An Introduction to Smale Spaces and their Homology

Robi Huq

Department of Mathematics

University of Colorado Boulder

Thesis Advisor:
Robin Deeley, Department of Mathematics

Honors Council Representative:
Nathaniel Thiem, Department of Mathematics

Outside Reader:
Michael Hermele, Department of Physics

Defended: April 7th, 2023

Abstract

Dynamics is often described as the evolution of spaces in time. This field is vast and often leads us to fascinating areas such as deterministic chaos, fractals, and applications to other fields of mathematics and science. Many dynamical systems are too chaotic or complicated to study their behavior through direct computation so we often use probabilistic or topological tools to understand them. This thesis will be primarily on the chaotic and hyperbolic systems known as Smale Spaces. Loosely speaking, hyperbolic in the context of dynamics is when an open ball expands in one direction and contracts in another. Their expansive/contractive behavior gives us a plethora of useful properties to study their behavior. In this thesis we will give a brief introduction to the main definitions of topological dynamics. Then, we shall follow Smale and Ruelle's work to give a rigorous introduction to Smale spaces and the main theorems that determine their structure. After this, we shall discuss one of the more important examples known as a shift of finite type which has a totally disconnected topology and can be used to construct a continuous, surjective, dynamic preserving map (factor map) onto any Smale space. We will then introduce Putnam's homology theory for Smale spaces. Finally, discuss new methods of computing this invariant using results obtained during an REU at CU Boulder in 2021 and results of Proietti and Yamashita.

Acknowledgments

This work on dynamical systems has had a major impact on my career and life choices. I have never learned so much in such a short amount of time. The tremendous experience of studying these systems has been a leading factor in my decision to pursue a PhD in mathematics. Words cannot express my appreciation for the people who supported me throughout the creation of this thesis and my undergraduate career.

First off, I would like to thank my thesis advisor, Robin Deeley. He taught my first analysis course, where I fell in love with mathematical proofs, and made an infamous course seem intuitive and fascinating. In the summer afterwards, he hosted an REU where I was exposed to how interdisciplinary and diverse mathematical research can be. Robin Deeley has been incredibly supportive and helpful throughout the time I've had at CU Boulder. I could not have asked for a better advisor. I would also like to thank the committee for overseeing this defense.

I would also like to thank the people who took part in the REU. The graduate students Rachel Chaiser, Maeve Coates-Welsh, Levi Lorenzo, Maggie Reardon, and Andrew M. Stocker have been incredibly helpful in leading discussions during the REU and cultivating an accepting community. Likewise, it was a pleasure problem solving and working with my fellow undergraduates, Annika Farhner, Jamal Giornoz, and Jose Oyola-Cortes.

Outside of the mathematical community, I would like to thank my family and friends. My mother and father have raised me to be curious and seek out difficult problems to solve. Moreover, my friends have always supported me in my academic pursuits. I would like to thank Nicholas Entin, Katie Jacobson, and Ella Stritzel for supporting me and my studies throughout undergraduate and specifically my work here.

My time as an undergraduate here has taught me that mathematics is an uniquely human field. It is the discovery of our collective imagination of hypothetical and theoretical ideas. The mathematical community at CU Boulder has been incredibly accepting and has encouraged me to further pursue mathematics, and I could not possibly list everyone who has helped me through this journey. Maybe the real mathematics was the friends we made along the way. I am grateful for all the teachers, classmates, mentors, and friends who have helped me on this journey.

Table of Contents

Introduction

Dynamical systems is one of the most diverse fields of mathematics. One can study solutions to differential equations, number theory, geometric properties, Hamiltonian mechanics, statistical mechanics, and countless more areas of research. The study of a repeated transformation or rule $f : X \to X$ can be applied to almost any field. However, this simple idea can lead to complicated theory and beautiful results. One such example are chaotic dynamical systems where points in X can map to every part of X as time moves forward. Another such example is when X is a fractal-like object and understanding the dynamics can be extraordinarily difficult. We can use topological tools to understand the behaviors of these systems.

An important question in topology is "When are two spaces homeomorphic." If two spaces are homeomorphic, then it is a lot easier to work with the simplest model of the space. We have developed tools like fundamental group and (co)homology to attempt to answer this question. A similar question can be asked in dynamical systems. When are there dynamic preserving invertible maps between dynamical systems. That is, when are two dynamical systems topological conjugate?

In this thesis, we will be studying a particular chaotic type of dynamical system known as a Smale space. The Smale space is a compact topological space X with discrete dynamics $f : X \to X$ such that sufficiently small balls simultaneously expand and contract. This behavior allows us to apply a multitude of tools to tame the chaos and understand Smale spaces. One particularly important example of Smale spaces is the shift of finite type which are the sets of bi-infinite sequences of edges along an infinite path of a graph. Shifts of finite type have an algebraic invariant known as the dimension group which can be generalized into Putnam's Homology for Smale spaces.

This thesis will start off with an introduction to topological dynamics. Following that, we will cover the main definitions, properties, and major theorems of the how Smale spaces behave. We will then focus on shifts of finite types and how they interact with dynamic preserving maps. Lastly, we will define Putnam's Homology for Smale spaces and discuss recent advancements in its computation.

1 Preliminaries

A dynamical system is loosely understood as "A system that changes in time." Of course, the sets that represents "time" can be different depending on the situation. One such example is a function that is applied to a set $\varphi : X \to X$ multiple times. As such, time is represented by $\mathbb{N}$. As another example, if we had a vector field on an manifold, M, we would get a flow $\varphi_t : M \to M$ which would represent a solution to an ordinary differential equation. In this case we would have some open subset of $\mathbb{R}$ as our time. We have have a multitude of representations of "time." As such this leads us to a general definition of a dynamical system.

Definition 1.1. Let X be a set and T be an abelian semi-group. We define a dynamical system as as a function $\varphi : T \times X \to X$ such that φ obeys a semi-group law $\varphi(s, \varphi(t, x)) = \varphi(s + t, x)$.

In this work, we will mostly be considering a discrete and invertible dynamical system by letting $T = \mathbb{Z}$. When the semi-group is $\mathbb{Z}$ we can view this type of dynamical system as (X, φ) where $\varphi : X \to X$ is an isomorphism of a category and X is an object of that category. For our purposes, φ will typically either be a homeomorphism or a diffeomorphism depending on whether X is topological space or a manifold.

In order to get a concrete understanding and to motivate some definitions, we will consider the following simple example.

Example 1.2. Consider the circle S^1 and the map $\varphi : S^1 \to S^1$ defined via $e^{2\pi i x} \mapsto e^{2\pi i(\alpha + x)}$ where $\alpha \in \mathbb{R}$. This map rotations all points by $2\pi\alpha$.

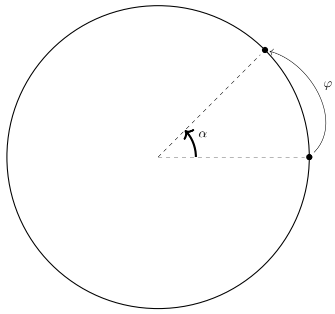


Figure 1: Circle Map of Example 1.2

Now, there are two possibilities, α can be rational or irrational. In the case where $\alpha = \frac{p}{q} \in \mathbb{Q}$, then for every point $\varphi^q(e^{2\pi x}) = e^{2\pi i(p+x)} = e^{2\pi i x}$. As we can see, after a finite number of iterations of φ, the point x returns to x. However, if α is irrational, there is no such integer that points can return after some number of iterations. Yet, if we follow any point $e^{2\pi i x} \in S^1$ the point will eventually get close to x.

More formally, let $\epsilon > 0$ and consider S^1 embedded in $\mathbb{C}$ as the unit circle. Let

$$B_\epsilon(x) = \{y \in S^1 : d(x, y) < \epsilon\}$$

be the open ball where d is the induced metric from the Euclidean metric of $\mathbb{C}$. Now for every $y \in S^1$ there exists $n \in \mathbb{N}$ such that the respective ball $B_\epsilon(y)$ has the property $\varphi^n(B_\epsilon(y)) \cap B_\epsilon(y) \neq \emptyset$

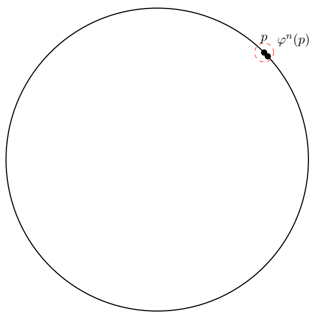


Figure 2: Non-wandering Point. For any sized ball, the point will return to it.

Example 1.2 highlights two important properties that related to points returning close to their starting position in our dynamical system. Returning to the general situation, we introduce the following definitions.

Definition 1.3. Let X be a set and $\varphi : X \to X$ be a bijection. We say x is a *periodic point* if there exists an integer n such that $\varphi^n(x) = x$. Moreover, define the set of periodic points of period n as

$$\mathrm{Per}_n(X, \varphi) = \{x \in X : \varphi^n(x) = n\}$$

And the set of all periodic points of any period as

$$\bigcup_{n\in\mathbb{N}} \mathrm{Per}_n(X,\varphi) = \mathrm{Per}(X,\varphi)$$

Definition 1.4. Let X be a topological space and $\varphi : X \to X$ be a homeomorphism. We say x is a *non-wandering point* if for every neighborhood U of x there exists an integer $n \geq 1$ such that $\varphi^n(U) \cap U \neq \emptyset$. Denote the set of non-wandering points as $\mathrm{NW}(X,\varphi)$.

Definition 1.5. Let X be a set and $\varphi : X \to X$ be a bijection. Take $x \in X$. Define the *orbit* of X to be

$$\mathrm{Orb}(x) = \{y \in X : \varphi^n(x) = y \text{ for some } n \in \mathbb{Z}\} = \{\dots, \varphi^{-1}(x), x, \varphi^1(x), \dots\}$$

Remark. Next, we can apply these definitions to the circle map in Example 1.2. If $\alpha = \frac{p}{q}$ is rational, then

$$S^1 = \mathrm{Per}_q(S^1,\varphi) = \mathrm{Per}(S^1,\varphi)$$

Since it is clear that $\mathrm{Per}(S^1,\varphi) \subseteq \mathrm{NW}(S^1,\varphi)$, it also follows that $\mathrm{NW}(S^1,\varphi) = S^1$. Moreover, due to every point being a periodic point with period q, we have

$$\mathrm{Orb}(x) = \{\varphi^{-q}(x), \dots \varphi^q(x)\}$$

which is finite.

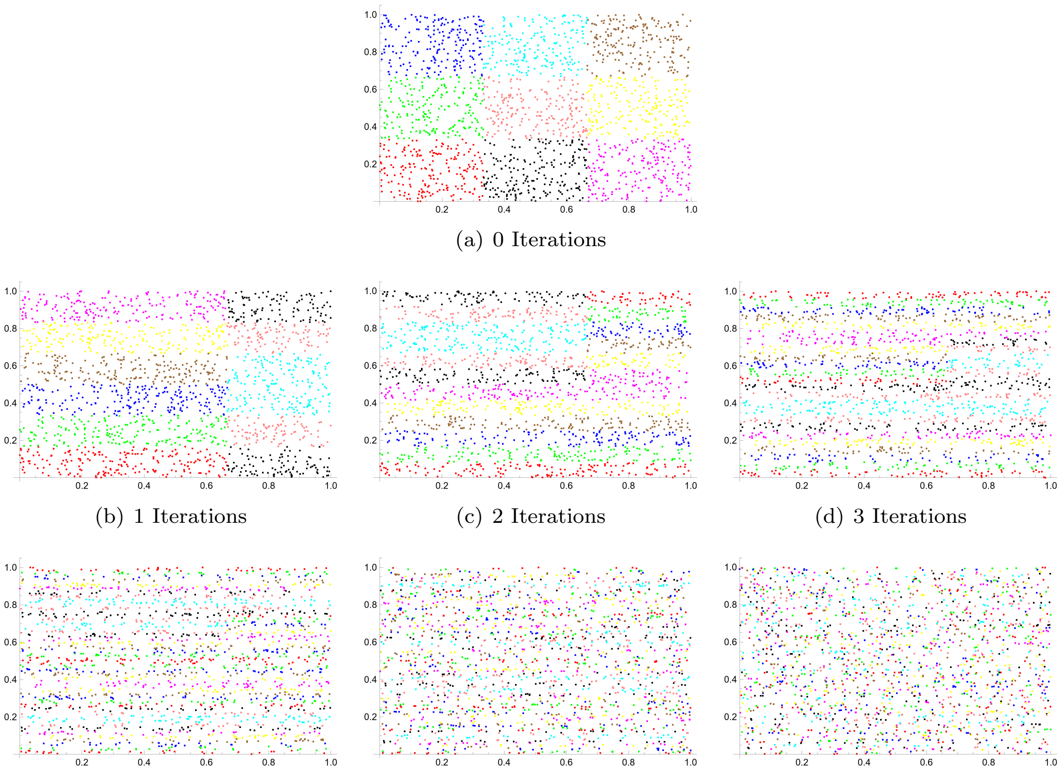
If α is irrational, then there is no integer n such that $\varphi^n(e^{2\pi ix}) = e^{2\pi ix}$ as it would contradict the irrationality of α. Thus we have $\mathrm{Per}(S^1,\varphi) = \emptyset$. Since there are no periodic points, $\mathrm{Orb}(e^{2\pi ix})$ must be infinite for each $e^{2\pi ix} \in S^1$. As discussed, for every $y \in S^1$, there exists $n \in \mathbb{N}$ such that $B_\epsilon(e^{2\pi ix})$ has the property $\varphi^n(B_\epsilon(e^{2\pi ix})) \cap B_\epsilon(e^{2\pi ix}) \neq \emptyset$ and thus every point is nonwandering and we once again have $\mathrm{NW}(S^1,\varphi) = S^1$.

For our next example, we shall investigate a more chaotic dynamical systems and define this chaotic behavior.

Example 1.6 (Baker's Map)**.** The baker's map is a map that emulates the idea of folding dough onto itself to mix ingredients. We define this map as $\varphi : [0,1]^2 \to [0,1]^2$

$$\varphi(x,y) = \begin{cases} \left(2x, \frac{1}{2}y\right) & 0 \leq x < \frac{1}{2} \\ \left(2(1-x), 1-\frac{1}{2}y\right) & \frac{1}{2} \leq x \leq 1 \end{cases}$$

To illustrate this map, we will take random points on $[0,1]^2$ and use Mathematica to apply the Baker's map and plot said points.



(a) 0 Iterations

(b) 1 Iterations (c) 2 Iterations (d) 3 Iterations

(e) 4 Iterations (f) 5 Iterations (g) 6 Iterations

Figure 3: Baker's Map Applied to Some Points

Notice if we take any open set of the form $(a,b)\times(c,d)\subseteq[0,1]^2$, then for any other open set $U\subseteq[0,1]^2$ after a suitable number of iterations, $\varphi^n\big((a,b)\times(c,d)\big)\cap U\neq\emptyset$. This motivates the idea of *mixing*.

Definition 1.7. Let X be a topological space and $f: X\to X$ be a homeomorphism. f is said to be *mixing* if for every open non-empty sets $U,V\subseteq X$, there exists a $N\in\mathbb{Z}$ such that $f^n(U)\cap V\neq\emptyset$ for all $n\geq N$

Lastly, we would like to define some notion of equivalence on dynamical systems. In other words, we want to define some sort of morphism between topological dynamical systems.

Definition 1.8. Let X and Y be topological spaces with $f: X\to X$ and $g: Y\to Y$ being homeomorphisms. A map between these dynamical systems $\varphi:(X,f)\to(Y,g)$ is defined to be a continuous function $\varphi: X\to Y$ such that $\varphi\circ f = g\circ\varphi$. Furthermore,

1. If φ is surjective, φ is known as a factor map.
2. If φ is a homeomorphism, φ is known as a topological conjugacy.

   If there exists a topological conjugacy between (X,f) and (Y,g), then (X,f) and (Y,g) are said to be topologically conjugate

A particularly useful case is when there the number of pre-images for every point in the codomain is bounded. We define this case as follows.

Definition 1.9. A map $\varphi : (X, f) \to (Y, g)$ is said to be finite-to-one if there exists $M \geq 1$ such that $|\varphi^{-1}(y)| \leq M$ for every $y \in Y$.

This case will prove quite useful in the later sections. But a simple result can be extracted on the periodic points

Proposition 1.10. *Suppose that $\varphi : (X, f) \to (Y, g)$ is finite-to-one. Then $x \in X$ is perodic if and only if $\varphi(x)$ also is.*

Proof. Suppose that $x \in X$ is periodic. Then there exists an n such that $f^n(x) = x$. Since φ is a map,$\varphi \circ f^n(x) = g^n \circ \varphi(x) = \varphi(x)$. Thus, $\varphi(x)$ is a periodic.

Conversely, suppose that $\varphi(x)$ is periodic. Take

$$\varphi(\{f^n(x) : n \in \mathbb{Z} \in\}) = \{g^n \circ \varphi(x) : n \in \mathbb{Z}\}$$

Since φ is finite to one, it follows that $\{f^n(x) : n \in \mathbb{Z}\}$ is finite if and only if $\varphi(\{f^n(x) : n \in \mathbb{Z}\})$ is. □

2 Smale Spaces

We will now introduce the main subject of study in this thesis. Instead of going straight to the definition, we shall introduce a simple example that allows us to visualize the main ideas and behaviors of a Smale space without making them seem overtly abstract. Once we have the motivation and definition, we shall go into the properties of their hyperbolic behavior where one set is expanding and other the contracting. We will also cover the most important theorems such as the decomposition theorem and the shadowing lemma. This section heavily relies off of Ruelle[13] and Putnams[10].

2.1 Motivating Example: Hyberbolic Toral Automorphism

We shall consider a famous example of a Smale space, the hyperbolic toral automorphism. This example often also goes by Arnold's Cat Map.

Example 2.1. Consider the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by the matrix multiplication:

$$f(v) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} v$$

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. An easy computation shows that the eigenvalues and corresponding eigenvectors are

$$v_1 = \begin{pmatrix} \frac{1+\sqrt{5}}{2} \\ 1 \end{pmatrix} \quad \omega_1 = \frac{3+\sqrt{5}}{2}$$

$$v_2 = \begin{pmatrix} \frac{1-\sqrt{5}}{2} \\ 1 \end{pmatrix} \quad \omega_2 = \frac{3-\sqrt{5}}{2}$$

Notice that $1 < \omega_1$ and $0 < \omega_2 < 1$. Moreover, $\omega_1^{-1} = \omega_2$. So let $\lambda = \omega_2 = \omega_1^{-1}$

Using linear algebra, we can determine the action of A on vectors in $\mathbb{R}^2$ through the eigenbasis of A. We can represent any vector in $\mathbb{R}^2$ as $c_1 v_1 + c_2 v_2$ for $c_1, c_2 \in \mathbb{R}$. By linearity,

$$f(c_1 v_1 + c_2 v_2) = c_1 f(v_1) + c_2 f(v_2) = c_1 \lambda^{-1} v_1 + c_2 \lambda v_2$$

Given that the eigenvalues are either greater than or less than one, we can the effect of f on the norm of an eigenvector to be.

$$\|c_1v_1\| < \|f(c_1v_1)\| = \lambda^{-1}\|c_1v_1\| \text{ and } \|c_2v_2\| > \|f(c_2v_2)\| = \lambda\|c_2v_2\|$$

In summary, f expands the eigenspace of ω_1 and contracts the eigenspace of ω_2

Now, we will use this map to induce a dynamical system on the torus T^2 which is a compact space. We will regard the torus as the usual quotient space $\mathbb{R}^2/\mathbb{Z}^2$. More explicitly, for $u_1, u_2 \in \mathbb{R}^2$

$$u_1 \sim u_2 \quad \text{if} \quad u_1 + \mathbf{n} = u_2 \quad \text{for some } \mathbf{n} \in \mathbb{Z}^2$$

We first want to show that f descends to a well defined function on the quotient space $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. Suppose we had $u_1 \sim u_2$. Since A consists of integer entries, $f(\mathbb{Z}^2) \subseteq \mathbb{Z}^2$

$$f\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} n \\ m \end{pmatrix}\right) = f\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + f\left(\begin{pmatrix} n \\ m \end{pmatrix}\right)$$

Since $f\left(\begin{pmatrix} n \\ m \end{pmatrix}\right) \in \mathbb{Z}^2$, it follows that $f\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) \sim f\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right)$.

Using this knowledge let $q : \mathbb{R}^2 \to T^2$ be the quotient space, we can define $\widetilde{f} : T^2 \to T^2$ by

$$\widetilde{f}(q(v)) = q(Av)$$

Since $u_1 \sim u_2$ implies that $f(u_1) \sim f(u_2)$, the map is well defined. Moreover, $\widetilde{f}$ is also invertible. Since $\det(A) = 1$, there exists an inverse

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Also since A^{-1} also integer entries, by the same argument, it induces a well defined map on T^2. Moreover, A^{-1} induces an inverse to $\widetilde{f}$ defined as $\widetilde{f}^{-1}(q(v)) = q(A^{-1}v)$.

$$\widetilde{f} \circ \widetilde{f}^{-1}(q(v)) = q(AA^{-1}v) = q(v)$$

$$\widetilde{f}^{-1} \circ \widetilde{f}(q(v)) = q(A^{-1}Av) = q(v)$$

This gives us a dynamical system $(T^2, \widetilde{f})$ where $\widetilde{f}$ is a diffeomorphism. Locally, that is for a small ϵ-ball of T^2, we have the same map $v \mapsto Av$. As such, the total derivative in local coordinates is simply $(D\widetilde{f})_x(v) = Av$. We can consider the eigenbasis from earlier and let $E^s_x = \text{span}\{v_2\}$ and $E^u_x = \text{span}\{v_1\}$ at each point, we get

$$T_xT^2 \cong E^s_x \oplus E^u_x$$

And if we recall the expanding/contracting behavior for the eigenbasis with $0 < \lambda < 1$:

$$\|(D\widetilde{f}^n)_x v\| < \lambda^n\|v\| \quad v \in E^u_x$$

$$\|(D\widetilde{f}^{-n})_x v\| < \lambda^n\|v\| \quad v \in E^s_x$$

Based on this expanding/contracting behavior, we can justly call E^s and E^u the stable and unstable subspaces respectively. Moreover, if we follow one of these tangent spaces, the property expanding/contracting behavior is maintained.

$$(D\widetilde{f}^n)_x E^s_x = E^s_{\widetilde{f}(x)}$$
$$(D\widetilde{f}^n)_x E^u_x = E^u_{\widetilde{f}(x)}$$

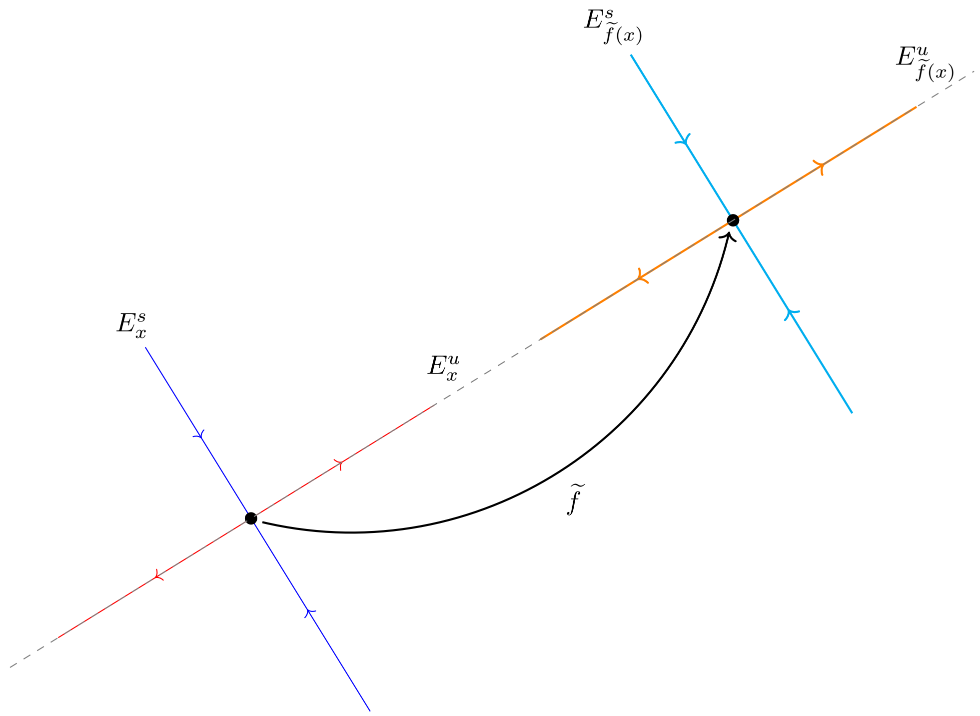


Figure 4: Decomposition of Hyperbolic Toral Automorphisms Tangent Spaces

The fact that the tangent space can be decomposed into a stable and unstable vector subspace is the underlying theme of hyperbolic dynamical systems when the underlying space is a manifold.

It is important to note that the majority of manifolds can not exhibit the same behavior as the hyperbolic toral automorphismin discussed in Example 2.1. However, there are often subsets of a manifold that do exhibit this behavior that do not have a manifold structure associated to them. These are known as hyperbolic sets. Smale originally kicked off this field by defining dynamical systems on non-wandering hyperbolic sets as Axiom A systems in [14]. More often than not, these hyperbolic sets were not manifolds and could be of a fractal like nature. Ruelle was able to extend the idea of a hyperbolic dynamical system in [13] to dynamical systems on a metric space with similar behavior which we now call Smale Spaces.

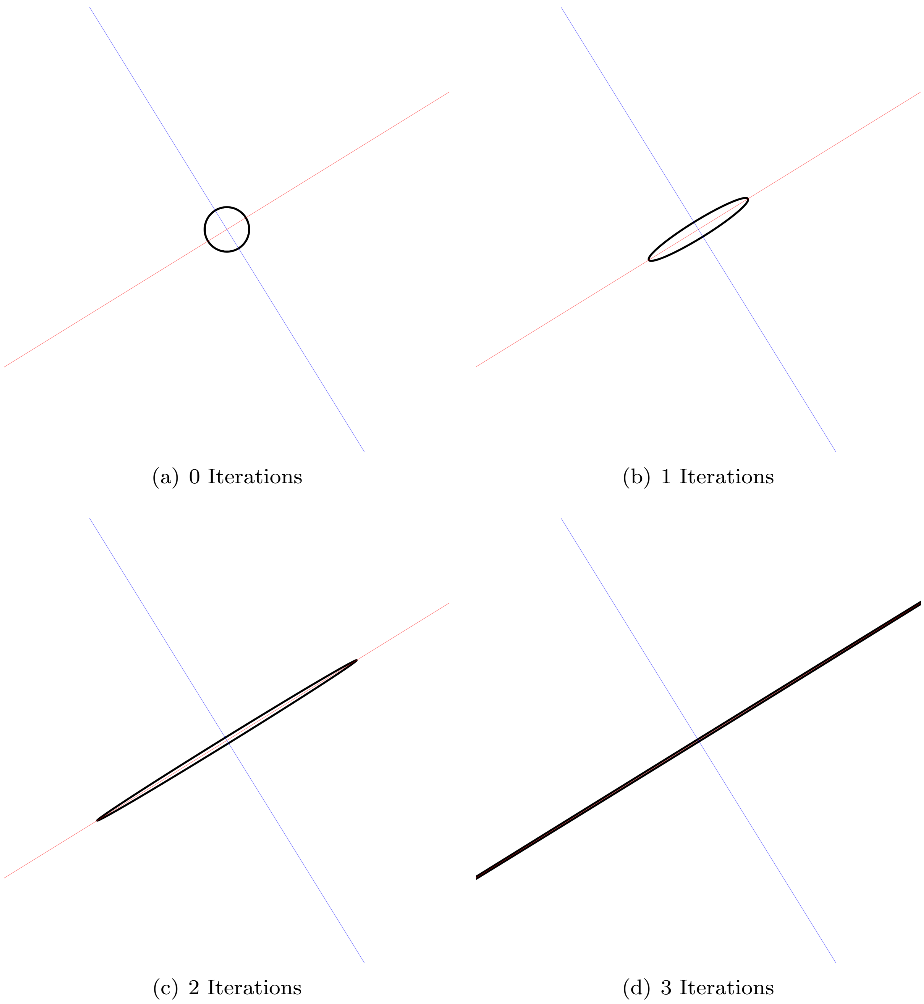
2.2 Smale Space Definition

The definition of a Smale space can be rather cumbersome and unintuitive because the axioms themselves never explicitly mention unstable and stable sets. So we will go through the axioms and attempt to illustrate the purpose of each one.

Let (X, d) be a metric spaces and $\varphi : X \to X$ be a homeomorphism. We want to define the correct axioms such that φ is hyperbolic. Recall in Example 2.1 that the hyperbolic behavior was captured in the tangent space. Unfortunately, a metric space need not have a tangent space, so we must make a generalization. Again, the hyperbolic behavior is local. So to capture this local behavior, let $\epsilon > 0$ and define

$$\Delta_\epsilon = \{(x, y) \in X : d(x, y) \leq \epsilon\} \tag{1}$$

The goal is to follow a small ball $B_\delta(x)$ through iterations of φ. To capture expanding/contracting behavior we must determine how the distance between points within $B_\delta(x)$ change under iterations of φ. If we return to Example 2.1 (the hyperbolic toral automorphism), we can plot the span of the stable/unstable vectors and look at the local behavior of a ball. We will label the span of the eigenvalues as red and blue.



(a) 0 Iterations (b) 1 Iterations (c) 2 Iterations (d) 3 Iterations

Figure 5: Hyperbolic Matrix effect on Ball

Now, given any two sufficiently close points x, y, there exists a third point z such that x and y lie on the stable/unstable spans respectively. $B_\delta(z)$ then follows the same process illustrated in Figure 5 while x and y stay on their respective lines. This leads us to the idea of the bracket map which in essence finds this point given two points close enough together. Following Ruelle[13], we will center the definition of the Smale Space around this Bracket axiom

Definition 2.2 (Smale Space)**.** Suppose (X, d) is a metric space, $\epsilon_X > 0$ is fixed, $f : X \to X$ is a homeomorphism, and $\Delta_{\epsilon_X} \subseteq X \times X$ as defined in (1) with the subspace topology. Let $[-,-]$ be a continuous function.

$$[-,-] : \Delta_{\epsilon_X} \to X$$

Such that the following axioms hold

Bracket Axioms

(B1) $[x, x] = x$

(B2) $[x, [y, z]] = [x, z]$

(B3) $[[x, y], z] = [x, z]$

(B4) $[f(x), f(y)] = f([x, y])$

Where in each of the cases above, the expression is defined

Contraction Axioms

There exists some $0 < \lambda_X < 1$

(C1) For y, z such that $x = [y, x] = [z, x]$ and $d(x, y), d(x, z) < \epsilon_X$,

$$d(f(y), f(z)) \leq \lambda_X d(y, z)$$

(C2) For y, z such that $x = [x, y] = [x, z]$ and $d(x, y), d(x, z) < \epsilon_X$,

$$d(f^{-1}(y), f^{-1}(z)) \leq \lambda_X d(y, z)$$

Then $(X, d, f, [-,-], \epsilon_X, \lambda_X)$ is known as a Smale Space. We will often refer to Smale spaces as simply (X, f) or $(X, d, f, [-,-])$ when appropriate.

To connect our motivation with our definition, we will informally consider the definition of the Smale space for next two pages. Let's try to visualize what this bracket map is doing. First consider Axioms **(C1)** and **(C2)**. Consider the point $x \in X$ reasonably close to $y, z \in X$ such that $x = [y, x] = [z, x]$. By the axiom, the distance between y and z will shrink and exhibit contracting behavior. Intuitively, this is because the bracket condition is implying that they are both in the contracting part of a neighborhood around x. Likewise, if we follow the conditions of **(C2)** instead, we have contracting behavior in the inverse direction, that is expanding behavior in the positive direction which is illustrated in Figure 6 for the hyperbolic toral automorphism.

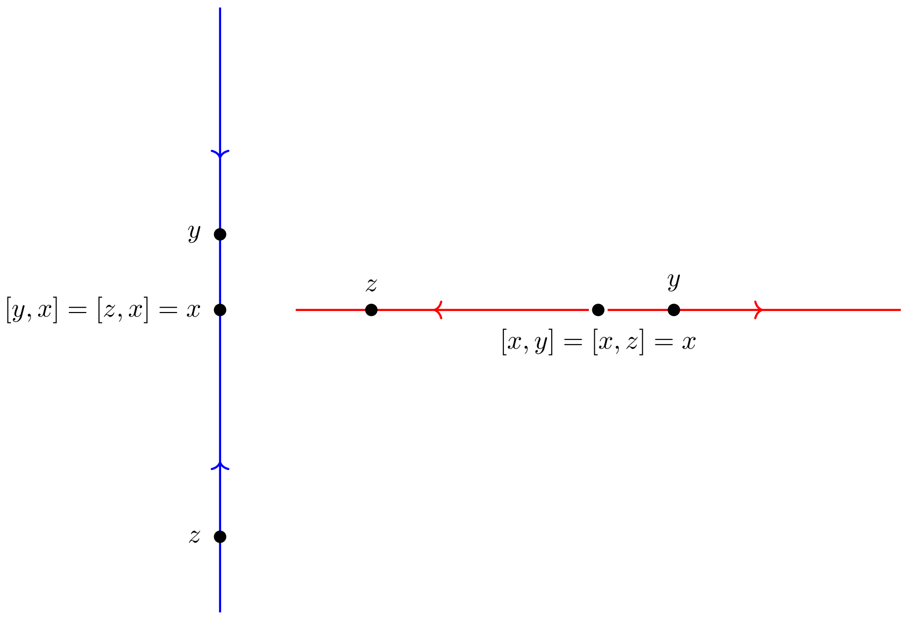


Figure 6: Axioms **C1** and **C2** in action

For the hyberbolic toral automorphism, the bracket map is the intersection of the span of E_x^s and E_y^u for the particular points input into the bracket. It can be illustrated in Figure 7 where the dotted lines are the part of span of the unstable/stable spaces.

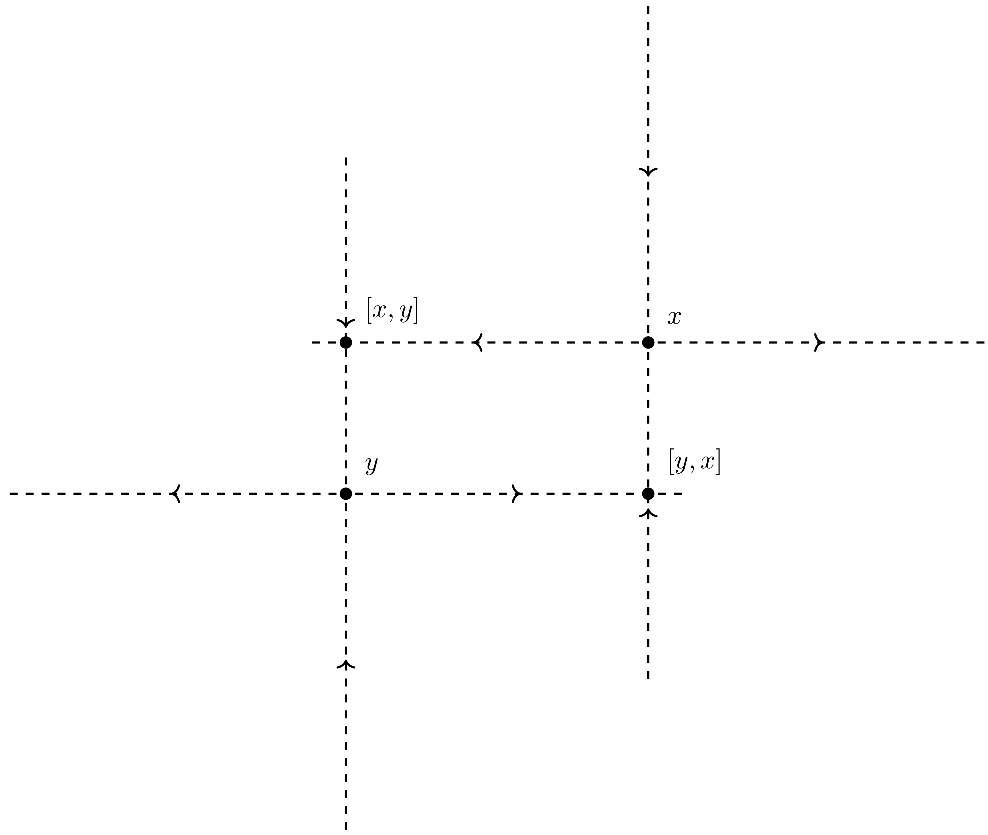


Figure 7: Bracket Map

Now building off of the idea that the bracket map encodes the hyperbolic behavior in the axioms, we shall now attempt to visualize Axioms **B1-B4**. Given $x, y \in X$, intuitively, $[x, y]$ is the point such that if we follow a small enough neighborhood, then x moves away from $[x, y]$ and y moves towards $[x, y]$ as we iterate the neighborhood through f. In other words, x is unstable with respect to $[x, y]$ and y is stable with respect to $[x, y]$, we can visualize the bracket as in Figure 7

Remark. It is important to note that the local stable and unstable sets (which we will rigorously define later) need not have a unique intersection point. The illustrations shown are meant for intuition purposely only.

We now return to consider Smale spaces and dynamical systems rigorously. Before, we talk about the stable/unstable sets of Smale spaces, we shall state a basic property that falls out of our definition for later proofs.

Proposition 2.3. *Let (X, f) be a Smale Space. Then*

1. $[x, y] = x$ *if and only if* $[y, x] = y$
2. $[x, y] = y$ *if and only if* $[y, x] = x$

Proof. To prove both these statements, one simply needs to apply the first three bracket axioms. Since the proofs are almost the same, we will only prove 1.

Suppose that $[x, y] = x$. Under the assumption we get

$$\begin{aligned} [y,x] &= [y,[x,y]] \\ &= [y,y] && \text{By axiom } \textbf{(B2)} \\ &= y && \text{By axiom } \textbf{(B1)} \end{aligned}$$

Which is what we wanted to show. Conversely, suppose that $[y, x] = y$. This implies

$$\begin{aligned} [x,y] &= [x,[y,x]] \\ &= [x,x] && \text{By axiom } \textbf{(B2)} \\ &= x && \text{By axiom } \textbf{(B1)} \end{aligned}$$

□

2.3 Stable/Unstable Behavior of Smale Spaces

Using the definition of the Smale space, we will now rigorously introduce the stable and unstable sets that we have been discussing informally. We will also rigorously justify, the illustrations from the previous subsection to demonstrate that we can use the stable/unstable behaviors to create a coordinate systems via the bracket map.

Definition 2.4 (Local Stable and Unstable Sets)**.** For $x \in X$, and $0 < \epsilon \leq \epsilon_X$, the local stable and unstable sets for this x respectively are defined as

$$X^s(x,\epsilon) = \{y \in X : d(x,y) < \epsilon, [y,x] = x\}$$

$$X^u(x,\epsilon) = \{y \in X : d(x,y) < \epsilon, [x,y] = x\}$$

Similar to what was discussed earlier in the previous section, $[x, y]$ picks out a stable point with respect to x and an unstable point with respect to y. More precisely, we have the following.

Proposition 2.5. *Let $x, y \in X$ for a Smale space (X, f) such that $d(x, y) < \epsilon_X$. Then for all $0 < \epsilon \leq \epsilon_X$*

1. *If $d(x, [x, y]) < \epsilon$, then $[x, y] \in X^s(x, \epsilon)$*
2. *If $d(y, [x, y]) < \epsilon$, then $[x, y] \in X^u(y, \epsilon)$*

Proof. We will assume that $d(x, [x, y]), d(y, [x, y]) < \epsilon \leq \epsilon_X$ and thus the bracket is defined. Now, take

$$\begin{aligned} [[x,y],x] &= [x,x] && \text{By axiom } \textbf{(B3)} \\ &= x && \text{By axiom } \textbf{(B1)} \end{aligned}$$

Since $d(x, [x, y]) < \epsilon_X$ and $[[x, y], x] = x$, by definition $[x, y] \in X^s(x, \epsilon_X)$. Likewise,

$$\begin{aligned} [y,[x,y]] &= [y,y] && \text{By axiom } \textbf{(B2)} \\ &= y && \text{By axiom } \textbf{(B1)} \end{aligned}$$

and once again $[x, y] \in X^s(y, \epsilon_X)$ □

Despite the expanding/contracting behavior not being directly in the definition of the local stable/unstable sets, these sets do indeed exhibit said behavior. The following lemma demonstrates that points move together forwards or backwards in time and can become arbitrarily close given enough time.

Lemma 2.6. *Let $(X, d, f, [-,-])$ be a Smale space. There exists a $\epsilon_1 > 0$ such that for all $0 < \epsilon \leq \epsilon_1$, and all $x, y \in X$*

1. $d(f^n(x), f^n(y)) < \epsilon$ *for all* $n \geq 0$ *if and only if* $y \in X^s(x, \epsilon)$
2. $d(f^n(x), f^n(y)) < \epsilon$ *for all* $n \leq 0$ *if and only if* $y \in X^u(x, \epsilon)$

Proof. We will only prove the first statement because the second statement requires an almost identical argument. First, using the continuity of $[-,-]$, there exists $0 < \epsilon_1 \leq \epsilon_X$ such that

$$d(x, y) < \epsilon_1 \implies d(x, [y, x]), d(x, [x, y]) < \epsilon_X$$

Take $0 < \epsilon \leq \epsilon_1$. Now fix some $x, y \in X$ such that $d(f^n(x), f^n(y)) < \epsilon$ for all $n \geq 0$. We want to show that $y \in X^s(x, \epsilon)$. Notice

$$d(f^n(x), f^n(y)) < \epsilon \leq \epsilon_X \qquad \text{By assumption}$$

Which implies $[f^n(y), f^n(x)]$ is well defined.

By axiom **(B2)** $x = [x, x] = [x, [y, x]]$. By axiom **(B4)**, we also have $f^n(x) = [f^n(x), f^n(x)] = [f^n(x), f^n([y, x])]$. Due to this property, we have

$$d(f^{-1} \circ f^n(x), f^{-1} \circ f^n([y, x])) \leq \lambda_X d(f^n x, f^n([y, x])) \qquad \text{By axiom } \mathbf{(C2)}$$

We can apply this process n times to get

$$\begin{aligned} d(x, [y, x]) &\leq \lambda_X^n d(f^n(x), f^n([y, x]) && \text{By } n \text{ applications of } \mathbf{C2} \\ &< \lambda_X^n \epsilon_X && \text{By choice of } \epsilon_1 \end{aligned}$$

Since this holds for any n, we can take n to infinity to get $d(x, [y, x]) = 0$ which of course implies $x = [y, x]$ and $y \in X^s(x, \epsilon)$.

Conversely, now assume that $y \in X^s(x, \epsilon)$. By definition $x = [y, x] = [x, x]$ By axiom **(C1)**, we have

$$d(f^n(y), f(x)) < \lambda_X^n d(y, z) < \epsilon \leq \epsilon_X$$

Thus, $[f^n(y), f^n(x)]$ is defined. Now, fix any n and apply axiom **(B4)** n times to get, $f^n(x) = [f^n(y), f^n(x)] = [f^n(x), f^n(x)]$. Then once again,

$$\begin{aligned} d(f^n(y), f^n(x)) &\leq \lambda_X^n d(y, x) && \text{By } n \text{ applications of axiom } \mathbf{(C1)} \\ &\leq \lambda_X^n \epsilon && \text{By definition of } X^s(x, \epsilon) \\ &< \epsilon \end{aligned}$$

Which is what we wanted to show.

□

These stable and unstable sets hold important topological information. When restricted to a small enough ϵ, $X^u(x,\epsilon) \times X^s(x,\epsilon)$ is a neighborhood of $x \in X$. This was illustrated in Figure 7 which we will rigorously state and prove in the following theorem.

Theorem 2.7. *There exists* $0 < \epsilon'_X \leq \frac{\epsilon_X}{2}$ *such that for every* $0 < \epsilon < \epsilon'_X$ *the bracket map restricted to*

$$[-,-] : X^u(x,\epsilon) \times X^s(x,\epsilon) \to X$$

is a homeomorphism onto its image which is open in X

Proof. Fix some $x \in X$. Since $[-,-]$ is continuous in both variables, both $[x,-]$ and $[-,x]$ are continuous. Thus, there exists some $0 < \delta \leq \epsilon_X$ such that $d(x,y) < \delta$ implies $d([x,x],[x,y]) < \epsilon_X/2$ and $d(x,y) < \delta$ implies $d([x,x],[y,x]) < \epsilon_X/2$. And of course by axiom **(B1)**, $[x,x] = x$. In summary, this gives us a δ such that

$$d(x,y) < \delta \leq \epsilon_X \implies d(x,[x,y]) < \frac{\epsilon_X}{2}$$
$$d(x,y) < \delta \leq \epsilon_X \implies d(x,[y,x]) < \frac{\epsilon_X}{2}$$

Once again, by definition of continuity for $[-,-]$, there exists $0 < \epsilon_X \leq \epsilon'_X/2$ such that

$$d(x,y), d(x,z) < \epsilon'_X \implies d(x,[y,z]) < \delta \leq \epsilon_X$$

Now, we wish to show that the bracket map restricted to $X^u(x,\epsilon) \times X^u(x,\epsilon)$ is well defined for any $0 < \epsilon \leq \epsilon'_X$. By definition, for $y \in X^u(x,\epsilon)$ and $z \in X^s(x,\epsilon)$, we have $d(x,y), d(x,z) < \epsilon \leq \epsilon'_X \leq \frac{\epsilon_X}{2}$. By the triangle inequality, $d(y,z) \leq d(x,y) + d(x,z) < \epsilon_X$ and thus $[y,z]$ is well defined.

We now wish to defined an inverse of $[-,-]$ restricted to $X^u(x,\epsilon) \times X^u(x,\epsilon)$. Define the following function

$$F(y) = ([y,x],[x,y])$$

which is continuous due to the bracket being continuous. Now, compose the bracket and F for $d(x,y) < \epsilon'_X$ to get

$$\begin{aligned} [-,-] \circ F(y) &= \big[[y,x],[x,y]\big] \\ &= \big[y,[x,y]\big] && \text{By axiom } \textbf{(B3)} \\ &= [y,y] && \text{By axiom } \textbf{(B2)} \\ &= y && \text{By axiom } \textbf{(B1)} \end{aligned}$$

Now, for the opposing composition, pick $y \in X^u(x,\epsilon)$ and $z \in X^s(x,\epsilon)$ to get

$$\begin{aligned} F([y,z]) &= \Big(\big[[y,z],x\big],\big[x,[y,z]\big]\Big) \\ &= \big([y,x],[x,z]\big) && \text{By axioms } \textbf{(B2)} \text{ and } \textbf{(B3)} \end{aligned}$$

By definition, of $X^u(x,\epsilon)$ and $X^s(x,\epsilon)$, $[x,y] = x$ and $[z,x] = x$. It follows that

$$= (y,z) \qquad \text{By Proposition 2.3}$$

Thus, $[-,-]$ restricted to $X^u(x,\epsilon) \times X^s(x,\epsilon)$ is a homeomorphism onto it's image. We now wish to show that the image of this map is an open set in X. Once again, let $y \in X^u(x,\epsilon)$ and $z \in X^s(x,\epsilon)$. Since $[-,-]$ is continuous with an inverse of F, we have

$$F(B([y,z],\delta'_{(y,z)})) \subseteq B(x,\epsilon - d(x,y)) \times B(x,\epsilon - d(x,z)) \subseteq X^u(x,\epsilon) \times X^s(x,\epsilon)$$

We can then apply $F^{-1} = [-,-]$ to determine that $B([y,z], \delta'_{(y,z)})$ is contained within the image of $[-,-]$. Now, take

$$\bigcup_{(y,z)\in X^u(x,\epsilon)\times X^s(x,\epsilon)} B([y,z], \delta'_{(y,z)})$$

To get the image of $[-,-]$ which of course is a union of open balls and is thus open. □

Theorem 2.8. *Let $(X, d, f, [-,-])$ be a Smale Space. There exists some ϵ_1 such that $d(x,y) \leq \epsilon_X$ and $d(x,[x,y]), d(y,[x,y]) < \epsilon_1$,*

$$\begin{aligned}\{[x,y]\} &= X^s(x,\epsilon_1) \cap X^u(y,\epsilon_1)\\ &= \bigcap_{n\in\mathbb{Z}} \{z \in X : d(f^{-n}(y), f^{-n}(z)), d(f^n(x), f^n(z)) < \epsilon_1\}\end{aligned}$$

Proof. Let ϵ_1 be the one defined in Lemma 2.6. We have the following set equivalence

$$\begin{aligned}&\bigcap_{n\in\mathbb{Z}} \{z \in X : d(f^{-n}(y), f^{-n}(z)), d(f^n(x), f^n(z)) < \epsilon_1\}\\ &= \bigcap_{n=0}^{\infty} \{z \in X : d(f^{-n}(y), f^{-n}(z)), d(f^n(x), f^n(z)) < \epsilon_1\}\\ &= \bigcap_{n=0}^{\infty} \Big(\{z \in X : d(f^{-n}(y), f^{-n}(z)) < \epsilon_1\} \cap \{z \in X : d(f^n(x), f^n(z)) < \epsilon_1\}\Big)\\ &= \left(\bigcap_{n=0}^{\infty} \Big(\{z \in X : d(f^{-n}(y), f^{-n}(z)) < \epsilon_1\}\right)\\ &\cap \left(\bigcap_{n=0}^{\infty} \Big(\{z \in X : d(f^n(x), f^n(z)) < \epsilon_1\}\right)\\ &= X^u(y,\epsilon_1) \cap X^s(x,\epsilon_1) \qquad \text{By Lemma 2.6}\end{aligned}$$

We now would like to show that the set above is a singleton. By Proposition 2.5, we must have $[x,y] \in X^u(y,\epsilon_1)$ and $[x,y] \in X^s(x,\epsilon_1)$. Thus $\{[x,y]\} \subseteq X^u(y,\epsilon_1) \cap X^s(x,\epsilon_1)$.

Now take $z \in X^u(y,\epsilon_1) \cap X^s(x,\epsilon_1)$. By definition, $[y,z] = y$ and $[z,x] = x$. Furthermore,

$$[z,y] = z \qquad [x,z] = z \qquad \text{By Proposition 2.3}$$

It then follows that

$$\begin{aligned}z &= [z,y]\\ &= [[x,z],y]\\ &= [x,y] \qquad \text{By axiom } \textbf{(B2)}\end{aligned}$$

Thus, it follows that $X^s(x,\epsilon_1) \cap X^u(y,\epsilon_1) = \{[x,y]\}$ which is what we wanted to show. □

Now, we shall return our attention to global stable/unstable sets. Recall the stretching of a ball in the hyperbolic toral automorphism in Figure 5. The ball rapidly flattens until it almost looks like a line. In Figure 5, we do not take the ball in the quiotent space. If we were to follow a ball on T^2, it would look like

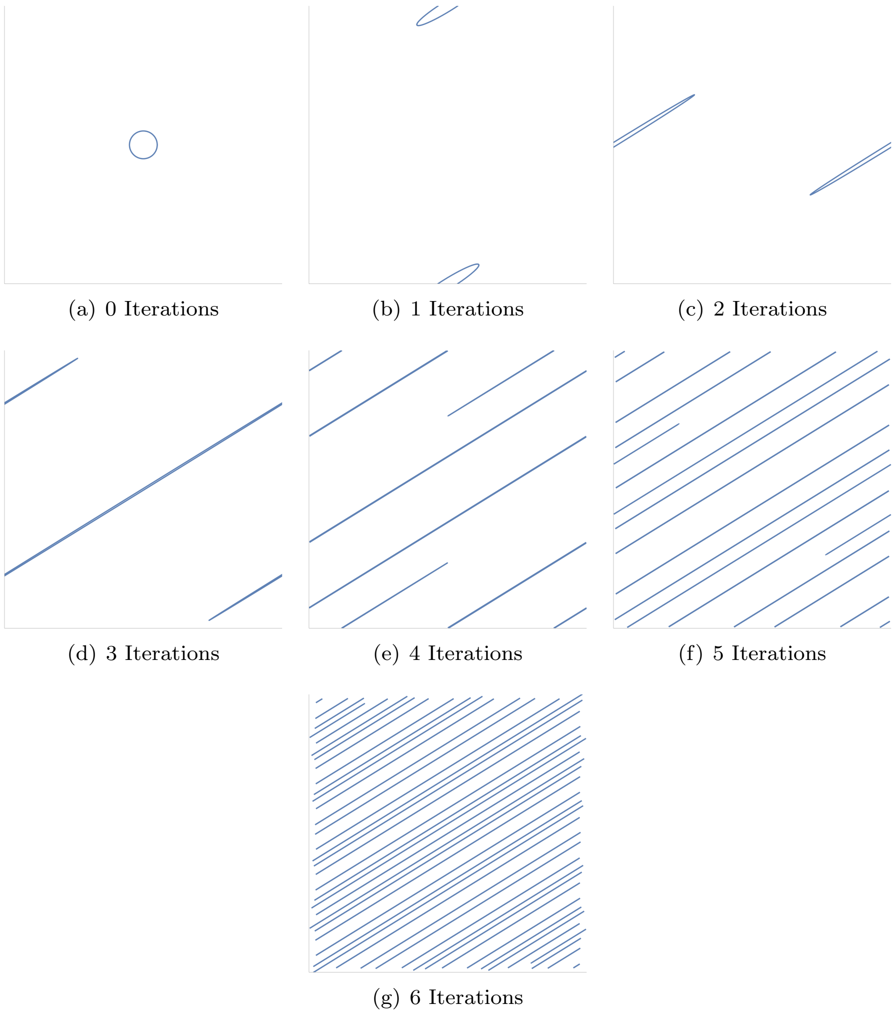
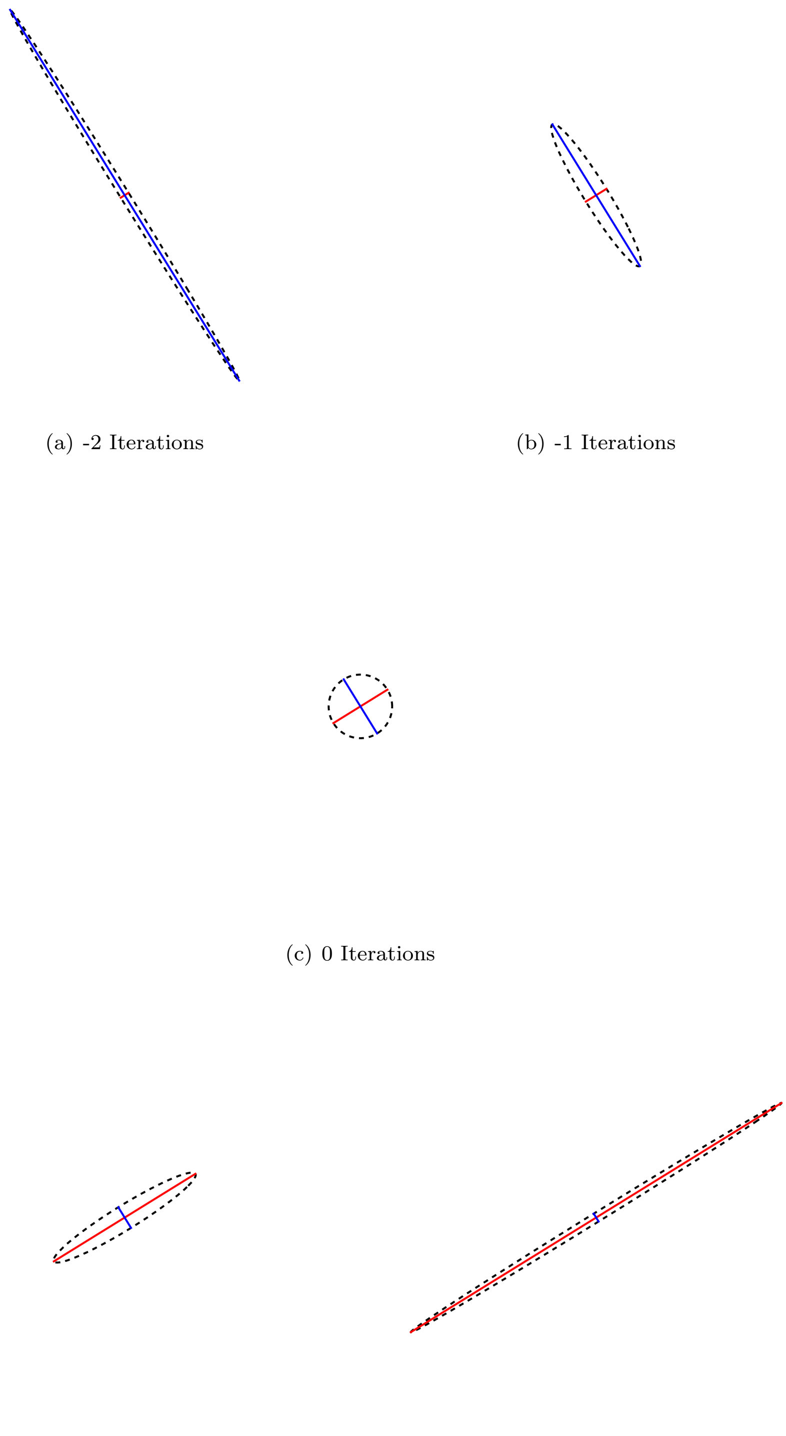


Figure 8: Hyperbolic Toral Automorphism on Ball

We can see the ball flatten itself extremely thin and wrap around the torus. Now, if we also focus on the ball itself, and the unstable/stable sets $(T^2)^u(x, \epsilon)$ and $(T^2)^s(x, \epsilon)$



(a) -2 Iterations

(b) -1 Iterations

(c) 0 Iterations

(d) 1 Iterations

(e) 2 Iterations

Figure 9: Hyperbolic Toral Automorphism Ball Stable/Unstable Growth

Notice that in Figure 9, we denote the stable local set $(T^2)^s(x,\epsilon)$ as blue and the unstable local set $(T^2)^u(x,\epsilon)$ as red. The entire ball $B(x,\epsilon)$ squashes down into into the other sets depending if the iterations increase or decrease. Then, this ball wraps around the entire torus as illustrated in 8. This brings up multiple properties we will further investigate. The first being, how does the local stable/unstable sets related to global stable/unstable behavior? This of course requires us to define global/unstable sets which we will do shortly. Then what would this global set look like? Would it always be dense in the entire space?

We shall now rigorously define the global stable/unstable set.

Definition 2.9 (Stably/Unstably Equivalence)**.** We define the following relations to be stably equivalent and unstably equivalent respectively

$$x \overset{s}{\sim} y \iff \lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$$

$$x \overset{u}{\sim} y \iff \lim_{n\to\infty} d(f^{-n}(x), f^{-n}(y)) = 0$$

Moreover, if two points are both stably and unstably equivalent, they are said to be *homoclinic*

$$x \overset{h}{\sim} y \iff \lim_{|n|\to\infty} d(f^n(x), f^n(y)) = 0$$

Definition 2.10 (Global Stable/Unstable Sets)**.** Take some $x \in X$, the global stable/unstable sets are defined to be the equivalence classes of $\overset{s}{\sim}$ and $\overset{u}{\sim}$ respectively. We will denote these by $X^s(x)$ and $X^u(x)$. Likewise, the equivalence class points homoclinic to x is denoted $X^h(x)$

The sets are crucial to understanding the behavior of a Smale space. One such example where the proof can be found in [11]. It gives a necessary and sufficient condition for when the Smale space is mixing.

Theorem 2.11. *(X, f) is a mixing Smale space if and only if the global unstable and stable sets $X^s(x)$ and $X^u(x)$ are dense for every $x \in X$ and any $x, y \in X$, $X^s(x) \cap X^u(y)$ is also dense in X*

Now, in order to draw a connection to the global and local behavior, we will rigorously investigate the stretching of the stable/unstable sets illustrated in Figure 9 with the following lemma

Lemma 2.12. *Let (X, f) be a Smale space, $0 < \epsilon \leq \epsilon_X$, and $x \in X$. Then*

$$f(X^s(x,\epsilon)) \subseteq X^s(f(x), \lambda_X\epsilon) \subseteq X^s(f(x),\epsilon) \tag{2}$$

$$f^{-1}(X^u(x,\epsilon)) \subseteq X^u(f^{-1}(x), \lambda_X\epsilon) \subseteq X^u(f^{-1}(x),\epsilon) \tag{3}$$

Proof. We will once again only prove one of the statements as the second one has am almost identical proof. Take any $f(y) \in f(X^s(x,\epsilon))$. By definition we have $d(x, y) < \epsilon$ and $[y, x] = x$. Since $d(f(y), f(x)) \leq \lambda_X d(y, x) < \lambda_X\epsilon < \epsilon_X$, $[f(y), f(x)]$ By axiom **(C1)** is defined. Moreover, by axiom **(B4)**, we also have $[f(y), f(x)] = f(x)$.

Since $d(f(x), f(y)) < \lambda_X\epsilon$ and $[f(y), f(x)] = f(x)$, it follows that $f(y) \in X^s(f(x), \lambda_X\epsilon) \subseteq X^s(f(x),\epsilon)$ which is what we wanted to show. □

Lemma 2.12 demonstrates that the local unstable/stable sets do indeed contradict/expand in the way we expect them to. As we take our iterations to infinity in the positive direction, the local unstable sets should stretch out to encapsulate the global unstable set. This connection between the global and local sets comes from the following proposition.

Proposition 2.13. *Let $x \in X$ and $0 < \epsilon \leq \epsilon_X$, then*

$$X^s(x) = \bigcup_{n=0}^{\infty} f^{-n}(X^s(f^n(x), \epsilon))$$

$$X^u(x) = \bigcup_{n=0}^{\infty} f^{n}(X^u(f^{-n}(x), \epsilon))$$

Proof. We will only show one of the properties above. Suppose that $y \in X^s(x)$ or in other words, $x \overset{s}{\sim} y$. Let ϵ_1 be from Lemma 2.6 and define $\epsilon' = \min\{\epsilon, \epsilon_1\}$. By definition, there exists $N \in \mathbb{N}$ such that

$$d(f^n(x), f^n(y)) < \epsilon' \qquad \text{for all } n \geq N$$

Hence, by Lemma 2.6, $f^N(y) \in X^s(f^N(x), \epsilon)$. Thus, $y \in f^{-N}(X^s(f^N(x), \epsilon))$. It follows that $X^s(x) \subseteq \bigcup_{n=0}^{\infty} f^{-n}(X^s(f^n(x), \epsilon))$.

Conversely, notice that from Lemma 2.12 that if $y \in X^s(x, \epsilon_X)$, then for every n, $f^n(y) \in X^s(f^n(x), \epsilon_X)$ and

$$d(f^n(y), f^n(x)) \leq \lambda_X^n d(x, y)$$

Taking $n \to \infty$ gives $\lim_{n\to\infty} d(f^n(y), f^n(x)) = 0$ and $x \sim y$. Thus we have $X^s(x, \epsilon_X) \subseteq X^s(x)$. This implies $f^{-n}(X^s(f^n(x), \epsilon)) \subseteq X^s(x)$. □

One immediate consequence of the previous proposition is that our bracket map is contained within the stable and unstable sets of their respect arguments. We get an analogous statement to Theorem 2.8, but this time for the global sets

Proposition 2.14. *For a Smale Space (X, f) and $x, y \in X$ such that $d(x, y) < \epsilon_X$, then $[x, y] \in X^s(x) \cap X^u(y)$*

Proof. From Proposition 2.5, we have $[x, y] \in X^s(x, \epsilon)$ and $[x, y] \in X^u(y, \epsilon)$ for small enough $0 \leq \epsilon \leq \epsilon_X$. It follows from Proposition 2.13 that $[x, y] \in X^s(x)$ and $[x, y] \in X^u(y)$ which is what we wanted to show. □

We would like to look at the structure of the global stable and unstable sets. Once again, recall the hyperbolic toral automorphism. We have

$$(T^2)^u(q(x), \epsilon) = \{q(x + tv_1) : |t| \leq \epsilon\}$$

$$(T^2)^s(q(x), \epsilon) = \{q(x + tv_2) : |t| \leq \epsilon\}$$

Just as we illustrated earlier, we get the global sets

$$(T^2)^u(q(x)) = \{q(x + tv_1) : t \in \mathbb{R}\}$$
$$(T^2)^s(q(x)) = \{q(x + tv_2) : t \in \mathbb{R}\}$$

and since both v_1 and v_2 have irrational components, the global stable and unstable sets are dense within T^2. Moreover, each of these sets are in one-to-one correspondence to $\mathbb{R}$ defined by $q(x + tv_1) \mapsto t$. This function is invertible and its inverse $t \mapsto\!\mapsto q(x + tv_1)$ is an immersion. It is clear that these global sets are non-embedded submanifolds which imply that we cannot use the subspace topology on these sets.

For a less smooth example, we shall introduce another famous example known as the n-solenoid.

Example 2.15 (n-Solenoid)**.** Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle. We shall denote elements of S^1 as equivalence classes using the notation $[z]_{\mathbb{Z}}$ for $z \in \mathbb{R}$. Let $n \geq 2$ and define the functions $g_n : S^1 \to S^1$

$$g_n([z]_{\mathbb{Z}}) = [nz]_{\mathbb{Z}}$$

Now, define the n-solenoid as the inverse limit:

$$S_n = \varprojlim(S^1, g_n) = \{(z_0, z_1, \dots) : g_n(z_{k+1}) = z_k, k \geq 1\}$$

With the metric

$$d((x_i)_{i\in n}, (y_i)_{i\in n}) = \sum_{i=0}^{\infty} \frac{1}{n^i} \inf\{|x_i - y_i + m| : m \in \mathbb{Z}\}$$

We can also define a function $f_n : S_n \to S_n$ via $f_n = \prod_{i\in\mathbb{N}} g_n$. So,

$$\begin{aligned}(z_0, z_1, z_2, \dots) &\overset{f_n}{\mapsto} (g_n(z_0), g_n(z_1), g_n(z_2), \dots)\\ &= (g_n(z_0), z_0, z_1, \dots)\end{aligned}$$

Define f_n^{-1} to be the shift to the left map. Notice that this is indeed an inverse.

$$\begin{aligned}(z_0, z_1, z_2, \dots) &\overset{f_n}{\mapsto} (g_n(z_0), z_0, z_1, \dots)\\ &\overset{f_n^{-1}}{\mapsto} (z_0, z_1, z_2, \dots)\end{aligned}$$

$$\begin{aligned}(z_0, z_1, z_2, \dots) &\overset{f_n^{-1}}{\mapsto} (z_1, z_2, z_3, \dots)\\ &\overset{f_n}{\mapsto} (g_n(z_1), g_n(z_2), g_n(z_3), \dots)\\ &= (z_0, z_1, z_2, \dots)\end{aligned}$$

The left shift is clearly continuous. Since f_n is the product of continuous functions restricted to S_n, it is also continuous. Thus f_n is a homeomorphism. Moreover, we can define the bracket map for $\epsilon_{S_n} = 1/2$,

$$[(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}] = \left(y_i + \frac{x_0 - y_0}{n^i}\right)_{i\in\mathbb{N}}$$

It is left as an exercise to show that S_n, f_n, ϵ_{S_X}, $\lambda < n^{-1}$, and $[-,-]$ above define a Smale space. Once that is done, the next reasonable step is to look at what kind of stable/unstable sets there are. So fix some $(x_i)_{i\in\mathbb{N}} \in S_n$ and consider the case in which $[(y_i)_{i\in\mathbb{N}}, (x_i)_{i\in\mathbb{N}}] = (x_i)_{i\in\mathbb{N}}$, we get

$$(x_i)_{i\in\mathbb{N}} = [(y_i)_{i\in\mathbb{N}}, (x_i)_{i\in\mathbb{N}}] = \left(x_i + \frac{y_0 - x_0}{n^i}\right)_{i\in\mathbb{N}}$$

By comparing each index, we get $x_i = x_i + \frac{y_0 - x_0}{n_i}$ which implies that $x_0 = y_0$. This gives us

$$S_n^s((x_i)_{i\in\mathbb{N}}, \epsilon) = \{(y_i)_{i\in\mathbb{N}} \in S_n : y_0 = x_0, d((x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}) < \epsilon\}$$

Notice if we fix $K \geq 1$

$$S^s_n((x_i)_{i\in\mathbb{N}}, n^{-K}) = \{(y_i)_{i\in\mathbb{N}} : y_i = x_i, 0 \leq i \leq K\}$$

Similarly, if $[(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}] = (x_i)_{i\in\mathbb{N}}$. We can note that

$$(x_i)_{i\in\mathbb{N}} = [(x_i)_{i\in\mathbb{N}}, (y_i)_{i\in\mathbb{N}}] = \left(y_i + \frac{x_0 - y_0}{n^i}\right)_{i\in\mathbb{N}}$$

And

$$S^u_n(x, \epsilon) = \{(y_i)_{i\in\mathbb{N}} \in S_n : y_k = x_k + t^{-nk}, |t| < \epsilon\}$$

Notice for the stable set, for each $i > K$, there are n possibilities for y_{K+1}, then each of these n possibilities yield n possibilities, meaning that there are n^n possible values for (y_{K+1}, y_{K+1}). Using this fact, one can show that $S^s_n((x_i)_{i\in\mathbb{N}}, n^{-K})$ is a Cantor set. Likewise, we can notice that $S^s_n(x, \epsilon)$ is in one-to-one correspondence with an interval. Proposition 2.13 tells us that the global unstable set is euclidean and the stable set is cantor which varies greatly from the topology of the solenoid itself.

There are a huge amount of solenoid examples. Wieler was able to come up with a condition in which one could construct a Smale space out of an inverse limit. Moreover, she discovered that a Smale space would have to be topologically conjugate to a space of this form if the local stable sets is totally disconnected. The following two theorems can be found in [15]

Theorem 2.16 (Wieler's Theorem A). *Let (Y, d) a compact metric space and $g : Y \to Y$ be continuous and surjective. If there exists constants $\beta > 0$, $K \geq 1$, and $0 < \gamma < 1$ such that*

1. *If $d(x, y) \leq \beta$*
$$d(g^K(x), g^K(y)) \leq \gamma^K d(g^{2K}(x), g^{2K}(y))$$
2. *For all $x \in Y$ and $0 < \epsilon \leq \beta$,*
$$g^K(B(g^K(x), \epsilon)) \subseteq g^{2K}(B(x, \gamma\epsilon))$$

Then

$$\left(\varprojlim(Y, g), \prod_{n\in\mathbb{N}} g\right)$$

is a Smale Space with totally disconnected local stable sets

Theorem 2.17 (Wieler's Theorem B). *Let (X, f) be an irreducible Smale space with totally disconnected local stable sets. Then (X, f) is topologically conjugate to a Smale space constructed as in Wieler's Theorem A.*

With the two examples above, one may note that the subspace topology on the global stable and unstable sets may not give us useful information due the density or behavior of the sets. We wish to define a different topology on $X^s(x)$ and $X^u(x)$ and we will rely off on union in Proposition 2.13 by using the the inductive limit topology of this union. We conclude this section by listing off the important point set topological properties of our global stable/unstable sets[11].

Theorem 2.18. *Let the topology of the global stable and unstable be define by the basis in 1,2. The following properties also follow.*

1. *$X^s(y, \epsilon)$ for $0 \leq \epsilon \leq \epsilon_X$ form a basis for the topology of $X^s(x)$*
2. *$X^u(y, \epsilon)$ for $0 \leq \epsilon \leq \epsilon_X$ form a basis for the topology of $X^u(x)$*
3. *$X^s(x)$ and $X^u(x)$ are locally compact and Hausdorff*
4. *A sequence $y_n \in X^s(x)$ converges to y if and only if $y_n \to y$ in X and $[y_n, y] = y$ for all $n \geq N$ for some N*
5. *A sequence $y_n \in X^u(x)$ converges to y if and only if $y_n \to y$ in X and $[y, y_n] = y$ for all $n \geq N$ for some N*
6. *$f : X^s(x) \to X^s(f(x))$ is a homeomorphism*
7. *$f : X^u(x) \to X^u(f(x))$ is a homeomorphism*

2.4 The Shadowing Lemma

In the previous section, we have built up the main structure of Smale spaces. These ideas and tools will prove invaluable for unraveling important properties of these systems. Dynamical systems, Smale spaces included, can often be quite chaotic. This makes dynamical difficult to simulate on computers. However, the extra structure of stable/instability allows us to extract information about these systems using the tools we just developed.

We will first introduce the idea of shadowing and pseudo-orbits which allow us to track the trajectory of points. Then, we will connect it the concept of mixing.

The idea of a pseudo-orbit is similar to the idea of an orbit. We will slightly reword the definition of a orbit from Definition 1.5 for some dynamical systems (X, f). An orbit is a countable set of points $\{x_n\}_{n\in\mathbb{Z}}$ such that $x_{n+1} = f(x_n)$. That is, for this countable set of points, the function f points towards the next point. We can then generalize this statement to a pseudo-orbit by allowing $f(x_n)$ to be close to x_{n+1}. To make a hazy analogy, one can image that a countable set of points $\{x_n\}$ want to be an orbit, but f is slightly inaccurate and f shoots x_n to a point nearby x_{n+1}. We illustrate this in Figure 10 by allowing red points to be $f(x_i)$ and black points being element of the sequence $\{x_n\}$

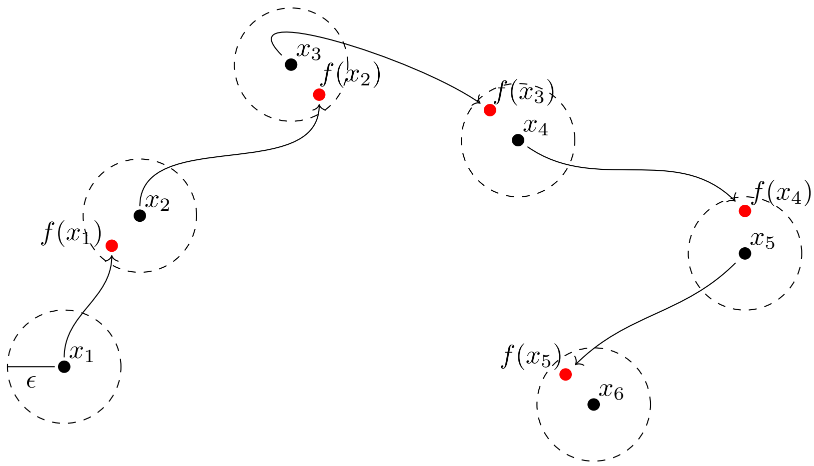


Figure 10: ϵ-Pseudo-orbit

Definition 2.19 (Interval in $\mathbb{Z}$)**.** $I = (a, b)$ is said to be an interval in $\mathbb{Z}$, if it is of the form

$$I = (a, b) = \{n \in \mathbb{Z} : a < n < b\}$$

For $a \in \mathbb{Z} \cup \{-\infty\}$ and $b \in \mathbb{Z} \cup \{\infty\}$

Definition 2.20 (ϵ-Pseudo-Orbit)**.** Let (X, d) be a metric space and $f : X \to X$ be a homeomorphism. Let $\epsilon > 0$. An ϵ-pseudo-orbit over a non-empty interval I over $\mathbb{Z}$ is a collection of points $\{x_n\}_{\in I}$ such that for each $n, n+1 \in I$,

$$d(f(x_n), x_{n+1}) \leq \epsilon$$

Of course, we can see that the $\mathrm{Orb}(x) = \{x_n\}_{n \in I}$ is an ϵ-psuedo orbit since $d(f(x_n), x_{n+1}) = d(x_{n+1}, x_{n+1}) = 0$.

Definition 2.21 (δ-Shadowing)**.** Let $\epsilon > 0$ and $\delta > 0$. Let x_n be an ϵ-pseudo orbit over I. We say that x_n is δ-shadowed by (the orbit of) x if for all $n \in I$

$$d(x_n, f^n(x)) \leq \delta$$

Pseudo orbits that are shadowed by an orbit are an incredible useful tool. We wish to prove that pseudo-orbits can be shadowed in Smale spaces. In order to do so, we must prove a statement about the behavior of the local stable set of points if we have the behavior of an pseudo-orbit. Namely, when $d(f(x_n), x_{n+1}) < \epsilon$.

Lemma 2.22. *Let $0 < \delta_1 \leq \epsilon_X$. There exists an $\epsilon > 0$ such that if $d(f(x), x') < \epsilon$, then for all $z \in X^s(x, \delta_1)$, $d(x', f(z)) \leq \epsilon_X$ and $[x', f(z)] \in X^s(x', \delta_1)$.*

Proof. Define the compact set

$$A = \{(x, y, z) : d(x, y), d(y, z) \leq \epsilon_X/2, [y, z] = z\}$$

Also define the function $g : A \to \mathbb{R}$ as

$$g(x, y, z) = d(x, [x, z]) - d(y, z)$$

Since both the bracket and the metric are continuous so is g. Moreover, since A is compact, g is uniformly continuous. If $x = y$ in A, then

$$\begin{aligned} g(y, y, z) &= d(y, [y, z]) - d(y, z) \\ &= d(y, z) - d(y, z) \\ &= 0 \end{aligned}$$

Thus, by uniformly continuity, there exists $0 < \epsilon < \delta_1(1 - \lambda_X)$ such that

$$d(x, y) < \epsilon \implies |h(x, y, z)| < \delta_1(1 - \lambda_X)$$

Now, take $z \in X^s(x, \delta_1)$, $x, x' \in X$ such that $d(f(x), x') < \epsilon$. It follows from the prior statements that

$$\begin{aligned} d(x', f(z)) &\leq d(x', f(x)) + d(f(x), f(z)) && \text{Triangle inequality} \\ &\leq \epsilon + \lambda_X d(x, z) && \text{Axiom } \mathbf{C1} \text{ and assumption} \\ &\leq \epsilon + \lambda_X \delta_1 && \text{Since } z \in X^s(x, \delta_1) \\ &\leq (1 - \lambda_X)\delta_1 + \lambda_X \delta_1 \\ &= \delta_1 \\ &\leq \epsilon_X \end{aligned}$$

This gives us that $[x', f(z)]$ is well defined and $d(x', f(z)) \leq \epsilon_X$ which is the first thing we wanted to show. Also notice that $d(x', f(x)), d(f(x), x) \leq \epsilon_X/2$ and $[f(x), f(z)] = f([x, z]) = z$. Thus, $(x', f(x), f(z)) \in A$. By our uniform continuity argument, since $d(x', f(x)) < \epsilon$, we get

$$\begin{aligned} |h(x', f(x), f(z))| &= |d(x', [x, f(x)]) - d(f(x), f(z))| \\ &< \delta_1(1 - \lambda_X) \end{aligned}$$

We can then use this inequality to get

$$\begin{aligned} d(x', [f(x), f(z)]) &\leq |d(x', [x, f(x)]) - d(f(x), f(z))| + d(f(x), f(z)) \\ &\leq \delta_1(1 - \lambda_X) + \lambda_X d(x, z) \\ &\leq \delta_1(1 - \lambda_X) + \lambda_X \delta_1 \\ &= \delta_1 \end{aligned}$$

And lastly, $[x', f(z)] \in X^s(x', \delta_1)$ from Proposition 2.5

□

Using this lemma, we can show that δ-shadowing is common in Smale spaces which attests to the accuracy of pseudo-orbits. When used in the differential scenario, the following theorem is often called the "Shadowing Lemma."

Theorem 2.23. *Let (X, f) be a Smale space. For any $\delta > 0$, there exists a $\epsilon > 0$ such that every ϵ-psuedo orbit in X is δ-shadowed by an orbit of X*

Proof. Let $\delta > 0$. We may choose a $0 \leq \delta_1 \leq \epsilon_X/2$ such that

$$[X^u(x, \delta_1), X^s(x, \delta_1)] \subseteq B(x, \delta)$$

For any $x \in X$. Given this δ_1, we can choose an $\epsilon > 0$ according to Lemma 2.22 and let x_n be an ϵ-pseduo-orbit in X

Consider the case in which $I = (a, b)$ is finite. Without loss of generality, suppose that $a < 0$ and $b > 0$. Now according the previous lemma, the following functions are well defined

$$g_{s,n} : X^s(x_n, \delta_1) \to X^s(x_{n+1}, \delta_1) \quad g_{s,n}(y) = [x_{n+1}, f(y)] \quad a < n < b - 1$$

$$g_{u,n} : X^u(x_n, \delta_1) \to X^s(x_{n-1}, \delta_1) \quad g_{u,n}(y) = [f^{-1}(y), x_{n-1}] \quad a + 1 < n < b$$

Now, let g_s and g_u be defined by the respective functions above as follows.

$$g_s : \bigcup_{n \in I} X^s(x_n, \delta_1) \to \bigcup_{n \in I} X^s(x_{n+1}, \delta_1) \quad g_u \bigcup_{n \in I} : X^u(x_n, \delta_1) \to \bigcup_{n \in I} X^s(x_{n-1}, \delta_1)$$

$$g_s|_{X^s(x_n, \delta_1)} = g_{s,n} \quad g_u|_{X^u(x_n, \delta_1)} = g_{u,n}$$

For each $a < n < b$ define

$$S_n = [h^{b-1+n}(X^u(x_{b-1}, \delta_1)), g^{1-a+n}(X^s(x_{a+1}, \delta_1))]$$

We clearly have for all $a < n < b$,

$$g_u^{b-1+n}(X^u(x_{b-1}, \delta_1)) \subseteq X^u(x_n, \delta_1) \quad g_s^{1-a+n}(X^s(x_{a+1}, \delta_1)) \subseteq X^s(x_n, \delta_1)$$

Moreover, from our initial choice of δ_1, $S_n \subseteq B(x_n, \delta)$.

We want to show that $f(S_n) = S_{n+1}$. Take any $y \in X^s(x_{a+1}, \delta_1)$ and $z \in X^u(x_{b-1}, \delta_1)$ and set $y' = g_s^{n-a-1}(y)$ and $z' = g_u^{n-b}(z)$. This gives us $[g_u(z'), y'] \in S_n$ and $[z', g_s(y')] \in S_{n+1}$. Any element of S_n and S_{n+1} can be found this way. All is that is left to show is that f takes one to the other. Through multiple usages of our assumptions, we get

$$\begin{aligned} f([g_u(z'), y'] &= f\left([[f^{-1}(z'), x_n], y']\right) \\ &= \left[f\left([f^{-1}(z'), x_n]\right), f(y')\right] \\ &= [[z', f(x_n)], f(y')] \\ &= [z', f(y')] \\ &= [z', [x_{n+1}, f(y')]] \\ &= [z', g(y')] \end{aligned}$$

However, this implies that any $x \in S_0$ will have an orbit that δ-shadows the pseudo-orbit.

Now consider the case in which $I = \mathbb{Z}$. Define the set

$$S^{(n)} = [g_u^{n-1}(X^u(x_{n-1}, \delta_1)), g_s^{n-1}(X^s(x_{1-n}, \delta_1))]$$

By the previous case, all elements of $S^{(n)}$ will δ-shadow the pseudo orbit over $(-n, n)$. Moreover, the same property holds for the closure $\overline{S^{(n)}}$. Thus, we have $S^{(n)} \supseteq S^{(n+1)} \supseteq S^{(n+2)} \supseteq \ldots$ for any n. This gives us $\bigcap_{n\in\mathbb{Z}} S^{(n)}$ is non-empty and will have elements that δ-shadow the pseudo-orbit over $\mathbb{Z}$. □

Due to computational error, the calculations of a computer on a dynamical system are pseudo orbits. The shadowing lemma tells us that these pseudo-orbits can still be somewhat trusted as they are shadowed by an orbit of (X, f).

2.5 Decompositions of Smale Spaces

Suppose we had two Smale spaces (X, f) and (Y, g). It is clear that if we take the disjoint union, then $(X \sqcup Y, f \sqcup g)$ is also a Smale space. An important question to ask is when can we do this in the opposing direction? That is, given a Smale space (X, f) does there exist a partition of X, $\{A_i\}_{i=1}^n$ such that each $(A_i, f|_{A_i})$ is a Smale space? In order to answer this question, we shall first introduce a definition that hints at the criterion of such a procedure.

Definition 2.24 (Irreducible)**.** Let X be a topological space and $f : X \to X$ be a homeomorphism. (X, f) is said to be irreducible if for every ordered pair of non-empty open sets $U, V \subseteq X$, there exists a positive integer n such that $f^n(U) \cap V \neq \emptyset$

Notice that this is a slightly weaker of version of mixing in Definition 1.7 where there only needs to be a single n in question. Notice that mixing, non-wandering, and irreducible all have similar definitions and have the following relationship.

$$\text{mixing} \implies \text{irreducible} \implies \text{non-wandering}$$

The converses of the above relationship do not hold. Two counter examples that demonstrate this are as follows

Example 2.25. Let (X, f) and (Y, g) are irreducible Smale spaces. Now, consider the Smale space $(X \sqcup Y, f \sqcup g)$. We will show that $(X \sqcup Y, f \sqcup g)$ is non-wandering. Take $x \in X \sqcup Y$ and suppose that $x \in X$. Take any non-empty open neighborhood of x, $U \sqcup V \subseteq X \sqcup g$ with

$U \subseteq X$ and $V \subseteq X$. Since f is non-wandering, there exists an n such that $f^n(U) \cap U \neq \emptyset$ which implies that $f^n \sqcup g^n(U \sqcup V) \cap (U \sqcup V) = f^n(U) \sqcup g^n(V) \neq \emptyset$. If $x \in Y$, the proof is similar and hence omitted. This shows that $(X \sqcup Y, f \sqcup g)$ is non-wandering. However, $(X \sqcup Y, f \sqcup g)$ is not irreducible.

Secondly, we can show that this Smale space is not irreducible. If we choose $U \subseteq X$ and $V \subseteq Y$, since $f^n(U) \cap Y = \emptyset$ for every n, then $f^n(U) \cap V = \emptyset$ for every n and we cannot be irreducible.

Let (X, f) is a mixing Smale space, we can define $F : X \times \mathbb{Z}/n\mathbb{Z} \to X \times \mathbb{Z}/n\mathbb{Z}$, define by $F(x, k) = (f(x), k + 1)$ with the metric induced from viewing $X \times \mathbb{Z}/n\mathbb{Z}$ as $\bigsqcup_{i=1}^{n} X$. Now, if we take $U \times k_U \subseteq X \times \mathbb{Z}/n\mathbb{Z}$ and $V \times k_V \subseteq X \times \mathbb{Z}/n\mathbb{Z}$, notice that for large enough N, and $i \geq N$ $f^n(U \times k_U) \cap V \times k_V \neq \emptyset$ if $k_U + i \equiv k_V \mod n$. Thus it is nonempty every n iterations and empty in the other cases. This mean that F is irreducible but not mixing.

What is incredible about the construction of the Smale spaces in Example 2.25 is that all non-wandering Smale spaces decompose into a finite set of disjoint irreducible Smale spaces. Similarly, an irreducible Smale space can be decomposed into a finite set of mixing Smale spaces where the function cyclically permutes through them.

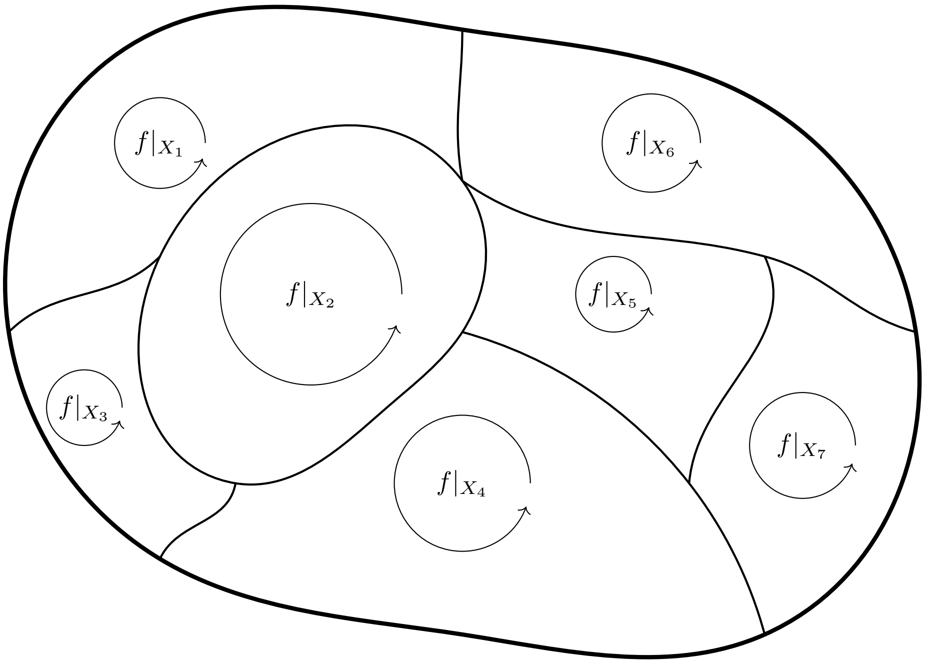


Figure 11: Decomposition of an Non-wandering Smale Space in Irreducible Sets

Theorem 2.26. *Let (X, f) be a non-wandering Smale space. Then there exists clopen, pairwise disjoint, f-invariant subsets unique up to relabeling $X_1, ..., X_n \subseteq X$ such that for each i,*

(1) $\bigsqcup_{i=1}^{n} X_i = X$

(2) $(X_i, f|_{X_i})$ *is irreducible*

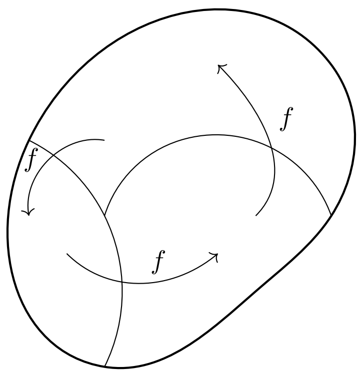


Figure 12: Decomposition of an Irreducible Smale Space into Cyclic Permuting sets

Theorem 2.27. *Let (X, f) be an irreducible Smale space. There exists clopen pairwaise disjoint sets $X_1, ..., X_n$ such that*

(1) The sets are cyclicly permuted by f

(2) (X_i, f^n) is mixing for every i

The proofs of these theorems are rather technical, but can be found in [13] and [1]. But the key is that we can focus on mixing Smale spaces as non-wandering and irreducible Smale spaces can be decomposed into such spaces.

2.6 Axiom A Systems

Historically, the hyperbolic behavior where we contract and expand in different directions was first observed on subsets of manifolds. Following Ruelle, we've managed to generalize to arbitrary metric spaces, the study of this flavor of dynamics is very much alive and we can apply the theory of Smale spaces to subsets of manifolds.

Definition 2.28. Let M be smooth Riemannian manifold and $f : M \to M$ be a diffeomorphism. Let $\Lambda \subseteq M$. Λ is said to be *hyperbolic* if it is closed and for each $x \in \Lambda$, $f(\Lambda) = \Lambda$, there exists a splitting

$$T_x M = E^u_x \oplus E^s_x$$

such that

(a) $Df(E^u_x) = E^u_{f(x)}$ and $D\,f(E^s_x) = E^s_{f(x)}$

(b) there exists constants $c > 0$ and $0 < \lambda < 1$ such that

$$\|Df^n(v)\| \leq c\lambda^n \|v\| \quad \text{when } v \in E^s_x,\, n \geq 0$$

and

$$\|Df^{-n}(v)\| \leq c\lambda^n \|v\| \quad \text{when } v \in E^u_x,\, n \geq 0$$

Remark. It is important to note that hyperbolic sets of M are often not submanifolds.

Proposition 2.29. *Let (M, g) be a smooth Riemannian manifold, $f : M \to M$ be a diffeomorphism and $\Lambda \subseteq M$ be hyberbolic. Then Λ is hyperbolic with respect to any Riemmanian metric on M. Furthermore, there exists a metric such that $c = 1$.*

Remark. Since there always exists a metric such that $c = 1$, whenever referring to a hyperbolic set, we will make the assumption that the metric has $c = 1$.

Recall that $\mathrm{NW}(X, f)$ denotes the set of non-wandering points in a dynamical system.

Definition 2.30. $f : M \to M$ is said to satisfy Axiom A if $\mathrm{NW}(M, f)$ is hyperbolic and $\mathrm{NW}(M, f) = \overline{\mathrm{Per}(M, f)}$

Many of the theorems we have already discussed were first proven for Axiom A systems. The bracket first being defined by Smale in [14]. It was known as the canonical coordinates before Smale spaces were defined.

Theorem 2.31. *If $f : M \to M$ is said to satisfy Axiom A, then $(\mathrm{NW}(M, f), f)$ is a Smale space.*

Example 2.32. Once again, we can refer to our poster child example, the hyperbolic toral automorphism. However, this time we can present it in more generality. Let $T^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-torus. Let A be a $n \times n$ invertible matrix with eigenvalues $\lambda_1, ..., \lambda_n$ such that $0 < |\lambda_i| \neq 1$ for all i and there exists at least one $|\lambda_j| > 1$ and at least one $|\lambda_k| < 1$. Then the induced map $\widetilde{f}$ on T^n once again has the splitting property

$$T_x T^n = E^u_x \oplus E^s_x$$

where E^u_x corresponds to the span of the eigenvalues with magnitude greater than 1, and E^s_x corresponds to the span of the eigenvalues with magnitude less than 1. Since the dervative is precisely the matrix A at this point, we get the expanding and contracting behavior of an axiom A system. Moreover, another important aspect is that the hyperbolic set is the entire manifold.

Definition 2.33. $f : M \to M$ is said to be an Anosov diffeomorphism if M itself is hyberbolic.

It is currently still an open problem to classify which manifolds can admit Anosov diffeomorphisms as the majority of manifolds cannot. A powerful theorem in this pursuit is the Franks-Newhouse Theorem[6]

Theorem 2.34 (Franks-Newhouse Theorem)**.** *Let $f : M \to M$ is an Anosov differomorphism on a compact Riemmannian manifold. If each such that each E^u_x is codimension one or each E^s_x is codimension one, then f is topologically conjugate to an hyperbolic toral automorphism.*

3 Shifts of Finite Type and Maps Between Smale Spaces

Shifts of finite type are crucial example of Smale Spaces. Not only do they have interesting properties and play a major role in the theory of Smale spaces, they are an important tool in all sorts of dynamical systems. There is an entire field known as a symbolic dynamics dedicated to the study of shift spaces. For many dynamical systems there exists a factor map from a shift of a finite type to the dynamical system. We recommend the reader recall that we defined maps for dynamical systems in Definition 1.8 which are continuous maps that preserve the dynamics. For a more in depth treatment of symbolic dynamics alone, we suggest the reader to take a look at [8]. We will study how shifts of finite type can be used in a way analogous to covering spaces in point set topology and discuss an algebraic invariant associated with the shift of finite type. The connection between this algebraic invariant for shifts of finite type lie in the fact that we can "disconnect" global stable and unstable sets through s/u-bijective maps which is the key to understanding Putnam's Homology Theory [10] in the next section.

3.1 Graphs and Their Shift Spaces

We will define a shift of finite slightly differently than how most symbolic dynamics books define shifts of finite types. A large part of our studies of shifts of finite type will be grounded in graph theory, so we will place a large emphasis on a graph theoretic perspective.

Due the variety of vocabulary used throughout graph theory, we shall introduce some basic definitions.

Definition 3.1. Let $G = (\mathcal{V}, \mathcal{E}, i, t)$ be a directed graph. The finite set $\mathcal{V}(G)$ is said to be the set of vertices. The finite set $\mathcal{E}(G)$ is said to be the set of edges.

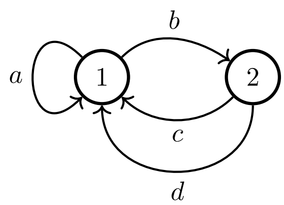
The functions $i : \mathcal{E}(G) \to \mathcal{V}(G)$ and $t : \mathcal{E}(G) \to \mathcal{V}(G)$ are call the initial state and terminal state functions.

Often, when the graph is understood, we will abbreviate $\mathcal{V}(G) = \mathcal{V}$ and $\mathcal{E}(G) = \mathcal{E}$. We will also suppress the i and t functions in the graph $G = (\mathcal{V}, \mathcal{E})$ when they are understood. Lastly, we will assume that all graphs are directed. Graphs are commonly illustrated using diagrams of arrows and nodes.

Example 3.2. Consider the following graph $G = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, 2\}$ and $\mathcal{E} = \{a, b, c, d\}$. Defined by

$$\begin{cases} (i(a), t(a)) = (1, 1) \\ (i(b), t(b)) = (1, 2) \\ (i(c), t(c)) = (2, 1) \\ (i(d), t(d)) = (2, 1) \end{cases}$$

We illustrate this as the following picture



From now on, we may simple draw a graph and then the set of vertices and edges is implied. Now, suppose we change the characters that label the graph above. In essence, the graph is still the same, but the sets change. So we must come up with some form of isomorphism between graphs and along the way we can discuss the maps between them. As the labeling does not matter, we will follow the convention that vertices are labeled alphabetically and edges are labeled numerically.

Definition 3.3. Let G and H be graphs. A graph homomorphism from G to H is defined to be two functions (both denoted as Φ)

$$\Phi : \mathcal{E}(G) \to \mathcal{E}(H) \quad \Phi : \mathcal{V}(G) \to \mathcal{V}(H)$$

such that

$$i(\Phi(e)) = \Phi(i(e)) \quad t(\Phi(e)) = \Phi(t(e))$$

We often write $\Phi : G \to H$ and

(a) If Φ is injective (in both edges and vertices), then it is said to be a graph embedding

(b) If Φ is bijective (in both edges and vertices), then it is said to be a graph isomorphism

(c) If there exists a graph isomorphism between G and H, they are said to be isomorphic which is denoted as $G \cong H$

Just like many other mathematical objects, we will consider graphs up to isomorphism and consider isomorphic graphs to be the same. Now, we will consider an important tool to analyze the behavior of graphs using linear algebra.

Definition 3.4. Let G be a graph with a vertex set $\mathcal{V}$. For $\alpha, \beta \in \mathcal{V}$, let

$$A_{\alpha\beta} = |\{e \in \mathcal{E} : i(e) = \alpha, t(e) = \beta\}|$$

If there are r vertices, this defines an $r \times r$ matrix known as the adjacency matrix $A = (A_{\alpha\beta})$, which is sometimes denoted A_G.

Likewise, if A is a $r \times r$ matrix, then the graph of A is the graph with vertex set $\mathcal{V} = \{1, \ldots, r\}$ and an edge set such that $[A_{\alpha\beta} = |\{e \in \mathcal{E} : i(e) = \alpha, t(e) = \beta\}|$

We can construct the adjacency matrix for Example 3.2. We simple count the edges to get

$$A_G = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$$

With some basic graph theory defined. We will now introduce the bi-infinite paths associated to a graph. There are two ways to do this, we can either track the edges or the vertices. If there are two distinct edges e and e' that initiate and terminate at the same vertices V_i and V_t, then there are two ways to go from V_i to V_t. Thus, a bi-infinite list of vertices is not enough to capture this path due to (V_i, V_t) introducing some ambiguity. So the edges are a more general approach to understanding bi-infinite paths.

Definition 3.5. Let $G = (\mathcal{E}, \mathcal{V})$ be a graph. The edge shift of G is defined as (Σ, σ), where

$$\Sigma = \{e = (e_n)_{n\in\mathbb{Z}} \in \mathcal{E}^{\mathbb{Z}} : t(e_n) = i(e_{n+1}) \text{ for all } n \in \mathbb{Z}\}$$

and $\sigma : \Sigma \to \Sigma$ is defined via

$$\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$$

Suppose that a graph G had no repeated edges, that is, no distinct edges $e_1, e_2 \in \mathcal{E}$ such that $i(e_1) = i(e_2)$ and $t(e_1) = t(e_2)$. In this case, every element of the edge shift would uniquely determine a bi-infinite sequence of vertices and there would be no ambiguity in a bi-infinite sequence of vertices. So, we can in a similar way defined as the vertex shift.

Definition 3.6. Let $G = (\mathcal{E}, \mathcal{V})$ be a graph with no repeated edges. The vertex shift of G is defined as (Σ, σ) where

$$\Sigma = \{v = (v_n)_{n\in\mathbb{Z}} \in \mathcal{V}^{\mathbb{Z}} : i^{-1}(v_n) \cap t^{-1}(v_{n+1}) \neq \emptyset \text{ for all } n \in \mathbb{Z}\}$$

$\sigma : \Sigma \to \Sigma$

$$\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$$

There is an easy understanding of distance within bi-infinite sequences. Let $\mathcal{A}$ be a finite non-empty set. If we define a metric on $\mathcal{A}^{\mathbb{Z}}$, we can consider the shift spaces to be the subspaces of $\mathcal{A}^{\mathbb{Z}}$ where $\mathcal{A}$ is either the vertex or edge set. Moreover, it is easy to see that the shift map is a homeomorphism under this metric, which is defined as follows.

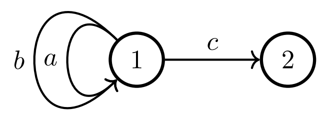
Proposition 3.7. *Let $\mathcal{A}$ be a finite non-empty set, the function $d : \mathcal{A}^{\mathbb{Z}} \times \mathcal{A}^{\mathbb{Z}} \to \mathbb{R}$ defines a metric.*

$$d((a_n)_{n\in\mathbb{Z}}, (b_n)_{n\in\mathbb{Z}}) = \inf\left\{\frac{1}{2^k} : k \geq 0,\, a_n = b_n \text{ for all } |n| < k\right\}$$

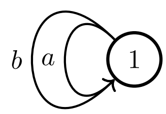
Moreover, the shift map $\sigma((a_n)_{n\in\mathbb{Z}}) = \sigma((a_{n+1})_{n\in\mathbb{Z}})$ is a homeomorphism with respect to the metric space topology.

Proposition 3.8. *Let $G = (\mathcal{E}, \mathcal{V})$ be a graph with no repeated edges. The vertex shift and edge shift are topological conjugate.*

Notice that bi-infinite paths must go in loops if we want them to be infinite on a finite vertex set. So consider the following graph



Let Σ be the associated edge shift of the graph above. For $(e_n)_{n\in\mathbb{Z}} \in \Sigma$, if $e_N = c$, then there does not exist a $e_{N+1} \in \mathcal{E}$ such that $t(e_N) = i(e_{N+1}) = 2$ It follows that we cannot have a $e_N = c$. Thus, the edge shift is equivalent to the edge shift of the graph constructed by removing the vertex 2.



$$\Sigma = \{a, b\}^{\mathbb{Z}}$$

A similar argument can be made for when the direction of c is reversed. This leads us to the following proposition.

Proposition 3.9. *Let G be a finite graph and Σ_G be the edge shift associated to G*

(a) Σ_G is non-empty if and only if there exists an ordered list of edges $(e_n)_{n=0}^N$ such that $i(e_0) = t(v_N)$ and $t(e_n) = i(e_{n+1})$. A list such as this is known as a cycle

(b) Let $v \in \mathcal{V}$ be a vertex such that $t^{-1}(v)$ or $i^{-1}(v)$ is empty. Let H be the graph obtained by deleting all vertices of this form and edges initiated or terminated at vertices of this form. Then $\Sigma_H = \Sigma_G$

Proof. First, we will show (a). By considering the contrapositive, suppose that there were no ordered lists of this form, then Σ_G must be empty since if there are no finite lists, then there cannot be infinite lists of this form due $\mathcal{E}$ being finite. Now, suppose there existed a $(e_n)_{n=0}^N$ of this form. Define the bi-infinite sequence $(\xi_k)_{k\in\mathbb{Z}}$ by $\xi_k = e_{k \mod N}$. Then we have $(\xi_k)_{k\in\mathbb{Z}} \in \Sigma_G$ and thus Σ_G is non-empty.

To show (b). We know that $\Sigma_H \subseteq \Sigma_G$. Let $(e_n)_{n\in\mathbb{Z}} \in \Sigma_G$. We will show that $(e_n)_{n\in\mathbb{Z}}$ consists of edges only in H .Take any $v \in \mathcal{V}$ such that $t^{-1}(v)$ or $i^{-1}(v)$ is empty. Let $t^{-1}(v)$ be empty.

Suppose for the sake of contradiction that there was a $\xi \in \mathcal{E}$ such that $i(\xi) = v$ and $e_k = \xi$ for some k. Then $i(e_k) = t(e_{k+1}) = v$ but his means that $e_{k+1} \in t^{-1}(v)$ which is non-empty and a contradiction. Likewise, if $i^{-1}(v)$ was empty, suppose for the sake of contradiction that there was a $\xi \in \mathcal{E}$ such that $t(\xi) = v$ and $e_k = \xi$ for some k. Then we have $i(e_{k-1}) = t(e_k) = v$ and once again $i^{-1}(v)$ is non-empty and this contradicts our assumption.

Thus $(e_n)_{n\in\mathbb{Z}}$ only consists of edges in Σ_H and thus $(e_n)_{n\in\mathbb{Z}} \in \Sigma_H$ and $\Sigma_H = \Sigma_G$ □

With some basic properties now introduced, we can now introduce the idea of the shift of finite type itself. The edge and vertex shifts were specific examples of such a space.

Definition 3.10. Let $\mathcal{A}$ be a finite non-empty set. Finite sequences $(a_1, \ldots, a_n)$ for $a_i \in A$ are known as words of $\mathcal{A}$. Let the forbidden words, $\mathcal{F}$ be a finite set of words of $\mathcal{A}$. The shift of finite type associated to $\mathcal{A}$ and $\mathcal{F}$ is defined as $(\Sigma_{\mathcal{F}}, \sigma)$ where

$$\Sigma_{\mathcal{F}} = \{a \in \mathcal{A}^{\mathbb{Z}} : \text{no finite subsequence of } a \text{ is in } \mathcal{F}\}$$

and σ is the shift map

$$\sigma((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}$$

It is easy to see that the edge and vertex shifts are shifts of finite type since we can disallow paths of edges/vertexes that do not exist in the graph. For a vertex shift, we can define $\mathcal{F} = \{(v, p) : i^{-1}(v) \cap t^{-1}(p) = \emptyset\}$. For the edge shift, $\mathcal{F} = \{ef : t(e) \neq i(f)\}$. These sets are both finite due to $\mathcal{V}$ and $\mathcal{E}$ being finite.

Theorem 3.11. *Every shift of finite type is topologically conjugate to an edge shift.*

Proof. Let $\mathcal{F}$ be a finite list of works of a finite non-empty set $\mathcal{A}$. Let M be an integer such that $M > \{\text{length}(w) : w \in \mathcal{F}\}$. We will construct the following graph $G = (\mathcal{V}, \mathcal{E})$,

$$\mathcal{V} = \{(a_1, \ldots, a_M) : (a_n)_{n\in\mathbb{Z}} \in \Sigma_{\mathcal{F}}\}$$

$$\mathcal{E} = \{(a_1, \ldots, a_{M+1}) : (a_n)_{n\in\mathbb{Z}} \in \Sigma_{\mathcal{F}}\}$$

In other literature, these are known as the sets of allowed M or $M + 1$ blocks. Now note that since $\mathcal{A}$ is finite , both $\mathcal{V}$ and $\mathcal{E}$ are finite. Also define

$$i(a_1, \ldots, a_{M+1}) = (a_1, \ldots, a_M)$$

$$t(a_1, \ldots, a_{M+1}) = (a_2, \ldots, a_{M+1})$$

This gives us our graph. Now, define the map from the shift of finite type to the edge shift, $\varphi : \Sigma_{\mathcal{F}} \to \Sigma_G$. via

$$\varphi\left((a_n)_{n\in\mathbb{Z}}\right)_k = (a_k, \ldots, a_{k+M})$$

It is clear from the definition that this is a continuous, injective open map with $\sigma \circ \varphi = \varphi \circ \sigma$. All that is left to show is that this map is surjective. Take any $(e_n)_{n\in\mathbb{Z}} \in \Sigma_G$. Since each edge is an $M + 1$ block, write this as

$$e_n = (b_n^1, \ldots, b_n^{M+1})$$

Now, let $a_n = a_n^1$ for each $n \in \mathbb{Z}$. We want to show that this defines a pre-image for $(e_n)_{n\in\mathbb{Z}}$

First, we must show that for any $0 \leq m \leq M$, $a_{n+m} = b_n^{m+1}$ for all n. This is true for $m = 0$ just by definition. Now, we shall make an inductive hypothesis and assume this is true for some

$m < M$. We want to show that it is true for $m+1$. Since we are taking elements from an edge shift, we have

$$t(e_n) = i(e_{n+1})$$
$$(b_n^2, \dots, b_n^{M+1}) = (b_{n+1}^1, \dots, b_{n+1}^M)$$

This shows that $b_n^{m+2} = b_{n+1}^{m+1} = a_{n+(m+1)}$. By induction, we have shown that $a_{n+m} = b_n^{m+1}$ for all n. Thus, we get

$$\varphi\left((a_n)_{n\in\mathbb{Z}}\right)_k = (a_k, \dots, a_{k+M}) = (b_k^1, \dots, b_k^{M+1}) = e_k$$

□

Now, that we have the basic definition down, we can define a bracket map on the shift of finite type to get a Smale space. Also, note that the following metric induced a totally discounted topology from the metric.

Theorem 3.12. *Let (Σ, σ) be a shift of finite type, d be the metric defined in Proposition 3.7. Let N be the length of the longest word in $\mathcal{F}$ (or 1 if $\mathcal{F} = \emptyset$). Define $\epsilon_\Sigma = 2^{-N}$ and $[-,-] : \Delta_{\epsilon_\Sigma} \to \Sigma$ as*

$$([a,b])_n = \begin{cases} a_n & n \leq 0 \\ b_n & n \geq 1 \end{cases}$$

Then $(\Sigma, d, \sigma, [-,-])$ is a Smale Space.

The prove is simple axiom checking and is thus omitted. Notice that we have constructed a totally disconnected Smale space. A remarkable fact is that the shifts of finite type are the only totally disconnected Smale space. This result can be found as Theorem 2.2.8 in [10]

Theorem 3.13. *Let (X, f) be a Smale space. Then (X, f) is topologically conjugate to a shift of finite type if and only if X is totally disconnected.*

3.2 Symbolic Representations of Dynamics

The importance of shift spaces extends beyond our discussion of Smale spaces. Shift spaces can be used to obtain a symbolic representation of many dynamical systems. In this subsection, we will extend our arguments to general dynamical systems.

Given a dynamical system on a topological space $\varphi : X \to X$, and take some point $x \in X$ and consider the orbit $\text{Orb}(x)$. Notice that if x is not a periodic point, $|\text{Orb}(x)| = \mathbb{Z}$ and can be viewed as a bi-infinite sequence. This property makes one wonder if one could leverage the work we did with the bi-infinite sequences in shifts of finite type. The issue is that there are simply too many points to do this with if X is infinite. To decrease the number of points we need to work with, we introduce the notion of a topological partition, It allows us to split up the space into a finite number of open sets. By creating these partitions, in good cases we can construct a factor map (recall Definition 1.8) from a shift of finite type to a different dynamical system,

Definition 3.14 (Topological Partition)**.** Let $\mathcal{P} = \{P_1, ..., P_n\}$ be a collection of disjoint open sets in a topological space X such that $\bigcup_{i=1}^n \overline{P_1} = X$. $\mathcal{P}$ is known as a *topological partition*

Remark. Note that a topological partition is not a partition of a space in the usual sense. There may be points that do not belong to any P_i.

Now, suppose we had a the topological partition $\mathcal{P} = \{P_1, ..., P_n\}$ of X where (X, φ) is a dynamical system and let $\mathcal{A} = \{1, ..., n\}$. We will construct a an shift of finite type out of these objects.

An r lettered word $w = a_1 a_2 ... a_r$ is going to be allowed if $\bigcap_{i=1}^{r} \varphi^{-i}(P_{a_i}) \neq \emptyset$. Intuitively, words are allowed if the sequence of partitions have points which can move to the next. If we take all the words not allowable by this definition, then we get a shift space. Together, we get the following.

Proposition 3.15. *Let (X, φ) be a dynamical system, $\mathcal{P} = \{P_1, ..., P_n\}$ be a topological partition of X. There exists a shift space $\Sigma_{X,\varphi,\mathcal{P}}$ defined by $\mathcal{A} = \{1, \ldots, n\}$ and*

$$\Sigma_{X,\varphi,\mathcal{P}} = \left\{ a \in \mathcal{A}^{\mathbb{Z}} : \textit{for any word in } a,\ (a_n)_{n=1}^{r},\ \bigcap_{i=1}^{r} \varphi^{-i}(P_{a_i}) \neq \emptyset \right\}$$

Given the shift space $\Sigma_{X,\varphi,\mathcal{P}}$, there is no guaranteed that this construction is easier to study than the original dynamical system as the shift space created may be overly simply or complicated. In fact, $(\Sigma_{X,\varphi,\mathcal{P}}, \sigma)$ may not even be a shift of finite type. So the question at hand is when is there a clear topological connection between X and $\Sigma_{X,\varphi,\mathcal{P}}$? Our goal is to define a factor map from $\Sigma_{X,\varphi,\mathcal{P}}$ to X. The key lies in the following proposition.

Proposition 3.16. *Let (X, φ) be dynamical system with a topological partition $\mathcal{P} = \{P_1, ..., P_n\}$. For any $(a_n)_{n\in\mathbb{Z}} \in \Sigma_{X,\varphi,\mathcal{P}}$ and $k \geq 0$, there exists non-empty open sets*

$$\mathcal{U}_k((a_n)_{n\in\mathbb{Z}}) = \bigcap_{i=-k}^{k} \varphi^{-i}(P_{a_i}) \subseteq X$$

Moreover, each $\overline{\mathcal{U}_n((a_n)_{n\in\mathbb{Z}})}$ is compact and

$$... \subseteq \overline{\mathcal{U}_3((a_n)_{n\in\mathbb{Z}})} \subseteq \overline{\mathcal{U}_2((a_n)_{n\in\mathbb{Z}})} \subseteq \overline{\mathcal{U}_1((a_n)_{n\in\mathbb{Z}})} \subseteq \overline{\mathcal{U}_0((a_n)_{n\in\mathbb{Z}})}$$

These sets allow us to define a factor map if $\bigcap_{k=0}^{\infty} \overline{\mathcal{U}_k((a_n)_{n\in\mathbb{Z}})}$ consists of a single point. We give this case a formal name.

Definition 3.17. If for every $a \in \Sigma_{X,\varphi,\mathcal{P}}$, $\bigcap_{k=0}^{\infty} \overline{\mathcal{U}_k(a)}$ consists of precisely one point, we call $\Sigma_{X,\varphi,\mathcal{P}}$ is a *symbolic representation* of (X, φ).

Furthermore, we call $\mathcal{P}$ a *Markov partition* of X if $\Sigma_{X,\varphi,\mathcal{P}}$ is a symbolic representation of (X, φ) and $\Sigma_{X,\varphi,\mathcal{P}}$ is a shift of finite type.

There are two main properties that come out of Definition 3.17. The first of which is in the following proposition which is more or less clear.

Proposition 3.18. *(X, φ) be dynamical system with a topological partition $\mathcal{P} = \{P_1, ..., P_n\}$. If $\Sigma_{X,\varphi,\mathcal{P}}$ is a symbolic representation of (X, φ), then $\Sigma_{X,\varphi,\mathcal{P}}$ is a closed subset of $\{1, \ldots, n\}^{\mathbb{Z}}$ that is invariant under σ*

The second and arguable more important property is that there is a canonical factor map from $\Sigma_{X,\varphi,\mathcal{P}}$ to X.

Theorem 3.19. *Suppose $\Sigma_{X,\varphi,\mathcal{P}}$ is a symbolic representation of (X, φ), then define $\pi : \Sigma_{X,\varphi,\mathcal{P}} \to X$ by letting $\pi(x)$ be the unique point in $\bigcap_{k=0}^{\infty} \overline{\mathcal{U}_k(x)}$. This map is a factor map, so in particular, the following diagram commutes.*

$$\begin{array}{ccc} \Sigma_{X,\varphi,\mathcal{P}} & \xrightarrow{\quad\sigma\quad} & \Sigma_{X,\varphi,\mathcal{P}} \\ \Big\downarrow{\scriptstyle\pi} & & \Big\downarrow{\scriptstyle\pi} \\ X & \xrightarrow{\quad\varphi\quad} & X \end{array}$$

Proof. First, we would like to show this is indeed a map between dynamical systems. In other words, the diagram above commutes and π is continuous. Take some $a \in \Sigma_{X,\varphi,\mathcal{P}}$ and notice that

$$\begin{aligned} \mathcal{U}_{k+1}(\sigma(a)) &= \bigcap_{i=-(k+1)}^{k+1} \varphi^{-i}(P_{\sigma(a)_i}) = \bigcap_{i=-(k+1)}^{k+1} \varphi^{-i}(P_{a_{i+1}}) \\ \varphi(\mathcal{U}_k(a)) &= \varphi\left(\bigcap_{i=-k}^{k} \varphi^{-i}(P_{a_k})\right) = \bigcap_{i=-k}^{k} \varphi^{-i+1}(P_{a_i}) = \bigcap_{i=-(k+1)}^{k-1} \varphi^{-i}(P_{a_{i+1}}) \\ \mathcal{U}_{k-1}(\sigma(a)) &= \bigcap_{i=-(k-1)}^{k-1} \varphi^{-i}(P_{\sigma(a)_i}) = \bigcap_{i=-(k-1)}^{k-1} \varphi^{-i}(P_{a_{i+1}}) \end{aligned}$$

It follows that

$$\mathcal{U}_{k+1}(\sigma(a)) \subseteq \varphi(\mathcal{U}_k(a)) \subseteq \mathcal{U}_{k-1}(\sigma(a))$$

Since φ is a homeomorphism, we get

$$\bigcap_{k=1}^{\infty} \overline{\mathcal{U}_{k+1}(\sigma(a))} \subseteq \varphi\left(\bigcap_{k=1}^{\infty} \overline{\mathcal{U}_k(a)}\right) \subseteq \bigcap_{k=1}^{\infty} \overline{\mathcal{U}_{k-1}(\sigma(a))}$$

And since this is a symbolic representation, the intersections above are each a singleton. Thus the above containment becomes

$$\{\pi(\sigma(a))\} \subseteq \varphi(\{\pi(a)\}) \subseteq \{\pi(\sigma(a))\}$$

This proves that $\pi \circ \sigma = \varphi \circ \pi$.

Now, we shall prove continuity. Since $\overline{\mathcal{U}_n(a)}$ is descending and intersection consists of a single point, $\lim_{n\to\infty} \operatorname{diam}\overline{\mathcal{U}_n(a)} = 0$. Suppose there was a sequence $\{a^k\}_{k=1}^{\infty} = \{(a_n^k)_{n\in\mathbb{Z}}\}_{k=1}^{\infty}$ such that $a^k \to a$. There exists a large enough N such that

$$\pi(a^k), \pi(a) \in \overline{\mathcal{U}_k(a)} \quad \text{for all } k \geq N$$

Thus,

$$d(\pi(a^k), \pi(a)) \leq \operatorname{diam}\overline{\mathcal{U}_k(a)} \quad \text{for all } k \geq N$$

Take $k \to \infty$ to give us $d(\pi(a^k), \pi(a)) \to 0$ which shows that π is continuous.

Lastly, we must show that π is surjective. Define the union of all the elements of the partition to be U. So,

$$U = \bigcup_{P\in\mathcal{P}} P$$

It is clear by definition of topological partition that

$$\overline{U} = \bigcup_{P\in\mathcal{P}} \overline{P} = X$$

Which means that U is dense in X. Moreover, since φ is a homeomorphism, the following sets are also dense and open

$$U_n = \bigcup_{k=-n}^{n} \varphi^k(U)$$

By the Baire Category Theorem, the following set is also dense and open

$$U_\infty = \bigcap_{n=0}^{\infty} U_n$$

From the way we defined this set, for any $x \in U_\infty$, there exists an $a \in \Sigma_{X,\varphi,\mathcal{P}}$ such that $x \in \bigcap_{n=0}^{\infty} \mathcal{U}_n(a)$ which of course implies that $y = \pi(x)$. Thus, $\pi(\Sigma_{X,\varphi,\mathcal{P}}) \supseteq U_\infty$. However, $\pi(\Sigma_{X,\varphi,\mathcal{P}})$ is compact and thus closed. Thus, since $\pi(\Sigma_{X,\varphi,\mathcal{P}})$ is closed and U_∞ is dense, we have

$$U_\infty \subseteq \pi(\Sigma_{X,\varphi,\mathcal{P}}) \subseteq X$$

$$\implies \overline{U_\infty} = X \subseteq \overline{\pi(\Sigma_{X,\varphi,\mathcal{P}})} = \pi(\Sigma_{X,\varphi,\mathcal{P}}) \subseteq X$$

This means that $\pi(\Sigma_{X,\varphi,\mathcal{P}}) = X$ □

Proposition 3.20. *If there exists a factor map $(X, f) \to (Y, g)$. Then for the following properties, if (X, f) has said property, so does (Y, g)*

1. *Irreducible*
2. *Mixing*

Proof. Suppose that $\pi : (X, f) \to (Y, g)$ is a factor map. Suppose that (X, f) is irreducible and take nonempty open $U, V \subseteq Y$. Then, $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are also non-empty subsets open sets since π is continuous. Since (X, f) is irreducible, there exists a positive integer n such that $f^n(\pi^{-1}(U)) \cap \pi^{-1}(V) \neq \emptyset$. Apply π to this to get $g^n(U) \cap V \neq \emptyset$. Thus, (Y, f) is irreducible. Almost the same argument works for the mixing case. □

Example 3.21. Recall the Baker's Map in Example 1.6. While this may not be a Smale space, this can still have a symbolic representation. The Markov's Partition is quite simple with

$$P_1 = \left[0, \frac{1}{2}\right) \times [0, 1] \quad P_2 = \left[\frac{1}{2}, 1\right) \times [0, 1]$$

This admits the full two shift $\Sigma_{I^2,f,\mathcal{P}} = \{1, 2\}^{\mathbb{Z}}$

Example 3.22. There exists a Markov Partition for our favorite example, the hyperbolic toral automorphism. This can be constructed using Figure 13. The dashed lines are follow the unstable direction and the dashed lines follow the stable direction. The construction is as follows:

1. Draw a line to the edge starting at $(0, 0)$ in the direction of the unstable direction
2. Draw a line until it intersects of line 1 starting at $(0, 1)$ down in the stable direction.
3. Draw a line until it intersects of line 1 starting at $(1, 0)$ up in the stable direction.
4. Draw a line until it intersects of line 2 starting at $(1, 1)$ up in the stable direction.
5. Continue line 1 to the other side of the torus until it intersects line 2

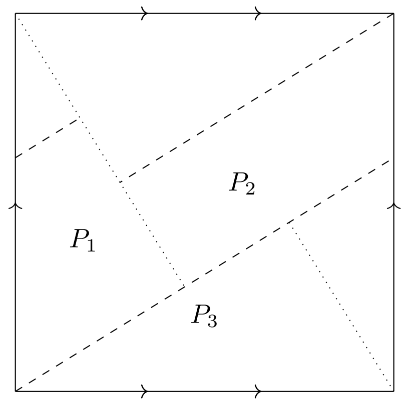


Figure 13: Markov Partition for Hyperbolic Toral Automorphism

The existence of a Markov partition is a highly non-trivial result. However, Theorem 3.12 of [1] gives a proof of the existence of a Markov partition for an irreducible Axiom A system. The proof relies on the Shadowing property and thus works for Smale spaces. Coupled with the decomposition theorem, this holds for non-wandering Smale Spaces

Theorem 3.23 (Bowen's Theorem)**.** *Let (X, f) be a non-wandering Smale Space. There exists Markov Partitions of an arbitrarily small diameter.*

This of course implies that there always exists a factor map from a shift of finite type to a Smale Space.

3.3 s/u-Bijective Maps Between Smale Spaces

In the previous subsection, we demonstrated that for every Smale Space (X, f), there exists a totally disconnected Smale Space that is, a shift of finite type (Σ, σ), and a factor map $\pi : (\Sigma, \sigma) \to (X, f)$. The fact that the shift of finite type is totally disconnected is an incredibly useful fact. However, it is not guaranteed that these maps preserve the stable unstable sets. That is elements of $\Sigma^s(a)$ to be in one-to-one correspondence to $X^s(\pi(a))$. Putnam noticed that if we were to constructing factor maps such that there are bijections between the stable/unstable sets, we could effectively "disconnected" the stable and unstable sets. Let us first state an important fact about maps.

Theorem 3.24. *Let (X, f) and (Y, g) be Smale spaces with a map*

$$\varphi : (X, f) \to (Y, g)$$

there exists an $\epsilon_\varphi > 0$ such that for all $x_1, x_2 \in X$ such that $d(x_1, x_2) < \epsilon_\varphi$, $[x_1, x_2]$ and $[\varphi(x_1), \varphi(x_2)]$ are defined and

$$[x_1, x_2] = [\varphi(x_1), \varphi(x_2)]$$

Following Fried[5], we define the case in which the restriction to the stable or unstable sets is an injection.

Definition 3.25. Suppose (X, f) and (Y, g) are Smale spaces with a map

$$\varphi : (X, f) \to (Y, g)$$

We say φ is s-resolving if for any $x \in X$, $\varphi|_{X^s(x)}$ is injective.
Likewise, we also say φ is u-resolving if for any $x \in X$, $\varphi|_{X^u(x)}$ is injective.

Theorem 3.26. *Let (X, f) and (Y, g) be Smale spaces such that*

$$\varphi : (Y, g) \to (X, f)$$

be an s-resolving map, there exists a $M \geq 1$ such that

(a) For every $x \in X$, there exists $y_1, \ldots, y_k \in Y$ such that $k \leq M$ and

$$\varphi^{-1}(X^u(x)) = \bigcup_{i=1}^{k} Y^u(y_i)$$

(b) For any $x \in X$, $|\varphi^{-1}(\{x\})| \leq M$. That is, φ is finite one.

Due to Putnam in Section 2.5 of [10], we can do one step better where we make the maps restricted to stable/unstable sets bijective.

Definition 3.27. Suppose (X, f) and (Y, g) are Smale spaces with a map

$$\varphi : (X, f) \to (Y, g)$$

We say φ is s-bijective if for any $x \in X$, $\varphi|_{X^s(x)}$ is bijective.
Likewise, we also say φ is u-bijective if for any $x \in X$, $\varphi|_{X^u(x)}$ is bijective.

These functions allow us to study the stable/unstable sets in the domain of the map. This can be incredible useful if the domain has much easier stable/unstable sets to study than the codomain. A useful fact that we will not show is the following.

Theorem 3.28. *Let (X, f) and (Y, g) be Smale spaces such that*

$$\varphi : (Y, g) \to (X, f)$$

be an s-resolving map. If (Y, g) is nonwandering, then φ is an s-bijective map.

Lastly, we shall define the precise concept of "disconnecting" the stable and unstable sets. When this happens, we have two s/u-bijective factor maps such that the domain has totally disconnected unstable/stable sets.

Definition 3.29. Suppose (X, f) is a Smale space. A s/u-bijective pair is a pair of space spaces (Y, g) and (Z, h) and factor maps.

$$\pi_s : (Y, g) \to (X, f) \quad \pi_u : (Z, h) \to (X, f)$$

Such that

(a) π_s is s-bijective and π_u is u-bijective

(b) For every $y \in Y$, $Y^s(y)$ is totally disconnected.

(c) For every $z \in Z$, $Z^u(z)$ is totally disconnected.

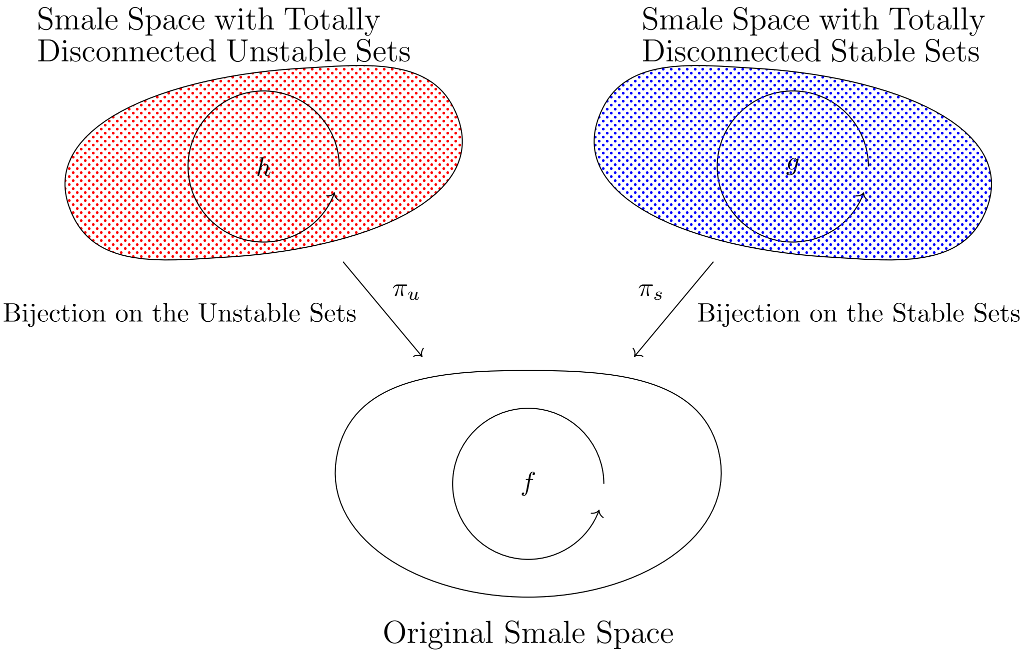


Figure 14: Disconnecting Stable/Unstable Sets

Just like our prior result with the existence of Markov Partitions, these s/u-bijective pairs also always exist, but a proof would be too technical, see [10] for details.

Theorem 3.30. *If (X, f) is a non-wandering Smale space, then there exists a s/u-bijective pair.*

3.4 The Dimension Group

Although quite explicit, shifts of finite type can be a little unwieldy as a topological dynamical system. For an example, it is actually incredibly difficult to determine if two shift of finite types are topologically conjugate. The problem is even more difficult for a general Smale space. One approach is the construction of a a functor to the category of groups analogous to topological invariants such as the homotopy groups or homologies.

One such relatively sensitive invariant for shifts of finite type which we will generalize later is known as the dimension group. There are multiple equivalent definitions for the dimension group: One with inductive limits, another of linear algebra, and one analogous to Cech cohomology. We will discuss each of these approaches.

Definition 3.31. Let A be a $n \times n$ integral matrix. The eventual range $\mathcal{R}_A$ is defined as

$$\mathcal{R}_A = \bigcap_{k=1}^{\infty} A^k \mathbb{Q}^k$$

Likewise, the eventual kernel $\mathcal{K}_A$ is defined to be

$$\mathcal{K}_A = \bigcup_{k=1} \ker A^k$$

Now, the following properties hold easily using linear algebra.

Proposition 3.32. *We have the following*

(a) If A is invertible on $A^k\mathbb{Q}^n$ for some k, then $\mathcal{R}_A = A^n\mathbb{Q}^n$ and $\mathcal{K}_A = \ker A^n$

(b) $\mathcal{R}_A$ is the largest subspace of $\mathbb{Q}^n$ that where A is invertible and $\mathcal{K}_A$ is the largest subspace such that that A is nilpotent

(c) $\mathbb{Q}^n \cong \mathcal{R}_A \oplus \mathcal{K}_A$

The first definition of the dimension group we will mention is defined in [8]. It relies directly from the eventual range.

Definition 3.33 (Dimension Group (Linear Algebra))**.** Let Σ_G be a shift of finite type with graph G and let A be the $n \times n$ adjacency matrix. The stable dimension group is defined to be

$$D^s(\Sigma_G, \sigma) = \left\{v \in \mathcal{R}_A : A^k v \in \mathbb{Z}^n \text{ for some } k \geq 0\right\}$$

Likewise, the unstable dimension group is

$$D^u(\Sigma_G, \sigma) = \left\{v \in \mathcal{R}_{A^T} : (A^T)^k v \in \mathbb{Z}^n \text{ for some } k \geq 0\right\}$$

The dimension group automorphism δ_A and δ_{A^T} is the restriction of A to $D^s(\Sigma_G, \sigma)$ and $D^u(\Sigma_G, \sigma)$ respectively.

However, this is not the only definition. Similar to eventual range and kernels, we define the dimension groups from inductive limits as in [2].

Definition 3.34 (Dimension Group (Limits))**.** Let Σ_G be a shift of finite type with graph G and let A be the $n \times n$ adjacency matrix. The stable dimension group is defined to be

$$D^s(\Sigma_G, \sigma) = \varinjlim_{A} \mathbb{Z}^n$$

Likewise, the unstable dimension group is

$$D^u(\Sigma_G, \sigma) = \varinjlim_{A^T} \mathbb{Z}^n$$

The dimension group automorphism $\delta_A([v, n]) = [Av, n]$ and $\delta_{A^T}([v, n]) = [A^T v, n]$ respectively.

Lastly, Kreiger introduced a definition that relies on the topological aspects of the shift of finite type in [7]. Interesting enough, this definition does not require the construction of a graph. Before we introduce the third definition, we must introduce some notation.

Definition 3.35. Let (Σ, σ) be shift of finite type, then define the following sets

$$\mathcal{K}^s(\Sigma, \sigma) = \{E \subseteq \Sigma^s(x) : x \in \Sigma, E \text{ is non-empty compact and open}\} \tag{4}$$

Likewise,

$$\mathcal{K}^u(\Sigma, \sigma) = \{E \subseteq \Sigma^u(x) : x \in \Sigma, E \text{ is non-empty compact and open}\} \tag{5}$$

Moreover, define an equivalence relation $\sim$ on $\mathcal{K}^s(\Sigma, \sigma)$ to be the smallest equivalence relation such that

(1) $E \sim F$ if $[E, F] = E$ and $[F, E] = F$ and they are both defined

(2) $E \sim F$ if and only if $\sigma(E) \sim \sigma(F)$

Likewise, define an equivalence relation $\sim$ on $\mathcal{K}^u(\Sigma, \sigma)$ to be the smallest equivalence relation such that

(1) $E \sim F$ if $[F, E] = E$ and $[E, F] = F$ and they are both defined

(2) $E \sim F$ if and only if $\sigma(E) \sim \sigma(F)$

Denote the set of equivalence classes as

$$\widetilde{\mathcal{K}^s} = \mathcal{K}^s / \sim$$

$$\widetilde{\mathcal{K}^s} = \mathcal{K}^u / \sim$$

The relation in Definition 3.35 relate non-empty compact open subsets of the global unstable/stable sets. Intuitively, this relation matches nearby unstable/unstable that have the same bracket points.

Definition 3.36 (Dimension Group (Compact stable/unstable sets))**.** Let $\mathbb{Z}\widetilde{\mathcal{K}}^s$ and $\mathbb{Z}\widetilde{\mathcal{K}}^u$ be free abelian groups. Let R^s and R^u be subgroups generated according to relations

$$R^s = \langle [E \cup F] - [E] - [F] | E, F, E \cup F \in \mathcal{K}^s, E \cap F = \emptyset \rangle$$
$$R^u = \langle [E \cup F] - [E] - [F] | E, F, E \cup F \in \mathcal{K}^u, E \cap F = \emptyset \rangle$$

Then define the stable/unstable dimension group as

$$D^s(\Sigma, \sigma) = \mathbb{Z}\widetilde{\mathcal{K}}^s / R^s$$
$$D^u(\Sigma, \sigma) = \mathbb{Z}\widetilde{\mathcal{K}}^u / R^u$$

We have defined three different groups in relation to a shift of finite type. They were defined using widely different information about the stable/unstable topologies and eventual ranges. The miraculous thing is, all these groups are isomorphic [10].

Theorem 3.37. *Let (Σ, σ) be a shift of finite type, then the dimension groups according to Definition 3.33, Definition 3.34, and Definition 3.36 are all isomorphic*

Now that we have an algebraic invariant on shifts of finite type. It is natural to ask if D^u and D^s functors? Unfortunately, a general map between shifts of finite type does not induce a natural homomorphism between dimension groups. However, s-bijective and u-bijective maps do!

Theorem 3.38. *Let (Σ, σ) and (Σ', σ) be shifts of finite type and let $\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$ be a factor map. Then we get the following*

1. If π is s-bijective, there is a natural homomorphism

$$\pi^s : D^s(\Sigma, \sigma) \to D^s(\Sigma', \sigma)$$

2. If π is u-bijective, there is a natural homomorphism

$$\pi^u : D^u(\Sigma, \sigma) \to D^u(\Sigma', \sigma)$$

3. *If π is s-bijective, there is a natural homomorphism*

$$\pi^{s*} : D^u(\Sigma', \sigma) \to D^u(\Sigma, \sigma)$$

4. *If π is u-bijective, there is a natural homomorphism*

$$\pi^{u*} : D^s(\Sigma', \sigma) \to D^s(\Sigma, \sigma)$$

5. id *induces the identity isomorphisms*

$$\mathrm{id}^u : D^u(\Sigma, \sigma) \to D^u(\Sigma', \sigma) \quad \mathrm{id}^{u*} : D^s(\Sigma', \sigma) \to D^s(\Sigma, \sigma)$$

$$\mathrm{id}^s : D^s(\Sigma, \sigma) \to D^s(\Sigma', \sigma) \quad \mathrm{id}^{s*} : D^u(\Sigma', \sigma) \to D^u(\Sigma, \sigma)$$

6. *If $\pi_1 : \Sigma_1 \to \Sigma_2$ and $\pi_2 : \Sigma_2 \to \Sigma_3$ are factor maps, then*

   (a) *If π_1 and π_2 are s-bijective*

$$(\pi_2 \circ \pi_1)^s = \pi_2^s \circ \pi_1^s \quad (\pi_2 \circ \pi_1)^{s^*} = \pi_1^{s^*} \circ \pi_2^{s^*}$$

   (b) *If π_1 and π_2 are u-bijective*

$$(\pi_2 \circ \pi_1)^u = \pi_2^u \circ \pi_1^u \quad (\pi_2 \circ \pi_1)^{u^*} = \pi_1^{u^*} \circ \pi_2^{u^*}$$

If we consider the category of shifts of finite types with only u-bijective (or s-bijective factor maps), we get a functor. For s-bijective maps, we get that D^s being a covariant functor and D^u is a contravariant functor. Likewise, for u-bijective maps, D^u is a covariant functor and D^s is a contravariant functor. This gives us the following result

Corollary 3.38.1. *If (Σ, σ) and (Σ', σ) are topologically conjugate, then $D^u(\Sigma, \sigma) \cong D^u(\Sigma', \sigma)$ and $D^s(\Sigma, \sigma) \cong D^s(\Sigma', \sigma)$*

Proof. This follows easily from a functor argument. Suppose $\varphi : (\Sigma, \sigma) \to (\Sigma, \sigma)$ is a topological conjugacy. Then φ and φ^{-1} are both s/u-bijective factor maps. Thus, the natural homomorphisms between dimension groups have an inverse. It follows that $D^u(\Sigma, \sigma) \cong D^u(\Sigma', \sigma)$ and $D^s(\Sigma, \sigma) \cong D^s(\Sigma', \sigma)$. □

This gives us a method if determining if two shifts of finite type are not topologically conjugate by showing they have different dimension groups. Following Putnam We shall construct a similar theory in the following section.

4 A Homology Theory for Smale Spaces

Putman was able to define a generalization of the dimension group for Smale spaces[10]. His Homology theory for Smale spaces gives us information about the periodic points as well as working as an invariant. In order to construct a homology theory for Smale spaces, one must come up with something analogous to a cell complex. The key ideas needed to construct this are using s/u-bijective pair to obtain shifts of finite type and and create a bi-complex using their dimension groups. We will leverage these facts in order to construct Putnam's Homology on Smale spaces.

4.1 The Construction of the Homology

In the previous section, we discussed an algebraic invariant for shift of finite type known as the dimension group. We also talked about the deep connection between shifts of finite types and general Smale space. Now, our goal is to extend the idea of an algebraic invariant to a non-wandering Smale spaces using the connection between Smale spaces and shifts of finite type.

The construction of the Homology is reliant off multiple usages of the dimension group. First, we shall define the following shift of finite type.

Definition 4.1. Suppose (X, f) is a Smale space and

$$\pi_s(Y, g) \to (X, f) \quad \pi_u : (Z, h) \to (X, f)$$

is a s/u-bijective pair. For each non-zero integer L, M, define the following

$$\Sigma_{L,M}(\pi) = \{(y_0, ..., y_L, z_0, ..., z_M) : y_l \in Y, z_m \in Z, \pi_s(y_l) = \pi_u(z_l)\, \forall 0 \leq l \leq L, 0 \leq m \leq m\}$$

And map

$$\sigma : \Sigma_{L,M} \to \Sigma_{L,M}(\pi)$$

$$(y_0, ..., y_L, z_0, ..., z_{M-1}) \mapsto (g(y_0), ..., g(y_L), h(z_0), ..., h(z_{M-1}))$$

Proposition 4.2. *[10]$\Sigma_{L,M}(\pi)$ is a shift of finite type.*

Definition 4.3. Define the families of maps for $L \geq 1$ and $0 \leq l \leq L$, define

$$\delta_{l,\cdot} : \Sigma_{L,M}(\pi) \to \Sigma_{L-1,M}$$

$$(y_0, ..., y_L, z_0, ..., z_M) \mapsto (y_0, ..., y_{l-1}, y_{l+1}, ..., y_{L-1}, z_0, ..., z_{M-1})$$

In other words, $\delta_{l,\cdot}$ deletes the y_l term.
Likewise, for $M \geq 1$ and $0 \leq m \leq M$, define

$$\delta_{\cdot,m} : \Sigma_{L,M}(\pi) \to \Sigma_{L,M-1}$$

$$(y_0, ..., y_L, z_0, ..., z_M) \mapsto (y_0, ..., y_{L-1}, z_0, ..., z_{l-1}, z_{l+1}, ..., z_{M-1})$$

In other words, $\delta_{\cdot,m}$ deletes the z_m term.

Proposition 4.4. *[10] $\delta_{l,\cdot}$ are a s-bijective factor maps and $\delta_{\cdot,m}$ is a u-bijective factor map.*

In essence, this allows us to capture the unstable and stable behavior of varying degrees of our Smale space in the shift space $(\Sigma_{L,M}, \sigma)$. Namely, the spaces $(\Sigma_{L,M}, \sigma)$ capture where the maps π_s and π_u are not one-to-one. Let us summarize some of the crucial results to construct Putnam's homology theory.

Suppose (X, f) is a Smale space and

$$\pi_s(Y, g) \to (X, f) \quad \pi_u : (Z, h) \to (X, f)$$

is a s/u-bijective pair. Then we have $Y^s(y)$ and $Z^u(z)$ totally disconnected for every $y \in Y$ and $z \in Z$. We also have shifts of finite types $(\Sigma_{L,M}, \sigma)$ according to Definition 4.1. Since each $(\Sigma_{L,M}, \sigma)$ is a shift of finite type, there are dimension groups $D^s(\Sigma_{L,M}, \sigma)$ and $D^u(\Sigma_{L,M}, \sigma)$ associated to these shift spaces. Lasting we have the maps $\delta_{l,\cdot}$ and $\delta_{\cdot,m}$ that deletes the y_l or z_m element respectively. By Proposition 4.4, $\delta_{l,\cdot}$ and $\delta_{\cdot,m}$ are s-bijective and u-bijective factor maps and by Theorem 3.38, these induce natural homomorphisms between dimension groups. Given these natural homomorphisms, we can define the double chain complex.

Definition 4.5 (Double Chain Complex For Smale Spaces)**.** Let (X, f) be a Smale space and let $(Y, f, \pi_s, Z, h, \pi_u)$ be an s/u-bijective pair. Then the double chain complex is defined as the following chain complex with δ maps corresponding to the s/u-bijective pair.

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle \delta^{s*}_{\cdot,m}} & & \uparrow{\scriptstyle \delta^{s*}_{\cdot,m}} & & \uparrow{\scriptstyle \delta^{s*}_{\cdot,m}} & & \\
0 & \longleftarrow & D^s(\Sigma_{0,2}) & \xleftarrow[\delta^s_{l,\cdot}]{} & D^s(\Sigma_{1,2}) & \xleftarrow[\delta^s_{l,\cdot}]{} & D^s(\Sigma_{2,2}) & \xleftarrow[\delta^s_{l,\cdot}]{} & \cdots \\
 & & \uparrow{\scriptstyle \delta^{s*}_{\cdot,m}} & & \uparrow{\scriptstyle \delta^{s*}_{\cdot,m}} & & \uparrow{\scriptstyle \delta^{s*}_{\cdot,m}} & & \\
0 & \longleftarrow & D^s(\Sigma_{0,1}) & \xleftarrow[\delta^s_{l,\cdot}]{} & D^s(\Sigma_{1,1}) & \xleftarrow[\delta^s_{l,\cdot}]{} & D^s(\Sigma_{2,1}) & \xleftarrow[\delta^s_{l,\cdot}]{} & \cdots \\
 & & \uparrow{\scriptstyle \delta^{s*}_{\cdot,m}} & & \uparrow{\scriptstyle \delta^{s*}_{\cdot,m}} & & \uparrow{\scriptstyle \delta^{s*}_{\cdot,m}} & & \\
0 & \longleftarrow & D^s(\Sigma_{0,0}) & \xleftarrow[\delta^s_{l,\cdot}]{} & D^s(\Sigma_{1,0}) & \xleftarrow[\delta^s_{l,\cdot}]{} & D^s(\Sigma_{2,0}) & \xleftarrow[\delta^s_{l,\cdot}]{} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Or likewise for the unstable double cell complex

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \delta^{u}_{\cdot,m}} & & \downarrow{\scriptstyle \delta^{u}_{\cdot,m}} & & \downarrow{\scriptstyle \delta^{u}_{\cdot,m}} & & \\
0 & \longrightarrow & D^u(\Sigma_{0,2}) & \xrightarrow[\delta^{u*}_{l,\cdot}]{} & D^u(\Sigma_{1,2}) & \xrightarrow[\delta^{u*}_{l,\cdot}]{} & D^u(\Sigma_{2,2}) & \xrightarrow[\delta^{u*}_{l,\cdot}]{} & \cdots \\
 & & \downarrow{\scriptstyle \delta^{u}_{\cdot,m}} & & \downarrow{\scriptstyle \delta^{u}_{\cdot,m}} & & \downarrow{\scriptstyle \delta^{u}_{\cdot,m}} & & \\
0 & \longrightarrow & D^u(\Sigma_{0,1}) & \xrightarrow[\delta^{u*}_{l,\cdot}]{} & D^u(\Sigma_{1,1}) & \xrightarrow[\delta^{u*}_{l,\cdot}]{} & D^u(\Sigma_{2,1}) & \xrightarrow[\delta^{u*}_{l,\cdot}]{} & \cdots \\
 & & \downarrow{\scriptstyle \delta^{u}_{\cdot,m}} & & \downarrow{\scriptstyle \delta^{u}_{\cdot,m}} & & \downarrow{\scriptstyle \delta^{u}_{\cdot,m}} & & \\
0 & \longrightarrow & D^u(\Sigma_{0,0}) & \xrightarrow[\delta^{u*}_{l,\cdot}]{} & D^u(\Sigma_{1,0}) & \xrightarrow[\delta^{u*}_{l,\cdot}]{} & D^u(\Sigma_{2,0}) & \xrightarrow[\delta^{u*}_{l,\cdot}]{} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Given this $\mathbb{Z} \times \mathbb{Z}$-graded double chain complex, there is a natural way way from homological algebra to create a $\mathbb{Z}$-graded chain complex known as the total complex.

Definition 4.6 (Homology of Smale Spaces)**.** Let (X, f) be a Smale space and let $(Y, f, \pi_s, Z, h, \pi_u)$ be an s/u-bijective pair. Using the double chain complex, define the total stable and unstable complexes as

$$C^s_n(X, f) = \bigoplus_{l-m=n} D^s(\Sigma_{l,m}, \sigma)$$

$$C^u_n(X, f) = \bigoplus_{l-m=n} D^u(\Sigma_{l,m}, \sigma)$$

The boundary maps are defined as follows. Let

$$\partial^s_n : C^s_n(X, f) \to C^s_{n-1}(X, f)$$

$$\partial^u_n : C^s_n(X, f) \to C^u_{n-1}(X, f)$$

defined via

$$\partial_n^s = \bigoplus_{l-m=n} d_{l,m}^s \quad \partial_n^u = \bigoplus_{l-m=n} d_{l,m}^u$$

Where

$$d_{l,m}^s : D^s(\Sigma_{l,m}) \to D^s(\Sigma_{l-1,m}) \oplus D^s(\Sigma_{l,m+1})$$
$$d_{l,m}^u : D^u(\Sigma_{l,m}) \to D^u(\Sigma_{l+1,m}) \oplus D^u(\Sigma_{l,m-1})$$

are defined as

$$d_{l,m}^s = \sum_{i=0}^{l} (-1)^i \delta_{i,\cdot}^s + \sum_{i=0}^{m+1} (-1)^{l+i} \delta_{\cdot,i}^{s*}$$
$$d_{l,m}^u = \sum_{i=0}^{m} (-1)^i \delta_{\cdot,i}^u + \sum_{i=0}^{l+1} (-1)^{m+i} \delta_{i,\cdot}^{s*}$$

This gives us a chain complexes

$$\dots \longleftarrow C_{n-1}^u(X,f) \xleftarrow{\partial_n^u} C_n^u(X,f) \xleftarrow{\partial_{n+1}^u} C_{n+1}^u(X,f) \longleftarrow \dots$$

Finally, the homology group is defined via

$$H_n^s(X,f) = \ker \partial_n^s / \operatorname{im} \partial_{n-1}^s$$
$$H_n^u(X,f) = \ker \partial_n^u / \operatorname{im} \partial_{n-1}^u$$

4.2 Properties of the Homology of Smale Spaces

At first, it might seem like the Homology seems like an arbitrary graded abelian group given a Smale space. In particular, it seems to depend on the choice of s/u-bijective pair. However, this invariant gives a plethora of useful properties about the dynamics of the system. We will avoid going into proving these facts as they can get incredibly technical. The details of many of these theorem can be found in Putnam's Memoir[10]. The first major result being that despite the fact that a s/u-bijective pair is required in the definition,

Theorem 4.7. *The homology is independent of choice of s/u-bijective pair*

Second off, there is is a question of functorality. Once again, just like the dimension group. There are induced homomorphisms between homology groups for *certain* maps. Namely, the u-bijective and s-bijective ones. These induced homomorphisms also demonstrate a functorial property.

Theorem 4.8. *Let (X,f) and (X',f') be non-wandering Smale spaces and let $\pi : (X,f) \to (X',f')$ be a factor map. Then we get the following*

1. *If π is s-bijective, there is a natural homomorphism*

$$\pi_*^s : H_*^s(X,f) \to H_*^s(X',f')$$

2. *If π is u-bijective, there is a natural homomorphism*

$$\pi_*^u : H_*^u(X,f) \to H_*^u(X',f')$$

3. *If π is s-bijective, there is a natural homomorphism*

$$\pi_*^{u*} : H_*^u(X', f') \to H_*^u(X, f)$$

4. *If π is u-bijective, there is a natural homomorphism*

$$\pi_*^{u*} : H_*^s(X', f') \to H_*^s(X, f)$$

5. id *induces the identity isomorphisms*

$$\mathrm{id}_*^u : H_*^u(X, f) \to H_*^u(X', f') \quad \mathrm{id}_*^{u*} : H_*^s(X', f') \to H_*^s(X, f)$$

$$\mathrm{id}_*^s : H_*^s(X, f) \to H_*^s(X', f') \quad \mathrm{id}_*^{s*} : H_*^u(X', f') \to H_*^u(X, f)$$

6. *If $\pi_1 : X_1 \to X_2$ and $\pi_2 : X_2 \to X_3$ are factor maps, then*

   (a) *If π_1 and π_2 are s-bijective*

$$(\pi_2 \circ \pi_1)_*^s = (\pi_2)_*^s \circ (\pi_1)_*^s \quad (\pi_2 \circ \pi_1)^{s^*} = (\pi_1)_*^{s*} \circ (\pi_2)_*^{s*}$$

   (b) *If π_1 and π_2 are u-bijective*

$$(\pi_2 \circ \pi_1)_*^u = (\pi_2)_*^u \circ (\pi_1)_*^u \quad (\pi_2 \circ \pi_1)_*^{u^*} = (\pi_1)_*^{u*} \circ (\pi_2)_*^{u*}$$

Once again, this functorality property implies that

Corollary 4.8.1. *If the non-wandering Smale spaces (X, f) and (X', f') are topologically conjugate, then $H^s(X, f) \cong H^s(X', f')$ and $H^u(X, f) \cong H^u(X', f')$*

Theorem 4.8 hints that this homology is the natural generalization of the dimension group. We can take this one step further by consider the homology of a shift of finite type.

Theorem 4.9. *Let (Σ, σ) be a shift a finite type. Then the homology of (Σ, σ) is*

$$H_n^s(\Sigma, \sigma) = \begin{cases} D^s(\Sigma, \sigma) & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad H_n^u(\Sigma, \sigma) = \begin{cases} D^u(\Sigma, \sigma) & n = 0 \\ 0 & \text{otherwise} \end{cases}$$

There are some more important properties to note. One being that there are only finitely many non-zero entries, secondly, each group has finite rank, finally disjoint unions work as expected

Theorem 4.10. *Let (X, f) be a non-wandering Smale space*

1. *There exists a M_0 and L_0 such that $H_n^s(X, f) = 0$ if $n \geq L_0$ or $n \leq M_0$ and $H_n^u(X, f) = 0$ if $n \geq M_0$ or $n \leq L_0$.*

2. *Each of $H_n^s(X, f)$ and $H_n^u(X, f)$ have finite rank. That is, $H_n^s(X, f) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $H_n^u(X, f) \otimes_{\mathbb{Z}} \mathbb{Q}$ are finite dimensional $\mathbb{Q}$-vector spaces.*

3. *Suppose (X, f) and (X', f') are non-wandering Smale spaces. Then the Smale space $(X \sqcup X', f \sqcup f')$ has the homology*

$$H_*^u(X \sqcup X', f \sqcup f') \cong H_*^u(X, f) \oplus H_*^s(X, f)$$

$$H_*^s(X \sqcup X', f \sqcup f') \cong H_*^s(X, f) \oplus H_*^s(X, f)$$

The last incredible property of this homology is that it directly gives us the number of points of period n. For a Smale space (X, f), we do this by considering the finite dimensional $\mathbb{Q}$-vector spaces $H_n^s(X, f) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then, we have linear maps defined by $f_n^{s^*} \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}}$. These linear maps each have a trace which we can take the alternating sum of. This gives us the following formula due to Putnam:

Theorem 4.11 (Lefschetz formula). *Let (X, f) be a non-wandering Smale Space. Then*

$$\begin{aligned}
|\mathrm{Per}_n(X, f)| &= \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{Tr}\Big(\big((f^{-1})_k^s \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}}\big)^n\Big) \\
&= \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{Tr}\Big(\big(f_k^u \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}}\big)^n\Big) \\
&= \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{Tr}\Big(\big(f_k^{s*} \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}}\big)^n\Big) \\
&= \sum_{k\in\mathbb{Z}} (-1)^k \operatorname{Tr}\Big(\big((f^{-1})_k^{u*} \otimes_{\mathbb{Z}} \mathrm{id}_{\mathbb{Q}}\big)^n\Big)
\end{aligned}$$

4.3 Computing the Homology of Smale Spaces

The theorem from the previous section give fundamental properties of the homology theory and what information we can extract from the Smale space using this theory. However, there still lies a major issue. Computing these groups from the definition alone is extremely involved and can be found in the examples of Putnam's original paper. Developing methods to compute the homology effectively is an active area of research.

Proietti and Yamashita were able to prove a theorem on the homology analogous to the Künneth theorem in [9]

Theorem 4.12. *Let (X, f) and (Y, g) be non-wandering Smale spaces. There exists a split exact sequence*

$$\begin{array}{c}
0 \\
\downarrow \\
\displaystyle\bigoplus_{a+b=k} H_a^s(X, f) \otimes_{\mathbb{Z}} H_b^s(Y, g) \\
\downarrow \\
H_k^s(X \times Y, f \times g) \\
\downarrow \\
\displaystyle\bigoplus_{a+b=k-1} \operatorname{Tor}\Big(H_a^s(X, f), H_b^s(Y, g)\Big) \\
\downarrow \\
0
\end{array}$$

We will now work to build a arsenal of examples so we can utilize the prior theorems. Our focus will be on manifolds and inverse limits since they have properties that allow us to exploit the (co)homology of the manifold itself or the space we are taking the inverse limit in order to compute Putnam's homology.

4.3.1 Anosov Diffeomorphisms

Suppose our non-wandering Smale space was a manifold and a diffeomorphism. Recall that this scenario is formally known as an Anosov diffeomorphism. We had this precise example in our motivating example, the hyperbolic toral automorphism. Proietti was able to draw a connection between the cohomology of the manifold as a topological space and Putnam's homology in [9]

Theorem 4.13. *Let (M, f) be an Anosov Diffeomorphism satifying the orientability condition (See Theorem 3.1 in [9]). If $M^u(x) \cong \mathbb{R}^n$ for any $x \in M$ for some fixed n, then*

$$H^s_*(M, f) \cong H^{n-*}(M)$$

Likewise, if If $M^s(x) \cong \mathbb{R}^m$ for any $x \in M$ for some fixed m, then

$$H^u_*(M, f) \cong H^{m-*}(M)$$

Example 4.14. Let (T^2, A) be the hyperbolic toral automorphism. We have $X^u(x) \cong \mathbb{R}$. Thus, Putnam's homology can easily be computed since we know that

$$H^*(T^2) = \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}^2 & * = 1 \\ \mathbb{Z} & * = 2 \\ 0 & \text{otherwise} \end{cases}$$

It follows that

$$H^s_*(T^2, A) \cong H^{1-*}(T^2) \cong \begin{cases} \mathbb{Z} & * = -1 \\ \mathbb{Z}^2 & * = 0 \\ \mathbb{Z} & * = 1 \\ 0 & \text{otherwise} \end{cases}$$

4.3.2 Solenoids Over Flat Manifolds

Let M be a closed compact flat Riemannian manifold. Suppose that $g : M \to M$ is an expanding endomorphism on M. That is, for some (and hence any) Riemannian metric on M, there exists $c > 0, \lambda > 1$ such that for any $v \in TM$, $c\lambda^k \|v\| \le \|(Dg^k)\, v)\|$.

Notice that $g : M \to M$ satisfies the conditions of Theorem 2.16 and we can construct a Smale Space

$$(X, f) = \left(\varprojlim(M, g), g^{\mathbb{N}}\right) \tag{6}$$

This Smale Space is particularly nice because we can relate the homology and cohomology of M as a topological space to compute Putnam's homology for the Smale Space (X, f). We did precisely that in a CU Boulder's 2021 Math REU with the results in [4].

Let $F : M \to N$ be an n-fold cover between closed manifolds. There are induced maps on their (co)homologies which we will label

$$F_* : H_*(M) \to H_*(N) \quad F^* : H^*(N) \to H^*(M)$$

Likewise, there are transfer maps associated to the homology and cohomology which we will label

$$\widetilde{F}_{\text{homology}} : H_*(N) \to H_*(M) \quad \widetilde{F}_{\text{cohomology}} : H^*(M) \to H^*(N)$$

One can read more about transfer maps in [12], but the main property we will use is the following proposition in the scenario where $N = M$.

Proposition 4.15. *Let $F : M \to N$ be an n-fold cover between closed manifolds. Then $\widetilde{F}_{\text{cohomology}} \circ F^* : H^*(M) \to H^*(M)$ is multiplication by n and $F_* \circ \widetilde{F}_{\text{homology}} : H^*(N) \to H^*(N)$ is also multiplication by n.*

The main theorem of [4] gives us a much easier way to compute Putnam's homology using the topology of the flat manifold that relies on the transfer map:

Theorem 4.16. *Let $g : M \to M$ an expanding endomorphism on a closed compact flat Riemannian manifold and (X, f) be the Smale space defined in (6). Then*

$$H^s_*(X, f) \cong \varinjlim(H^*(M), \widetilde{g}_{\text{cohomology}})$$
$$H^u_*(X, f) \cong \varinjlim(H_*(M), \widetilde{g}_{\text{homology}})$$

This gives us some immediate consequences that are quite easy to prove.

Corollary 4.16.1. *Let $g : M \to M$ an expanding endomorphism on a closed compact flat Riemannian manifold and (X, f) be the Smale space defined in (6). Moreover suppose that M is connected. Then*

$$H^s_0(X, f) \cong H^u_0(X, f) \cong \mathbb{Z}\left[\frac{1}{n}\right]$$

$$H^u_{\dim M}(X, f) \cong \begin{cases} \mathbb{Z} & M \text{ is orientable} \\ 0 & M \text{ is not orientable} \end{cases}$$

$$H^s_{\dim M}(X, f) \cong \begin{cases} \mathbb{Z} & M \text{ is orientable} \\ \mathbb{Z}/2\mathbb{Z} \text{ or } 0 & M \text{ is not orientable} \end{cases}$$

Proof. Since M is connected $H^0(M) \cong H_0(M) \cong \mathbb{Z}$ and both g_* and g^* are the identity. Thus, by Proposition 4.15, $\widetilde{g}_{\text{cohomology}} \circ g^0 = \widetilde{g}_{\text{cohomology}}$ and $g_0 \circ \widetilde{g}_{\text{homology}} = \widetilde{g}_{\text{homology}}$ are both multiplication by n. Thus, by Theorem 4.16, we get

$$H^s_0(X, f) \cong \varinjlim(H^0(M), \widetilde{g}_{\text{cohomology}}) \cong \varinjlim(\mathbb{Z}, - \times n) \cong \mathbb{Z}\left[\frac{1}{n}\right]$$

Likewise,

$$H^u_0(X, f) \cong \varinjlim(H_0(M), \widetilde{g}_{\text{homology}}) \cong \varinjlim(\mathbb{Z}, - \times n) \cong \mathbb{Z}\left[\frac{1}{n}\right]$$

Which is the first result we wanted to show.

Now, suppose that M is not orientable, then $H_{\dim M}(M) \cong \{0\}$ then our results follows for homology follows. If $H^{\dim M}(M) \cong \{0\}$, then once again the result follows. If $H^{\dim M}(M) \cong \mathbb{Z}/2\mathbb{Z}$, then then $\widetilde{g}_{\text{cohomology}}$ is either the zero map or identity. This gives us

$$\varinjlim(H^{\dim M}(M), \widetilde{g}_{\text{cohomology}}) = 0 \quad \text{if } \widetilde{g}_{\text{cohomology}} \text{ is the zero map}$$
$$\varinjlim(H^{\dim M}(M), \widetilde{g}_{\text{cohomology}}) = 2/2\mathbb{Z} \quad \text{if } \widetilde{g}_{\text{cohomology}} \text{ is the identity}$$

Now consider the case where M is orientable, then $H^{\dim M}(M) \cong H_{\dim M}(M) \cong \mathbb{Z}$ and the induced homomorphisms $g_{\dim M}$ and $g^{\dim M}$ are multiplication by n since g is an n-fold covering map. By the same arguments for the zeroth homology, we get the transfer maps being the identity and

$$H^s_{\dim M}(X,f) \cong H^u_{\dim M}(X,f) \cong \varinjlim(\mathbb{Z}, \mathrm{id}) \cong \mathbb{Z}$$

□

Let us demonstrate the power of these theorems with a few computations.

Example 4.17. Let $S^1 = \mathbb{R}/\mathbb{Z}$ be the circle and (S_n, f) be the Smale space associated to the n-fold cover $[z]_{\mathbb{Z}} \mapsto [nz]_{\mathbb{Z}}$ for $n \geq 2$. Recall that

$$H_*(S^1) \cong H^*(S^1) \cong \begin{cases} \mathbb{Z} & * = 0,1 \\ \{0\} & \text{otherwise} \end{cases}$$

It follows that $H^s_k(S_n, f) \cong H^u_k(S_n, f) = 0$ for $k \neq 1, 0$ from having trivial homologies. Likewise, it follows immediately from Corollary 4.16.1

$$H^s_*(S_n, f) \cong H^u_*(S_n, f) \cong \begin{cases} \mathbb{Z}\left[\frac{1}{n}\right] & * = 0 \\ \mathbb{Z} & * = 1 \\ \{0\} & \text{otherwise} \end{cases}$$

since S^1 is connected and 1-dimensional.

Example 4.18. Let us consider a more complicated example. Let $K = \mathbb{R}^2/\sim$ be the Klein bottle. Consider 9-fold covering endomorphism induced by the matrix

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

On $\mathbb{R}^2$ which induce an expanding endomorphsim on the quotient space that can be visualized as the following in Figure 15

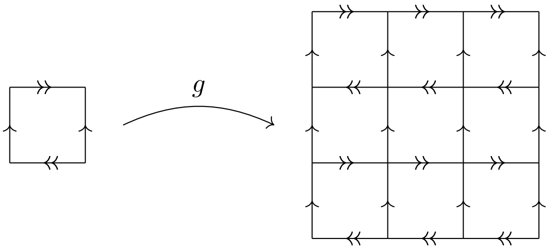


Figure 15: 9-fold-cover on the Klein bottle

Let (K, f) be the associated Smale space. Now recall some topological facts

$$H_*(K) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & * = 1 \\ 0 & \text{otherwise} \end{cases} \qquad H^*(K) \cong \begin{cases} \mathbb{Z} & * = 0,1 \\ \mathbb{Z}/2\mathbb{Z} & * = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$g_* = \begin{cases} \mathrm{id} & * = 0 \\ - \times 3 & * = 1 \end{cases} \quad g^* = \begin{cases} \mathrm{id} & * = 0,2 \\ - \times 3 & * = 1 \end{cases}$$

Of course, once again by Proposition 4.15, we get

$$\widetilde{g}_{\text{homology}} = \begin{cases} - \times 9 & * = 0 \\ - \times 3 & * = 1 \end{cases} \quad \widetilde{g}_{\text{cohomology}} = \begin{cases} - \times 9 & * = 0, 2 \\ - \times 3 & * = 1 \end{cases}$$

Of course, multiplying by an odd number is simply the identity on $\mathbb{Z}/2\mathbb{Z}$. This gives us

$$H^s(K, f) = \varinjlim (H^*(K), \widetilde{f}_{\text{cohomology}}) \cong \begin{cases} \mathbb{Z}\left[\frac{1}{9}\right] & * = 0 \\ \mathbb{Z}\left[\frac{1}{3}\right] & * = 1 \\ \mathbb{Z}/2\mathbb{Z} & * = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$H^u(K, f) = \varinjlim (H_*(K), \widetilde{f}_{\text{homology}}) \cong \begin{cases} \mathbb{Z}\left[\frac{1}{9}\right] & * = 0 \\ \mathbb{Z}\left[\frac{1}{3}\right] \oplus \mathbb{Z}/2\mathbb{Z} & * = 1 \\ 0 & \text{otherwise} \end{cases}$$

Also, note that $\mathbb{Z}\left[\frac{1}{9}\right] \cong \mathbb{Z}\left[\frac{1}{3}\right]$. This gives us our final result.

$$H^s(K, f) \cong \begin{cases} \mathbb{Z}\left[\frac{1}{3}\right] & * = 0, 1 \\ \mathbb{Z}/2\mathbb{Z} & * = 2 \\ 0 & \text{otherwise} \end{cases} \quad H^u(K, f) \cong \begin{cases} \mathbb{Z}\left[\frac{1}{3}\right] & * = 0 \\ \mathbb{Z}\left[\frac{1}{3}\right] \oplus \mathbb{Z}/2\mathbb{Z} & * = 1 \\ 0 & \text{otherwise} \end{cases}$$

4.3.3 p/q Solenoid

To continue with the theme of inverse limits and exploiting the (co)homology of the original topological space. We will now consider an example where we take two successive inverse limits. First, we will note a theorem from algebraic topology that follows from the countuity of Čech cohomology on paracompact spaces.

Theorem 4.19. *If $g : Y \to Y$ is a locally expansive homeomorphism on a metric space then*

$$H^*\left(\varprojlim (Y, g)\right) \cong \varinjlim (H^*(Y), g^*)$$

Example 4.20. Consider the n-solenoid once again. Then, we have $g^0 = \text{id}$ and $g^1 = - \times n$ which of course gives

$$H^*(S_n) \cong \begin{cases} \mathbb{Z} & * = 0 \\ \mathbb{Z}\left[\frac{1}{n}\right] & * = 1 \\ 0 & \text{otherwise} \end{cases}$$

We will now construct and consider Putnam's homology of the p/q-solenoid. Let p and q be coprime positive integers greater than 1. Let $g : S^1 \to S^1$ be defined as

$$g([z]_{\mathbb{Z}}) = g([qz]_{\mathbb{Z}})$$

We have our usual construction of the q-solenoid Smale Space

$$(S_q, \varphi) = \left(\varprojlim (S^1, g), g^{\mathbb{N}}\right)$$

Now, we can define another locally expansive homeomorphism on S_q defined as $\gamma : S_q \to S_q$

$$([z_0]_{\mathbb{Z}}, [z_1]_{\mathbb{Z}}, [z_2]_{\mathbb{Z}}, \dots) \overset{\gamma}{\longmapsto} ([pz_1]_{\mathbb{Z}}, [pz_2]_{\mathbb{Z}}, [pz_3]_{\mathbb{Z}}, \dots)$$

Which is a p-fold-covering map. Moreover, γ is a locally expansive homeomorphism on S_q, and satisfies Wieler's criterion in Theorem 2.16. Thus, we get another Smale space

$$(S_{p/q}, f) = \left(\left(\varprojlim(S_q, \gamma)\right), \gamma^{\mathbb{N}}\right)$$

We call this the p/q-solenoid. We will not prove it here, but the statement of Proposition 4.15 and Theorem 4.16 can be generalized for $f : S_{p/q} \to S_{p/q}$. Hence, we can use the same method to for computing Putnam's homology. Thus, we once again have

$$\widetilde{\gamma}_{\text{cohomology}} \circ \gamma^* = - \times p$$

The induced homomorphisms is given by

$$\gamma^* = \begin{cases} \text{id} & * = 0 \\ \times p/q & * = 1 \end{cases}$$

Since p and q are coprime, this forces the transfer maps to be

$$\widetilde{\gamma}_{\text{cohomology}} = \begin{cases} \times p & * = 0 \\ \times q & * = 1 \end{cases}$$

Thus,

$$H_0^s(S_{p/q}, f) \cong \varinjlim(H^0(S_q), \widetilde{\gamma}_{\text{cohomology}}) \cong \varinjlim(\mathbb{Z}, - \times p) \cong \mathbb{Z}\left[\frac{1}{p}\right]$$

$$H_1^s(S_{p/q}, f) \cong \varinjlim(H^1(S_q), \widetilde{\gamma}_{\text{cohomology}}) \cong \varinjlim\left(\mathbb{Z}\left[\frac{1}{q}\right], - \times q\right) \cong \mathbb{Z}\left[\frac{1}{q}\right]$$

Giving us

$$H_*^s(S_{p/q}, f) \cong \begin{cases} \mathbb{Z}\left[\frac{1}{p}\right] & * = 0 \\ \mathbb{Z}\left[\frac{1}{q}\right] & * = 1 \\ 0 & \text{otherwise} \end{cases}$$

A significantly more involved computation can be found in [3] that relies on constructing a Markov partition.

Appendix A Inverse and Inductive Limits

Definition A.1 (Inductive/Direct Limit)**.** Suppose we had a directed set $(I, \leq)$ and a family of sets $\{A_i\}_{i \in I}$ with an algebraic structure (Groups, Modules, Rings, ect). Also suppose there existed a family of homomorphisms $\{f_{ij} : A_i \to A_j : i, j \in I, i \leq j\}$ such that

(1) $f_{ii} = \text{id}_{A_i}$

(2) $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$

Then the direct limit is defined as

$$\varinjlim A_i = \bigsqcup_{i \in I} A_i / \sim$$

Where $\sim$ is the equivalence relation

$$x_i \sim x_j \iff \text{there exists } k \geq i, j \text{ such that } f_{ik}(x_i) = f_{jk}(x_j)$$

Proposition A.2. *Let ι_j be the canonical functions*

$$\iota_j : A_j \to \varinjlim A_j \quad x \mapsto [x]$$

There exists an unique group structure on $\lim_{\longrightarrow} A_i$ such that ι_j are homomorphisms.

Proposition A.3. *If $0 \to A_i \to B_i \to C_i \to 0$ is a short exact sequence, then so is*

$$0 \to \varinjlim A_i \to \varinjlim B_i \to \varinjlim C_i \to 0$$

Definition A.4 (Inverse/Projective Limit)**.** Suppose we had a directed set $(I, \leq)$ and a family of sets $\{A_i\}_{i \in I}$ with an algebraic structure (Groups, Modules, Rings, ect). Also suppose there existed a family of homomorphisms $\{f_{ij} : A_j \to A_i : i, j \in I, i \leq j\}$ (note the order of the homomorphisms are flipped) such that

(1) $f_{ii} = \mathrm{id}_{A_i}$

(2) $f_{ki} = f_{ij} \circ f_{jk}$ for all $i \leq j \leq k$

We define the inverse/projective limit as

$$\varprojlim A_i = \left\{ a \in \prod_{i \in I} A_i : a_i = f_{ij}(a_j) \text{ for all } i \leq j \right\}$$

Proposition A.5. *The inverse limit is a subgroup/module/ring of the direct product. In other words*

$$\varprojlim A_i \leq \prod_{i \in I} A_i$$

Proposition A.6. *The inductive and inverse limits relate to one another by the property*

$$\mathrm{Hom}\left(\varinjlim X_i, Y\right) = \varprojlim \mathrm{Hom}\left(X_i, Y\right)$$

Remark. There is a special case of the inductive/inverse limit. Suppose, we had single set A with an endomorphism $f : A \to A$. We can defined the family $\{A_i\}_{i \in \mathbb{N}}$ such that $A_i = A$. Then we construct an inverse/inductive limit by defining the families of homomorphisms $f_{ij} : A \to A$ as $f_{ij} = f^{j-i}$ for the inductive limit or conversely $f_{ij} = f^{i-j}$ for the inverse limit. Since we use the special case so much, we will use a separate notation

$$\varinjlim(A, f) = \bigsqcup_{n \in \mathbb{N}} A / \sim$$

Where $a \sim b$ if there exists some positive integers n, m such that $f^n(a) = f^m(b)$. Moreover, we can denote equilivent relations as $[a, n]$ for $a \in A$ and $n \in \mathbb{N}$.

$$\varprojlim(A, f) = \left\{ a \in A^{\mathbb{N}} : a_i = f^n(a_j) \text{ for some } n, \text{ for all } i \leq j \right\}$$

References

[1] Rufus Bowen. *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, volume 470 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, revised edition, 2008. With a preface by David Ruelle, Edited by Jean-René Chazottes.

[2] Mike Boyle, Brian Marcus, and Paul Trow. *Resolving maps and the dimension group for shifts of finite type*. Memoirs of the American Mathematical Society. American Mathematical Society, Providence, RI, December 1987.

[3] NIGEL D. BURKE and IAN F. PUTNAM. Markov partitions and homology for n/m-solenoids. *Ergodic Theory and Dynamical Systems*, 37(3):716–738, 05 2017. Copyright - © Cambridge University Press, 2015; Last updated - 2017-04-04.

[4] Rachel Chaiser, Maeve Coates-Welsh, Robin J. Deeley, Annika Farhner, Jamal Giornozi, Robi Huq, Levi Lorenzo, Jose Oyola-Cortes, Maggie Reardon, and Andrew M. Stocker. Invariants for the smale space associated to an expanding endomorphism of a flat manifold. *Münster Journal of Mathematics*, 16(1):177–199, 2023.

[5] David Fried. Finitely presented dynamical systems. *Ergodic Theory and Dynamical Systems*, 7(4):489–507, 1987.

[6] KOICHI HIRAIDE. A simple proof of the franks–newhouse theorem on codimension-one anosov diffeomorphisms. *Ergodic Theory and Dynamical Systems*, 21(3):801–806, 2001.

[7] Wolfgang Krieger. On dimension functions and topological Markov chains. *Invent. Math.*, 56(3):239–250, 1980.

[8] Douglas Lind and Brian Marcus. *An introduction to symbolic dynamics and coding*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 2021. Second edition [of 1369092].

[9] Valerio Proietti and Makoto Yamashita. Homology and k-theory of dynamical systems. iii. beyond totally disconnected case, 2022.

[10] Ian F. Putnam. A homology theory for Smale spaces. *Mem. Amer. Math. Soc.*, 232(1094):viii+122, 2014.

[11] Ian F. Putnam. Lecture notes on smale spaces, July 2015.

[12] Fred W. Roush. *Transfer in generalized cohomology theories*. Pure and Applied Mathematics (Budapest). Akadémiai Kiadó, Budapest, 1999.

[13] David Ruelle. *Thermodynamic Formalism: The Mathematical Structure of Equilibrium Statistical Mechanics*. Cambridge Mathematical Library. Cambridge University Press, 2 edition, 2004.

[14] S. Smale. Differentiable dynamical systems. *Bull. Amer. Math. Soc.*, 73:747–817, 1967.

[15] Susana Wieler. Smale spaces via inverse limits. *Ergodic Theory Dynam. Systems*, 34(6):2066–2092, 2014.