapter, and present a representative theorem to illustrate the nature of the main results in the given chapter. For full discussion of the main results, see the containing chapters.

1.2.1 Chapter 2: Preliminaries

In Chapter 2, we discusses preliminary results and local theory for (NLS) and (emNLS).

1.2.2 Chapter 3: Global Dynamics of Electromagnetic Nonlinear Schrödinger Equations Below the Ground State

Chapter 3 consists of the paper [15]. For certain (emNLS) equations, we demonstrate a dichotomy classification for initial data with mass energy below that of the ground state.

A given magnetic potential has an associated trapping component B_τ. Results on Strichartz estimates, such as those in [16], place a smallness condition on $\left\| |x|^2 B_\tau \right\|_{L^\infty(\mathbb{R}^N)}$. We obtain a dichotomy result for purely magnetic (emNLS) in dimension $N \geq 3$. By purely magnetic, we mean (emNLS) with $A_0 \equiv 0$. The norm dichotomy portion of the result holds for general potentials inducing good local theory, and we obtain dichotomy with blow up when $\left\| |x|^2 B_\tau \right\|_{L^\infty(\mathbb{R}^N)}$ is sufficiently small. The main result of Chapter 3 is the folowing theorem.

Theorem 1.2.1. *There exists a constant C_τ so that if $A \in L^2_{loc}(\mathbb{R}^n; \mathbb{R}^n)$ is a potential inducing satisfactory local theory, then we have the following dichotomy result: Suppose $u_0 \in H^1_A(\mathbb{R}^N)$ is such that*

$$E_A[u_0]^{s_c} \left\| u_0 \right\|^{2(1-s_c)}_{L^2(\mathbb{R}^N)} < E[u_Q]^{s_c} \left\| u_Q \right\|^{2(1-s_c)}_{L^2(\mathbb{R}^n)} . \tag{1.2.1}$$

- *If*

$$\left\| \nabla_A u_0 \right\|^{s_c}_{L^2(\mathbb{R}^N)} \left\| u_0 \right\|^{1-s_c}_{L^2(\mathbb{R}^N)} < \left\| \nabla u_Q \right\|^{s_c}_{L^2(\mathbb{R}^N)} \left\| u_Q \right\|^{1-s_c}_{L^2(\mathbb{R}^N)} , \tag{1.2.2}$$

  then the solution u is global and

$$\left\| \nabla_A u(t) \right\|^{s_c}_{L^2(\mathbb{R}^N)} \left\| u_0 \right\|^{1-s_c}_{L^2(\mathbb{R}^N)} < \left\| \nabla u_Q \right\|^{s_c}_{L^2(\mathbb{R}^N)} \left\| u_Q \right\|^{1-s_c}_{L^2(\mathbb{R}^N)} , \tag{1.2.3}$$

  for all time $t \in \mathbb{R}$.

- *If*

$$\|\nabla_A u_0\|_{L^2(\mathbb{R}^N)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^N)}^{1-s_c} > \|\nabla u_Q\|_{L^2(\mathbb{R}^N)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^N)}^{1-s_c}, \tag{1.2.4}$$

then

$$\|\nabla_A u(t)\|_{L^2(\mathbb{R}^N)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^N)}^{1-s_c} > \|\nabla u_Q\|_{L^2(\mathbb{R}^N)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^N)}^{1-s_c}, \tag{1.2.5}$$

on the full interval of existence, and if

$$\left\| |x|^2 B_\tau \right\|_{L^\infty(\mathbb{R}^N)} < C_\tau, \tag{1.2.6}$$

and u_0 has finite variance, the solution blows up in finite time.

In addition to this theorem, we obtain dichotomy results for equations with both electric and magnetic potentials.

1.2.3 Chapter 4: Global Dynamics of Nonlinear Schrödinger Equations Above the Ground State

Chapter 4 consists of the paper [45]. We study solutions of (NLS) above the ground state, and obtain scattering and blow up results.

We consider finite variance initial data above the ground state, and show that large angular momentum can be sufficient to obtain global existence and blow up results. Our results include the following theorem regarding scattering in both time directions for large energy initial data.

Theorem 1.2.2. *Consider $u_0 \in H^1(\mathbb{R}^N)$, with $|x|u_0 \in L^2(\mathbb{R}^N)$ and*

$$\left[E[u_0] - \frac{|2\boldsymbol{M}[u_0]|^2}{32\int |x|^2|u_0|^2\,dx}\right] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q]\, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}. \tag{1.2.7}$$

If

$$\|\nabla |u_0|\|_2^2 \, \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < \|\nabla u_Q\|_2^2 \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \tag{1.2.8}$$

then the solution u with initial data u_0 exists for all time and scatters in H^1 in both time directions.

We prove this theorem in a more general form which includes blow up results. We also obtain results for global behavior in one time direction which improve on existing results.

1.2.4 Chapter 5: Momentum Based Modulations of Initial Data

In Chapter 5, we continue our study of solutions of (NLS) above the ground state. We introduce countably many subspaces of $H^1(\mathbb{R}^2)$, and establish improved dichotomy results for these spaces. The main theorem of Chapter 5 can be stated in the following way.

Theorem 1.2.3. *Consider (NLS) with $N = 2$, and $0 < s_c$. There exists a sequence of subspaces $\{\mathcal{H}_k\}_{k=1}^{\infty} \subset H^1(\mathbb{R}^2)$, and a sequence $\{\gamma_k\} \subset (1, \infty)$ with $\gamma_k \to \infty$ as $k \to \infty$, so that if $u_0 \in \mathcal{H}_k$, and*

$$E[u_0] \, \|u_0\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} < \gamma_k E[u_Q] \, \|u_Q\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}},$$

the following statements hold.

- *If*

  $$\|\nabla u_0\|_{L^2(\mathbb{R}^2)}^2 \|u_0\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} < \gamma_k \|\nabla u_Q\|_{L^2(\mathbb{R}^2)}^2 \|u_Q\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}},$$

  then the corresponding solution u satisfies

  $$\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 \|u\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} < \gamma_k \|\nabla u_Q\|_{L^2(\mathbb{R}^2)}^2 \|u_Q\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}}$$

  for all time, and u exists globally in both time directions.

- *If*

  $$\|\nabla u_0\|_{L^2(\mathbb{R}^2)}^2 \|u_0\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} > \gamma_k \|\nabla u_Q\|_{L^2(\mathbb{R}^2)}^2 \|u_Q\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}},$$

  then the corresponding solution u satisfies

  $$\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 \|u\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} > \gamma_k \|\nabla u_Q\|_{L^2(\mathbb{R}^2)}^2 \|u_Q\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}}$$

  on the full interval of existence. If we also have $|x|u_0 \in L^2(\mathbb{R}^2)$, then u blows up in finite time in both time directions.

Chapter 2

Preliminaries

We will now introduce background for our study of (NLS) and (emNLS). We will also provide some preliminary results used in subsequent chapters.

2.1 Weak derivatives, and spaces for (NLS) and (emNLS)

This section collects definitions and results regarding weak differentiation and electromagnetic Sobolev spaces. We review pointwise a.e. results for weak derivatives, and discuss electromagnetic weak differentiation. In this section, and throughout this thesis, we will assume that all magnetic potentials satisfy the condition $A \in L^2_{\text{loc}}(\mathbb{R}^N; \mathbb{R}^N)$.

2.1.1 Distributions and weak derivatives

The space of test functions, $C_c^\infty(\mathbb{R}^N)$, is the space of compactly supported, infinitely differentiable functions in $\mathbb{R}^N$ equipped with the following topology: a sequence $\{\phi_j\} \subset C_c^\infty(\mathbb{R}^N)$ converges to a $\phi \in C_c^\infty(\mathbb{R}^N)$ if there is some compact set K, so that $\phi, \phi_1, \phi_2, \ldots$ are supported in K, and so that for each finite sequence $\{l_j\}_{j=1}^m \subset \{1, \ldots, N\}$,

$$\lim_{j\to\infty} \|\partial_{l_1} \cdots \partial_{l_m} (\phi_j - \phi)\|_{L^\infty(\mathbb{R}^N)} = 0.$$

A distribution, ψ, is a continuous linear functional on $C_c^\infty(\mathbb{R}^N)$. If there is a measurable function f, such that for each $\phi \in C_c^\infty(\mathbb{R}^N)$,

$$\psi(\phi) = \int f\overline{\phi},$$

then we call f a distribution, and identify f with ψ. Hence, each $L^1_{\text{loc}}(\mathbb{R}^N)$ function is a distribution.

If f is a classically differentiable function with $\partial_j f \in L^1_{\text{loc}}(\mathbb{R}^N)$, then $\partial_j f$ is a distribution which acts on $\phi \in C_c^\infty(\mathbb{R}^N)$ by

$$\begin{aligned}
(\partial_j f)(\phi) &= \int \partial_j f \overline{\phi} \\
&= -\int f \overline{\partial_j \phi} \\
&= -f(\partial_j \phi).
\end{aligned}$$

More generally, for any distribution ψ, and partial derivative ∂_j, the map $\phi \mapsto -\psi(\partial_j \phi)$ is a distribution, and this property is used to define a distributional derivative of ψ, $\partial_j \psi$, by action on $\phi \in C_c^\infty(\mathbb{R}^N)$,

$$\partial_j \psi(\phi) = -\psi(\partial_j \phi).$$

We let $W^{1,1}_{\text{loc}}(\mathbb{R}^N)$ denote the set of functions in $L^1_{\text{loc}}(\mathbb{R}^N)$ with distributional derivatives also in $L^1_{\text{loc}}(\mathbb{R}^N)$. In the case $f \in W^{1,1}_{\text{loc}}(\mathbb{R}^N)$, we refer to distributional differentiation as weak differentiation.

Special care must be taken in defining magnetic derivatives of distributions. In particular, for a general distribution $f \in L^1_{\text{loc}}(\mathbb{R}^N)$, potential $A \in L^2_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^N)$, and index $j \in \{1,\ldots,N\}$, it may not be true that the function $A_j f$ defines a distribution. If we restrict to $f \in L^2_{\text{loc}}(\mathbb{R}^N)$, then the condition $A \in L^2_{\text{loc}}(\mathbb{R}^N;\mathbb{R}^N)$ implies that $A_j f$ will indeed be a distribution, and we can define the distributional magnetic derivative

$$D_j = \partial_j + iA_j.$$

We define the magnetic gradient by

$$\nabla_A = \begin{bmatrix} D_1 \\ \vdots \\ D_N \end{bmatrix},$$

and the magnetic Laplacian by

$$\Delta_A = \nabla_A \cdot \nabla_A,$$

when this operator produces a distribution.

The following theorem collects certain properties of weak differentiation used in this thesis. Each of these properties is given in [42].

Theorem 2.1.1 (Properties of weak derivatives)**.** *Let $A \in L^2_{loc}(\mathbb{R}^N;\mathbb{R}^N)$ be some potential.*

Derivative of the absolute value and Kato's, or the diamagnetic, inequality: *Suppose $u \in W^{1,1}_{loc}(\mathbb{R}^N)$. Then we have both*

$$(\nabla |u|)(x) = \begin{cases} Re\left[\frac{\overline{u}(x)}{|u(x)|}\nabla u\right] & u(x) \neq 0 \\ 0 & u(x) = 0, \end{cases}$$

and

$$|\nabla|u|| \leq |\nabla u| \tag{2.1.1}$$

almost everywhere. The inequality (2.1.1) is Kato's inequality.

If $u \in W^{1,1}_{loc}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, then

$$|\nabla|u|| \leq |\nabla_A u|$$

almost everywhere.

Gradients on level sets: *If $u \in W^{1,1}_{loc}(\mathbb{R}^N)$, then $\nabla u = 0$ almost everywhere on any level set of u. That is, given any $c \in \mathbb{C}$, and Lebesgue measure μ,*

$$\mu\left(\{u = c\} \cap \{\nabla u \neq 0\}\right) = 0.$$

2.1.2 Sobolev spaces and electromagnetic variants

The space $H^1(\mathbb{R}^N)$ is the space of functions $f \in L^2(\mathbb{R}^N)$ such that each first distributional derivative is given by a function in $L^2(\mathbb{R}^N)$, and is a Hilbert space with norm

$$\|f\|_{H^1(\mathbb{R}^N)} = \left(\|f\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla f\|_{L^2(\mathbb{R}^N)}^2\right)^{1/2}.$$

The magnetic space, $H_A^1(\mathbb{R}^N)$ is the space of functions $f \in L^2(\mathbb{R}^N)$ such that each first magnetic distributional derivative is given by a function in $L^2(\mathbb{R}^N)$, and is a Hilbert space with norm

$$\|f\|_{H_A^1(\mathbb{R}^N)} = \left(\|f\|_{L^2(\mathbb{R}^N)}^2 + \|\nabla_A f\|_{L^2(\mathbb{R}^N)}^2\right)^{1/2},$$

having $C_c^\infty(\mathbb{R}^N)$ as a dense subset.

Note that since we consider $f \in L^2(\mathbb{R}^N)$, the magnetic distributional derivatives are indeed well defined.

A useful consequence of the above statement in the context of (emNLS) is the fact that for functions in $H_A^1(\mathbb{R}^N)$, one can take two distributional magnetic derivatives, and hence the magnetic Laplacian is well defined as a distribution.

This property is also useful for discussing the space $H_A^{-1}(\mathbb{R}^N)$, which is a representation of the dual space $H_A^1(\mathbb{R}^N)$.

We first recall $H^{-1}(\mathbb{R}^N)$. Note that for each function u in $L^2(\mathbb{R}^N) = (L^2(\mathbb{R}^N))^*$, the map

$$v \mapsto \langle u, v\rangle_{L^2(\mathbb{R}^N)}$$

is a bounded linear functional on $H^1(\mathbb{R}^N)$ (not arising from the Riesz representation theorem for $H^1(\mathbb{R}^N)$). This is exactly the inclusion $(L^2(\mathbb{R}^N))^* \hookrightarrow (H^1(\mathbb{R}^N))^*$. Since the Hilbert space $L^2(\mathbb{R}^N)$ densely embeds into the Hilbert space $H^1(\mathbb{R}^N)$, we can express the full space $(H^1(\mathbb{R}^N))^*$ as the closure of the dense subset $(L^2(\mathbb{R}^N))^*$. It can be shown that elements of this extension can be represented by elements of the form $u - \Delta u$ with $u \in H^1(\mathbb{R}^N)$.

The same discussion as above is used to construct a space $H_A^{-1}(\mathbb{R}^N)$ which is isometric to $(H^1(\mathbb{R}^N))^*$, with elements which are represented by distributions of the form $u - \Delta_A u$ with $u \in H_A^1(\mathbb{R}^N)$, and satisfying the identity inclusion map $L^2(\mathbb{R}^N) = (L^2(\mathbb{R}^N))^* \xrightarrow{\text{Id}} H_A^{-1}(\mathbb{R}^N)$.

2.1.3 Time dependent spaces

Solutions of (NLS) and (emNLS) are functions $u(t) : I \to X$ where $I \subset \mathbb{R}$ is a time interval and X is a suitable space. We now define the spaces solutions will live in. For an interval $I \subset \mathbb{R}$, and a Banach space X, we will consider spaces $C(I; X)$, and $C^1(I; X)$, which are respectively the spaces of continuous or once continuously differentiable maps from I to X. Given $u \in C^1(I; X)$, differentiation is defined in the standard way,

$$\partial_t u(t) = \lim_{s \to 0} \frac{u(t+s) - u(t)}{s}.$$

If X is a separable space, as all those we consider will be, and $I = [T_0, T_1]$ is a closed interval, then making use of Pettis' theorem (e.g. [24]), a function, $u \in C([T_0, T_1]; X)$, is integrable, with the integral defined in the following manner. Integration of simple functions is defined as normal. So a function $\phi : [T_0, T_1] \to X$, of the form

$$\phi(t) = \sum_{k=1}^{M} \psi_k \chi_{E_k}(t),$$

where each $\psi_k \in X$ is some fixed function, and each $E_k \subset [T_0, T_1]$ is some measurable set, has integral

$$\int_{T_0}^{T_1} \phi(t) dt = \sum_{k=1}^{M} \psi_k \mu(E_k).$$

Moving on to integration in $C([T_0, T_1]; X)$, given any $u \in C([T_0, T_1]; X)$, there is some sequence of simple functions, $\{\phi_j\}$, such that

$$\lim_{j \to \infty} \int_{T_0}^{T_1} \|\phi(t) - \phi_j(t)\|_X \, dt = 0.$$

Using this sequence, the integral of u is defined by the limit

$$\int_{T_0}^{T_1} u(t) dt = \lim_{j \to \infty} \int_{T_0}^{T_1} \phi_j(t) dt,$$

which is independent of the approximating sequence.

2.2 Solutions of equations

We are now prepared to discuss our notion of solutions for (NLS) and (emNLS).

2.2.1 Defining solutions of linear equations

We begin by considering linear evolution equations in a rather general context. This treatment roughly follows [8] (see also [48]). In what follows, $\langle\, ,\, \rangle$ will always denote the $L^2(\mathbb{R}^N)$ inner product, and extensions of this inner product to duality pairings (e.g. the $H^1(\mathbb{R}^N)$, $H^{-1}(\mathbb{R}^N)$ pairing). Suppose we have an operator B which maps $D(B)$, a dense subset of $L^2(\mathbb{R}^N)$, to $L^2(\mathbb{R}^N)$, such that B is nonpositive and self adjoint.

We begin by discussing solutions to the homogeneous equation

$$i\partial_t u + Bu = 0. \tag{2.2.1}$$

Stone's Theorem guarantees the existence of a strongly continuous linear semigroup e^{itB} so that for $u_0 \in D(B)$, $t \mapsto e^{itB}u_0$ solves (2.2.1) in $L^2(\mathbb{R}^N)$.

Define a norm

$$\|u\|_{L^2_B(\mathbb{R}^N)} = \left(\|u\|^2_{L^2(\mathbb{R}^N)} - \langle Bu, u\rangle\right)^{1/2},$$

and let $L^2_B(\mathbb{R}^N) \subset L^2(\mathbb{R}^N)$ be the closure of $D(B)$ with respect to this norm. The spectral theorem implies that we can extend B to a self adjoint operator $B : L^2_B(\mathbb{R}^N) \to (L^2_B(\mathbb{R}^N))^*$, and the linear semigroup e^{itB} is a strongly continuous collection of isometries on all of the spaces $L^2_B(\mathbb{R}^N)$, $L^2(\mathbb{R}^N)$, $(L^2_B(\mathbb{R}^N))^*$. Furthermore, for $u_0 \in L^2_B(\mathbb{R}^N)$, the function $t \mapsto e^{itB}u_0$ solves (2.2.1) in $(L^2_B(\mathbb{R}^N))^*$.

Consider the inhomogeneous equation

$$i\partial_t u + Bu + f = 0. \tag{2.2.2}$$

with $f \in C([0,T]; (L^2_B(\mathbb{R}^N))^*)$. For $u \in C([0,T]; L^2_B(\mathbb{R}^N)) \cap C^1([0,T]; (L^2_B(\mathbb{R}^N))^*)$, every term of (2.2.2) is well defined in $(L^2_B(\mathbb{R}^N))^*$, and if f and u are functions with these regularity conditions, we will say that u solves (2.2.2) if u solves the equation in $(L^2_B(\mathbb{R}^N))^*$.

A more general notion of solution for (2.2.2) will be useful for constructing solutions to the nonlinear problem. Suppose $f \in C([0,T]; (L^2_B(\mathbb{R}^N))^*)$, and

$$u \in C([0,T]; L^2_B(\mathbb{R}^N)) \cap C^1([0,T]; (L^2_B(\mathbb{R}^N))^*)$$

solves (2.2.2). Then defining u_0 by $u_0 = u(0) \in L^2_B(\mathbb{R}^N)$, u satisfies the integral equation

$$u(t) = e^{itB}u_0 + \int_0^t e^{i(t-s)B} f(s)ds, \tag{2.2.3}$$

in $C([0,T];(L^2_B(\mathbb{R}^N))^*)$, and also in

$$C([0,T];L^2_B(\mathbb{R}^N)) \cap C^1([0,T];(L^2_B(\mathbb{R}^N))^*),$$

since

$$e^{itB}u_0 \in C([0,T];L^2_B(\mathbb{R}^N)) \cap C^1([0,T];(L^2_B(\mathbb{R}^N))^*).$$

This integral equation is Duhamel's formula (see [48]).

More generally, given any $u_0 \in L^2_B(\mathbb{R}^N)$, and $f \in C([0,T];(L^2_B(\mathbb{R}^N))^*)$, the function u defined by (2.2.3) is a well defined $C([0,T];(L^2_B(\mathbb{R}^N))^*)$ function. For $u_0 \in L^2_B(\mathbb{R}^N)$, and $f \in C([0,T];(L^2_B(\mathbb{R}^N))^*)$, we use (2.2.3) as the definition of solution for (2.2.2). If u solves (2.2.2) in the integral sense, and $u \in C([0,T];L^2_B(\mathbb{R}^N)) \cap C^1([0,T];(L^2_B(\mathbb{R}^N))^*)$, then u also solves (2.2.2) in $C([0,T];(L^2_B(\mathbb{R}^N))^*)$, so there is no contradiction between our notions of solution.

In general, for a solution u arising from (2.2.3), establishing further regularity on u comes down to establishing sufficient regularity on the term $\int_0^t e^{i(t-s)B} f(s)ds$. As an example, if we consider the restriction on the inhomogeneous term, $f \in C([0,T];L^2_B(\mathbb{R}^N)))$, then for any u_0, the solution u will have $C([0,T];L^2_B(\mathbb{R}^N)) \cap C^1([0,T];(L^2_B(\mathbb{R}^N))^*)$ regularity, though this condition on f is quite restrictive.

For our nonlinear equations (NLS) and (emNLS), the integral formulation (2.2.3) is useful, with the inhomogeneous term f replaced by the nonlinearity. Consider (emNLS) for example. If $u \in C([0,T];H^1_A(\mathbb{R}^N)) \cap C^1([0,T];H^{-1}_A(\mathbb{R}^N))$, we say that u is a $C([0,T];H^1_A(\mathbb{R}^N)) \cap C^1([0,T];H^{-1}_A(\mathbb{R}^N))$ solution of (emNLS) if

$$u(t) = e^{it\Delta_A}u_0 + \int_0^t e^{i(t-s)\Delta_A}(|u|^{p-1}u)ds. \tag{2.2.4}$$

Note that both sides of (2.2.4) are well defined since Sobolev embedding implies

$$|u|^{p-1}u \in C([0,1];H^{-1}_A(\mathbb{R}^N)),$$

and that the assumption

$$u \in C([0,T]; H_A^1(\mathbb{R}^N)) \cap C^1([0,T]; H_A^{-1}(\mathbb{R}^N))$$

implies that u solves (emNLS) in the sense of distributions [48][Theorem 2.4].

One method for approaching the nonlinear equation involves using Duhamel's formula (2.2.3) to construct solutions of certain inhomogeneous equations. In this context, it is important to establish sufficient regularity of the resulting solutions. As noted above, this requires control on the term $\int_0^t e^{i(t-s)B} f(s)ds$. In the case of (NLS), this control is established through the classical Strichartz estimates. For emNLS, for example, D'Ancona, Fanelli, Vega, and Visciglia [16] established Strichartz estimates for a class of electromagnetic potentials.

2.3 A note about self adjointness for the electromagnetic Laplacian

In order to make use of the above linear theory for (emNLS) with nontrivial A_0, a single operator B can be constructed by grouping the electric potential from the time derivative with the Laplacian. That is, we can take

$$B = \Delta_A - A_0.$$

The theory above requires self adjointness on the operator. This is often given as an abstract condition on the potentials A_0, A. The following two theorems demonstrate existence of some potentials generating such a self adjoint operator.

The first is the Leinfelder-Simader theorem [14, 41].

Theorem 2.3.1. *If $A \in L^4_{loc}(\mathbb{R}^N; \mathbb{R}^N)$ with $div\, A \in L^2_{loc}(\mathbb{R}^N; \mathbb{R}^N)$, and $A_0 \in L^2_{loc}(\mathbb{R}^N)$ with $A_0 \geq 0$, then the operator*

$$-\Delta_A + A_0$$

is essentially self adjoint.

One can obtain self adjoint operators with negative valued potentials A_0 by introducing what are known as Kato class functions. For $N \geq 3$, we define the Kato class K to consist of real valued measurable functions V satisfying

$$\lim_{r\downarrow 0}\sup_{x_0\in\mathbb{R}^N}\int_{y\in B_{x_0}(r)}\frac{|V(y)|}{|y-x_0|}dy,$$

where $B_{x_0}(r)$ is the ball of radius r centered at x_0. This class can be used to produce self adjoint operators, as the following theorem [14] demonstrates.

Theorem 2.3.2. *If $A_0\in L^2_{loc}(\mathbb{R}^N)$, and $(A_0)_-\in K$, where $(A_0)_-(x)=|\min\{0,A_0\}|$, then $-\Delta+A_0$ is essentially self adjoint.*

2.4 Local theory of the nonlinear equations

We state a classical theorem regarding local wellposedness of (NLS) in our setting [50].

Theorem 2.4.1. *Suppose $0<s_c<1$. For initial data $u_0\in H^1(\mathbb{R}^N)$, there exists a unique $u\in C((T_*,T^*);H^1(\mathbb{R}^N))$ with $u(x,0)=u_0(x)$, solving (NLS), such that if $T^*<\infty$,*

$$\lim_{t\uparrow T^*}\|\nabla u(t)\|_{L^2(\mathbb{R}^N)}=\infty,$$

and if $T_*>-\infty$,

$$\lim_{t\downarrow T_*}\|\nabla u(t)\|_{L^2(\mathbb{R}^N)}=\infty.$$

In addition, for $T'<T^*$, there is some $r>0$ so that if $B_{u_0}(r)$ denotes the set of $v_0\in H^1(\mathbb{R}^N)$ satisfying $\|u_0-v_0\|_{H^1(\mathbb{R}^N)}<r$, then the map $v_0\mapsto v$, from $B_{u_0}(r)$ to $C((0,T');H^1(\mathbb{R}^N))$, taking initial data to solutions of (NLS), is continuous.

Furthermore, the solution u satisfies the conservation laws

$$E[u(t)]=E[u_0],\quad \textit{and}$$

$$\|u(t)\|^2_{L^2(\mathbb{R}^N)}=\|u_0\|^2_{L^2(\mathbb{R}^N)},$$

on the full time interval (T_*,T^*). If $|x|u_0\in L^2(\mathbb{R}^N)$, then on the full time interval (T_*,T^*), we also have $|x|u(t)\in L^2(\mathbb{R}^N)$, and the map $t\mapsto V[u](t)=\int|x|^2|u(x,t)|^2\,dx$ is twice differentiable, *satisfying*

$$V_t[u](t) = 2\int \boldsymbol{x}\cdot\ \mathcal{P}\left[u(x,t)\right]\,dx,$$

$$V_{tt}[u](t) = 8\left\|\nabla u(t)\right\|_{L^2(\mathbb{R}^N)}^2 - \frac{4N(p-1)}{p+1}\left\|u(t)\right\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}.$$

We will place an abstract assumption on electromagnetic potentials, that (emNLS) will have this same satisfactory local theory. Explicitly, for dimension $N \geq 3$ and p such that $0 < s_c < 1$, we will say that the electromagnetic potentials A_0, A generate satisfactory local theory if they satisfy the following assumption

Assumption 2.4.1. We say the potentials A_0, A generate satisfactory local theory if the following holds.

For initial data $u_0 \in H^1_A(\mathbb{R}^N)$, there exists a unique $u \in C((T_*, T^*); H^1_A(\mathbb{R}^N))$ with $u(x,0) = u_0(x)$, solving (emNLS), such that if $T^* < \infty$, $\lim_{t\uparrow T^*}\left\|\nabla_A u(t)\right\|_{L^2(\mathbb{R}^N)} = \infty$, and if $T_* > -\infty$, $\lim_{t\downarrow T_*}\left\|\nabla_A u(t)\right\|_{L^2(\mathbb{R}^N)} = \infty$. In addition, for $T' < T^*$, there is some $r > 0$ so that if $B_{u_0}(r)$ denotes the set of $v_0 \in H^1_A(\mathbb{R}^N)$ satisfying $\|u_0 - v_0\|_{H^1_A(\mathbb{R}^N)} < r$, then the map $v_0 \mapsto v$, from $B_{u_0}(r)$ to $C((0,T'); H^1_A(\mathbb{R}^N))$, taking initial data to solutions of (emNLS), is continuous.

Furthermore, the solution u satisfies the conservation laws

$$E_A[u(t)] = E_A[u_0], \quad \text{and}$$

$$\left\|u(t)\right\|_{L^2(\mathbb{R}^N)}^2 = \left\|u_0\right\|_{L^2(\mathbb{R}^N)}^2,$$

on the full time interval (T_*, T^*). If $|x|u_0 \in L^2(\mathbb{R}^N)$, then $|x|u(t) \in L^2(\mathbb{R}^N)$ on the full time interval, and the map $t \mapsto V[u](t) = \int |x|^2|u(x,t)|^2\,dx$ is twice differentiable, satisfying

$$V_t[u] = 2\int \mathbf{x}\cdot\ \mathcal{P}\left[u(x,t)\right]\,dx,$$

$$V_{tt}[u] = 4N(p-1)E_A[u] - 2(N(p-1)-4)\left\|\nabla_A u\right\|_{L^2(\mathbb{R}^N)}^2 - 2N(p-1)\int A_0\,|u|^2\,dx$$

$$-4\int |x|\,\partial_r A_0\,|u|^2\ dx + 8\int |x|\,\mathrm{Im}(uB_\tau\cdot\overline{\nabla_A u})\,dx,$$

where B_τ is a vector, depending on A, defined in Chapter 3.

Solutions of (NLS) additionally satisfy the following conservation laws.

Theorem 2.4.2. *Given initial data $u_0 \in H^1(\mathbb{R}^N)$, and the corresponding solution u, for all t in the interval of existence, u satisfies the conservation laws*

$$\textit{Linear Momentum:} \quad \boldsymbol{P}[u](t) := \int \mathcal{P}[u(x,t)]\, dx = \boldsymbol{P}[u_0], \quad \textit{and}$$

$$\textit{Angular Momentum:} \quad \boldsymbol{M}[u](t) := \int \mathcal{P}[u(x,t)] \wedge \boldsymbol{x}\, dx = \boldsymbol{M}[u_0],$$

where

$$\mathcal{P}[u] := 2\, Im\, [\overline{u}\, \nabla u]\,,$$

and $\wedge$ is a generalization of the cross product to any dimension, described in Chapter 4.

Chapter 3

Global Dynamics of Electromagnetic Nonlinear Schrödinger Equations Below the Ground State

3.1 Introduction

We consider the electromagnetic nonlinear Schrödinger equation (emNLS) and investigate global existence versus blow-up in finite time. Mathematically, emNLS is a natural generalization of the NLS equation when one replaces regular derivatives by covariant derivatives. In physics, the equation has connections to the quantum electrodynamics and Bose-Einstein condensates, e.g., see [22, 43].

In this article, we are interested in the focusing equation and making progress in obtaining a general understanding of the global theory similar to what is known for the standard NLS equation. Here, "standard" refers to the case when all the potentials are zero.

In that case,

$$i\partial_t u + \Delta u + |u|^{p-1}u = 0, \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{3.1.1}$$

where $u = u(x,t)$ is complex-valued and $p > 1$. The solutions of this equation satisfy several conservation laws. In particular, the L^2 norm, which is mass/charge, is conserved, and we also have conservation of energy

$$E[u] = \frac{1}{2}\left\|\nabla u\right\|^2_{L^2(\mathbb{R}^n)} - \frac{1}{p+1}\left\|u\right\|^{p+1}_{L^{p+1}(\mathbb{R}^n)}. \tag{3.1.2}$$

The NLS equation has scaling: $u_\lambda(x,t) = \lambda^{\frac{2}{p-1}}u(\lambda x, \lambda^2 t)$ is a solution if $u(x,t)$ is as well.

This scaling produces a scale-invariant Sobolev norm $\dot{H}^{s_c}(\mathbb{R}^n)$ with

$$s_c = \frac{n}{2} - \frac{2}{p-1}. \tag{3.1.3}$$

The NLS equation admits a soliton solution, which we call u_Q (see Section 3.2.1). This solution acts as a threshold for when solutions exhibit a dichotomy between scattering and blow-up in both time directions. For a precise statement of this fact, we consider the scale invariant quantities

$$E[u]^{s_c} \|u\|_{L^2(\mathbb{R}^n)}^{2(1-s_c)},$$

$$\|\nabla u\|_{L^2(\mathbb{R}^n)}^{s_c} \|u\|_{L^2(\mathbb{R}^n)}^{(1-s_c)}.$$

The set of initial data satisfying $E[u]^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{2(1-s_c)} < E[u_Q]^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{2(1-s_c)}$ has the following dichotomy.

Theorem 3.1.1. *[19, 34, 35] Let $u_0 \in H^1(\mathbb{R}^n)$ and let the critical exponent s_c satisfy $0 < s_c < 1$. Suppose*

$$E[u_0]^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{2(1-s_c)} < E[u_Q]^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{2(1-s_c)}. \tag{3.1.4}$$

- *If*

$$\|\nabla u_0\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{1-s_c} < \|\nabla u_Q\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{1-s_c}, \tag{3.1.5}$$

  then the solution u is global in both time directions and

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{1-s_c} < \|\nabla u_Q\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{1-s_c}, \tag{3.1.6}$$

  for all time $t \in \mathbb{R}$. Moreover, the solution scatters in $H^1(\mathbb{R}^n)$.

- *If*

$$\|\nabla u_0\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{1-s_c} > \|\nabla u_Q\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{1-s_c}, \tag{3.1.7}$$

  then

$$\|\nabla u(t)\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{1-s_c} > \|\nabla u_Q\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{1-s_c}, \tag{3.1.8}$$

and if $|x|\, u_0 \in L^2$ *or* u *is radial, then the solution blows up in finite time in both time directions.*

Remark 3.1.2. For further results regarding the blow up scenario, see e.g. [36].

In this chapter, we wish to establish a dichotomy result regarding global existence and blow-up for the electromagnetic equation.

There is extensive literature on the existence of smoothing and Strichartz estimates for emNLS (see for example [16, 23, 27]). Several authors have also studied wellposedness and scattering [9, 17, 40]. Much less is known, however, about blow-up and formation of singularities. To our knowledge, the only articles considering blow-up are the works of Goncalves Ribero [30], Garcia [26], Kieffer and Loss [39], and Dinh [18]. The first two articles consider negative energy blow-up using the virial identity argument. The work of Kieffer and Loss [39] develops a modification of the virial identity argument, with a new conserved quantity related to the angular momentum. All articles consider constant magnetic field with the exception of Garcia, who besides considering constant magnetic field, considers potentials with a non-constant magnetic field, but having a zero trapping component (see below). The articles [26, 30, 39] focus on blow-up results, while [18] considers both global wellposedness and blow-up and obtains dichotomy results in the spirit of [34]. Among these articles, most have focused on the case with no electric potential. However, Garcia [26] considers a class of both magnetic and electric potentials.

The goal of the present chapter is to extend the results of [18, 26, 30, 39] to potentials with non-constant magnetic field and with nonzero trapping component. Moreover, the results address both global wellposedness and blow-up in finite time for positive or negative energy, and are stated as a dichotomy result.

We work in the Coulomb gauge. The proof of global existence can go through similar to the standard NLS equation. The main obstacle comes while establishing the blow-up. This is due to having extra terms in the virial identity (see (3.2.19)). In the case of the purely magnetic equation, this can be handled by a careful analysis of the remaining error term.

We also include results for the full electromagnetic equation. To obtain the dichotomy statement, the result requires smallness of the potentials, and a smaller bound on the mass-energy, i.e., the bound (3.1.19). If we are only interested in a global wellposedness result, then no smallness is required, and we can get as close as we want to an electromagnetic analog of the ground state quantity. We state this as a corollary.

3.1.1 Main results

Let $n \geq 3$. Consider the focusing magnetic nonlinear Schrödinger equation, given by

$$\begin{cases} iD_t u + \Delta_A u = -\left|u\right|^{p-1} u, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(0,x) = u_0(x), \end{cases} \tag{3.1.9}$$

where

$$u : \mathbb{R}^{n+1} \mapsto \mathbb{C},$$

$$A_\alpha : \mathbb{R}^n \mapsto \mathbb{R}, \quad \alpha = 0, \cdots, n,$$

$$D_\alpha = \partial_\alpha + iA_\alpha, \quad \alpha = 0, \cdots, n, \quad D_t = D_0,$$

$$\Delta_A = D^2 = D_j D_j = (\partial_j + iA_j)(\partial_j + iA_j) = \Delta + 2iA \cdot \nabla + i\nabla \cdot A - \left|A\right|^2.$$

The Greek indices range from 0 to n, and Roman indices range from 1 to n. We sum over repeated indices. We use $\nabla_A u$ to denote

$$(D_1 u, \ldots, D_n u)$$

and sometimes write (A_0, A) to mean $(A_0, A_1, \ldots, A_n)$. The power p satisfies

$$1 + \frac{4}{n} < p < 1 + \frac{4}{n-2}. \tag{3.1.10}$$

The equation emNLS is scaling invariant (see Section 3.2.2 below) and (3.1.10) is equivalent to the critical Sobolev exponent s_c satisfying the intercritical range

$$0 < s_c < 1.$$

We note that this correspondence of the range of s_c and p is the same as in the case of the standard NLS equation.

We next define the spaces adapted to our magnetic NLS setting. Let

$$H_A^1(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \nabla_A f \in L^2(\mathbb{R}^n)\},$$

and

$$\Sigma_A(\mathbb{R}^n) = \{f \in H_A^1(\mathbb{R}^n) : |x|\, f \in L^2(\mathbb{R}^n)\}.$$

The energy (or Hamiltonian) for emNLS is given by

$$E_A[u] = \frac{1}{2}\left\|\nabla_A u\right\|_2^2 - \frac{1}{p+1}\left\|u\right\|_{p+1}^{p+1} + \frac{1}{2}\int A_0\,|u|^2\,dx,$$

and is conserved when the equation has a satisfactory local theory for data in H_A^1.

In the case of 3 dimensions, for a magnetic potential A, the magnetic field can be identified with $\operatorname{curl} A$. It was observed in [25] that

$$B_\tau = \frac{x}{|x|} \times \operatorname{curl} A$$

is an obstruction to dispersion. The term B_τ is called the **trapping component** of the magnetic field.

In higher dimensions, the trapping component can be generalized to

$$B_\tau^T = \frac{x^T}{|x|}(F_{jk}),$$

where (F_{jk}) is a matrix with the (j,k) entry given by $F_{jk} = \partial_j A_k - \partial_k A_j$, so the k'th entry of B_τ is

$$(B_\tau)_k = \frac{x_j}{|x|}F_{jk}. \tag{3.1.11}$$

Our main result is the following theorem.

Theorem 3.1.3. *Let $n \geq 3$ and $0 < s_c < 1$. There exists a constant C_τ so that if $A \in L^2_{loc}(\mathbb{R}^n;\mathbb{R}^n)$ is a potential inducing satisfactory local theory, then we have the following dichotomy result: Suppose $u_0 \in H_A^1(\mathbb{R}^n)$ is such that*

$$E_A[u_0]^{s_c}\left\|u_0\right\|_{L^2(\mathbb{R}^n)}^{2(1-s_c)} < E[u_Q]^{s_c}\left\|u_Q\right\|_{L^2(\mathbb{R}^n)}^{2(1-s_c)}. \tag{3.1.12}$$

- *If*

$$\|\nabla_A u_0\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{1-s_c} < \|\nabla u_Q\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{1-s_c}, \tag{3.1.13}$$

then the solution u is global and

$$\|\nabla_A u(t)\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{1-s_c} < \|\nabla u_Q\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{1-s_c}, \tag{3.1.14}$$

for all time $t \in \mathbb{R}$.

- *If*

$$\|\nabla_A u_0\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{1-s_c} > \|\nabla u_Q\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{1-s_c}, \tag{3.1.15}$$

then

$$\|\nabla_A u(t)\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{1-s_c} > \|\nabla u_Q\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{1-s_c}, \tag{3.1.16}$$

on the full interval of existence, and if

$$\left\||x|^2 B_\tau\right\|_{L^\infty(\mathbb{R}^n)} < C_\tau, \tag{3.1.17}$$

and u_0 has finite variance, the solution blows up in finite time.

Remark 3.1.4. Observe that for potentials satisfying (3.1.17) this theorem is the magnetic analog of the global well-posedness and blow-up portion of Theorem 3.1.1.

Remark 3.1.5. Under the assumption (3.1.12), the expressions (3.1.13) and (3.1.15) are the only possibilities. This can be seen, for example, in the proof of Lemma 3.3.2.

Remark 3.1.6. We note that the theorem applies to both positive and negative energy. For negative energy, the applicable scenario is only the blow-up scenario.

Remark 3.1.7. We can show that the theorem holds for the value

$$C_\tau = \min\left(\frac{(n-2)(n(p-1)-4)}{16\, C_{p,n} \|u_Q\|_2^{\frac{(1-s_c)}{s_c}} \|\nabla u_Q\|_2}, \frac{(n-2)(n(p-1)-4)}{16} \right),$$

where the constant $C_{p,n}$ is obtained in Lemma 3.3.3. This value is not sharp.

Remark 3.1.8. The Coulomb gauge and the norm in (3.1.17) is inspired by assumptions needed to obtain Strichartz estimates. In general, smallness assumptions on B_τ are needed to obtain Strichartz estimates and show local wellposedness. For example, in [16] to obtain Strichartz estimates in dimensions greater than or equal to 4, a smallness condition on $\left\| |x|^2 B_\tau \right\|_\infty$ is needed (possibly with a different constant from (3.1.17)), while in 3 dimensions, this appears as a smallness condition on $\left\| |x|^{3/2} B_\tau \right\|_{L_r^2 L^\infty(S_r)}$, where

$$\|f\|^2_{L_r^2 L^\infty(S_r)} = \int_0^\infty \sup_{|x|=r} |f|^2 \, dr, \tag{3.1.18}$$

see further details on this in Appendix A.

If we include the electric potential, we obtain the following results.

Theorem 3.1.9. *Let $n \geq 3$, $0 < s_c < 1$, and $u_0 \in H^1_A(\mathbb{R}^n)$ such that*

$$E_A[u_0] \, \|u_0\|_{L^2(\mathbb{R}^n)}^{2\frac{(1-s_c)}{s_c}} < (1-a_0)^{1+\frac{2}{s_c(p-1)}} E[u_Q] \, \|u_Q\|_{L^2(\mathbb{R}^n)}^{2\frac{(1-s_c)}{s_c}} - C(A_0, B_\tau), \tag{3.1.19}$$

where

$$C(A_0, B_\tau) = \frac{2a_0 + a_1 + 2b_0}{n(p-1)} x_* \quad \text{and} \quad x_* = (1-a_0)^{\frac{2}{s_c(p-1)}} \|\nabla u_Q\|_2^2 \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \tag{3.1.20}$$

and a_0, a_1, b_0 are constants appearing in (3.2.21), (3.2.22), and (3.2.23), respectively.

If A, A_0 are potentials inducing satisfactory local theory, then we have the following dichotomy result:

- *If*

$$\|\nabla_A u_0\|^{s_c}_{L^2(\mathbb{R}^n)} \|u_0\|^{1-s_c}_{L^2(\mathbb{R}^n)} < (1-a_0)^{\frac{1}{p-1}} \|\nabla u_Q\|^{s_c}_{L^2(\mathbb{R}^n)} \|u_Q\|^{1-s_c}_{L^2(\mathbb{R}^n)}, \tag{3.1.21}$$

  then the solution u is global and

$$\|\nabla_A u(t)\|^{s_c}_{L^2(\mathbb{R}^n)} \|u_0\|^{1-s_c}_{L^2(\mathbb{R}^n)} < (1-a_0)^{\frac{1}{p-1}} \|\nabla u_Q\|^{s_c}_{L^2(\mathbb{R}^n)} \|u_Q\|^{1-s_c}_{L^2(\mathbb{R}^n)}, \tag{3.1.22}$$

  for all time $t \in \mathbb{R}$.

- *If*

$$\|\nabla_A u_0\|^{s_c}_{L^2(\mathbb{R}^n)} \|u_0\|^{1-s_c}_{L^2(\mathbb{R}^n)} > (1-a_0)^{\frac{1}{p-1}} \|\nabla u_Q\|^{s_c}_{L^2(\mathbb{R}^n)} \|u_Q\|^{1-s_c}_{L^2(\mathbb{R}^n)}, \tag{3.1.23}$$

and if

$$n(p-1)a_0 + 2a_1 + 4b_0 < n(p-1) - 4 = s_c(p-1), \tag{3.1.24}$$

then

$$\|\nabla_A u(t)\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{1-s_c} > (1-a_0)^{\frac{1}{p-1}} \|\nabla u_Q\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{1-s_c}, \tag{3.1.25}$$

on the full interval of existence, and if u_0 has finite variance the solution blows up in finite time.

Remark 3.1.10. We can consider arbitrarily small $C(A_0, B_\tau)$ by taking potentials with sufficiently small constants a_0, a_1, b_1.

The proof of the above theorem implies the following corollary.

Corollary 3.1.11. Let $n \geq 3$, $0 < s_c < 1$, and A, A_0 be potentials inducing satisfactory local theory. Suppose $u_0 \in H_A^1$ is such that

$$E_A[u_0] \|u_0\|_{L^2(\mathbb{R}^n)}^{2\frac{(1-s_c)}{s_c}} < (1-a_0)^{1+\frac{2}{s_c(p-1)}} E[u_Q] \|u_Q\|_{L^2(\mathbb{R}^n)}^{2\frac{(1-s_c)}{s_c}}. \tag{3.1.26}$$

If

$$\|\nabla_A u_0\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{1-s_c} < (1-a_0)^{\frac{1}{p-1}} \|\nabla u_Q\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{1-s_c}, \tag{3.1.27}$$

then the solution u is global and

$$\|\nabla_A u(t)\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_0\|_{L^2(\mathbb{R}^n)}^{1-s_c} < (1-a_0)^{\frac{1}{p-1}} \|\nabla u_Q\|_{L^2(\mathbb{R}^n)}^{s_c} \|u_Q\|_{L^2(\mathbb{R}^n)}^{1-s_c}, \tag{3.1.28}$$

for all time $t \in \mathbb{R}$.

The paper is organized as follows. In Section 3.2 we discuss preliminaries regarding the standard NLS as well as emNLS. Then, in Section 3.3 we prove our main lemmas, including a remainder identity for Kato's inequality which provides control of the trapping term, as well as refined bounds for the kinetic energy of solutions, and bounds regarding the proximity of a solution to the ground state. In Section 3.4 we prove Theorem 3.1.3 by applying the remainder identity for Kato's inequality to reduce the problem to the control of a nonlinear error term, which we then bound using refined control of kinetic energy. In Section 3.5 we demonstrate a way to modify dichotomy arguments for NLS to obtain results for emNLS, and prove Theorem 3.1.9. Finally, in Appendix A, we provide a technique for constructing potentials which satisfy the abstract conditions appearing in this article and other literature.

3.2 Preliminaries

3.2.1 Gagliardo-Nirenberg inequality

We start with recalling the Gagliardo-Nirenberg inequality [5, p.168], [8, Theorem 1.3.7],

$$\|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \leq C_{GN} \|\nabla u\|_{L^2(\mathbb{R}^n)}^{\frac{n(p-1)}{2}} \|u\|_{L^2(\mathbb{R}^n)}^{2-\frac{(n-2)(p-1)}{2}}, \tag{3.2.1}$$

which holds for $1 \leq p \leq \frac{n+2}{n-2}$, which includes the case $0 < s_c < 1$.

From Kato's diamagnetic inequality [42], we have the following estimate

$$|\nabla |u|| \leq |\nabla_A u|. \tag{3.2.2}$$

Combining (3.2.1) with (3.2.2), we obtain the following magnetic version

$$\|u\|_{L^{p+1}(\mathbb{R}^n)}^{p+1} \leq C_{GN} \|\nabla_A u\|_{L^2(\mathbb{R}^n)}^{\frac{n(p-1)}{2}} \|u\|_{L^2(\mathbb{R}^n)}^{2-\frac{(n-2)(p-1)}{2}}. \tag{3.2.3}$$

To simplify the notation, we write $\|\cdot\|_{L^p(\mathbb{R}^n)}$ simply as $\|\cdot\|_p$. The following equivalent form of the exponents in (3.2.3) will be useful

$$\|u\|_{p+1}^{p+1} \leq C_{GN} \|\nabla_A u\|_2^2 \left(\|\nabla_A u\|_2^{s_c} \|u\|_2^{1-s_c}\right)^{p-1}. \tag{3.2.4}$$

We are concerned with the case $0 < s_c < 1$, which we assume in the sequel. The Gagliardo-Nirenberg inequality (3.2.1) admits a positive smooth H^1 minimizer, Q, unique up to scaling and translation satisfying (via Pokhozhaev identities, e.g., see [34, 49, 53])

$$\|\nabla Q\|_2 = \|Q\|_2 \,, \tag{3.2.5}$$

$$\|Q\|_{p+1}^{p+1} = \frac{p+1}{2} \|Q\|_2^2 \,, \tag{3.2.6}$$

$$C_{GN} = \frac{p+1}{2\,\|Q\|_2^{p-1}}. \tag{3.2.7}$$

A soliton solution to NLS can be written as

$$u_Q(x,t) = e^{i\lambda t} Q(\alpha x),$$

where

$$\alpha = \frac{\sqrt{n(p-1)}}{2} \qquad \text{and} \qquad \lambda = 1 - \frac{(n-2)(p-1)}{4} = (1-s_c)\frac{p-1}{2}.$$

We have

$$\|u_Q\|_{p+1}^{p+1} = \frac{1}{\alpha^n} \|Q\|_{p+1}^{p+1} = \frac{p+1}{2\alpha^n} \|Q\|_2^2 \,, \tag{3.2.8}$$

as well as

$$\|\nabla u_Q\|_2^2 = \frac{\alpha^2}{\alpha^n} \|Q\|_2^2 \,, \tag{3.2.9}$$

and

$$E[u_Q] = \frac{1}{2\alpha^n}(\alpha^2 - 1) \|Q\|_2^2 \,. \tag{3.2.10}$$

Finally, for future reference we note

$$1 + \frac{s_c(p-1)}{2} = \frac{n(p-1)}{4}. \tag{3.2.11}$$

3.2.2 Scale invariant quantities and minimal energy function

If u solves (emNLS), then

$$u_\lambda(x,t) = \lambda^{\frac{2}{p-1}} u(\lambda x, \lambda^2 t)$$

solves (emNLS) with potentials

$$(A_{0,\lambda}(x,t), A_\lambda(x,t)) = (\lambda^2 A_0(\lambda x, \lambda^2 t), \lambda A(\lambda x, \lambda^2 t)).$$

We consider the scale invariant quantities

$$\|\nabla_A u\|_2^{2s_c} \|u\|_2^{2(1-s_c)} \qquad \text{and} \qquad E_A[u]^{s_c} \|u\|_2^{2(1-s_c)},$$

and similarly to [34], we note that by (3.2.3) and (3.2.11)

$$\begin{aligned} E_A[u] \|u\|_2^{2\frac{(1-s_c)}{s_c}} &= \frac{1}{2} \|\nabla_A u\|_2^2 \|u\|_2^{2\frac{(1-s_c)}{s_c}} - \frac{1}{p+1} \|u\|_{p+1}^{p+1} \|u\|_2^{2\frac{(1-s_c)}{s_c}} \\ &\geq \frac{1}{2} \|\nabla_A u\|_2^2 \|u\|_2^{2\frac{(1-s_c)}{s_c}} - \frac{C_{GN}}{p+1} \|\nabla_A u\|_2^2 \|u\|_2^{2\frac{(1-s_c)}{s_c}} \left(\|\nabla_A u\|_2^{s_c} \|u\|_2^{1-s_c}\right)^{p-1} \\ &= \frac{1}{2} \|\nabla_A u\|_2^2 \|u\|_2^{2\frac{(1-s_c)}{s_c}} - \frac{C_{GN}}{p+1} \left(\|\nabla_A u\|_2^2 \|u\|_2^{2\frac{(1-s_c)}{s_c}}\right)^{\frac{n(p-1)}{4}}. \end{aligned} \tag{3.2.12}$$

This motivates the definition of the *minimal energy function* (cf. [34], [37])

$$P(x) = \frac{1}{2}x - \frac{C_{GN}}{p+1} x^{\frac{n(p-1)}{4}}. \tag{3.2.13}$$

In particular, P is called the minimal energy function due to the bound that follows from (3.2.12)

$$P(\|\nabla_A u\|_2^2 \|u\|_2^{2\frac{(1-s_c)}{s_c}}) \leq E_A[u] \|u\|_2^{2\frac{(1-s_c)}{s_c}}. \tag{3.2.14}$$

Equality in (3.2.14) holds for the NLS soliton solution u_Q, namely, we have

$$P(\|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}) = E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}. \tag{3.2.15}$$

In addition, the function P is concave down on $[0, \infty)$ and attains the maximum of

$$E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \tag{3.2.16}$$

at

$$x = \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}. \tag{3.2.17}$$

Besides the magnetic energy $E_A[u]$, we introduce **Kato energy**,

$$E_K[u] = \frac{1}{2} \|\nabla |u|\|_2^2 - \frac{1}{p+1} \|u\|_{p+1}^{p+1}.$$

Note, due to the positivity of Q, we have

$$E_K[u_Q] = E[u_Q].$$

Next, just like above, but using the Gagliardo–Nirenberg inequality (3.2.1) applied to $|u|$ we can show a similar bound to (3.2.14)

$$P\left(\|\nabla|u|\|_2^2 \, \|u\|_2^{2\frac{(1-s_c)}{s_c}}\right) \le E_K[u] \, \|u\|_2^{2\frac{(1-s_c)}{s_c}} . \tag{3.2.18}$$

3.2.3 Virial identity

The virial identity for the standard NLS is well-known [29,52,56]. For a general linear emNLS it was derived first in [25]. Including nonlinearity is an easy extension (see e.g. [13]), and it can be stated as follows

$$\begin{aligned} \partial_t^2 \int |x|^2 |u|^2 \, dx = 4n(p-1)E_A[u] - 2(n(p-1)-4) \|\nabla_A u\|^2 - 2n(p-1) \int A_0 |u|^2 \, dx \\ - 4 \int_{\mathbb{R}^n} |x| \, \partial_r A_0 \, |u|^2 \, dx + 8 \int_{\mathbb{R}^n} |x| \operatorname{Im}(u B_\tau \cdot \overline{\nabla_A u}) \, dx. \end{aligned} \tag{3.2.19}$$

If the electromagnetic potentials are all identically zero, then the third, fourth and fifth terms are not present, and we recover the virial identity for the standard NLS.

We note that even in the purely magnetic equation, an extra term involving the trapping term B_τ (the last term in (3.2.19)) remains. Therefore, besides being an obstruction to dispersion, this term, in some sense, can also be an obstruction to virial blow-up arguments.

3.2.4 Magnetic Hardy's inequality

We recall a magnetic Hardy's inequality (see e.g. [25, A.1]). If $u \in H_A^1(\mathbb{R}^n)$ and $n \ge 3$, then

$$\left\| \frac{u}{|x|} \right\|_2 \le \frac{2}{n-2} \|\nabla_A u\|_2 . \tag{3.2.20}$$

3.2.5 Conditions on the potentials

If we include the electric potential, we can still work with the same norm for B_τ as before, and then we also need bounds on the electric potential and its derivative. More specifically, in this setting we consider potentials A, A_0 so that, with notation $f_- = \max\{-f, 0\}$, the estimates

$$\int A_{0,-} |u|^2 \, dx \leq a_0 \left\| \nabla_A u \right\|_2^2 , \tag{3.2.21}$$

$$\int_{\mathbb{R}^n} (\partial_r A_0)_- \, |x| \, |u|^2 \; dx \leq a_1 \left\| \nabla_A u \right\|^2 , \tag{3.2.22}$$

$$\int_{\mathbb{R}^n} |x| \, \mathcal{I}m(u B_\tau \cdot \overline{\nabla_A u}) \; dx \leq b_0 \left\| \nabla_A u \right\|_2^2 , \tag{3.2.23}$$

hold for all $u \in H^1_A(\mathbb{R}^N)$ with constants $0 \leq a_0 < 1$, and $0 \leq a_1, b_0 < \infty$.

As mentioned before, we are interested in the potentials considered in the article [16], as well as the conditions on potentials in [25]. The conditions in [16, 25] are imposed on the potentials in order to obtain weak dispersion and Strichartz estimates. By [16, Lemma 2.3], we have $0 < a_0 < 1$.

Next, using (3.2.20), we can take

$$b_0 = \left\| |x|^2 \, B_\tau \right\|_\infty \frac{2}{n-2}. \tag{3.2.24}$$

The condition on B_τ for $n \geq 4$ in [16, 25] is

$$\left\| |x|^2 \, B_\tau \right\|^2_{L^\infty_x} + 2 \left\| |x|^3 \, (\partial_r A_0)_+ \right\|_{L^\infty_x} < \frac{2}{3}(n-1)(n-3). \tag{3.2.25}$$

Hence in that case,

$$b_0 \leq \frac{4}{3} \frac{(n-1)(n-3)}{n-2}.$$

This can be compared with bounds in (3.1.17). We discuss connections between these conditions and those necessary for Strichartz estimates in 3D in Appendix A.

Finally, having nonzero $(\partial_r A_0)_-$ is the place where we need to impose an additional condition that does not appear in the work of [16, 25]. There, a smallness condition on $(\partial_r A_0)_+$ is needed. We use the condition (3.2.22) to establish a finite time blow-up. Therefore, if we are only after global wellposedness, the condition (3.2.22) would not be needed (see Corollary 3.1.11).

3.3 Main lemmas

There are several technical lemmas we need to establish.

3.3.1 Remainder identity for Kato's inequality

Here we establish a remainder identity for Kato's inequality on H^1_A.

Lemma 3.3.1 (Remainder identity for Kato's inequality)**.** *For $u \in H^1_A$, the identity*

$$\|\nabla_A u\|_2^2 - \|\nabla |u|\|_2^2 = \int_{u\neq 0} \Big| Im\Big[\frac{\overline{u}}{|u|}\nabla_A u\Big]\Big|^2 dx$$

is satisfied.

Proof. Given $u \in H^1_A$, we have pointwise [42, Thm 6.17]

$$|\partial_j |u||^2 = \begin{cases} \Big|\mathrm{Re}\Big[\frac{\overline{u}}{|u|}\partial_j u\Big]\Big|^2 & u \neq 0 \\ 0 & u = 0 \end{cases}$$

$$= \begin{cases} \Big|\mathrm{Re}\Big[\frac{\overline{u}}{|u|}D_j u\Big]\Big|^2 & u \neq 0 \\ 0 & u = 0. \end{cases}$$

Furthermore by [42, Thm 6.19] we have that a.e.

$$\Big|D_j u\Big|^2 = \begin{cases} \Big|\frac{\overline{u}}{|u|}D_j u\Big|^2 & u \neq 0 \\ 0 & u = 0. \end{cases}$$

It follows that a.e.

$$|\nabla_A u|^2 - |\nabla |u||^2 = \begin{cases} \Big|\mathrm{Im}\Big[\frac{\overline{u}}{|u|}\nabla_A u\Big]\Big|^2 & u \neq 0 \\ 0 & u = 0. \end{cases}$$

From here the result follows by integration, which completes the proof. □

3.3.2 Double Cut Lemma

We show that the set of the initial data strictly below the ground state is the union of two disjoint open sets such that within each set we can also compare the size of $\|\nabla_A u\|_2$ to $\|\nabla |u|\|_2$.

Lemma 3.3.2 (Double Cut Lemma)**.** *For a potential $A \in L^2_{loc}(\mathbb{R}^n;\mathbb{R}^n)$, consider*

$$\mathcal{R} = \{u \in H^1_A(\mathbb{R}^N) : E_A[u]^{s_c} \|u\|_2^{2(1-s_c)} < E[u_Q]^{s_c} \|u_Q\|_2^{2(1-s_c)}\}.$$

There are two disjoint sets $\mathcal{R}_1, \mathcal{R}_2$ with $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ such that for each $u \in \mathcal{R}_1$,

$$\|\nabla|u|\|_2^{s_c} \|u\|_2^{1-s_c} \leq \|\nabla_A u\|_2^{s_c} \|u\|_2^{1-s_c} < \|\nabla u_Q\|_2^{s_c} \|u_Q\|_2^{1-s_c}, \tag{3.3.1}$$

and for each $u \in \mathcal{R}_2$,

$$\|\nabla u_Q\|_2^{s_c} \|u_Q\|_2^{1-s_c} < \|\nabla|u|\|_2^{s_c} \|u\|_2^{1-s_c} \leq \|\nabla_A u\|_2^{s_c} \|u\|_2^{1-s_c}. \tag{3.3.2}$$

Proof. Recall the minimal energy function from Section 3.2.2

$$P(x) = \frac{1}{2}x - \frac{C_{GN}}{p+1} x^{\frac{n(p-1)}{4}},$$

where for $u \in H^1_A$, the inequality

$$P(\|\nabla_A u\|_2^2 \|u\|_2^{2\frac{(1-s_c)}{s_c}}) \leq E_A[u] \|u\|_2^{2\frac{(1-s_c)}{s_c}}$$

holds by (3.2.12). Now, since for $u \in \mathcal{R}$ we have

$$E_A[u] \|u\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$

and by (3.2.15)

$$P(\|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}) = E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$

it must be that if $u \in \mathcal{R}$, we either have

$$\|\nabla_A u\|_2^{s_c} \|u\|_2^{1-s_c} < \|\nabla u_Q\|_2^{s_c} \|u_Q\|_2^{1-s_c} \tag{3.3.3}$$

or

$$\|\nabla_A u\|_2^{s_c} \|u\|_2^{1-s_c} > \|\nabla u_Q\|_2^{s_c} \|u_Q\|_2^{1-s_c}. \tag{3.3.4}$$

We now show that we can further describe the set $\mathcal{R}$ as a union of $\mathcal{R}_1$ and $\mathcal{R}_2$. By Kato's inequality, to establish the full result it is sufficient to show that each u satisfying (3.3.4) also satisfies

$$\|\nabla|u|\|_2^{s_c} \|u\|_2^{1-s_c} > \|\nabla u_Q\|_2^{s_c} \|u_Q\|_2^{1-s_c}. \tag{3.3.5}$$

Suppose we have $u \in H^1_A$ such that

$$E_A[u]^{s_c} \|u\|_2^{2(1-s_c)} < E[u_Q]^{s_c} \|u_Q\|_2^{2(1-s_c)},$$

and (3.3.4). Then

$$E_A[u] \|u\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$

which with (3.3.4) implies

$$\|u\|_{p+1}^{p+1} \|u\|_2^{2\frac{(1-s_c)}{s_c}} > \|u_Q\|_{p+1}^{p+1} \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}.$$

Applying the Gagliardo-Nirenberg inequality (3.2.1) on the left hand side and using that Q is the optimizer, and so is u_Q by scale invariance (i.e., applying equality in (3.2.1) to the right-hand side above), we obtain

$$C_{GN} \|\nabla|u|\|_2^{\frac{n(p-1)}{2}} \|u\|_2^{2-\frac{(n-2)(p-1)}{2}} \|u\|_2^{2\frac{(1-s_c)}{s_c}} > C_{GN} \|\nabla u_Q\|_2^{\frac{n(p-1)}{2}} \|u_Q\|_2^{2-\frac{(n-2)(p-1)}{2}} \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}.$$

Then rewriting the exponents, as in (3.2.4), gives

$$\|\nabla|u|\|_2^2 \left(\|\nabla|u|\|_2^{s_c} \|u\|_2^{1-s_c}\right)^{p-1} \|u\|_2^{2\frac{(1-s_c)}{s_c}} > \|\nabla u_Q\|_2^2 \left(\|\nabla u_Q\|_2^{s_c} \|u_Q\|_2^{1-s_c}\right)^{p-1} \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}.$$

Manipulating the exponents further, we see the above is exactly

$$\left(\|\nabla|u|\|_2^{s_c} \|u\|_2^{1-s_c}\right)^{(p-1)+\frac{2}{s_c}} > \left(\|\nabla u_Q\|_2^{s_c} \|u_Q\|_2^{1-s_c}\right)^{(p-1)+\frac{2}{s_c}},$$

which in turn implies (3.3.5) as claimed.

$\square$

3.3.3 Proximity estimate for kinetic energy above the ground state

In this section, we establish the main estimate, which will be used in our blow-up argument.

For $u \in \mathcal{R}_2$, as defined in Lemma 3.3.2, we consider two non-negative quantities, which are useful for measuring how far u is from being a ground state. The first quantity arises from the remainder u in Kato's inequality and is given by

$$\mathcal{F}_{RK}(u) = \|\nabla_A u\|_2^2 \|u\|_2^{2\frac{(1-s_c)}{s_c}} - \|\nabla|u|\|_2^2 \|u\|_2^{2\frac{(1-s_c)}{s_c}}.$$

The second quantity arises from the kinetic energy of u and is given by

$$\mathcal{F}_{KE}(u) = \|\nabla_A u\|_2^2 \, \|u\|_2^{2\frac{(1-s_c)}{s_c}} - \|\nabla u_Q\|_2^2 \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} .$$

The Double Cut Lemma 3.3.2 equation (3.3.2) implies the bound

$$\mathcal{F}_{RK}(u) \leq \mathcal{F}_{KE}(u), \tag{3.3.6}$$

when $u \in \mathcal{R}_2$. In fact, what the lemma below shows is that the geometry of $\mathcal{R}_2$ additionally provides a bound

$$\mathcal{F}_{RK}(u) \lesssim \left(\mathcal{F}_{KE}(u)\right)^2 ,$$

which holds on $\mathcal{R}_2$ and is stronger than the linear bound (3.3.6) when $\mathcal{F}_{KE}$ is small.

Lemma 3.3.3 (Ground State Proximity Bound)**.** *There exists a constant $C_{p,n} > 0$ such that*

$$\left(\mathcal{F}_{RK}(u)\right)^{1/2} \leq C_{p,n} \, \mathcal{F}_{KE}(u) \tag{3.3.7}$$

for $u \in \mathcal{R}_2$.

Furthermore, the inequality holds for the following values of $C_{p,n}$.

- *For $n(p-1) \leq 8$,*

$$C_{p,n} = \left(\frac{16(p+1)}{C_{GN} n(p-1) \left(n(p-1) - 4\right)} \right)^{-1/2} \left(\|\nabla u_Q\|_2^2 \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \right)^{\frac{n(p-1)-8}{8}} .$$

- *For $n(p-1) > 8$,*

$$C_{p,n} = 1 + \left(\frac{16(p+1)}{C_{GN} n(p-1) \left(n(p-1) - 4\right)} \right)^{-1/2} \left(1 + \|\nabla u_Q\|_2^2 \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \right)^{\frac{n(p-1)-8}{8}} .$$

Before we begin the proof of Lemma 3.3.3 we make some remarks. We also state a simple Calculus lemma that is used in the proof.

The bound of Lemma 3.3.3 arises from the geometry of $\mathcal{R}_2$. The minimal energy function P introduced in Section 3.2.2 (3.2.13) provides a description of the relevant geometry. Prior to establishing the desired bound, we define a rigid transformation of the minimal energy function into a form better suited to this context.

Recall, $P(x)$ is concave down on $[0,\infty)$, attaining its maximum of $E[u_Q]\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}$ at $x_* = \|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}$. We define a transformation, P^*, which recenters the curve P to the ground state.

$$
\begin{aligned}
P^*(x) &:= P(x_*) - P\left(x + x_*\right) \\
&= E[u_Q]\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} - P\left(x + \|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right).
\end{aligned}
$$

This transformation is defined so that P^* is concave up on $[-x_*,\infty)$ and increasing on $[0,\infty)$, and such that $P^*(0) = (P^*)'(0) = 0$.

We now recall the bound (3.2.18), namely,

$$
E_K[u]\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} \geq P\left(\|\nabla|u|\|_2^2\,\|u\|_2^{2\frac{(1-s_c)}{s_c}}\right).
$$

Under the transformation of P to P^*, this bound implies

$$
P(x_*) - E_K[u]\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} \leq P^*\left(\|\nabla|u|\|_2^2\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} - x_*\right),
$$

and applying the main energy assumption (3.1.12) to the left hand side, with the explicit value of $P(x_*)$, results in the bound

$$
\frac{1}{2}\,\|u\|_2^{2\frac{(1-s_c)}{s_c}}\left[\|\nabla_A u\|_2^2 - \|\nabla|u|\|_2^2\right] \leq P^*\left(\|\nabla|u|\|_2^2\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} - x_*\right), \tag{3.3.8}
$$

which by (3.2.2) and the definition of $\mathcal{F}_{RK}(u)$ and $\mathcal{F}_{KE}(u)$ gives

$$
\frac{1}{2}\mathcal{F}_{RK}(u) \leq P^*(\mathcal{F}_{KE}(u)). \tag{3.3.9}
$$

We now consider the inverse of P^* on $[0,\infty)$, denoted by $(P^*)^{-1}$. Since P^* is strictly increasing in $[0,\infty)$, the inverse $(P^*)^{-1}$ is well defined and increasing. Applying $(P^*)^{-1}$ to both sides of (3.3.9) results in

$$(P^*)^{-1}\left(\frac{1}{2}\mathcal{F}_{RK}(u)\right) \leq \mathcal{F}_{KE}(u). \tag{3.3.10}$$

To establish the Lemma 3.3.3, we use a simple Calculus fact that we state without proof.

Lemma 3.3.4. *Let $b \in (0,\infty]$, $f : [0,b) \to [0,\infty)$ be twice differentiable with well-defined inverse, f^{-1}, and such that $f(0) = f'(0) = 0$. If $f''(x) \leq C$ for some $C > 0$ for all $x \in [0,b)$, then $f(x) \leq \frac{C}{2}x^2$ for all $x \in [0,b)$, and $f^{-1}(x) \geq \sqrt{\frac{2}{C}}\sqrt{x}$ for all $x \in [0, f(b))$.*

We are now ready to start the proof.

Proof of Lemma 3.3.3. We consider three cases.

Case 1: $n(p-1) = 8$**.** In this case, by completing the square, we can see $P^* : [0,\infty) \to [0,\infty)$ is simply a quadratic function

$$P^*(x) = \frac{C_{GN}}{p+1}x^2,$$

so

$$(P^*)^{-1}(x) = \sqrt{\frac{p+1}{C_{GN}}}\sqrt{x},$$

and (3.3.10) implies

$$\sqrt{\frac{p+1}{2C_{GN}}}\left(\mathcal{F}_{RK}(u)\right)^{\frac{1}{2}} \leq \mathcal{F}_{KE}(u),$$

which gives (3.3.7) as needed.

Case 2: $n(p-1) < 8$**.** We begin by computing

$$\begin{aligned}
\partial_x^2 P^*(x) &= -\partial_x^2 P\left(x + \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right) \\
&= \frac{C_{GN}}{p+1}\left(\frac{n(p-1)}{4}\right)\left(\frac{n(p-1)}{4} - 1\right)\left(x + \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)^{\frac{n(p-1)-8}{4}} \\
&\leq \frac{C_{GN}}{p+1}\left(\frac{n(p-1)}{4}\right)\left(\frac{n(p-1)}{4} - 1\right)\left(\|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)^{\frac{n(p-1)-8}{4}},
\end{aligned}$$

for $x \in [0, \infty)$. Then by Lemma 3.3.4

$$(P^*)^{-1}(x) \geq \left(\frac{2(p+1)}{C_{GN}\left(\left(\frac{n(p-1)}{4}\right)^2 - \frac{n(p-1)}{4}\right)\left(\|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)^{\frac{n(p-1)-8}{4}}} \right)^{1/2} x^{1/2}$$

$$= \sqrt{2}\left[\left(\frac{16(p+1)}{C_{GN} n(p-1)\left(n(p-1)-4\right)}\right)^{-1/2} \left(\|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)^{\frac{n(p-1)-8}{8}}\right]^{-1} x^{1/2},$$

and (3.3.10) implies (3.3.7) in this case.

Case 3: $n(p-1) > 8$. In this case, we consider $\mathcal{F}_{KE}(u) < 1$ and $\mathcal{F}_{KE}(u) \geq 1$. If $\mathcal{F}_{KE}(u) \geq 1$, then (3.3.6) implies (3.3.7).

Next, if $\mathcal{F}_{KE}(u) < 1$, we observe that in $[0, 1)$ we have

$$\partial_x^2 P^*(x) = -\partial_x^2 P\left(x + \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)$$

$$\leq \frac{C_{GN}}{p+1}\left(\frac{n(p-1)}{4}\right)\left(\frac{n(p-1)}{4} - 1\right)\left(1 + \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)^{\frac{n(p-1)-8}{4}},$$

and by Lemma 3.3.4 for all $x \in [0, P^*(1))$

$$(P^*)^{-1}(x) \geq \sqrt{2}\left[\left(\frac{16(p+1)}{C_{GN} n(p-1)\left(n(p-1)-4\right)}\right)^{-1/2} \left(1 + \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)^{\frac{n(p-1)-8}{8}}\right]^{-1} x^{1/2}.$$

We now note that since $\mathcal{F}_{KE}(u) < 1$, by (3.3.9), $\frac{1}{2}\mathcal{F}_{RK}(u) \in [0, P^*(1))$, so by Lemma 3.3.4, we obtain

$$(P^*)^{-1}\left(\frac{1}{2}\mathcal{F}_{RK}(u)\right) \geq$$

$$\left[\left(\frac{16(p+1)}{C_{GN} n(p-1)\left(n(p-1)-4\right)}\right)^{-1/2} \left(1 + \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)^{\frac{n(p-1)-8}{8}}\right]^{-1} \left(\mathcal{F}_{RK}(u)\right)^{1/2}.$$

Hence, using (3.3.10) as above, we obtain

$$
\begin{aligned}
&(\mathcal{F}_{RK}(u))^{1/2} \\
&\leq \left[\left(\frac{16(p+1)}{C_{GN} n(p-1)\left(n(p-1)-4\right)} \right)^{-1/2} \left(1 + \|\nabla u_Q\|_2^2 \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \right)^{\frac{n(p-1)-8}{8}} \right] \mathcal{F}_{KE}(u) \\
&\leq \left[1 + \left(\frac{16(p+1)}{C_{GN} n(p-1)\left(n(p-1)-4\right)} \right)^{-1/2} \left(1 + \|\nabla u_Q\|_2^2 \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \right)^{\frac{n(p-1)-8}{8}} \right] \mathcal{F}_{KE}(u),
\end{aligned}
$$

as claimed. □

3.3.4 Bound for the term involving trapping

The following lemma builds on the previous lemmas and is a key component of the proof of the blow-up.

Lemma 3.3.5. *If $\left\| |x|^2 B_\tau \right\|_\infty$ is small enough, and $u \in \mathcal{R}_2$, then*

$$
8 \|u\|_2^{2\frac{(1-s_c)}{s_c}} \; Im \int |x| u B_\tau \cdot \overline{\nabla_A u} \; dx \leq \; 2(n(p-1)-4)\, \mathcal{F}_{KE}(u).
$$

Proof. Let $u \in \mathcal{R}_2$, then $\mathcal{F}_{KE}(u) > 0$. We can consider the integral over the union of the sets where $u = 0$ and when $u \neq 0$. If the integral is taken over the set where $u = 0$, then the inequality holds since we are considering the intercritical case. Therefore, we can assume $u \neq 0$. We proceed to bound the integral over that set. Since B_τ is real-valued we have

$$
\begin{aligned}
\left| \mathrm{Im} \int |x| u B_\tau \cdot \overline{\nabla_A u} \; dx \right| &= \left| \int |x| \, B_\tau \cdot \mathrm{Im}\left(u \overline{\nabla_A u} \right) \; dx \right| \\
&\leq \left\| |x|^2 B_\tau \right\|_\infty \left\| \frac{u}{|x|} \right\|_2 \left\| \mathrm{Im}\left(\frac{u}{|u|} \overline{\nabla_A u} \right) \right\|_2 \\
&\leq \frac{2}{n-2} \left\| |x|^2 B_\tau \right\|_\infty \|\nabla_A u\|_2 \left\| \mathrm{Im}\left(\frac{u}{|u|} \overline{\nabla_A u} \right) \right\|_2 ,
\end{aligned}
$$

by Hardy's inequality (3.2.20). Then by Lemma 3.3.1 and (3.3.6) we have

$$
\begin{aligned}
&\|u\|_2^{2\frac{(1-s_c)}{s_c}} \left| 8\mathrm{Im} \int |x| u B_\tau \cdot \overline{\nabla_A u}\, dx \right| \\
&\le \|u\|_2^{\frac{(1-s_c)}{s_c}} \frac{16}{n-2} \left\| |x|^2 B_\tau \right\|_\infty \|\nabla_A u\|_2 \left[\mathcal{F}_{RK}(u)\right]^{1/2} \\
&= \frac{16}{n-2} \left\| |x|^2 B_\tau \right\|_\infty \Big[\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \|\nabla u_Q\|_2^2 + \|u\|_2^{2\frac{(1-s_c)}{s_c}} \|\nabla_A u\|_2^2 \\
&\quad - \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \|\nabla u_Q\|_2^2 \Big]^{1/2} \left[\mathcal{F}_{RK}(u)\right]^{1/2} \\
&\le \frac{16}{n-2} \left\| |x|^2 B_\tau \right\|_\infty \left[\|u_Q\|_2^{\frac{(1-s_c)}{s_c}} \|\nabla u_Q\|_2 \right] \left[\mathcal{F}_{RK}(u)\right]^{1/2} + \frac{16}{n-2} \left\| |x|^2 B_\tau \right\|_\infty \mathcal{F}_{KE}(u) \\
&= I + II.
\end{aligned}
$$

To complete the proof we show

$$
I \le (n(p-1)-4)\mathcal{F}_{KE}(u),
$$

and

$$
II \le (n(p-1)-4)\mathcal{F}_{KE}(u).
$$

By Lemma 3.3.3 there is a constant $C_{p,n}$ such that

$$
\left[\mathcal{F}_{RK}(u)\right]^{1/2} \le C_{p,n}\, \mathcal{F}_{KE}(u).
$$

It follows

$$
\begin{aligned}
I &= \frac{16}{n-2} \left\| |x|^2 B_\tau \right\|_\infty \left[\|u_Q\|_2^{\frac{(1-s_c)}{s_c}} \|\nabla u_Q\|_2 \right] \left[\mathcal{F}_{RK}(u)\right]^{1/2} \\
&\le \left[\frac{16\, C_{p,n}}{(n-2)(n(p-1)-4)} \|u_Q\|_2^{\frac{(1-s_c)}{s_c}} \|\nabla u_Q\|_2 \left\| |x|^2 B_\tau \right\|_\infty \right] \cdot \left[(n(p-1)-4)\mathcal{F}_{KE}(u) \right].
\end{aligned}
$$

So we obtain the desired bound if

$$
\left\| |x|^2 B_\tau \right\|_\infty \le \frac{(n-2)(n(p-1)-4)}{16\, C_{p,n} \|u_Q\|_2^{\frac{(1-s_c)}{s_c}} \|\nabla u_Q\|_2}.
$$

Next, if

$$\left\||x|^2 B_\tau\right\|_\infty \leq \frac{(n-2)(n(p-1)-4)}{16},$$

then

$$II = \frac{16}{n-2}\left\||x|^2 B_\tau\right\|_\infty \mathcal{F}_{KE}(u) \leq (n(p-1)-4)\mathcal{F}_{KE}(u),$$

as needed. $\square$

3.4 Proof of Theorem 3.1.3

We break the proof of the theorem into two main parts. We begin with the global existence.

3.4.1 Proof of Theorem 3.1.3: global existence

The reasoning for global existence is the same as in [34]. The details are as follows. The conservation of mass and energy, (3.2.14), and (3.1.12) imply for all time

$$P(\|\nabla_A u(t)\|_2^2 \|u(t)\|_2^{2\frac{(1-s_c)}{s_c}}) \leq E_A[u(t)]\, \|u(t)\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q]\, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$

and hence,

$$\|\nabla_A u(t)\|_2^2 \|u(t)\|_2^{2\frac{(1-s_c)}{s_c}} \neq \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}. \tag{3.4.1}$$

Then, if (3.1.13) holds, by continuity of $\|\nabla_A u(t)\|_2$ in t, we must have (3.1.14). Then the standard blow-up criterion implies that the solution must exist for all time.

3.4.2 Proof of Theorem 3.1.3: blow-up

Now, suppose (3.1.15) holds. Then (3.1.16) holds by (3.4.1) and the continuity of $\|\nabla_A u(t)\|_2$. Next, from the virial identity (3.2.19) we get

$$\|u\|_2^{2\frac{(1-s_c)}{s_c}} \partial_t^2 \int |x|^2|u|^2 dx = 4n(p-1)E_A[u]\, \|u\|_2^{2\frac{(1-s_c)}{s_c}} - 2(n(p-1)-4)\, \|\nabla_A u\|_2^2 \|u\|_2^{2\frac{(1-s_c)}{s_c}}$$

$$+ \|u\|_2^{2\frac{(1-s_c)}{s_c}} 8\mathrm{Im} \int |x| u B_\tau \cdot \overline{\nabla_A u} dx.$$

From (3.2.9) and (3.2.10) we have

$$4n(p-1)E[u_Q]\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} - 2(n(p-1)-4)\,\|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} = 0, \tag{3.4.2}$$

which allows us to rephrase the virial identity as

$$\begin{aligned}\|u\|_2^{2\frac{(1-s_c)}{s_c}}\,\partial_t^2\int |x|^2|u|^2dx &= 4n(p-1)\left(E_A[u]\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} - E[u_Q]\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)\\ &\quad - 2(n(p-1)-4)\left(\|\nabla_A u\|_2^2\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} - \|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)\\ &\quad + 8\,\|u\|_2^{2\frac{(1-s_c)}{s_c}}\,\mathrm{Im}\int |x|uB_\tau\cdot\overline{\nabla_A u}dx.\end{aligned}$$

From (3.1.12) we have there exists $\varepsilon > 0$ such that for all time

$$4n(p-1)\left(\|u\|_2^{2\frac{(1-s_c)}{s_c}}\,E[u] - \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\,E[u_Q]\right) < -\varepsilon.$$

Then by definition of $\mathcal{F}_{KE}(u)$, it follows

$$\|u\|_2^{2\frac{(1-s_c)}{s_c}}\,\partial_t^2\int |x|^2|u|^2dx < 8\,\|u\|_2^{2\frac{(1-s_c)}{s_c}}\,\mathrm{Im}\int |x|uB_\tau\cdot\overline{\nabla_A u}dx - 2(n(p-1)-4)\,\mathcal{F}_{KE}(u) - \varepsilon.$$

From Lemma 3.3.5 we have

$$8\,\|u\|_2^{2\frac{(1-s_c)}{s_c}}\,\mathrm{Im}\int |x|uB_\tau\cdot\overline{\nabla_A u}\;\le\;2(n(p-1)-4)\,\mathcal{F}_{KE}(u),$$

so then the blow-up follows by the classical virial identity argument.

3.5 Proofs with the electric potential

In this section we prove Theorem 3.1.9. We consider the most interesting case, namely, when $A_{0,-}\neq 0$, and $(\partial_r A_0)_- \neq 0$. If $A_{0,-}=0$, and $(\partial_r A_0)_-=0$, the statements of the theorems, and the proofs would be essentially identical to the purely magnetic case. It will be clear from the proof below why this is the case.

The main ideas are as follows. To obtain global existence, we bound the energy similarly as in (3.2.12)

$$
\begin{aligned}
E_A[u] \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}} &\geq \frac{1}{2} \left\|\nabla_A u\right\|_2^2 \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}} - \frac{C_{GN}}{p+1} \left(\left\|\nabla_A u\right\|_2^2 \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}} \right)^{\frac{n(p-1)}{4}} \\
&\quad - \frac{1}{2} \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}} \int A_{0,-} \left|u\right|^2 dx \\
&\geq \frac{1}{2} \left\|\nabla_A u\right\|_2^2 \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}} - \frac{C_{GN}}{p+1} \left(\left\|\nabla_A u\right\|_2^2 \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}} \right)^{\frac{n(p-1)}{4}} \\
&\quad - \frac{a_0}{2} \left\|\nabla_A u\right\|_2^2 \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}},
\end{aligned}
\tag{3.5.1}
$$

by (3.2.21). Based on this, we modify the definition of the minimal energy function to be

$$
F(x) = \frac{1}{2}(1-a_0)x - \frac{C_{GN}}{p+1} x^{\frac{n(p-1)}{4}}.
$$

A computation shows that F achieves the maximum at $(1-a_0)^{\frac{2}{s_c(p-1)}} \left\|\nabla u_Q\right\|_2^2 \left\|u_Q\right\|_2^{2\frac{(1-s_c)}{s_c}}$ with the value of $(1-a_0)^{1+\frac{2}{s_c(p-1)}} E[u_Q] \left\|u_Q\right\|_2^{2\frac{(1-s_c)}{s_c}}$ (c.f. (3.2.17) and (3.2.16), respectively).

It follows that if (3.1.19) and (3.1.21) hold (or if (3.1.26) and (3.1.27) hold), then again by the continuity of $\left\|\nabla_A u(t)\right\|_2$ in t, (3.1.22) (and (3.1.28)) holds for all time, and the solution exists globally. This concludes the proof of global existence for Theorem 3.1.9 and the proof of Corollary 3.1.11.

3.5.1 Proof of blow-up in Theorem 3.1.9

On the other hand, if (3.1.23) holds, then again by continuity and (3.5.1), (3.1.25) follows. To obtain blowup for finite variance, and given the smallness condition (3.1.24), the virial identity, (3.2.19), gives us

$$
\begin{aligned}
\|u\|_2^{2\frac{(1-s_c)}{s_c}} \partial_t^2 \int |x|^2|u|^2dx &= 4n(p-1)E_A[u]\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} - 2(n(p-1)-4)\,\|\nabla_A u\|_2^2\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} \\
&\quad - 2n(p-1)\int A_0\,|u|^2\,dx\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} \\
&\quad - 4\int_{\mathbb{R}^n} |x|\,\partial_r A_0\,|u|^2\ dx\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} \\
&\quad + 8\int_{\mathbb{R}^n} |x|\,\mathcal{I}m(uB_\tau\cdot\overline{\nabla_A u})\ dx\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} \\
&\leq 4n(p-1)E_A[u]\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} - 2(n(p-1)-4)\,\|\nabla_A u\|_2^2\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} \\
&\quad + 2n(p-1)\int A_{0,-}\,|u|^2\,dx\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} \\
&\quad + 4\int_{\mathbb{R}^n} |x|\,(\partial_r A_0)_-\,|u|^2\ dx\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} \\
&\quad + 8\int_{\mathbb{R}^n} |x|\,\mathcal{I}m(uB_\tau\cdot\overline{\nabla_A u})\ dx\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} .
\end{aligned}
$$

Therefore, by (3.2.21), (3.2.22), (3.2.23)

$$
\begin{aligned}
\|u\|_2^{2\frac{(1-s_c)}{s_c}} \partial_t^2 \int |x|^2|u|^2dx &\leq 4n(p-1)E_A[u]\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} \\
&\quad - 2(n(p-1)-4)\,\|\nabla_A u\|_2^2\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} \\
&\quad + (2n(p-1)a_0 + 4a_1 + 8b_0)\,\|\nabla_A u\|_2^2\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} ,
\end{aligned}
$$

and by (3.1.24), when collected, all the coefficients of $\|\nabla_A u\|^2$ above add up to be negative, and we can use that by (3.1.23), $\|\nabla_A u\|_2^2\,\|u\|_2^{2\frac{(1-s_c)}{s_c}} > (1-a_0)^{\frac{2}{s_c(p-1)}}\,\|\nabla u_Q\|_2^2\,\|u_Q\|_2^{\frac{2(1-s_c)}{s_c}}$. This gives

$$\|u\|_2^{2\frac{(1-s_c)}{s_c}}\ \partial_t^2 \int |x|^2|u|^2dx \le 4n(p-1)E_A[u]\,\|u\|_2^{2\frac{(1-s_c)}{s_c}}$$

$$- \Bigg(2(n(p-1)-4)(1-a_0)^{\frac{2}{s_c(p-1)}}\,\|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}$$

$$- (2n(p-1)a_0+4a_1+8b_0)(1-a_0)^{\frac{2}{s_c(p-1)}}\,\|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\Bigg).$$

Next, by (3.1.19), we have there exists $\varepsilon > 0$ such that

$$\|u\|_2^{2\frac{(1-s_c)}{s_c}}\ \partial_t^2 \int |x|^2|u|^2dx \le -\varepsilon + 4n(p-1)(1-a_0)^{1+\frac{2}{s_c(p-1)}}E[u_Q]\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}$$

$$- 4(2a_0+a_1+2b_0)x_*$$

$$- 2(n(p-1)-4)(1-a_0)^{\frac{2}{s_c(p-1)}}\,\|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}$$

$$+ (2n(p-1)a_0+4a_1+8b_0)(1-a_0)^{\frac{2}{s_c(p-1)}}\,\|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}.$$

Then (3.4.2) allows us to make some cancellations and obtain

$$\|u\|_2^{2\frac{(1-s_c)}{s_c}}\ \partial_t^2 \int |x|^2|u|^2dx < -\varepsilon - 2a_0(n(p-1)-4)(1-a_0)^{\frac{2}{s_c(p-1)}}\,\|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}$$

$$- 4(2a_0+a_1+2b_0)x_*$$

$$+ (2n(p-1)a_0+4a_1+8b_0)(1-a_0)^{\frac{2}{s_c(p-1)}}\,\|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}$$

$$= -\varepsilon,$$

as needed.

Chapter 4

Global Dynamics of Nonlinear Schrödinger Equations Above the Ground State

4.1 Introduction

We consider the intercritical, focusing, nonlinear Schrödinger equation,

$$\begin{cases} iu_t + \Delta u = -|u|^{p-1}u, \\ u(x,t) = u_0(x), \end{cases} \tag{NLS}$$

with unknown $u : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$. The equation has scaling invariance

$$u \mapsto \lambda^{2/(p-1)} u(\lambda x, \lambda^2 t), \qquad \lambda > 0,$$

and we consider $N \geq 2$ [†] and p such that this scaling is mass supercritical and energy subcritical. That is, defining

$$s_c = \frac{N}{2} - \frac{2}{p-1},$$

the scaling is critical in $\dot{H}^{s_c}(\mathbb{R}^N)$ with

$$0 < s_c < 1.$$

We refer to solutions of NLS simply as solutions, and explicitly state when we discuss solutions of different equations.

Here, for initial data lying below a certain threshold, global dynamics are very well understood. Above this threshold, it is known that global dynamics are more complex, but apart from

[†] We establish new control of solutions based on angular momentum. In the case $N = 1$, angular momentum is degenerate.

this fact, much remains unknown. We will elaborate on the difficulties and overview existing results below, after we set up some necessary notation.

In this chapter, all function spaces will be spatial, over $\mathbb{R}^N$. We will omit the base space from description of our function spaces, for example writing L^q for $L^q(\mathbb{R}^N)$, and we simplify notation for norms, writing $\|\cdot\|_q$ for $\|\cdot\|_{L^q}$.

We consider H^1 initial data, for which there is extensive classical theory (e.g. [8, 50]). For initial data $u_0 \in H^1$, the equation is locally well-posed. Given a solution u, the map $t \mapsto \|\nabla u\|_2^2$ is continuous. Given a maximal interval of existence, (T_*, T^*), for a solution u, T^* is finite if and only if $\lim_{t\uparrow T^*} \|\nabla u(t)\|_2^2 = \infty$, in which case we say u blows-up in finite positive time. Similarly, T_* is finite if and only if $\lim_{t\downarrow T_*} \|\nabla u(t)\|_2^2 = \infty$, in which case we say u blows-up in finite negative time.

Furthermore, a solution u satisfies several conservation laws on the full interval of existence. These include

$$\begin{aligned}
\text{Mass:} \quad & \|u(t)\|_2^2 = \|u_0\|_2^2, \\
\text{Energy:} \quad & E[u](t) \coloneqq \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} = E[u_0], \\
\text{Linear Momentum:} \quad & \mathbf{P}[u](t) \coloneqq \int \mathcal{P}[u(x,t)]\,dx = \mathbf{P}[u_0], \\
\text{Angular Momentum:} \quad & \mathbf{M}[u](t) \coloneqq \int \mathcal{P}[u(x,t)] \wedge \mathbf{x}\,dx = \mathbf{M}[u_0],
\end{aligned}$$

where

$$\mathcal{P}[u] \coloneqq 2\,\mathrm{Im}\,[\overline{u}\,\nabla u]$$

is the momentum density.

In our definition of angular momentum, the wedge of vectors

$$\langle v_1, \ldots, v_N\rangle \wedge \langle w_1, \ldots, w_N\rangle$$

can be thought of as a vector with $\binom{N}{2}$ components, given by

$$v_j w_k - v_k w_j$$

for $j < k$. In the case of 3 dimensions, the angular momentum is typically defined by the vector

$$\mathbf{M}^*[u](t) \coloneqq -\int \mathbf{x} \times \mathcal{P}[u(x,t)]\, dx.$$

There is an equivalence between these definitions. In particular, the vector $\mathbf{M}^*[u](t)$ has the same components as $\mathbf{M}[u](t)$ up to a sign, and in 3 dimensions, our results can be formulated with either definition with no change to the statement of the theorems.

Conservation of angular momentum in general dimension follows component-wise in the same manner as for three dimensions.

4.1.1 The soliton solution

As shown in [53] (see also [34]), NLS admits a soliton solution

$$u_Q(x,t) = e^{i\lambda t} Q(\alpha x)$$

where $Q : \mathbb{R}^N \to \mathbb{R}_{\geq 0}$ is the ground state solution for the equation

$$\frac{N(p-1)}{4} \Delta Q - \left(1 - \frac{(N-2)(p-1)}{4}\right) Q + Q^p = 0,$$

$\alpha = \frac{\sqrt{N(p-1)}}{2}$, and $\lambda = 1 - \frac{(N-2)(p-1)}{4}$. The Gagliardo–Nirenberg inequality,

$$\|u\|_{p+1}^{p+1} \leq C_{GN} \|\nabla u\|_2^{\frac{N(p-1)}{2}} \|u\|_2^{2-\frac{(N-2)(p-1)}{2}}, \tag{4.1.1}$$

which holds for $u \in H^1$, is optimized by Q, and has sharp constant

$$C_{GN} = \frac{p+1}{2\|Q\|_2^{p-1}}.$$

Furthermore, Q decays exponentially (see [4, 51]). By construction, Q satisfies

$$\|\nabla Q\|_2^2 = \|Q\|_2^2,$$

and this together with Pohozaev's identity implies

$$\|\nabla Q\|_2^2 = \frac{2}{p+1}\|Q\|_{p+1}^{p+1}.$$

The soliton u_Q also optimizes the Gagliardo–Nirenberg inequality, so that

$$\|u_Q\|_{p+1}^{p+1} = C_{GN}\|\nabla u_Q\|_2^{\frac{N(p-1)}{2}}\|u_Q\|_2^{2-\frac{(N-2)(p-1)}{2}},$$

and by computation using the identities of Q, we obtain identities

$$E[u_Q] = \frac{N(p-1)-4}{2N(p-1)}\|\nabla u_Q\|_2^2, \tag{4.1.2}$$

$$C_{GN} = \frac{2(p+1)}{N(p-1)\left(\|\nabla u_Q\|_2^{s_c}\|u_Q\|_2^{1-s_c}\right)^{p-1}}. \tag{4.1.3}$$

4.1.2 Variance and other important quantities

At a given time t, solutions have spatial variance

$$V[u](t) \coloneqq \int |x|^2|u(x,t)|^2\,dx.$$

If $V[u_0]$ is finite, then $V[u](t)$ is finite for the full interval of existence and satisfies the well-known identities [50]

$$V_t[u](t) = 2\int \mathbf{x}\cdot\ \mathcal{P}\left[u(x,t)\right]\,dx,$$

$$V_{tt}[u](t) = 8\|\nabla u(t)\|_2^2 - \frac{4N(p-1)}{p+1}\|u(t)\|_{p+1}^{p+1},$$

where the identity for $V_{tt}[u](t)$ is the classical virial identity.

By either direct computation using properties of u_Q stated above, or else the stationary properties of the soliton,

$$8\|\nabla u_Q\|_2^2 - \frac{4N(p-1)}{p+1}\|u_Q\|_{p+1}^{p+1} = 0. \tag{4.1.4}$$

In what follows, given initial data $u_0 \in H^1$, expressions such as $V_t[u_0]$ will refer to $V_t[u](0)$ where u is the solution with initial data u_0.

The quantity $\|\nabla u(t)\|_2^2$ will be referred to as kinetic energy. We also consider the quantity $\|\nabla|u|(t)\|_2^2$, which will be referred to as Kato kinetic energy. An identity first noted in [15] and presented in Lemma 4.2.2 demonstrates an aspect of the connection between Kato kinetic energy and momentum. This connection has been exploited to obtain blow-up results for electromagnetic nonlinear Schrödinger equations [15], and will be used in our main result. Also significant will be the Kato energy of a solution, which is defined by

$$E_K[u](t) \coloneqq \frac{1}{2}\left\|\nabla\left|u(t)\right|\right\|_2^2 - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1},$$

and is not a conserved quantity.

Kato kinetic energy and Kato energy are introduced in [15] for the electromagnetic nonlinear Schrödinger equation, and we do not know of its presence in prior literature for NLS. However, as we demonstrate in this chapter, Kato kinetic energy reveals important structure regarding the mass profile, $|u_0|$, of initial data, and we will contextualize many previous results in terms of Kato energy and Kato kinetic energy. For existing results, this contextualization is a natural alternative to the originally presented context, and does not modify the results, as is discussed further in Remark 4.1.5.

4.1.3 At and below the ground state

Questions of global existence and blow-up at and below the ground state have been extensively studied (e.g. [7, 10, 19, 20, 28, 31, 32, 34–36]). We review certain results which are relevant to our work and are established, in their current generality, in [32] and [7].

With respect to the scale invariant quantities

$$\text{Mass-Energy}: \quad E[u(t)]\,\|u(t)\|_2^{2\frac{(1-s_c)}{s_c}},$$

$$\text{Mass-Kinetic Energy}: \quad \|\nabla u(t)\|_2^2\,\|u(t)\|_2^{2\frac{(1-s_c)}{s_c}},$$

the soliton u_Q provides a threshold for global existence and blow-up. We say that initial data u_0

is below the ground state if u_0 has smaller mass-energy than the soliton u_Q, so

$$E[u_0]\,\|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q]\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}.$$

The study of initial data at and below the ground state has yielded very satisfactory results. Global for mass and energy critical equations in works such as [37, 53] influenced the work on the intercritical equation which we describe below. Holmer and Roudenko obtained a dichotomy result in [34] which, in a sense, interpolated between the results of [37] and [53] for the mass and energy critical equations. The full resolution for the case of finite variance,[†] achieved in theorems of Duyckaerts, Holmer, and Roudenko [19], Guevara [32], and Campos, Farah, and Roudenko [7].

Theorem 4.1.1 (Below the Ground State [32])**.** *Assume,* $0 < s_c < 1$*, and* $u_0 \in H^1$ *with* $|x|u_0 \in L^2$ *such that*

$$E[u_0]\,\|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q]\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}.$$

- *If*
$$\|\nabla u_0\|_2^2\,\|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < \|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$
then the solution u exists globally and scatters in both time directions.

- *If*
$$\|\nabla u_0\|_2^2\,\|u_0\|_2^{2\frac{(1-s_c)}{s_c}} > \|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$
then the solution u blows-up in finite time in both time directions.

Remark 4.1.2. The energy assumption implies that u_0 can not satisfy

$$\|\nabla u_0\|_2^2\,\|u_0\|_2^{2\frac{(1-s_c)}{s_c}} = \|\nabla u_Q\|_2^2\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}.$$

The above theorem shows that for finite variance initial data below the ground state, there is a true dichotomy with global and blow-up solutions classified by mass-kinetic energy.

[†] The case of infinite variance, while having well developed theory, is to some extent not fully resolved. This is due to remaining questions regarding an a priori infinite time blow-up scenario (see [36]).

Theorem 4.1.3 ('Threshold Solutions': At the Ground State [7]). *Assume $0 < s_c < 1$. The theory of solutions at the ground state can be presented in two parts.*

Part 1:

There exist radial solutions u_Q^+ and u_Q^- to NLS in H^1 satisfying the following properties

- $\left\|u_Q^-(t)\right\|_2^2 = \left\|u_Q^+(t)\right\|_2^2 = \|u_Q\|_2^2, \quad E[u_Q^-](t) = E[u_Q^+](t) = E[u_Q],$
- $\left\|\nabla u_Q^-(0)\right\|_2^2 < \|\nabla u_Q\|_2^2 < \left\|\nabla u_Q^+(0)\right\|_2^2,$
- *In positive time, u_Q^- converges to the soliton u_Q in the sense that there are $C, \alpha > 0$ such that*

  $$\left\|u_Q^-(t) - u_Q\right\|_{H^1} \leq Ce^{-\alpha t},$$

  and in negative time u_Q^- exists globally and scatters.

- *In positive time, u_Q^+ converges to the soliton u_Q in the sense that there are $C, \alpha > 0$ such that*

  $$\left\|u_Q^+(t) - u_Q\right\|_{H^1} \leq Ce^{-\alpha t},$$

  and u_Q^+ blows-up in finite negative time.

Part 2:

Suppose $u_0 \in H^1$ with $|x|u_0 \in L^2$ satisfies

$$E[u_0] \, \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} = E[u_Q] \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}.$$

- *If*

  $$\|\nabla u_0\|_2^2 \, \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < \|\nabla u_Q\|_2^2 \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$

  then either

  * *the solution satisfies $u = u_Q^-$ up to symmetry, or*
  * *the solution exists globally and scatters in both time directions.*

- *If*

$$\|\nabla u_0\|_2^2 \, \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} = \|\nabla u_Q\|_2^2 \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$

then the solution satisfies $u = u_Q$ *up to symmetries.*

- *If*

$$\|\nabla u_0\|_2^2 \, \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} > \|\nabla u_Q\|_2^2 \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$

then either

  * *the solution satisfies* $u = u_Q^+$ *up to symmetry, or*
  * *the solution blows-up in finite time in both time directions.*

This theorem shows that once the threshold is reached, the dichotomy seen under the ground state begins to break down, where in addition to the soliton, there is a single special solution as an exception to scattering and a single special solution as an exception to blow-up. In all, there are five possibilities for behavior of solutions at the ground state. Later, we will see that above the ground state the classification property initial data below the ground state enjoys breaks down even further.

4.1.4 Above the ground state

Note that at and below the ground state, for the case of finite variance, the theorems of Guevara [32], and of Campos, Farah, and Roudenko [7], completed the classification of initial data based on general global behavior of solutions.

Above the ground state, there is considerably less known. Here we will review relevant results.[†]

[†] Some of these results are proved for specific NLS equations, commonly the 3 dimensional cubic equation. At times these restrictions seem to be for convenience, with arguments generalizing to more general intercritical equations, though there are times when these restrictions are important for arguments. Here we wish to review what has been done for any case of intercritical equation, so for discussion of background results we will avoid mention of such restrictions.

We begin with a remark concerning linear momentum. Given $v \in \mathbb{R}^N$, the Galilean transformation

$$u(x,t) \mapsto u_v(x,t) = e^{ix\cdot v}e^{-it|v|^2}u(x-2vt,t)$$

allows us to transform between inertial reference frames, with u_v being a solution if u is, and with

$$\mathbf{P}[u_v] = \mathbf{P}[u] + 2v.$$

For nontrivial u, the Galilean transformation preserves $\|u(t)\|_2^2$, and by choosing v with large enough magnitude, $E[u_v](t)$ can be taken to be arbitrarily large. This allows for classification of global behavior for certain solutions arbitrarily far above the ground state by taking initial data below the ground state and performing an appropriate Galilean transformation. In this chapter, we only consider results which are not vacuous under the assumption on linear momentum, $\mathbf{P}[u_0] = 0$.

As stated above, there are existing results which indicate that the problem above the ground state is more delicate than below the ground state. Nakanishi and Schlag [46] identify at least nine different possibilities for global dynamics, each attained by infinitely many initial data, including scattering in one time direction and blow-up in the other (see also [10, 21], for explicit examples).

It is natural to separately consider results for initial data which is close to the ground state, and results where initial data is allowed to be far from the ground state.

There are two results we know of imposing some restriction on distance of initial data from the ground state. Nakanishi and Schlag [46] obtain results for initial data which is close to the ground state in mass-energy. Beceanu [2] obtains obtains results for certain initial data which is close to the ground state in norm.

We now review what is known for initial data permitted to be arbitrarily far from the ground state. Given initial data $u_0 \in H^1$ with $|x|u_0 \in L^2$, a rewriting of the standard virial identity blow-up argument for positive energy implies that if

$$\left[E[u_0] - \frac{|V_t[u_0]|^2}{32V[u_0]}\right] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} \leq 0, \tag{4.1.5}$$

and

$$V_t[u_0] < 0,$$

then the solution u with initial data u_0 blows-up in finite positive time (see [53, Theorem 4.2], [50]). Duyckaerts and Roudenko [21] improve on this by establishing results after raising the right hand side of the inequality (4.1.5) from 0 to the ground state. These results include the establishment of both solutions which are global in positive time, and solutions which blow-up in finite positive time. We now recount this result in the intercritical case [21].

Theorem 4.1.4. *Suppose* $0 < s_c < 1$, *and consider* $u_0 \in H^1$ *with* $|x|u_0 \in L^2$. *Furthermore, assume*

$$\left[E[u_0] - \frac{|V_t[u_0]|^2}{32V[u_0]}\right] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} \leq E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}. \tag{4.1.6}$$

- *(Existence and Scattering) If*

$$\|\nabla |u_0|\|_2^2 \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \tag{4.1.7}$$

  and

$$V_t(0) \geq 0,$$

  then the solution u *with initial data* u_0 *exists globally forward in time and scatters forward in time.*

- *(Blow-Up) If*

$$\|\nabla |u_0|\|_2^2 \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} > \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \tag{4.1.8}$$

  and

$$V_t(0) \leq 0,$$

  then the solution u *with initial data* u_0 *blows-up in finite positive time.*

Remark 4.1.5. This result is formulated somewhat differently, but equivalently, in its original paper. In conditions (4.1.7) and (4.1.8), the original paper considers the quantity $\|\cdot\|_{p+1}^{p+1}$ where we consider Kato kinetic energy. By a computation (see [15, 'Double Cut Lemma'] as well as the proofs of Corollary 4.2.8 or Theorem 4.1.10 in this chapter), if $w \in H^1$ with $E_K[w] \|w\|_2^{2\frac{(1-s_c)}{s_c}} \leq E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}$, then

$$\|w\|_{p+1}^{p+1}\|w\|_2^{2\frac{(1-s_c)}{s_c}} < \|u_Q\|_{p+1}^{p+1}\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \text{ and } \|\nabla|w|\|_2^2\|w\|_2^{2\frac{(1-s_c)}{s_c}} < \|\nabla u_Q\|_2^2\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \tag{4.1.9}$$

are equivalent, and

$$\|w\|_{p+1}^{p+1}\|w\|_2^{2\frac{(1-s_c)}{s_c}} > \|u_Q\|_{p+1}^{p+1}\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \text{ and } \|\nabla|w|\|_2^2\|w\|_2^{2\frac{(1-s_c)}{s_c}} > \|\nabla u_Q\|_2^2\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \tag{4.1.10}$$

are equivalent. In fact, if $E[w]\|w\|_2^{2\frac{(1-s_c)}{s_c}} \leq E[u_Q]\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}$, then all of

$$\|w\|_{p+1}^{p+1}\|w\|_2^{2\frac{(1-s_c)}{s_c}} < \|u_Q\|_{p+1}^{p+1}\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$

$$\|\nabla|w|\|_2^2\|w\|_2^{2\frac{(1-s_c)}{s_c}} < \|\nabla u_Q\|_2^2\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$

$$\|\nabla w\|_2^2\|w\|_2^{2\frac{(1-s_c)}{s_c}} < \|\nabla u_Q\|_2^2\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$

are equivalent, and likewise for the reverse inequalities.

The energy assumption, (4.1.6) implies, using Lemma 4.2.5 in this chapter, that

$$E_K[u_0]\|u_0\|_2^{2\frac{(1-s_c)}{s_c}} \leq E[u_Q]\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$

and hence the equivalences (4.1.9) and (4.1.10) apply. This also relates to the connection between quantities used for below ground state and above ground state results as noted in [21].

Remark 4.1.6. The improvement that Theorem 4.1.4 makes on the positive energy virial identity result, improving (4.1.5) to (4.1.6), has a parallel structure to the improvement that Theorems 4.1.1 and 4.1.3 jointly make by raising the negative energy blow-up condition

$$E[u_0]\|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < 0$$

to

$$E[u_0]\|u_0\|_2^{2\frac{(1-s_c)}{s_c}} \leq E[u_Q]\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}.$$

Theorem 4.1.4 provides the most important context for the main global existence and blow-up results of this chapter. For our result in one time direction, Theorem 4.1.10 below, we extend Theorem 4.1.4 by obtaining results after further weakening (4.1.6) to

$$\left[E[u_0] - \frac{|V_t[u_0]|^2 + |2\mathbf{M}[u_0]|^2}{32V[u_0]}\right] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} \leq E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} .$$

Incorporation of the term

$$\frac{|2\mathbf{M}[u_0]|^2}{32V[u_0]} \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} \tag{4.1.11}$$

causes this inequality to provide a weaker condition, which allows us to establish results for initial data which is, in a sense, further from the ground state.

In fact, we find that the term (4.1.11) provides a control of solutions which is less time dependent than

$$\frac{|V_t[u_0]|^2}{32V[u_0]} \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} ,$$

especially for the case of global existence. In Theorem 4.1.7, consideration of the term (4.1.11) on its own produces a global existence result which is a direct extension of global existence below the ground state, weakening the energy assumption from

$$E[u_0] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} ,$$

to

$$\left[E[u_0] - \frac{|2\mathbf{M}[u_0]|^2}{32V[u_0]}\right] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} , \tag{4.1.12}$$

with no further restrictions. In particular, this is a global result which has no dependence on time direction, and hence attains one of the goals of this chapter. In addition, this global result places no restriction on $V_t[u_0]$. We also use the energy condition (4.1.12) to obtain a new result concerning blow-up in two time directions.

For initial data far from the ground state, in addition to the results of [21] discussed above, there have been a few other results for blow-up, as well as one other result concerning global existence. Both [21] and [33] present new criteria for blow-up. In [33], Holmer, Platte, and Roudenko also construct new blow-up solutions for the 3d cubic equation using numerical computations. This shows that there is initial data with mass-Kato kinetic energy below the ground state which blows-up in at least one time direction. The only other result, not discussed above, for initial data above the ground state regards global existence, and can be found in [10], where Cazenave and Weissler demonstrate a phase modulation which allows for construction of solutions of arbitrarily large mass-energy (and arbitrarily large mass-Kato kinetic energy) which exist globally in at least one time direction.

The result of [10] demonstrates that there is initial data of any mass-Kato kinetic energy which exists and even scatters forward in time. One can ask if there is a similar result regarding blow-up. Surprisingly, even though reversing the phase modulation from [10] produces attractive candidates for finite time blow-up, there has only been a partial result in this direction, which is the construction of blow-up solutions found in [33].

4.1.5 Main results

We are now prepared to state our main theorems. We begin with our result regarding global existence and blow-up in both time directions.

Theorem 4.1.7. *Suppose $N \geq 2$, and $0 < s_c < 1$, and consider initial data $u_0 \in H^1$ with $|x|u_0 \in L^2$, such that*

$$E[u_0] \left\|u_0\right\|_2^{2\frac{(1-s_c)}{s_c}} > E[u_Q] \left\|u_Q\right\|_2^{2\frac{(1-s_c)}{s_c}}, \tag{4.1.13}$$

and

$$\left[E[u_0] - \frac{|2\boldsymbol{M}[u_0]|^2}{32V[u_0]}\right] \left\|u_0\right\|_2^{2\frac{(1-s_c)}{s_c}} \leq E[u_Q] \left\|u_Q\right\|_2^{2\frac{(1-s_c)}{s_c}}. \tag{4.1.14}$$

- *(Existence) If*

$$\left\|\nabla |u_0|\right\|_2^2 \left\|u_0\right\|_2^{2\frac{(1-s_c)}{s_c}} < \left\|\nabla u_Q\right\|_2^2 \left\|u_Q\right\|_2^{2\frac{(1-s_c)}{s_c}} \tag{4.1.15}$$

  then the solution u with initial data u_0 exists for all time and scatters in H^1 in both time directions.

- *(Blow-Up) If*

$$\|\nabla|u_0|\|_2^2 \, \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} > \|\nabla u_Q\|_2^2 \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \tag{4.1.16}$$

and either,

  * $V_t[u_0] = 0$ *and*

$$\frac{|2\boldsymbol{M}[u_0]|^2}{32V[u_0]} \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} > P^*\left(\frac{8}{N(p-1)-4}\left(E[u_0]\,\|u_0\|_2^{2\frac{(1-s_c)}{s_c}} - E[u_Q]\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)\right)$$
$$+\left(E[u_0]\,\|u_0\|_2^{2\frac{(1-s_c)}{s_c}} - E[u_Q]\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right), \tag{4.1.17}$$

  where $P^*(x)$ is defined in Lemma 4.2.6, or

  * *for some $\lambda > 1$ small enough so that*

$$\left[E[u_0] - \frac{|2\boldsymbol{M}[u_0]|^2}{32\lambda V[u_0]}\right] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q]\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},$$

  u_0 satisfies the nonlinear condition

$$\frac{|2\boldsymbol{M}[u_0]|^2}{32V[u_0]} \|u_0\|_2^{2\frac{(1-s_c)}{s_c}}$$
$$> \lambda\, G\left(E[u_0]\,\|u_0\|_2^{2\frac{(1-s_c)}{s_c}} - E[u_Q]\,\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},\ \frac{|V_t[0]|^2}{(\lambda-1)V[0]}\,\|u_0\|_2^{2\frac{(1-s_c)}{s_c}}\right) \tag{4.1.18}$$

  where

$$G(x,y) = P^*\left(\frac{1}{2N(p-1)-8}\,(16x+y)\right) + x,$$

then the solution u blows-up in finite time in both time directions.

Remark 4.1.8. There exist infinitely many initial data with $E[u_0]\,\|u_0\|_2^{2\frac{(1-s_c)}{s_c}}$ arbitrarily large and with $\frac{|2\mathbf{M}[u_0]|^2}{32V[u_0]}\,\|u_0\|_2^{2\frac{(1-s_c)}{s_c}}$ also sufficiently large for u_0 to satisfy the hypotheses of Theorem 4.1.7.

Remark 4.1.9. There exists scattering theory for initial data which is small in the critical norm. One can construct data which is small in critical norm and has large mass-energy and mass-kinetic energy by choosing suitable $\widehat{u_0}$. Our result and previous work above the ground state (e.g. [21]) do not require smallness of critical norm and the bound on mass-Kato kinetic energy does not impose a bound on the critical norm.

For discussion of the function P^* and conditions (4.1.17) and (4.1.18), see Remark 4.1.13 and Lemma 4.2.6.

We also obtain the following result for behavior in one time direction.

Theorem 4.1.10. *Suppose $N \geq 2$, and $0 < s_c < 1$, and consider initial data $u_0 \in H^1$ with $|x|u_0 \in L^2$, such that*

$$E[u_0]\, \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} > E[u_Q]\, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \tag{4.1.19}$$

and

$$\left[E[u_0] - \frac{|V_t[u_0]|^2 + |2\boldsymbol{M}[u_0]|^2}{32V[u_0]}\right] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} \leq E[u_Q]\, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}. \tag{4.1.20}$$

- *(Existence) If*

$$\|\nabla |u_0|\|_2^2\, \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < \|\nabla u_Q\|_2^2\, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \tag{4.1.21}$$

  and

$$V_t[u_0] \geq 0, \tag{4.1.22}$$

  then the solution u with initial data u_0 exists for all positive time and scatters in H^1 forward in time.

- *(Blow-Up) If*

$$\|\nabla |u_0|\|_2^2\, \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} > \|\nabla u_Q\|_2^2\, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \tag{4.1.23}$$

  and

$$V_t[u_0] < 0, \tag{4.1.24}$$

  then if u_0 satisfies the nonlinear condition

$$\frac{|2\boldsymbol{M}[u_0]|^2}{32V[u_0]}\, \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} > F\left(\left(E[u_0] - \frac{|V_t[u_0]|^2}{32V[u_0]}\right) \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} - E[u_Q]\, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right) \tag{4.1.25}$$

  where

$$F(x) = \begin{cases} x + P^*\left(\frac{8}{N(p-1)-4}\, x\right) & x \geq 0 \\ 0 & x \leq 0, \end{cases}$$

and $P^*(x)$ is defined in Lemma 4.2.6, then the solution u with initial data u_0 blows-up in finite positive time.

Remark 4.1.11. Note that if (4.1.20) and (4.1.21), we know that the solution exists globally and scatters in at least one time direction. On the other hand, in the case of blow up, strict inequality in the assumption $V_t[u_0] < 0$ is technical, and with a slight modification of the nonlinear assumption (4.1.25), in the spirit of Theorem 4.1.7, one can obtain a theorem which guarantees blow up in at least one time direction.

Remark 4.1.12. When (4.1.13) in Theorem 4.1.7 or (4.1.19) in Theorem 4.1.10 do not hold, the initial data is at or below the ground state and, as noted above, global dynamics have been previously resolved.

Remark 4.1.13. In the case

$$\left[E[u_0] - \frac{|V_t[u_0]|^2}{32V[u_0]}\right] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} \leq E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \tag{4.1.26}$$

the energy assumption, (4.1.19) implies that (4.1.25) holds trivially, and Theorem 4.1.10 reduces exactly to Theorem 4.1.4. Hence Theorem 4.1.10 presents a new result in the case

$$\left[E[u_0] - \frac{|V_t[u_0]|^2}{32V[u_0]}\right] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} - E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} > 0.$$

The nonlinear condition (4.1.25) asserts that when the failure to satisfy (4.1.26) is large, in the sense that

$$\left[E[u_0] - \frac{|V_t[u_0]|^2}{32V[u_0]}\right] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} - E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}$$

is large, then in order to obtain blow-up, the angular momentum must also be large, in the sense that

$$\frac{|2\mathbf{M}[u_0]|^2}{32V[u_0]} \|u_0\|_2^{2\frac{(1-s_c)}{s_c}}$$

must be large.

The nonlinear condition (4.1.17) is simply (4.1.25) with $V_t[u_0] = 0$. The condition (4.1.18) serves a similar function to (4.1.25), but is adapted to a somewhat more delicate setting. In particular, while the definite sign on $V_t[u_0]$ contributes to blow-up for results in one time direction. The loss of definite sign in the two directional case makes $V_t[u_0]$ an obstruction. This obstruction accounts for the changes in the nonlinear condition.

The global existence and blow-up results, Theorems 4.1.7 and 4.1.10, are obtained by using bootstrap methods. The key property allowing us to overcome the breakdown of standard dichotomy arguments is that while the mass-energy of the solutions that we consider is above the ground state, the mass-Kato energy remains below the ground state. The mass-Kato energy can be thought of as mass-energy with a modification by momentum, and can be analyzed with methods analogous to the case of mass-energy.

Compared to the prior work for large mass-energy, the main theorem of Duyckaerts and Roudenko in [21], our main theorems take a different perspective. In particular, Duyckaerts and Roudenko are able to reduce control of (4.1.6) and other key time dependent properties of the solution to the control of certain functions defined by a solution. In our results, we control such properties of solutions directly. Since initial data considered by Duyckaerts and Roudenko satisfies the hypothesis of Theorem 4.1.10, our proof offers a somewhat alternate perspective for this prior result.

We also establish the following result of independent interest regarding the quantity $\|\nabla|u|\|_2^2$.

Theorem 4.1.14. *The map from $H^1(\mathbb{R}^N;\mathbb{C})$ into $\mathbb{R} \geq 0$ given by $u \mapsto \|\nabla|u|\|_2^2$ is continuous.*

A major difficulty in establishing Theorem 4.1.14 is that in general, for small v, one can control

$$|\nabla|u+v| - \nabla|u|| \tag{4.1.27}$$

by $|\nabla u|$, but not by smallness of v. More so, one has worse control of (4.1.27) when $|u|$ is small.

To obtain this continuity result, we in essence analyze specific properties of functions $\frac{1}{|u|}$ for $u \in H^1(\mathbb{R}^N)$ by constructing a decomposition of the measure space $\mathbb{R}^N$ based on interacting dyadic decompositions of functions.

4.2 Kato kinetic energy and the continuity result

We begin with the proof of Theorem 4.1.14, and follow with general discussion of Kato kinetic energy.

4.2.1 Proof of Theorem 4.1.14

We consider $u \in H^1$. From Kato's inequality, we have that Kato kinetic energy is bounded by kinetic energy. That is,

$$\|\nabla|u|\|_2^2 \leq \|\nabla u\|_2^2 .$$

Theorem 4.1.14 will allow us to upgrade from boundedness to continuity.

An elementary lemma will be useful for establishing this result.

Lemma 4.2.1. *If $z, w \in \mathbb{C}$ with $z \neq 0$, and $w \neq 0$ then*

$$\left| \frac{z+w}{|z+w|} - \frac{z}{|z|} \right| \leq 2\, min \left\{ 1, \frac{|w|}{|z|} \right\}.$$

This lemma follows from rearrangement and the triangle inequality.

We are now prepared to prove Theorem 4.1.14.

Proof. Consider $u \in H^1$ and $\{v_n\} \subset H^1$ with $v_n \overset{H^1}{\to} 0$. The result will follow if we show that

$$\int |\nabla |u+v_n||^2 - |\nabla |u||^2 \to 0.$$

Step I: *Initial Bounds.*

Note that, using Kato's inequality (see Theorem 2.1.1),

$$\left| \int |\nabla |u+v_n||^2 - |\nabla |u||^2 \right| \leq \int \Big| \left(\nabla |u+v_n| + \nabla |u| \right) \cdot \left(\nabla |u+v_n| - \nabla |u| \right) \Big|$$

$$\leq \left(2\, \|\nabla u\|_2 + \|\nabla v_n\|_2 \right) \left(\int \Big| \nabla |u+v_n| - \nabla |u| \Big|^2 \right)^{1/2}.$$

To complete the proof it is sufficient to show that

$$\nabla|u+v_n| - \nabla|u| \overset{L^2}{\to} 0.$$

We split the problem into problems of establishing sufficient bounds with restriction to the sets

$$E_1^n = \left\{x \in \mathbb{R}^N : u(x) = 0, \text{ or } u(x)+v_n(x) = 0, \text{ or } v_n(x) = 0\right\},$$

and

$$E_2^n = \left\{x \in \mathbb{R}^N : u(x) \neq 0, \text{ and } u(x)+v_n(x) \neq 0, \text{ and } v_n(x) \neq 0\right\}.$$

Step II: *Bounds on E_1^n.*

First consider the case with $u = 0$. Note that, by Theorem 2.1.1, $\partial_j u = 0$ a.e. on the set where $u = 0$. Then, using Kato's inequality we have the a.e. pointwise estimate

$$|\nabla|u+v_n| - \nabla|u|| \leq |\nabla v_n|.$$

Now we consider the case with $u \neq 0$ and $u + v_n = 0$. On this set, $\nabla u = -\nabla v_n$ a.e. and so we also have that pointwise a.e.

$$\begin{aligned} |\nabla\,|u+v_n| - \nabla\,|u|| &\leq |\nabla u| \\ &= |\nabla v_n|. \end{aligned}$$

Finally, if $u \neq 0$, $u + v_n \neq 0$ and $v_n = 0$, then, using 2.1.1,

$$\begin{aligned} |\nabla|u+v_n| - \nabla|u|| &= \left|\mathrm{Re}\left[\frac{\overline{u+v_n}}{|u+v_n|}\nabla(u+v_n)\right] - \mathrm{Re}\left[\frac{\overline{u}}{|u|}\nabla u\right]\right| \\ &\leq |\nabla v_n|. \end{aligned}$$

Since $v_n \overset{H^1}{\to} 0$, to complete the proof it is now sufficient to establish bounds on E_2^n.

Step III: *Setting Up for Convergence on E_2^n.*

From here on, we consider $x \in \mathbb{R}^N$ satisfying the conditions of E_2^n with no further comment, and establish bounds for each partial derivative.

Making using of Lemma 4.2.1, we have the a.e. pointwise estimate

$$|\partial_j |u+v_n| - \partial_j |u|| = \left| \mathrm{Re}\left[\frac{\overline{u+v_n}}{|u+v_n|} \partial_j (u+v_n) \right] - \mathrm{Re}\left[\frac{\overline{u}}{|u|} \partial_j u \right] \right|$$

$$\leq \left| \mathrm{Re}\left[\left(\frac{\overline{u+v_n}}{|u+v_n|} - \frac{\overline{u}}{|u|} \right) \partial_j u \right] \right| + |\partial_j v_n|$$

$$\leq 2 \min\left\{1, \frac{|v_n|}{|u|}\right\} |\partial_j u| + |\partial_j v_n| .$$

Since $v_n \xrightarrow{H^1} 0$, the problem is reduced to showing that

$$\min\left\{1, \frac{|v_n|}{|u|}\right\} |\partial_j u| \xrightarrow{L^2} 0.$$

Step IV: *Decomposition Sets.*

We now decompose $E_2^n \subset \mathbb{R}^N$ into sets defined using level sets of u and v_n.

Consider $M \in \mathbb{Z}$ and define the following sets and functions which are independent of v_n.

$$\Omega_M := \left\{ x \in E_2^n : 2^M \leq |u(x)| < 2^{M+1} \right\},$$

$$f_M : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad f_M(\lambda) := \sup_{\substack{A \subset \Omega_M \\ \mu(A) \leq \lambda}} \|\partial_j u\|^2_{L^2(A)},$$

where μ is standard Lebesgue measure.

Now we define sets which depend on both u and v_n. Given $M \in \mathbb{Z}$, consider $L \in \mathbb{Z}$ with $L \leq M$, and define

$$\Omega_{M,L}^{v_n} := \begin{cases} \left\{x \in \Omega_M : 2^{L-1} \leq |v_n(x)| < 2^L\right\}, & L < M, \\ \\ \left\{x \in \Omega_M : 2^{L-1} \leq |v_n(x)|\right\}, & L = M. \end{cases}$$

Step V *Convergence on Decomposition Sets.*

The definition of $\Omega_{M,L}^{v_n}$ provides the bound

$$\sum_{M \in \mathbb{Z}} \sum_{\substack{L \in \mathbb{Z} \\ L \leq M}} 2^{2L-2} \mu\left(\Omega_{M,L}^{v_n}\right) \leq \|v_n\|_2^2,$$

and hence

$$\mu\left(\Omega_{M,L}^{v_n}\right) \leq 2^{-2L}\left(4\|v_n\|_2^2\right).$$

Now,

$$\left\|\min\left\{1, \frac{|v_n|}{|u|}\right\}|\partial_j u|\right\|_{L^2(E_2^n)}^2$$

$$\leq \sum_{M \in \mathbb{Z}} \sum_{\substack{L \in \mathbb{Z} \\ L \leq M}} 2^{2(L-M)} \|\partial_j u\|_{L^2(\Omega_{M,L}^{v_n})}^2$$

$$\leq \sum_{M \in \mathbb{Z}} \sum_{\substack{L \in \mathbb{Z} \\ L \leq M}} 2^{2(L-M)} \left[f_M\left(\mu\left(\cup_{\substack{J \in \mathbb{Z} \\ L \leq J \leq M}} \Omega_{M,J}^{v_n}\right)\right) - f_M\left(\mu\left(\cup_{\substack{J \in \mathbb{Z} \\ L+1 \leq J \leq M}} \Omega_{M,J}^{v_n}\right)\right)\right]$$

$$= \sum_{M \in \mathbb{Z}} \sum_{\substack{L \in \mathbb{Z} \\ L \leq M}} 2^{2(L-M)} \left[f_M\left(\sum_{L \leq J \leq M} \mu\left(\Omega_{M,J}^{v_n}\right)\right) - f_M\left(\sum_{L+1 \leq J \leq M} \mu\left(\Omega_{M,J}^{v_n}\right)\right)\right].$$

For fixed M, the dominated convergence theorem implies that

$$\sum_{\substack{L \in \mathbb{Z} \\ L \leq M}} 2^{2(L-M)} \left[f_M\left(\sum_{L \leq J \leq M} \mu\left(\Omega_{M,J}^{v_n}\right)\right) - f_M\left(\sum_{L+1 \leq J \leq M} \mu\left(\Omega_{M,J}^{v_n}\right)\right)\right] \to 0$$

as $n \to \infty$. Another application of the dominated convergence theorem implies that

$$\sum_{M\in\mathbb{Z}} \sum_{\substack{L\in\mathbb{Z}\\ L\leq M}} 2^{2(L-M)} \left[f_M \left(\sum_{L\leq J\leq M} \mu\left(\Omega^{v_n}_{M,J}\right)\right) - f_M \left(\sum_{L+1\leq J\leq M} \mu\left(\Omega^{v_n}_{M,J}\right)\right)\right] \to 0$$

as $n \to \infty$. Hence there exist sufficient bounds on $\left\| \min\left\{1, \frac{|v_n|}{|u|}\right\} |\partial_j u| \right\|^2_{L^2(E^n_2)}$ to imply desired convergence restricted to E^n_2

As noted above, this is sufficient to establish the convergence,

$$\nabla|u+v_n| - \nabla|u| \xrightarrow{L^2} 0,$$

completing the proof.

□

4.2.2 Kato kinetic energy in NLS

We will now discuss some properties of Kato kinetic energy quantity as they relate to solutions of NLS. We begin by demonstrating an identity for H^1 functions which has connections to momentum in NLS. This identity is presented in [15] in the more general setting of covariant derivatives, and is there used to establish dichotomy below the ground state for magnetic nonlinear Schrödinger equations. Here we include the proof for the more specialized case of the standard gradient.

Lemma 4.2.2 (Remainder Identity for Kato's Inequality). *For $u \in H^1$, the identity*

$$\int_{u\neq 0} \Big| \frac{\mathcal{P}[u]}{2|u|}\Big|^2 dx = \int_{u\neq 0} \Big| Im\Big[\frac{\overline{u}}{|u|}\nabla u\Big]\Big|^2 dx$$

$$= \|\nabla u\|_2^2 - \|\nabla|u|\|_2^2$$

is satisfied.

Proof. Given $u \in H^1$, we have pointwise, using Theorem 2.1.1,

$$|\partial_j |u||^2 = \begin{cases} \left|\mathrm{Re}\left[\frac{\overline{u}}{|u|}\partial_j u\right]\right|^2 & u \neq 0 \\ 0 & u = 0. \end{cases}$$

Furthermore, again using Theorem 2.1.1 we have that a.e.

$$\left|\partial_j u\right|^2 = \begin{cases} \left|\frac{\overline{u}}{|u|}\partial_j u\right|^2 & u \neq 0 \\ 0 & u = 0. \end{cases}$$

It follows that a.e.

$$|\nabla u|^2 - |\nabla |u||^2 = \begin{cases} \left|\mathrm{Im}\left[\frac{\overline{u}}{|u|}\nabla u\right]\right|^2 & u \neq 0 \\ 0 & u = 0. \end{cases}$$

From here the result follows by integration. ☐

Remark 4.2.3. Pointwise, for $u \neq 0$, we have

$$\begin{aligned} \left|\mathrm{Im}\left[\frac{\overline{u}}{u}\nabla u\right]\right| &\leq \left|\frac{\overline{u}}{u}\nabla u\right| \\ &= |\nabla u|, \end{aligned}$$

and so it is natural to interpret $\mathrm{Im}\left[\frac{\overline{u}}{u}\nabla u\right]$ as

$$\mathrm{Im}\left[\frac{\overline{u}}{u}\nabla u\right] = \begin{cases} \mathrm{Im}\left[\frac{\overline{u}}{u}\nabla u\right] & u \neq 0 \text{ and } \nabla u \neq 0 \\ 0 & \nabla u = 0. \end{cases} \tag{4.2.1}$$

As this is defined a.e. and equal to

$$\begin{cases} \mathrm{Im}\left[\frac{\overline{u}}{|u|}\nabla u\right] & u \neq 0 \\ 0 & u = 0, \end{cases}$$

if we take (4.2.1) as a definition, it is clear from the proof that the identity

$$|\nabla u|^2 - |\nabla |u|\,|^2 = \left|\mathrm{Im}\left[\frac{\overline{u}}{|u|}\nabla u\right]\right|^2$$

holds a.e. without need for integration.

This identity provides us with a property relating to our energy assumption which will be very useful for the proofs of Theorems 4.1.10 and 4.1.7.

Theorem 4.2.4. *For $u_0 \in H^1$ with $|x|u_0 \in L^2$,*

$$\frac{|V_t[u_0]|^2 + |2\boldsymbol{M}[u_0]|^2}{16V[u_0]} \leq \|\nabla u_0\|_2^2 - \|\nabla |u_0|\|_2^2 \,,$$

and each of the energy conditions (4.1.6), (4.1.14), or (4.1.20) implies

$$E_K[u_0]\, \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} \leq E[u_Q]\, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} .$$

The main ingredient used to prove Theorem 4.2.4 is the following lemma.

Lemma 4.2.5. *For $u \in H^1$ with $|x|u \in L^2$,*

$$\left(\int |x \cdot \frac{\mathcal{P}[u]}{2}| dx\right)^2 + \left(\frac{\boldsymbol{M}[u]}{2}\right)^2 \leq \left(\int |x|^2 |u|^2 dx\right) \left(\|\nabla u\|_2^2 - \|\nabla |u|\|_2^2\right).$$

Proof. First,

$$\begin{aligned}\left(\int |x \cdot \frac{\mathcal{P}[u]}{2}| dx\right)^2 &= \left(\int |x||u| \left(\frac{x}{|x|} \cdot \frac{\mathcal{P}[u]}{2|u|}\right) dx\right)^2 \\ &\leq \left(\int |x|^2|u|^2 dx\right) \left(\int \left|\frac{x}{|x|} \cdot \frac{\mathcal{P}[u]}{2|u|}\right|^2 dx\right).\end{aligned}$$

Similarly,

$$\left(\int |x \wedge \frac{\mathcal{P}[u]}{2}| dx\right)^2 \leq \left(\int |x|^2|u|^2 dx\right) \left(\int \left|\frac{x}{|x|} \wedge \frac{\mathcal{P}[u]}{2|u|}\right|^2 dx\right).$$

By a direct computation,

$$\left|\frac{x}{|x|} \cdot \frac{\mathcal{P}[u]}{2|u|}\right|^2 + \left|\frac{x}{|x|} \wedge \frac{\mathcal{P}[u]}{2|u|}\right|^2 = \left|\frac{\mathcal{P}[u]}{2|u|}\right|^2,$$

and the inequality follows by Lemma 4.2.2. ∎

With Lemma 4.2.5 established, Theorem 4.2.4 follows immediately.

4.2.3 A nonlinear inequality below the ground state

Our main results for NLS will require a nonlinear inequality which holds when $|u|$ is below the ground state.

For this we will introduce the minimal energy function, $P : \mathbb{R}_{\geq 0} \to \mathbb{R}$ by

$$P(x) = \frac{1}{2}x - \frac{C_{GN}}{p+1}x^{\frac{n(p-1)}{4}},$$

so that the Gagliardo–Nirenberg (4.1.1) inequality implies

$$E[u] \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}} \geq P\left(\left\|\nabla u\right\|_2^2 \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}}\right).$$

Kenig and Merle [37] used a similar function to establish results for energy critical NLS, and Holmer and Roudenko [34] use such a function in establishing results for intercritical NLS below the ground state. By the Gagliardo–Nirenberg inequality and Kato's inequality, the minimal energy function also satisfies

$$E_K[u] \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}} \geq P\left(\left\|\nabla \left|u\right|\right\|_2^2 \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}}\right), \tag{4.2.2}$$

and this property was used in [15] to obtain results for electromagnetic NLS.

The function P is concave down on $[0, \infty)$, and a computation with (4.1.2) and (4.1.3) shows that P attainins its maximum of

$$E[u_Q] \left\|u_Q\right\|_2^{2\frac{(1-s_c)}{s_c}}$$

at

$$\left\|\nabla u_Q\right\|_2^2 \left\|u_Q\right\|_2^{2\frac{(1-s_c)}{s_c}}.$$

Using (4.1.2), the concavity of P implies that on $\left[0, \left\|\nabla u_Q\right\|_2^2 \left\|u_Q\right\|_2^{2\frac{(1-s_c)}{s_c}}\right]$,

$$P(x) \geq \frac{N(p-1)-4}{2N(p-1)}x.$$

So if

$$\left\|\nabla \left|u\right|\right\|_2^2 \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}} \leq \left\|\nabla u_Q\right\|_2^2 \left\|u_Q\right\|_2^{2\frac{(1-s_c)}{s_c}},$$

then

$$E_K[u] \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}} \geq P\left(\left\|\nabla \left|u\right|\right\|_2^2 \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}}\right) \geq \frac{N(p-1)-4}{2N(p-1)} \left\|\nabla \left|u\right|\right\|_2^2 \left\|u\right\|_2^{2\frac{(1-s_c)}{s_c}}. \tag{4.2.3}$$

Lemma 4.2.6. *Define* $P^* : \left[- \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \infty \right) \to \mathbb{R}$ *by*

$$P^*(x) = E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} - P\left(x + \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \right).$$

Then we have

$$E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} - E_K[u] \|u\|_2^{2\frac{(1-s_c)}{s_c}} \le P^* \left(\|\nabla |u| \|_2^2 \|u\|_2^{2\frac{(1-s_c)}{s_c}} - \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \right). \quad (4.2.4)$$

This lemma follows immediately by (4.2.2) and construction of P^*.

Remark 4.2.7. For the function P^* defined in Lemma 4.2.6, in application for blow up we will only be concerned with the restriction $P^* : \mathbb{R}_{\geq 0} \to \mathbb{R}$. Here P^* is increasing, and satisfies

$$P^*(0) = \frac{d}{dx} P^*(0) = 0.$$

Furthermore, for this restriction of P^*, near 0, P^* grows like x^2, and at infinity P^* grows like $x^{\frac{N(p-1)}{4}}$.

Note for intuition, that in the case $s_c \le \frac{N}{4}$, we have $\frac{N(p-1)}{4} \le 2$, and a computation using the second derivative of $P^*(x)$ shows that for P^* restricted to $\mathbb{R}_{\geq 0}$, the inequality (4.2.4) holds (but is weaker) if we replace $P^*(x)$ by a function $F(x)$, defined by

$$F(x) = \frac{C_{GN}}{p+1} x^{\frac{N(p-1)}{4}}.$$

Making use of Lemma 4.2.6 and [21, Theorem 3.7], we obtain the following sufficient condition for scattering.

Corollary 4.2.8. *Given* $u_0 \in H^1$ *with*

$$\|\nabla |u_0| \|_2^2 \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < \|\nabla u_Q\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \quad (4.2.5)$$

and corresponding solution u *satisfying* $u(0) = u_0$*, defined on* $[0, T^*)$ *with* T^* *being maximal, if*

$$\sup_{t \in [0, T^*)} E_K[u(t)] \|u(t)\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \quad (4.2.6)$$

then $T^* = \infty$ *and* u *scatters forward in time in* H^1.

Remark 4.2.9. We do not need the full power of Lemma 4.2.6. It will be apparent that control such as that which appears in [37, Lemma 3.4] for the energy critical case is sufficient. As such, this is essentially just a rephrasing of [21, Theorem 3.7] suited to our problem.

Proof. If u_0 satisfies the hypothesis of Corollary 4.2.8, then (4.2.5) and (4.2.6) imply that (4.2.5) holds for the full positive interval of existence, and $T^* = \infty$. With Lemma 4.2.6, this implies

$$\sup_{t\in[0,\infty)} P^*\left(\|\nabla |u_Q|\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} - \|\nabla |u(t)|\|_2^2 \|u(t)\|_2^{2\frac{(1-s_c)}{s_c}}\right) > 0,$$

and by continuity of P^*,

$$\sup_{t\in[0,\infty)} \|\nabla |u(t)|\|_2^2 \|u(t)\|_2^{2\frac{(1-s_c)}{s_c}} < \|\nabla |u_Q|\|_2^2 \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}.$$

Using

$$\begin{aligned}
&\sup_{[0,\infty)} \left[\|u\|_{p+1}^{p+1} \|u\|_2^{2\frac{(1-s_c)}{s_c}}\right] \\
&\leq \sup_{t\in[0,\infty)} \left[\left(\frac{\|u_Q\|_{p+1}^{p+1} \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}}{\|\nabla u_Q\|_2^{\frac{N(p-1)}{2}} \|u_Q\|_2^{2-\frac{(N-2)(p-1)}{2}} \|u_Q\|_2^{2\frac{1-s_c}{s_c}}}\right) \|\nabla u\|_2^{\frac{N(p-1)}{2}} \|u\|_2^{2-\frac{(N-2)(p-1)}{2}} \|u\|_2^{2\frac{1-s_c}{s_c}}\right] \\
&= \sup_{t\in[0,\infty)} \left[\left(\frac{\|\nabla u\|_2^{\frac{N(p-1)}{2}} \|u\|_2^{2-\frac{(N-2)(p-1)}{2}} \|u\|_2^{2\frac{1-s_c}{s_c}}}{\|\nabla u_Q\|_2^{\frac{N(p-1)}{2}} \|u_Q\|_2^{2-\frac{(N-2)(p-1)}{2}} \|u_Q\|_2^{2\frac{1-s_c}{s_c}}}\right) \|u_Q\|_{p+1}^{p+1} \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right] \\
&= \sup_{t\in[0,\infty)} \left[\left(\frac{\|\nabla u\|_2^{\frac{N(p-1)}{2}} \|u\|_2^{2-\frac{(N-2)(p-1)}{2}} \|u\|_2^{2\frac{1-s_c}{s_c}}}{\|\nabla u_Q\|_2^{\frac{N(p-1)}{2}} \|u_Q\|_2^{2-\frac{(N-2)(p-1)}{2}} \|u_Q\|_2^{2\frac{1-s_c}{s_c}}}\right)\right] \|u_Q\|_{p+1}^{p+1} \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \\
&< \|u_Q\|_{p+1}^{p+1} \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}},
\end{aligned}$$

we can directly apply [21, Theorem 3.7], and u scatters forward in time in H^1. □

4.3 Global existence and blow-up

In this section we prove the main results regarding global existence and blow-up for solutions to NLS. We begin with a lemma which collects properties which are classical or follow directly from above arguments, and which we will use freely in this section.

Lemma 4.3.1. *Given a solution u, of* (NLS), *the following properties hold.*

(1) The map

$$t \mapsto \|\nabla |u|\|_2^2$$

is continuous on the full interval of existence.

(2) The solution u blows up at positive time T^* if and only if

$$\lim_{t \uparrow T^*} \|\nabla |u|\|_2^2 = \infty.$$

(3) If, on the positive interval of existence, the bound $V[u](t) \leq G(t)$ holds, where G is a function so that for some positive T^*, and the limit $\lim_{t \uparrow T^*} G(t) = 0$ holds, then u blows up in positive time, at or before the time T^*.

Proof. Property (1) follows from continuity of the map $t \mapsto \|\nabla u\|_2^2$, and the continuity result, Theorem 4.1.14.

Property (2) follows from Kato's inequality in one direction, and the Gagliardo–Nirenberg inequality and conservation of energy in the other direction..

Property (3) is classical. □

We will also make frequent use of Theorem 4.2.4, at times without comment.

4.3.1 Proof of Theorem 4.1.10

Here we prove Theorem 4.1.10, which establishes results in one time direction.

Proof. In this proof we will assume (4.1.19) without further comment.

For initial data satisfying conditions for global existence or blow up in Theorem 4.1.4, the arguments in this proof will apply and some steps will not be needed. Hence, our proof provides a different perspective on Theorem 4.1.4.

To simplify the steps involving the nonlinear condition (4.1.25), on first reading it is useful to consider only the case

$$\left[E[u_0] - \frac{|V_t[u_0]|^2}{32V[u_0]}\right] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} - E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} > 0. \tag{4.3.1}$$

This is in fact sufficient to establish the result, as when (4.3.1) fails, Theorem 4.1.4 can be applied.

As noted in Theorem 4.2.4, if u satisfies (4.1.20), then u also satisfies

$$E_K[u] \|u\|_2^{2\frac{(1-s_c)}{s_c}} \leq E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \tag{4.3.2}$$

and that if (4.1.20) is strict, then (4.3.2) is strict as well.

We begin with the case of global existence and scattering. The goal will be to show that the hypothesis of Corollary 4.2.8 hold. In order to do this, we will use a bootstrap argument to show that a stronger set of conditions hold on a maximal forward interval of existence.

We now introduce assumptions we will use for our bootstrap argument, which are stronger than what appears in the hypothesis of Theorem 4.1.10.

Assumption 4.3.3. We say $w \in H^1$ satisfies Assumption 4.3.3 if w satisfies (with u or u_0 replaced with w where necessary):

– $$\left[E[w] - \frac{|V_t[w]|^2 + |2\mathbf{M}[w]|^2}{32V[w]}\right] \|w\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \tag{4.3.4}$$

– (4.1.21), and

– $$V_t[w] > 0. \tag{4.3.5}$$

We will establish the following with u as the solution with initial data u_0.

(1) If u_0 satisfies (4.1.20), (4.1.21) and (4.1.22), then there is some $\varepsilon > 0$ so that on the open interval $(0, \varepsilon)$, u satisfies Assumption 4.3.3.

(2) If $u(T)$ satisfies Assumption 4.3.3, then there is some $\varepsilon > 0$ so that on the interval $[T, T+\varepsilon)$, u satisfies Assumption 4.3.3.

(3) If u satisfies Assumption 4.3.3 on an open interval (T_1, T_2), then u can be extended to $(T_1, T_2]$, and $u(T_2)$ satisfies Assumption 4.3.3.

If (1), (2), and (3) hold, then global existence follows immediately by a bootstrap argument. It is sufficient to prove (1) and (3).

Proof of (1): Assume u_0 satisfies (4.1.20), (4.1.21), and (4.1.22). Using the virial identity, (4.1.4), and (4.2.3), since u_0 satisfies (4.1.19), (4.1.20), and (4.1.21),

$$\begin{aligned} V_{tt}[u_0] &= 8\,\|\nabla u_0\|_2^2 - \frac{4N(p-1)}{p+1}\,\|u_0\|_{p+1}^{p+1} \\ &\geq 8\left(\|\nabla u_0\|_2^2 - \|\nabla |u_0|\|_2^2\right) + 4N(p-1)\left(E_K[u] - \frac{N(p-1)-4}{2N(p-1)}\,\|\nabla\,|u|\,\|_2^2\right) \\ &\geq 8\left(\|\nabla u_0\|_2^2 - \|\nabla |u_0|\|_2^2\right) && (4.3.6) \\ &> 0, && (4.3.7) \end{aligned}$$

where the final inequality, (4.3.7), follows from (4.1.19) and (4.1.20). In the sequel, we will make use of the intermediate inequality 4.3.6 as well as the final inequality (4.3.7).

We compute, using the notation from Section 4.1.2 and conservation of angular momentum,

$$\begin{aligned} &\partial_t\left[\frac{|V_t[u_0]|^2 + |2\mathbf{M}[u_0]|^2}{V[u_0]}\right] && (4.3.8) \\ &= \frac{V_t[u_0]}{V[u_0]}\left[2V_{tt}[u_0] - \left(\frac{|V_t[u_0]|^2 + |2\mathbf{M}[u_0]|^2}{V[u_0]}\right)\right]. \end{aligned}$$

Using (4.3.7) and Theorem 4.2.4,

$$\left[2V_{tt}[u_0] - \left(\frac{|V_t[u_0]|^2 + |2\mathbf{M}[u_0]|^2}{V[u_0]}\right)\right] > 0. \qquad (4.3.9)$$

Now, we can choose sufficiently small ε so that on $(0,\varepsilon)$, using (4.1.22) and (4.3.7), u satisfies (4.3.5). Using strict inequality in (4.1.21), with ε chosen sufficiently small, u also satisfies (4.1.21). Using strict inequality in both (4.3.9) and (4.3.7), for ε sufficiently small, u further satisfies

$$\partial_t \left[\frac{|V_t[u]|^2 + |2\mathbf{M}[u]|^2}{V[u]} \right] > 0. \tag{4.3.10}$$

Together with (4.1.20), we have that u satisfies (4.3.4). Hence, u satisfies Assumption 4.3.3 on $(0,\varepsilon)$ and the proof of (1) is complete.

Proof of (3): Assume that u satisfies assumption 4.3.3 on (T_1,T_2). Using (4.1.21) and blow up criterion, we can extend u to $(T_1,T_2]$. As in the proof of (1), on (T_1,T_2) u satisfies (4.3.7) and (4.3.10). Since u satisfies these as well as (4.3.4) and (4.3.5) on (T_1,T_2), $u(T_2)$ also satisfies (4.3.4) and (4.3.5). Since u satisfies (4.1.21) on (T_1,T_2), $u(T_2)$ satisfies

$$\|\nabla\, |u(T_2)|\,\|_2^2 \, \|u(T_2)\|_2^{2\frac{(1-s_c)}{s_c}} \leq \|\nabla u_Q\|_2^2 \, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}.$$

Furthermore, since (4.3.4) implies the strict inequality $E_K[u(T_2)]\, \|u(T_2)\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q]\, \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}$, $u(T_2)$ satisfies (4.1.21). Hence $u(T_2)$ satisfies Assumption 4.3.3 and the proof of (3) is complete.

We are now prepared to apply Corollary 4.2.8. We note that the preceding bootstrap argument would be sufficient to establish global existence. While the hypothesis of Corollary does not hold for the solution u on all of $[0,\infty)$, by (4.3.4) at $u(\varepsilon)$, (4.3.10) on $[\varepsilon,\infty)$, Theorem 4.2.4, the hypothesis does hold for u on $[\varepsilon,\infty)$ with initial data $u(\varepsilon)$. This implies that the solution with initial data $u(\varepsilon)$ scatters. Scattering for the solution with initial data u_0 is immediate.

We now move on to the case of blow-up. We begin by restating the hypothesis of the theorem for blow-up in context of a general function $w \in H^1$.

Assumption 4.3.11. We say $w \in H^1$ satisfies Assumption 4.3.11 if w satisfies all of (4.1.20), (4.1.23), (4.1.24), and (4.1.25), where for each condition u or u_0 is replaced with w as necessary. In particular, (4.1.25) becomes

$$\frac{|2\mathbf{M}[w]|^2}{32V[w]} \|w\|_2^{2\frac{(1-s_c)}{s_c}} > F\left(\left(E[w] - \frac{|V_t[w]|^2}{32V[w]}\right)\|w\|_2^{2\frac{(1-s_c)}{s_c}} - E[u_Q]\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right). \tag{4.3.12}$$

By another bootstrap argument, to establish the blow-up result it is sufficient to show the following.

(1) If $u(T)$ satisfies Assumption 4.3.11, then there is some $\varepsilon > 0$ so that on the interval $[T, T+\varepsilon)$, u satisfies Assumption 4.3.11.

(2) If u is defined on an interval $(T_1, T_2]$, and u satisfies Assumption 4.3.11 on the open interval (T_1, T_2), then $u(T_2)$ satisfies Assumption 4.3.11.

(3) If there is some T such that for any t in the interval of existence of u with $t > T$, $u(t)$ satisfies Assumption 4.3.11, then u blows-up in finite positive time.

To complete the theorem we must establish (1), (2), and (3). Note that if (1) and (2) both hold, then the hypothesis of (3) is satisfied.

Before we establish (1), (2), and (3), we perform some computations. Suppose some fixed w satisfies Assumption 4.3.11. Then

$$\frac{V_t[w]}{V[w]} < 0,$$

and using (4.1.2), (4.1.4), (4.1.23), and Remark 4.1.5,

$$\begin{aligned}
&\left(2V_{tt}[w] - \left(\frac{|V_t[w]|^2 + |2\mathbf{M}[w]|^2}{V[w]}\right)\right)\|w\|_2^{2\frac{(1-s_c)}{s_c}} \\
&= \left(32E[w] - \frac{|V_t[w]|^2 + |2\mathbf{M}[w]|^2}{V[w]} - \frac{8(N(p-1)-4)}{p+1}\|w\|_{p+1}^{p+1}\right)\|w\|_2^{2\frac{(1-s_c)}{s_c}} \\
&\leq 32E[u_Q]\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} - \frac{8(N(p-1)-4)}{p+1}\|w\|_{p+1}^{p+1}\|w\|_2^{2\frac{(1-s_c)}{s_c}} \\
&< 0.
\end{aligned}$$

Together with (4.3.8), we have

$$\partial_t\left[\frac{|V_t[w]|^2+|2\mathbf{M}[w]|^2}{32V[w]}\right]>0. \tag{4.3.13}$$

We continue by computing

$$\begin{aligned}
&\partial_t\left[\frac{|V_t[w]|^2}{V[w]}\right]\|w\|_2^{2\frac{(1-s_c)}{s_c}}\\
&=\frac{V_t[w]}{V[w]}\left[32E[w]-\frac{|V_t[w]|^2}{V[w]}-\frac{8(N(p-1)-4)}{p+1}\|w\|_{p+1}^{p+1}\right]\|w\|_2^{2\frac{(1-s_c)}{s_c}}.
\end{aligned}$$

Note that we once again have $E_K[w]\|w\|_2^{2\frac{(1-s_c)}{s_c}}\le E[u_Q]\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}$. Using Lemma 4.2.5 and Lemma 4.2.6, and in the last step, (4.1.25), we have

$$\begin{aligned}
&\|w\|_{p+1}^{p+1}\|w\|_2^{2\frac{(1-s_c)}{s_c}}-\|u_Q\|_{p+1}^{p+1}\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\\
&=\frac{p+1}{2}\left(\|\nabla|w|\|_2^2\|w\|_2^{2\frac{(1-s_c)}{s_c}}-\|\nabla u_Q\|_2^2\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)\\
&\quad+(p+1)\left(E[u_Q]\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}-E_K[w]\|w\|_2^{2\frac{(1-s_c)}{s_c}}\right)\\
&\ge\frac{p+1}{2}\left(\|\nabla|w|\|_2^2\|w\|_2^{2\frac{(1-s_c)}{s_c}}-\|\nabla u_Q\|_2^2\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)\\
&>\frac{4(p+1)}{N(p-1)-4}\left(\left(E[w]-\frac{|V_t[w]|^2}{32V[w]}\right)\|w\|_2^{2\frac{(1-s_c)}{s_c}}-E[u_Q]\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left[32E[w]-\frac{|V_t[w]|^2}{V[w]}-\frac{8(N(p-1)-4)}{p+1}\|w\|_{p+1}^{p+1}\right]\|w\|_2^{2\frac{(1-s_c)}{s_c}}\\
&=\left(32E[w]-\frac{|V_t[w]|^2}{V[w]}\right)\|w\|_2^{2\frac{(1-s_c)}{s_c}}-32E[u_Q]\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\\
&\quad-\frac{8(N(p-1)-4)}{p+1}\left(\|w\|_{p+1}^{p+1}\|w\|_2^{2\frac{(1-s_c)}{s_c}}-\|u_Q\|_{p+1}^{p+1}\|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}\right)\\
&<0,
\end{aligned}$$

and so

$$\partial_t \left[\frac{|V_t[w]|^2}{V[w]} \right] > 0. \tag{4.3.14}$$

Finally, note that

$$\partial_t \left[\sqrt{V[w]} \right] = \frac{V_t[w]}{2\sqrt{V[w]}}$$

$$= -\frac{1}{2} \left(\frac{|V_t[w]|^2}{V[w]} \right)^{\frac{1}{2}}.$$

Proof of (1): Application of (4.3.13), along with strict inequalities implies that (1) holds.

Proof of (2): Assume u is defined on some interval $(T_1, T_2]$, and that Assumption 4.3.11 holds on (T_1, T_2). We immediately have that $u(T_2)$ satisfies (4.1.20). In fact, by (4.3.13), we have that on $(T_1, T_2]$, $u(t)$ satisfies (4.1.20) with strict inequality. This strict inequality implies that $u(T_2)$ satisfies (4.1.23). Now, note that since u exists at T_2, $V[u]$ is bounded below by some positive value on $[T_1, T_2]$, and using (4.3.14), we have that

$$\lim_{t \uparrow T_2} |V_t[w]|^2 > 0,$$

and $u(T_2)$ satisfies (4.1.24). To see that $u(T_2)$ satisfies (4.1.25), note that F is an increasing function, and (4.1.24) implies

$$\partial_t \left[\frac{|2\mathbf{M}[\mathbf{u}]|^2}{V[u]} \right] > 0,$$

while (4.3.14) implies

$$\partial_t \left[\left(E[w] - \frac{|V_t[w]|^2}{32V[w]} \right) \|w\|_2^{2\frac{(1-s_c)}{s_c}} - E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}} \right] < 0.$$

Hence $u(T_2)$ satisfies (4.1.25), and (2) holds.

Proof of (3): Without loss of generality, assume u is defined at $t = 0$, and satisfies Assumption 4.3.11 for all positive time in its interval of existence. Since

$$\partial_t \left[\frac{|V_t[u](t)|^2}{32V[u](t)} \right] > 0,$$

and

$$V_t[u](t) < 0,$$

we have

$$\partial_t \left[\frac{V_t[u](t)}{\sqrt{V[u](t)}} \right] < 0,$$

$$\frac{V_t[u](t)}{\sqrt{V[u](t)}} < \frac{V_t[u](0)}{\sqrt{V[u](0)}},$$

and

$$\partial_t \left[\sqrt{V[u](t)} \right] < \frac{V_t[u](0)}{2\sqrt{V[u](0)}}$$

$$< 0.$$

The nonnegativity of $\sqrt{V[u](t)}$ implies that u blows-up in finite positive time by a contradiction argument (or by the uncertainty principle). This completes the proof of blow-up. □

4.3.2 Proof of Theorem 4.1.7

We will now prove Theorem 4.1.7, establishing global existence and blow-up in both time directions.

Proof. We establish the result forward in time under conditions which are invariant under time reversal. Results in both time directions follow from time reversal. Throughout the proof we will assume (4.1.19) without further comment.

We begin by showing global existence. Assume u_0 is initial data satisfying (4.1.14) and (4.1.15).

If $V_t[u](0) \geq 0$, then the result follows by Theorem 4.1.10.

If $V_t[u](0) < 0$, then by strict inequality, there is a sufficiently small interval $[0, T)$ so that on $[0, T)$, u satisfies $V_t[u](t) < 0$, and hence (4.1.14), and also satisfies (4.1.15). Now suppose $[0, S)$ is any interval on which a solution satisfies (4.1.14) and (4.1.15). By (4.1.15), u can be extended to $[0, S]$. Either there is some $t \in [0, S]$ such that $V_t[u](t) \geq 0$, and we can apply Theorem 4.1.10, or else $V_t[u](t) < 0$ on all of $[0, S]$ and

$$\frac{|2\mathbf{M}[u_0]|^2}{32V[u](S)} > \frac{|2\mathbf{M}[u_0]|^2}{32V[u](0)}, \tag{4.3.15}$$

so (4.1.14) holds with strict inequality. This implies that (4.1.15) also holds for $u(S)$, and global existence follows by bootstrap.

Consider u on $[\varepsilon, \infty)$, where (4.1.14) holds with strict inequality for $u(\varepsilon)$. An application of (4.3.15) implies that for $t > \varepsilon$,

$$E_K[u(t)]\, \|u(t)\|_2^{2\frac{(1-s_c)}{s_c}} < E_K[u(\varepsilon)]\, \|u(\varepsilon)\|_2^{2\frac{(1-s_c)}{s_c}},$$

which, applying Corollary 4.2.8, shows that u scatters forward in time.

We now prove the blow-up result.

We assume u_0 satisfies (4.1.14), and (4.1.16), and we will take $\lambda > 1$ to be fixed for the remainder of the proof. If $V_t[u_0] < 0$ and u_0 satisfies (4.1.18), then the proof is complete by application of Theorem 4.1.10. If $V_t[u_0] = 0$ and u_0 satisfies (4.1.17), then (4.1.17) implies that $V_{tt}[u_0] < 0$, and by strict inequalities of the conditions there is some small time where $u(t)$ satisfies the blow-up conditions of Theorem 4.1.10. Finally, assume that $V_t[u_0] > 0$ and that u_0 satisfies (4.1.18). For the remainder of the proof, we will always assume u is a solution such that u_0 satisfies (4.1.14), (4.1.16), and (4.1.18) with $V_t[u_0] > 0$.

We consider the following time dependent assumption.

Assumption 4.3.16. Given $u(t)$ for some $t \in [0, (\lambda - 1)\frac{V[u_0]}{V_t[u_0]})$, we say that $u(t)$ satisfies Assumption

4.3.16 if $u(t)$ satisfies

– (4.1.16)

– $$V[u](t) \leq V[u_0] + tV_t[u_0], \tag{4.3.17}$$

– $$V_t[u](t) \leq V_t[u_0] - t\frac{|V_t[u_0]|^2}{(\lambda-1)V[u_0]}, \tag{4.3.18}$$

This assumption holds trivially for u_0. Notice that if $u\left((\lambda-1)\frac{V[u_0]}{V_t[u_0]}\right)$ satisfies Assumption 4.3.16, then $u\left((\lambda-1)\frac{V[u_0]}{V_t[u_0]}\right)$ satisfies (4.1.14), (4.1.16), (4.1.17), and $V_t\left[u\left((\lambda-1)\frac{V[u_0]}{V_t[u_0]}\right)\right] = 0$, which we have already demonstrated imply blow-up. So in order to establish blow-up for u, it is sufficient to show that the following hold.

(1) If, for some $0 \leq T < (\lambda-1)\frac{V[u_0]}{V_t[u_0]}$, $u(T)$ satisfies Assumption 4.3.16, then there is some $\varepsilon > 0$ so that $u(t)$ satisfies 4.3.16 on $[T, T+\varepsilon)$.

(2) If, for some $0 < S \leq (\lambda-1)\frac{|V_t[u_0]|^2}{V[u_0]}$, u is defined on $[0, S]$ and satisfies Assumption 4.3.16 on $[0, S)$, then $u(S)$ satisfies Assumption 4.3.16.

We begin by establishing (2). The only property which needs to be shown is (4.1.16), which follows from (4.3.17) and the strict inequality

$$\left[E[u_0] - \frac{|2\mathbf{M}[u_0]|^2}{32\lambda V[u_0]}\right] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} < E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}.$$

Now we move on to the proof of (1). Assume $u(T)$ satisfies Assumption 4.3.16 at some $0 \leq T < (\lambda-1)\frac{V[u_0]}{V_t[u_0]}$. Then $V[u](T) < \lambda V[u_0]$, which, with (4.1.18), implies

$$\begin{aligned} \frac{|2\mathbf{M}[u_0]|^2}{32V[u](t)} \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} &\geq \frac{|2\mathbf{M}[u_0]|^2}{32\lambda V[u_0]} \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} \\ &> G\left(E[u_0] \|u_0\|_2^{2\frac{(1-s_c)}{s_c}} - E[u_Q] \|u_Q\|_2^{2\frac{(1-s_c)}{s_c}}, \frac{|V_t[0]|^2}{(\lambda-1)V[0]} \|u_0\|_2^{2\frac{(1-s_c)}{s_c}}\right), \end{aligned}$$

and hence, the virial identity and use of Lemma 4.2.6 as in the proof of Theorem 4.1.10,

$$V_{tt} < -\frac{|V_t[u_0]|^2}{(\lambda - 1)V[u_0]}.$$

This is sufficient to imply we can choose $\varepsilon > 0$ so that u satisfies Assumption 4.3.16 on $[T, T+\varepsilon)$. Hence (1) holds and u blows-up in finite time. $\square$

Chapter 5

Momentum Based Modulations of Initial Data

In this chapter we define generalizations of radial and axisymmetric functions, and construct subspaces of $H^1(\mathbb{R}^N)$ defined by certain modulations of these functions. We construct these spaces in general dimension for future study of excited states. We then consider these spaces in the special case of $N = 2$, and establish improved dichotomy results for (NLS) on these spaces.

5.1 Introduction

Cazenave and Weissler [10] identify that rotational properties of a solution have an impact on wellposedness and blow up. Given initial data $u(x) = e^{i\phi(x)}\,|u(x)|$, rotational properties are given by ϕ, and for sufficiently smooth data, more specifically by $\nabla\phi$. Initial data satisfying $u_0(x) = |u_0(x)|$ has momentum which is uniformly zero, and the momentum of a solution with a given mass profile is determined by the function ϕ. Here we consider modulations of ϕ. These modulations inherently preserve the mass profile.

For complex valued u, the radial quadratic modulation

$$e^{i|x|^2}u(x)$$

has seen a fair amount of study, and direct applications to wellposedness and blow-up have been obtained [10, 21, 33]. We are primarily concerned with the angular modulation

$$e^{ik\theta}u(x)$$

for $k \in \mathbb{Z}$, where $\theta = \arctan\left(\frac{x_2}{x_1}\right)$. We retain this notation for θ throughout this section.

The modulation $e^{ik\theta}u(x)$ has a connection to (emNLS), as a gauge transformation taking (NLS) to (emNLS) with certain Aharonov–Bohm potentials. There are close connections between results and spaces described in this chapter, and the *vortex solitons*, as described in works such as [3], as well as [1] and references therein. Additionally, vortex solutions and the results in this chapter have close connections to construction of radial standing waves for (emNLS) with Aharonov–Bohm potential, by Clapp and Szulkin in [12].

5.2 Statement of higher dichotomy

In this chapter we will let $H^1_r(\mathbb{R}^2)$ denote the space of radial $H^1(\mathbb{R}^2)$ functions. For a measurable function f, and a set of measurable functions, $\mathcal{B}$, let $f\mathcal{B}$ be the set of measurable functions defined by

$$f\mathcal{B} = \{fu : u \in \mathcal{B}\}.$$

We are now prepared to state our higher dichotomy theorem.

Theorem 5.2.1. *Consider (NLS) with $N = 2$, and $0 < s_c$. For $k \geq 1$ let $\mathcal{H}_k$ be the subspace of $H^1(\mathbb{R}^2)$ defined by*

$$\mathcal{H}_k = e^{ik\theta} H^1_r(\mathbb{R}^2) \cap H^1(\mathbb{R}^2).$$

There exists a sequence $\{\gamma_k\}$ with $\gamma_k \to \infty$ as $k \to \infty$, so that if $u_0 \in \mathcal{H}_k$, and

$$E[u_0] \left\|u_0\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} < \gamma_k E[u_Q] \left\|u_Q\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}},$$

the following statements hold.

- *If*

$$\left\|\nabla u_0\right\|_{L^2(\mathbb{R}^2)}^2 \left\|u_0\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} < \gamma_k \left\|\nabla u_Q\right\|_{L^2(\mathbb{R}^2)}^2 \left\|u_Q\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}},$$

  then the corresponding solution u satisfies

$$\left\|\nabla u\right\|_{L^2(\mathbb{R}^2)}^2 \left\|u\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} < \gamma_k \left\|\nabla u_Q\right\|_{L^2(\mathbb{R}^2)}^2 \left\|u_Q\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}}$$

  for all time, and u exists globally in both time directions.

- *If*
$$\|\nabla u_0\|_{L^2(\mathbb{R}^2)}^2 \|u_0\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} > \gamma_k \|\nabla u_Q\|_{L^2(\mathbb{R}^2)}^2 \|u_Q\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}},$$
then the corresponding solution u satisfies
$$\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 \|u\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} > \gamma_k \|\nabla u_Q\|_{L^2(\mathbb{R}^2)}^2 \|u_Q\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}}$$
on the full interval of existence. If we also have $|x|u_0 \in L^2(\mathbb{R}^2)$, then u blows up in finite time in both time directions.

Remark 5.2.2. Recall that $s_c = \frac{N}{2} - \frac{2}{p-1}$, and so in the case of $N = 2$, given any $1 < p < \infty$, we automatically have
$$s_c = 1 - \frac{2}{p-1} < 1.$$

5.3 Definitions and initial results

Here we define classes of initial data related to the angular modulation
$$e^{ik\theta}u(x)$$
in general dimension. In particular, these spaces are generalizations of $e^{ik\theta}H_r^1(\mathbb{R}^2) \cap H^1(\mathbb{R}^2)$. We demonstrate weighted Sobolev type embeddings for these spaces.

5.3.1 Definition of semiradial Sobolev spaces as well as modulated semiradial Sobolev spaces

Let $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ be a partition of $\{1, \ldots, N\}$, and notate $\alpha_j = \{\alpha_{j,1}, \ldots, \alpha_{j,m}\}$. We define
$$x_{\alpha_j} = (x_{\alpha_{j,1}}, \ldots, x_{\alpha_{j,m}}).$$
For a function $u : \mathbb{R}^N \to \mathbb{C}$, for $\{a_j\}_{j=1}^N \subset \mathbb{R}$, we will use the notation
$$u(x_1 = a_1, x_2 = a_2, \ldots, x_N = a_N) = u(a_1, \ldots, a_N) \tag{5.3.1}$$

irrespective of the order of inputs on the left hand side of (5.3.1), so that e.g.

$$u(x_2 = 2, x_3 = 3, x_1 = 1) = u(1, 2, 3).$$

We now generalize radial functions and axisymmetric functions. For a partition α, we consider functions which are symmetric in each α_j.

Definition 5.3.1. Given a partition, α, of $\mathbb{R}^N$, for each $x \in \mathbb{R}^N$, define a point $x^* \in \mathbb{R}^N$ corresponding to x by $x^*_{\alpha_{j,1}} = |x_{\alpha_j}|$, and $x^*_{\alpha_{j,k}} = 0$ for $j \in \{1, \dots, |\alpha|\}$, and $k \geq 2$. We say a function u on $\mathbb{R}^N$ is *semiradial in* α if, for any $x \in \mathbb{R}^N$,

$$u(x) = u(x^*).$$

In this case, using the notation $\mathbb{R}^M_+ = \{(x_1, \dots, x_M) \in \mathbb{R}^M : x_1, \dots x_M \geq 0\}$, we identify $u = u_{\mathbb{R}^N} : \mathbb{R}^N \to \mathbb{C}$ with a function

$$u(r) = u_{\mathbb{R}^{|\alpha|}} = u(r_1, \dots, r_{|\alpha|}) : \mathbb{R}^{|\alpha|}_+ \to \mathbb{C},$$

defined by

$$u_{\mathbb{R}^{|\alpha|}}(r) = u_{\mathbb{R}^N}(r^*),$$

with $r^* \in \mathbb{R}^N$ defined by $r^*_{\alpha_{j,1}} = r_j$, and $r^*_{\alpha_{j,k}} = 0$ for $j \in \{1, \dots, |\alpha|\}$, and $k \geq 2$. We remark that for α_j satisfying $|\alpha_j| = 1$, it is reasonable to remove the domain restriction $r_j \geq 0$. In the sequel we will use this identification of $u_{\mathbb{R}^N}$ and $u_{\mathbb{R}^{|\alpha|}}$ freely, sometimes without a subscript to indicate domain.

To consider some examples, a function on $\mathbb{R}^5$ is semiradial in $\{\{1, 2\}, \{3, 4, 5\}\}$ if the function only depends on the two values

$$|x_{\{1,2\}}| = |(x_1, x_2)| = \left(x_1^2 + x_2^2\right)^{1/2}$$

and

$$|x_{\{3,4,5\}}| = |(x_3, x_4, x_5)| = \left(x_3^2 + x_4^2 + x_5^2\right)^{1/2}.$$

A function on $\mathbb{R}^N$ is radial if and only if it is semiradial in $\{\{1,\ldots,N\}\}$, and a function on $\mathbb{R}^3$ is axisymmetric about the x_3 axis if and only if it is semiradial in $\{\{x_1,x_2\},\{x_3\}\}$.

We define the spaces $H^1_\alpha(\mathbb{R}^N)$ and $L^p_\alpha(\mathbb{R}^N)$ as the spaces consisting of functions which are semiradial in α, and which are also in $H^1(\mathbb{R}^N)$ or $L^p(\mathbb{R}^N)$, respectively.

For a partition α with $\alpha_1 = \{1,2\}$, and $k \in \mathbb{Z}$, we define the space $H^1_{\alpha,k}(\mathbb{R}^N)$ to be the functions

$$e^{ik\theta}u(x)$$

which are in $H^1(\mathbb{R}^N)$, such that u is semiradial in α. For the special case of $H^1_{\{\{1,2\}\},k}(\mathbb{R}^2)$, we will use notation $\mathcal{H}_k = H^1_{\{\{1,2\}\},k}(\mathbb{R}^2)$.

In general, for $u \in H^1_{\alpha,k}(\mathbb{R}^N)$, we will define u^* to be the function, semiradial in α, such that $u(x) = e^{ik\theta}u^*(x)$. Note that for $u \in H^1_{\alpha,k}(\mathbb{R}^N)$, $\nabla e^{ik\theta} \perp \nabla u^*$, $u^* \in H^1_\alpha(\mathbb{R}^N)$, and pointwise a.e.

$$|\nabla u|^2 = |\nabla u^*|^2 + \left| \frac{k}{|x_{\{1,2\}}|} u^* \right|^2. \tag{5.3.2}$$

For $u, v \in H^1_{\alpha,k}(\mathbb{R}^N)$, we have

$$\langle u, v\rangle_{L^2(\mathbb{R}^N)} = \langle u^*, v^*\rangle_{L^2(\mathbb{R}^N)}.$$

We also have, using identified points, the a.e. pointwise identity

$$|\nabla_{x_1,\ldots,x_N} u^*_{\mathbb{R}^N}(x)| = |\nabla_{r_1,\ldots r_{|\alpha|}} u^*_{\mathbb{R}^{|\alpha|}}(r)|.$$

5.3.2 Weighted Sobolev inequalities for semiradial data

An estimate of Strauss implies a weighted $L^\infty(\mathbb{R}^N)$ bound on radial $H^1(\mathbb{R}^N)$ functions,

$$|u(x)| \lesssim |x|^{-\frac{N-1}{2}} \|u\|_{H^1(\mathbb{R}^N)}.$$

We generalize this to weighted Sobolev estimates on the semiradial spaces $H^1_\alpha(\mathbb{R}^N)$ and the modulated semiradial spaces $H^1_{\alpha,k}(\mathbb{R}^N)$. In particular, we prove the following result.

Theorem 5.3.2. *Consider $k \geq 1$, and a partition α such that $\alpha_1 = \{1, 2\}$, and*

$$\min_{\alpha_j \in \alpha, j \geq 2} |\alpha_j| \geq 3,$$

and define

$$p^*_{|\alpha|} = \frac{2\,|\alpha|}{|\alpha| - 2}.$$

If $|\alpha| \geq 3$, then the inequality

$$\left\| |x_{\alpha_1}|^{\frac{|\alpha_1|-1}{|\alpha|}} \cdots \left|x_{\alpha_{|\alpha|}}\right|^{\frac{\left|\alpha_{|\alpha|}\right|-1}{|\alpha|}} u \right\|_{L^{p^*_{|\alpha|}}(\mathbb{R}^N)} \lesssim \|u\|_{\dot{H}^1(\mathbb{R}^N)} \tag{5.3.3}$$

holds for $u \in H^1_{\alpha,k}(\mathbb{R}^N)$. For $|\alpha| = 2$, given any $2 < q < \infty$, the inequality

$$\left\| |x_{\alpha_1}|^{(|\alpha_1|-1)\left(\frac{1}{2}-\frac{1}{q}\right)} |x_{\alpha_2}|^{(|\alpha_2|-1)\left(\frac{1}{2}-\frac{1}{q}\right)} u \right\|_{L^q(\mathbb{R}^N)} \lesssim \|u\|_{H^1(\mathbb{R}^N)} \tag{5.3.4}$$

holds for all $u \in H^1_{\alpha,k}(\mathbb{R}^N)$.

If α is instead some partition with

$$\min_{\alpha_j \in \alpha} |\alpha_j| \geq 3,$$

then if $|\alpha| \geq 3$, the estimate (5.3.3) holds on $H^1_\alpha(\mathbb{R}^N)$, or if $|\alpha| = 2$, the estimate (5.3.4) holds on $H^1_\alpha(\mathbb{R}^N)$.

We establish these estimates by passing to lower dimensional weighted spaces and applying Sobolev inequalities.

Proof. We begin with the case $|\alpha| \geq 3$. If $u \in H^1_\alpha(\mathbb{R}^N)$ satisfies the hypothesis of the theorem, including

$$\min_{\alpha_j \in \alpha} |\alpha_j| \geq 3,$$

we have

$$\left\| \left|x_{\alpha_1}\right|^{\frac{|\alpha_1|-1}{|\alpha|}} \cdots \left|x_{\alpha_{|\alpha|}}\right|^{\frac{\left|\alpha_{|\alpha|}\right|-1}{|\alpha|}} u_{\mathbb{R}^N} \right\|_{L^{p^*_{|\alpha|}}(\mathbb{R}^N)} \cong \left\| r_1^{\frac{|\alpha_1|-1}{2}} \cdots r_{|\alpha|}^{\frac{\left|\alpha_{|\alpha|}\right|-1}{2}} u_{\mathbb{R}^{|\alpha|}} \right\|_{L^{p^*_{|\alpha|}}(\mathbb{R}^{|\alpha|}_+)}$$

$$\lesssim \left\| r_1^{\frac{|\alpha_1|-1}{2}} \cdots r_{|\alpha|}^{\frac{\left|\alpha_{|\alpha|}\right|-1}{2}} u_{\mathbb{R}^{|\alpha|}} \right\|_{\dot{H}^1(\mathbb{R}^{|\alpha|}_+)} \tag{5.3.5}$$

$$\lesssim \|u_{\mathbb{R}^N}\|_{\dot{H}^1(\mathbb{R}^N)}, \tag{5.3.6}$$

where Sobolev embedding for dimension $|\alpha|$ is used in line (5.3.5), and the application of Hardy's inequality

$$\left\| \frac{1}{|x_{\alpha_j}|} v \right\|_{L^2(\mathbb{R}^N)} = \left\| \left\| \frac{1}{|x_{\alpha_j}|} v \right\|_{L^2_{\alpha_j}(\mathbb{R}^{|\alpha_j|})} \right\|_{L^2_{\{1,\ldots,N\}\setminus\alpha_j}(\mathbb{R}^{N-|\alpha_j|})}$$

$$\lesssim \left\| \left\| \nabla_{\alpha_j} v \right\|_{L^2_{\alpha_j}(\mathbb{R}^{|\alpha_j|})} \right\|_{L^2_{\{1,\ldots,N\}\setminus\alpha_j}(\mathbb{R}^{N-|\alpha_j|})}$$

$$= \left\| \nabla_{\alpha_j} v \right\|_{L^2(\mathbb{R}^N)},$$

valid for $|\alpha_j| \geq 3$, is used in line (5.3.6) after application of the product rule.

For $u \in H^2_{\alpha,k}(\mathbb{R}^N)$ satisfying the hypothesis of the theorem, a modification of the above argument using (5.3.2) demonstrates

$$\left\| \left|x_{\{1,2\}}\right|^{\frac{1}{|\alpha|}} \left|x_{\alpha_2}\right|^{\frac{|\alpha_2|-1}{|\alpha|}} \cdots \left|x_{\alpha_{|\alpha|}}\right|^{\frac{\left|\alpha_{|\alpha|}\right|-1}{|\alpha|}} u_{\mathbb{R}^N} \right\|_{L^{p^*_{|\alpha|}}(\mathbb{R}^N)} \lesssim \left\| r_1^{\frac{1}{2}} r_2^{\frac{|\alpha_2|-1}{|\alpha|}} \cdots r_{|\alpha|}^{\frac{\left|\alpha_{|\alpha|}\right|-1}{2}} u^*_{\mathbb{R}^{|\alpha|}} \right\|_{\dot{H}^1(\mathbb{R}^{|\alpha|}_+)}$$

$$\lesssim \left(\|u^*_{\mathbb{R}^N}\|^2_{\dot{H}^1(\mathbb{R}^N)} + \left\| \frac{1}{|x_{\{1,2\}}|} u^*_{\mathbb{R}^N} \right\|^2_{L^2(\mathbb{R}^N)} \right)$$

$$\lesssim \|u_{\mathbb{R}^N}\|_{\dot{H}^1(\mathbb{R}^N)}.$$

The proof of the theorem for $|\alpha| = 2$ with α satisfying either given hypothesis follows by the above arguments, replacing the Sobolev embedding $\dot{H}^1(\mathbb{R}^{|\alpha|}) \hookrightarrow L^{p^*_{|\alpha|}}(\mathbb{R}^{|\alpha|})$ with the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$. ∎

5.4 Ground states, and proof of Theorem 5.2.1

In this section, we prove Theorem 5.2.1. Recall our previous notation $\mathcal{H}_k = H^1_{\{\{1,2\}\},k}(\mathbb{R}^2)$. We begin by establishing preliminary results regarding Weinstein functionals acting on $\mathcal{H}_k$. Though we work with $N = 2$, we reference standard objects which are typically defined in general dimension. For ease of reading, we will at times still use N to refer to dimension.

5.4.1 Existence and properties of minimizers

Consider the following Weinstein functional

$$W(u) = \frac{\|\nabla u\|_{L^2(\mathbb{R}^N)}^{\frac{N(p-1)}{2}} \|u\|_{L^2(\mathbb{R}^N)}^{2-\frac{(N-2)(p-1)}{2}}}{\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}}, \tag{5.4.1}$$

defined in general dimension on $H^1(\mathbb{R}^N) \setminus \{0\}$. The Gagliardo–Nirenberg inequality,

$$\|u\|_{L^{p+1}(\mathbb{R}^N)}^{p+1} \leq C_{GN} \|\nabla u\|_{L^2(\mathbb{R}^N)}^{\frac{N(p-1)}{2}} \|u\|_{L^2(\mathbb{R}^N)}^{2-\frac{(N-2)(p-1)}{2}},$$

implies that W is bounded from below, and if C_{GN} is a sharp constant, then

$$\frac{1}{C_{GN}} = \inf_{u \in H^1(\mathbb{R}^N)} W(u).$$

Note that, restricted to the set $\mathcal{H}_k \subset H^1(\mathbb{R}^2)$, the Gagliardo–Nirenberg inequality holds with some sharp constant $C_{GN,k}$, possibly distinct from the sharp constant for the full set $H^1(\mathbb{R}^2)$, so that

$$\frac{1}{C_{GN,k}} = \inf_{u \in \mathcal{H}_k} W(u) \geq \frac{1}{C_{GN}}.$$

The following lemma asserts that W restricted to $\mathcal{H}_k$ attains a minimum.

Lemma 5.4.1. *There exists some $\phi_k \in \mathcal{H}_k$, so that*

$$W(\phi_k) = \frac{1}{C_{GN,k}}.$$

In general, we will use ϕ_k to denote an $\mathcal{H}_k$ function satisfying $W(\phi_k) = \frac{1}{C_{GN,k}}$, with scaling or other restrictions stated when necessary. We will establish Lemma 5.4.1 with only a slight variation to construction of minimizers in [53].

Proof. Let $\{\psi_n\}$ be a minimizing sequence of the functional W in $\mathcal{H}_k$. By scaling and Kato's inequality, we can take this sequence so that for each n,

$$\|\psi_n\|_{L^2(\mathbb{R}^2)} = \|\nabla\psi_n\|_{L^2(\mathbb{R}^2)} = 1, \qquad\qquad \psi_n(x) = e^{ik\theta}\psi_n^*(x) = e^{ik\theta}|\psi_n^*(x)|.$$

Note that by the compact embedding $H_r^1(\mathbb{R}^2) \hookrightarrow L^{p+1}(\mathbb{R}^2)$ [53], the sequence $\{\psi_n^*\}$ is precompact in $L^{p+1}(\mathbb{R}^2)$, and in particular, after passing to a subsequence, $\{\psi_n^*\}$ is Cauchy in $L^{p+1}(\mathbb{R}^2)$. Further, note that for any $n, m \in \mathbb{Z}_+$,

$$\begin{aligned} |\psi_n - \psi_m| &= \left| e^{ik\theta}\psi_n^* - e^{ik\theta}\psi_m^* \right| \\ &= |\psi_n^* - \psi_m^*| \,. \end{aligned}$$

This implies that $\{\psi_n\}$ is in fact also Cauchy in $L^{p+1}(\mathbb{R}^2)$. Hence we can pass to a subsequence of $\{\psi_n\}$ which is weakly convergent in $H^1(\mathbb{R}^2)$, and strongly convergent in $L^{p+1}(\mathbb{R}^2)$. It is readily verified that there is a single function $\phi \in \mathcal{H}_k$ so that

$$\psi_n \to \phi \text{ weakly in } H^1(\mathbb{R}^2),\ L^2(\mathbb{R}^2) \text{ and } \dot{H}^1(\mathbb{R}^2), \text{ and strongly in } L^{p+1}(\mathbb{R}^2).$$

This weak convergence implies $\|\phi\|_{L^2(\mathbb{R}^2)} \leq 1$, and $\|\nabla\phi\|_{L^2(\mathbb{R}^2)} \leq 1$. Note that

$$\begin{aligned} C_{GN,k} &= \lim_{n\to\infty} W(\psi_n) \\ &= \frac{1}{\lim_{n\to\infty} \|\psi_n\|_{L^{p+1}(\mathbb{R}^2)}^{p+1}} \\ &= \frac{1}{\|\phi\|_{L^{p+1}(\mathbb{R}^2)}^{p+1}}. \end{aligned}$$

So

$$
\begin{aligned}
1 &\geq \|\nabla\phi\|_{L^2(\mathbb{R}^2)}^{\frac{N(p-1)}{2}} \|\phi\|_{L^2(\mathbb{R}^2)}^{2-\frac{(N-2)(p-1)}{2}} \\
&= W(\phi)\, \|\phi\|_{L^{p+1}(\mathbb{R}^2)}^{p+1} \\
&\geq C_{GN,k}\, \|\phi\|_{L^{p+1}(\mathbb{R}^2)}^{p+1} \\
&= 1.
\end{aligned}
$$

Hence $\|\nabla\phi\|_{L^2(\mathbb{R}^2)} = \|\phi\|_{L^2(\mathbb{R}^2)} = 1$, and

$$W(\phi) = C_{GN,k},$$

completing the proof. $\square$

The following lemma establishes convergence of $C_{GN,k}$ to zero as $k \to \infty$.

Lemma 5.4.2. *The constants $C_{GN,k}$ satisfy*

$$C_{GN} > C_{GN,1} > C_{GN,2} > C_{GN,3} > \ldots,$$

and

$$C_{GN,k} \to 0 \text{ as } k \to \infty.$$

Proof. First note that the existence of minimizers and (5.3.2) implies

$$C_{GN} > C_{GN,1} > C_{GN,2} > C_{GN,3} > \ldots.$$

Let $\{\phi_k\}$ be minimizers of W over $\{\mathcal{H}_k\}$ with $\|\phi_k\|_{L^2(\mathbb{R}^2)} = \|\nabla\phi_k\|_{L^2(\mathbb{R}^2)} = 1$. It is sufficient to show that

$$\lim_{k\to\infty} \|\phi_k\|_{L^{p+1}(\mathbb{R}^2)}^{p+1} = 0.$$

The a.e. pointwise inequality

$$k\left|\frac{\phi_k}{x}\right| \lesssim |\nabla\phi_k|,$$

following from (5.3.2), implies that for any $r > 0$,

$$\begin{aligned}\|\phi_k\|_{L^2(\{|x|<r\})} &\leq \frac{r}{k} \left\| k \left| \frac{\phi_k}{x} \right| \right\|_{L^2(\{|x|<r\})} \\ &\lesssim \frac{r}{k}.\end{aligned}$$

Hence

$$\sup_{r>0} \lim_{k\to\infty} \|\phi_k\|_{L^2(\{|x|<r\})} = 0. \tag{5.4.2}$$

Note that each $\phi_k^* \in H_r^1(\mathbb{R}^2)$, and so if x_0 is some point with $|x_0|$ sufficiently large, we have the bound

$$\begin{aligned}\|\phi_k\|_{L^2(B_{x_0}(1))} &\lesssim \frac{1}{|x_0| - 1} \|\phi_k\|_{L^2(\{|x_0|-1-1<|x|<|x_0|+1\})} \\ &\leq \frac{1}{|x_0| - 1}.\end{aligned}$$

This, together with (5.4.2), implies that

$$\lim_{k\to\infty} \sup_{y_0\in\mathbb{R}^2} \|\phi_k\|_{L^2(B_{y_0}(1))} = 0.$$

By a theorem of Lions [54][Lemma 1.21], this implies

$$\lim_{k\to\infty} \|\phi_k\|_{L^{p+1}(\mathbb{R}^2)}^{p+1} = 0,$$

and the proof is complete. □

5.4.2 Proof of Theorem 5.2.1

We now prove the main theorem of this chapter.

Proof of Theorem 5.2.1. For $u_0 \in \mathcal{H}_k$, by symmetry, the corresponding solution u of (NLS) is in $\mathcal{H}_k$ for all time. Hence, given $\mathcal{H}_k$ initial data, on the full interval of existence, the improved Gagliardo–Nirenberg inequality

$$\|u\|_{L^{p+1}(\mathbb{R}^2)}^{p+1} \leq C_{GN,k} \|\nabla u\|_{L^2(\mathbb{R}^2)}^{\frac{N(p-1)}{2}} \|u\|_{L^2(\mathbb{R}^2)}^{2-\frac{(N-2)(p-1)}{2}}$$

holds.

Now,

$$E[u]\left\|u\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} \geq \frac{1}{2}\left\|\nabla u\right\|_{L^2(\mathbb{R}^2)}^2\left\|u\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} - \frac{C_{GN,k}}{p+1}\left(\left\|\nabla u\right\|_{L^2(\mathbb{R}^2)}^2\left\|u\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}}\right)^{\frac{N(p-1)}{4}}.$$

Recalling that

$$\frac{1}{2}x - \frac{C_{GN}}{p+1}x^{\frac{N(p-1)}{4}}$$

attains a maximum value of

$$E[u_Q]\left\|u_Q\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}}$$

at

$$x = \left\|\nabla u_Q\right\|_{L^2(\mathbb{R}^2)}^2\left\|u_Q\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}},$$

a computation shows that

$$\frac{1}{2}x - \frac{C_{GN,k}}{p+1}x^{\frac{N(p-1)}{4}}$$

attains a maximum value of

$$\left(\frac{C_{GN}}{C_{GN,k}}\right)^{\frac{4}{N(p-1)-4}} E[u_Q]\left\|u_Q\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}}$$

at

$$x = \left(\frac{C_{GN}}{C_{GN,k}}\right)^{\frac{4}{N(p-1)-4}} \left\|\nabla u_Q\right\|_{L^2(\mathbb{R}^2)}^2\left\|u_Q\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}}.$$

We will show that the theorem holds with

$$\gamma_k = \left(\frac{C_{GN}}{C_{GN,k}}\right)^{\frac{4}{N(p-1)-4}}.$$

Given $u_0 \in \mathcal{H}_k$ with

$$E[u_0]\left\|u_0\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} < \left(\frac{C_{GN}}{C_{GN,k}}\right)^{\frac{4}{N(p-1)-4}} E[u_Q]\left\|u_Q\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}},$$

this implies that if

$$\left\|\nabla u_0\right\|_{L^2(\mathbb{R}^2)}^2\left\|u_0\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} < \left(\frac{C_{GN}}{C_{GN,k}}\right)^{\frac{4}{N(p-1)-4}} \left\|\nabla u_Q\right\|_{L^2(\mathbb{R}^2)}^2\left\|u_Q\right\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}},$$

then

$$\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 \|u\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} < \left(\frac{C_{GN}}{C_{GN,k}}\right)^{\frac{4}{N(p-1)-4}} \|\nabla u_Q\|_{L^2(\mathbb{R}^2)}^2 \|u_Q\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}}$$

on the full interval of existence (and u exists for all time), and if

$$\|\nabla u_0\|_{L^2(\mathbb{R}^2)}^2 \|u_0\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} > \left(\frac{C_{GN}}{C_{GN,k}}\right)^{\frac{4}{N(p-1)-4}} \|\nabla u_Q\|_{L^2(\mathbb{R}^2)}^2 \|u_Q\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}},$$

then

$$\|\nabla u\|_{L^2(\mathbb{R}^2)}^2 \|u\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} > \left(\frac{C_{GN}}{C_{GN,k}}\right)^{\frac{4}{N(p-1)-4}} \|\nabla u_Q\|_{L^2(\mathbb{R}^2)}^2 \|u_Q\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}}$$

on the full interval of existence.

If we also have both

$$\|\nabla u_0\|_{L^2(\mathbb{R}^2)}^2 \|u_0\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} > \left(\frac{C_{GN}}{C_{GN,k}}\right)^{\frac{4}{N(p-1)-4}} \|\nabla u_Q\|_{L^2(\mathbb{R}^2)}^2 \|u_Q\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}},$$

and $|x|u_0 \in L^2(\mathbb{R}^2)$, then we can apply the virial identity to obtain

$$\begin{aligned}
\|u\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} V_{tt}[u] &= 4N(p-1)E[u] \|u\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} - 2(N(p-1)-4) \|\nabla u\|_2^2 \|u\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} \\
&\leq 4N(p-1)E[u] \|u\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} - 4N(p-1) \left(\frac{C_{GN}}{C_{GN,k}}\right)^{\frac{4}{N(p-1)-4}} E[u_Q] \|u_Q\|_{L^2(\mathbb{R}^2)}^{2\frac{1-s_c}{s_c}} \\
&< 0,
\end{aligned}$$

and u blows up in finite time in both time directions. This completes the proof of the main theorem for fixed k. Note that since $s_c > 0$, we have $N(p-1) > 4$, and so the divergence of

$$\gamma_k = \left(\frac{C_{GN}}{C_{GN,k}}\right)^{\frac{4}{N(p-1)-4}}$$

as $k \to \infty$ follows from Lemma 5.4.2. $\square$

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Appendix A

On Potentials in 3 Dimensions

The results in Chapter 3 hold for potentials providing sufficient local theory and smallness condition on $\left\| |x|^2 B_\tau \right\|_\infty$.

We have interest in the potentials considered by D'Ancona, Fanelli, Vega, and Visciglia [16], for which they have developed Strichartz estimates. In dimensions 4 and greater, they consider potentials with a smallness condition on $\left\| |x|^2 B_\tau \right\|_\infty$, similar to our assumption. In dimension 3, this is replaced with a smallness condition on $\left\| |x|^{3/2} B_\tau \right\|_{L^2_r L^\infty(S_r)}$, where the $L^2_r L^\infty(S_r)$ norm is given by (3.1.18). Due to the differing assumptions, we discuss construction of potentials in dimension 3, which imply existence of potentials having nonzero trapping component, for which Strichartz estimates hold, and for which our results hold. We also note that these constructions have several straightforward generalizations.

Consider a function $f : \mathbb{R}^+ \to \mathbb{R}$ such that $f(|x|^2) \in C^1_{loc}(\mathbb{R}^N \setminus \{0\})$. We construct a potential

$$A \coloneqq f(|x|^2)\,(-x_2, x_1, 0)\,.$$

For $x \neq 0$, we have

$$\nabla \cdot A = 0.$$

If we also assume $|x|\, f(|x|^2) \in L^4_{loc}(\mathbb{R}^N)$, then the Leinfelder-Simander Theorem implies that Δ_A is essentially self adjoint on $C_0^\infty(\mathbb{R}^N)$. The potential A has trapping component

$$B_\tau = \frac{2\left(f(|x|^2) + |x|^2 f'(|x|^2)\right)}{|x|}\,(-x_2, x_1, 0)\,.$$

In particular, for $\alpha \in \mathbb{R}$, and $f(r) = \frac{1}{r^\alpha}$, we have

$$B_\tau = \frac{2(1-\alpha)}{|x|^{2\alpha+1}}(-x_2, x_1, 0).$$

By gluing, given any $\beta \neq 1$, and $\gamma \neq 1$, we can construct a potential A such that β is minimal satisfying

$$\left\| |x|^\beta A \right\|_{L^\infty(B(1))} < \infty,$$

$\beta + 1$ is minimal satisfying

$$\left\| |x|^{\beta+1} B_\tau \right\|_{L^\infty(B(1))} < \infty,$$

γ is minimal satisfying

$$\| |x|^\gamma A \|_{L^\infty(\mathbb{R}^N \setminus B(1))} < \infty\,,$$

and $\gamma + 1$ is minimal satisfying

$$\| |x|^{\gamma+1} B_\tau \|_{L^\infty(\mathbb{R}^N \setminus B(1))} < \infty\,.$$

It is straightforward that there exist similar constructions respecting the norm $\| |x|^\alpha B_\tau \|_{L^2_r L^\infty(S_r)}$ rather than $\| |x|^\alpha B_\tau \|_{L^\infty(B(1))}$. In other words, we can construct potentials with relative freedom over the strength of singularity at 0 as well as decay rate, and for $N \geq 3$ the main theorem holds for a nontrivial class of magnetic potentials.